Embeddings of groups into Banach spaces

Thèse

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Pierre-Nicolas Jolissaint

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Prof. Alain Valette	Directeur de thèse
Dr Antoine Gournay	Rapporteur (TU Dresden)
Dr Anastasia Khukhro	Rapporteur (Université de Neuchâtel)
Prof. Nicolas Monod	Rapporteur (EPF Lausanne)

Institut de mathématiques, Université de Neuchâtel,
Rue Emile-Argand 11, CH-2000 Neuchâtel

UNIVERSITÉ DE NEUCHÂTEL

Faculté des sciences
Secrétariat-décanat de Faculté
Rue Emile-Argand 11
2000 Neuchâtel - Suisse
Tél: + 41 (0)32 718 2100
E-mail: secretariat.sciences@unine.ch

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Monsieur Pierre-Nicolas JOLISSAINT

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sur le rapport des membres du jury composé comme suit:

- Prof. Alain Valette, Université de Neuchâtel, directeur de thèse
- Dr Antoine Gournay, TU Dresden, D
- Dr Anastasia Khukhro, Université de Neuchâtel
- Prof. Nicolas Monod, EPF Lausanne

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Le Doyen, Prof. B. Colbois

B. Colbois.

Résumé

L'objectif de cette thèse est de faire des liens entre les propriétés algébriques et géométriques des groupes. Une façon d'étudier un espace métrique quelconque est de le représenter comme un sous-ensemble d'un espace de Banach dont la géométrie est bien comprise. En premier lieu, nous étudions les plongements bi-Lipschitz de graphes de Cayley de groupes finis dans les espaces $L^p[0,1]$. En particulier, nous donnons une borne inférieure pour la distortion de tels plongements à l'aide d'invariants de graphes comme le diamètre, la régularité et le trou spectral. Dans un deuxième temps, nous étudions le comportement de plongements qui préservent la géométrie à grande échelle des espaces métriques infinis. Pour ce faire, nous calculons l'exposant de compression de certaines extensions HNN ainsi que d'espaces métriques obtenus comme réunion de graphes finis. Nous nous intéressons ensuite à la 1-cohomologie à valeur dans les G-modules unitaires, dans le cas où G est un groupe agissant transitivement sur les sommets d'un arbre régulier ainsi que sur son bord. Nous parvenons à calculer explicitement les fonctions conditionnellement de type négatif associées aux 1-cocycles non-bornées sur G. Enfin, dans le dernier chapitre de cette thèse, nous donnons une condition de non-annulation de l'espace de 1-cohomologie bornée pour certains Γ-modules de Banach, pour Γ un groupe dénombrable discret.

Mots clés: Géométrie métrique; Distortion euclidienne; Trou spectral; Exposant de compression; Box space; 1-cohomologie; Arbre; Fonction conditionnellement de type négatif.

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I have nothing to offer but blood, toil, tears and sweat.

Winston Churchill

Contents

Chapter 1

Introduction

At the end of the nineties, remarkable works of Higson and Kasparov ([HK97]) and Tu ([Tu99]) showed that the Baum-Connes conjecture holds for groups with the Haagerup property. In brief, the Baum-Connes conjecture asserts that a so-called assembly map, which relates two specific objects built from a group, a geometrical one and an analytical one, is an isomorphism. Implying several famous results in functional analysis and in topology (such as the conjecture of idempotents and the Novikov conjecture), the Baum-Connes conjecture has drawn a lot of interest.
Recall that a second countable, locally compact group G has the Haagerup property if it admits a proper affine isometric action on some Hilbert space $\mathcal{H}$. Equivalently, this amounts to saying that we can find a map $F : G \to \mathcal{H}$ which is proper[1] and equivariant[2], relatively to the affine isometric G-action on $\mathcal{H}$. On the non-equivariant side, Yu showed ([Yu00]) that the coarse Baum-Connes conjecture holds for groups admitting a coarse embedding into a Hilbert space. This was then extended to the class of groups admitting a coarse embedding into a uniformly convex Banach space by Kasparov and Yu ([KY06]). Recall that a coarse embedding of a metric space (X, d) into a Banach space $(Y, \|\cdot\|)$ is a map $F : X \to Y$ for which we can find real-valued, non-decreasing control functions $\rho_\pm$ such that $\rho_\pm(t)$ tend to infinity with t and that satisfy the following inequalities

$$\rho_-(d(x, y)) \leq \|F(x) - F(y)\| \leq \rho_+(d(x, y)),$$

for all $x, y \in X$. We would like to stress that the coarse Baum-Connes conjecture concerns metric spaces and that it can be seen as a non-equivariant counterpart of the Baum-Connes conjecture, when restricted to finitely generated groups, seen as metric spaces. Although the embeddability condition

[1]The inverse image of any ball of finite radius is relatively compact.
[2]For any $x \in G$, the map satisfies $F(g \cdot x) = g \cdot F(x)$, for all $g \in G$.

seems weak, it is still strong enough to deduce the Novikov conjecture, when applied to the fundamental group of a compact manifold. These two lines of research motivated for us the study of equivariant and non-equivariant coarse embeddings of groups and metric spaces into Banach spaces.

In order to find potential counterexamples for the coarse Baum-Connes conjecture, finding obstructions to coarse embeddability became a motivating challenge. Gromov constructed the first example of a metric space that does not embed into any Hilbert space ([Gro00]). His construction relies on a simple idea. One takes a disjoint union of an appropriate sequence of finite graphs, called expander graphs, and one defines any metric on this union which induces the same metric on each of the graphs. It is worth noting that expander graphs embed badly into Hilbert spaces, in the sense that they only admit bi-Lipschitz embeddings with the highest possible order of distortion by a famous result of Bourgain ([Bou85]). This motivates the study of bi-Lipschitz embeddings of finite metric spaces into Hilbert spaces.

The purpose of this work consists in studying maps $F : X \to Y$, where (X, d) and (Y, δ) are metric spaces, by estimating as well as possible the behaviour of the so-called control functions $\rho_\pm$, which are real-valued and satisfy

$$\rho_-(d(x, y)) \leq \delta(F(x), F(y)) \leq \rho_+(d(x, y)),$$

for all $x, y \in X$. Several classically studied embeddings belong to this class. For instance, if ρ_+ and ρ_- are linear, then, the map F is bi-Lipschitz. In a similar fashion, if ρ_+ is affine, then F is called large-scale Lipschitz. To measure the degree of dissimilarity between X and Y, one can define an invariant over a class of embeddings with control functions of the same type. For instance, let us consider a bi-Lipschitz embedding $F : X \to Y$. We define the distortion of F, denoted by $dist(F)$, to be the infimum of the quotient $\frac{\rho_+(1)}{\rho_-(1)}$ over all pairs of linear control functions for F. Clearly, this quantity is always greater or equal to 1, and the equality occurs when F is a dilation, meaning that Y contains an isometric copy of X, up to a multiplicative factor. By extension, we define the distortion of X in Y as the infimum of the distortion $dist(F)$, over the class of bi-Lipschitz maps $F : X \to Y$. We will denote this number by $c_Y(X)$. If no such bi-Lipschitz embedding exists, then we set $c_Y(X)$ to be infinite.

As bi-Lipschitz embeddings are often too rigid to study infinite metric spaces, it is possible to define a quasi-isometry invariant for metric spaces, based on large-scale Lipschitz maps. In [GK04], Guentner and Kaminker introduced the notion of compression exponent. Let us consider a large-scale Lipschitz map $F : X \to Y$. The compression exponent of F, denoted by $R(F)$, is the supremum of the exponents $\alpha \in [0, 1]$ over the class of lower control functions for F of the form $\rho_-(t) = At^\alpha - B$, where $A, B > 0$. By extension, we define the compression of X inside Y as the supremum of the compression $R(F)$, over the class of large-scale Lipschitz maps $F : X \to Y$. We will denote this number by $\alpha^*_Y(X)$. We observe that, if X quasi-isometrically embeds into Y, then $\alpha^*_Y(X) = 1$. Similarly, when $X = G$ is a group endowed with a left invariant metric and Y is a Banach space endowed with an affine isometric G-action, we define the equivariant compression exponent of G in Y as the supremum of $R(F)$ over the class of large-scale Lipschitz maps $F : G \to Y$ which are G-equivariant. We denote this number $\alpha^\natural_Y(G)$. Interestingly, these notions are all linked to (weak forms of) amenability. For instance, it is well-known that amenable groups admit a proper affine isometric action on some Hilbert space ([BCV95]), and therefore, have the Haagerup property. Although the converse implication is strongly wrong, Guentner and Kaminker proved ([GK04]) that if a group admits an equivariant coarse embedding into a Hilbert space with a compression exponent strictly greater than 1/2, then it is amenable. A similar result holds in the non-equivariant setting. Namely, if a group admits a coarse embedding into a Hilbert space with a compression exponent strictly greater than 1/2, then the group has the so-called Property A, which is a non-equivariant weakening of amenability. We recall that Property A was designed by Yu in [Yu00] to provide a sufficient condition for a metric space to coarsely embed into a Hilbert space, and therefore, satisfies the coarse Baum-Connes conjecture. In conclusion, apart from having a better understanding of the metric on a group, computing precisely these exponents for groups possibly allows us to deduce strong properties of these groups, by providing new group invariants.

We will mainly focus on the case where X is either a graph endowed with the shortest path metric, or a group endowed with a left invariant metric. As target spaces, we will take various familiar normed spaces (for instance, Banach spaces such as Hilbert spaces or L^p-spaces). Viewing groups as objects with a metric nature and analysing their embeddings into Banach spaces will be the content of the first part of this text.

In the last chapter of our thesis, we will forget about the possible metric structure of groups and study their uniformly bounded representations[3]. It is easy to produce such representations by intertwining any unitary representation with an invertible operator of the Hilbert space on which the unitary representation acts. However, it is surprisingly difficult to find other examples, that is, representations which are not similar to any unitary one. The first

[3]A uniformly bounded representation of a topological group G on a Hilbert space $\mathcal{H}$ is a continuous (for the strong operator topology on $\mathcal{B}(\mathcal{H})$) homomorphism $\pi : G \to \mathcal{B}(\mathcal{H})$ such that the image $\pi(G)$ is a bounded subset of $\mathcal{B}(\mathcal{H})$ for the operator norm.

example was provided by Ehrenpreis and Mautner ([EM55]) who exhibited a uniformly bounded representation of $SL_2(\mathbb{R})$ which fails to be unitarisable[4]. In the 1980's and in the early 1990's, several uniformly bounded representations were discovered for free groups ([PS86]), groups acting on trees ([Val90]) and Coxeter groups ([Jan93], [Jan02]). More precisely, using the fact that any of these groups acts on some combinatorial metric space X, these authors were able to build explicit holomorphic families[5] $(\pi_z)_{z\in\mathbb{D}}$ of uniformly bounded representations, acting on the same Hilbert space $\mathcal{H}$, and it was shown that, for any $z \in \mathbb{D}$, the function $g \mapsto z^{L(g)}$ is a coefficient of the uniformly bounded representation π_z. Here, L is a length function on the group obtained from the combinatorial distance on X. This was then used to deduce the weak amenability with Cowling-Haagerup constant 1 for these groups ([Val93]). Guentner and Higson later generalised this approach to groups acting properly on a finite-dimensional CAT(0)-cubical complex ([GH10]). Our main motivation to study uniformly bounded representations is the so-called Dixmier unitarisability problem. In 1950, generalising the result of de Sz. Nagy on $\mathbb{Z}$ ([dSN47]), Day ([Day50]) and Dixmier ([Dix50]) showed that amenable groups are unitarisable, that is, any uniformly bounded representation is unitarisable. Dixmier also asks whether the converse holds, namely, if a unitarisable group is amenable. Although partial answers in that direction were proved (see for instance [Pis98]), this question is still open. Since any group containing a non-abelian free group (as a closed subgroup) is necessarily non-unitarisable (as well as non-amenable), it is necessary to look at the strange class of groups which are non-amenable and without free subgroups, in the hope of providing a counterexample. Monod and Ozawa showed that some Burnside groups are non-unitarisable ([MO10]), hence giving examples of non-unitarisable groups without free subgroups (the first examples of such groups were exhibited shortly earlier by Epstein and Monod in [EM09]). The proof of Monod and Ozawa amounts to showing the existence of some bounded map $b : G \to \mathcal{B}(\mathcal{H})$ satisfying a certain derivation property. More precisely, let π be a unitary representation of a group G. A bounded map $b : G \to \mathcal{B}(\mathcal{H}_\pi)$ satisfying

$$b(gh) = \pi(g)b(h) + b(g)\pi(h),$$

for all $g, h \in G$, is called a derivation. It is easy to produce uniformly bounded representation using derivations. Moreover, these new representa-

[4] A representation is unitarisable if it is similar to a unitary one.

[5] A family of representations $(\pi_z)_{z\in\mathbb{D}}$ is holomorphic if, for any group element g and for any pair of vectors $\xi, \eta \in \mathcal{H}$, the complex-valued function defined on $\mathbb{D}$ by $\phi(z) = \langle\pi_z(g)\xi, \eta\rangle$ is holomorphic.

tions are unitarisable if and only if the derivation we start with is inner[6]. To summarize, if a group admits a non-inner bounded derivation, then it is non-unitarisable. The purpose of the last chapter of this text is to study such maps and to give a sufficient condition for a group to admit non-inner bounded derivations.
Before stating the main results of our thesis, we discuss briefly its structure. In Chapter 2, we prove a general lower bound for the distortion of any bi-Lipschitz embeddings of finite graphs into L^p-spaces in terms of graph invariants such as the diameter, the degree and the spectral gap. In Chapter 3, we prove an upper bound to estimate the compression of coarse embeddings of box spaces of finite graphs into the L^p-spaces, and compute the L^p-compression exponent of a particular family of HNN extensions. In Chapter 4, we deal with equivariant coarse embeddings of subgroups of the full automorphism group of a regular tree into Hilbert spaces. We give a general upper bound for the compression exponent of such embeddings and then show it is optimal. Finally, in Chapter 5, we first discuss the Dixmier problem and its link with some bounded cohomology group. We then study closely bounded derivations on groups and give a special criterion to prove the existence of non-inner derivations and we apply this method to free groups.

1.1 Bi-Lipschitz embeddings

The main results of Chapter 2 give a lower bound for the distortion of bi-Lipschitz embeddings of a finite graph into L^p-spaces in terms of the p-spectral gap $\lambda_1^{(p)}(X)$, the average degree k and the maximal displacement[7] of the graph $D(X)$. Recall that, for $1 \leq p < \infty$ and for any finite graph $X = (V, E)$, the p-spectral gap is defined as

$$\lambda_1^{(p)}(X) = \inf \left\{ \frac{\frac{1}{2}\sum_{x\in V}\sum_{y:\, y\sim x} |f(x) - f(y)|^p}{\inf_{\alpha\in\mathbb{R}} \sum_{x\in V} |f(x) - \alpha|^p} \right\},$$

where the infimum is taken over all $f \in \ell^p(V)$ such that f is not constant.

[6]An inner derivation is a map of the form $g \mapsto [\pi(g), T]$, for some bounded operator $T \in \mathcal{B}(\mathcal{H}_\pi)$.

[7]For a finite graphs $X = (V, E)$, the maximal displacement of X is defined as $D(X) = \max_{\alpha\in \mathrm{Sym}(V)} \min_{x\in V} d(x, \alpha(x))$.

Our first result is :

Theorem 1.1.1. *(Theorem 2.2.1) Let X be a finite, connected graph of average degree k. Then, for $1 \leq p < \infty$,*

$$D(X) \left(\frac{\lambda_1^{(p)}(X)}{k \ 2^{p-1}} \right)^{\frac{1}{p}} \leq c_p(X).$$

For vertex-transitive graphs, the maximal displacement coincides with the diameter. Therefore, the result here above takes the form :

Corollary 1.1.2. *(Corollary 2.2.2) Let X be a finite, connected, vertex-transitive graph. Then for $1 \leq p < \infty$:*

$$diam(X) \left(\frac{\lambda_1^{(p)}(X)}{k \ 2^{p-1}} \right)^{\frac{1}{p}} \leq c_p(X),$$

where k is the degree of each vertex.

After having proved these two results, we apply our lower bounds to different sequences of finite graphs (cyclic graphs, hypercubes, sequences of expanders, etc...) to recover numerous well-known results and to get new lower bounds.

1.2 Coarse embeddings

In Chapter 3, we study the behaviour of large-scale Lipschitz embeddings of some graph-like metric spaces into L^p-spaces. Our first contribution is a lower bound for the L^p-compression exponent of box spaces of finite graphs, by developing the techniques appearing in Chapter 2.

Proposition 1.2.1. *(Proposition 3.3.4) Let $(X_n)_{n \geq 1}$ be an increasing sequence of finite, connected graphs with bounded degree. Let k_n be the average degree of $X_n = (V_n, E_n)$. We denote by X the box space formed by the sequence $(X_n)_n$. The p-compression exponent of the box space X admits the following upper bound:*

$$\alpha_p^*(X) \leq \frac{1}{p} \ \liminf_{n \to \infty} \frac{\log \left(\lambda_1^{(p)}(X_n)^{-1} \right)}{\log \log |V_n|},$$

for any $1 \leq p < \infty$.

We recall that an important example of box space is provided by graphs obtained by taking an increasing sequence of finite quotients of some fixed finitely generated, residually finite group Γ. It was shown that if such a box space built from a group Γ has a Hilbert compression exponent strictly greater than 0, then the group Γ has the Haagerup property (see Proposition 11.26 in [Roe03]). This connects the result here above with Chapter 4.
Our second contribution is the exact computation of the L^p-compression exponent of some particular HNN extensions. In order to state our result, we need to introduce some notations. Let $\mathfrak{N}$ be a locally compact compactly generated group and let G be a closed subgroup of $\mathfrak{N}$. Let $i_1, i_2 : H \to G$ be two inclusions of a group H onto open subgroups of finite index in G, and assume i_1 and i_2 are conjugated by an automorphism φ of $\mathfrak{N}$. The $\mathfrak{N}$-BS group Γ is then the HNN extension $\mathrm{HNN}(G, i_1(H), i_2(H))$ whose presentation is given by $\langle S, t | R, t i_1(h) t^{-1} = i_2(h)\ \forall h \in H\rangle$, where $G = \langle S|R\rangle$.

Theorem 1.2.2. *(Theorem 3.4.1) Let $\mathfrak{N}$ be a connected Lie group, let G be a closed cocompact subgroup of $\mathfrak{N}$ and let Γ be an HNN extension as above. Then, for all $p \geq 1$, $\alpha_p^*(\Gamma) = 1$.*

In particular, this class of groups contains the Baumslag-Solitar groups.

1.3 Equivariant embeddings

In Chapter 4, we prove a sharp upper bound for G-equivariant embeddings of the $(q+1)$-regular tree $\mathcal{T}_{q+1} = (V, E)$, for $q \geq 2$, into a Hilbert space, where G is a closed and non-compact subgroup of $\mathrm{Aut}(\mathcal{T}_{q+1})$ acting transitively on both the set of vertices and on the boundary of the tree $\mathcal{T}_{q+1}$.

Theorem 1.3.1. *(Theorem 4.2.3) Assume that G acts on a Hilbert space $\mathcal{H}$ by affine isometries. Then, any G-equivariant map $F : V \to \mathcal{H}$ such that $F(x_0) = 0$ for some $x_0 \in V$ satisfies:*

$$\|F(x)\|^2 \leq Ad(x, x_0) - B + Bq^{-d(x,x_0)},$$

where x_1 is any vertex adjacent to x_0,

$$A = \frac{(q+1)\|F(x_1)\|^2}{q-1} \quad \text{and} \quad B = \frac{2q\|F(x_1)\|^2}{(q-1)^2}.$$

Furthermore, if the F is harmonic and non-constant, then the equality holds.

The second result shows that the previous theorem is optimal by proving the existence of such an equivariant and harmonic map.

Proposition 1.3.2. *(Proposition 4.2.4) There exists a Hilbert space $\mathcal{H}$ endowed with an affine isometric G-action and a map $F : V \to \mathcal{H}$ which is G-equivariant, non-constant and harmonic.*

Combining this with a result of Nebbia ([Neb12]), we deduce that this map is essentially unique. Also, this allows us to characterize pure negative type functions in the positive cone of negative type functions on G, denoted by $\mathrm{CL}(G)$.

Proposition 1.3.3. *(See Corollary 4.3.7) Choose a basepoint $x_0 \in V$. For $g \in G$, we set $|g| := d(x_0, gx_0)$. Then, the function on G defined by*

$$g \mapsto \Psi(gx_0, x_0) = |g| + \frac{2q}{q^2 - 1}(q^{-|g|} - 1)$$

is the unique (up to multiplication by a positive scalar) pure negative type function in $\mathrm{CL}(G)$ which is unbounded on G and identically 0 on G_{x_0}, the stabiliser of x_0.

1.4 Cohomology in Banach algebras

In Chapter 5, we discuss the Dixmier problem and its link with bounded cohomology of groups, taking value in Banach algebras. Let Γ be a countable and discrete group and let π be a uniformly bounded representations acting on a Banach space V. We introduce the notion of *uniform almost intertwiners* for uniformly bounded representations acting on Banach spaces.

Definition 1.4.1. *(Definition 5.2.1) The representation π admits **uniform almost intertwiners** if for all $\epsilon > 0$, there exists $T \in \mathcal{B}(V)$ such that*

$$\sup_{g \in \Gamma} \|[\pi(g), T]\| < \epsilon \|T\|_q.$$

Here, $\|\cdot\|_q$ is the seminorm defined on $\mathcal{B}(V)$ by

$$\|T\|_q = \inf_{Q \in \pi(\Gamma)'} \|T + Q\|,$$

where the infimum runs over all bounded operators Q which commute with π.

We will denote by $D^\infty(\Gamma, \pi)$ the space of all bounded derivations with respect to π, and by $I(\Gamma, \pi)$ the space of all inner derivations. We endow $D^\infty(\Gamma, \pi)$ with the supremum norm and give a necessary and sufficient condition for $I(\Gamma, \pi)$ to be closed inside $D^\infty(\Gamma, \pi)$.

Proposition 1.4.2. *(Proposition 5.2.2) The space of inner derivations $I(\Gamma, \pi)$ is closed in $D^\infty(\Gamma, \pi)$ for the topology induced by the supremum norm if and only if the representation π does not have uniform almost intertwiners.*

Then, we give some hereditary properties (by direct sum and by induced representation) of uniform almost intertwiners. We finish the chapter by analysing the space of bounded derivations with respect to the left regular representation of the free group $\mathbb{F}_\infty$. In particular, we show that the quotient space

$$H^1_d(\mathbb{F}_\infty, \lambda) = D^\infty(\mathbb{F}_\infty, \lambda)/I(\mathbb{F}_\infty, \lambda)$$

is infinite dimensional, and that the left regular representation of $\mathbb{F}_\infty$ admits uniform almost intertwiners.

Proposition 1.4.3. *(Proposition 5.3.5) The vector space $H^1_d(\mathbb{F}_\infty, \lambda)$ is infinite dimensionial.*

Proposition 1.4.4. *(Proposition 5.3.8) The left regular representation λ of $\mathbb{F}_\infty$ admits uniform almost intertwiners.*

Chapter 2

Distortion of finite metric spaces

This chapter is dedicated to the study of bi-Lipschitz embeddings of finite metric spaces into vector normed-spaces. As source space, we will take the underlying metric space of a finite, connected graph $X = (V, E)$, where the distance is the graph metric. As target space, we will consider only $L^p = L^p([0, 1])$. The main result of this chapter provides a lower bound on the p-distortion of finite graphs in terms of the p-spectral gap, the average degree and the maximal displacement of the graph.
This chapter is organized as follows : we give the necessary definitions in Section 2.1 to state and prove the main results of the present chapter in Section 2.2. We estimate the maximal displacement in Section 2.3 to apply the main results to various families of graphs in Section 2.4. Then, we compare our main result with similar results in Section 2.5. We end up this chapter with a remark on antipodal maps.
This chapter is strongly inspired from the joint work with Alain Valette [JV14] and all the results stated here, except Corollary 2.4.7, Proposition 2.4.9, 2.5.3 and 2.6.1, appeared in [JV14].

2.1 Definitions and background

2.1.1 Basic notations of graph theory

We state a few basic definitions and notations of graph theory that we will need throughout this text. Readers who are familiar with graph theory is encouraged to skip the present subsection.
A **graph** X is a couple (V, E), where V is the set of vertices and $E \subset V \times V$

is the set of edges. We will write $X = (V, E)$, and by abuse of notation, we will sometimes identify X with the vertex set V. If $(x, y) \in E$, we will write $x \sim y$. This will be called the adjacency relation. A graph is **unoriented** if for any edge $(x, y) \in E$, then the pair (y, x) also belongs to E. A **loop** is an edge of the form (x, x), for some $x \in V$. The **degree** $\deg(x)$ of a vertex $x \in V$ is the number of vertices which are adjacent to x. The graph is said to be **regular** (or k-regular), if the degree of all the vertices is the same, equal to some number k. The graph is said to be **locally finite** if all the vertices have finite degree. Furthermore, if there is a constant K such that $\sup_{x\in X} \deg(x) < K$, then the graph is said to have **bounded degree**.
Let us explain how to see graphs as metric objects. A **path** in the graph $X = (V, E)$ is a (possibly infinite) sequence of consecutive adjacent vertices $(x_k)_{k=1}^n$, that is, $x_k \sim x_{k+1}$, for all k. In this case, the number $n \in \mathbb{N}$, will be refered to as the **length of the path**, and we will say that the path links (or joins) x_0 to x_n. A graph will be called **connected** if, any two vertices can be joined by a path.
Finally, an unoriented, connected graph without loops is naturally endowed with a metric, the so-called **graph metric** (or shortest path metric). The distance between two vertices x, y is the infimum of the length, over all paths linking x and y. As an important example of graph, we recall the following definition.

Definition 2.1.1. *Let G be a group. Let S be a subset of G. We define the **Cayley graph** of G with respect to S, denoted by $X = (G, S)$, as the graph with vertex-set G and endowed with the adjacency relation given by $g \sim gs$, for all $g \in G$, $s \in S$. We define the left Cayley graph of G with respect to S similarly, by using the adjacency relation given by $g \sim sg$.*

It is easy to see that the chosen set $S \subset G$ has a a great influence on the different properties of a Cayley graph $X = (G, S)$.

1. The graph X is regular, of degree $|S|$. It is locally finite if and only if S is finite.

2. The graph X is connected if and only if S is a generating set for G.

3. The graph X is unoriented if and only if S is symmetric, that is, $S = S^{-1}$.

4. The graph X is without loops if and only if the identity element of G does not belong to S.

5. If X is unoriented, connected and without loops, then the shortest path metric on X coincides with the word-length metric on G with respect to S.

Concerning the last claim stated here above, let us recall the construction of the word-length metric on groups. Let G be a locally compact, compactly generated group. Let S be a compact generating set for G. If we assume furthermore that S is symmetric ($S = S^{-1}$), then the word-length of an element $g \in G$ relatively to S is the integer $|g|_S = \inf\{n \in \mathbb{N} \ : \ g \in S^n\}$. The **word-length metric** on G with respect to S is the left invariant metric obtained by setting $d_S(g,h) = |g^{-1}h|_S$, for $g, h \in G$.
Let $X = (V, E)$ be a graph. An **automorphism** α of X is a bijection of V preserving the graph structure, that is, $x \sim y$ if and only if $\alpha(x) \sim \alpha(y)$, for all $x, y \in V$. The group of all automorphisms of X is denoted by $\mathrm{Aut}(X)$. A graph is **vertex-transitive** if $\mathrm{Aut}(X)$ acts transitively on V. As a group acts transitively by left multiplication on any of its Cayley graphs, such graphs are always vertex-transitive.

2.1.2 Distortion of metric spaces

We start with the definition of distortion.

Definition 2.1.2. *Let (X, d) and (Y, δ) be two metric spaces. Let $F : X \to Y$ be an injective map of X into Y. We define the* ***distortion*** *of F as*

$$dist(F) = \sup_{x,y\in X, x\neq y} \frac{\delta(F(x),F(y))}{d(x,y)} \cdot \sup_{x,y\in X, x\neq y} \frac{d(x,y)}{\delta(F(x),F(y))},$$

where the first supremum is the Lipschitz constant $\|F\|_{Lip}$ of F, and the second supremum is the Lipschitz constant $\|F^{-1}\|_{Lip}$ of F^{-1}. In the case where X is finite, supremum can be changed into maximum. The least distortion with which X can be embedded into Y is denoted $c_Y(X)$, namely

$$c_Y(X) = \inf\{dist(F) : \ F : X \hookrightarrow Y\}.$$

If the target space Y is fixed and if there is no risk of confusion, we will refer to this number as the ***distortion of*** *X.*

In this case where the target space is $L^p = L^p([0,1])$, we write $c_p(X) = c_{L^p}(X)$. The quantity $c_2(X)$ is also known as the **Euclidean distortion of** X.
As a first estimation, we note that $c_p(X) \leq diam(X)$, for any metric space,

as shown by the embedding $F : X \to \ell^p(X) : x \mapsto \delta_x$. Since for several classes of finite graphs X, the diameter of X behaves like $\log |X|$ (see the remark below Proposition 2.3.3), we deduce that $c_p(X) \leq O(\log |X|)$, for any finite graph X in such a class. The following fundamental result due to Bourgain generalizes this fact.

Theorem 2.1.3. *(Bourgain, [Bou85]) Let $1 \leq p < \infty$. Let (X, d) be a finite metric space. Then,*

$$c_p(X) = O(\log |X|).$$

The aim of this chapter is to obtain lower bounds for the distortion c_p of finite graphs. To state our results, we introduce an invariant of finite metric spaces and we recall two invariants of graphs.

Definition 2.1.4. *Let (X, d) be a finite metric space. For α a permutation of the set X, we introduce the* **displacement** *of α :*

$$\rho(\alpha) = \min_{x \in X} d(\alpha(x), x).$$

The **maximal displacement** *of X is $D(X) = \max_{\alpha \in \mathrm{Sym}(X)} \rho(\alpha)$.*

We stress that, in the definition of the maximal displacement of a metric space X, we need to consider all permutations of the set X, and not to restrict ourself to isometries of X. The next two definitions are well-known (at least in the case $p = 2$).

Definition 2.1.5. *For $1 < p < \infty$, the* ***p-Laplacian*** *$\Delta_p : \ell^p(V) \to \ell^p(V)$ is an operator defined by the formula*

$$\Delta_p f(x) = \sum_{x \sim y} \left(f(x) - f(y)\right)^{[p]},$$

($f \in \ell^p(V), x \in V$), where $a^{[p]} = |a|^{p-1} sign(a)$ and $\sim$ denotes the adjacency relation on V.

It is worth noting that for $p = 2$, the operator $\Delta_2 = \Delta$ corresponds to the standard linear discrete Laplacian. That is,

$$\Delta f(x) = \deg(x) f(x) - (Af)(x) = \deg(x) f(x) - \sum_{y \sim x} f(x),$$

where A is the **adjacency matrix** of the graph X.

Definition 2.1.6. *Let $1 \leq p < \infty$. The p-**spectral gap** of $X = (V, E)$ is defined by*

$$\lambda_1^{(p)}(X) = \inf \left\{ \frac{\frac{1}{2}\sum_{x\in V}\sum_{y:\, y\sim x} |f(x) - f(y)|^p}{\inf_{\alpha\in\mathbb{R}} \sum_{x\in V} |f(x) - \alpha|^p} \right\},$$

where the infimum is taken over all $f \in \ell^p(V)$ such that f is not constant.

We say that λ is an eigenvalue of Δ_p if we can find $f \in \ell^p(V)$ such that $\Delta_p f = \lambda f^{[p]}$. When $p \neq 1$, it is known that the p-spectral gap is the smallest positive eigenvalue of Δ_p (see [GN12] or [BH09]). The spectral gap is closely related to the following graph invariant. Recall that, for a collection of vertices $A \subset V$, the set ∂A denotes the **boundary** of A, and it consists in vertices $x \in V \setminus A$ which are adjacent to some vertex $v \in A$.

Definition 2.1.7. *Let $X = (V, E)$ be a graph. The **Cheeger constant** is defined by*

$$h_C(X) = \inf \frac{|\partial A|}{\min\{|A|, |V \setminus A|\}},$$

where the infimum is taken over all non-empty subsets A of V.

Roughly speaking, the Cheeger constant quantifies how fast the information spreads in the graph.
We end up this section with two results that will be proved useful in order to estimate the spectral gaps of general graphs, and Cayley graphs. The first result is a p-version of the standard Cheeger-Buser inequalities in the case of graphs.

Theorem 2.1.8. *([BH09], Theorem 4.3) Let $X = (V, E)$ be a finite graph. Then, for any $p \in [1, \infty)$, we have*

$$\left(\frac{2}{\max_{x\in V} \deg(x)}\right)^{p-1} \left(\frac{h_C(X)}{p}\right)^p \leq \lambda_1^{(p)}(X) \leq 2^{p-1} h_C(X).$$

In particular, we can use this result to get a lower bound of the p-spectral gap in terms of the 2-spectral gap of a graph, which is easier to estimate in general. Indeed, we get the following lower bound

$$\lambda_1^{(p)}(X) \geq \left(\frac{2}{\max_{x\in V} \deg(x)}\right)^{p-1} \left(\frac{\lambda_1^{(2)}(X)}{2p}\right)^p.$$

The next result relates the 2-spectral gap of Cayley graphs to the representations of the underlying group. It is especially powerful when applied to finite abelian groups. We refer to Chapter 7 of [KS11].

Proposition 2.1.9. *Let G be a finite group and let S be a symmetric, generating set for G.*

1. *The Laplace operator Δ associated to the Cayley graph $X = (G,S)$ coincides with the oprator $|S| - \sum_{s\in S}\rho(s)$ acting on $\ell^2(G)$, where ρ is the right regular representation*[1] *of G.*

2. *Furthermore, if G is abelian, then, the spectrum of Δ is given by the set*
$$\left\{\sum_{s\in S}(1-\chi(s)) \ : \ \chi\in\widehat{G}\right\},$$
where $\widehat{G}$ is the dual group of G. In particular, we have
$$\lambda_1^{(2)}(X) = \min_{\chi\in\widehat{G}\setminus\{1\}}\sum_{s\in S}(1-\chi(s)).$$

2.2 Main results

The main results of this chapter are :

Theorem 2.2.1. *Let X be a finite, connected graph of average degree k. Then*
$$D(X)\left(\frac{\lambda_1^{(p)}(X)}{k\ 2^{p-1}}\right)^{\frac{1}{p}} \le c_p(X),$$
for $1\le p<\infty$.

For vertex-transitive graphs, this takes the form :

Corollary 2.2.2. *Let X be a finite, connected, vertex-transitive graph. Then for $1\le p<\infty$:*
$$diam(X)\left(\frac{\lambda_1^{(p)}(X)}{k\ 2^{p-1}}\right)^{\frac{1}{p}} \le c_p(X),$$
where k is the degree of each vertex.

[1]For any discrete group G, it is the homomorphism $\rho : G \to \mathcal{U}(\ell^2(G))$ defined by $[\rho(g)f](x) = f(xg)$, for any $g, x\in G$, and any $f\in\ell^2(G)$.

2.2.1 Proof of Theorem 2.2.1

We start with a lemma.

