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On a Question of Dusa McDuff

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1 Introduction

We endow Euclidean space $\mathbb{R}^{2n}$ with the standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i. \tag{1.1}$$

A C^∞-smooth embedding φ of an open subset U of $\mathbb{R}^{2n}$ into $\mathbb{R}^{2n}$ is called *symplectic* if $\varphi^*\omega_0 = \omega_0$. An embedding of an arbitrary subset S of $\mathbb{R}^{2n}$ into another subset S' of $\mathbb{R}^{2n}$ is called symplectic if it extends to a symplectic embedding of a neighbourhood of S into $\mathbb{R}^{2n}$. We denote by $B^{2n}(\pi r^2)$ the closed $2n$-dimensional ball of radius r, and by $Z^{2n}(\pi)$ the closed $2n$-dimensional symplectic cylinder

$$Z^{2n}(\pi) = B^2(\pi) \times \mathbb{R}^{2n-2}. \tag{1.2}$$

The following theorem is one of the most fundamental results in symplectic topology.

Nonsqueezing theorem (Gromov [3]). *The ball $B^{2n}(a)$ symplectically embeds into the cylinder $Z^{2n}(\pi)$ only if $a \le \pi$.* □

Note that a symplectic embedding preserves the Euclidean volume form $(1/n!)\omega_0^n$, and that for any $a > 0$ there exists a volume preserving embedding of $B^{2n}(a)$ into $Z^{2n}(\pi)$. Gromov's nonsqueezing theorem, therefore, demonstrates that the symplectic structure of a map is much more rigid than the volume preserving structure. It implies that the

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group of symplectic diffeomorphisms of $\mathbb{R}^{2n}$ is C^0-closed in the group of all diffeomorphisms of $\mathbb{R}^{2n}$, see [5, Chapter 2.2] or [8, Chapter 12.2].

In view of the nonsqueezing theorem, we fix $a \in \,]0, \pi]$. We recall that the simply connected hull $\widehat{T}$ of a subset T of $\mathbb{R}^2$ is the union of its closure $\overline{T}$ and the bounded components of $\mathbb{R}^2 \setminus \overline{T}$. We denote by μ the Lebesgue measure on $\mathbb{R}^2$, and we set $\widehat{\mu}(T) = \mu(\widehat{T})$. It is well known that the nonsqueezing theorem is equivalent to each of the identities

$$\begin{aligned} a &= \inf_{\varphi} \mu\big(p\big(\varphi\big(B^{2n}(a)\big)\big)\big), \\ a &= \inf_{\varphi} \widehat{\mu}\big(p\big(\varphi\big(B^{2n}(a)\big)\big)\big), \end{aligned} \tag{1.3}$$

where φ varies over all symplectic embeddings of $B^{2n}(a)$ into $Z^{2n}(\pi)$ and where $p : Z^{2n}(\pi) \to B^2(\pi)$ is the projection, see [2] and [10, Corollary B.10]. Following [7, Section 3], we consider sections of the image $\varphi(B^{2n}(a))$ instead of its projection, and define

$$\begin{aligned} \sigma(a) &= \inf_{\varphi} \sup_{x} \mu\big(p\big(\varphi\big(B^{2n}(a)\big) \cap D_x\big)\big), \\ \widehat{\sigma}(a) &= \inf_{\varphi} \sup_{x} \widehat{\mu}\big(p\big(\varphi\big(B^{2n}(a)\big) \cap D_x\big)\big), \end{aligned} \tag{1.4}$$

where φ again varies over all symplectic embeddings of $B^{2n}(a)$ into $Z^{2n}(\pi)$, and where $D_x \subset Z^{2n}(\pi)$ denotes the disc $D_x = B^2(\pi) \times \{x\}$, $x \in \mathbb{R}^{2n-2}$. Clearly,

$$\sigma(a) \le \widehat{\sigma}(a) \le a. \tag{1.5}$$

It is also well known that the nonsqueezing theorem is equivalent to the identity

