Published in Physical Review A 59, issue 1, 773-777, 1999
which should be used for any reference to this work

Noise considerations for locking to the center of a Lorentzian line

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Many areas of science require locking a probe signal to the center of a Lorentzian line. In this paper we experimentally explore the limits to the resolution for locking a sine-wave modulated probe signal to a Lorentzian line. Surprisingly we find that in some cases relatively high-frequency phase modulation (PM) noise on the probe signal (even 100 times the linewidth of the resonance) can significantly degrade the stability of the lock. The 6.835-GHz hyperfine transition in ^{87}Rb was used as a test bed for these studies because it affords very high resolution for investigating this effect. The details presented here should prove useful for testing a complete theory of this effect and give some insight into the requirements of PM noise of local oscillators to be locked to a Lorentzian line.

I. INTRODUCTION

In this paper we explore the limits to the resolution for locking a probe signal to a Lorentzian resonance line resulting from phase modulation (PM) noise on the local oscillator or, more generally, PM noise on the total resonant field. This is a universal problem caused by the shape of the resonance and is independent of the way in which the line shape is realized [1]. The 6.835-GHz hyperfine transition in ^{87}Rb is a very convenient test bed for these studies because the linewidth of this resonance in a typical buffer gas cell is approximately 1 kHz, the line can be optically pumped with available diode lasers, and the signal-to-noise ratio is extremely high. This results in very high resolution for investigating this universal effect.

Kramer [2] pointed out that PM noise in the interrogating source at the even harmonics of the sine-wave modulation frequency degrades the resolution for locking a probe signal to a resonance. Various analyses have been done for frequency standards operating either in pulsed [3] or continuous [4,5] mode. None of these analyses treats the practical case investigated here, namely, the sensitivity to PM noise at Fourier frequencies that are large compared to the nominal linewidth for sine-wave modulation and demodulation.

References [4, 5] investigate the ‘‘quasistatic’’ case where the modulation frequency f_m is very small compared to the linewidth. This approach can be calculated analytically, but is expected to break down when the modulation frequency approaches the linewidth. The limits to the resolution or fractional frequency stability due to PM noise at even harmonics of the modulation frequency in the quasi-static approximation, with square-wave modulation and sine-wave demodulation, are given by [4]

$$\sigma_y(\tau)_{\text{PM noise}}=\left[\sqrt{\sum_{n=1}^{\infty} C_{2n}^2 S_\phi(2nf_m)}\right]\tau^{-1/2}. \qquad (1)$$

For square-wave modulation and sine-wave demodulation, the general formula for C_{2n} and the first few coefficients are given by

$$C_{2n}=\frac{2n}{(2n-1)(2n+1)}\frac{f_m}{\nu_0} \quad \text{(general)},$$

$$C_2=\frac{2}{3}\frac{f_m}{\nu_0} \quad \text{(second harmonic)},$$

$$C_4=\frac{4}{15}\frac{f_m}{\nu_0}=0.4C_2 \quad \text{(fourth harmonic)}, \qquad (2)$$

$$C_6=\frac{6}{35}\frac{f_m}{\nu_0}\approx 0.26C_2 \quad \text{(sixth harmonic)},$$

$$C_8=\frac{8}{63}\frac{f_m}{\nu_0}\approx 0.19C_2 \quad \text{(eight harmonic)},$$

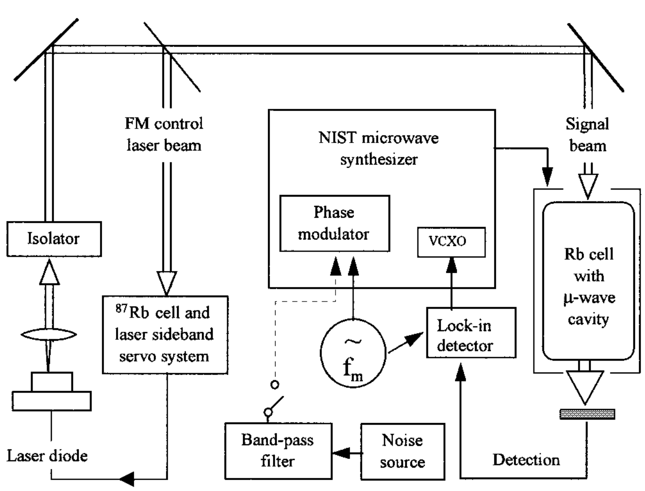

FIG. 1. Block diagram of the Rb clock. The role of the noise source is described in the text (Sec. II).