Lemma 2.2.3. *Let $X = (V, E)$ be a finite, connected graph. Let $1 \leq p < \infty$.*

1. *Let α be any permutation of V. For $F : V \to \ell^p(\mathbb{N})$:*

$$\sum_{x \in V} \|F(x) - F(\alpha(x))\|_p^p \leq 2^p \sum_{x \in V} \|F(x)\|_p^p.$$

2. *Fix an arbitrary orientation on the edges. For every $F : V \to \ell^p(\mathbb{N})$, there exists $G : V \to \ell^p(\mathbb{N})$ such that $dist(G) = dist(F)$ and*

$$\sum_{x \in V} \|G(x)\|_p^p \leq \frac{1}{\lambda_1^{(p)}(X)} \sum_{e \in E} \|G(e^+) - G(e^-)\|_p^p.$$

Proof: *1.* Define a linear operator T on $\ell^p(V, \ell^p(\mathbb{N}))$ by setting

$$(TF)(x) = F(\alpha(x)).$$

Clearly, $\|T\| = 1$. Then, in the formula to be proved, the LHS is $\|(I - T)F\|_p^p$. Hence, the result immediately follows from the fact that the operator norm of $T - I$ is at most 2, by the triangle inequality.
2. We proceed as in the proof of Theorem 3 in [GN12]. Let $\{u_n\}_{n \in \mathbb{N}}$ be the standard basis vectors in $\ell^p(\mathbb{N})$. Write $F(x) = \sum_{n \in \mathbb{N}} F_n(x) u_n$, for all $x \in V$; we denote by $\alpha_n \in \mathbb{R}$ a projection of F_n on the subspace of constant functions in $\ell^p(V)$. It satisfies :

$$\inf_{\alpha \in \mathbb{R}} \|F_n - \alpha\|_p = \|F_n - \alpha_n\|_p.$$

By the proof of Theorem 3 in [GN12], the sum $w := \sum_{n \in \mathbb{N}} \alpha_n u_n$ belongs to $\ell^p(\mathbb{N})$. Defining $G(x) = F(x) - w$, so that $G_n(x) = F_n(x) - \alpha_n$, we observe that distortions of G and F are equal. Recalling the definition of $\lambda_1^{(p)}(X)$, we have for every n:

$$\sum_{x \in V} |G_n(x)|^p \leq \frac{1}{\lambda_1^{(p)}(X)} \sum_{e \in E} |G_n(e^+) - G_n(e^-)|^p.$$

Taking the sum over n, we get the result. □

We deduce the following Poincaré-type inequality for finite graphs :

Proposition 2.2.4. *Let $X = (V, E)$ and let $1 \leq p < \infty$ be a finite, connected graph with average degree k. Fix an arbitrary orientation of the edges. Then, for any permutation α of V and any embedding $F : V \to \ell^p(\mathbb{N})$, we have :*

$$\frac{1}{|V|2^p}\sum_{x\in V}\|F(x)-F(\alpha(x))\|_p^p \leq \frac{k}{2|E|\lambda_1^{(p)}(X)}\sum_{e\in E}\|F(e^+)-F(e^-)\|_p^p.$$

Proof: Let $F : V \to \ell^p(\mathbb{N})$ be any embedding, let G be the translation of F as in Lemma 2.2.3 and let k be the average degree of X. Combining both statements of Lemma 2.2.3 with the fact that $|E| = \frac{k|V|}{2}$, we obtain the desired equality for the map G. Since $\|G(x) - G(y)\|_p^p = \|F(x) - F(y)\|_p^p$, for all $x, y \in V$, the same inequality must hold for F. □

Proposition 2.2.5. *Let $X = (V, E)$ be a finite connected graph with average degree k. For any permutation α of V and any embedding $F : V \to \ell^p(\mathbb{N})$, we have:*

$$\rho(\alpha)\left(\frac{\lambda_1^{(p)}(X)}{k\ 2^{p-1}}\right)^{\frac{1}{p}} \leq dist(F).$$

Proof: Clearly, we may assume that α has no fixed point. Then:

$$\frac{1}{\|F^{-1}\|_{Lip}^p} = \min_{x\neq y}\frac{\|F(x)-F(y)\|_p^p}{d(x,y)^p} \leq \min_{x\in V}\frac{\|F(x)-F(\alpha(x))\|_p^p}{d(x,\alpha(x))^p}$$

$$\leq \frac{1}{\rho(\alpha)^p}\min_{x\in V}\|F(x)-F(\alpha(x))\|_p^p \leq \frac{1}{\rho(\alpha)^p|V|}\sum_{x\in V}\|F(x)-F(\alpha(x))\|_p^p$$

$$\leq \frac{2^{p-1}k}{\lambda_1^{(p)}(X)\rho(\alpha)^p|E|}\sum_{e\in E}\|F(e^+)-F(e^-)\|_p^p \text{ (by Proposition 2.2.4)}$$

$$\leq \frac{2^{p-1}k}{\lambda_1^{(p)}(X)\rho(\alpha)^p}\max_{x\sim y}\|F(x)-F(y)\|_p^p = \frac{2^{p-1}k}{\lambda_1^{(p)}(X)\rho(\alpha)^p}\|F\|_{Lip}^p,$$

where the last equality comes from the fact that the above maximum is attained for adjacent points in the graph (see for instance Claim 3.2 in [LM00]). Re-arranging and taking p-th roots, we get the result. □

Proof of Theorem 2.2.1: Theorem 2.2.1 for embeddings $V \to \ell^p(\mathbb{N})$ immediately follows from Proposition 2.2.5. Now, since ℓ^p embeds isometrically in L^p, we clearly have $c_p(X) \leq c_{\ell^p}(X)$. Actually $c_p(X) = c_{\ell^p}(X)$, since for every map $F : V \to L^p$ and every $\varepsilon > 0$, we can find a finite measurable partition $[0,1] = \bigcup_{j=1}^{k} \Omega_j$ and, for each $x \in V$, a step function $H(x)$ which is constant on each Ω_j, such that $\|F(x) - H(x)\|_p < \varepsilon$ for $x \in V$. Denoting by m the Lebesgue measure on $[0,1]$, the embedding $G : V \to \ell^p\{1,...,k\} : x \mapsto (H(x)|_{\Omega_j} m(\Omega_j)^{1/p})_{1\leq j\leq k}$ then satisfies $\|G(x) - G(y)\| = \|H(x) - H(y)\|_p$ for every $x, y \in V$, hence the distortion of G is $\delta(\varepsilon)$-close to the one of F, where $\delta(\epsilon) \to 0$ for $\varepsilon \to 0$. □

2.3 Estimates on the maximal displacement

From the definition of the invariant $D(X)$, we have $D(X) \leq diam(X)$. The equality holds if and only if the graph X admits an **antipodal map**, i.e. a permutation α of the vertices such that $d(x, \alpha(x)) = diam(X)$ for every $x \in V$. The existence of an antipodal map is a fairly strong condition. Recall that the **radius** of X is $\min_{x\in V} \max_{y\in V} d(x,y)$, so that the existence of an antipodal map implies that the radius is equal to the diameter of X. The converse is false however, a counter-example was provided by G. Paseman. A necessary and sufficient condition for X to admit an antipodal map was provided by R. Bacher. For a subset $S \subset V$, set

$$\mathcal{A}(S) = \{v \in V : \exists w \in S, d(v,w) = diam(X)\}$$

The graph X admits an antipodal map if and only if $|\mathcal{A}(S)| \geq |S|$ for every $S \subset V$. For all this, see [Mat].
The proof of Corollary 2.2.2 follows immediately from Theorem 2.2.1 and the next lemma:

Lemma 2.3.1. *Finite, connected, vertex-transitive graphs admit antipodal maps.*

Proof: For S a finite subset of the vertex set of some graph Y, denote by $\Gamma(S)$ the set of vertices adjacent to at least one vertex of S. It is classical that, if Y is a regular graph, then the inequality $|\Gamma(S)| \geq |S|$ holds[2]. Now, let $X = (V, E)$ be a finite, connected, vertex-transitive graph. Define the *antipodal graph* X^a as the graph with vertex set V, with x adjacent to y whenever the distance between x and y in X, is equal to $diam(X)$. By vertex-transitivity of X, the graph X^a is regular. Now observe that, for $S \subset V$, the

[2]Recall the easy argument: assuming that Y is k-regular, count in two ways the edges joining S to $\Gamma(S)$; as edges emanating from S, there are $k|S|$ of them; as edges entering $\Gamma(S)$, there are at most $k|\Gamma(S)|$ of them.

set $\Gamma(S)$ in X^a is exactly the set $\mathcal{A}(S)$ defined above. By regularity of X^a and the observation beginning the proof, we therefore have $|\mathcal{A}(S)| \geq |S|$ for every $S \subset V$, and Bacher's result applies. □

Remark 2.3.2. *For Cayley graphs, there is a direct proof of the existence of antipodal maps. Indeed, let G be a finite group, and let X be a Cayley graph of G with respect to some symmetric, generating set S; use right multiplications by generators to define X, so that the distance d is left invariant. Let $g \in G$ be any element of maximal word length with respect to S. Then $\alpha(x) = xg$ (right multiplication by g) is an antipodal map.*

For arbitrary graphs, we have:

Proposition 2.3.3. *For finite, connected graphs X with maximal degree $k \geq 3$:*

$$D(X) = \Omega(\log |X|).$$

Proof: For a positive integer $r > 0$, the number of vertices in X at distance at most r from a given vertex, is at most the number of vertices in the ball of radius r in the k-regular tree, i.e.

$$1 + k + k(k-1) + k(k-1)^2 + ... + k(k-1)^{r-1} = \frac{k(k-1)^r - 2}{k-2}.$$

For $r = [\log_{k-1}(\frac{|V|}{6})]$, we have $\frac{k(k-1)^r-2}{k-2} < \frac{|V|}{2}$. Let Y be the graph with the same vertex set V as X, where two vertices are adjacent if their distance in X is at least $\log_{k-1}(\frac{|V|}{6})$. The preceding computation shows that, in the graph Y, every vertex has degree at least $\frac{|V|}{2}$. By G.A. Dirac's theorem (see e.g. Theorem 2 in Chapter IV of [Bol79]), Y admits a Hamiltonian circuit. Let $\alpha \in Sym(V)$ be the cyclic permutation of V defined by this Hamiltonian circuit. Then $\rho(\alpha) \geq \log_{k-1}(\frac{|V|}{6})$, which concludes the proof. □

The following inequality due to Alon and Milman (see Theorem 2.7 in [AM85]) shows that Proposition 2.3.3 is essentially the best possible. For any connected graph $X = (V, E)$ with degree bounded by k, we have

$$diam(X) \leq 2\sqrt{\frac{2k}{\lambda_1^{(2)}(X)}}\ \log_2 |X|,$$

We now observe that, for families of non-vertex-transitive k-regular graphs, the maximal displacement can be much smaller than the diameter (compare with Lemma 2.3.1). We thank the referee of a first version of the paper [JV14] for suggesting this construction.

Proposition 2.3.4. *Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(n) = \Omega(n)$ and $f(n) = o(8^n)$. There exists a family $(X_n)_{n\geq 1}$ of 3-regular graphs such that:*

1. $|X_n| = \Theta(8^n)$;
2. $diam(X_n) = \Theta(f(n))$;
3. $D(X_n) = \Theta(n)$.

Proof: Let $(Y_n)_{n\geq 1}$ be a family of 3-regular graphs with $|Y_n| = \Theta(8^n)$ and $diam(Y_n) = \Theta(n)$ (such a family is constructed e.g. in Theorem 5.13 of Morgenstern [Mor94]). Let Z_n be the product of the cycle $C_{2f(n)}$ with the one-edge graph (so that Z_n is 3-regular on $4f(n)$ vertices). Let $\{y_1, y_2\}$ (resp. $\{z_1, z_2\}$) be an edge in Y_n (resp. Z_n). We "stitch" Y_n and Z_n by replacing the edges $\{y_1, y_2\}$ and $\{z_1, z_2\}$ by edges $\{y_1, z_1\}$ and $\{y_2, z_2\}$, and define X_n as the resulting 3-regular graph. Clearly $|X_n| = \Theta(8^n)$.
Observe that, since every edge in Z_n belongs to some 4-cycle, the distance in X_n between any two vertices in Y_n will differ by at most 5 from the original distance in Y_n; and similarly for vertices in Z_n. So:

$$f(n) = diam(Z_n) \leq diam(X_n) \leq diam(Y_n) + diam(Z_n) + 5,$$

hence $diam(X_n) = \Theta(f(n))$.
Finally, let α be any permutation of the vertices of X_n. Since the overwhelming majority of vertices belongs to Y_n, we find a vertex x such that x and $\alpha(x)$ are both in Y_n. Then

$$\rho(\alpha) \leq d_{X_n}(x, \alpha(x)) \leq d_{Y_n}(x, \alpha(x)) + 5 \leq diam(Y_n) + 5,$$

hence $D(X_n) = O(n)$. The equivalence $D(X_n) = \Theta(n)$ then follows from Proposition 2.3.3. □

2.4 Applications

We give a series of consequences of Theorem 2.2.1 and Corollary 2.2.2.

2.4.1 Expanders

We start with a fundamental definition.

Definition 2.4.1. *A sequence $(X_n)_n$ of finite graphs is a* ***family of expanders*** *if the following two conditions are satisfied :*

1. *The sequence of spectral gaps is bounded below, that is,*

$$\inf_{n\geq 1} \lambda_1^{(2)}(X_n) > 0.$$

2. *The family of graphs is increasing, that is,*

$$\lim_{n\to\infty} |V_n| = \infty.$$

The expanders show that Bourgain's upper bound is asymptotically tight.

Corollary 2.4.2. *([LLR95],[Mat97]) Let $(X_n)_{n\geq 1}$ be a family of expanders with bounded degree. For every $p \geq 1$, we have $c_p(X_n) = \Omega(\log |X_n|)$.*

Proof: If $(X_n)_n$ is a family of expanders, then by Theorem 2.1.8, the sequence $(\lambda_1^{(p)}(X_n))_n$ is bounded away from 0. So the result follows straight from Theorem 2.2.1 together with Proposition 2.3.3. □

2.4.2 Cycles

We will denote by C_n, the graph consisting in the cycle of length n. This graph is isomorphic to the Cayley graph of the cyclic group of order n (seen additively), with generating set $S = \{\pm 1\}$.

Corollary 2.4.3. *(Linial-Magen [LM00]) For n even: $c_2(C_n) = \frac{n}{2}\sin\frac{\pi}{n}$.*

Proof: We apply Corollary 2.2.2 with $k = 2$, and $diam(C_n) = \frac{n}{2}$, and $\lambda_1^{(2)}(C_n) = 4\sin^2\frac{\pi}{n}$ (by a direct computation using Proposition 2.1.9, or see Example 1.5 in [Chu97]): so $c_2(C_n) \geq \frac{n}{2}\sin\frac{\pi}{n}$. For the converse inequality, it is an easy computation that the embedding of C_n as a regular n-gon in $\mathbb{R}^2$, has distortion $\frac{n}{2}\sin\frac{\pi}{n}$. □

2.4.3 The hypercube H_d

The hypercube H_d is the set of d-tuples of 0's and 1's, endowed with the Hamming distance ρ_H. It counts the number of digits that differ between two strings and it can be decribed as

$$\rho_H(x,y) = \sum_{j=1}^{d} (x_j + y_j \mod 2),$$

for $x, y \in \mathbb{F}_2^d$. Note that the hypercube is isometric to the Cayley graph of $\mathbb{F}_2^d$ with respect to the standard basis $e_1, \ldots, e_d$.

Corollary 2.4.4. *(Enflo [Enf69])* $c_2(H_d) = \sqrt{d}$

Proof: For H_d, we have $k = d$, and $diam(H_d) = d$, and $\lambda_1^{(2)}(H_d) = 2$ (by a direct computation using Proposition 2.1.9, or see Example 1.6 in [Chu97]): so $c_2(H_d) \geq \sqrt{d}$ by Corollary 2.2.2. For the converse inequality, it is easy to see that the canonical embedding of H_d into $\mathbb{R}^d$, has distortion $\sqrt{d}$. □

2.4.4 Quotients of the hypercube

We recall some definitions from coding theory. A **code** C is a vector subspace of $\mathbb{F}_2^d$, endowed with the Hamming distance ρ_H. For a code C, the **weight** of C is $w(C) = \min_{x \in C \setminus \{0\}} \rho_H(x, 0)$. The **dual code** of C is

$$C^\perp = \{y \in \mathbb{F}_2^d \ : \ x \cdot y = 0, \ \forall x \in C\},$$

with $x \cdot y = \sum_{j=1}^{d} x_j y_j$. It is easy to see that, whenever C is a code, $C^\perp$ is also a code, and $|C^\perp| = 2^{d - \dim(C)}$ (see Chapter 1 of [HP03]).
As in Section 3.1 of [KN06] and in Section 4 of [NR09], we consider quotients $\mathbb{F}_2^d / C^\perp$, viewed as Cayley graphs with respect to the image of the canonical generating set of $\mathbb{F}_2^d$ via the natural quotient map, that is,

$$\{e_j + C^\perp \ : \ 1 \leq j \leq d\}.$$

Lemma 2.4.5. *([NR09] Proposition 4.1) If C is a code in $\mathbb{F}_2^d$, then*

$$\lambda_1^{(2)}(\mathbb{F}_2^d) = 2w(C).$$

Proof: Here, the group $\widehat{\mathbb{F}_2^d/C^\perp}$ identifies with C, via the group isomorphism $C \to \widehat{\mathbb{F}_2^d/C^\perp} : x \mapsto \chi_x$, where

$$\chi_x(y + C^\perp) = (-1)^{x \cdot y}.$$

Applying Proposition 2.1.9, we obtain

$$\begin{aligned} \lambda_1^{(2)}(\mathbb{F}_2^d/C^\perp) &= \min_{x \in C \setminus \{0\}} \sum_{j=1}^{d} (1 - (-1)^{x \cdot e_j}) \\ &= 2 \min_{x \in C \setminus \{0\}} \sum_{j=1}^{d} x_j \\ &= 2w(C), \end{aligned}$$

which proves the result. □

Lemma 2.4.6. *(Proposition 3.5 [NR09]) If C is a code in $\mathbb{F}_2^d$ such that* $\dim(C) \geq \frac{d}{4}$*, then*

$$diam(\mathbb{F}_2^d/C^\perp) = \Omega(\log |\mathbb{F}_2^d/C^\perp|).$$

Proof : Let $\delta \in (\frac{1}{d}, \frac{1}{2})$ be a parameter, to be specified later. By a counting argument, we will show that, for an appropriate choice of δ, we can always find an element of $\mathbb{F}_2^d/C^\perp$ being outside of the ball of radius δd, for all d large enough. This will imply that the diameter of the Cayley graph is greater than δd. Since we certainly have $d \geq \log_2 |\mathbb{F}_2^d/C^\perp|$, we can conclude that $diam(\mathbb{F}_2^d/C^\perp) \geq \delta d \geq \delta \log_2 |\mathbb{F}_2^d/C^\perp|$, for all d large enough, and this proves the desired result.
To prove the claim, we first observe that, since $\mathbb{F}_2^d/C^\perp$ is an abelian group endowed with a system of d generators of order 2, the number of distinct words of length n is at most $\binom{d}{n}$. Then, the number of distinct words of length at most δd in $\mathbb{F}_2^d/C^\perp$ is bounded above by

$$\sum_{n=1}^{\delta d} \binom{d}{n} \leq 2\nu(\delta)^d \sqrt{\delta d},$$

where $\nu(\delta) = [\delta^\delta (1-\delta)^{1-\delta}]^{-1}$ (see formula (3) in Lemma 3.2 [KN06]). Set $\delta = 10^{-2}$. Then, for any $d \geq 100$, it is easy to check that both quantities $\nu(\delta)^d$ and $2\sqrt{\delta d}$ are strictly smaller that $2^{\frac{d}{8}}$. In this case, we can conclude that the size of the ball of radius δd is strictly smaller than $2^{\frac{d}{4}}$. Since the code C satisfies $\dim(C) \geq \frac{d}{4}$, we have $|\mathbb{F}_2^d/C^\perp| \geq 2^{\frac{d}{4}}$. Therefore, this ball cannot contain every element of $\mathbb{F}_2/C^\perp$, proving the claim. $\square$

As a corollary, we obtain the existence of a family of arbitrary large regular abelian Cayley graphs whose p-distortion is logarithmic in the number of vertices. Again, this shows that Bourgain's upper bound is asymptotically optimal.

Corollary 2.4.7. *(Corollary 3.5 [KN06]) Let $1 \leq p < \infty$. There exists a sequence of codes $(C_d)_{d\geq 1}$, with $C_d \subset \mathbb{F}_2^d$, satisfying*

$$c_p(\mathbb{F}_2^d/C_d^\perp) = \Omega(\log |\mathbb{F}_2^d/C_d^\perp|).$$

Proof : For any code C, the p-spectral gap

$$\lambda_1^{(p)}(\mathbb{F}_2^d/C^\perp) \geq \left(\frac{2}{d}\right)^{p-1} \left(\frac{w(C)}{p}\right)^p,$$

by Theorem 2.1.8 and Lemma 2.4.5. Applying Theorem 2.2.2, we obtain the lower bound

$$c_p(\mathbb{F}_2^d/C^\perp) \geq \frac{diam(\mathbb{F}_2^d/C^\perp) w(C)}{dp}.$$

Thus, in view of Lemma 2.4.6, it is enough to find codes C_d such that $\dim(C_d) \geq \frac{d}{4}$ and $w(C_d) = \Omega(d)$. It is classical that such codes exist (see Lemma 15.6.2 [Mat02]), and we are done. $\square$

2.4.5 Cayley graphs of $SL_n(q)$

Let q be a fixed prime, and let Y_n be the Cayley graph of $SL_n(q)$ (where $n \geq 2$) with respect to the following set of 4 generators : $S_n = \{A_n^{\pm 1}, B_n^{\pm 1}\}$ and

$$A_n = \begin{pmatrix} 1 & 1 & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}; \quad B_n = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ (-1)^{n-1} & & & & 0 \end{pmatrix}.$$

Proposition 2.4.8. $c_2(Y_n) = \Omega(n^{1/2}) = \Omega((\log |Y_n|)^{1/4})$.

The interest of the family $(Y_n)_{n\geq 2}$ comes from the fact that it is known NOT to be an expander family : see Proposition 3.3.3 in [Lub10].

Proof : Since $|SL_n(q)| \approx q^{n^2-1}$, we have $diam(Y_n) = \Omega(n^2)$ (actually it is a result by Kassabov and Riley [KR07] that $diam(Y_n) = \Theta(n^2)$). On the other hand, from Kassabov's estimates for the Kazhdan constant $\kappa(SL_n(\mathbb{Z}), S_n)$ (see [Kas05], and also the Introduction of [KR07]), we have: $\kappa(SL_n(\mathbb{Z}), S_n) = \Omega(n^{-3/2})$. If X is a Cayley graph of a finite quotient of a Kazhdan group G, with respect to a finite generating set $S \subset G$, then $\lambda_1^{(2)}(X) \geq \frac{\kappa(G,S)^2}{2}$ (see [Lub10], Proposition 3.3.1 and its proof). From this we get: $\sqrt{\lambda_1^{(2)}(Y_n)} = \Omega(n^{-3/2})$ and therefore $c_2(Y_n) = \Omega(n^{1/2})$ by Corollary 2.2.2. □

2.4.6 Coxeter groups

We say that W is a **Coxeter group** if there exists a generating set $S = \{s_1, s_2, \dots\}$ such that W admits the following presentation

$$\langle S \mid (s_i s_j)^{m_{i,j}} = 1\rangle,$$

with $m_{i,j} \in \{2, 3, \dots\} \cup \{\infty\}$ and $m_{i,i} = 2$, for all i. If $m_{i,j} = \infty$, this means that there is no relation between s_i and s_j. The couple (W, S) is called a **Coxeter system**, and, in the case where S is finite, we define the **rank** of (W, S) as the integer $|S|$. By the classification of finite Coxeter groups, the following examples are fundamental (see for instance [Hum90] for all this). We will denote by A_n, $(n \geq 2)$, the Coxeter group of rank n, with the generators satisfying the relations according to the integers $m_{i,j}$ given by

$$m_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 3, & \text{if } j = i + 1 \text{ and } 1 \leq i \leq n - 1, \\ 2, & \text{otherwise.} \end{cases}$$

The group A_n is isomorphic to $\mathrm{Sym}(n + 1)$, generated by the transpositions $s_i = (i\ i + 1)$. We will denote by B_n the Coxeter group of rank n, with the generators satisfying the relations according to the integers $m_{i,j}$ given by

$$m_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 3, & \text{if } j = i + 1 \text{ and } 1 \leq i \leq n - 2, \\ 4, & \text{if } i = n - 1 \text{ and } j = n, \\ 2, & \text{otherwise.} \end{cases}$$

To realize B_n as a matrix group, we can take the generators s_i to be the matrices (acting on $\mathbb{R}^n$) that permute the i-th and $(i+1)$-th coordinates, for all $1 \leq i \leq n-2$, and we can take s_n to be the matrix that changes the sign of the n-th coordinate. The group B_n is isomorphic to the semi-direct product $\mathrm{Sym}(n) \ltimes \mathbb{Z}_2^n$. Finally, we will denote by D_n the Coxeter group of rank n, with the generators satisfying the relations according to the integers $m_{i,j}$ given by

$$m_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 3, & \text{if } j = i+1 \text{ and } 1 \leq i \leq n-2, \\ 3, & \text{if } i = n-2 \text{ and } j = n, \\ 2, & \text{otherwise.} \end{cases}$$

It can be shown that D_n is a subgroup of B_n of index 2.
We can now state the main result of this subsection.

Proposition 2.4.9. *Let X_n be a Cayley graph associated with a Coxeter group of type A_n, B_n or D_n with respect to its standard generating set. Then, for $1 \leq p < \infty$, we have the upper bound*

$$c_p(X_n) = \begin{cases} O\left(n^{2-\frac{2}{p}}\right), & \text{for } p \in [1,2], \\ O(n \log n), & \text{for } p > 2, \end{cases}$$

and for $p = 2$, we also have the lower bound

$$c_2(X_n) = \Omega(\sqrt{n}).$$

Proof : The upper bound follows from [BJS88], where it is proved that, for any $1 \leq p < \infty$, and for any Coxeter system (W, S) (not necessarily finite), there exists an embedding $f : W \to \ell^p(\mathbb{N})$ satisfying

$$d_S(x, y) = \|f(x) - f(y)\|_p^p,$$

for all $x, y \in W$ (where d_S is the word metric with respect to the generating set S). If W is finite, then this embedding is bi-Lipschitz and of distortion $diam(W)^{1-\frac{1}{p}}$. From Theorem 2.15 in [Ron89] and p.44 of [Hum90], we get $diam(X_n) = \Theta(n^2)$, and this shows that $c_p(X_n) = O(n^{2-\frac{2}{p}})$. Then, by remarking that Bourgain's upper bound, which is of the order of $O(n \log n)$, is better for $p > 2$, we obtain the result. To prove the lower bound, we apply Theorem 2.2.2 with the estimate $\lambda_1^{(2)}(X_n) = \Theta(n^{-2})$ (see [Kas11]). □

2.4.7 Lamplighters over discrete tori

We first recall that, if G is a finite group, the lamplighter group of G is the wreath product $C_2 \wr G$, i.e. the semi-direct product of the additive group of all subsets of G (endowed with symmetric difference) with G acting by shifting indices. Take $G = C_n^d$ and denote by $\{\pm e_j \ : \ 1 \leq j \leq d\}$ the standard symmetric generating set for C_n^d, and denote by W_n^d the Cayley graph of the lamplighter group $C_2 \wr C_n^d$, with respect to the generating set

$$S = \{(\{0\}, 0)\} \cup \{(\emptyset, \pm e_j) : 1 \leq j \leq d\},$$

(so that W_n^d is $(2d+1)$-regular). We will prove the following :

Proposition 2.4.10. $c_2(W_n^d) = \begin{cases} \Omega\left(\frac{n}{\sqrt{\log(n)}}\right), & \textit{for } d = 2, \\ \Omega(n^{\frac{d}{2}}), & \textit{for } d \geq 3. \end{cases}$

Proof : Once again we apply Corollary 2.2.2. Let us define the matrix M on $C_2 \wr C_n^d$ given by

$$M_{[(f,a),(g,b)]} = \begin{cases} \frac{1}{4} & \text{if } (f,a) = (g,b); \\ \frac{1}{4} & \text{if } a = b \text{ and } f = g + \delta_a; \\ \frac{1}{16d} & \text{if } a = b \pm e_j \text{ and } f(z) = g(z), \forall z \notin \{a, b\}; \\ 0 & \text{otherwise.} \end{cases}$$

($a, b \in C_n^d$ and $f, g : C_n^d \to \{0, 1\}$). Then M is the transition matrix of the lazy random walk on $C_2 \wr C_n^d$ analysed by Peres and Revelle in Theorem 1.1 of [PR04]. Using their estimation of the relaxation time of M, we deduce that the spectral gap of M behaves as $\Theta(\frac{1}{n^d})$ for $d \geq 3$ and as $\Theta(\frac{1}{n^2 \log(n)})$ for the case $d = 2$. By standard comparison theorems (see e.g. Theorems 3.1 and 3.2 in [Woe00]), the Dirichlet forms for M and for the Laplace operator on W_n^d are bi-Lipschitz equivalent ; moreover the Lipschitz constants do not depend on n (since the comparison can be made on the group $C_2 \wr \mathbb{Z}^d$, of which our lamplighters are quotients). So, we find $\lambda_1^{(2)}(W_n^2) = \Theta(n^{-2} \log(n)^{-1})$ and $\lambda_1^{(2)}(W_n^d) = \Theta(n^{-d})$ for $d \geq 3$. Furthermore, since the diameter of a regular graph is at least logarithmic in the number of vertices, we have $diam(W_n^d) = \Omega(n^d)$, so we apply Corollary 2.2.2 to get:

$$c_2(W_n^d) = \begin{cases} \Omega\left(\frac{n}{\sqrt{\log(n)}}\right) & \text{for } d = 2, \\ \Omega\left(n^{\frac{d}{2}}\right) & \text{for } d \geq 3. \end{cases}$$

□

However, this method does not give a good estimate for the case $d = 1$, as we will see at the end of this section.

2.4.8 The limits of the method

We give examples of Cayley graphs for which the lower bound of the Euclidean distortion given by Corollary 2.2.2 is not tight.

Products of cycles

Let us consider the product of 2 cycles $C_n \times C_N$, where n, N are even integers such that $n < N$. It is clear that it corresponds to the Cayley graph of the additive group $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ with generating set $S = \{(\pm 1, 0), (0, \pm 1)\}$. Since for the product of finite abelian groups G, H, we can identify the dual of $G \times H$ as $\{\chi \cdot \eta \,:\, \chi \in \hat{G}, \eta \in \hat{H}\}$, it is easy to see that $\lambda_1(C_n \times C_N) = 4\sin^2 \frac{\pi}{N}$, by Proposition 2.1.9. As the diameter is equal to $\frac{n+N}{2}$, we get the lower bound

$$c_2(C_n \times C_N) \geq \frac{(n+N)\sin\frac{\pi}{N}}{2\sqrt{2}}.$$

On the other hand, it is known from [LM00] that the normalized trivial embedding of $C_n \times C_N$ into $\mathbb{C}^2$ gives the optimal embedding. Namely, defining

$$\phi : C_n \times C_N \to \mathbb{C}^2 : (k, l) \mapsto \left(\frac{\exp\frac{2\pi i k}{n}}{2\sin\frac{\pi}{n}}, \frac{\exp\frac{2\pi i l}{N}}{2\sin\frac{\pi}{N}} \right)$$

we have

$$c_2(C_n \times C_N) = dist(\phi).$$

Since $\|\phi(x) - \phi(y)\| \leq 1$ for every $x, y \in C_n \times C_N$, we have to estimate

$$\|\phi^{-1}\|_{Lip} = \max_{k \leq \frac{n}{2}, l \leq \frac{N}{2}} \frac{k+l}{\sqrt{\frac{sin^2\frac{\pi k}{n}}{sin^2\frac{\pi}{n}} + \frac{sin^2\frac{\pi l}{N}}{sin^2\frac{\pi}{N}}}}.$$

By taking $k = \frac{n}{2}$ and $l = \frac{N}{2}$, we get

$$dist(\phi) \geq \frac{n+N}{2\sqrt{\sin^{-2}\frac{\pi}{n} + \sin^{-2}\frac{\pi}{N}}}.$$

Since it is always the case that

$$\sqrt{\frac{1}{\sin^{-2}\frac{\pi}{n} + \sin^{-2}\frac{\pi}{N}}} > \frac{\sin\frac{\pi}{N}}{\sqrt{2}},$$

we conclude that the lower bound given by Corollary 2.2.2 is not sharp in this case.

Lamplighter groups over the discrete circle

Here we consider the graphs W_n^1 associated with the lamplighter groups $C_2 \wr C_n$, with respect to the generating S described in Subsection 2.4.7. It is known from [ANV10] that $c_2(W_n^1) = \Theta(\sqrt{\log(n)})$. By way of contrast, let us check that $diam(W_n^1)\sqrt{\lambda_1^{(2)}(W_n^1)} = O(1)$. Let us first estimate the spectral gap. Let us consider the homomorphism χ, which factors through the epimorphism $C_2 \wr C_n \to C_n$, given by $\chi(A,k) = e^{2\pi i k/n}$. Here we get

$$\begin{aligned}\lambda_1^{(2)}(W_n^1) &\leq \sum_{s\in S}(1-\chi(s)) \\ &= 2 - 2\cos(2\pi/n) = 4\sin^2(\pi/n).\end{aligned}$$

Hence, $\lambda_1^{(2)}(W_n^1) = O(\frac{1}{n^2})$. On the other hand, by Theorem 1.2 in [Par92], the word length of $(A,k) \in C_2 \wr C_n$ is equal to $|A| + \ell(A,k)$, where $\ell(A,k)$ is the length of the shortest path in the cycle C_n, going from 0 to k and containing A. For $A = C_n$, we have $\ell(C_n,k) = n-1+\min(|n-1-k|,|k-1|)$. From this, it is clear that $diam(W_n^1) \leq 3n$. This finishes the proof of the claim.

2.5 Comparison with similar inequalities

Lower bounds of spectral nature on $c_2(X)$, can be traced back to [LLR95]. At least two other inequalities (see [GN12, NR09]) linking the distortion, the p-spectral gap and other graph invariants have been published. In this section, we compare them to Theorem 2.2.1.

2.5.1 Grigorchuk-Nowak Inequality

In their paper [GN12], Grigorchuk and Nowak introduce the following invariant for finite metric spaces.

Definition 2.5.1. *Let X be a finite metric space. Given $0 < \epsilon < 1$ define the constant $\rho_\epsilon(X) \in [0,1]$, called the* ***volume distribution****, by the relation*

$$\rho_\epsilon(X) = \min\left\{\frac{diam(A)}{diam(X)} \ : \ A \subset X \text{ such that } |A| \geq \epsilon|X|\right\}.$$

Theorem 2.5.2. *([GN12] Theorem 3) Let X be a connected graph of degree bounded by k and let $1 \leq p < +\infty$. Then, for every $0 < \epsilon < 1$,*

$$\frac{(1-\epsilon)^{\frac{1}{p}}\rho_\epsilon(X)}{2^{\frac{1}{p}}}\; diam(X) \left(\frac{\lambda_1^{(p)}(X)}{k\; 2^{p-1}}\right)^{\frac{1}{p}} \leq c_p(X).$$

It is easy to see that, when the graph satisfies $D(X) = diam(X)$ (this is the case for vertex-transitive graphs, by Lemma 2.3.1), then this result is weaker than Theorem 2.2.1, since the factor $\frac{(1-\epsilon)^{\frac{1}{p}}\rho_\epsilon(X)}{2^{\frac{1}{p}}}$ is strictly smaller than 1. We end up this subsection by an improvement of Proposition 7 in [GN12].

Proposition 2.5.3. *Let (X, d) be a finite metric space that admits an antipodal map.*

1. *The volume distribution of X satisfies $\rho_\epsilon(X) = 1$, for all $\epsilon > \frac{1}{2}$.*
2. *Moreover, if we assume that (X, d) is isometric to a connected graph endowed with the graph metric, then $\rho_{\frac{1}{2}}(X) \geq \frac{diam(X)-1}{diam(X)}$.*

Proof : Let α be any antipodal map for X. For the first claim, we need to prove that, if $A \subset X$ satisfies $|A| > \frac{|X|}{2}$, then $diam(A) = diam(X)$. By contraposition, let α be an antipodal map and let $A \subset X$ be a subset satisfying $diam(A) < diam(X)$. It follows that $\alpha(A) \subset A^c$ and, recalling that α is a bijection, we can conclude that $|A| \leq \frac{|X|}{2}$.
To prove the second claim, let $B \subset X$ be a subset satisfying $|B| = \frac{|X|}{2}$. If $\alpha(B) \cap B$ is not empty, then B has maximal diameter. If not, then we can decompose X as a disjoint union of B and $\alpha(B)$. Since X is endowed with the structure of a (connected) graph, there exist $x, y \in B$ such that $d(x, \alpha(y)) = 1$. Then, by the triangle inequality, we conclude that $diam(B) \geq diam(X) - 1$, and this proves the second claim. □

The example of the hypercube H_d shows that this result is optimal. Finally, using Theorem 2.5.2 and Proposition 2.5.3, we obtain the following corollary.

Corollary 2.5.4. *Let X be a connected graph of degree bounded by k and let $1 \leq p < +\infty$. If X admits an antipodal map, then,*

$$\frac{diam(X)}{2^{\frac{2}{p}}} \left(\frac{\lambda_1^{(p)}(X)}{k\; 2^{p-1}}\right)^{\frac{1}{p}} \leq c_p(X).$$

2.5.2 Newman-Rabinovich Inequality

The second result, due to Newman-Rabinovich [NR09], holds for $p = 2$:

Proposition 2.5.5. *([NR09] Proposition 3.2) Let $X = (V, E)$ be a k-regular graph. Then,*

$$\sqrt{\frac{(|V|-1)\lambda_1^{(2)}(X)}{|V|\ k}\ avg(d^2)} \leq c_2(X),$$

where $avg(d^2) = \frac{1}{|V|(|V|-1)} \sum_{x,y \in V} d(x,y)^2$.

In the following, we will compute the term $avg(d^2)$ for the cycle C_n and for the hypercube H_d in order to give explicitly the LHS term of the inequality due to Newman and Rabinovich. First, it is true that for a vertex-transitive graph $X = (V, E)$, we have

$$\sum_{y,x \in V} d(x,y)^2 = |V| \sum_{j=1}^{diam(X)} j^2 |S(x_0, j)|,$$

where x_0 is an arbitrary point in X and $S(x_0, j)$ is the sphere of radius j, centered in x_0. By taking $n \geq 4$ and even, we clearly have

$$\sum_{x,y \in C_n} d(x,y)^2 = n \left(2 \sum_{j=1}^{\frac{n}{2}-1} j^2 + \frac{n^2}{4} \right) = \frac{n^2(n^2+2)}{12}.$$

Therefore, we get $\sqrt{\frac{n^2+2}{6}} \sin \frac{\pi}{n}$ as lower bound for $c_2(C_n)$, which is strictly weaker than Corollary 2.4.3. On the other hand, for the hypercube H_d, by the same argument, we have

$$avg(d^2) = \frac{1}{2^d(2^d-1)} \sum_{x,y \in H_d} d(x,y)^2 = \frac{1}{2^d-1} \sum_{j=1}^{d} j^2 \binom{d}{j}.$$

Since $\sum_{j=1}^{d} j^2 \binom{d}{j} < d^2 2^{d-1}$ for $d \geq 2$, we conclude that Corollary 2.4.4 gives a better lower bound for $c_2(H_d)$.

