CONTROL THEORY AND IRRATIONALITY OF DIRICHLET SERIES

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ABSTRACT. In this paper we present some results on the irrationality of certain series, including some Dirichlet series from a control theory viewpoint.

1. INTRODUCTION

The aim of this paper is to analyse some irrationality problems arising from certain convergent series from a control theory viewpoint.

The common way to prove the irrationality of a certain number is to provide some rational approximations that simultaneously are good approximations and with relatively small denominators. In a certain sense a number can be proved to be irrational when rational approximations with sufficiently small denominators exist.

The motivation of this paper is to provide a control theory viewpoint in this sense: starting from a partial sum of a series (a state), so just a finite number of beginning terms of it, we provide a way to control the denominators of certain rational approximations, by suitably acting on the subsequent coefficients of the series.

One of the main problems in number theory is given a sequence a_n for $n = 1, 2, \dots$, and $s \in \mathbb{C}$ is the number

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

an irrational number ? Series of this form are called Dirichlet series. When $a_n = 1$ for every n the above series is the Riemann zeta function, and the sum of the series have been proved irrational for s even and for $s = 3$, altough is conjectured to be irrational for every integer $s \geq 2$.

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For $a_n = (-1)^{n+1}$ and $s = 1$ the series converges to $\log 2$, which is an irrational number. Other examples of sequence a_n correspond to other well known values of the corresponding Dirichlet series.

In this paper we analyse some of these examples and we will mainly interested in the irrationality of Dirichlet series with $|a_n| = 1$.

2. Control theory and convergence as reachability

Theorem 2.1. *Let $s = 2$. For every $\alpha \in (2 - \zeta(2), \zeta(2))$, there exist a sequence $\{a_n\}_{n\in\mathbb{N}}$ with $a_n = 1$ or -1 and such that*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} = \alpha.$$

Proof. Let $a_1 = 1$. For $n \geq 2$, define by recurrence:

$$a_n = \begin{cases} 1 & \text{if } \displaystyle\sum_{k=1}^{n-1} \frac{a_k}{k^2} < \alpha \\ \\ -1 & \text{else} \end{cases}$$

It easy to prove that the above sequence verifies the theorem. Note that:

$$(2 - \zeta(2), \zeta(2)) = (1 - \sum_{k=2}^{\infty} \frac{1}{k^2}, 1 + \sum_{k=2}^{\infty} \frac{1}{k^2}).$$

The theorem is a consequence of the fact that for every $n \geq 2$ one has:

$$\frac{1}{n^2} < \sum_{k=n+1}^{\infty} \frac{1}{k^2}. \tag{1}$$

To prove prove (1) let

$$f(n) = \frac{1}{n^2} - \sum_{k=n+1}^{\infty} \frac{1}{k^2}.$$

We will prove that $f(n) < 0$ for $n \geq 2$. We have

$$f(2) = 1/4 - \sum_{k=3}^{\infty} \frac{1}{k^2} = \frac{3}{2} - \zeta(2) \simeq -0.145.$$

Moreover:

$$f(3) = 1/9 - \sum_{k=4}^{\infty} \frac{1}{k^2} = \frac{5}{4} + \frac{2}{9} - \zeta(2) \simeq -0.173.$$

For $n \geq 3$ we have:

$$f(n+1) - f(n) = \frac{2n^2 - (n+1)^2}{n^2(n+1)^2} > 0.$$

So for $n \geq 3$ the function $f(n)$ is increasing and its limit is 0. Since $f(3) < 0$, this proves that $f(n) < 0$ for every $n \geq 3$.

□

The precedent theorem can be stated in terms of the control theory. Classical notation from the theory of control is used (see for exemple [1]). Let the system $\Sigma = (\mathcal{T}, \mathcal{X}, \mathcal{U}, \phi)$ be defined by

(i) the time set $\mathcal{T} = \mathbb{N} \cup \{\infty\}$
(ii) the state space $\mathcal{X} =]2 - \zeta(2), \zeta(2)[$
(iii) the input-value space $\mathcal{U} = \{-1, +1\}$
(iv) the transition map ϕ is defined on the subset of

$$\mathcal{D}_\phi = \{(\tau, \sigma, x, \omega) | \sigma, \tau \in \mathcal{T}, \sigma \leq \tau, x \in \mathcal{X}, \omega \in \mathcal{U}^{[\sigma,\tau)}\}$$

for which $\omega(n)$ is defined for $n = \sigma, \sigma+1, \ldots, \tau-1$ by

$$\omega(n) = \begin{cases} 1 & \text{if } x + \sum_{k=\sigma}^{n-1} \frac{\omega(k)}{k^2} < \alpha \\ -1 & \text{else} \end{cases}$$