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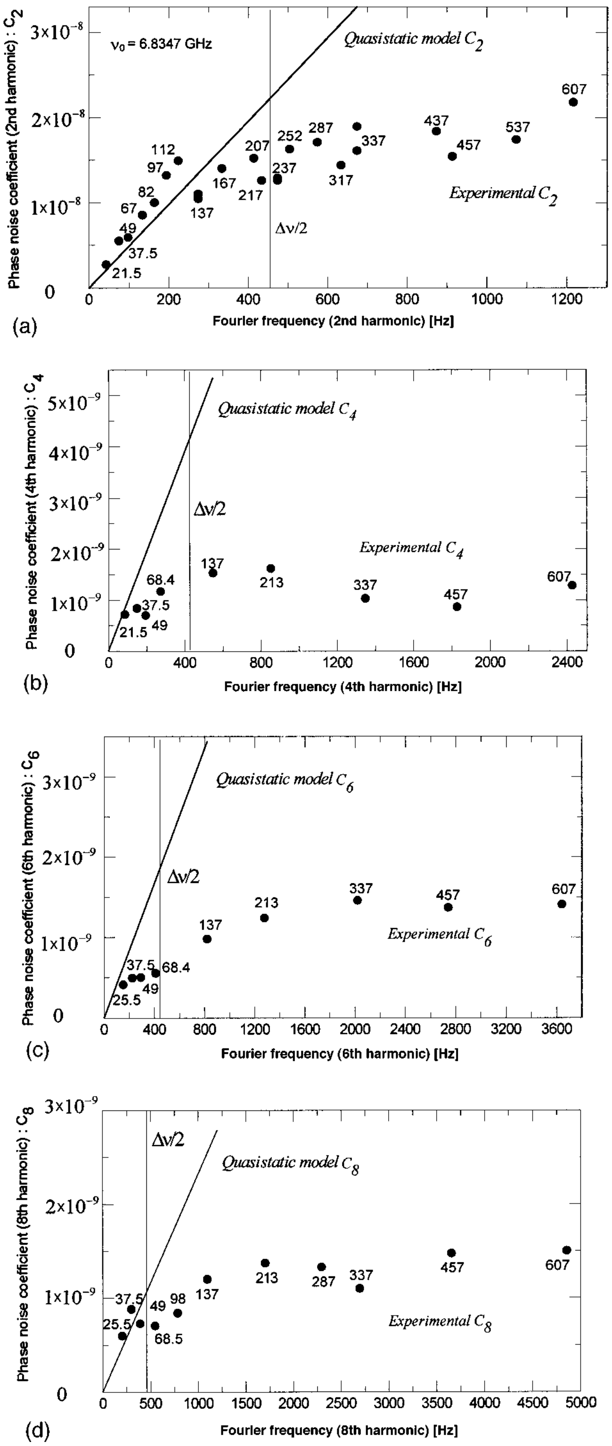

FIG. 2. PM noise coefficients (a) C_2, (b) C_4, (c) C_6, and (d) C_8 at $\nu_0 = 6.8347$ GHz. The modulation frequency is indicated. $\Delta\nu$ (=900 Hz) is the atomic resonance width.

where ν_0 is the center frequency of the Lorentzian resonance line. Equations (1) and (2) are useful for estimating the effect of the synthesizer PM noise, but cannot provide a precise evaluation for our device where we use sine-wave modulation, sine-wave demodulation, and a modulation frequency f_m (300 Hz) equal to one-third of the resonance linewidth [full width at half maximum (FWHM) equal to 900 Hz]. We have experimentally obtained the coefficients of Eq. (1) by measuring the white frequency noise contribution to the fractional frequency stability, $\sigma_y(\tau)_{\text{PM noise}}$ resulting from PM