2.5.3 Linial-Magen-Naor Inequality

Finally, we mention for completeness a remarkable result, of a different nature, due to Linial, Magen and Naor [LMN02] :

Theorem 2.5.6. *([LMN02], Theorem 1.3) There is a universal constant $C > 0$ such that, for every k-regular graph X with girth g:*

$$c_2(X) \geq \frac{Cg}{\sqrt{\min\{g, \frac{k}{\lambda_1^{(2)}(X)}\}}}.$$

Observe however that, for the family $(H_d)_{d\geq 2}$ of hypercubes, the right-hand side of the inequality remains bounded, while $c_2(H_d) = \sqrt{d}$ by Corollary 2.4.4.

2.6 A remark on antipodal maps

In most of the examples we considered for which an antipodal map was explicitly known, this permutation was in fact a graph automorphism (e.g. the cycle, hypercube and its quotients). However, antipodal maps are not graph automorphisms in general. The next proposition gives a helpful criterion to provide such examples of antipodal maps.

Proposition 2.6.1. *Let $X = (G, S)$ be a Cayley graph of a finite group G, with a symmetric generating set S, not containing the identity.*

1. *If there exists a unique element w_0 of maximal length relative to S, then, $w_0^{-1} = w_0$ and there is a unique antipodal map α on X.*
2. *Furthemore, the map α is a graph automorphism if and only if w_0 normalises S, that is,*
$$w_0 S w_0 = S.$$

Proof : By vertex-transitivity, for any $x \in V$, there is a unique antipodal vertex y. Therefore, the map defined by $\alpha(x) = xw_0$ is antipodal and is unique. It also follows that $w_0^{-1} = w_0$. To prove the second claim, we observe that α is an automorphism if and only if, for all $x_0 \in G$ and $s_0 \in S$, we can find $s \in S$ so that $\alpha(x_0 s_0) = \alpha(x_0)s$, which is equivalent to $s_0 w_0 = w_0 s$, and we are done. □

It is well-known that finite Coxeter groups satisfy the assumptions of Proposition 2.6.1. However, it is easy to give examples of Cayley graphs without any antipodal map being a graph automorphism. Let $n \geq 2$. Recall that we denote by W_{2n}^1 the Cayley graph of the group $C_2 \wr C_{2n}$ with the standard generating set. Using the notations of Subsection 2.4.7, it is easy to see that the element $w_0 = (C_{2n}, n)$ has maximal length, and that the generator $(\{0\}, 0)$ conjugated by w_0 is $(\{n\}, 0)$, and does not belong to the chosen generating set.

Chapter 3

Coarse embeddings

This chapter is devoted to the study of coarse embeddings of infinite metric spaces into $L^p([0,1])$ spaces. In the first section, we give the necessary definitions about coarse geometry to introduce the L^p-compression exponent. In order to get the reader familiar with this concept, we state some known results in the second section. In Section 3.3, we present a new method, based on Proposition 3.3.4, to estimate the L^p-compression exponent of box spaces and apply this to box spaces of families of finite graphs encountered in Chapter 2. This leads to new proofs of Corollary 3.3.5 and Corollary 3.3.6. (these results already appeared in the joint work [JV14]).
Finally, in the last section, we study the behaviour of compression exponent under group constructions by stating some known results and then, by computing the L^p-compression of some particular HNN extensions. Those results were obtained as a collaboration with Thibault Pillon and published in [JP13].

3.1 Definitions

In [GK04], Guentner and Kaminker introduced a numerical quasi-isometry invariant called compression exponent to characterize how close to a quasi-isometry a coarse embedding can be. Roughly speaking, it gives a way of quantifying how well a metric space sits inside another. We recall some basic definitions of large-scale geometry.

Definition 3.1.1. *Let* (X, d) *and* (Y, δ) *be two metric spaces. A map* $F : X \to Y$ *is a* ***coarse embedding*** *of* X *into* Y *if there exist non-decreasing functions* $\rho_-, \rho_+ : \mathbb{R} \to \mathbb{R}$ *so that the following conditions hold:*

1. $\rho_-(d(x, y)) \leq \delta(F(x), F(y)) \leq \rho_+(d(x, y))$*, for all* $x, y \in X$*;*

2. $\lim_{t\to+\infty} \rho_-(t) = +\infty$*.*

Coarse embeddings are the maps that preserve the large scale geometry of metric spaces. In the notations of the definition above, if we take two points in X which are far away from each other, then the two conditions imply that their images under F have to be far away from each other in Y. To study the behaviour of such embeddings, we look at specific coarse maps and try to estimate their control functions $\rho_\pm$ as well as possible.

Definition 3.1.2. *Let* $F : X \to Y$ *be a coarse embedding with control functions* $\rho_\pm$*. Then* F *is said to be :*

1. ***Large-scale Lipschitz****, if* ρ_+ *can be taken of the form* $t \mapsto At + B$*, for some constants* $A, B \geq 0$*.*

2. *A* ***quasi-isometric embedding****, if* $\rho_\pm(t) = C^{\pm 1}t \pm D$*, for some constants* $C, D \geq 0$*.*

3. *A* ***quasi-isometry****, if* F *is a quasi-isometric embedding and if* $F(X)$ *is* ***quasi-dense*** *in* Y*, that is, there is a constant* $C \geq 0$ *such that* $\delta(y, F(X)) \leq C$*, for all* $y \in Y$*.*

Clearly, if X is a uniformly 1-discrete metrice space, then, the upper control function ρ_+ of any coarse embedding $F : X \to Y$ may be replaced by a linear function. Indeed, such an embedding F has to satisfy $\|F(x) - F(y)\| \leq \rho_+(1)d(x, y)$, for all $x, y \in X$. It is also easy to see that, if a compactly generated, locally compact group G is endowed with two word-length metrics relatively to two distinct compact, symmetric generating sets S_1 and S_2, then the identity map $(G, d_{S_1}) \to (G, d_{S_2})$ is a quasi-isometry. To study groups, it is therefore natural to introduce invariants which are preserved by quasi-isometries, as the compression exponent defined below.

Definition 3.1.3. *If* $F : X \to Y$ *is a large-scale Lipschitz map, we can define the* ***compression exponent*** *of* F *(or simply* ***compression****), denoted* $R(F)$*,* *as the supremum taken over all* $\alpha \in [0,1]$ *such that there exist* $C, D \geq 0$ *satisfying*

$$\delta(F(x), F(y)) \geq C \cdot d(x,y)^{\alpha} - D, \quad \forall x, y \in X.$$

Then, the ***compression exponent*** $\alpha^*_Y(X)$ *corresponds to the supremum of* $R(F)$ *over all large-scale Lipschitz maps* $F : X \to Y$.

In the sequel, we will consider only $L^p = L^p([0,1])$ as target space. In this case, we write $\alpha^*_p(X) = \alpha^*_{L^p}(X)$ and we will refer to $\alpha^*_p(X)$ (resp. α^*_2) as the L^p **compression of** X (resp. **Hilbert compression**). It is clear that the L^p compression is invariant under quasi-isometry. It is worth noting that, by Corollary 14 in [Bau14], we have $\alpha^*_{\ell^p}(X) = \alpha^*_{L^p}$, for any locally finite metric space X.
Quasi-isometries arise naturally via actions of groups on metric spaces. For its importance in geometric group theory and its use in the next sections, we state one of the various versions of the so-called Schwarz Lemma. We recall the necessary definitions.

Definition 3.1.4. *A metric space* (X, d) *is* ***coarsely geodesic*** *if there is a constant* $c \geq 0$ *such that, for any points* $x, y \in X$*, we can find a map* $f : [0,a] \to X$ *satisfying* $f(0) = x, f(a) = y$ *and*

$$|s - t| - c \leq d(f(x), f(y)) \leq |s - t| + c,$$

for all $s, t \in [0,a]$.

Definition 3.1.5. *Let* G *be a locally compact group acting by isometries on* (X, d).

1. *The action of* G *is (metrically)* ***proper*** *if, for all* $r \geq 0$ *and all* $x \in X$*, the set*
$$\{g \in G \ : \ B(x,r) \cap B(g \cdot x, r) \neq \emptyset\}$$
is relatively compact in G.

2. *The action of* G *is* ***cocompact*** *if there is a compact subset* $K \subset X$ *such that* $G \cdot K = X$

Here is a version of Schwarz Lemma for geometric actions of locally compact groups (for a proof, see Lemma 2 in [Sal11]). Recall that if G is a compactly generated locally compact group (e.g. a finitely generated group), we view it as a metric space with the word metric.

Theorem 3.1.6. *(Schwarz Lemma) Let G be a locally compact group and let X be a coarsely geodesic metric space. If G acts continuously, properly, cocompactly and by isometries on X, then G is compactly generated and, for any fixed $x_0 \in X$, the associated orbit map $G \to X$ sending $g \mapsto g \cdot x_0$ is a quasi-isometry.*

3.2 Examples

We now give a (non-exhaustive) list of metric spaces for which the L^p compression is known. Most of the groups we will consider are compactly generated. In this case, we will see them endowed with the word-length metric relative to a compact symmetric generating set, except when stated otherwise.

1. Clearly, if Y is a subspace of a metric space X, then, for any p, we have $\alpha_p^*(Y) \geq \alpha_p^*(X)$. As a consequence, since L^2 embeds isometrically in L^p for any $p \geq 1$, we deduce that $\alpha_p^*(X) \geq \alpha_2^*(X)$, as observed in [NP11]. Moreover, since $\ell^p(\mathbb{N})$ isometrically embed into $L^p([0,1])$, we have $\alpha_{\ell^p(\mathbb{N})}^*(X) \leq \alpha_p^*(X)$, for any metric space X. In fact, as shown by Corollary 14 in [Bau14], for any $p \in [1, +\infty]$ and for any locally finite metric space X, we have $\alpha_{\ell^p(\mathbb{N})}^*(X) = \alpha_p^*(X)$.

2. Let (X, d) and (Y, δ) be metric spaces. The product set $X \times Y$ is endowed with the metric $d_{X \times Y}((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + \delta(y_1, y_2)$. It is possible to show that $\alpha_p^*(X \times Y) = \min\{\alpha_p^*(X), \alpha_p^*(Y)\}$, (Proposition 4.1 [GK04]).

3. Since the group $\mathbb{Z}^n$, endowed with the ℓ^1 metric, can be bi-Lipschitzly embedded into $\ell^2(\{1, 2, \dots, n\})$ with distortion $\sqrt{n}$, we immediately get $\alpha_2^*(\mathbb{Z}^n) = 1$. More generally, it is known that groups with polynomial growth have Hilbert compression 1 ([Ass83]). Hence, they also have L^p compression 1 for any $p \geq 1$.

4. Let $\mathcal{T} = (V, E)$ be a simplicial tree[1] and let $1 \leq p < \infty$. Let us choose a basepoint $x_0 \in V$. We define an embedding $F : V \to \ell^p(V)$ by

$$F(y) = \sum_{j=0}^{d(x_0, y)} \delta_{x_j},$$

[1] A *tree* is a connected graph without cycles.

for $y \in V$, where the sequence of vertices $(x_j)_{j=0}^{d(x_0,y)}$ is the unique geodesic path between x_0 and $x_{d(x_0,y)} = y$. It is easy to see that

$$\|F(x) - F(y)\|_p = d(x,y)^{1/p},$$

for all $x, y \in V$. Therefore, F is a coarse embedding with control functions $\rho_\pm(t) = t^{1/p}$. In fact, if $p = 2$, it is possible to modify this embedding to obtain another one with control function

$$\rho_t(t) = \frac{t}{\sqrt{\log t} \log\log t}$$

by [BS08]. Thus, trees and free groups have L^p compression 1, for all $p \geq 1$.

5. More generally, Tessera showed ([Tes11]) that $\alpha_p^*(X) = 1$, for all $1 \leq p < \infty$, for X being : a tree, a finitely generated hyperbolic group or a connected Lie group.

6. Since any Coxeter group admits an isometric embedding in a finite product of locally finite trees ([DJ99]) and since trees have L^p compression 1, we get $\alpha_p^*(G) = 1$, for any Coxeter group G and for any $p \geq 1$.

7. Finite dimensional CAT(0) cube complexes and groups that can be quasi-isometrically embedded into those complexes have Hilbert compression exponent 1 ([CN05]). This applies to finitely generated right-angled Artin groups.

8. Another important example is the computation of the Hilbert compression of Richard Thompson's group F in [AGS06], where it is shown $\alpha_2^*(F) = 1/2$.

9. Motivated by understanding the structure of operator algebras built from groups, namely the reduced group C^*-algebra $C_r^*(G)$, Guentner and Kaminker showed in [GK04] that, if a finitely generated group G satisfies $\alpha_2^*(G) > \frac{1}{2}$, then G has Property A, or equivalently, that the algebra $C_r^*(G)$ is exact.

10. By making clever use of expanders, it was shown in [ADS09] that, for any number $\alpha \in [0,1]$, there exists a finitely generated group G such that $\alpha_2^*(G) = \alpha$.

3.3 Compression of box spaces

In this section, we will use the techniques developed in Chapter 2 to estimate the behaviour of control functions ρ_- of coarse embeddings of a particular type of metric spaces obtained as disjoint union of finite graphs, called box spaces. Their importance goes back to Gromov who used this elementary construction to give the first example of a metric space which does not admit any coarse embedding into any Hilbert space. We will re-prove this fact in Proposition 3.3.3.

Definition 3.3.1. *Let $(X_n, d_n)_n$ be a sequence of metric spaces. Let us denote by d the metric on $X = \sqcup_n X_n$, the disjoint union of the X_n, defined by*

$$d(x,y) = \begin{cases} d_n(x,y), & \text{if } x, y \in X_n; \\ \max\{diam(X_n), diam(X_m)\}, & \text{if } x \in X_n, y \in X_m, \text{ and } n \neq m. \end{cases}$$

The metric space (X,d) is called the ***box space*** *of $(X_n)_n$.*

Lemma 3.3.2. *Let $(X_n)_{n\geq 1}$ be a sequence of finite, connected graphs $X_n = (V_n, E_n)$. Let $D(X_n)$ be the maximal displacement and let k_n be the average degree of X_n. We denote by X the box space formed by the sequence $(X_n)_n$.*
If $F : X \to \ell^p(\mathbb{N})$ is a coarse embedding, then $\rho_\pm$, the control functions of F, satisfy :

$$\rho_-\left(D(X_n)\right) \leq \left(\frac{2^{p-1}k_n}{\lambda_1^{(p)}(X_n)}\right)^{\frac{1}{p}} \rho_+(1),$$

for all $n \geq 1$.

Proof : Let $F : X \to \ell^p(\mathbb{N})$ be a coarse embedding of the box space formed by the family $(X_n)_n$. For all $n \geq 1$, pick α_n a permutation of V_n realizing the maximal displacement of the graph X_n, that is, $d(\alpha_n(x), x) \geq D(X_n)$, for all $x \in V_n$. Since the lower control function ρ_- is non-decreasing, we get

$$\|F(x) - F(\alpha_n(x))\|_p \geq \rho_-(D(X_n)),$$

for any $x \in V_n$. Furthermore, for any edge $e \in E_n$, we have

$$\|F(e^+) - F(e^-)\|_p \leq \rho_+(1).$$

Applying inequality of Proposition 2.2.4 to the restriction of the map F to the graph X_n, we obtain

$$\frac{\rho_-(D(X_n))^p}{2^p} \leq \frac{k_n \rho_+(1)^p}{2\lambda_1^{(p)}(X_n)}.$$

Re-arranging and taking p-th roots, we obtain the desired result. □

We obtain another proof of a celebrated result of Gromov, that asserts box spaces of expanders do not embed coarsely into any Hilbert space ([Gro03]). In fact, expanders do not embed coarsely into any ℓ^p-space, as it is shown in Proposition 11.30 [Roe03].

Proposition 3.3.3. *(Gromov) If $(X_n)_{n\geq 1}$ is a family of expander graphs with bounded degree, then X, the box space formed by the sequence $(X_n)_{n\geq 1}$ does not coarsely embed into $\ell^p(\mathbb{N})$, for any $1 \leq p < \infty$.*

Proof : By contradiction, assume that there exists a coarse embedding $F : X \to \ell^p(\mathbb{N})$ with control functions $\rho_\pm$. Setting $\mu = \inf_{n\geq 1} \lambda_1^{(p)}(X_n)$, there exist constants α and k such that

$$\rho_-(\alpha \log |V_n|)^p \leq \frac{k\rho_+(1)^p}{\mu}$$

by Lemma 3.3.2. Since $|V_n|$ tends to infinity, the LHS of this expression tends to infinity, while the RHS remains bounded. This is a contradiction. □

We give another consequence of Lemma 3.3.2. Recall that, for functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$, we write $f \succ g$ if for any $\epsilon > 0$ there exists a compact set $K \subset \mathbb{R}^+$ such that $f \leq \epsilon g$ outside K.

Proposition 3.3.4. *Let $(X_n)_{n\geq 1}$ be an increasing sequence of finite, connected graphs. Let $D(X_n)$ be the maximal displacement and let k_n be the average degree of $X_n = (V_n, E_n)$. We denote by X the box space formed by the sequence $(X_n)_n$. Let $1 \leq p < \infty$. If $F : X \to \ell^p(\mathbb{N})$ is a coarse embedding whose lower control function ρ_- satisfies $\rho_-(t) \succ t^\beta$, with $\beta > 0$, then, we have :*

$$\beta \leq \frac{1}{p} \liminf_{n\to+\infty} \frac{\log\left(\frac{2^{p-1}k_n\rho_+(1)}{\lambda_1^{(p)}(X_n)}\right)}{\log\left(D(X_n)\right)}.$$

In particular, if the sequence $(X_n)_n$ *has bounded degree, we obtain the following upper bound for the compression exponent of the box space* X *:*

$$\alpha_p^*(X) \leq \frac{1}{p} \liminf_{n\to\infty} \frac{\log\left(\lambda_1^{(p)}(X_n)^{-1}\right)}{\log\log|V_n|}.$$

Proof : The first inequality is a direct consequence of Lemma 3.3.2. If we assume furthermore that the graphs $(X_n)_n$ have bounded degree, then we can suppose $\inf_{n\geq 1}\lambda_1^{(p)}(X_n) = 0$ (otherwise, the box space X would contain a sequence of expanders and would not be coarsely embeddable into any ℓ^p, by Proposition 3.3.3). By the first part of the proposition, we have

$$\beta \leq \frac{1}{p} \inf_{n\geq m} \frac{\log\left(M\lambda_1^{(p)}(X_n)^{-1}\right)}{\log\left(\frac{\log|V_n|}{M}\right)},$$

for all m and M large enough, since the displacement is logarithmic in the number of vertices for a family of graphs with bounded degree by Proposition 2.3.3. As m goes to infinity, we get the desired upper bound. Then, we finish the proof by using the fact that $\alpha^*_{\ell^p(\mathbb{N})}(X) = \alpha_p^*(X)$ (Corollary 14 of [Bau14]). □

As application, we can estimate the compression of some box spaces.

Corollary 3.3.5. *Using the notations of Proposition 2.4.8,* $\alpha_2^*(\sqcup_{n\geq 2} Y_n) \leq \frac{3}{4}$.

Proof: Since this is a sequence of 4-regular graphs satisfying $|Y_n| = \Theta(q^{n^2-1})$ and $\lambda_1^{(2)}(Y_n)^{-1} = O(n^3)$ (see the proof of Proposition 2.4.8), we get :

$$\liminf_{n\to\infty} \frac{\log n^3}{\log\log q^{n^2-1}} = \frac{3}{2},$$

which yields the result, by Proposition 3.3.4. □

Corollary 3.3.6. *Using the notations of Corollary 2.4.4,* $\alpha_2^*(\sqcup_{d\geq 2} H_d) = \frac{1}{2}$.

Proof: The first inequality $\alpha_2(\sqcup_{d\geq 1} H_d) \leq \frac{1}{2}$ is obvious from Proposition 3.3.4. To prove the converse inequality, we simply remark that $\sqcup_{d\geq 2} H_d$ admits a bi-Lipschitz embedding into $\ell^1(\mathbb{N})$ and the conclusion follows from

Proposition 2.6 in [GK04]. Indeed, let $\phi_k : \sqcup_{d\geq 2} H_d \to \mathbb{R}^{k+1} : x \mapsto \phi_k(x)$, where

$$\phi_k(x) = \begin{cases} (diam(H_k), x), & \text{if } x \in H_k; \\ 0, & \text{otherwise.} \end{cases}$$

Then, ϕ, defined by $\phi = \oplus_{d\geq 2}\phi_d$, where the target space is endowed with the ℓ^1 norm, is a bi-Lipschitz embedding. □

The interest of this particular box space is due to the fact that it is the first explicit example of a metric space without Yu's property A which is coarsely embeddable into a Hilbert space (see [Now07]).

3.4 Compression exponent and group constructions

3.4.1 Known results

The behaviour of L^p compression under group constructions have been studied in some cases only. Here is again a (non-exhaustive) list of known results.

1. Let G and H be finitely generated groups. As we saw in the first set of examples, $\alpha_p^*(G \times H) = \min\{\alpha_p^*(G), \alpha_p^*(H)\}$.
2. Hilbert compression has been computed for some wreath products. The first example is given by taking wreath products of $\mathbb{Z}$ with itself several times successively. More precisely, let us define $\mathbb{Z}_{(k)}$ inductively as follows : $\mathbb{Z}_{(1)} = \mathbb{Z}$ and $\mathbb{Z}_{(k+1)} = \mathbb{Z}_{(k)} \wr \mathbb{Z}$, for $k \geq 1$. In [NP08], it was shown that $\alpha_2^*(\mathbb{Z}_{(k)}) = \frac{1}{2-2^{1-k}}$.
3. For $d \geq 2$, we have $\alpha_2^*(C_2 \wr \mathbb{Z}^d) \geq \frac{1}{d}$ by [Tes11], where C_2 denotes the cyclic group of order 2. There is even equality in the cases $d = 1, 2$ ([NP08]).
4. Another result involving wreath products is the group $C_2 \wr \mathbb{F}_n$. It was shown that $\alpha_2^*(C_2 \wr \mathbb{F}_n) = \frac{1}{2}$, for $n \geq 2$, (see [dCSV08] and [NP11]).
5. If we assume furthermore that F is a finite subgroup of both G and H, then, for all $p \geq 1$, Corollary 4.3 of [Hum11] shows :
   (a) $\alpha_p^*(G *_F H) = \min\{\alpha_p^*(G), \alpha_p^*(H)\}$,
   (b) $\alpha_p^*(HNN(G, F, \Theta)) = \alpha_p^*(G)$.

The examples here above should convince the reader that it is hopeless to find a general result that estimates the compression of the extension of some group G by some group H, knowing the compression of the G and H only.
The rest of this chapter is dedicated to the computation of the L^p compression for some HNN extensions of a group G relatively to a subgroup F, which is not necessarily finite, and thus contrasting with the last example given in the list above.

3.4.2 L^p compression of certain HNN extensions

This subsection consists in a joint work ([JP13]).
In [GJ03] the authors introduced the notion of an $\mathfrak{N}$-BS group. These are groups arising as HNN extensions satisfying properties similar to Baumslag-Solitar groups. In order to prove the Haagerup property for such groups, they developed a framework that we shall heavily rely on and that we now recall.
Let $\mathfrak{N}$ be a locally compact compactly generated group and let G be a closed subgroup of $\mathfrak{N}$. Let $i_1, i_2 : H \to G$ be two inclusions of a group H onto open subgroups of finite index in G, and assume i_1 and i_2 are conjugated by an automorphism φ of $\mathfrak{N}$. The $\mathfrak{N}$-BS group Γ is then the HNN extension $\mathrm{HNN}(G, i_1(H), i_2(H))$ whose presentation is given by

$$\langle S, t | R, t i_1(h) t^{-1} = i_2(h)\ \forall h \in H \rangle,$$

where $G = \langle S|R \rangle$.

Theorem 3.4.1. *Let $\mathfrak{N}$ be a connected Lie group, let G be a closed cocompact subgroup of $\mathfrak{N}$ and let Γ be an HNN extension as above. Then, for all $p \geq 1$, $\alpha_p^*(\Gamma) = 1$.*

The strategy to prove Theorem 3.4.1 is to construct a metric space M on which Γ acts continuously, properly, cocompactly and by isometries, so that, M and Γ are quasi-isometric by the Schwarz Lemma.
We first construct the desired space M and we give a quantitative comparison between two natural metrics on M. Then, we end this section with the proof of Theorem 3.4.1 and treat some concrete examples.

Fibred product

Let $\tilde{\mathfrak{N}} = \mathfrak{N} \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathfrak{N}$ by iterations of φ and let $j_{\mathfrak{N}} : \Gamma \to \tilde{\mathfrak{N}}$ be the homomorphism defined by $g \mapsto (g, 0)$ for $g \in G$ and $t \mapsto (1, 1)$. Then consider T, the Bass-Serre tree associated with the HNN extension Γ and denote by $j_T : \Gamma \to \mathrm{Aut}(T)$ the monomorphism induced by the action of Γ on T.
For later use, recall the following result from [GJ03].

Theorem 3.4.2. *The homomorphism $j : \Gamma \to \tilde{\mathfrak{N}} \times \mathrm{Aut}(T)$, $g \mapsto (j_{\mathfrak{N}}(g), j_T(g))$, is injective and has closed image. In particular, it is an isomorphism onto its image.*

Following Proposition 2.1 in [CdCMT], we will define a metric space Y on which $\tilde{\mathfrak{N}}$ acts continuously, properly, cocompactly and by isometries. Endow $\tilde{\mathfrak{N}}$ with a left invariant Riemannian metric. For each coset $L_i = \mathfrak{N} \times \{i\}$ of $\mathfrak{N}$ in $\tilde{\mathfrak{N}}$ we consider a strip $L_i \times [0, 1]$, equipped with the product Riemannian metric, and attach it to $\tilde{\mathfrak{N}}$ by identifying $(l, 0)$ to l and $(l, 1)$ to $l \cdot (1, 1)$. Denote by Y the space obtained in this way. Y has a natural shortest-path metric induced by the riemannian metric on each of the strips. Furthermore, Y is naturally homeomorphic (but not necessarily isometric!) to $\mathfrak{N} \times \mathbb{R}$. Using this obvious parametrization, $\tilde{\mathfrak{N}}$ acts on Y by $(n, k) \cdot (y, s) = (n\varphi^k(y), k + s)$, for $(n, k) \in \tilde{\mathfrak{N}}$ and $(y, s) \in Y$. As in Proposition 2.1 in [CdCMT], Y is a locally compact, geodesic metric space on which $\tilde{\mathfrak{N}}$ acts continuously, properly, cocompactly and by isometries. We denote by b the projection map $(y, s) \mapsto s$.
Let us recall briefly the construction of the Bass-Serre tree T of Γ. It is an oriented graph whose vertices are the left cosets Γ/G and the edges correspond to the left cosets $\Gamma/i_1(H)$. The edge $\gamma/i_1(H)$ is directed from $\gamma t^{-1}G$ to γG. As the $i_k(H)$ are of finite index in G, T is locally finite. Then, by construction, Γ acts naturally on T by left multiplication.
Let $p : \Gamma \to \mathbb{Z}$ be the homomorphism defined on the generators by $p(t) = 1$ and $p(g) = 0$, for every $g \in G$. Since the vertices of T correspond to the left cosets of G in Γ, we can define a map c on the vertices of T by $c(xG) = p(x)$ and extend it to the metric tree T by affine interpolation. This allows us to define the fibred product M :

$$M = \{(x, y) \in T \times Y \,:\, c(x) = b(y)\}.$$

The subspace M is Γ-invariant for the diagonal action of Γ on $T \times Y$. Indeed, for all $x \in T$, $c(g \cdot x) = c(x)$ if $g \in G$ and $c(t \cdot x) = c(x) + 1$. In a similar fashion, for all $y \in Y$, $b(g \cdot y) = b(y)$ if $g \in G$ and $b(t \cdot y) = b(y) + 1$. Hence,

if $c(x) = b(y)$, it implies that $c(\gamma \cdot x) = b(\gamma \cdot y)$ for any $\gamma \in \Gamma$.
We endow $T \times Y$ with the product metric, namely,

$$d((x, y), (x', y')) = d_T(x, x') + d_Y(y, y').$$

Lemma 3.4.3. *The subspace M is path-connected. Furthermore, denoting by d_M the shortest-path metric induced by d on M, the metrics d and d_M are bilipschitz equivalent.*

Proof : First, observe that, for any point $y = (n, s) \in Y$, the path

$$\alpha_y : \mathbb{R} \to Y : \ u \mapsto (n, u + s)$$

is a geodesic such that $\alpha_y(0) = y$ and $b(\alpha_y(u)) = b(y) + u$, $\forall u \in \mathbb{R}$. Similarly, for any point $x \in T$ one can chose a geodesic path $\beta_x : \mathbb{R} \to T$ such that $\beta_x(0) = x$ and $c(\beta_x(u)) = c(x) + u$. Let $(x_0, y_0), (x_1, y_1) \in M$. We will build a path linking those points in two steps. For the first one, let $\sigma : [0, d_T(x_0, x_1)] \to T$ be the geodesic from x_0 to x_1. Let θ_1 be the path defined by

$$\theta_1(u) = (\sigma(u), \alpha_{y_0}(c(\sigma(u)) - b(y_0)).$$

The left component links x_0 to x_1, while the right component starts from y_0 and ends at a certain point y_2. Moreover, the path θ_1 is contained in M. Indeed, for all $u \in [0, d_T(x_0, x_1)]$, we have:

$$\begin{aligned} b(\alpha_{y_0}(c(\sigma(u)) - b(y_0))) &= b(y_0) + c(\sigma(u)) - b(y_0) \\ &= c(\sigma(u)). \end{aligned}$$

So, θ_1 connects (x_0, y_0) to a point $(x_1, y_2) \in M$ satisfying

$$b(y_2) = c(x_1) = b(y_1).$$

For the second step, we will find a path in M between (x_1, y_2) and (x_1, y_1). In a similar way, let $\tilde{\sigma} : [0, d_Y(y_2, y_1)] \to Y$ be a geodesic path linking y_2 to y_1 in Y. Then, it is easy to check that the path

$$\theta_2 : [0, d_Y(y_2, y_1)] \to M, \theta_2(u) = (\beta_{x_1}(b(\tilde{\sigma}(u)) - c(x_1)), \tilde{\sigma}(u))$$

does the job. This shows that M is path-connected. Now, the inequality $d \leq d_M$ being immediate, we need to analyze the length of the path we just considered in order to finish the proof. Denoting by $L(\theta_j)$ the length of the path θ_j, we get the following estimates:

$$L(\theta_1) \leq 2d_T(x_0, x_1)$$

and

$$L(\theta_2) \leq 2d_Y(y_2, y_1) \leq 2(d_Y(y_0, y_1) + d_Y(y_1, y_2)).$$

By construction, $d_Y(y_0, y_2) \leq d_T(x_0, x_1)$. We can conclude:

$$\begin{aligned} d_M((x_0, y_0), (x_1, y_1)) &\leq L(\theta_1) + L(\theta_2) \\ &\leq 2d_T(x_0, x_1) + 2d_Y(y_0, y_1) + 2d_Y(y_1, y_2) \\ &\leq 4d_T(x_0, x_1) + 2d_Y(y_0, y_1) \\ &\leq 4 \cdot d((x_0, x_1), (y_0, y_1)). \end{aligned}$$

□

Proof of Theorem 3.4.1 and Applications

As before, let Γ be an $\mathfrak{N}$-BS group with G a closed cocompact subgroup of $\mathfrak{N}$. In order to apply the Schwarz Lemma, we prove that the action of Γ is proper and cocompact.

Lemma 3.4.4. *The action of* Γ *on* $T \times Y$ *is proper.*
That is, for all $(x, y) \in T \times Y$*, there exists an* $r > 0$ *so that*

$$\{\gamma \in \Gamma \ : \ \gamma \cdot B((x, y), r) \cap B((x, y), r) \neq \emptyset\}$$

is relatively compact in Γ*.*

In particular, as M is a closed subset of $T \times Y$, we get immediately the following corollary.

Corollary 3.4.5. *The action of* Γ *on the fibred product* M *is proper.*

Proof of Lemma 3.4.4 : The action of $Aut(T) \times \tilde{\mathfrak{N}}$ on $T \times Y$ is proper, therefore, by Theorem 3.4.2, it is also the case for the action of the closed subgroup $j(\Gamma)$. As M is Γ-invariant and closed in $T \times Y$, we can conclude. □

Lemma 3.4.6. *The action of* Γ *on* M *is cocompact.*

Proof : It is enough to see that, for any sequence $(x_k, y_k)_k \subset M$, we can find a sequence $(\gamma_k)_k \subset \Gamma$ so that the sequence $(\gamma_k \cdot (x_k, y_k))_k$ converges. Since Γ acts transitively on the edges of T, we can assume that the sequence $(x_k)_k$ belongs to the edge $[G, tG]$. This implies that $0 \leq c(x_k) = b(y_k) \leq 1$, for all but possibly finitely many k, so that the sequence $(y_k)_k$ is contained in the strip of Y corresponding to the coset $\mathfrak{N}$ in $\tilde{\mathfrak{N}}/\mathfrak{N}$. Using the fact that the action of G on $\mathfrak{N}$ is cocompact, we can multiply by elements of G in such a way that the sequence $(y_k)_k$ converges. But since G stabilizes the vertex G in T, this process maintains the sequence $(x_k)_k$ inside the edges adjacent to G. Since there are only finitely many of these, the sequence $(x_k, y_k)_k$ converges up to extracting a subsequence. This concludes the proof. □

We are now able to prove Theorem 3.4.1.

Proof of Theorem 3.4.1 : Firstly, we shall show that $\alpha_p^*(\Gamma) \geq \alpha_p^*(\tilde{\mathfrak{N}})$. Indeed, by Schwarz Lemma, Γ is quasi-isometric to (M, d_M), which is quasi-isometric to (M, d) by Lemma 3.4.3. Moreover, Y is quasi-isometric to $\tilde{\mathfrak{N}}$. Hence,

$$\alpha_p^*(\Gamma) = \alpha_p^*(M) \geq \alpha_p^*(T \times Y) \text{ and } \alpha_p^*(Y) = \alpha_p^*(\tilde{\mathfrak{N}}).$$

Then, the lower bound follows from the propositions:

- For a tree T, $\alpha_p^*(T) = 1$, for all $p \geq 1$. (See Theorem 2.6 in [BS08])
- For two metric spaces X and X', the compression of $X \times X'$ is the minimum of the compressions of the factors. (See [GK04])

Finally, we conclude the proof by noting that $\alpha_p^*(\tilde{\mathfrak{N}}) = 1$, which follows from the following propositions:

- Any semi-direct product of a connected Lie group with $\mathbb{Z}$ is quasi-isometric to a connected Lie group. (By an unpublished result of Y. Cornulier)
- Let $\mathcal{K}$ be a connected Lie group. Then, $\alpha_p^*(\mathcal{K}) = 1$, for all $p \geq 1$. (See [Tes11])

□

We remark that, if $\mathfrak{N}$ is a soluble connected Lie group, then Cornulier's result is a simple consequence of a lemma of Mostow. Here is a short proof that we owe to Alain Valette. In this case, $\tilde{\mathfrak{N}}$ is soluble and Noetherian (i.e. every closed subgroup is compactly generated). A lemma of Mostow (see

Lemma 5.2 in [Mos71]) asserts that there exist a compact normal subgroup K of $\tilde{\mathfrak{N}}$ and a soluble almost connected Lie group $\mathcal{M}$ such that the quotient $\tilde{\mathfrak{N}}/K$ is isomorphic to $\mathcal{L}$, where $\mathcal{L}$ is a cocompact, closed subgroup of $\mathcal{M}$. Then, the connected component of unity $\mathcal{M}^0$ is quasi-isometric to $\tilde{\mathfrak{N}}$. Indeed, on the one hand, by compactness, $\tilde{\mathfrak{N}}$ is quasi-isometric to $\tilde{\mathfrak{N}}/K$ and by Mostow we deduce that $\tilde{\mathfrak{N}}$ is quasi-isometric to $\mathcal{L}$. On the other hand, by cocompactness, $\mathcal{L}$ is quasi-isometric to $\mathcal{M}$ and, since $\mathcal{M}$ has only finitely many connected components, it is quasi-isometric to $\mathcal{M}^0$.
In particular, Theorem 3.4.1 allows us to cover all the examples appearing in [GJ03].

Corollary 3.4.7. *The following groups have compression* 1*.*

1. *The Baumslag-Solitar groups*

$$BS^p_q = \langle x, t \mid x^p = tx^q t^{-1} \rangle = HNN(\mathbb{Z}, \mathbb{Z}, p\cdot, q\cdot),$$

   with parameters $p, q \in \mathbb{Z}_+$. (For a different proof, see also [dCV].)