For every τ, σ, x the function ω is uniquely determined by the above definition and the transition map can be defined by:

$$\phi(\tau, \sigma, x, \omega) = x + \sum_{k=\sigma}^{n-1} \frac{\omega(k)}{k^2}.$$

The discrete system Σ apply a control that is decreasing on time t and whose absolute value is $1/t^2$, and that goes toward the target value α, which is an equilibrium point at infinity.

3. From a sequence to an irrational number

In the previous section we proved that for every α in a certain interval, there exist a Dirichlet series with coefficients 1 or -1 that converges to α. In particular this is true for every irrational number in that interval.

In this section we will start from suitable explicit sequences a_n with $a_n = 1$ or -1 and we prove that the corresponding Dirichlet series converges to an irrational numeber.

Theorem 3.1. *Let* $a_1, \ldots, a_{n_0}$ *be arbitrarily defined in* $\{-1, 1\}$. *Let* $\varepsilon > 0$ *and define recursively* n_h *by*

$$n_{h+1} = 1 + [\ \mathrm{lcm}(1, 2, \ldots, n_h)]^{\frac{1+\varepsilon}{2}}.$$

Let $a_{n_h+1} = 1$. *For* $n = n_h + 2, n_h + 3, \ldots, n_{h+1}$ *define:*

$$a_n = \begin{cases} +1 & \text{if} \quad -\dfrac{1}{n^2} + \displaystyle\sum_{k=n_h+1}^{n-1} \frac{a_k}{k^2} \leq 0 \\ \\ -1 & \text{else} \end{cases}$$

The number α *defined by*

$$\alpha = \sum_{k=1}^{\infty} \frac{a_k}{k^2} \tag{2}$$

is irrational.

Proof. The series (2) defining α is an absolute convergent series, so in particular it is convergent. For every $h = 0, 1, \ldots$ consider the rational approximations p_h/q_h of α defined by

$$\frac{p_h}{q_h} = \sum_{k=1}^{n_h} \frac{a_k}{k^2} \in \mathbb{Q}.$$

We assume that $(p_h, q_h) = 1$ and q_h positive. According to that, it is $q_h \leq \mathrm{lcm}\ (1, 2, 3, \ldots, n_h)^2$.

Note that $p_{h+1}/q_{h+1} > p_h/q_h$. This is a consequence of the fact that for every $n = n_h + 1, n_h + 2, \ldots, n_{h+1}$, is by definition

$$\sum_{k=n_h+1}^{n} \frac{a_k}{k^2} > 0.$$

This means that

$$\frac{p_h}{q_h} < \alpha$$

and

$$\alpha = \lim_{h \to \infty} \frac{p_h}{q_h}.$$

Moreover we have:

$$0 \neq \left| \alpha - \frac{p_h}{q_h} \right| = \left| \sum_{k=n_h+1}^{\infty} \frac{a_k}{k^2} \right| \leq \left| \sum_{k=n_h+1}^{n_{h+1}} \frac{a_k}{k^2} \right| + \left| \sum_{k=n_{h+1}+1}^{\infty} \frac{a_k}{k^2} \right|$$

Note that

$$\left|\sum_{k=n_h+1}^{n_{h+1}} \frac{a_k}{k^2}\right| \leq \frac{1}{n_{h+1}^2}$$

and that

$$\left|\sum_{k=n_{h+1}+1}^{\infty} \frac{a_k}{k^2}\right| \leq \frac{1}{n_{h+1}^2},$$

so

$$0 \leq \left|\alpha - \frac{p_h}{q_h}\right| \leq \frac{2}{n_{h+1}^2} \leq \frac{2}{q_h^{1+\varepsilon}}. \tag{3}$$

On the other hand if $\alpha = M/N$ for certain $M, N \in \mathbb{N}$, since $p_h/q_h \neq \alpha$, we should have

$$\left|\alpha - \frac{p_h}{q_h}\right| \geq \frac{1}{Nq_h},$$

and this is in contraddiction with (3) for sufficiently large h. So α is irrational. □

References

[1] E.D. Sontag, *Mathematical Control Theory, Deterministic Finite Dimensional Systems*, Second Edition, Springer, New York, 1998.

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