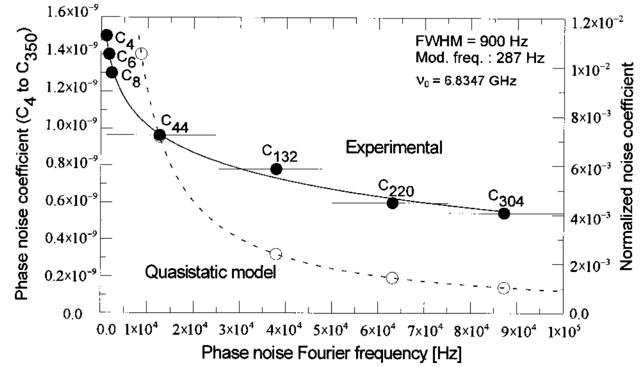

FIG. 3. High-order PM noise coefficients. The left-hand scale is the absolute resolution for our ^{87}Rb clock and the right-hand scale is a normalized ($C_{2n} \times 2\nu_0$ divided by the FWHM) scale for estimating this effect in other systems.

noise at the various f_m harmonics. In some cases we found that PM noise at Fourier frequencies 100 times the nominal linewidth dominated the instability of locking to the resonance line. This result is at variance with the quasistatic model. At present we are not aware of a complete theory for PM noise at Fourier frequencies that are very large compared to the resonance linewidth. The data presented here show that the form of the quasistatic theory is preserved for noise at large offset frequencies, just that the coefficients vary much more slowly with harmonic number than predicted. The sum can be significantly larger than predicted by the quasistatic model. These details should therefore prove useful for testing a complete theory of this effect and give some insight into the PM noise requirements of local oscillators to be locked to a resonance line.

II. EXPERIMENTAL SETUP AND RESULTS

Figure 1 describes our experimental setup. Briefly, an optical microwave double resonance experiment is performed

TABLE I. Phase noise at 100 MHz of two different synthesizers for our ^{87}Rb.

Fourier frequency (Hz)	$S_\phi(f)$ (rad^2/Hz)	
	Synthesizer 1	Synthesizer 2
10	4×10^{-13}	4×10^{-13}
50	2.6×10^{-13}	2.6×10^{-13}
100	1.6×10^{-13}	1.6×10^{-13}
300	2×10^{-14}	3.2×10^{-14}
600	1.2×10^{-14}	6.4×10^{-15}
1×10^3	1×10^{-14}	1.2×10^{-15}
3×10^3	1×10^{-14}	1.2×10^{-15}
10×10^3	1×10^{-14}	1.2×10^{-15}
30×10^3	1×10^{-14}	1×10^{-16}
100×10^3	1×10^{-14}	1×10^{-16}
300×10^3	1×10^{-14}	1×10^{-16}
1×10^6	1×10^{-14}	1×10^{-16}
3×10^6	$<5\times10^{-15}$	$<5\times10^{-16}$

TABLE II. Limit to short-term frequency stability of Rb gas-cell clock at 1 s due to the microwave PM noise calculated from Eq. (1), the PM noise of column 2 of Table I and the coefficients of Eq. (2) and Figs. 2 and 3.

Harmonic contribution in Eq. (1)	2nd 574 Hz	4th 1148 Hz	4th–350th 1–100 kHz	350th–3500th 0.1–1 MHz	Total limit
quasistatic (theory)	2.1×10^{-13}	8×10^{-14}	1.2×10^{-13}	$<1\times10^{-14}$	2.4×10^{-13}
Dynamic (Expt.)	1.3×10^{-13}	1×10^{-14}	7×10^{-14}	1.4×10^{-13}	2×10^{-13}

in which a microwave local oscillator is locked to the ^{87}Rb hyperfine resonance. The three main components of the test bed (physics package, laser, and microwave synthesizer) have been described in [6]. A distributed-Bragg-reflection laser is used to optically pump a rubidium vapor in a buffer-gas cell. The laser is stabilized to a saturated absorption line of a separate ^{87}Rb evacuated cell. Sideband locking at 12 MHz is used [7].