2. *Torsion free, finitely presented abelian-by-cyclic groups.*

3. *Let $\mathfrak{N}$ be a homogeneous nilpotent Lie group. So, it admits a dilating automorphism φ. Suppose that $\mathfrak{N}$ contains a discrete, cocompact subgroup G which is invariant by φ. Then, for any finite index subgroup H in G, the extension $HNN(G, H, i_1, \varphi_{|H})$, where i_1 is the canonical injection, has compression* 1*.*

Proof : The Bausmslag-Solitar groups BS^p_q can be seen as HNN extension of $\mathbb{Z}$ with itself, considering the inclusions $i_1, i_2 : \mathbb{Z} \to \mathbb{Z}$ defined by $i_1(n) = pn$ and $i_2(n) = qn$. Then, apply Theorem 3.4.1 with the automorphism of $\mathfrak{N} = \mathbb{R}$ given by $\varphi(x) = \frac{p}{q}x$.
For the second class of examples, it is known (see for instance [FM00]) that these groups are HNN extensions of $\mathbb{Z}^n$ with itself with respect to $i_1, i_2 : \mathbb{Z}^n \to \mathbb{Z}^n$, where i_1 is the identity and $i_2 \in GL_n(\mathbb{Z})$. Again, apply Theorem 3.4.1 with the automorphism of $\mathfrak{N} = \mathbb{R}^n$ given by i_2^{-1}.
The case of the last class of examples is clear by construction. □

Remark 3.4.8. *It is important to note that the assumption about finite presentation is necessary to treat the second class of examples. Indeed, the wreath product $\mathbb{Z} \wr \mathbb{Z}$ is torsion free, abelian-by-cyclic and finitely generated. However, it is computed in [ANP09] that $\alpha_2^*(\mathbb{Z} \wr \mathbb{Z}) = \frac{2}{3}$.*

Remark 3.4.9. *In the case where* $\mathfrak{N}$ *is a generic compactly generated locally compact (not necessarily a connected Lie group), it is also possible to find a space* Y *admitting a geometric action of* $\tilde{\mathfrak{N}}$*, by Proposition 2.1 in [CdCMT]. However, it is not clear how to endow* Y *with a natural fibration* b *compatible with the semi-direct product structure on* $\tilde{\mathfrak{N}}$ *in order to generalize Theorem 3.4.1.*

Chapter 4

Equivariant embeddings

This chapter is dedicated to the study of equivariant embeddings of spaces and groups into Banach spaces. To be more specific, let G be a group and take X to be some G-set and take Y to be a Banach space endowed with a certain G-action. We will consider maps $F : X \to Y$ which are G-equivariant, that is, $F(gx) = gF(x)$, for all $x \in X$ and $g \in G$. If we assume furthermore, that X is a metric space, then we can look at G-equivariant coarse embeddings of X into Y and study the behaviour of their control functions. We will mainly focus on the case where G acts on X by isometries and on Y by affine isometric transformations.
This chapter is organised as follows : In the first section, we state the main definitions concerning equivariant compression, affine isometric actions and 1-cohomology taking values in a unitary representation and then, we give a list of examples of equivariant embeddings of groups into L^p spaces. Then, we define negative type functions and discuss their relationship with affine isometric actions on Hilbert spaces, and, in particular, this leads us to the definitions of Property (T) and Haagerup property. We end Section 4.1 with a condition to deduce the amenability of a group action on a metric space. For the rest of the chapter, we focus our attention on groups acting on trees. We state our main results in Section 4.2 and we prove them in Section 4.3. Taking a group G acting transitively on the set of vertices and on the boundary of a regular tree $\mathcal{T}_{q+1}$, we first give an upper bound for the growth of cocycles with values in any unitary representation of the group G. Our second result shows the existence of a harmonic cocycle with optimal growth. As an application, we get a description of functions conditionally of negative type on G which are unbounded. Finally, Section 4.4 is divided into two subsections. In the first one, we decompose the so-called Haagerup cocycle by projecting it onto two orthogonal G-invariant subspaces and compute the norms of these projected cocycles. In the last subsection of this chapter,

we study an interesting family of kernels conditionally of negative type. We finish by summarizing the content of this chapter by a classification of pure negative type functions on groups acting transitively on both $\mathcal{T}_{q+1}$ and on its boundary $\partial\mathcal{T}_{q+1}$.
Aside from Proposition 4.1.12, all the results appearing in the present chapter are part of the joint work with Antoine Gournay [GJ15].

4.1 Definitions and background

In this section, we start by defining the equivariant compression exponent. Then, we give the necessary informations about representation theory of groups and affine group actions on Banach spaces as well as their interpretation in first cohomology group taking values in an isometric representation. Here, G will always denote a locally compact group, except stated otherwise.

Definition 4.1.1. *Let G be a group acting isometrically on a metric space (X,d) and on some Banach space Y. The* ***equivariant compression*** *of X into Y relatively to G is the number, denoted by $\alpha_Y^G(X)$, that corresponds to the supremum of the compression $R(F)$ over all large-scale Lipschitz maps $F : X \to Y$ which are G-equivariant.*

Once again, as target space, we will consider only L^p spaces and we will write $\alpha_p^\natural(X) = \alpha_{L^p}^G(X)$, when there is no risk of confusion as to the group we consider. In particular, we will write $\alpha_p^\natural(\Gamma)$ for the equivariant compression of a finitely generated group Γ acting on itself.

Definition 4.1.2. *Let Y be a Banach space. Let us denote the space of bounded linear operators $Y \to Y$ by $\mathcal{B}(Y)$ and the group of* ***invertible isometric operators*** *by $\mathcal{O}(Y)$. It consists of all invertible bounded linear operators $T \in \mathcal{B}(Y)$ which are norm-preserving, that is,*

$$\|Tx\| = \|x\|,$$

for all $x \in Y$.
A (linear) ***isometric representation*** *is a group homomorphism $\pi : G \to \mathcal{O}(Y)$ which is* ***strongly continuous****, namely, the map*

$$G \to Y : g \mapsto \pi(g)x,$$

is continuous for any $x \in Y$.
A representation is called ***unitary*** *(resp.* ***orthogonal****) in the case where $Y = \mathcal{H}$ is a complex (resp. real) Hilbert space.*

The next definition fixes the notations concerning cocycles and affine isometric actions.

Definition 4.1.3. *Let π be an isometric, linear representation of a group G on a Banach space Y.*

1. *A* ***1-cocycle*** *with values in π is a continuous fonction $b : G \to Y$ satisfying the so-called cocycle relation, that is,*

$$b(gh) = \pi(g)b(h) + b(g),$$

   for every $g, h \in G$.

2. *A 1-cocycle of the form*

$$g \mapsto (1 - \pi(g))v,$$

   for some vector $v \in Y$, is called a ***1-coboundary****.*

3. *We denote by $Z^1(G, \pi)$ the space of all 1-cocycles and by $B^1(G, \pi)$ the subspace of 1-coboundaries with values in π. The* ***first cohomology group with coefficients in*** *π is defined as the quotient space*

$$H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi).$$

Note that a map b satisfies the cocycle relation if and only if G acts by affine isometries on Y by $\alpha(g)v = \pi(g)v + b(g)$. We will say that π (resp. b) is the linear (resp. the translation) part of the affine isometric action α, and use the self-explanatory notation $\alpha = (\pi, b)$.

4.1.1 Examples

1. Clearly, by definition, the following inequality holds $\alpha_Y^G(X) \leq \alpha_Y^*(X)$, for any metric space X and any Banach space Y.

2. Let Γ be a finitely generated amenable group. Then, $\alpha_p^\natural(\Gamma) = \alpha_p^*(\Gamma)$, for any $p \in [1, \infty)$, as proved in [NP11]. This implies that, for instance, groups with polynomial growth have equivariant Hilbert compression equal to 1, as seen in Chapter 3. We also deduce from the Theorem of Naor and Peres that, in the class of finitely generated amenable groups, $\alpha_p^\natural$ is a quasi-isometry invariant. However, this does not hold in general. Indeed, in [CAPV14], the authors exhibited two quasi-isometric groups with different equivariant compression exponents.

3. As for non-equivariant compression, a sufficiently good embedding allows to deduce information on the group. Let Γ be a finitely generated group. If $\alpha_p^\natural(\Gamma) > \frac{1}{2}$, for some $p \in [2, \infty)$, then Γ is amenable ([NP08]). For some time, it was unknown if amenable groups with arbitrary small compression exist. In fact, there exist solvable groups with L^p compression 0 ([Aus11]).

4. The following result can be found in [NP08] and formalizes the fact that amongst the class of L^p spaces, embeddings into L^2 are the worst. For any finitely generated group Γ and for all $p > 1$, we have $\alpha_p^\natural(\Gamma) \geq \alpha_2^\natural(\Gamma)$.

5. From the previous two claims, it is easy to show that non-abelian free groups have equivariant L^p-compression equal to $\frac{1}{2}$, for all $p \geq 2$. By non-amenability, we deduce that $\alpha_p^\natural(\mathbb{F}_n) \leq 1/2$, for all $p \geq 2$. Then, the following embedding shows that this bound is sharp for $p = 2$. Let us introduce some notations that will be used later on (see p. 90 of [BdlHV08]). Let $X = (V, \mathbb{E})$ be a locally finite graph, where $\mathbb{E}$ denotes the set of **oriented** edges. Each edge $e \in \mathbb{E}$ has a source $s(e) \in V$ and a range $r(e) \in V$. There is an obvious fixed-point free involution $e \mapsto \bar{e}$ on $\mathbb{E}$ with $s(\bar{e}) = r(e)$ and $r(\bar{e}) = s(e)$, for all $e \in \mathbb{E}$. The set of all pairs $\{e, \bar{e}\}$ is the set of **geometric edges** of the graph X. We denote by $\ell^2_{\text{alt}}(\mathbb{E})$ the real Hilbert space of those maps $\xi : \mathbb{E} \to \mathbb{R}$ satisfying $\xi(\bar{e}) = -\xi(e)$ and such that $\sum_{e \in \mathbb{E}} |\xi(e)|^2 < \infty$. This vector space is endowed with the inner product

$$\langle \xi, \eta \rangle = \frac{1}{2} \sum_{e \in \mathbb{E}} \xi(e)\eta(e).$$

For two vertices $x, y \in V$, we define the **signed characteristic function** of the geodesic $[x, y]$ by

$$\chi_{x \to y}(e) = \begin{cases} 1, & \text{if } e \text{ is on } [x,y] \text{ and } e \text{ points from } x \text{ to } y, \\ -1, & \text{if } e \text{ is on } [x,y] \text{ and } e \text{ points from } y \text{ to } x, \\ 0, & \text{otherwise,} \end{cases}$$

In the case where $X = \mathcal{T}$ is a tree, then a simple calculation shows that
$$\|\chi_{x\to y}\|_{\ell^2_{\mathrm{alt}}(\mathbb{E})} = \sqrt{d(x,y)}.$$
Now, let G be a closed subgroup of $\mathrm{Aut}(\mathcal{T})$ for the topology of pointwise convergence on vertices. Let π be the orthogonal representation of G on $\ell^2_{\mathrm{alt}}(\mathbb{E})$ induced by the action of G on $\mathcal{T} = (V, \mathbb{E})$, that is,
$$(\pi(g)\xi)(e) = \xi(g^{-1}e),$$
for all $\xi \in \ell^2_{\mathrm{alt}}(\mathbb{E})$, $e \in \mathbb{E}$ and $g \in G$. Let $x_0 \in V$ be fixed. The **Haagerup cocycle** is defined by $b : G \to \ell^2_{\mathrm{alt}}(\mathbb{E})$ with
$$b(g) = \chi_{x_0 \to gx_0},$$
It is easy to check that b satisfies the cocycle relation with respect to π. By a previous observation, we have the following identity:
$$\|b(g)\|_{\ell^2_{\mathrm{alt}}(\mathbb{E})} = \sqrt{d(x_0, gx_0)}.$$
This proves that, if G acts properly on $\mathcal{T}$, then b is a proper cocycle and, therefore, G acts properly by affine isometries on a Hilbert space. In particular, if $G = \mathbb{F}_n$ is any non-abelian free group endowed with the word-length metric and if we see it acting on its Cayley tree, then the associated Haagerup cocycle has compression 1/2. From all this, we deduce $\alpha_2^\natural(\mathbb{F}_n) = 1/2$. Finally, as $\alpha_2^\natural(\mathbb{F}_n) \leq \alpha_p^\natural(\mathbb{F}_n)$, we conclude that $\alpha_p^\natural(\mathbb{F}_n) = 1/2$, for all $2 \leq p < \infty$. Recall that $\alpha_p^*(\mathbb{F}_n) = 1$, as mentioned in Section 3.2.

6. Let Γ be a Coxeter group. The embedding $F : \Gamma \to \ell^p$ of [BJS88] satisfying $d(x,y) = \|F(x) - F(y)\|_p^p$, for all $x, y \in \Gamma$, is equivariant. Therefore, $\alpha_2^\natural(\Gamma) \geq 1/2$. In case Γ is not amenable, then this bound is tight for all $p \geq 2$. If Γ is amenable, then $\alpha_p^\natural(\Gamma) = \alpha_p^*(\Gamma) = 1$, for all $p \in [1, \infty)$, as shown by the first set of examples in Chapter 3.

7. In [Yu05], Yu showed that any hyperbolic group Γ acts properly by affine isometries on some ℓ^p space for $p \gg 1$. A refinement of this work ([Bou]) shows that $\alpha_p^\natural(\Gamma) \geq \frac{1}{p}$. It is worth noting that some hyperbolic groups also have property (T). As such, any of their affine actions on ℓ^2 has a globally fixed point.

8. Theorem 1.10 of [dCV] gives the equivariant compression for any Baumslag-Solitar group G. For $p \geq 1$, we have
$$\alpha_p^\natural(G) = \begin{cases} 1, & \text{if } G \text{ is amenable,} \\ \max\{1/p, 1/2\}, & \text{if } G \text{ is not amenable.} \end{cases}$$

9. The coarse embedding of Thompson's group F exhibited in [AGS06] being equivariant, this shows $\alpha_2^\natural(F) = 1/2$. It is worth noting that, a slightly better embedding, for instance with control function ρ_- satisfying $\lim_{t\to\infty} \frac{\sqrt{t}}{\rho_-(t)} = 0$, would imply the amenability of F (see Theorem 1.2 in [dCTV07]).

4.1.2 Negative type functions

In this section, we introduce kernels and functions conditionally of negative type. We also make the correspondence between orbits of affine actions and functions conditionally of negative type on groups. We end up this section by defining Property (T) and Haagerup property for groups. We refer to the Appendix C of [BdlHV08] for the proofs and more details about conditionally negative type functions.

Definition 4.1.4. *A kernel $\Psi : W \times W \to \mathbb{R}$ on a set W is said to be* ***conditionally of negative type*** *if it satisfies the following properties:*

1. $\Psi(x, x) = 0$, *for all* $x \in W$;

2. $\Psi(x, y) = \Psi(y, x)$, *for all* $x, y \in W$;

3. *For any* $x_1, \ldots, x_n \in W$ *and for any* $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ *satisfying* $\sum_{i=1}^n \alpha_i = 0$, *we have*
$$\sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \Psi(x_i, x_j) \leq 0.$$

It is fairly elementary to produce examples of such kernels. Indeed, take $f : W \to \mathcal{H}$ to be any map with range $\mathcal{H}$, a Hilbert space. Then, a direct computation shows that the kernel defined by $\Psi(x, y) = \|f(x) - f(y)\|^2$ is conditionally of negative type. Interesting examples arise when the set W is endowed with a G-action and when the kernel is G-invariant, i.e. $\Psi(gx, gy) = \Psi(x, y)$, for all $x, y \in W$ and $g \in G$. Considering kernels on G which are invariant for the action on itself leads us to the next definition.

Definition 4.1.5. *A continuous function $\psi : G \to \mathbb{R}$ on a topological group G is said to be* ***conditionally of negative type*** *if the kernel on G defined by $(g, h) \mapsto \psi(g^{-1}h)$ is conditionally of negative type.*

Going back to the preceding example, if the set W is a G-space and if the map f satisfies $\|f(gx) - f(gy)\| = \|f(x) - f(y)\|$, for all $x, y \in W$ and all $g \in G$, then, chosing a basepoint $x_0 \in W$, the associated function ψ on G defined by

$$\psi(g) = \|f(gx_0) - f(x_0)\|^2$$

is conditionally of negative type. In particular, if b is a continuous cocycle for some unitary representation π of a group G, then the function on G given by $g \mapsto \|b(g)\|^2$ is conditionally of negative type. This example is essentially universal, by the so-called GNS construction (see p.63 in [dlHV89] for a proof).

Proposition 4.1.6. *(GNS construction) Let ψ be a function conditionally of negative type on G. There exist an orthogonal representation π_ψ of G acting on a real Hilbert space $\mathcal{H}_\psi$ and a cocycle $b_\psi \in Z^1(G, \pi_\psi)$ satisfying $\psi(g) = \|b_\psi(g)\|^2$, for all $g \in G$. Moreover, the triple $(\mathcal{H}_\psi, \pi_\psi, b_\psi)$ is unique up to isomorphism.*

We also note that, if $b \in Z^1(G, \pi)$, then the image of b corresponds to the orbit of 0 under the affine action $\alpha = (\pi, b)$, that is, $b(g) = \alpha(g)0$. This explains why cocycles are sometimes referred to as orbital maps.
The following result fills the gap between the geometrical and the algebraic aspects of affine actions on Hilbert spaces. Its proof is a direct consequence of the GNS construction and the Lemma of the centre.

Proposition 4.1.7. *Let ψ be a conditionally of negative type function on G, let $(\mathcal{H}_\psi, \pi_\psi, b_\psi)$ be its associated GNS triple and let $\alpha_\psi = (\pi_\psi, b_\psi)$ be the corresponding affine action. The following properties are equivalent :*

1. *The function ψ is bounded on G;*
2. *The map b_ψ is a coboundary;*
3. *The affine action α_ψ has a fixed point;*
4. *The affine action α_ψ has only bounded orbits.*

In particular, we deduce that an affine isometric action $\alpha = (\pi, b)$ is unbounded if and only if b is not a coboundary, and, furthermore, the affine action α is proper if and only if the function on G given by $g \mapsto \|b(g)\|$ is proper.

From all this, it is natural to identify groups for which any affine isometric action on a Hilbert space has a fixed point. These groups have the so-called Property (T). Before stating the precise definition, we recall some important notions of representation theory.

Definition 4.1.8. *Let π be a unitary (or orthogonal) representation of G.*

1. *The representation π **strongly contains** the trivial representation 1_G (or, π has **invariant vectors**) if, there exists a non-zero vector $\xi \in \mathcal{H}$ so that $\pi(g)\xi = \xi$, for all $g \in G$. In this case, we write $1_G \subset \pi$.*

2. *The representation π **weakly contains** the trivial representation 1_G (or, π has **almost invariant vectors**) if, for any compact subset $Q \subset G$ and for any $\epsilon > 0$, there exists a vector $\xi \in \mathcal{H}$ so that*

$$\sup_{g \in Q} \|\pi(g)\xi - \xi\| < \epsilon\|\xi\|.$$

   In this case, we write $1_G \prec \pi$.

3. *The representation π is C_0 if all of its matrix coefficients belong to $C_0(G)$, that is, all the functions of the form $\varphi_{\xi,\eta} : g \mapsto \langle \pi(g)\xi, \eta \rangle$, for some $\xi, \eta \in \mathcal{H}_\pi$, vanish at infinity.*

Definition 4.1.9. *Let G be a second countable, locally compact group. The group G has **Kazhdan's Property (T)** (or simply, **Property (T)**), if it satisfies one of the following equivalent conditions :*

1. *Every strongly continuous unitary representation that weakly contains the trivial representation also contains it strongly;*

2. *Every continuous, conditionally negative type function on G is bounded;*

3. *Every continuous, affine isometric action of G on a Hilbert space has a fixed point.*

The first condition, which was originally introduced by D. Kazhdan in the 1960s, means that the trivial representation 1_G is isolated in $\widehat{G}$, the unitary dual of G, for the Fell topology. This explains the notation "(T)", where the letter T stands for trivial. We remark that, in the above definition, the equivalence between the last two claims is proved by Proposition 4.1.7.

Property (T) has strong structural consequences. Amongst other things, a group G with Property (T) is compactly generated and is unimodular. If G is discrete, then its abelianisation $G/[G,G]$ is finite.
Examples of groups with Property (T) are : compact groups, $SL_{n+1}(\mathbb{Z})$, $SL_{n+1}(\mathbb{R})$ and $Sp(n,1)$, the real rank 1 simple Lie groups of isometries of a quaternionic hermitian form of signature $(n,1)$, for $n \geq 2$. By way of contrast, the next definition is a strong negation of Property (T).

Definition 4.1.10. *Let G be a second countable, locally compact group. The group G has the* ***Haagerup Property*** *(or, is* ***a-T-menable****), if it satisfies one of the following equivalent conditions :*

1. *There exists a strongly continuous unitary C_0-representation which contains the trivial representation weakly;*
2. *There exists a continuous function on G which is conditionally of negative type and proper;*
3. *There exists a continuous, affine isometric action of G on a Hilbert space which is metrically proper.*

This class of groups is quite large and contains : amenable groups, all groups with $\alpha_2^\natural > 0$ encountered in the set of examples appearing in Subsection 4.1.1 (free groups and, more generally, groups acting properly on trees, Coxeter groups, Baumslag-Solitar groups, Thompson's group F, ...) and the Lie groups $SO(n,1)$ and $SU(n,1)$, which are isometry groups of the n-dimensional real and complex hyperbolic spaces respectively.

4.1.3 Amenable actions via equivariant compression

We give a sufficent condition to prove that an action of a group on a graph is amenable. As already mentioned, it is possible to show that a finitely generated group has Property A (resp. is amenable) by providing a sufficiently good embedding (resp. equivariant embedding) into a Hilbert space, by results of Guentner and Kaminker ([GK04]). We get a slighty more general result. If we suppose that a discrete countable group Γ acts by isometries on a graph with bounded degree and by affine isometries on a Hilbert space, then, we can deduce that the Γ-action on the graph is amenable, providing a sufficiently good Γ-equivariant embedding. The proof of our result is based on the proof of Theorem 4.1 in [dCTV07].

Definition 4.1.11. *Let Γ be a discrete group and let X be a countable set. Assume that Γ acts on X. We say that the action of Γ on X is* ***amenable*** *if it satisfies the so-called Følner condition, that is, for all $\epsilon > 0$ and for all finite set $Q \subset \Gamma$, we can find a finite set $F \subset X$ such that*

$$|(\gamma \cdot F)\triangle F| < \epsilon |F|,$$

for all $\gamma \in Q$.

Note that, the action on itself of an amenable group is amenable. In fact, any action of an amenable group on a countable set is amenable.
For functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$, we write $f \prec g$ if for any $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}^+$ such that $f \leq \epsilon g$ outside K.

Proposition 4.1.12. *Let $X = (V, E)$ be a graph with bounded degree. Let $\mathcal{H}$ be a Hilbert space and let Γ be a countable discrete group. Assume that Γ acts both on X and $\mathcal{H}$ by isometries. We suppose furthermore that there exists a Γ-equivariant coarse embedding $F : X \to \mathcal{H}$ with control functions $\rho_\pm$ satisfying*

$$\rho_-(d(x, y)) \leq \|F(x) - F(y)\| \leq \rho_+(d(x, y))$$

for all $x, y \in X$, where $\rho_- : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing function with the following growth condition : $\rho_-(t) \succ t^{1/2}$. Then, under these assumptions, the action of Γ on X is amenable.
In particular, if Γ is a finitely generated group and X is its Cayley graph, then $\alpha_2^\natural(\Gamma) > 1/2$ implies that Γ is amenable.

To prove this proposition, we will need the following characterisation of amenable actions, appearing as Theorem 1.1 in [KT08].

Theorem 4.1.13. *Let Γ be a discrete group and let X be a countable set. Assume that Γ acts on X. Let us denote by λ_X the natural unitary representation of Γ acting on $\ell^2 X$. Then, the action of Γ on X is amenable if and only if $\lambda_X \succ 1_\Gamma$.*

Proof of Proposition 4.1.12: For any $t > 0$, let us define the kernel on $X \times X$ by

$$k_t(x, y) = \exp(-t\|F(x) - F(y)\|^2),$$

for all $x, y \in X$. Since F is Γ-equivariant, we note that k_t is invariant under the diagonal action of Γ on $X \times X$. Let K_t be the linear operator acting on $\ell^2 X$ with matrix $(k_t(x,y))_{x,y\in X}$. We first show that, for any $t>0$, the operator K_t is bounded. Since the graph X has bounded degree, there exists $a>0$ such that any sphere of radius n has at most e^{an} elements. That is,

$$|S_x(n)| \leq e^{an},$$

for all $x\in X$ and $n\in\mathbb{N}$. Now, fix $t>0$. By the hypothesis on ρ_-, there exists n_0 (depending on t and a) such that, for all $n\geq n_0$, we have

$$(a+1)n \leq t\rho_-(d(x,y))^2 \leq t\|F(x)-F(y)\|^2,$$

for all $x,y\in X$ satisfying $d(x,y)=n$. Therefore, for all $x_0\in X$ and for all $n\geq n_0$, we get :

$$\begin{aligned}\sum_{y\in S_{x_0}(n)} k_t(y,x_0) &\leq |S_{x_0}(n)|e^{-(a+1)n}\\ &\leq e^{an}e^{-(a+1)n}\\ &= e^{-n}.\end{aligned}$$

We deduce that

$$\sum_{n\geq n_0}\sum_{y\in S_{x_0}(n)} k_t(y,x_0) \leq \sum_{n\geq n_0} e^{-n} = C < \infty,$$

which implies that, for any fixed x_0, the sum $\sum_{y\in X} k_t(y,x_0)$ is finite. Moreover, the upper bound C depends only on t and on the maximum degree of the graph (via the constant a). However, it does not depend on x_0. Hence,

$$\sup_{x\in X}\sum_{y\in X} k_t(y,x) \leq C'.$$

for some constant C'. The kernel k_t being symmetric, we can exchange the roles of x and y here above. Applying the Schur test, we get that K_t is a bounded operator of norm at most C'. Since k_t is a positive-definite kernel, K_t is a positive operator and we have

$$k_t(x,y) = \langle K_t\delta_y, \delta_x\rangle,$$

for all $x,y\in X$. Henceforth, there exists a unique positive square-root $Q_t\in\mathcal{B}(\ell^2 X)$ for K_t. Recall that $Q_t^*=Q_t$ and $Q_t^2=K_t$. It also has the following nice property : if $T\in\mathcal{B}(\ell^2 X)$ commutes with K_t, then it commutes with Q_t. Let us see how we will use this fact. Let $x, y \in X$ and $\gamma \in \Gamma$. We have :

$$\begin{aligned}\langle \lambda_X(\gamma) K_t \delta_y, \delta_x \rangle &= \langle K_t \delta_y, \lambda_X(\gamma^{-1}) \delta_x \rangle \\ &= k_t(\gamma^{-1} x, y) \\ &= k_t(x, \gamma y) \\ &= \langle K_t \lambda_X(\gamma) \delta_y, \delta_x \rangle,\end{aligned}$$

which implies that λ_X commutes with K_t and, consequently, with Q_t.
We can now prove that λ_X almost has invariant vectors. Chose any $x_0 \in X$. The operators Q_t being bounded, $Q_t \delta_{x_0}$ is a vector in $\ell^2 X$ and we have

$$\begin{aligned}\|Q_t \delta_{x_0}\|^2 &= \langle Q_t \delta_{x_0}, Q_t \delta_{x_0} \rangle \\ &= \langle K_t \delta_{x_0}, \delta_{x_0} \rangle \\ &= k_t(x_0, x_0) \\ &= 1.\end{aligned}$$

Moreover, for any γ, we have :

$$\begin{aligned}\langle \lambda_X(\gamma) Q_t \delta_{x_0}, Q_t \delta_{x_0} \rangle &= \langle Q_t \delta_{\gamma x_0}, Q_t \delta_{x_0} \rangle \\ &= \langle K_t \delta_{\gamma x_0}, \delta_{x_0} \rangle \\ &= k_t(\gamma x_0, x_0) \\ &= \exp(-t\|F(\gamma x_0) - F(x_0)\|^2).\end{aligned}$$

Finally, let $\epsilon > 0$ and let $S \subset \Gamma$ be a finite subset. Taking $t > 0$ small enough so that

$$\exp(-t\|F(\gamma x_0) - F(x_0)\|^2) > 1 - \epsilon,$$

for all $\gamma \in S$, and setting $\xi = Q_t \delta_{x_0}$, we get

$$\sup_{\gamma \in S} |1 - \langle \lambda_X(\gamma) \xi, \xi \rangle| < \epsilon,$$

and we can conclude that $1_\Gamma \prec \lambda_X$ and the action is amenable.
In the case X is the Cayley graph of Γ, then, the representation λ_X coincides with the left regular representation on Γ and the proof shows that $1_\Gamma \prec \lambda_\Gamma$ and Γ is amenable. □

Although this result fits nicely in the theory, it seems difficult to apply to get new and non-trivial examples of amenable actions. Here, non-trivial should be understood as when the group Γ is not amenable, or, more generally, when the action is not clearly amenable. Indeed, if some orbit Γx_0 is a quasi-isometric embedding of Γ into X, then, by composing with the map F, we would obtain a Γ-equivariant embedding of Γ into a Hilbert space with control function $\rho_-(t) \succ t^{1/2}$, and, therefore Γ would be amenable.

4.2 Main Results

Let $q \geq 2$ and let $\mathcal{T}_{q+1}$ be the homogeneous $(q+1)$-regular tree and let $\partial\mathcal{T}_{q+1}$ be its boundary. In this section, we consider a group G of automorphims of $\mathcal{T}_{q+1}$ acting transitively both on the vertices and on the boundary of $\mathcal{T}_{q+1}$. Firstly, we study the behaviour of control functions of G-equivariant coarse embeddings $F : V \to \mathcal{H}$. We give an optimal behaviour for the norm of F and we show it is attained when F is harmonic in an appropriate sense. As a consequence, we determine the unique pure functions conditionally of negative type on G.

Definition 4.2.1. *For a locally finite, non-oriented graph $X = (V, E)$, we will denote by $\mathcal{L}$ the (normalised)* ***Laplace operator*** *defined as*

$$\mathcal{L}f(x) = \left(\frac{1}{\deg(x)} \sum_{y \sim x} f(y) \right) - f(x),$$

for any $f \in \ell^2(V)$ and $x \in V$. More generally, for any Hilbert space $\mathcal{H}$, this defines an operator acting on maps $F : V \to \mathcal{H}$. It is therefore natural to call such a map F ***harmonic*** *if $\mathcal{L}F$ is identically zero.*

Remark 4.2.2. *The Laplace operator Δ appearing in Chapter 2 is linked to $\mathcal{L}$ in the following way : $\Delta = -k\mathcal{L}$, for all k-regular graphs. We note that both operators have the same kernel, therefore, the (usual) notion of harmonicity stays unchanged. We justify our decision to consider $\mathcal{L}$ instead of Δ in the present chapter by our wish of being consistent with the notations of the book [Woe00], which we will refer to at several occasions.*

The first result of this chapter is the following theorem.

Theorem 4.2.3. *Let G be a closed non compact subgroup of* $\mathrm{Aut}(\mathcal{T}_{q+1})$*, with $q \geq 2$. Suppose that G acts transitively on the vertices and on the boundary of $\mathcal{T}_{q+1}$. Assume furthermore that G acts on a Hilbert space $\mathcal{H}$ by affine isometries. Then, any G-equivariant map $F : V \to \mathcal{H}$ such that $F(x_0) = 0$ for some $x_0 \in V$ satisfies:*

$$\|F(x)\|^2 \leq Ad(x, x_0) - B + Bq^{-d(x,x_0)},$$

where x_1 is any vertex adjacent to x_0,

$$A = \frac{(q+1)\|F(x_1)\|^2}{q-1} \quad \text{and} \quad B = \frac{2q\|F(x_1)\|^2}{(q-1)^2}.$$

Furthermore, if the F is harmonic and non-constant, then the equality occurs.

The second result shows that it is optimal by proving the existence of such an equivariant and harmonic map.

Proposition 4.2.4. *Let G be a closed non compact subgroup of* $\mathrm{Aut}(\mathcal{T}_{q+1})$*, with $q \geq 2$. Suppose that G acts transitively on the vertices and on the boundary of $\mathcal{T}_{q+1}$. Then, there exists a Hilbert space $\mathcal{H}$ endowed with an affine isometric G-action and a map $F : V \to \mathcal{H}$ which is G-equivariant, non-constant and harmonic.*

Combining this with a result of Nebbia, we deduce that this map is essentially unique.

Proposition 4.2.5. *(See Corollary 4.3.7) Let G be as in Theorem 4.2.3 and choose a basepoint $x_0 \in V$. For $g \in G$, we set $|g| := d(x_0, gx_0)$. Then, the function on G defined by*

$$g \mapsto \Psi(gx_0, x_0) = |g| + \frac{2q}{q^2-1}(q^{-|g|} - 1)$$

is the unique (up to multiplication by a positive scalar) pure negative type function in $\mathrm{CL}(G)$*, the positive cone of negative type functions on G, which is unbounded on G and identically 0 on G_{x_0}.*

We finally point out that, in Corollary 4.4.6, we will give a concrete realization of the cocycle associated to the function of negative type here above as the projection of the Haagerup cocycle onto an appropriate G-invariant closed subspace of $\ell^2_{\mathrm{alt}}(\mathbb{E})$.

4.3 Proofs of the main results

The proofs of Theorem 4.2.3 and of Proposition 4.2.4 presented here are adapted from Section 3.3 of [BdlHV08]. In their text, the authors study the behaviour of G-equivariant maps from X into a Hilbert space $\mathcal{H}$, with $X = G/K$ being an irreducible Riemannian symmetric space of rank one and (G, K) being a Gelfand pair. In comparison, by taking G an appropriate subgroup of $\mathrm{Aut}(\mathcal{T}_{q+1})$ and K a maximal compact subgroup of G, we also get a Gelfand pair (G, K), a natural identification $\mathcal{T}_{q+1} = G/K$ and the regular tree $\mathcal{T}_{q+1}$ plays the role of symmetric space in our case.

4.3.1 Proof of Theorem 4.2.3

We start with a lemma.

Lemma 4.3.1. *Let $X = (V, E)$ be a locally finite graph. Let $F : V \to \mathcal{H}$ be any map.*

1. *For any $x \in V$, the following identity holds:*
$$\mathcal{L}(\|F\|^2)(x) = \|\nabla_x F\|^2_{\mathcal{H}} + 2\Re\langle \mathcal{L}F(x), F(x)\rangle_{\mathcal{H}},$$
where
$$\|\nabla_x F\|^2 = \|\nabla_x F\|^2_{\mathcal{H}} = \frac{1}{\deg(x)} \sum_{y\sim x} \|F(x) - F(y)\|^2.$$

2. *Let G be a group acting on X by automorphisms and on $\mathcal{H}$ by affine transformations. Then, for any G-equivariant map $F : V \to \mathcal{H}$, the quantity $\|\nabla_x F\|$ is constant along each orbit of $x \in V$, that is,*
$$\|\nabla_{gx} F\| = \|\nabla_x F\|, \quad \forall x \in V, g \in G.$$

Proof : The proof of the first identity is a simple calculation. Since we have
$$\|\nabla_x F\|^2 = \|F(x)\|^2 + \frac{1}{\deg(x)} \sum_{y\sim x} \|F(y)\|^2 - \frac{2}{\deg(x)} \sum_{y\sim x} \Re\langle F(y), F(x)\rangle,$$
and
$$\langle \mathcal{L}F(x), F(x)\rangle = -\|F(x)\|^2 + \frac{1}{\deg(x)} \sum_{y\sim x} \langle F(y), F(x)\rangle,$$

it readily follows that

$$\begin{aligned}\|\nabla_x F\|^2 + 2\Re\langle \mathcal{L}F(x), F(x)\rangle &= -\|F(x)\|^2 + \frac{1}{\deg(x)}\sum_{y\sim x}\|F(y)\|^2 \\ &= \mathcal{L}\|F\|^2(x).\end{aligned}$$

To prove the second statement, we recall that the equivariance implies

$$\begin{aligned}\|F(gx) - F(y)\| &= \|\alpha(g)F(x) - F(y)\| \\ &= \|F(x) - \alpha(g^{-1})F(y)\| \\ &= \|F(x) - F(g^{-1}y)\|,\end{aligned}$$

for all $g \in G$, $x, y \in V$, where α is the map corresponding to the G-action on $\mathcal{H}$. Hence, we obtain

$$\begin{aligned}\|\nabla_{gx} F\|^2 &= \frac{1}{\deg(x)}\sum_{y\sim gx}\|F(gx) - F(y)\|^2 \\ &= \frac{1}{\deg(x)}\sum_{y\sim gx}\|F(x) - F(g^{-1}y)\|^2 \\ &= \|\nabla_x F\|^2.\end{aligned}$$

□

Lemma 4.3.2. *Let G be a group acting transitively on $\partial\mathcal{T}_{q+1}$ and let us denote by G_{x_0} the satbiliser of x_0. If the map $F : V \to \mathcal{H}$ satisfies $F(x_0) = 0$ for some point x_0 and if F is G-equivariant with respect to some affine isometric action $\alpha = (\pi, b)$, where b is zero on G_{x_0}, then both $\|F\|$ and $\langle \mathcal{L}F, F\rangle$ are radial.*

Proof : Let $x, y \in V$ be two vertices at distance r from x_0. Since G_{x_0} acts transitively on any sphere about x_0, we can find $h \in G_{x_0}$ so that $hx = y$. By the hypothesis on b, α acts by unitary operators when restricted to the stabilizer of x_0. Using the equivariance of F, it is straightforward that

$$\|F(y)\| = \|\pi(h)F(x)\| = \|F(x)\|.$$

Moreover, using the G-equivariance of $\mathcal{L}$ with respect to π, we obtain

$$\begin{aligned}\langle \mathcal{L}F(y), F(y)\rangle &= \langle \mathcal{L}F(hx), F(hx)\rangle \\ &= \langle \pi(h)\mathcal{L}F(x), \pi(h)F(x)\rangle \\ &= \langle \mathcal{L}F(x), F(x)\rangle.\end{aligned}$$

□

For the rest of the section, we will assume that G, F and $\alpha = (\pi, b)$ satisfy the hypotheses of Theorem 4.2.3. Select one vertex on each sphere of radius n and denote it by x_n. We can make two remarks.

1. Since G acts transitively on V, then $\|\nabla_x F\| = \|\nabla_{x_0} F\| = \|F(x_1)\|$.

2. We recall that a non compact closed subgroup of $\mathrm{Aut}(\mathcal{T}_{q+1})$ acts transitively on $\partial\mathcal{T}_{q+1}$ if and only if there exists $y \in V$ so that the stabilizer G_y acts transitively on $\partial\mathcal{T}_{q+1}$ (see Proposition 10.1 in [FTN91]). If the group G acts transitively on V, then the latter condition is also equivalent to having all the vertex-stabilizers acting transitively on $\partial\mathcal{T}_{q+1}$. This means that under the hypotheses of Theorem 4.2.3, the subgroup G_{x_0} acts transitively on $\partial\mathcal{T}_{q+1}$.

Now, set $\varphi(n) = \|F(x_n)\|^2$. For radial maps, the Laplace operator takes a very simple form :

$$\mathcal{L}\varphi(0) = \varphi(1) - \varphi(0),$$

and

$$\mathcal{L}\varphi(n) = \frac{q}{q+1}\varphi(n+1) - \varphi(n) + \frac{1}{q+1}\varphi(n-1),$$

for any $n \geq 1$. Using Lemma 4.3.1 and setting $R_F(n) = 2\Re\langle \mathcal{L}F(x_n), F(x_n)\rangle$, we obtain the relations for all $n \geq 1$,

$$\frac{q}{q+1}\varphi(n+1) - \varphi(n) + \frac{1}{q+1}\varphi(n-1) = \varphi(1) + R_F(n), \tag{4.1}$$

with initial conditions $\varphi(0) = 0$ and $\varphi(1) = \|F(x_1)\|^2$.
Let us find a general solution to this second order linear recurrence equation. Set

$$\psi(n+1) = \varphi(n+1) - \varphi(n),$$

for all $n \geq 0$. We can express the left-hand side of the relation (4.1) using ψ:

$$\begin{aligned}\frac{q}{q+1}\varphi(n+1) - \varphi(n) + \frac{1}{q+1}\varphi(n-1) &= \frac{q}{q+1}(\varphi(n+1) - \varphi(n)) \\ &\quad -\frac{1}{q+1}(\varphi(n) - \varphi(n-1)) \\ &= \frac{q}{q+1}\psi(n+1) - \frac{1}{q+1}\psi(n).\end{aligned}$$

For all $n \geq 1$, we obtain the new relation:

$$\psi(n+1) = \frac{1}{q}\psi(n) + \frac{(q+1)\|F(x_1)\|^2}{q} + \frac{q+1}{q}R_F(n),$$

with initial condition $\psi(1) = \|F(x_1)\|^2$. By iterating this relation, we get

$$\psi(n+1) = \frac{(q+1)\|F(x_1)\|^2}{q-1} - \frac{2\|F(x_1)\|^2}{(q-1)q^n} + (q+1)\sum_{j=1}^{n} \frac{R_F(n+1-j)}{q^j}.$$

To proceed, we need a crucial negativity result.

Lemma 4.3.3. *Under the hypotheses of Theorem 4.2.3, we have*

$$\Re\langle \mathcal{L}F(x), F(x)\rangle \leq 0,$$

for all $x \in V$.

Let us postpone the proof of this lemma to the end of the section and let us show how to finish the proof of Theorem 4.2.3. Firstly, Lemma 4.3.3 implies the following inequality

$$\psi(n+1) \leq \frac{(q+1)\|F(x_1)\|^2}{q-1} - \frac{2\|F(x_1)\|^2}{(q-1)q^n},$$

with equality if F is harmonic. Replacing ψ by φ, we get

$$\varphi(n+1) \leq \varphi(n) + \frac{(q+1)\|F(x_1)\|^2}{q-1} - \frac{2\|F(x_1)\|^2}{(q-1)q^n},$$

Iterating this inequality, we obtain the desired upper bound. Once again, if F is harmonic, then the equality occurs, and we are done. □

Proof of Lemma 4.3.3

The end of the present section is dedicated to the proof of Lemma 4.3.3. The main steps of the proof are described in the next lemma.

Lemma 4.3.4. *We assume that the group G and the map F satisfy the hypotheses of Theorem 4.2.3. Also, we write K for the compact subgroup G_{x_0} and dk for the normalised Haar measure on G so that the subgroup K has measure 1. Fix $x \in V$ and let g and s in G be such that $x = g^{-1}x_0$ and sx_0 is adjacent to x_0.*

1. *For any vertex $y \in V$ adjacent to x_0, the map F satisfies the following integral formula :*

$$\int_K F(g^{-1}ky)dk = \frac{1}{q+1}\sum_{u\sim x} F(u).$$

2. *Furthermore, if we denote by* $\alpha = (\pi, b)$ *the affine* G*-action on* $\mathcal{H}$*, then*

$$\mathcal{L}F(x) = \int_K \pi(g^{-1}k)b(s)dk.$$

3. *We also have*

$$\langle \mathcal{L}F(x), F(x)\rangle = -\langle P_K b(s), b(g)\rangle ,$$

   where $P_K = \int_K \pi(k)dk$ *is the orthogonal projection onto the space of* $\pi(K)$*-fixed vectors in* $\mathcal{H}$*.*

4. *In particular, desintegrating* π *and* b *as direct integrals over some measure space* (Z, ν)*, we obtain*

$$\langle \mathcal{L}F(x), F(x)\rangle = -\int_Z \langle P_z b_z(s), b_z(g)\rangle d\nu(z),$$

   where, for almost all $z \in Z$*,* π_z *is an irreducible unitary representation of* G*,* b_z *is a* π_z*-cocycle, and*

$$P_z = \int_K \pi_z(k)dk$$

   is the orthogonal projection onto the space of $\pi_z(K)$*-invariant vectors.*

5. *Finally, for any irreducible unitary representation* σ *and for any cocycle* $w \in Z^1(G, \sigma)$*, we have*

$$\langle Pw(s), w(g)\rangle \geq 0,$$

   where $P = \int_K \sigma(k)dk$ *denotes the orthogonal projection onto the space of* $\sigma(K)$*-invariant vectors.*

Clearly, Lemma 4.3.3 follows directly from the last two claims above.

Proof of Claim 1 : Fix $x \in V$. Since G acts on V transitively, there exists $g \in G$ so that $x = g^{-1}x_0$. Let $y_1, \ldots, y_{q+1}$ be the $q+1$ neighbours of x_0. Using the fact that the action of G_{x_0} on the sphere of radius 1 centered at x_0 is transitive, we can find $h_j \in G_{x_0}$ so that $h_j y_j = y_1$, for $j = 1, \ldots, q+1$. We remark that the cosets given by $h_j^{-1}(G_{x_0} \cap G_{y_1})$ are all distinct and that the subgroup $G_{x_0} \cap G_{y_1}$ has index $q+1$ in G_{x_0}. Normalising the Haar measure on

G so that the compact subgroup G_{x_0} has measure 1, we obtain the following relation:

$$\begin{aligned}\int_{G_{x_0}} F(g^{-1}ky_1)dk &= \sum_{j=1}^{q+1}\int_{G_{x_0}\cap G_{y_1}} F(g^{-1}h_j^{-1}ky_1)dk \\ &= \sum_{j=1}^{q+1} F(g^{-1}y_j)\int_{G_{x_0}\cap G_{y_1}} dk \\ &= \frac{1}{q+1}\sum_{j=1}^{q+1} F(g^{-1}y_j) \\ &= \frac{1}{q+1}\sum_{u\sim x} F(u).\end{aligned}$$

In particular, we see that the integral on the left-hand side does not depend on the choice of the neighbour of x_0.

Proof of Claim 2 : Again, by transitivity of the G-action, we can find $s \in G$ so that $sx_0 = y_1$. It is straightforward to see that b, the translation part of the G-action on $\mathcal{H}$, factors through a G-equivariant map $G/K \to \mathcal{H}$ and that it coincides with F. Namely, $F(hx_0) = b(h)$, for all $h \in G$. In particular, $F(x) = b(g^{-1})$. By the cocycle relation, we have

$$\begin{aligned}F(g^{-1}ky_1) - F(g^{-1}kx_0) &= F(g^{-1}ksx_0) - F(g^{-1}kx_0) \\ &= b(g^{-1}ks) - b(g^{-1}k) \\ &= \pi(g^{-1}k)b(s) + b(g^{-1}k) - b(g^{-1}k) \\ &= \pi(g^{-1}k)b(s).\end{aligned}$$

Hence, we obtain the desired integral formula for the Laplace operator applied to F:

$$\begin{aligned}\mathcal{L}F(x) &= \left(\frac{1}{q+1}\sum_{y\sim x} F(y)\right) - F(x) \\ &= \left(\int_K F(g^{-1}ksx_0)dk\right) - F(g^{-1}x_0) \\ &= \int_K \left(F(g^{-1}ksx_0) - F(g^{-1}x_0)\right) dk \\ &= \int_K \pi(g^{-1}k)b(s)dk.\end{aligned}$$

Proof of Claim 3 : Hence, applying Claim 2, we directly get :

$$\begin{aligned}
\langle \mathcal{L}F(x), F(x)\rangle &= \left\langle \int_K \pi(g^{-1}k)b(s)dk, b(g^{-1})\right\rangle \\
&= \int_K \langle \pi(g^{-1}k)b(s), b(g^{-1})\rangle dk \\
&= \int_K \left\langle \pi(k)b(s), \pi(g)b(g^{-1})\right\rangle dk \\
&= -\int_K \langle \pi(k)b(s), b(g)\rangle\, dk \\
&= -\left\langle \int_K \pi(k)b(s)dk, b(g)\right\rangle \\
&= -\langle P_K b(s), b(g)\rangle\, .
\end{aligned}$$

Proof of Claim 4 : Now, let us desintegrate π and b as direct integrals. We write

$$\pi = \int_Z^{\oplus} \pi_z d\nu(z),$$

and

$$b = \int_Z^{\oplus} b_z d\nu(z),$$

for some measure space (Z,ν). Recall that, for almost all $z \in Z$, π_z is an irreducible unitary representation of G and b_z is a π_z-cocycle. Thus, we deduce from Claim 3 :

$$\langle \mathcal{L}F(x), F(x)\rangle = -\int_Z \langle P_z b_z(s_0), b_z(g)\rangle d\nu(z).$$

Proof of Claim 5 : As G acts transitively on $\partial\mathcal{T}_{q+1}$, the couple (G,K) forms a Gelfand pair (see chapter II, section 4 in [FTN91]) and we treat three cases. If the irreducible representation σ is not spherical, then $P = 0$ and the inner-product is 0. If $\sigma = 1_G$ is the trivial representation, then $w = 0$ and the inner-product is again equal to 0. Indeed, under the assumptions on G, we have that $H^1(G, 1_G) = Hom(G, \mathbb{C}) = 0$ (see p. 5 of [Neb12])[1]. Finally, if we suppose that σ is spherical and non-trivial, then there exists a $\sigma(K)$-invariant vector η such that

$$w(h) = (\sigma(h) - 1)\eta, \ \forall h \in G.$$

Up to rescaling w, we can assume that η has norm 1. As the space of $\sigma(K)$-invariant vectors is one-dimensional, P is a rank-one operator. So, we can write

$$P\xi = \langle \xi, \eta \rangle \eta,$$

for all $\xi \in \mathcal{H}$. This yields

$$\begin{aligned} \langle Pw(s), w(g) \rangle &= \langle P(\sigma(s) - 1)\eta, (\sigma(g) - 1)\eta \rangle \\ &= \langle (\sigma(s) - 1)\eta, \eta \rangle \langle \eta, (\sigma(g) - 1)\eta \rangle \\ &= (\langle \sigma(s)\eta, \eta \rangle - 1)(\langle \eta, \sigma(g)\eta \rangle - 1) \\ &= (\phi(s) - 1)\left(\phi(g^{-1}) - 1\right), \end{aligned}$$

where ϕ is the (normalised) positive-definite function of σ associated to η. In particular, ϕ is a spherical function, that is, a radial eigenfunction of the normalised adjacency operator on $\mathcal{T}_{q+1}$ and $\phi(x_0) = 0$. This function being positive-definite and radial, it is therefore real-valued. By Cauchy-Schwarz, we conclude that $\phi(h) - 1 \leq 0$, for all $h \in G$ and therefore, the scalar product we began with is always positive or null. This ends the proof of the Claim 5. □

4.3.2 Proof of Proposition 4.2.4

The existence of an equivariant and harmonic map follows from a general argument due to Shalom. We recall it here briefly and we refer to Chapter 3 of [BdlHV08] for more details. Let G be a group as in Theorem 4.2.3. Fixing a based-vertex $x_0 \in V$, we recall that (G, K) is a Gelfand pair, with $K = G_{x_0}$. Moreover, G is locally compact and compactly generated. As in the proof of Lemma 4.3.3, $H^1(G, 1_G) = 0$. Since G does not have property (T), there exists a non-trivial irreducible unitary representation π admitting an unbounded, continuous cocycle b (see Corollaire 1 in [LSV04]). As K is

[1] We recall the argument briefly for the reader's convenience. Let $\phi \in Hom(G, \mathbb{C})$ be a continuous homomorphism. The group G is generated by the vertex-stabilizer G_{x_0} and by the edge-stabilizer $G_{[x_0,y_1]}$. Both subgroups being compact, their images under ϕ are compact subgroups of $\mathbb{C}$. Therefore, ϕ is identically 0.

compact, we can find a cocycle vanishing on K which is in the cohomology class of b. Hence, we can assume that b is identically 0 on K and that b factors through a G-equivariant map on G/K, which we denote by F. By Proposition 3.3.7 in [BdlHV08], the map F satisfies the following mean value property:

$$\int_K F(g_0 k g_0^{-1} g K) dk = F(g_0 K),$$

for all $g, g_0 \in G$. As b is not a coboundary, we remark that F is not constant. Seen as a map on V, we claim that F is harmonic.

Proof of Proposition 4.2.4 : Fix $g_0 \in G$. For every $g \in G$, we will write ${}_gF(x) = F(gx)$. For any $x \in G/K$, we set

$$\begin{aligned}\left(\int_K {}_{g_0k}F dk\right)(x) &= \int_K \left({}_{g_0k}F\right)(x) dk \\ &= \int_K F(g_0 k x) dk.\end{aligned}$$

For $x \in G/K$ fixed, the integral exists, as we are just integrating a continuous map over a compact set. Hence, the map $\int_K {}_{g_0k}F dk : G/K \to \mathcal{H}$ is well defined. We claim that it is constant. Indeed, by the mean value property, we have

$$\begin{aligned}\left(\int_K {}_{g_0k}F dk\right)(x) &= \int_K \left({}_{g_0k}F\right)(x) dk \\ &= \int_K F(g_0 k x) dk \\ &= \int_K F(g_0 k g_0^{-1}(g_0 x)) dk \\ &= F(g_0 K).\end{aligned}$$

Since $\mathcal{L}$ annihilates the constants, we deduce that $\mathcal{L}\left(\int_K {}_{g_0k}F dk\right) \equiv 0$. Now, using the linearity of $\mathcal{L}$ and the fact that it commutes with the G-action, we get:

$$\begin{aligned}\mathcal{L}\left(\int_K {}_{g_0k}F dk\right) &= \int_K \mathcal{L}\left({}_{g_0k}F\right) dk \\ &= \int_K {}_{g_0k}\left(\mathcal{L}F\right) dk.\end{aligned}$$

Therefore, one has

$$\begin{aligned}
(\mathcal{L}F)(g_0x_0) &= \int_K (\mathcal{L}F)(g_0x_0)dk \\
&= \int_K (\mathcal{L}F)(g_0kx_0)dk \\
&= \mathcal{L}\left(\int_K {}_{g_0k}F dk\right)(x_0) \\
&= 0.
\end{aligned}$$

Since $g_0 \in G$ is arbitrary and the action of G on V is transitive, we conclude that $\mathcal{L}F$ vanishes everywhere on V and we are done. □

In fact, the irreducible representation π is unique and $H^1(G,\pi)$ has complex dimension 1 (see [Neb12], where this representation is denoted by σ^-). Furthermore, π being non-spherical, it has no non-trivial K-invariant vector. In particular, this implies that $H^1(G,\pi)$ is isomorphic to $Z^1_K(G,\pi)$, the space of 1-cocycles which vanish on the compact subgroup K. Therefore, Proposition 4.2.4 gives an alternative description of the unique unbounded, cocycle which is identically 0 on K and which appears in Nebbia's paper.

Corollary 4.3.5. *The cocycle F of Proposition 4.2.4 is a representative of the only non-trivial cohomology class of the unique irreducible unitary representation of G which has a non-vanishing cohomology group in degree 1.*

4.3.3 First proof of Proposition 4.2.5

In this subsection, we will see that Proposition 4.2.4 gives rise to an interesting example of a kernel which is conditionally of negative type on the set of vertices of a regular tree and, then, we turn to the study of pure negative type functions on groups acting on regular trees. It is well-known that, for a group G, the set of all functions conditionally of negative type forms a convex, positive cone, denoted by $\mathrm{CL}(G)$. We say that a function $\psi \in \mathrm{CL}(G)$ is **pure** if it lies on an extremal ray of $\mathrm{CL}(G)$. As an application of Proposition 4.2.4, we describe the pure functions which are conditionally of negative type, for groups satisfying the hypotheses of Proposition 4.2.4.

We start with a result characterizing pure negative type functions in terms of their associated representations (via the GNS construction).

Theorem 4.3.6. *(Théorème 1, [LSV04]) Let G be a topological group.*

1. *Let ψ be a function conditionally of negative type on G and let $(\pi_\psi, \mathcal{H}_\psi, b_\psi)$ be its associated GNS triple. If ψ is pure, then the orthogonal representation π_ψ is irreducible.*
2. *Let π be an irreducible orthogonal representation and let b be any 1-cocycle for the representation π. Then, the function of negative type ψ associated with b is pure.*

This yields:

Corollary 4.3.7.
1. *The kernel defined on the set of vertices of $\mathcal{T}_{q+1}$ by*
$$\Psi : (x,y) \mapsto d(x,y) - \frac{2q}{q^2-1} + \frac{2q^{1-d(x,y)}}{q^2-1}$$
is conditionally of negative type.
2. *Let G be as in Theorem 4.2.3 and choose a basepoint $x_0 \in V$. For $g \in G$, we set $|g| := d(x_0, gx_0)$. Then, the function on G defined by*
$$g \mapsto \Psi(gx_0, x_0) = |g| + \frac{2q}{q^2-1}(q^{-|g|} - 1)$$
is the unique (up to multiplication by a positive scalar) pure negative type function in $\mathrm{CL}(G)$ which is unbounded on G and identically 0 on G_{x_0}.

Proof : To prove the first claim, we simply remark that
$$\Psi(x,y) = \|F(x^{-1}y)\|^2,$$
where the map F is as in Proposition 4.2.4 and is normalised so that
$$\|F(x_1)\|^2 = \frac{q-1}{q+1}.$$
Clearly, the function here above is conditionally of negative type and pure, by Corollary 4.3.5 and Theorem 4.3.6. Let us check the uniqueness. Let ψ be a function conditionally of negative type which vanishes on G_{x_0} and which is unbounded on G. Then, by Theorem 4.3.6, we know that b_ψ, the cocycle associated with ψ, is unbounded, pure and that it vanishes on G_{x_0}. By Corollary 4.3.5, we conclude that b_ψ is the unique non-trivial cocycle associated with π and the result follows. □

4.4 Cone of negative type functions

4.4.1 Decomposition of the Haagerup cocycle

By way of contrast, we give an example of a natural negative type function on G which is not pure. More precisely, we will decompose the orthogonal representation associated to the Haagerup cocycle and give a concrete realization of the pure negative type function described in Corollary 4.3.7.
To do this, we need first to fix some notations (see Example 5 of Subsection 4.1.1 and Chapter 1 of [Woe00]). Let $X = (V, E)$ be a locally finite graph. We define $\ell^2(V, m)$ to be the Hilbert space of square-summable functions on V endowed with the inner product

$$(f, g) = \sum_{x \in V} f(x)g(x)m(x),$$

where $m(x)$ is the degree of the vertex x. Note that, since X is locally finite, the quantity $m(x)$ is finite for every $x \in V$. Now we can define two operators connecting $\ell^2(V, m)$ with the space $\ell^2_{\text{alt}}(\mathbb{E})$. Let $\nabla : \ell^2(V, m) \to \ell^2_{\text{alt}}(\mathbb{E})$ be the **gradient**, defined by

$$(\nabla f)(e) = f(e_+) - f(e_-).$$

It is straightforward from the definition that $(\nabla f)(\overline{e}) = -(\nabla f)(e)$, for all $e \in \mathbb{E}$. We also define the **divergence** $\nabla^* : \ell^2_{\text{alt}}(\mathbb{E}) \to \ell^2(V, m)$ as the adjoint of ∇, that is,

$$(\nabla^* u)(x) = \frac{1}{\deg(x)} \sum_{y \sim x} u(y, x),$$

for all $u \in \ell^2_{\text{alt}}(\mathbb{E})$ and $x \in V$. The link between the Laplace operator $\mathcal{L}$ and the operators ∇ and ∇^* is given by the following formula:

$$\mathcal{L} = -\nabla^* \nabla.$$

Lemma 4.4.1. *(Poincaré lemma on trees) Let $\mathcal{T} = (V, \mathbb{E})$ be a tree and fix a vertex x_0. For any map $\xi : \mathbb{E} \to \mathbb{R}$ such that $\xi(\overline{e}) = -\xi(e)$, for all $e \in \mathbb{E}$, there is a unique function $\tilde{\xi} : V \to \mathbb{R}$ such that $\nabla\tilde{\xi} = \xi$ and $\tilde{\xi}(x_0) = 0$.*

Proof : Set $\tilde{\xi}(x_0) = 0$. Let $n \geq 1$. Let x_n be a vertex at distance n from x_0. Let $(x_j)_{j=0}^n$ be the chain of vertices forming the unique geodesic path linking x_0 to x_n in V. Set

$$\tilde{\xi}(x_n) = \sum_{j=0}^{n-1} \xi(x_j, x_{j+1}).$$

It is easy to see that $\nabla\tilde{\xi} = \xi$. □

Recall that a function on the vertices $\eta : V \to \mathbb{R}$ is harmonic if $\mathcal{L}\eta = 0$. In the case where $X = \mathcal{T}$ is a tree, using the previous lemma, a map $\xi \in \ell^2(\mathbb{E})$ belongs to $\ker \nabla^*$ if and only if $\tilde{\xi}$ is harmonic. Therefore, it is natural to say that a map on the edges $\xi \in \ell^2_{\text{alt}}(\mathbb{E})$ is **harmonic** if $\xi \in \ker \nabla^*$. This remark suggests the following orthogonal decomposition:

$$\ell^2_{\text{alt}}(\mathbb{E}) = \ker \nabla^* \oplus \overline{\text{im}\nabla},$$

since $(\ker \nabla^*)^\perp = \overline{\text{im}\nabla}$. We can give a more precise description of the projection onto $\overline{\text{im}\nabla}$. Recall first that, if the graph is non amenable, then $\text{im}\nabla$ is closed and the Laplace operator $\mathcal{L}$ is invertible. Furthermore, its inverse is the operator $-G$, with G being the **Green kernel** of X (see p.14 of [Woe00] for a precise definition, or Lemma 4.4.4 for a characterization). Now, let $\xi = h + \nabla k$, with $h \in \ker \nabla^*$ and $k \in \ell^2(V, m)$. Then, $\nabla^*\xi = \nabla^*\nabla k = -\mathcal{L}k$. Hence, we obtain $k = G\nabla^*\xi$. This forces to define Q, the orthogonal projection onto $\text{im}(\nabla)$, by

$$Q(\xi) = \nabla G \nabla^* \xi.$$

We can deduce that the "harmonic part" of an element $\xi \in \ell^2_{\text{alt}}(\mathbb{E})$ is given by $(1 - Q)\xi$. Now, we can study the decomposition of the Haagerup cocycle along the two subspaces $\ker \nabla^*$ and $\text{im}\nabla$.
Recall that, for any group acting on a tree, one can consider the Haagerup cocycle (Example 5 of Section 4.1.1). It is the cocycle $b : G \to \ell^2_{\text{alt}}(\mathbb{E})$ associated with π, the natural representation of G acting on the Hilbert space $\ell^2_{\text{alt}}(\mathbb{E})$ and, chosing a based-vertex x_0, it satisfies

$$b(g) = \chi_{x_0 \to gx_0},$$

where $\chi_{x \to y}$ is the signed characteristic function of the geodesic in $\mathcal{T}$, joining x to y. As already mentioned, it satisfies

$$\|b(g)\|_{\ell^2_{\text{alt}}(\mathbb{E})} = \sqrt{d(x_0, gx_0)}.$$

Under the condition that G acts properly on $\mathcal{T}$, this proves that b is a proper cocycle and that G has the Haagerup property.
Since $\ell^2_{\text{alt}}(\mathbb{E})$ can be decomposed into an orthogonal sum of two (closed) G-invariant subspaces, then we can conclude that the representation π is not irreducible. Hence, we obtain the following corollary.

Corollary 4.4.2. *The negative type function $g \mapsto d(x_0, gx_0)$ is not pure on G.*

Remark 4.4.3. *Corollary 4.4.2 also follows from the uniqueness (up to scalar multiplication) of unbounded pure negative type functions on G, by Corollary 4.3.7.*

We will show in the sequel that the projection of the cocycle b onto $\ker \nabla^*$ is still proper. To do so, we will prove that $\|Q\chi_{x\to y}\|_{\ell^2_{\mathrm{alt}}(\mathbb{E})}$ is bounded, indenpendently of x and y. Here is a useful lemma allowing us to estimate the operator norm of the Green kernel.

Lemma 4.4.4. *Let $X = (V, E)$ be a graph and let P be the normalised adjacency operator acting on $\ell^2(V, m)$, namely, the operator whose matrix coefficients $p(x, y)$ are*

$$p(x,y) = \begin{cases} \frac{1}{\deg(x)}, & \text{if } x \sim y, \\ 0, & \text{otherwise}, \end{cases}$$

1. *If $\|P\| < 1$, then the series $\sum_{n\geq 0} P^n$ defines a bounded operator and we have the equalities*

$$\mathcal{L}^{-1} = -\sum_{n\geq 0} P^n = -G.$$

   In particular, $\|G\| \leq \frac{1}{1-\|P\|}$.

2. *(Theorem (11.1), [Woe00]) If X is a graph with all vertices of valency bounded by $q+1$, then $\|P\| \geq \frac{2\sqrt{q}}{q+1}$, with equality if $X = \mathcal{T}_{q+1}$.*

For $x, y \in V$, let us estimate the norm of the harmonic part of $\chi_{x\to y}$. Firstly, it is a general fact that, for any $\xi \in \ell^2_{\mathrm{alt}}(\mathbb{E})$, we have

$$\|(1-Q)\xi\| \leq \|\xi\|,$$

and

$$\|(1-Q)\xi\|^2 = \|\xi\|^2 - \|Q(\xi)\|^2.$$

Now, let us compute $Q\chi_{x\to y}$. It is easy to check that

$$\nabla^*(\chi_{x\to y}) = \frac{\delta_y}{\deg(y)} - \frac{\delta_x}{\deg(x)}.$$

We get

$$\begin{aligned} \|\nabla^*\chi_{x\to y}\|^2_{\ell^2(V,m)} &= \left\|\frac{\delta_y}{\deg(y)} - \frac{\delta_x}{\deg(x)}\right\|^2_{\ell^2(V,m)} \\ &= \frac{1}{\deg(y)} + \frac{1}{\deg(x)}. \end{aligned}$$

Using Lemma 4.4.4, we obtain

$$\begin{aligned}\|(1-Q)\chi_{x\to y}\|^2 &= \|\chi_{x\to y}\|^2 - \|Q(\chi_{x\to y})\|^2\\ &\geq d(x,y) - \|\nabla\|^2\|G\|^2\|\nabla^*\chi_{x\to y}\|^2\\ &\geq d(x,y) - \frac{2}{(1-\|P\|)^2}\left(\frac{1}{\deg(x)}+\frac{1}{\deg(y)}\right)\\ &\geq d(x,y) - \frac{4}{(1-\|P\|)^2},\end{aligned}$$

since $\|\nabla\| = \|\nabla^*\| = \sqrt{2}$.

In particular, the orthogonal projection of the cocycle b onto $\ker\nabla^*$ is still proper and its compression exponent is $\frac{1}{2}$. Indeed, for any $g \in \mathrm{Aut}(\mathcal{T})$, we have

$$\|(1-Q)b(g)\|^2 \geq d(x_0, gx_0) - \frac{4}{(1-\|P\|)^2}.$$

If $\mathcal{T} = \mathcal{T}_{q+1}$ is the homogeneous $(q+1)$-regular tree, then one can compute $Q(\chi_{x\to y})$ explicitely in order to give precisely its norm and to find some examples of negative type functions on G. We will prove the following lemma.

Lemma 4.4.5. *Let $q \geq 2$ be an integer and let $\mathcal{T}_{q+1}$ be the homogeneous $(q+1)$-regular tree. Then, for any $x, y \in V$, we have:*

1. $$|(Q\chi_{x\to y})(e)| = \begin{cases} \dfrac{q^{-d(y,e)}+q^{-d(x,e)}}{q+1}, & \text{if } e \text{ is on the geodesic } [x,y],\\ \left|\dfrac{q^{-d(y,e)}-q^{-d(x,e)}}{q+1}\right|, & \text{otherwise,}\end{cases}$$

2. *Moreover,*
$$\|Q(\chi_{x\to y})\|^2_{\ell^2_{\mathrm{alt}}(\mathbb{E})} = \frac{2q}{q^2-1}(1-q^{-d(x,y)}).$$

3. *Let G be a group acting isometrically on $\mathcal{T}_{q+1}$ and fix $x_0 \in V$. Then, the function conditionally of negative type on G given by*
$$g \mapsto \frac{2q}{q^2-1}(1-q^{-|g|})$$
is bounded and it tends to $\frac{2q}{q^2-1}$ as g goes to infinity.

We can deduce:

Corollary 4.4.6. *Let G be a subgroup of $\mathrm{Aut}(\mathcal{T}_{q+1})$ acting properly on $\mathcal{T}_{q+1}$. Fix a basepoint $x_0 \in V$, set $|g| = d(x_0, gx_0)$, for $g \in G$, and write $\tilde{b}$ for the projection of the Haagerup cocycle onto the closed invariant subspace $\ker\nabla^*$, that is, $\tilde{b}(g) = (1-Q)\chi_{x_0\to gx_0}$. We have:*

1. *The map* $\tilde{b}$ *is a proper cocycle with respect to a subrepresentation of the natural unitary representation acting on* $\ell^2_{\mathrm{alt}}(\mathbb{E})$. *Moreover, the cocycle* $\tilde{b}$ *satisfies the following estimate*

$$\|\tilde{b}(g)\| \geq |g|^{\frac{1}{2}} - \frac{2q}{q^2-1},$$

for all $g \in G$.

2. *Assume furthermore that* G *is a closed subgroup of* $\mathrm{Aut}(\mathcal{T}_{q+1})$ *acting transitively on both* $\mathcal{T}_{q+1}$ *and* $\partial\mathcal{T}_{q+1}$. *Then, the function conditionally of negative type associated to* $\tilde{b}$

$$g \mapsto |g| + \frac{2q}{q^2-1}(q^{-|g|}-1),$$

is pure in $\mathrm{CL}(G)$.

3. *The cocycle* $\tilde{b}$ *coincides with the harmonic proper cocycle* F *of Proposition 4.2.4 and Corollary 4.3.5.*

Proof of Corollary 4.4.6 using Lemma 4.4.5 : The first claim is clear. The second one follows from the fact that this negative type function is a multiple of the negative type function appearing in Corollary 4.3.7 and from the uniqueness of the latter. The last claim is a direct consequence of *2.* □

Before proving Lemma 4.4.5, we recall that the Green kernel G takes a particularly simple form on $\mathcal{T}_{q+1}$.

Lemma 4.4.7. *(Lemma (1.23), [Woe00]) Let* $q \geq 2$ *and let* G *be the Green kernel defined on the homogeneous* $(q+1)$*-regular tree* $\mathcal{T}_{q+1}$. *If we denote by* $(G(x,y))_{x,y\in V}$ *the associated matrix of* G, *then*

$$G(x,y) = \frac{q^{1-d(x,y)}}{q-1},$$

for all $x, y \in V$.