To evaluate the effect of the microwave PM noise on the stability, we added band-limited PM noise that was narrow compared to f_m and centered on the second (or fourth, sixth, and eighth harmonics) by turning the switch on in Fig. 1 and centering the frequency of the adjustable passband filter to the desired harmonics. We then measured the stability as a function of the PM noise when f_m was varied from 21.5 to 607 Hz. The results are given in Fig. 2. For low modulation frequencies, there is good qualitative and even quantitative agreement between the quasistatic predictions and the experimental results. The coefficients for the added white frequency noise increase linearly with the modulation frequency, with a slope slightly different from that predicted, probably because the theory was established for square-wave modulation while we used sine-wave modulation. When $2nf_m$ ($n=1,2,3,\ldots$) becomes large compared to the resonance half-width, the coefficients stop increasing linearly and instead appear to become nearly constant.

With this technique, we have also measured the effect of PM noise centered on the odd harmonics of f_m. The noise added to the first harmonic produced an instability 40 times lower than for the second. PM noise added at the third harmonic has a coefficient 5 times lower than C_4 and actually smaller than the coefficient of the 350th harmonic. Thus we conclude that the general form of the quasistatic model in Eq. (1) is correct, namely, only noise at the even harmonics of the modulation makes a significant contribution to the added white frequency noise. As discussed below, the amplitude and variation of the coefficients with harmonic number seem to be much different from those predicted by the quasistatic model.

A second type of measurement was performed to evaluate the effect of higher harmonics. The modulation frequency was fixed at 287 Hz. The bandpass filter after the noise source was adjusted so that, instead of one harmonic, the noise spectrum (typically 25 kHz wide) included a large number of harmonics. An estimate of the average coefficients could then be obtained by measuring the clock stability as a function of the noise power. The results are given in Fig. 3 in terms of an absolute instability for our ^{87}Rb clock and as a function of the fractional linewidth of the resonance FWHM divided by $2\nu_0$. This normalized scale can be used to estimate this universal effect in many areas of science.

In Figs. 2 and 3 we observe that the magnitude of the coefficients decreases by about one order of magnitude between C_2 and C_4, but then decreases very slowly. In fact, the coefficient is reduced only about 50% from C_8 to C_{304}. If we use the quasistatic formula (2), we obtain a 97% reduction between C_8 and C_{304}. However, the quasistatic approach is not valid for high harmonics since $2nf_m \gg \Delta\nu$. The consequence of the slow decrease of high-order coefficients is that broadband PM noise in the synthesizer signal (or more generally the total resonant field) can significantly affect the clock stability [because of the high number of terms that contribute to the instability; cf. Eq. (1)].

A. Limit due to microwave synthesizer PM noise

We measured the power spectral density of PM noise $S_\phi(f)$ of our synthesizer at 100 MHz by using the ''three-cornered-hat, cross-correlation technique'' [8]. The results are shown in column 2 of Table I.

The effect of the microwave PM noise on the clock instability can be calculated from Eq. (2) and Fig. 3. Table II compares the results obtained with the coefficients corresponding to the quasistatic model to the coefficients measured in the ''dynamic'' case. Even though the two total limits are very close, we can observe a significant difference regarding the origin of these two limits. In the quasistatic case the contributions from the low-order harmonics are dominant and the effect of the broadband noise is negligible. In the dynamic case, the effect of the low-order harmonics is much lower, but the broadband noise contribution is not negligible and is actually even larger than the effect for the second harmonic.

If we include the effects of the photocurrent noise ($1.2\times10^{-13}\tau^{-1/2}$), the light shift ($2\times10^{-13}\tau^{-1/2}$, [9,10,6]), and PM noise in the probe signal ($2\times10^{-13}\tau^{-1/2}$), we obtain a limit for short-term fractional frequency stability $\sigma_y(\tau)=3\times10^{-13}\tau^{-1/2}$. This is equal to the measured performance.