Proof of Lemma 4.4.5 : Since $\nabla^*\chi_{x\to y} = \frac{1}{q+1}(\delta_y - \delta_x)$, we need to compute $\nabla G\delta_x$. Let $e \in \mathbb{E}$ be an oriented edge. Using the description of the Green kernel, we get

$$\begin{aligned}
(\nabla G\delta_x)(e) &= (G\delta_x(e_+)) - (G\delta_x)(e_-) \\
&= G(e_+,x) - G(e_-,x) \\
&= \frac{q}{q-1}\left(q^{-d(e_+,x)} - q^{-d(e_-,x)}\right).
\end{aligned}$$

Clearly, $|d(e_+, x) - d(e_-, x)| = 1$. Setting $d(x, e) = \min\{d(x, e_-), d(x, e_+)\}$ (this is simply the natural distance between x and the geometric edge associated with e in the geometric realisation of $\mathcal{T}_{q+1}$), we immediately obtain:

$$(\nabla G\delta_x)(e) = \begin{cases} -q^{-d(x,e)}, & \text{if } d(x,e) = d(x, e_-), \\ q^{-d(x,e)}, & \text{if } d(x,e) = d(x, e_+) \end{cases}$$

It is easy to see that $(\nabla G\delta_x)(e)$ and $(\nabla G\delta_y)(e)$ have the same sign if and only if x and y belong to the same connected component of [2] $\mathcal{T}_{q+1} \setminus \{mid(e)\}$, which happens exactly when e does not lie on the geodesic $[x, y]$. This shows the first claim.
From the first claim, we deduce:

$$\begin{aligned} \|Q\chi_{x\to y}\|^2_{\ell^2_{\mathrm{alt}}(\mathbb{E})} &= \frac{1}{2}\sum_{e\in\mathbb{E}} |(Q\chi_{x\to y})(e)|^2 \\ &= \sum_{e\in E} |(Q\chi_{x\to y})(e)|^2 \\ &= \frac{1}{(q+1)^2}\sum_{e\in E} |\left(\nabla G\delta_y - \nabla G\delta_x\right)(e)|^2. \end{aligned}$$

To compute the last sum, we will decompose the set of geometric edges. First of all, let $m = d(x, y)$ and let $\{z_j\}_{j=0}^m$ be the set of vertices describing the geodesic $[x, y]$, with $z_0 = x$ and $z_m = y$. Let T_0 be the subgraph which is induced on the connected component of $\mathcal{T}_{q+1} \setminus \{z_1\}$ containing x. For $1 \le j \le m-1$, let T_j be the subgraph which is induced on the connected component of $\mathcal{T}_{q+1} \setminus \{z_{j-1}, z_{j+1}\}$ containing z_j. Finally, let T_m be the subgraph which is induced on the connected component of $\mathcal{T}_{q+1} \setminus \{z_{m-1}\}$ containing y. We remark that for all j, the graph T_j is a subtree of $\mathcal{T}_{q+1}$ with root z_j. With these notations, a geometric edge e belongs either to one of the T_j, for some j, or e lies on $[x, y]$. Thus, we get:

$$\begin{aligned} \|Q\chi_{x\to y}\|^2_{\ell^2_{\mathrm{alt}}(\mathbb{E})} &= \frac{1}{(q+1)^2}\left(\left(\sum_{e\in[x,y]} \left|q^{-d(y,e)} + q^{-d(x,e)}\right|^2\right)\right. \\ &\quad \left. + \left(\sum_{j=0}^{m}\sum_{e\in T_j} \left|q^{-d(y,e)} - q^{-d(x,e)}\right|^2\right)\right). \end{aligned}$$

To compute the first sum, let us denote by e_j the edge (z_j, z_{j+1}). There-

[2] Here, $mid(e)$ denotes the median point of e in the geometric realisation of $\mathcal{T}_{q+1}$.

fore, we have

$$\begin{aligned}
\sum_{e\in[x,y]} \left|q^{-d(y,e)} + q^{-d(x,e)}\right|^2 &= \sum_{j=0}^{m-1} \left|q^{-d(y,e_j)} + q^{-d(x,e_j)}\right|^2 \\
&= \sum_{j=0}^{m-1} \left|q^{-(m-j-1)} + q^{-j}\right|^2 \\
&= \sum_{j=0}^{m-1} q^{-2j} + q^{-2(m-j-1)} + 2q^{m-1} \\
&= \frac{2m}{q^{m-1}} + 2\sum_{j=0}^{m-1} \frac{1}{q^{2j}} \\
&= \frac{2m}{q^{m-1}} + 2\frac{1-q^{-2m}}{1-q^{-2}}.
\end{aligned}$$

Secondly, let us compute the sum over the edges belonging to the subtree T_0. For any edge e in T_0, we notice that $d(y,e) = m + d(x,e)$. Since the number of edges in T_0 which are at distance k to x is equal to q^{k+1}, for $k \geq 0$, we have:

$$\begin{aligned}
\sum_{e\in T_0} \left|q^{-d(y,e)} - q^{-d(x,e)}\right|^2 &= \sum_{k\geq 0} \sum_{\substack{e\in T_0: \\ d(x,e)=k}} |q^{-k} - q^{-k-m}|^2 \\
&= \left(1-q^{-m}\right)^2 \sum_{k\geq 0} q^{-k+1} \\
&= \left(1-q^{-m}\right)^2 \frac{q}{1-q^{-1}}.
\end{aligned}$$

By symmetry, the same is true for the sum over T_m. That is:

$$\sum_{e\in T_m} \left|q^{-d(y,e)} - q^{-d(x,e)}\right|^2 = \left(1-q^{-m}\right)^2 \frac{q}{1-q^{-1}}.$$

Finally, we need to compute the sum over the edges belonging to T_j for $1 \leq j \leq m-1$. For any edge e in T_j, we notice that $d(x,e) = d(x,z_j) + d(z_j,e) = j + d(z_j,e)$ and $d(y,e) = d(y,z_j) + d(z_j,e) = m - j + d(z_j,e)$. Since the number of edges in T_j which are at distance k to z_j is equal to $(q-1)q^k$, for $k \geq 0$, we have:

$$\begin{aligned}
\sum_{e \in T_j} \left| q^{-d(y,e)} - q^{-d(x,e)} \right|^2 &= \sum_{k \geq 0} \sum_{\substack{e \in T_j: \\ d(z_j,e)=k}} |q^{-m+j-k} - q^{-j-k}|^2 \\
&= \sum_{k \geq 0} \sum_{\substack{e \in T_j: \\ d(z_j,e)=k}} q^{-2k} |q^{-m+j} - q^{-j}|^2 \\
&= \left(q^{-m+j} - q^{-j}\right)^2 \sum_{k \geq 0} (q-1) q^k q^{-2k} \\
&= \left(q^{-m+j} - q^{-j}\right)^2 (q-1) \sum_{k \geq 0} q^{-k} \\
&= q \left(q^{-2(m-j)} + q^{-2j} - 2q^{-m}\right).
\end{aligned}$$

We can compute the sum over all the T_j, for $1 \leq j \leq m-1$:

$$\begin{aligned}
\sum_{j=1}^{m-1} \sum_{e \in T_j} \left| q^{-d(y,e)} - q^{-d(x,e)} \right|^2 &= \sum_{j=1}^{m-1} q \left(q^{-2(m-j)} + q^{-2j} - 2q^{-m}\right) \\
&= -\frac{2(m-1)}{q^{m-1}} + 2q \sum_{j=1}^{m-1} q^{-2j} \\
&= -\frac{2(m-1)}{q^{m-1}} + \frac{2}{q} \frac{1 - q^{2-2m}}{1 - q^{-2}} \\
&= -\frac{2(m-1)}{q^{m-1}} + 2 \frac{q^{-1} - q^{1-2m}}{1 - q^{-2}}
\end{aligned}$$

Since we have

$$
\begin{aligned}
\frac{1}{2}\sum_{e\in E}|\left(\nabla G\right)\left(\left(\delta_y-\delta_x\right)(e)\right|^2 &= \frac{1}{q^{m-1}}+\frac{1-q^{-2m}+q^{-1}-q^{1-2m}}{1-q^{-2}}+\frac{q(1-q^{-m})^2}{1-q^{-1}}\\
&= \frac{1}{q^{m-1}}+\frac{2+q^{-1}+q-2q^{-m}-2q^{1-m}}{1-q^{-2}}\\
&= \frac{-1-2q^{-1}-q^{-2}+q^{m-2}+2q^{m-1}+q^m}{(1-q^{-2})q^{m-1}}\\
&= \frac{(q^m-1)(1+q^{-1})^2}{(1-q^{-1})(1+q^{-1})q^{m-1}}\\
&= \frac{(q^m-1)(1+q^{-1})}{(1-q^{-1})q^{m-1}}\\
&= \frac{(q^m-1)(q+1)}{(q-1)q^{m-1}},
\end{aligned}
$$

we deduce finally that

$$
\begin{aligned}
\|Q\chi_{x\to y}\|^2 &= \frac{1}{(q+1)^2}\sum_{e\in E}|(\nabla G)(\delta_y-\delta_x)|^2\\
&= \frac{2}{(q+1)^2}\cdot\frac{(q^m-1)(q+1)}{(q-1)q^{m-1}}\\
&= \frac{2(q^m-1)}{(q^2-1)q^{m-1}}\\
&= \frac{2q}{q^2-1}\left(1-q^{-m}\right),
\end{aligned}
$$

which proves the second claim. The last claim being straightforward, the proof is done. □

4.4.2 Classification of negative type functions

We start by giving an interesting family of examples of kernels conditionally of negative type on trees. Let $\mathcal{T}=(V,E)$ be any tree. It was shown by Valette (Theorem 1, [Val92]) that, for any function $\psi: V\to[0,1]$ satisfying the condition

$$\psi(x)\leq\frac{1}{\deg(x)},$$

for all $x \in V$ (with the convention that $\psi(x) = 0$ if $\deg(x) = \infty$), then, the kernel defined by

$$\Psi(x,y) = \begin{cases} 0, & \text{if } x = y, \\ d(x,y) - \frac{\psi(x)+\psi(y)}{2}, & \text{if } x \neq y, \end{cases}$$

is negative definite on V.
We address the following question. Let G be a subgroup of $\mathrm{Aut}(\mathcal{T})$. When is Ψ a G-invariant kernel?
We will answer this question in a special case.

Proposition 4.4.8. *Let $q \geq 2$ and let G be a closed subgroup of $\mathrm{Aut}(\mathcal{T}_{q+1})$ acting transitively on both V and $\partial\mathcal{T}_{q+1}$. Then, a kernel Ψ defined as above is G-invariant if and only if the function ψ used to construct Ψ is constant.*

Proof : Clearly, the kernel Ψ is G-invariant if and only if the function ψ satisfies the following condition

$$\psi(gx) + \psi(gy) = \psi(x) + \psi(y), \tag{4.2}$$

for any $x, y \in V$ and $g \in G$. Since the stabilizer of any vertex acts transitively on any sphere about any point, it is straightforward to see that ψ has to take at most 2 values. Indeed, let us fix vertex x_0. Recall the standard bipartition of V given by V_e and V_o. The set V_e (resp. V_o) consists of vertices which are at even (resp. odd) distance of x_0. Let u, v be both in the same subset of the bipartition. Then, $d(u,v)$ is even and the median point of the geodesic $[u,v]$ is a certain vertex z. We can find $g \in \mathrm{stab}(z)$ sending u on v. By condition (4.2) we obtain

$$\begin{aligned} \psi(u) - \psi(v) &= \psi(u) - \psi(gu) \\ &= \psi(gz) - \psi(z) \\ &= 0, \end{aligned}$$

which implies that $\psi(u) = \psi(v)$. To finish the proof, consider any geodesic segment of length 2 formed by vertices $(v_j)_{j=0}^2$. Since G acts doubly transitively on V, we can find an element g such that $gv_j = v_{j+1}$, for $j = 0, 1$. We observe that $d(v_1, gv_2) = 2$ and this forces $\psi(gv_2) = \psi(v_1)$. Again, by condition (4.2), we have

$$\begin{aligned}\psi(v_0) &= \frac{1}{2}\left(\psi(v_0)+\psi(v_2)\right)\\ &= \frac{1}{2}\left(\psi(gv_0)+\psi(gv_2)\right)\\ &= \frac{1}{2}\left(\psi(v_1)+\psi(gv_2)\right)\\ &= \psi(v_1),\end{aligned}$$

and therefore, ψ is constant. □

Corollary 4.4.9. *Let G and Ψ be as in Proposition 4.4.8. The only negative type functions on G induced by the negative type kernels Ψ are of the form*

$$g \mapsto d(x_0, gx_0) - \alpha,$$

for some constant $\alpha \in [0, \frac{1}{q+1}]$.

These functions are not pure in general, as we observe from the next two results. We summarize the content of this section in the following Corollary.

Corollary 4.4.10. *Let G be a closed non compact subgroup of $\mathrm{Aut}(\mathcal{T}_{q+1})$, with $q \geq 2$. Suppose that G acts transitively on the vertices and on the boundary $\partial\mathcal{T}_{q+1}$. Let ψ be a function conditionally of negative type on G. Suppose that ψ is pure in $\mathrm{CL}(G)$ and that it vanishes on the stabilizer of some vertex x_0. We have the following alternative :*

1. *The function ψ is bounded on G and then it is of the form*

   $$\psi(g) = \|\xi\|^2 - \langle \pi_\psi(g)\xi, \xi\rangle,$$

   where π_ψ is the irreducible unitary representation associated with ψ via the GNS construction, and ξ is a $\pi_\psi(G_{x_0})$-fixed vector (which is unique, up to scalar multiplication).

2. *The function ψ is unbounded and then it is of the form*

   $$\psi(g) = C\left(|g| + \frac{2q}{q^2-1}(q^{-|g|}-1)\right),$$

   where $|g| = d(gx_0, x_0)$ and C is a positive constant.

Proof : The only thing left to prove is the first claim. It is a general fact that $\psi(g) = \|(\pi_\psi(g) - 1)\xi\|^2$, for some G_{x_0}-invariant vector $\xi \in \mathcal{H}_\psi$, using the GNS construction. By developing the norm and remarking that the coefficient $\langle \pi_\psi(\cdot)\xi, \xi\rangle$ is real-valued, we get $\psi(g) = 2\|\xi\|^2 - 2\langle \pi_\psi(\cdot)\xi, \xi\rangle$. Finally, since the representation π_ψ is spherical, then the space of G_{x_0}-invariant vectors has dimension one, and the result follows. □

For the sake of completeness, we prove the following well-known Proposition.

Proposition 4.4.11. *Let $q \geq 2$ be an integer (possibly infinite). Let G be a closed non compact subgroup of* $\mathrm{Aut}(\mathcal{T}_{q+1})$*. Suppose that G acts transitively on the vertices and on the boundary $\partial\mathcal{T}_{q+1}$. Fix a vertex x_0, and set $|g| = d(gx_0, x_0)$, for $g \in G$. Then, the function $g \mapsto |g|$ is pure in* $\mathrm{CL}(G)$ *if and only if $q = \infty$.*

Proof : Let us show that, in the case $q = \infty$, then the representation π acting on $\ell^2_{alt}(\mathbb{E})$ is irreducible. Let us fix a geometric edge $a = \{a_0, a_1\} \in E$. Firstly, we note that π is equivalent to the quasi-regular representation λ_{G/G_a}, where G_a is the stabilizer of a. By a theorem of Mackey (see Theorem 2.1 in [BdlH97]), we need to show that the commensurator of G_a in G is exactly G_a. Recall that the commensurator of G_a in G, denoted by $\mathrm{Com}_G(G_a)$, is the set of elements $g \in G$ such that the subgroup $G_a \cap G_{a'}$ has finite index in both G_a and $G_{a'}$, where a' is the edge satisfying $a' = ga$. Clearly, G_a is contained in $\mathrm{Com}_G(G_a)$.
To prove the other inclusion, let $g \in G \setminus G_a$ and set $a' = ga$ and $a'_i = ga_i$, for $i = 0, 1$. We will see that $G_a \cap G_{a'}$ has not finite index in G_a. We can suppose that the geodesic $[a_0, a'_0]$ is contained in the geodesic $[a_1, a'_1]$. Since the tree is of infinite degree, then, for all $k \geq 2$, there exists $a'_k \in V$ such that $a'_0 \sim a'_k$ and $a'_k \neq a'_j$, for all $j \geq k - 1$. Using the transitivity of the action on $\partial\mathcal{T}_\infty$, for every k, we can find $\tilde{g}_k \in G_{a_0}$ sending a' to the edge $\{a'_0, a'_k\}$. It is easy to see that $\tilde{g}_k \in G_a$, for all k, and that the cosets $\tilde{g}_k\,(G_a \cap G_{a'})$ are pairwise different. This ends the proof. □

Chapter 5

Cohomology in Banach algebras

This chapter is connected to the rest of this work in the following way. Let G be a group and let π and σ be two unitary representations of G acting on the same Hilbert space $\mathcal{H}$. We will study embeddings of G into the Banach space of all bounded operators on $\mathcal{H}$ which satisfy a certain derivation relation. To be more precise, an embedding $b : G \to \mathcal{B}(\mathcal{H})$ is a bounded derivation for π if it is bounded and if it satisfies the so-called derivation relation relatively to the pair (π, σ). Namely,

$$b(gh) = \pi(g)b(h) + b(g)\sigma(h),$$

for all $g, h \in G$, and the image of G inside $\mathcal{B}(\mathcal{H})$ via b is bounded for the operator norm. Maps of the form $g \mapsto \pi(g)T - T\sigma(g)$, for some $T \in \mathcal{B}(\mathcal{H})$ are easily seen to satisfy the derivation relation and are bounded. Those derivations are called inner. It is possible to show that, in numerous cases, all bounded derivations, for a given pair of representations, are inner. In this chapter, we provide one condition to ensure the existence of non-inner bounded derivations relatively to (π, σ). In this case, we say that the pair of representations (π, σ) admits uniform almost intertwiners. We warn the reader that, as opposed to the other chapters, we do not care about these embeddings from a metric point of view.

As the interest of showing the existence of non-inner bounded derivations for groups is not immediately apparent, we feel the need of a more concrete motivation. That is the reason why we will start this chapter with a section about the so-called Dixmier problem. Let us consider a group G and a unitary representation π. Taking any invertible operator S on the Hilbert space $\mathcal{H}_\pi$, one can define a new homomorphism $\tilde{\pi} : G \to \mathcal{B}(\mathcal{H}_\pi)$ by

$$g \mapsto S^{-1}\pi(g)S.$$

We note that this representation is not unitary in general. Yet, its norm satisfies the following property : $\|\tilde{\pi}(g)\| \leq \|S\|\|S^{-1}\|$, for any $g \in G$. We call such a representation uniformly bounded. A group is called unitarisable if any uniformly bounded representation is similar to a unitary representation, that is, of the form of $\tilde{\pi}$ here above. Day and Dixmier proved in 1950 that any amenable group is unitarisable. The Dixmier problem consists in deciding whether the converse holds. As we will see, it is possible to produce uniformly bounded representations which are not similar to unitary ones using bounded derivations which are not inner. This procedure allows us to prove the non-unitarisability of certain groups.
This chapter is organised as follows : We start the first section by an introduction to the Dixmier problem. We give the relevant definitions concerning unitarisability and known results in Subsection 5.1.1. We then define bounded derivations in Subsection 5.1.2 and show how to use them to produce uniformly bounded representations. Finally, in Subsection 5.1.3, we recall a concept of cohomology for Banach algebras and discuss its link with bounded derivations. This allows us to state several known results concerning the cohomology of various Banach algebras related to a discrete group Γ (e.g. $\ell^1\Gamma$, the reduced C^*-algebra of Γ, etc...). In Section 5.2, we introduce a topology on the space of bounded derivations relatively to a pair of representations. The first new result of this chapter appears in Subsection 5.2.1. We define the concept of uniform almost intertwiners for a pair of representations (π, σ) and we prove that, the pair (π, σ) admits uniform almost intertwiners if and only if the space of inner derivations is not closed inside the space of all bounded derivations relatively to (π, σ). Then, in Subsection 5.2.2, we see how this property behaves under natural constructions of representations. In Subsection 5.2.3, we use the techniques developed in this chapter to explicitely construct uniformly bounded and non-unitarisable representations. As an application, in Section 5.3, we study the space of bounded derivations relatively to the left regular representation of free groups. We first recall the definition of contraction operators on trees and use them to produce non-inner derivations. This gives a first proof of the non-unitarisability of non-abelian free groups. In Subsection 5.3.2, we prove the second main result of this chapter. We show that the left regular representation of the free group $\mathbb{F}_\infty$ admits uniform almost intertwiners. We finish this chapter by showing that a natural sequence of operators acting on $\mathcal{B}(\ell^2\mathbb{F}_q)$, with $q \geq 2$ finite, does not lead a sequence of uniform almost intertwiners.
Our original results are contained in Section 5.2 and 5.3. We do not claim any originality for the results appearing in Section 5.1. In particular, Corollary 5.1.23 is classical and Proposition 5.1.24 should be known to experts, even though we are not able to provide a reference.

5.1 Motivation and bounded cohomology

5.1.1 The Dixmier Problem

In this subsection, we present the Dixmier problem and give an account on known results as well as on its current state. This is largely inspired by the nice survey by Pisier [Pis05].

Definition 5.1.1. *Let G be a topological group, let V be a Banach space and let $\pi \in \mathrm{Hom}(G, \mathcal{B}(V))$ be a continuous homomorphism (with respect to the strong operator topology).*

1. *The representation π is* ***uniformly bounded*** *if*

$$|\pi| = \sup_{g \in G} \|\pi(g)\| < \infty.$$

2. *In the case where $V = \mathcal{H}$ is a Hilbert space, the representation π is* ***unitarisable*** *if π is equivalent to a unitary representation. Namely, there exists a continuous and invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that the representation σ defined by*

$$\sigma(g) = S\pi(g)S^{-1}$$

is unitary. Such an operator S is called a ***unitariser*** *for π.*

3. *By extension, a group G is said to be* ***unitarisable*** *if every uniformly bounded representation of G is unitarisable.*

We start with a couple of hereditary results about unitarisability.

Proposition 5.1.2. *(Proposition 0.5, [Pis05]) Let G be a discrete group and let H be a subgroup.*

1. *If G is unitarisable, then H is unitarisable.*
2. *Assume that H is normal in G. Then, G is unitarisable if and only if both H and G/H are unitarisable.*

The following result shows why it is sensible to restrict our attention to countable groups.

Proposition 5.1.3. *(Corollary 0.10, [Pis05]) Let G be a discrete group and let π be a uniformly bounded representation of G. Then, the representation π is unitarisable if and only if its restriction to any countable subgroup H of G is unitarisable.*
In particular, if all the countable subgroups of a discrete group G are unitarisable, then G is unitarisable.

Until now, the largest class of groups which are known to be unitarisable is given by amenable groups. This result is attributed to Dixmier and Day, generalizing a previous argument on $\mathbb{Z}$ to all (non-necessarily discrete) amenable groups. Before stating the result, we recall some (of the many!) equivalent formulations of amenable groups.

Definition 5.1.4. *Let Γ be a discrete and countable group. The group Γ is **amenable** if it satisfies one of the following equivalent conditions :*

1. *The left regular representation λ of Γ weakly contains the trivial representation;*
2. *The group Γ admits a sequence of **Følner sets**, that is, there exists a sequence $(F_i)_i$ of non-empty, finite subsets of Γ satisfying*

$$\frac{|(F_i \cdot \gamma) \triangle F_i|}{|F_i|} \to 0,$$

   as $i \to \infty$, for any $\gamma \in \Gamma$;

3. *The group Γ admits a right invariant **mean**, that is, there exists a linear functional $m : \ell^\infty\Gamma \to \mathbb{R}$ such that :*
   (a) *$m(1_\Gamma) = 1$;*
   (b) *If $f \in \ell^\infty\Gamma$ and $f \geq 0$, then $m(f) \geq 0$;*
   (c) *$m(\rho(\gamma)f) = m(f)$, for any $\gamma \in \Gamma$ and $f \in \ell^\infty\Gamma$, where ρ is the right regular representation of Γ acting on $\ell^\infty\Gamma$.*

Theorem 5.1.5. *(Day, Dixmier) Let Γ be a discrete and countable group. If Γ is amenable, then Γ is unitarisable.*

Until the works of Mautner and Ehrenpreis showing that $SL_2(\mathbb{R})$ is not unitarisable in 1955, it was not clear whether non-unitarisable groups exist. Apart from $SL_2(\mathbb{R})$, it is possible to show that non-abelian free groups and some Burnside's groups also are non-unitarisable. In particular, by Proposition 5.1.2, we observe that any discrete group containing a non-unitarisable subgroup is non-unitarisable. We note that, all these groups are not amenable. It is therefore sensible to ask whether unitarisability implies amenability. This is what we call the **Dixmier problem**.
Question : *Is a unitarisable group necessarily amenable ?*
Before proving Theorem 5.1.5, we give a necessary and sufficient condition for a uniformly bounded representation to be unitarisable. It involves the notion of contragredient representation.

Definition 5.1.6. *Let $\pi : G \to \mathcal{B}(\mathcal{H})$ be a uniformly bounded representation. We define $\overline{\pi}$ the **contragredient** of π by setting:*

$$\overline{\pi}(g) = \pi(g^{-1})^*,$$

for all $g \in G$.

The following result is well-known, but we provide a proof for the reader's convenience.

Proposition 5.1.7. *A uniformly bounded representation $\pi : G \to \mathcal{B}(\mathcal{H})$ is unitarisable if and only if there exists an invertible and positive operator T which intertwines π and its contragredient, that is, if we have:*

$$\overline{\pi}(g)T = T\pi(g), \tag{5.1}$$

for all $g \in G$. Moreover, if we denote by S the unique positive square root of T, then the representation $\sigma(g) = S\pi(g)S^{-1}$ is unitary.

Proof : Let us suppose that π is unitarisable. Then, there is an invertible operator $A \in \mathcal{B}(\mathcal{H})$ such that $A\pi(\cdot)A^{-1} = u(\cdot)$, for some unitary representation u. For all $g \in G$, we get

$$\begin{aligned} A\pi(g)A^{-1} &= u(g) \\ &= u(g^{-1})^* \\ &= (A\pi(g^{-1})A^{-1})^* \\ &= (A^*)^{-1}\pi(g^{-1})^*A^*. \end{aligned}$$

By setting $T = A^*A$, this implies that

$$\overline{\pi}(g)T = T\pi(g).$$

Since T is positive and invertible, this ends the first part of the proof. Now, let us suppose that there exists a positive and invertible operator T so that (5.1) holds. Then, there is a unique positive and invertible operator S satisfying $S^2 = T$. Defining the representation $\sigma(g) = S\pi(g)S^{-1}$, it is easy to see that σ is unitary, which implies that π is unitarisable. □

We now prove that discrete amenable groups are unitarisable. We follow the argument of Cowling published in [EFST79].

Proof of Theorem 5.1.5 : Let π be a uniformly bounded representation of Γ acting on some Hilbert space $\mathcal{H}$. We will construct a positive and invertible operator A which intertwines π and its contragredient. Firstly, we recall briefly how to build a right invariant mean on Γ using Følner sets. Let $(F_i)_i$ be a sequence of non-empty, finite sets of Γ satisfying

$$\frac{|(F_i \cdot \gamma) \triangle F_i|}{|F_i|} \to 0,$$

as $i \to \infty$, for any $\gamma \in \Gamma$. For all i, let m_i be the probability measure on Γ corresponding to the normalised counting measure associated with the set F_i. By chosing ω, any non-principal ultrafilter on $\mathbb{N}$, we obtain a right invariant mean[1] m on Γ by setting

$$m(f) = \int_\Gamma f(x)\,dm(x) = \lim_{i\to\omega} \int_\Gamma f(x)\,dm_i(x),$$

for any $f \in \ell^\infty\Gamma$. We can now define the desired operator A on $\mathcal{B}(\mathcal{H})$ by

$$A = \int_\Gamma \pi(\gamma)^*\pi(\gamma)\,dm(\gamma),$$

which means that for any $\xi, \eta \in \mathcal{H}$, we have

$$\langle A\xi, \eta\rangle = \int_\Gamma \langle \pi(\gamma)^*\pi(\gamma)\xi, \eta\rangle\,dm(\gamma).$$

Note that, for any fixed ξ and η, the map $\varphi_{\xi,\eta} : \Gamma \to \mathbb{C}$ defined by

$$\varphi_{\xi,\eta}(\gamma) = \langle \pi(\gamma)^*\pi(\gamma)\xi, \eta\rangle$$

[1]Recall that m is *right invariant* if $m(\rho(\gamma)f) = m(f)$, for all $f \in \ell^\infty\Gamma$ and $\gamma \in \Gamma$, where ρ is the natural right-representation acting on $\ell^\infty\Gamma$ as $\rho(\gamma)\delta_x = \delta_{x\gamma^{-1}}$.

is bounded. Therefore, the expression $\int_\Gamma \varphi_{\xi,\eta}\, dm(\gamma)$ here above is well-defined. Let us check that the operator A is bounded and that we can indeed permute the scalar product and the integration relative to the mean dm. Using Fubini's theorem, for any fixed ξ and η, and any i, we have

$$\begin{aligned}
\left|\left\langle \int_\Gamma \pi(\gamma)^*\pi(\gamma)\xi\, dm_i(\gamma), \eta \right\rangle\right| &= \left|\int_\Gamma \langle \pi(\gamma)^*\pi(\gamma)\xi, \eta\rangle \, dm_i(\gamma)\right| \\
&\leq \int_\Gamma |\langle \pi(\gamma)^*\pi(\gamma)\xi, \eta\rangle| \, dm_i(\gamma) \\
&\leq \int_\Gamma |\pi|^2 \|\xi\| \|\eta\| \, dm_i(\gamma) \\
&\leq |\pi|^2 \|\xi\| \|\eta\|.
\end{aligned}$$

The upper bound being independent of i, we can pass to the limit and therefore permute the integral and the scalar product. Moreover, we obtain that A is continuous with operator norm bounded by $|\pi|^2$. We claim that the operator A is positive and invertible. Indeed, for any vector η, we have

$$\langle \pi(\gamma)^*\pi(\gamma)\eta, \eta\rangle \geq \frac{\|\eta\|^2}{|\pi|^2}$$

for any $\gamma \in \Gamma$, and this implies

$$\langle A\eta, \eta\rangle \geq \frac{\|\eta\|^2}{|\pi|^2}$$

by positivity of the mean m. This also shows that A is invertible. Finally, the only thing left to show is that A intertwines π and $\overline{\pi}$. Fix $\eta, \xi \in \mathcal{H}$ and $g \in \Gamma$. We have

$$\begin{aligned}
\langle \overline{\pi}(g)A\xi, \eta\rangle &= \langle A\xi, \overline{\pi}(g)^*\eta\rangle \\
&= \int_\Gamma \langle \overline{\pi}(g)\pi(\gamma)^*\pi(\gamma)\xi, \eta\rangle \, dm(\gamma) \\
&= \int_\Gamma \langle \pi(\gamma g^{-1})^*\pi(\gamma)\xi, \eta\rangle \, dm(\gamma) \\
&= \int_\Gamma \langle \pi(\gamma)^*\pi(\gamma g)\xi, \eta\rangle \, dm(\gamma g) \\
&= \langle A\pi(g)\xi, \eta\rangle,
\end{aligned}$$

where the last equality is due to the right invariance of the mean m. Since g, ξ and η are arbitrary, the operator A interwtines π and $\overline{\pi}$, and we are done. □

We now state some known results providing partial solutions to the Dixmier problem. The first one gives the equivalence between unitarisability and amenability in the class of almost connected locally compact groups.

Proposition 5.1.8. *(Remark 0.8, [Pis05]) Let G be an almost connected locally compact group, that is, the quotient G/G_e is compact, where G_e is the connected component of G containing the identity. If G is unitarisable, then G is amenable.*

Roughly speaking, the next theorem, proved by Pisier, says that, if for every uniformly bounded representation π of some discrete group one can find a unitariser with norm controlled by $|\pi|$, then this group must be amenable.

Theorem 5.1.9. *(Pisier, [Pis98]) Let G be a discrete group. The following are equivalent :*

1. *The group G is amenable.*
2. *For any uniformly bounded representation π of G, there exists a unitariser S for π satisfying :*

$$\|S\|\|S^{-1}\| \leq |\pi|^2.$$

We end this subsection with the following striking result.

Theorem 5.1.10. *(Monod-Ozawa, [MO10]) Let G be a discrete group. The following are equivalent:*

1. *The group G is amenable.*
2. *The wreath product $A \wr G$ (i.e. the lamplighter group over the space G) is unitarisable for some infinite abelian group A.*
3. *The wreath product $A \wr G$ is unitarisable for all amenable groups A.*

In particular, this implies that if unitarisability of groups is a property which is stable under taking semi-direct products, then unitarisability is equivalent to amenability. However, we do not even know if it holds for direct products.

5.1.2 Bounded derivations

In this subsection, Γ will always be a countable discrete group.

Definition 5.1.11. *Let π and σ be two uniformly bounded representations of a group Γ acting on the same Banach space V.*

1. *A* ***derivation*** *with respect to π and σ is a map $b : \Gamma \to \mathcal{B}(V)$ satisfying the relation*
$$b(gh) = \pi(g)b(h) + b(g)\sigma(h),$$
for all $g, h \in \Gamma$.

2. *A derivation b is* ***bounded*** *if*
$$\|b\| = \sup_{g\in\Gamma} \|b(g)\| < \infty.$$

3. *A derivation b is* ***inner*** *if there exists a bounded operator $P \in \mathcal{B}(V)$ such that*
$$b(g) = \pi(g)P - P\sigma(g),$$
for all $g \in \Gamma$. We remark that such a derivation is automatically bounded, since $\|b(g)\| \leq \|P\|(|\pi| + |\sigma|)$, for all g.

4. *We will denote by $D(\Gamma, \pi, \sigma)$ the space of derivations, by $D^\infty(\Gamma, \pi, \sigma)$ the space of bounded derivations and by $I(\Gamma, \pi, \sigma)$ the subspace of inner derivations. Finally, we will use the following notation for the quotient space*
$$H^1_d(\Gamma, \pi, \sigma) = D^\infty(\Gamma, \pi, \sigma)/I(\Gamma, \pi, \sigma).$$

In all the notations above, we will omit the reference to the group Γ, when there is no risk of confusion.
To get used to the derivation relation, we state here a couple of simple properties.

Proposition 5.1.12. *Let $b \in D(\Gamma, \pi, \sigma)$ be a derivation. Then:*

1. *$b(e) = 0$;*

2. *$b(g^{-1}) = -\pi(g^{-1})b(g)\sigma(g^{-1})$;*

3. *Moreover, if $\sigma = \pi$ and $b(g) \in \pi(\Gamma)'$ for all $g \in \Gamma$, then $b(gx) = b(xg)$, for all $x \in \Gamma$.*

Proof : The chain of equalities $b(e) = b(e^2) = \pi(e)b(e) + b(e)\sigma(e) = 0$ proves the first item. To prove the second one, we use the relation $b(g^{-1}g) = \pi(g^{-1})b(g) + b(g^{-1})\sigma(g) = 0$ and, then we isolate $b(g^{-1})$, in order to get the desired result.
The last claim follows directly from the derivation identity. □

Now, we see how to build new representations using derivations. The following Proposition and its proof are taken from [Pis01], Lemma 4.5.

Proposition 5.1.13. *Let π and σ be unitary representations of a group Γ. We define a map $\pi_b : \Gamma \to GL(\mathcal{H} \oplus \mathcal{H})$ by setting for all $g \in \Gamma$:*

$$\pi_b(g) = \begin{pmatrix} \pi(g) & b(g) \\ 0 & \sigma(g) \end{pmatrix},$$

where b is a map from Γ into $\mathcal{B}(\mathcal{H})$. Then:

1. *The map π_b defines a homomorphism if and only if b is a derivation.*
2. *The map π_b defines a uniformly bounded representation if and only if b is a bounded derivation.*
3. *The map π_b defines a unitarisable representation if and only if b is an inner derivation. Moreover, in this case, π_b is similar to the unitary representation $\pi \oplus \sigma$.*

Proof of Proposition 5.1.13 : To prove the first item, it is enough to compute the matrix multiplication $\pi_b(g)\pi_b(h)$ and to compare its top right entry with the corresponding one in the matrix representing $\pi_b(gh)$. Then, the derivation relation follows easily.
The proof of the second claim relies on the following estimate

$$\sqrt{\|b(g)\|^2 + 1} \leq \|\pi_b(g)\| \leq \sqrt{1 + (1 + \|b(g)\|)^2}.$$

Firstly, we have:

$$\begin{aligned} \|\pi_b(g)\|^2 &= \sup_{\|x\|^2+\|y\|^2=1} \left(\|\pi(g)x + b(g)y\|^2 + \|\sigma(g)y\|^2\right) \\ &\geq \sup_{\|y\|=1} \|b(g)y\|^2 + 1 \\ &= \|b(g)\|^2 + 1, \end{aligned}$$

and the first inequality follows. Now, let x and y be vectors of norm at most 1. We get $\|\pi(g)x + b(g)y\| \leq 1 + \|b(g)\|$, for any g, which implies the second inequality.
Finally, let us prove the third claim of the proposition. If there is a bounded operator P so that b is of the form $b(g) = \pi(g)P - P\sigma(g)$, for all $g \in \Gamma$, then, the (bounded) operator T defined by

$$T = \begin{pmatrix} 1 & P \\ 0 & 1 \end{pmatrix},$$

whose inverse is

$$T^{-1} = \begin{pmatrix} 1 & -P \\ 0 & 1 \end{pmatrix},$$

intertwines π_b and $\pi \oplus \sigma$. In particular, π_b is unitarisable. Now, assume that π_b is unitarisable. By Proposition 5.1.7, there is a positive and invertible operator $T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ so that $T\pi_b = \overline{\pi}_b T$. Recall that the operator T can be written as a 2×2 matrix whose entries are operators $T_{i,j} \in \mathcal{B}(\mathcal{H})$. Since T is self-adjoint, $T_{i,i} = T_{i,i}^*$ and $T_{i,j} = T_{j,i}^*$. Thus, the identity (5.1) of Proposition 5.1.7 gives the equality between the operators :

$$\begin{pmatrix} T_{1,1}\pi(g^{-1})^* + T_{1,2}b(g^{-1})^* & T_{1,2}\sigma(g^{-1})^* \\ T_{1,2}^*\pi(g^{-1})^* + T_{2,2}b(g^{-1})^* & T_{2,2}\sigma(g^{-1})^* \end{pmatrix}$$

and

$$\begin{pmatrix} \pi(g)T_{1,1} + b(g)T_{1,2}^* & \pi(g)T_{1,2} + b(g)T_{2,2} \\ \sigma(g)T_{1,2}^* & \sigma(g)T_{2,2} \end{pmatrix}.$$

By comparing the (2,2)-entries, we see that $T_{2,2}$ commutes with σ. By comparing the (1,2)-entries and isolating $b(g)$, we get

$$b(g) = \pi(g)(-T_{1,2}T_{2,2}^{-1}) - (-T_{1,2}T_{2,2}^{-1})\sigma(g),$$

where we used the fact that $T_{2,2}$ is invertible. Since all the coefficients of T are bounded operators, $-T_{1,2}T_{2,2}^{-1} \in \mathcal{B}(\mathcal{H})$. Hence, b is an inner derivation and π_b is similar to $\pi \oplus \sigma$. □

In view of Proposition 5.1.13, there is a straightforward way of producing uniformly bounded but non-unitarisable representations.

Corollary 5.1.14. *Let π and σ be two unitary representations of a group Γ acting on the same Hilbert space. Let b be a derivation with respect to the*

pair (π, σ). Then, the representation π_b defined as in Proposition 5.1.13 is uniformly bounded and non-unitarisable if and only b is a bounded and non-inner derivation.
In particular, if a group Γ admits a pair of representations (π, σ) such that the inclusion of $I(G, \pi, \sigma)$ inside $D^\infty(\Gamma, \pi, \sigma)$ is strict, then Γ is not unitarisable.

As a consequence of the theorem of Dixmier, we deduce the well-known fact that amenable groups have only inner derivations.

Corollary 5.1.15. *Let Γ be an amenable group and let π and σ be unitary representations of Γ. Then, any bounded derivation with respect to the pair (π, σ) is inner. This can be summarized as*

$$H^1_d(\Gamma, \sigma, \pi) = 0.$$

Proof : This is direct from Theorem 5.1.5 and from Proposition 5.1.13. □

Remark 5.1.16. *In the rest of this chapter, given a derivation $b \in D^\infty(\Gamma, \pi, \sigma)$, we will write π_b for the uniformly bounded representation obtained by the construction of Proposition 5.1.13.*

5.1.3 Cohomology in Banach algebras

In this subsection, we will consider a more general setting which will help us to deduce some known results about the space of bounded derivations.

General setting

Let $\mathcal{A}$ and X be Banach algebras over $\mathbb{C}$. We say that X is a **Banach $\mathcal{A}$-module** if it is a two-sided $\mathcal{A}-$module and there is a positive real number K such that $\|ax\| \leq K\|a\|\|x\|$ and $\|xa\| \leq K\|x\|\|a\|$, for all $a \in \mathcal{A}$ and $x \in X$. For example, for X we can take the space of all bounded operators on a Hilbert space $\mathcal{H}$, and for $\mathcal{A}$, any subalgebra of $\mathcal{B}(\mathcal{H})$.

Definition 5.1.17. *Let $\mathcal{A}$ and X be Banach algebras over $\mathbb{C}$, and suppose that X is a Banach $\mathcal{A}$-module.*

1. *A* ***bounded*** 1***-cocycle*** *(or, simply, a* ***bounded cocycle****) is a continuous linear map* $\phi : \mathcal{A} \to X$ *satisfying*

$$\phi(ab) = a\phi(b) + \phi(a)b,$$

   for every $a, b \in \mathcal{A}$*.*

2. *A bounded cocycle of the form*

$$a \mapsto ax - xa,$$

   for some $x \in X$*, is called a* ***bounded*** 1***-coboundary*** *(or, simply, a* ***coboundary****). Such a cocycle will be denoted by* ∂_x*.*

3. *We denote by* $Z_b^1(\mathcal{A}, X)$ *the space of all bounded* 1*-cocycles and by* $B^1(\mathcal{A}, X)$ *the subspace of bounded* 1*-coboundaries. The* ***first bounded cohomology group of*** $\mathcal{A}$ ***with coefficients in*** X *is defined as the quotient space*

$$H_b^1(\mathcal{A}, X) = Z_b^1(\mathcal{A}, X)/B^1(\mathcal{A}, X).$$

The dual X^* of X becomes a Banach $\mathcal{A}-$module if we define

$$\begin{aligned} \langle ya, x \rangle &= \langle y, ax \rangle, \\ \langle ay, x \rangle &= \langle y, xa \rangle, \text{ for all } a \in \mathcal{A}, x \in X, y \in X^*. \end{aligned}$$

Even if X^* has a natural structure of $\mathcal{A}$-module, we will consider the action induced by duality to compute $H_b^1(\mathcal{A}, X^*)$, if not stated otherwise.

Bounded cohomology in degree 1 for groups

Let Γ be a discrete countable group. Let $\ell^1\Gamma$ be the algebra of ℓ^1 functions on Γ. Recall that the product is given by the convolution:

$$(f * g)(x) = \sum_{\gamma \in \Gamma} f(x\gamma^{-1})g(\gamma),$$

for $f, g \in \ell^1\Gamma, x \in \Gamma$. In particular, for any $a, b \in \Gamma$, we have:

$$\delta_a * \delta_b = \delta_{ab}.$$

Now, let π and σ be two uniformly bounded representations of Γ acting on the same Banach space V. Then, we can see $\mathcal{B}(V)$, the space of bounded operators on V, as a Banach $\ell^1\Gamma$-module. Indeed, any uniformly bounded representation π extends to a homomorphism $\pi : \ell^1\Gamma \to \mathcal{B}(V)$ by setting

$$\pi(x) = \sum_{\gamma\in\Gamma} x_\gamma \pi(\gamma)$$

for $x = \sum_{\gamma\in\Gamma} x_\gamma \delta_\gamma \in \ell^1\Gamma$. Thus, we can let $\ell^1\Gamma$ act on $\mathcal{B}(V)$ in a natural way:

$$xT = \pi(x)T \quad \text{and} \quad Tx = T\sigma(x),$$

for $x \in \ell^1\Gamma$ and $T \in \mathcal{B}(V)$. Moreover, these actions satisfy

$$\max\{\|xT\|, \|Tx\|\} \leq C\|T\|\|x\|_1,$$

where $C = |\pi||\sigma|$. An important example that we will consider is the case of the left regular representation λ acting isometrically on $\ell^p\Gamma$. By applying the above construction, $\ell^1\Gamma$ acts on $\mathcal{B}(\ell^p\Gamma)$ on both sides.
The next lemma gives the equivalence between the space of bounded cocycles relatively to some $\ell^1\Gamma$-module $\mathcal{B}(V)$ and the space of bounded derivations on the group Γ with values in $\mathcal{B}(V)$. This equivalence appeared in Chapter 2 of [Joh72].

Lemma 5.1.18. *Let Γ be a countable discrete group. Let π and σ be two uniformly bounded representations acting on a Banach space V, so that $\mathcal{B}(V)$ becomes a Banach $\ell^1\Gamma$-module. Then, there is an isomorphism between the spaces $H^1_b(\ell^1\Gamma, \mathcal{B}(V))$ and $H^1_d(\Gamma, \pi, \sigma)$.*

Proof : Define $\Phi : D^\infty(\Gamma, \pi, \sigma) \to Z^1_b(\ell^1\Gamma, \mathcal{B}(V))$ by setting

$$\Phi(b)(x) = \sum_\gamma x_\gamma b(\gamma),$$

for any $x \in \ell^1\Gamma$ and $b \in D^\infty(\Gamma, \pi, \sigma)$. Clearly, the map Φ is linear. It is also bounded since

$$\|\Phi(b)(x)\| \leq \sum_\gamma |x_\gamma|\|b(\gamma)\| \leq \|b\|\|x\|_1,$$

for all $b \in D^\infty(\Gamma, \pi, \sigma)$ and $x \in \ell^1\Gamma$. Let us check that $\Phi(b)$ satisfies the cocycle identity. For any $x, y \in \Gamma$, it is easy to see that:

$$\begin{aligned}
\Phi(b)(\delta_x * \delta_y) &= \Phi(b)(\delta_{xy}) \\
&= b(xy) \\
&= \pi(x)b(y) + b(x)\sigma(y) \\
&= \pi(x)\Phi(b)(\delta_y) + \Phi(b)(\delta_x)\sigma(y) \\
&= x\Phi(b)(\delta_y) + \Phi(b)(\delta_x)y.
\end{aligned}$$

Using the linearity of Φ, we get that $\Phi(b) \in Z_b^1(\ell^1\Gamma, \mathcal{B}(V))$. Now, we can prove that Φ maps inner derivations to coboundaries. Let $T \in \mathcal{B}(V)$ and $x \in \ell^1\Gamma$. We easily have :

$$\begin{aligned} \Phi(\partial_T)(x) &= \sum_\gamma x_\gamma \partial_T(\gamma) \\ &= \sum_\gamma x_\gamma \left(\pi(\gamma)T - T\sigma(\gamma)\right) \\ &= \left(\sum_\gamma x_\gamma \pi(\gamma)\right) T - T \left(\sum_\gamma \sigma(\gamma)\right) \\ &= \pi(x)T - T\sigma(x). \end{aligned}$$

Finally, we see that Φ is surjective. Let $\alpha \in Z_b^1(\ell^1\Gamma, \mathcal{B}(V))$. We can define b, a derivation with respect to the pair (π, σ), by setting $b(\gamma) = \alpha(\delta_\gamma)$. Clearly, $\Phi(b) = \alpha$. Therefore, Φ is an isomorphism. □

The next result, also due to Johnson, gives a characterization of the amenability of a group in terms of its first bounded cohomology group relatively to dual modules.

Theorem 5.1.19. *(Theorem 2.5, [Joh72]) Let Γ be a discrete group. Then, Γ is amenable if and only if $H_b^1(\ell^1\Gamma, X^*) = 0$, for any Banach $\ell^1\Gamma$-module X.*

By extension, it is natural to define the following notion of amenability for Banach algebras.

Definition 5.1.20. *A Banach algebra $\mathcal{A}$ is **amenable** if $H_b^1(\mathcal{A}, X^*) = 0$, for any Banach $\mathcal{A}$-module X.*

We gather some information concerning amenability of some special Banach algebras associated with groups. We recall the following definitions.

Definition 5.1.21. *1. A C^*-**algebra** is a $*$-Banach algebra[2] $\mathcal{A}$ such that*

$$\|x^*x\| = \|x\|^2,$$

[2] Recall that a $*$-Banach algebra $\mathcal{A}$ is a Banach algebra over $\mathbb{C}$ endowed with an involution $^* : \mathcal{A} \to \mathcal{A}$ satisfying $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(\mu x)^* = \overline{\mu}x^*$, for all $x, y \in \mathcal{A}$, and $\mu \in \mathbb{C}$.

for all $x \in \mathcal{A}$. Recall that any C^*-algebra can be realized concretely as a norm-closed subalegbra of $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.

2. *For a countable and discrete group Γ, the* ***reduced C^*-algebra of*** *Γ, denoted by $C_r^*(\Gamma)$, is the C^*-algebra obtained by completing $\ell^1\Gamma$ with respect to the C^*-norm defined as follows for $f \in \ell^1\Gamma$,*

$$\|f\|_{C_r^*(\Gamma)} = \|\lambda(f)\|.$$

Equivalently, $C_r^*(\Gamma)$ coincides with the closed linear span of

$$\lambda(\Gamma) = \{\lambda(\gamma) \;:\; \gamma \in \Gamma\},$$

in $\mathcal{B}(\ell^2\Gamma)$.

3. *For a countable and discrete group Γ, the* ***maximal C^*-algebra of*** *Γ, denoted by $C_{max}^*(\Gamma)$, is the C^*-algebra obtained by completing $\ell^1\Gamma$ with respect to the C^*-norm defined as follows for $f \in \ell^1\Gamma$,*

$$\|f\|_{C_{max}^*(\Gamma)} = \sup \left\| \sum_\gamma f(\gamma)\pi(\gamma) \right\|,$$

where the supremum runs over all unitary representations of Γ.

Theorem 5.1.22. *Let Γ be a discrete group. The following are all equivalent:*

1. *The group Γ is amenable;*
2. *The Banach algebra $\ell^1\Gamma$ is amenable;*
3. *The C^*-algebra $C_r^*(\Gamma)$ is amenable;*
4. *$C_{max}^*(\Gamma) = C_r^*(\Gamma)$.*

In particular, we can reprove the following corollary (compare with Corollary 5.1.15.)

Corollary 5.1.23. *Let Γ be a discrete amenable group. Then,*

$$H_b^1(\ell^1\Gamma, B(\ell^2\Gamma)) = 0.$$

Proof : We will see is that $\mathcal{B}(\ell^2\Gamma)$ is in fact a dual of a Banach $\ell^1\Gamma$-module. For a separable Hilbert space $\mathcal{H}$, let us denote $\mathcal{C}^1(\mathcal{H})$ be the subspace of $\mathcal{B}(\mathcal{H})$ consisting of trace-class operators. It is well-known that $B(\mathcal{H})$ and $\mathcal{C}^1(\mathcal{H})$ form a dual pair, where the pairing is given by

$$\langle T, A\rangle = Tr(AT),\ A \in \mathcal{C}^1(\mathcal{H}), T \in \mathcal{B}(\mathcal{H}).$$

The isomorphism can be realised by the map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{C}^1(\mathcal{H})^* : T \mapsto \Psi_T$ defined by

$$\Psi_T(A) = Tr(AT),\ A \in \mathcal{C}^1(\mathcal{H}).$$

In the case $\mathcal{H} = \ell^2\Gamma$, this map is $\ell^1\Gamma$-equivariant, that is, $\Psi(aT) = a\Psi(T)$, for all $a \in \ell^1\Gamma$ and $T \in \mathcal{B}(\ell^2\Gamma)$. Recall that the action on the left-hand side is the natural action induced by the embedding $\lambda : \ell^1\Gamma \to \mathcal{B}(\ell^2\Gamma)$ and that the action on the right-hand side corresponds to the dual action induced by the natural Banach $\ell^1\Gamma$-module structure of $\mathcal{C}^1(\ell^2\Gamma)$. Let $x \in \mathcal{C}^1(\ell^2\Gamma)$. We have:

$$\begin{aligned} \Psi(aT)(x) &= \Psi_{\lambda(a)T}(x) \\ &= Tr(x\lambda(a)T) \\ &= \Psi_T(x\lambda(a)) \\ &= \Psi(T)(x\lambda(a)) \\ &= [\lambda(a)\Psi(T)](x). \end{aligned}$$

Using equivariance, the map Ψ clearly gives an isomorphism between the spaces $H_b^1(\ell^1\Gamma, \mathcal{B}(\ell^2\Gamma))$ and $H_b^1(\ell^1\Gamma, \mathcal{C}^1(\ell^2\Gamma)^*)$. Finally, by amenability of Γ, we get that $H_b^1(\ell^1\Gamma, \mathcal{C}^1(\ell^2\Gamma)^*) = 0$, which ends the proof. □

It is possible to improve the last result by a direct computation using invariant means on amenable groups. The proof is based on an average argument appearing in Section 0.4 of [SS95].

Proposition 5.1.24. *Let Γ be a discrete amenable group. Then,*

$$H_b^1(\ell^1\Gamma, \mathcal{B}(\ell^p\Gamma)) = 0,$$

for any $p \in (1, \infty)$.

Proof : Let $b \in D^\infty(\Gamma, \lambda)$ be a bounded derivation with respect to the left regular representation λ acting by isometries on $\ell^p\Gamma$. We shall see that b is inner by finding a bounded operator T satisfying $b(\gamma) = [\lambda(\gamma), T]$, for all $\gamma \in \Gamma$. The operator T will be defined by taking an appropriate average over Γ of bounded operators acting on $\ell^p\Gamma$. This strategy is closely related to the proof of Theorem 5.1.5, from which we recall the notations. Taking a sequence of Følner sequence for Γ indexed by $i \in \mathbb{N}$, let m_i be the normalised counting measure associated with the i-th Følner set. As before, by chosing ω, any non-principal ultrafilter on $\mathbb{N}$, we obtain a right invariant mean m on Γ. We can now define the desired operator T.

$$T = \int_\gamma \lambda(\gamma^{-1})b(\gamma)\, dm(\gamma),$$

meaning that for $\xi \in \ell^p\Gamma$, we have

$$(T\xi)(x) = \int_\Gamma [\lambda(\gamma^{-1})b(\gamma)\xi](x)\, dm(\gamma),$$

for all $x \in \Gamma$. For the rest of the proof, we set $\varphi(x) = \lambda(x^{-1})b(x)$ and $\varphi_\xi(x,\gamma) = [\varphi(x)\xi](\gamma)$. We need to check that the operator T is bounded. Let us fix $\xi \in \ell^p\Gamma$. By denoting $\langle\cdot,\cdot\rangle$ the natural pairing between $\ell^p\Gamma$ and $\ell^q\Gamma$, for $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\langle T\xi, \eta\rangle = \lim_{i\to\omega}\langle \int_\Gamma \varphi(\gamma)\xi\, dm_i(\gamma), \eta\rangle,$$

for all $\eta \in \ell^q\Gamma$. For all $i \in \mathbb{N}$ and $\eta \in \ell^q\Gamma$, using Fubini's theorem, we obtain:

$$\begin{aligned}
|\langle \int_\Gamma \varphi(\gamma)\xi\, dm_i(\gamma), \eta\rangle| &= \left|\sum_{x\in\Gamma}\int_\Gamma (\varphi(\gamma)\xi)(x)\eta(x)\, dm_i(\gamma)\right| \\
&= \left|\int_\Gamma \sum_{x\in\Gamma} (\varphi(\gamma)\xi)(x)\eta(x)\, dm_i(\gamma)\right| \\
&\leq \int_\Gamma \left|\sum_{x\in\Gamma} (\varphi(\gamma)\xi)(x)\eta(x)\right| dm_i(\gamma) \\
&\leq \int_\Gamma \|b\|\|\xi\|_p\|\eta\|_q\, dm_i(\gamma) \\
&= \|b\|\|\xi\|_p\|\eta\|_q,
\end{aligned}$$

which is independent of i. Since $\|T\xi\|_p = \sup_{\|\eta\|_q=1}|\langle T\xi,\eta\rangle|$ and by passing to the limit, we see that $T\xi$ belongs to $\ell^p\Gamma$ and is of norm at most $\|b\|\|\xi\|_q$. As ξ is arbitrary, we conclude that T is bounded and its operator-norm is at most $\|b\|$.

To finish the proof, we need to show that $b(\gamma) = [\lambda(\gamma), T]$, for all $\gamma \in \Gamma$. Firstly, using the derivation relation, we have

$$\begin{aligned} b(\gamma) &= \lambda(x^{-1})b(x\gamma) - \lambda(x^{-1})b(x)\lambda(\gamma) \\ &= \lambda(\gamma)\lambda(\gamma^{-1}x^{-1})b(x\gamma) - \lambda(x^{-1})b(x)\lambda(\gamma), \end{aligned}$$

for all $x, \gamma \in \Gamma$. By right invariance of m, we obtain:

$$\begin{aligned} \int_\Gamma \lambda(\gamma^{-1}x^{-1})b(x\gamma)\,dm(x) &= \int_\Gamma (\check{\lambda}\cdot b)(x\gamma)\,dm(x) \\ &= \int_\Gamma (\check{\lambda}\cdot b)(x)\,dm(x) \\ &= T, \end{aligned}$$

which is independent of the chosen γ. Therefore, this leads us to

$$\begin{aligned} b(\gamma) &= \int_\Gamma b(\gamma)\,dm(x) \\ &= \int_\Gamma \left(\lambda(\gamma)\left(\check{\lambda}\cdot b\right)(x\gamma) - \left(\check{\lambda}\cdot b\right)(x)\lambda(\gamma)\right)\,dm(x) \\ &= \lambda(\gamma)\left(\int_\Gamma (\check{\lambda}\cdot b)(x)\,dm(x)\right) - \left(\int_\Gamma (\check{\lambda}\cdot b)(x)\,dm(x)\right)\lambda(\gamma) \\ &= [\lambda(\gamma), T], \end{aligned}$$

which proves that the derivation b is inner. □

Important examples of C^*-algebras are provided by von Neumann algebras. We recall the definitions of von Neumann algebra as well as the so-called group von Neumann algebra.

Definition 5.1.25. *1. A* ***von Neumann algebra*** $\mathcal{M}$ *on a Hilbert space* $\mathcal{H}$ *is a unital* **-subalgebra of* $\mathcal{B}(\mathcal{H})$ *such that*

$$\mathcal{M}'' = \mathcal{M},$$

where, $\mathcal{M}''$ *denotes the bicommutant* [3] *of* $\mathcal{M}$.
Equivalently, by von Neumann's bicommutant Theorem, $\mathcal{M}$ *is a von Neumann algebra if and only if* $\mathcal{M}$ *is strongly closed* [4] *in* $\mathcal{B}(\mathcal{H})$.

[3] For a subset Q of $\mathcal{B}(\mathcal{H})$, the commutant Q' of Q corresponds to the set of all bounded operators that commute with every operator in Q. Thus, the bicommutant of Q is defined as $Q'' = (Q')'$.

[4] A net of operators $(T_i)_i$ in $\mathcal{B}(\mathcal{H})$ converges to 0 in the strong topology if, for any $\xi \in \mathcal{H}$, the net $\|T_i\xi\|$ tends to 0.

2. *For a countable and discrete group* Γ*, the (left)* ***group von Neumann algebra of*** Γ*, denoted by* $L(\Gamma)$*, is defined as*

$$L(\Gamma) = \lambda(\Gamma)'' = \{\lambda(\gamma) \ : \ \gamma \in \Gamma\}''.$$

Equivalently, $L(\Gamma)$ *coincides with the closure of the linear span of* $\lambda(\Gamma)$ *in* $\mathcal{B}(\ell^2\Gamma)$ *for the strong topology.*
Similarly, on can define the right group von Neumann algebra $R(\Gamma)$ *as the von Neumann algebra generated by the right regular representation* ρ*. These algebras are related to each other by the following identity*

$$\lambda(\Gamma)' = R(\Gamma).$$

To illustrate the use of the different Banach algebras we introduced in this section, we give an immediate consequence of Lemma 5.1.18.

Corollary 5.1.26. *Let* $\mathcal{A}$ *be one of the following algebra :* $C^*_{max}(\Gamma)$*,* $C^*_r(\Gamma)$ *or* $L(\Gamma)$*. If* $H^1_b(\mathcal{A}, \mathcal{B}(\ell^2\Gamma))$ *does not vanish, then* $H^1_d(\Gamma, \lambda)$ *is non-trivial. In particular,* Γ *is not unitarisable.*

Proof : Let $\phi \in Z^1_b(\mathcal{A}, \mathcal{B}(\ell^2\Gamma))$ be a non-trivial bounded cocycle. We can define a derivation $b \in D^\infty(\Gamma, \lambda)$, which is not inner, by setting $b(\gamma) = \phi(\lambda(\gamma))$. As ϕ is bounded, we have $\|\phi(x)\| \leq \|\phi\|\|x\|$, for all $x \in \mathcal{A}$. In particular, we get:

$$\|b(\gamma)\| = \|\phi(\lambda(\gamma))\| \leq \|\phi\|\|\lambda(\gamma)\| = \|\phi\|.$$

Therefore, b is uniformly bounded, of norm at most $\|\phi\|$. By Proposition 5.1.13, the representation λ_b is uniformly bounded and non-unitarisable. Hence, the group Γ is non-unitarisable.
□

We end this section with a couple of results showing that the non-vanishing of the space $H^1_b(\mathcal{A}, X)$ is rather unusual.

1. (Theorem 1, [Sak66]) Let $\mathcal{M}$ be a von Neumann algebra. Then, any derivation $b : \mathcal{M} \to \mathcal{M}$ is inner, that is,

$$H^1_b(\mathcal{M}, \mathcal{M}) = 0.$$

2. Using Ryll-Nardzewski's fixed point theorem, one can prove the following (see Theorem 3.4, [Joh72]). Let Γ be a discrete group and let X be reflexive Banach $\ell^1\Gamma$-module. Then,

$$H_b^1(\ell^1\Gamma, X) = 0.$$

3. (Corollary 11.4.2, [Mon01]) Let Γ be a finitely generated group and let X be a Banach $\ell^1\Gamma$-module such that X^* is separable. Then,

$$H_b^1(\ell^1\Gamma, X^*) = 0.$$

4. As an application of the preceding item, let Γ be a countable and discrete group. Since $\ell^1\Gamma$ is separable with dual $\ell^\infty\Gamma$, then,

$$H_b^1(\ell^1\Gamma, \ell^\infty\Gamma) = 0.$$

5. By way of contrast, Example 10.3.3 in [Mon01] shows that for any discrete, non-elementary Gromov hyperbolic group, the space

$$H_b^1(\ell^1\Gamma, (\ell^\infty\Gamma)/\mathbb{C})$$

is infinite dimensional.

5.2 Topology on the space of bounded derivations

In this section, we will endow the space of bounded derivations with a natural topology and we will give a condition under which the space of inner derivations is closed. We then study basic hereditary properties.

5.2.1 Uniform almost intertwiners

Let Γ be a countable discrete group. Let π and σ be two uniformly bounded representations of G acting on the same Banach space V. We endow the space of bounded derivations $D^\infty(\Gamma, \pi, \sigma)$ with the topology induced by the supremum norm, that is,

$$\|b\| = \sup_{g\in\Gamma} \|b(g)\|, \quad b \in D^\infty(\Gamma, \pi, \sigma).$$

This norm is natural at least for the reason that it turns $D^\infty(\pi,\sigma)$ into a Banach space. In order to identify the subspace of inner derivations appropriately, we state a few definitions. We define the set of **intertwiners** for the pair (π,σ) as

$$Int(\pi,\sigma)=\{Q\in\mathcal{B}(V):\pi(g)Q=Q\sigma(g),\forall g\in\Gamma\}.$$

It is the space of all bounded operators on V that intertwine π and σ. We remark that in the case $\pi=\sigma$, then this space is simply the **commutant** of π, that is,

$$Int(\pi,\pi)=\pi(\Gamma)'.$$

When there is no risk of confusion, we write $Int(\pi,\sigma)$ instead of $Int(\Gamma,\pi,\sigma)$. Let $\partial:\mathcal{B}(V)\to D^\infty(\Gamma,\pi,\sigma)$ be the linear map defined by

$$\partial_P(g)=\pi(g)P-P\sigma(g),\ \text{for all } g\in\Gamma,$$

for $P\in\mathcal{B}(V)$. We will refer to ∂ as a **coboundary map**. We remark that the image of ∂ is exactly $I(\Gamma,\pi,\sigma)$ and its kernel coincides with $Int(\pi,\sigma)$. Therefore, we get the natural identification

$$I(\Gamma,\pi,\sigma)\cong\mathcal{B}(V)/Int(\pi,\sigma).$$

Finally, this identification suggests to introduce the following norm. We will denote by $\|\cdot\|_q$ the quotient semi-norm on $\mathcal{B}(V)$ (which becomes a norm on $\mathcal{B}(V)/Int(\pi,\sigma)$) defined by

$$\|T\|_q=\inf_{Q\in Int(\pi,\sigma)}\|T-Q\|,$$

for $T\in\mathcal{B}(V)$.
Here comes the main definition of the section.

Definition 5.2.1. *We say that the pair (π,σ) has* ***uniform almost intertwiners*** *if for all $\epsilon>0$, there exists $T\in\mathcal{B}(V)$ such that*

$$\sup_{g\in\Gamma}\|\pi(g)T-T\sigma(g)\|<\epsilon\|T\|_q.$$

For a fixed $\epsilon>0$, an operator satisfying the equality here above is a called a ***uniform ϵ-intertwiner****.*
We also say that a representation π admits ***uniform almost intertwiners*** *if the pair (π,π) does.*

We remark that an uniform ϵ-intertwiner for a pair (π, σ) cannot intertwine π and σ.
The following proposition is closely related to a well-known result of Guichardet. We give an adapted proof for the reader's convenience (see Proposition 2.12.2 in [BdlHV08]).

Proposition 5.2.2. *Let π and σ be uniformly bounded representations of a group Γ. Suppose that π and σ act on the same Banach space V. Then, the space of inner derivations $I(\Gamma, \pi, \sigma)$ is closed in $D^\infty(\Gamma, \pi, \sigma)$ for the topology induced by the supremum norm if and only if the pair (π, σ) does not have uniform almost intertwiners.*

Proof : Following the proof of the theorem of Guichardet, we look at the coboundary map

$$\partial : (\mathcal{B}(V)/Int(\pi, \sigma), \|\cdot\|_q) \to D^\infty(\Gamma, \pi, \sigma) : [T] \mapsto \partial_T,$$

where $\partial_T(g) = \pi(g)T - T\sigma(g)$. This is a well-defined, linear, injective and continuous map whose image is $I(\pi, \sigma)$. Let us check the continuity of ∂. Let $T \in \mathcal{B}(V)$. For any $\epsilon > 0$, there exists $Q_\epsilon \in Int(\pi, \sigma)$ such that

$$\|T - Q_\epsilon\| < \|T\|_q + \epsilon.$$

Moreover, since $\partial_T = \partial_{T-Q_\epsilon}$, we obtain

$$\begin{aligned} \|\partial_T(g)\| &= \|\partial_{T-Q_\epsilon}(g)\| \\ &\leq 2\|T - Q_\epsilon\| \\ &< 2\|T\|_q + 2\epsilon, \end{aligned}$$

for all $g \in \Gamma$. Hence, ∂ is of norm at most 2.
Now, if we assume that $I(\pi, \sigma)$ is closed, then $I(\pi, \sigma)$ is a Banach space for the supremum norm and it follows from the open mapping theorem that the operator $\partial : \mathcal{B}(V) \to I(\pi, \sigma)$ is bicontinuous. The continuity of the map ∂^{-1} is equivalent to saying that there exists a constant $\beta > 0$ such that

$$\sup_g \|\partial_T(g)\| \geq \beta\|T\|_q,$$

for all $T \in \mathcal{B}(V)$. Thus, the pair (π, σ) does not have uniform almost intertwiners.
For the converse implication, assume that the pair (π, σ) does not have uniform almost intertwiners. Let $(T_i)_i$ be a net in $\mathcal{B}(V)/Int(\pi, \sigma)$ such that ∂_{T_i}

converges to some derivation $b \in D^\infty(\pi, \sigma)$. We can extract a subsequence $(T_n)_n$ such that

$$\lim_n \|\partial_{T_n} - b\| = 0.$$

We get :

$$\|\partial_{T_n} - \partial_{T_m}\| \geq \beta \|T_n - T_m\|_q,$$

which proves that (T_n) is Cauchy and hence, converging to some T. Then it is clear that $\partial_T = b$, so we can conclude that $I(\pi, \sigma)$ is closed. □

Remark 5.2.3. *If, for a group* Γ*, we can find a pair of unitary representations* (π, σ) *such that* $I(\pi, \sigma)$ *is not closed in* $D^\infty(\pi, \sigma)$*, then the inclusion of* $I(\pi, \sigma)$ *inside* $D^\infty(\pi, \sigma)$ *is strict. Therefore, this allows us to build a uniformly bounded representation* π_b *which is not unitarisable, by Corollary 5.1.14.*

Remark 5.2.4. *Proposition 5.2.2 implies that, if a pair of representations* (π, σ) *does not admit uniform almost intertwiners, then the natural seminorm induced on* $H^1_d(\Gamma, \pi, \sigma)$ *is indeed a norm.*
In particular, Proposition 5.2.2 can be seen as a special case of Theorem 2.3 in [MM85].

5.2.2 Hereditary results

We prove that the property of admitting uniform almost intertwiners is preserved by taking direct sums and by inducing representations from a subgroup to the ambient group.

Proposition 5.2.5. *Let* π *be a uniformly bounded representation of a group* Γ*. Suppose that* π *admits uniform almost intertwiners. Then, for any uniformly bounded representation* σ *of* Γ*, the direct sum* $\pi \oplus \sigma$ *admits uniform almost intertwiners.*

Proof : Let $\epsilon > 0$ and let T be a uniform ϵ-intertwiner. Then, the operator $\tilde{T}$ defined by

$$\tilde{T} = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H}_\pi \bigoplus \mathcal{H}_\sigma),$$

is a uniform ϵ-intertwiner for $\pi \oplus \sigma$. Indeed, if $R \in \mathcal{B}(\mathcal{H}_\pi \bigoplus \mathcal{H}_\sigma)$, with entries $(R_{i,j})_{i,j}$, belongs to the commutant of $\pi \oplus \sigma$, then

$$\begin{aligned} \|\tilde{T} + R\| &\geq \|T + R_{1,1}\| \\ &\geq \|T\|_q, \end{aligned}$$

where $R_{1,1}$ commutes with π. Denoting by $\|\cdot\|_{\tilde{q}}$ the norm on the quotient of $\mathcal{B}(\mathcal{H}_\pi \bigoplus \mathcal{H}_\sigma)$ by the commutant of $\pi \oplus \sigma$, we have obtained $\|\tilde{T}\|_{\tilde{q}} \geq \|T\|_q$. The following easy computation ends the proof:

$$\begin{aligned} \|[(\pi \oplus \sigma)(\gamma), \tilde{T}]\| &= \|[\pi(\gamma), T]\| \\ &< \epsilon\|T\|_q \\ &\leq \epsilon\|\tilde{T}\|_q, \end{aligned}$$

for all $\gamma \in \Gamma$. □

Proposition 5.2.6. *Let Γ be a discrete countable group, let H be a subgroup of Γ and let $\pi : H \to \mathcal{B}(\mathcal{H})$ be a uniformly bounded representation of H. If we suppose that π admits uniform almost intertwiners, then the induced representation $\mathrm{Ind}_H^\Gamma \pi$ admits uniform almost intertwiners.*

Proof : We recall the construction of the induced representation $\mathrm{Ind}_H^\Gamma \pi$. Write $\Gamma = \sqcup_{j \geq 1} s_j H$, where the $s_j \in \Gamma$ is a set of representatives of the cosets. We can suppose $s_1 = e$ without loss of generality. For $x \in \Gamma$, the operator $\mathrm{Ind}_H^\Gamma \pi(x)$ is defined by the matrix with entries in $\mathcal{B}(\mathcal{H})$:

$$\mathrm{Ind}_H^\Gamma \pi(x)_{i,j} = \begin{cases} \pi(s_i^{-1} x s_j), & \text{if } s_i^{-1} x s_j \in H, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\mathrm{Ind}_H^\Gamma \pi : \Gamma \to \mathcal{B}(\bigoplus_{j \geq 1} \mathcal{H})$ is a well-defined uniformly bounded representation. In particular, for any $x \in \Gamma$ and for any fixed j, there exists a unique i so that $s_i^{-1} x s_j \in H$ (see Theorem 2.8 in [Pis01] for more details). Therefore, we denote σ_x the permutation of $\mathbb{N}^*$ satisfying $s_{\sigma_x(j)}^{-1} x s_j \in H$, for all $j \geq 1$. We also observe that for any $h \in H$, we have

$$\mathrm{Ind}_H^\Gamma \pi(h)_{1,j} = \mathrm{Ind}_H^\Gamma \pi(h)_{j,1} = \begin{cases} \pi(h), & \text{if } i = j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

This implies that, if $R \in \mathcal{B}(\bigoplus_{j \geq 1} \mathcal{H})$ commutes with $\mathrm{Ind}_H^\Gamma \pi$, then the diagonal entry $R_{1,1}$ belongs to the commutant $\pi(H)'$. So, if T is a uniform ϵ-intertwiner for π, we claim that the operator Q defined by

$$Q_{i,j} = \begin{cases} T, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

is a uniform ϵ-intertwiner for $\mathrm{Ind}_H^\Gamma \pi$. Indeed, we first observe that, given $R \in \mathrm{Ind}_H^\Gamma \pi(\Gamma)'$, we have

$$\begin{aligned} \|Q + R\| &\geq \|T + R_{1,1}\| \\ &\geq \|T\|_q. \end{aligned}$$

Therefore, $\|T\|_q \leq \|Q\|_q$. Moreover, it is easy to compute

$$[\mathrm{Ind}_H^\Gamma \pi(x), Q]_{i,j} = \begin{cases} [\pi(s_i^{-1} x s_j), T], & \text{if } i = \sigma_x(j), \\ 0, & \text{otherwise,} \end{cases}$$

for any $x \in \Gamma$. Let $v = (v_j)_j \in \bigoplus_{j\geq 1} \mathcal{H}$ and $x \in \Gamma$. We have :

$$\begin{aligned} \|[\mathrm{Ind}_H^\Gamma \pi(x), Q]v\|^2 &= \sum_{j\geq 1} \| \left([\mathrm{Ind}_H^\Gamma \pi(x), Q]v\right)_j \|^2 \\ &= \sum_{j\geq 1} \| \left([\pi(s_j^{-1} x s_{\sigma_x^{-1}(j)}), T] v_{\sigma_x^{-1}(j)}\right)_j \|^2 \\ &< \epsilon^2 \|T\|_q^2 \|v\|^2. \end{aligned}$$

Hence,

$$\|[\mathrm{Ind}_H^\Gamma \pi(x), Q]\| < \epsilon \|Q\|_q,$$

which finishes the proof. □

5.2.3 Construction of non-unitarisable representations

We give an explicit construction to build uniformly bounded, non-unitarisable representations.

We start with a straightforward lemma, which states that a direct sum of uniformly bounded representations is uniformly bounded if and only if the sequence of representations is uniformly bounded in a uniform way (!).

Lemma 5.2.7. *Let $\pi_k : \Gamma \to \mathcal{B}(\mathcal{H})$ be a sequence of uniformly bounded representations of a group Γ. If the family $(\pi_k)_k$ is uniformly bounded in norm, that is, there is a constant $C > 0$ such that*

$$\sup_{k\in\mathbb{N}} |\pi_k| = \sup_{k\in\mathbb{N}} \sup_{g\in\Gamma} \|\pi_k(g)\| \leq C,$$

then we can define a uniformly bounded representation $\Pi : \Gamma \to \mathcal{B}(\bigoplus_k \mathcal{H})$ by setting

$$\Pi(g)v = \bigoplus_{k\in\mathbb{N}} \pi_k(g) v_k,$$

for any $v = (v_k)_k$*, where* $v_k \in \mathcal{H}$ *and* $\sum_k \|v_k\|^2 < \infty$*. In particular,*

$$\|\Pi\| = \sup_{k\in\mathbb{N}} |\pi_k| \leq C.$$

We give a necessary and sufficient condition for such a direct sum to be unitarisable.

Lemma 5.2.8. *Let* Γ *be a group and let* $(\pi_k)_k$ *be a sequence of representations. The direct sum* $\Pi = \bigoplus_k \pi_k$ *is unitarisable if and only if* π_k *is unitarisable for all* k *and if there exists a sequence* $(S_k)_k$ *of invertible operators such that:*

1. *The representations defined by* $u_k(g) = S_k \pi_k(g) S_k^{-1}$ *are unitary;*
2. *There is a constant* $C > 0$ *so that* $\|S_k\|, \|S_k^{-1}\| \leq C$*, for all* k*.*

In this case, the representation Π *is similar to* $U = \bigoplus_k u_k$*.*

The proof of Lemma 5.2.8 relies on a careful study of positive unitarisers and we will need the following fact that we recall now.