B. Improving the microwave synthesizer

Since the clock instability is due to both the PM noise at the second harmonic and the broadband noise, it is necessary to reduce the amplitude and the spectral width of the PM

TABLE III. Limit to short-term frequency stability of Rb gas-cell clock at 1 s due to the microwave PM noise calculated from Eq. (1), the PM noise of column 3 of Table I, and the coefficients of Eq. (2) and Figs. 2 and 3.

Harmonic contribution	2nd 574 Hz	4th 1148 Hz	4th–42nd 1–12 kHz	Total limit	With notch on 2nd harmonic
quasistatic (theory)	1.6×10^{-13}	3×10^{-14}	4×10^{-14}	1.6×10^{-13}	4×10^{-14}
dynamic (Expt.)	1×10^{-13}	4×10^{-15}	1×10^{-14}	1×10^{-13}	1×10^{-14}

noise in order to significantly diminish this effect in this case. The use of a notch filter at the second harmonic alone would not be sufficient to reach the intrinsic frequency stability [11,12].

In synthesizer 1 the output frequency of 6.8 GHz was obtained by doubling a low-noise 5-MHz quartz oscillator to 10 MHz, applying the frequency modulation, and multiplying the output to 6.8 GHz. The broadband PM noise of this synthesizer is primarily limited by the noise floor of the modulator. Several changes were made to reduce the broadband noise in synthesizer 2. First, the 5-MHz quartz oscillator was used to phase lock a low-PM-noise 100-MHz quartz oscillator [6]. Second, the modulator was moved to 100 MHz. The broadband PM noise of this second approach is reduced by a factor of at least 50 over the previous design, as shown in column 3 of Table I. The resulting limit for the short-term stability of the clock is now dominated by PM noise at $2f_m$ as shown in Table III.

From this analysis and Ref. [5], we find that the added white frequency noise due to PM noise in the interrogation signal derived from synthesizer 2 is significantly smaller than the expected frequency stability of our present clock of $2.5\times10^{-13}\tau^{-1/2}$ for $1s<\tau<25$ s. If a notch filter at the second harmonic of the modulation were used, our present synthesis would support a clock operating near $1\times10^{-14}\tau^{-1/2}$ [10]. See Table III.

III. SUMMARY

We have studied the limits to the resolution for locking a probe signal to a Lorentzian resonance line due to PM noise on the probe signal. Coefficients for the degradation in the white frequency noise have been experimentally determined for PM noise at Fourier frequencies up to the 350th harmonic or approximately 200 times the half linewidth of the resonance. These data confirm the general form of the quasistatic model given in Eq. (1), except that coefficients for noise above the half linewidth of the resonance are smaller than predicted. Surprisingly, the coefficients for even harmonics larger than 10 times the resonance bandwidth fall very slowly with increasing harmonic number. As a result, PM noise-induced white frequency noise at high modulation frequencies can dominate the ultimate resolution of locking the probe signal to a Lorentzian resonance. Using these data enabled us to refine our microwave synthesis so that it now contributes only $1\times10^{-13}\tau^{-1/2}$ to the overall instability of a ^{87}Rb clock. The use of a notch filter [10] at $2f_m$ would reduce the effect by a factor of 10.

We can use the results of these experiments to estimate how these effects would scale in a resonance with a different linewidth. For $2nf_m$ less than or equal to the half linewidth, we can use Eq. (2) to estimate the coefficients. For $2nf_m$ greater than the half linewidth, we need to use the coefficient data in Fig. 3. The data presented in this paper should also prove helpful in testing a complete dynamic theory of this effect.

ACKNOWLEDGMENTS

We thank Francisco Garcia for assistance in testing the microwave synthesizer and Andrea De Marchi, Roland Barillet, Claude Audoin, Steve Jefferts, Hugh Robinson, and Jon Shirley for helpful discussions.

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