Lemma 5.2.9. *Let* T *be a positive and invertible operator on the direct sum of Hilbert spaces* $\bigoplus_k \mathcal{H}$*. If we denote by* $T_{i,j}$ *the linear maps appearing as the entries of the matrix representing* T*, then for all* i, j*, the operators* $T_{i,j}$ *are bounded on* $\mathcal{H}$ *and the operators* $T_{i,i}$ *appearing on the diagonal of* T *are also positive and invertible. Furthermore, they satisfy* $\|T_{i,j}\| \leq \|T\|$ *and* $\|T_{i,i}^{-1}\| \leq \|T^{-1}\|$*.*

Proof : It is well-known that if T is positive and invertible, then there exists a constant $\delta > 0$ such that

$$\langle Tx, x\rangle \geq \delta \|x\|^2,$$

for all $x \in \oplus_k \mathcal{H}$. This easily implies that the same equality holds for the operators on the diagonal. Hence, the operators $T_{i,i}$ are positive, invertible and $\|T_{i,i}^{-1}\| \leq \|T^{-1}\|$. The rest of the proof is an easy exercice. $\square$

Proof of Lemma 5.2.8 : Firstly, suppose there is a sequence $(S_k)_k$ of operators satisfying conditions *1* and *2*, here above. We will prove that Π is unitarisable. Set $S = \bigoplus_k S_k$. It is a bounded, invertible operator on $\bigoplus_k \mathcal{H}$, with inverse $S^{-1} = \bigoplus_k S_k^{-1}$. Since for all g we have

$$\begin{aligned} S\pi(g)S^{-1} &= \bigoplus_k S_k \pi_k(g) S_k^{-1} \\ &= \bigoplus_k u_k(g) \\ &= U(g), \end{aligned}$$

we get the result. Secondly, if we assume that Π is unitarisable, then there exists T a positive and invertible operator over $\bigoplus \mathcal{H}$ intertwining Π and its contragredient. Representing $T = (T_{i,j})$ as an infinite matrix, it is easy to see that $T_{k,k}$ intertwine the representations π_k with their contragredients. By Lemma 5.2.9, for every k, the operator $T_{k,k}$ is positive and invertible. Hence, there is a unique invertible square root S_k of T_k and it satisfies $\|S_k\|^2 \leq \|T\|$, which gives the desired uniform bound on $\|S_k\|$. A similar argument shows that $\|S_k^{-1}\|^2 \leq \|T^{-1}\|$. So, all the representations π_k are similar to a unitary representation with an intertwining operator of norm at most $\|T\|$ and we are done. □

The rest of this section is devoted to apply Lemma 5.2.8 to appropriate sequences of uniformly bounded representations. We will see that, taking a sequence of $(T_n)_n$ of uniform almost intertwiners for some representation π, then, setting $b_n = \partial_{T_n}$ and taking the direct sum of the representations π_{b_n} built as in Proposition 5.1.13, we obtain a concrete uniformly bounded representation which is not unitarisable.

Lemma 5.2.10. *Let π and σ be unitary representations of a group Γ and suppose that they act on the same Hilbert space $\mathcal{H}$. Let $T \in \mathcal{B}(\mathcal{H})$ and denote by b the inner derivation obtained by T, that is, $b(g) = \pi(g)T - T\sigma(g)$. Let π_b be a representation defined as in Proposition 5.1.13. Then, for any invertible operator S intertwining π_b and $\pi \oplus \sigma$, we have*

$$\|T\|_q \leq \|S\|\|S^{-1}\|.$$

Proof : Let S be an invertible map such that $S\pi_b = (\pi \oplus \sigma)S$. Without loss of generality we can suppose that S is positive. As in the proof of Proposition 5.1.13, we get

$$b(g) = \pi(g)(-S_{1,2}S_{2,2}^{-1}) - (-S_{1,2}S_{2,2}^{-1})\sigma(g),$$

for all g. It implies that $T + S_{1,2}S_{2,2}^{-1} \in Int(\pi, \sigma)$. Therefore, we obtain

$$\begin{aligned} \|T\|_q &= \inf_{Q \in Int(\pi,\sigma)} \|T - Q\| \\ &\leq \|S_{1,2}S_{2,2}^{-1}\| \\ &\leq \|S\|\|S^{-1}\|, \end{aligned}$$

which ends the proof. □

Corollary 5.2.11. *Let π and σ be two unitary representations of a group Γ. Assume that the pair (π, σ) admits uniform almost intertwiners. Then, there exists a sequence of operators $(T_n)_n$ such that, setting $b_n = \partial_{T_n}$, the direct sum*

$$\Pi = \bigoplus_n \pi_{b_n}$$

is a uniformly bounded and non-unitarisable representation.

Proof : For all $n \in \mathbb{N}^*$, let S_n be a uniform $(1/n)$-intertwiner. We normalise them appropriately by setting

$$T_n = \frac{n}{\|S_n\|_q} S_n,$$

so that $\|\partial_{T_n}\| \leq 1$ and $\|T_n\|_q = n$, for all n. The first condition ensures that the family of representations $(\pi_{b_n})_n$ is uniformly bounded in norm, and thus, Π is uniformly bounded. The second condition implies that the family $(\pi_n)_n$ is not unitarisable with uniform norm and, therefore, Π is not unitarisable. □

Remark 5.2.12. *If, for a group Γ, there exists a pair (π, σ) that admits uniform almost intertwiners, then we can deduce by the theorem of Pisier (see Theorem 5.1.9) that Γ is not amenable. Indeed, using the notations of the proof here above, if S intertwines the representations $\pi \oplus \sigma$ and $\pi_{\partial_{T_n}}$, then, by Lemma 5.2.10, we get $n = \|T_n\|_q \leq \|S\|\ \|S^{-1}\|$. However, we proved that $|\pi_{\partial_{T_n}}|^2 \leq 5$, by the proof of Corollary 5.2.11 and the proof of Proposition 5.1.13). Therefore, taking n to infinity strongly contradicts Condition 2. of Theorem 5.1.9.*

5.3 Non-unitarisability of free groups

In this section, we provide several different proofs of the non-unitarisability of free groups.

5.3.1 Contraction operators

Let $\mathcal{T} = (V, E)$ be a tree. Let us fix an origin $x_0 \in V$. We denote by d the shortest path metric on V and by $[x, y]$ the unique geodesic segment between two vertices x, y. For $x \in V \setminus \{x_0\}$, we denote by $\overline{x} \in V$ the only vertex belonging to $[x_0, x]$ such that $d(x, \overline{x}) = 1$. This allows us to define the following operator on $C_c(V)$, the space of functions with finite support on V,

$$P\delta_u = \begin{cases} \delta_{\overline{u}}, & \text{if } u \in V \setminus \{x_0\}, \\ 0, & \text{if } u = x_0, \end{cases}$$

This operator (and its variations) will play an important role in the sequel and it will allow us to give a concrete example of bounded derivation.
In this section, we will restrict our attention to the case of $\mathcal{T}_{2q} = (V, E)$, the $(2q)$-regular tree, with $q \geq 1$ (possibly infinite). We will see it as the Cayley tree of the free group $\mathbb{F}_q$, with generators denoted by x_j, $j \in \mathbb{N}$, and we will identify the vertex set V with $\mathbb{F}_q$. Recall that the graph metric d coincides with the word-length metric on $\mathbb{F}_q$ and we will write $|x|$ for the length of an element. In particular, for any non-trivial $u \in \mathbb{F}_q$, the element $\overline{u}$ simply corresponds to the reduced word u to which we deleted the last letter.
For later use, it is convenient to write A_q (resp. A_∞) for the set of indices $\{1, 2, \cdots, q\}$, when q is finite (resp. $\mathbb{N} \cup \{\infty\}$, when $q = \infty$). For any subset $X \subset \mathbb{F}_q$, we also define $C_c(X)$ to be the space of complex-valued functions on X with finite support.

Definition 5.3.1. *Let $q \in \mathbb{N} \cup \{\infty\}$.*

1. *For any $j \in A_q$, the operator $P_j : C_c(\mathbb{F}_q) \to C_c(\mathbb{F}_q)$ is called the **operator of contraction** relatively to j and is defined to be :*

$$P_j\delta_u = \begin{cases} \delta_{\overline{u}}, & \textit{if the reduced word } u \textit{ ends with the letter } x_j^{\pm 1}, \\ 0, & \textit{otherwise}, \end{cases}$$

2. *Similarly, for any subset $A \subset A_q$, we define P_A, the **operator of contraction** relatively to A, by*

$$P_A = \sum_{j \in A} P_j$$

Equivalently, for any $u \in \mathbb{F}_q$, we have

$$P_A\delta_u = \begin{cases} \delta_{\overline{u}}, & \textit{if the reduced word } u \textit{ ends with } x_j^{\pm 1}, \textit{ for some } j \in A, \\ 0, & \textit{otherwise}, \end{cases}$$

In the case where $A = A_q$, we will simply write P instead of P_A.

The following lemma is an easy exercise (see the remarks in p. 295 of [PS86]).

Lemma 5.3.2. *Let $q \in \mathbb{N} \cup \{\infty\}$ and let $A \subset A_q$. Let P_A be the contraction operator relatively to A. Then, the operator P_A is bounded if and only if the set A is finite. Moreover, in this case, we have $\|P_A\| = \sqrt{2|A|}$.*

Proof : Suppose first that A is a finite set. Let $f \in C_c(V)$. Let $\{v_j\}_{j=1}^L \subset V$ and $\{\alpha_j\}_{j=1}^L \subset \mathbb{C}$ such that $f = \sum_{j=1}^L \alpha_j \delta_{v_j}$. We can assume that e is not in the support of f and that the vertices v_j are pairwise different. For $k \in \{1, \ldots, n\}$, let $\{u_k\}_{k=1}^n = \{\overline{v_j}\}_{j=1}^L$ be such that n is minimal. Set

$$J_k = \{j : \ \overline{v_j} = u_k \text{ and } v_j \text{ ends with the letter } x_i^{\pm 1}, \text{ for some } i \in A\}.$$

Since A is finite, we have $\max_k |J_k| \leq 2|A|$. We get

$$P_A f = \sum_{k=1}^n \left(\sum_{j \in J_k} \alpha_j \right) \delta_{u_k},$$

and

$$\|P_A f\|_2^2 = \sum_{k=1}^n \left| \sum_{j \in J_k} \alpha_j \right|^2 .$$

For each k, we define the standard inner product on the (finite-dimensional) space of complex-valued functions on J_k and we denote it by $\langle \cdot, \cdot \rangle_{(k)}$. Put $\alpha^{(k)}$ to be the function on J_k defined by $\alpha^{(k)}(j) = \alpha_j$. With those notations and using Cauchy-Schwarz, we have:

$$\begin{aligned}
|\sum_{j \in J_k} \alpha_j|^2 &= |\sum_{j \in J_k} \alpha^{(k)}(j)|^2 \\
&= |\langle \alpha^{(k)}, 1 \rangle_{(k)}|^2 \\
&\leq \|1\|_{(k)}^2 \|\alpha^{(k)}\|_{(k)}^2 \\
&\leq |J_k| \sum_{j \in J_k} |\alpha^{(k)}(j)|^2 \\
&\leq 2|A| \sum_{j \in J_k} |\alpha_j|^2.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\|P_A f\|_2^2 &= \sum_{k=1}^{n} |\sum_{j\in J_k} \alpha_j|^2 \\
&\leq 2|A| \sum_{k=1}^{n} \sum_{j\in J_k} |\alpha_j|^2 \\
&= 2|A| \sum_{j=1}^{N} |\alpha_j|^2 \\
&= 2|A| \|f\|_2^2.
\end{aligned}$$

So, we can deduce that P_A is bounded and $\|P_A\| \leq \sqrt{2|A|}$. It is easy to check this is optimal. Indeed, taking

$$f = \sum_{j\in A} \left(\delta_{x_j} + \delta_{x_j^{-1}}\right),$$

we get that $\|f\|_2 = \sqrt{2|A|}$ and $\|P_A f\|_2 = 2|A|$, which proves the desired result. □

Finally, for any $y \in V$, we will denote by S_A and S_A^* some shifting operators acting on $C_c([e,y])$ defined as follows. We write $\{e = y_0, y_1, \dots, y_n = y\}$ for the set of vertices belonging to the geodesic segment $[e,y]$. Then, we set

$$S_A \delta_{y_k} = \delta_{y_{k-1}},$$

if $1 \leq k \leq n$ and if the reduced word y_k ends with the letter $x_j^{\pm 1}$, for some $j \in A$. We set $S_A \delta_{y_k} = 0$, otherwise. By computing the formal adjoint of S_A, it is natural to define :

$$S_A^* \delta_{y_k} = \delta_{y_{k+1}},$$

if $0 \leq k \leq n-1$ and if the reduced word y_{k+1} ends with the letter $x_j^{\pm 1}$, for some $j \in A$. We set $S_A^* \delta_{y_k} = 0$, otherwise.
The following crucial result is a slight generalization of a lemma due to Pytlik and Szwarc (see Lemma 1 in [PS86]). To familiarize ourself with contraction operators, we give a proof.

Lemma 5.3.3. *Let $q \in \mathbb{N} \cup \{\infty\}$ and let $\mathcal{T}_{2q}$ be the Cayley graph of the free group $\mathbb{F}_q$. Fix $A \subset A_q$. Let $\gamma \in \mathbb{F}_q$. Then:*

1. *On $C_c(\mathbb{F}_q \setminus [e,\gamma])$, the following holds*
$$\lambda(\gamma)P_A\lambda(\gamma^{-1}) = P_A;$$
2. *On $C_c([e,\gamma])$, the following holds*
$$P_A = S_A;$$
3. *On $C_c([e,\gamma])$, the following holds*
$$\lambda(\gamma)P_A\lambda(\gamma^{-1}) = S_A^*;$$
4. *For $f \in C_c(\mathbb{F}_q)$, the following holds*
$$(P_A - \lambda(\gamma)P_A\lambda(\gamma^{-1}))f = (S_A - S_A^*)(f \cdot \chi_{[e,\gamma]});$$
5. *Moreover, the operator $P_A - \lambda(\gamma)P_A\lambda(\gamma^{-1})$ is bounded on $\ell^2\mathbb{F}_q$ and its norm satisfies*
$$\|P_A - \lambda(\gamma)P_A\lambda(\gamma^{-1})\| \leq 2.$$

Proof : For *1*, let $u \in \mathbb{F}_q \setminus [e,\gamma]$. Then, we can find reduced words g, c and r such that $\gamma = gc$ and $u = gr$, with $|r| \geq 1$. This implies $\gamma^{-1}u = c^{-1}r$ and it ends with the letter $x_j^{\pm 1}$, with $j \in A$, if and only if the word u does. Morever, since $\gamma\ (\overline{c^{-1}r}) = \overline{u}$, we obtain the equality $\lambda(\gamma)P_A\lambda(\gamma^{-1})\delta_u = P_A\delta_u$. Let $u \in [e,\gamma]$. The second point being obvious, let us check *3*. Assume that the geodesic segment $[e,\gamma]$ is described by
$$\{\gamma_0 = y_0 = e, \gamma_1 = y_1, \gamma_2 = y_1y_2, \dots, \gamma_n = y_1y_2\dots y_n\},$$
where the words γ_k are all reduced and y_j are generators. Now, let $u \in [e,\gamma]$. If $u = \gamma$, then we clearly have $\lambda(\gamma)P_A\lambda(\gamma^{-1})\delta_u = S_A^*\delta_u = 0$. Therefore, we turn to the case where $u = \gamma_k$, for $0 \leq k \leq n-1$. Again, we can find a reduced word r such that $\gamma = \gamma_k r$, with $|r| \geq 1$. This implies that $\gamma^{-1}u = r^{-1}$ and that it ends with the letter y_{k+1}^{-1}. In particular, $\overline{\gamma^{-1}u} = y_n^{-1}\dots y_{k+1}^{-1}$. Thus, if $y_{k+1} = x_j^{\pm 1}$, for some $j \in A$, we obtain
$$\begin{aligned}\lambda(\gamma)P_A\lambda(\gamma^{-1})\delta_{\gamma_k} &= \lambda(\gamma)P_A\delta_{\gamma^{-1}u}\\ &= \delta_{\gamma(\overline{\gamma^{-1}u})}\\ &= \delta_{\gamma_{k+1}}.\end{aligned}$$
which coincides with $S_A^*\delta_u$. Similarly, if $\gamma^{-1}u$ ends with the letter y_{k+1}^{-1} and $y_{k+1} = x_j^{\pm 1}$ is such that j is not in A, then δ_u belongs to the kernel of both

P_A and $\lambda(\gamma)P_A\lambda(\gamma^{-1})$. This finishes the proof of *3*.
The proof of *4* is a straightforward corollary of the previous points.
Finally, since S_A and S_A^* are shifting operators, and, as such, they are of norm at most 1, the item *5* is a consequence of the triangle inequality applied to *4*. □

The next lemma shows that the contraction operators defined on trees with infinite degree give rise to non-inner derivations.

Lemma 5.3.4. *Let $A \subset A_\infty$ and consider P_A the contraction operator acting on $C_c(\mathbb{F}_\infty)$.*

1. *The map $\partial_{P_A} : \mathbb{F}_\infty \to \mathcal{B}(\ell^2\mathbb{F}_\infty)$ defined by*
$$\gamma \mapsto \partial_{P_A}(\gamma) = [\lambda(\gamma), P_A],$$
is a bounded derivation for the left regular representation λ, that is, $\partial_{P_A} \in D^\infty(\mathbb{F}_\infty, \lambda)$.

2. *The derivation ∂_{P_A} is inner if and only if A is finite.*

3. *Let B be a subset of A_∞. Then, the derivations ∂_{P_A} and ∂_{P_B} are equivalent in $H^1_d(\mathbb{F}_\infty, \lambda)$ if and only if the symmetric difference $A \triangle B$ is finite.*

Proof : The first claim is a direct consequence of Lemma 5.3.3. Moreover, if A is finite, then the derivation ∂_{P_A} is inner, since P_A is bounded. To end the proof of the second claim, we need to show that, if A is infinite, then the derivation ∂_{P_A} is not inner. Indeed, we need to check that ∂_{P_A} is not in the image of $\mathcal{B}(\ell^2\mathbb{F}_\infty)$ via the coboundary map ∂. By contradiction, suppose there exists $T \in \mathcal{B}(\ell^2\mathbb{F}_\infty)$ such that $\partial_{P_A} = \partial_T$. This implies that the characteristic function $1_{A\cup A^{-1}}$ is a coefficient of some unitary representation of $\mathbb{F}_\infty$, which is a contradiction, by Lemma 2.4. of [Pis01]. For the reader's convenience, we recall the argument here[5]. By Proposition 5.1.13, if ∂_{P_A} is an inner derivation, we deduce that the representation $\lambda_{\partial_{P_A}}$ is equivalent to the unitary representation $\lambda \oplus \lambda$ and that the operator
$$\tilde{T} = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \in \mathcal{B}(\ell^2\mathbb{F}_\infty \oplus \ell^2\mathbb{F}_\infty),$$

[5]Note that we made a slight abuse of notations. Here, we see $A \cup A^{-1}$ as the subset of $\mathbb{F}_\infty$ consisting of the generators $x_j^{\pm 1}$ for which $j \in A$.

is an intwertiner. On the one hand, taking $x = (0, -\delta_e)$ and $y = (\delta_e, 0)$, we get

$$\begin{aligned} \langle \lambda_{\partial_{P_A}}(\gamma)x, y\rangle &= -\langle \partial_{P_A}(\gamma)\delta_e, \delta_e\rangle \\ &= \langle P_A\delta_\gamma, \delta_e\rangle \\ &= 1_{A\cup A^{-1}}(\gamma), \end{aligned}$$

for any $\gamma \in \mathbb{F}_\infty$. On the other hand, we have

$$\begin{aligned} \langle \lambda_{\partial_{P_A}}(\gamma)x, y\rangle &= \langle \tilde{T}^{-1}(\lambda\oplus\lambda)(\gamma)\tilde{T}x, y\rangle \\ &= \left\langle (\lambda\oplus\lambda)(\gamma)\tilde{T}x, \left(\tilde{T}^{-1}\right)^* y\right\rangle. \end{aligned}$$

This shows that $1_{A\cup A^{-1}}$ is a coefficient of a unitary representation of $\mathbb{F}_\infty$. To get the desired contradiction, for all $j \in A$, set

$$a_j = \Re((\lambda\oplus\lambda)(x_j)) = \frac{1}{2}\left((\lambda\oplus\lambda)(x_j) + (\lambda\oplus\lambda)(x_j)^*\right).$$

Let us enumerate the elements of A as $A = \{j_1, j_2, \dots\}$ so that we can define the following operator for all n:

$$R_n = \prod_{k=1}^{n}(1 + \frac{i}{\sqrt{n}}a_{j_k}).$$

Since the operators a_j are of norm at most 1 for all j, we get

$$\begin{aligned} \left\|1 + \frac{i}{\sqrt{n}}a_j\right\|^2 &= \left\|\left(1 + \frac{i}{\sqrt{n}}a_j\right)^*\left(1 + \frac{i}{\sqrt{n}}a_j\right)\right\| \\ &= \left\|1 + \frac{a_j^2}{n}\right\| \\ &\le 1 + \frac{1}{n}, \end{aligned}$$

and therefore

$$\begin{aligned} \|R_n\|^2 &\le \prod_{k=1}^{n}\left\|1 + \frac{i}{\sqrt{n}}a_{j_k}\right\|^2 \\ &\le \left(1 + \frac{1}{n}\right)^n \le e. \end{aligned}$$

We deduce that the sequence $(R_n)_n$ is uniformly bounded in norm. We observe that for all n, there exists ψ_n, a finitely supported function on $\mathbb{F}_\infty$, such that

$$R_n = \left(\frac{i}{\sqrt{n}} \sum_{k=1}^{n} a_{j_k} \right) + \left(\sum_{|\gamma| \neq 1} \psi_n(\gamma)(\lambda \oplus \lambda)(\gamma) \right).$$

Moreover, for the vectors $\tilde{x} = \tilde{T}x$ and $\tilde{y} = \left(\tilde{T}^{-1}\right)^* y$, it is easy to compute the scalar-product

$$\langle R_n \tilde{x}, \tilde{y} \rangle = i\sqrt{n},$$

by using the expression of R_n here above and the fact that :

$$\langle a_j \tilde{x}, \tilde{y} \rangle = 1.$$

Hence, the following inequality gives a contradiction

$$\begin{aligned} \sqrt{n} &= |\langle R_n \tilde{x}, \tilde{y} \rangle| \\ &\leq e\|\tilde{x}\|\|\tilde{y}\| \end{aligned}$$

as n tends to infinity. This shows that $1_{A \cup A^{-1}}$ is not a coefficient of some unitary representation of $\mathbb{F}_\infty$, and, as a corollary, a bounded operator T such that $\partial_{P_A} = \partial_T$ cannot exist. This ends the proof of the second claim.
To prove the third claim, we decompose $A \triangle B$ as $A_1 \sqcup B_1$, where $A_1 = A \setminus (A \cap B)$ and $B_1 = B \setminus (A \cap B)$. Since $\partial_{P_A} - \partial_{P_B} = \partial_{P_{A_1} - P_{B_1}}$, the derivation $\partial_{P_A} - \partial_{P_B}$ is inner if and only if both A_1 and B_1 are finite. □

Proposition 5.3.5. *The vector space $H^1_d(\mathbb{F}_\infty, \lambda)$ is infinite dimensionial.*

Proof : Let N be a positive integer. For $k \in \{0, 1, \ldots, N-1\}$, write A_k for the sets of non-negative integers which are congruent to k modulo N. Then, it is easy to see that the family of bounded derivations $(\partial_{P_{A_k}})_{k=0}^{N-1}$ is linearly independent. For instance, $\partial_{P_{A_0}}$ cannot be a linear combination of the other derivations, since δ_{x_N} belongs to the kernel of $\partial_{P_{A_k}}(x_N^{-1})$, for every $k \geq 1$ and $\partial_{P_{A_0}}(x_N^{-1})\delta_{x_N} = \delta_{x_N^{-1}}$. □

As a corollary, we obtain the following well-known result.

Corollary 5.3.6. *The free group $\mathbb{F}_\infty$ admits infinitely many uniformly bounded representations which are not unitarisable. In particular, any non-abelian free group is not unitarisable.*

Proof : Consider the derivation ∂_{P_A}, where $A \subset A_\infty$ is an infinite set. By Lemma 5.3.4, the derivation ∂_{P_A} is not inner. By Proposition 5.1.13, the representation $\lambda_{\partial_{P_A}}$ is uniformly bounded and not unitarisable.
We can conclude the proof of the corollary by seeing $\mathbb{F}_\infty$ as a subgroup of $\mathbb{F}_2$ and using Proposition 5.1.2. □

We would like to stress that we really need to use Proposition 5.1.2 to conclude the proof here above. It would be natural to consider the restriction to $\mathbb{F}_2$ of the given non-unitarisable representation λ_{∂_P} of $\mathbb{F}_\infty$. However, as we will see in Subsection 5.3.3, this restriction is unitarisable.
To this point, we only considered contractions with respect to the identity e as basepoint. It is possible to change of basepoint to obtain other contractions. Fix $x_0 \in \mathbb{F}_\infty$. Now, for any $u \in \mathbb{F}_\infty$, define $r(u)$ to be the unique element of $\mathbb{F}_\infty$ which belongs to the geodesic $[x_0, u]$ in $\mathcal{T}_\infty$ and that is adjacent to u. For instance, in the case $x_0 = e$, we get $r(u) = \overline{u}$, for any u. We denote by P_{x_0} the operator acting on $C_c(\mathbb{F}_\infty)$ by

$$P_{x_0}\delta_u = \begin{cases} \delta_{r(u)}, & \text{if } u \neq x_0 \\ 0, & \text{otherwise,} \end{cases}$$

It is easy to check that P_{x_0} is unbounded on $\ell^2\mathbb{F}_\infty$ and that it satisfies

$$P_{x_0} = \lambda(x_0)P\lambda(x_0^{-1}).$$

In particular, the commutator $[\lambda(\gamma), P_{x_0}]$ is bounded in norm, independently of γ. This implies that $\partial_{P_{x_0}}$ represents a bounded derivation with respect to λ. However, the operator $P - P_{x_0}$ being bounded, we deduce that the derivations ∂_P and $\partial_{P_{x_0}}$ represent the same class in $H^1_d(\mathbb{F}_\infty, \lambda)$. Hence, we proved :

Proposition 5.3.7. *For any $x, y \in \mathbb{F}_\infty$, the derivations ∂_{P_x} and ∂_{P_y} are equivalent in $H^1_d(\mathbb{F}_\infty, \lambda)$.*

5.3.2 Uniform almost intertwiners for free groups

In this subsection, we will show that the left regular representation of $\mathbb{F}_\infty$ admits uniform almost intertwiners. This will imply that the space of inner derivations $I(\mathbb{F}_\infty, \lambda)$ is not closed inside $D^\infty(\mathbb{F}_\infty, \lambda)$ for the natural topology introduced in Section 5.2.

Proposition 5.3.8. *The left regular representation λ of $\mathbb{F}_\infty = \langle x_1, x_2, \dots \rangle$ admits uniform almost intertwiners.*
More precisely, for every k, let $A_k = \{1, 2, \dots, k\}$. Then, the sequence of contraction operators $(P_{A_k})_{k\geq 1}$ satisfies :

1. $\|[\lambda(\gamma), P_{A_k}]\| \leq 2$, *for all* $\gamma \in \mathbb{F}_\infty$;
2. $\|P_{A_k}\|_q = \Theta(\sqrt{k})$.

As a corollary, we obtain :

Corollary 5.3.9. *The left regular representation of $\mathbb{F}_2$ admits uniform almost intertwiners.*

Proof of Corollary 5.3.9 using Proposition 5.3.8: Firstly, we see $\mathbb{F}_\infty$ as a subgroup of $\mathbb{F}_2$. For instance, take a and b two generators for $\mathbb{F}_2$. Then, the set $\{a^n b^n : n > 0\}$ generates the free group $\mathbb{F}_\infty$. Since $\lambda_{\mathbb{F}_\infty}$ admits uniform almost intertwiners, then the induced representation $\mathrm{Ind}_{\mathbb{F}_\infty}^{\mathbb{F}_2} \lambda_{\mathbb{F}_\infty}$ also admits uniform almost intertwiners, by Proposition 5.2.6. It is well-known that it coincides with the left regular representation of the full group, which, in this case, is $\lambda_{\mathbb{F}_2}$. This concludes the proof. □

To prove that the quotient semi-norm of the operators P_{A_k} tend to infinity with k, we need the following lemma.

Lemma 5.3.10. *Let $k \geq 2$. Set the two functions*

$$h_1 = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \delta_{x_j} \text{ and } h_2 = \delta_e.$$

Then, for all $f \in C_c(\mathbb{F}_\infty)$, we have

$$\max_{l=1,2} \|(P_{A_k} + \rho(f))h_l\|_2 \geq \frac{\sqrt{k}}{2}.$$

Proof : Let f be a finitely supported function on $\mathbb{F}_\infty$. Firstly, h_1 and h_2 both have norm 1. Secondly, it is easy to see that

$$P_{A_k} h_1 = \sqrt{k}\delta_e \text{ and } P_{A_k} h_2 = 0.$$

Computing $\rho(f)h_1$, we get:

$$\begin{aligned}
\rho(f)h_1 &= \sum_{x\in\mathbb{F}_\infty}\sum_{j=1}^{k}\frac{f(x)}{\sqrt{k}}\rho(x)\delta_{x_j}\\
&= \sum_{x\in\mathbb{F}_\infty}\sum_{j=1}^{k}\frac{f(x)}{\sqrt{k}}\delta_{x_jx^{-1}}\\
&= \sum_{u\in\mathbb{F}_\infty}\left(\sum_{j=1}^{k}\frac{f(u^{-1}x_j)}{\sqrt{k}}\right)\delta_u.
\end{aligned}$$

Let us treat two cases. If we suppose that f satisfies

$$\left|\sum_{j=1}^{k}\frac{f(x_j)}{\sqrt{k}}\right| \leq \frac{\sqrt{k}}{2},$$

we obtain :

$$\begin{aligned}
\|(P_{A_k}+\rho(f))h_1\|_2 &\geq |\langle(P_{A_k}+\rho(f))h_1,\delta_e\rangle|\\
&= \left|\sqrt{k}+\frac{1}{\sqrt{k}}\sum_{j=1}^{k}f(x_j)\right|\\
&\geq \left|\sqrt{k}-\left|\frac{1}{\sqrt{k}}\sum_{j=1}^{k}f(x_j)\right|\right|\\
&\geq \frac{\sqrt{k}}{2},
\end{aligned}$$

which concludes the first case. Now, let us suppose that

$$\left|\sum_{j=1}^{k}\frac{f(x_j)}{\sqrt{k}}\right| > \frac{\sqrt{k}}{2}.$$

Using Cauchy-Schwarz inequality, we have:

$$\begin{aligned}
\left|\frac{1}{\sqrt{k}}\sum_{j=1}^{k}f(x_j)\right| &= |\langle f,h_1\rangle|\\
&\leq \|f\|_2.
\end{aligned}$$

which gives us the lower bound $\frac{\sqrt{k}}{2} < \|f\|_2$. Hence, we get

$$\begin{aligned}
\|(P_{A_k}+\rho(f))h_2\|_2 &= \|\rho(f)\delta_e\|_2\\
&= \|f\|_2\\
&\geq \frac{\sqrt{k}}{2},
\end{aligned}$$

which ends the proof of Lemma 5.3.10. □

Proof of Proposition 5.3.8 : For the proof of *1*, see Lemma 5.3.3 and *3* follows from Lemma 5.3.2. Let us show the second point. Let $k \geq 2$. We can find T_k in the commutant of the left regular representation $\lambda(\mathbb{F}_\infty)'$ satisfying

$$\|P_{A_k}\|_q \geq \|P_{A_k} - T_k\| - \frac{1}{2}.$$

Since the commutant of λ coincides with the von Neumann algebra generated by the right regular representation ρ, and since $R(\mathbb{F}_\infty)$ is the closure of the linear span of $\rho(\mathbb{F}_\infty)$ in $\mathcal{B}(\ell^2\mathbb{F}_\infty)$ for the strong operator topology, there exists $f_k \in C_c(\mathbb{F}_\infty)$ so that

$$\max_{l=1,2} \|(T_k - \rho(f_k))h_l\|_2 < \frac{1}{2},$$

where $h_1 = \frac{1}{\sqrt{k}} \sum_{j=1}^k \delta_{x_j}$ and $h_2 = \delta_e$. We obtain:

$$\begin{aligned} \|P_{A_k}\|_q &\geq \|P_{A_k} - T_k\| - \frac{1}{2} \\ &\geq \max_{l=1,2} \|(P_{A_k} - T_k)h_l\|_2 - \frac{1}{2} \\ &\geq \max_{l=1,2} \|(P_{A_k} - \rho(f_k))h_l\|_2 - 1 \\ &\geq \frac{\sqrt{k}}{2} - 1. \end{aligned}$$

Hence, $\|P_{A_k}\|_q$ tends to infinity with k. □

Remark 5.3.11. *Clearly, the sequence of bounded derivations $(\partial_{A_k})_{k\geq 2}$, that appears in Proposition 5.3.8, converges to ∂_P in the following sense. For any $f \in C_c(\mathbb{F}_\infty)$ and any $\gamma \in \mathbb{F}_\infty$, there exists N so that*

$$\partial_{P_{A_k}}(\gamma)f = \partial_P(\gamma)f,$$

for all $k \geq N$. However, we observe that

$$\sup_{\gamma\in\mathbb{F}_\infty} \|\partial_P(\gamma) - \partial_{P_{A_k}}(\gamma)\| \geq 1,$$

for all k. Indeed, for $k \geq 2$, take any generator x_j such that $j \in A \setminus A_k$. Then, we clearly have $(\partial_{P_{A_k}}(x_j) - \partial_P(x_j))\delta_e = \delta_e$. Therefore, this implies that ∂_P is not an adherent point of the sequence $(\partial_{P_{A_k}})_{k\geq 2}$ in the space $D^\infty(\mathbb{F}_\infty, \lambda)$ for the topology we are considering.

5.3.3 A remark on the powers of the contraction operator on locally finite trees

The goal of the present subsection is to convice the reader that it is relatively difficult to find explicit examples of non-inner bounded derivations on finitely generated groups. In particular, it would be interesting to find concrete examples of non-trivial bounded derivations on $\mathcal{B}(\ell^2\mathbb{F}_r)$, when r is finite. We consider two natural constructions and show that they do not lead to the desired result.
It is clear that the property of being non-unitarisable does not pass to subgroup (think of $\mathbb{Z}$ sitting inside $\mathbb{F}_\infty$). However, one can ask if the restriction of a uniformly bounded, non-unitarisable representation to a non-amenable subgroup is still non-unitarisable. Here is an example showing that it is not always the case.

Proposition 5.3.12. *Let P be the contraction operator defined on $\mathcal{T}_\infty$. Let λ_{∂_P} be the associated uniformly bounded and non-unitarisable representation of $\mathbb{F}_\infty = \langle x_1, x_2, \dots \rangle$. Let $S \subset \mathbb{N}$ be any finite set. Then, the restriction of λ_b to the subgroup $\mathbb{F}_S$, generated by $\{x_s : s \in S\}$, is unitarisable.*
In particular, the restriction of the derivation ∂_P to the subgroup $\mathbb{F}_S$ is inner.

Proof : It is not difficult to see that for $\gamma \in \mathbb{F}_S$:

$$[\lambda(\gamma), P] = [\lambda(\gamma), P_S].$$

In particular, this shows that the restriction of the derivation ∂_P to the subgroup $\mathbb{F}_S$ is inner, since P_S is bounded. It also implies that the restriction of the representation λ_b to the subgroup $\mathbb{F}_S$ is unitarisable. $\square$

Let us fix a finite $r \geq 2$ and consider the free group $\mathbb{F}_r = \langle x_1, x_2, \dots, x_r \rangle$. In view of Lemma 5.3.3, it is natural to ask whether $(P^n)_{n\geq 1}$, where P is the contraction operator defined on the $(2r)$-regular Cayley tree of $\mathbb{F}_r$, forms a sequence of uniform almost intertwiners for the left regular representation of $\mathbb{F}_r$. We will see that it is not the case. More precisely, we will prove :

Proposition 5.3.13. *For all $n \geq 1$, we have :*

$$\frac{1}{\|P^n\|_q} \sup_{\gamma \in \mathbb{F}_r} \|[\lambda(\gamma), P^n]\| \geq \frac{1}{\sqrt{2r}}.$$

Proposition 5.3.13 will directly follow from the next lemma.

Lemma 5.3.14. *For $n \geq 1$, we have :*

$$(2r-1)^{\frac{n-1}{2}} \leq \sup_{\gamma \in \mathbb{F}_r} \|[\lambda(\gamma), P^n]\| \leq 4\sqrt{r}\ (2r-1)^{\frac{n}{2}}.$$

Proof : By Lemma 5.3.2, we easily get the upper bound. Indeed, let $\gamma \in \mathbb{F}_r$. We obtain:

$$\begin{aligned} \|[\lambda(\gamma), P^n]\| &\leq 2\|P^n\| \\ &= 2(2r)^{\frac{1}{2}}(2r-1)^{\frac{n}{2}}. \end{aligned}$$

Let us show the lower bound. Let $n \geq 1$ and $r \geq 2$ be fixed. We will write W for the set of words $v \in \mathbb{F}_r$ which do not start with the letter x_1^{-1} and which are of length $n-1$. It is clear that $P^n 1_W = 0$ and $|W| = (2r-1)^{n-1}$, so that $\|1_W\|_2 = \sqrt{|W|} = (2r-1)^{\frac{n-1}{2}}$. Moreover, it is easy to see that

$$\lambda(x_1^{-1})P^n\lambda(x_1)1_W = |W|\delta_{x_1^{-1}}.$$

Therefore, we have

$$\|[\lambda(x_1), P^n]\| \geq \sqrt{|W|},$$

which proves the lemma. □

Proof of Proposition 5.3.13 :

Since $\|P^n\|_q \leq \|P^n\|$, we immediately get

$$\begin{aligned} \frac{1}{\|P^n\|_q} \sup_{\gamma \in \mathbb{F}_r} \|[\lambda(\gamma), P^n]\| &\geq \frac{(2r-1)^{\frac{n-1}{2}}}{\|P^n\|} \\ &= \frac{1}{\sqrt{2q}}. \end{aligned}$$

□

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