$$\widehat{\sigma}(\pi) = \pi. \tag{1.6}$$

Indeed, the nonsqueezing theorem implies that for every symplectic embedding φ of $B^{2n}(\pi)$ into $Z^{2n}(\pi)$, there exists $x \in \mathbb{R}^{2n-2}$ such that $\varphi(B^{2n}(\pi)) \cap D_x$ contains the unit circle $S^1 \times \{x\}$, see [6, Lemma 1.2]. Therefore, on her search for symplectic rigidity phenomena beyond the nonsqueezing theorem, McDuff asked in [7] for lower bounds of the function $\sigma(a)$, and whether $\sigma(a) \to \pi$ as $a \to \pi$. It was known to Polterovich that $\sigma(a)/a \to 0$ as $a \to 0$, see again [7]. We will prove the following theorem.

Theorem 1.1. For the functions $\sigma(a)$ and $\widehat{\sigma}(a)$ it holds true that

(i) $\sigma(a) = 0$ for all $a \in \,]0, \pi]$;
(ii) $\widehat{\sigma}(a) = 0$ for all $a \in \,]0, \pi[$. □

The symplectic embeddings in the definitions of $\sigma(a)$ and $\widehat{\sigma}(a)$ were not further specified. Following a suggestion of Polterovich, we next ask whether the vanishing phenomenon, described by Theorem 1.1, persists if we restrict ourselves to symplectic embeddings which are close to the identity mapping in a symplectically relevant sense. We denote by $\mathcal{H}_c(2n)$ the set of smooth functions $H : \mathbb{R}^{2n} \to \mathbb{R}$ whose support is a compact subset of $Z^{2n}(\pi)$. For $H \in \mathcal{H}_c(2n)$, we define the Hamiltonian vector field X_H through the identities

$$\omega_0(X_H(z), \cdot) = dH(z), \quad z \in \mathbb{R}^{2n}, \tag{1.7}$$

and denote by ϕ_H the time-1-map of the flow generated by X_H. Moreover, we set

$$\|H\| = \sup_{z \in \mathbb{R}^{2n}} H(z) - \inf_{z \in \mathbb{R}^{2n}} H(z). \tag{1.8}$$

For each $a \in]0, \pi]$ we define

$$\begin{aligned} \sigma_H(a) &= \inf_H \left\{ \sup_x \mu\big(p\big(\phi_H\big(B^{2n}(a)\big) \cap D_x\big)\big) + \|H\| \right\}, \\ \widehat{\sigma}_H(a) &= \inf_H \left\{ \sup_x \widehat{\mu}\big(p\big(\phi_H\big(B^{2n}(a)\big) \cap D_x\big)\big) + \|H\| \right\}, \end{aligned} \tag{1.9}$$

where H varies over $\mathcal{H}_c(2n)$. Clearly, $\sigma(a) \le \sigma_H(a)$ and $\widehat{\sigma}(a) \le \widehat{\sigma}_H(a)$. In particular, $\widehat{\sigma}_H(\pi) = \pi$.

Theorem 1.2. For the functions $\sigma_H(a)$ and $\widehat{\sigma}_H(a)$ it holds true that

(i) $\sigma_H(a) = 0$ for all $a \in]0, \pi]$;
(ii) $\widehat{\sigma}_H(a) = 0$ for all $a \in]0, \pi[$. □

In order to see Theorem 1.2 in its right perspective we set

$$\mathrm{Ham}_c\left(Z^{2n}(\pi)\right) = \{\phi_H \mid H \in \mathcal{H}_c(2n)\}, \tag{1.10}$$

and define the energy $E(\phi)$ of $\phi \in \mathrm{Ham}_c(Z^{2n}(\pi))$ by

$$E(\phi) = \inf\{\|H\| \mid \phi = \phi_H \text{ for some } H \in \mathcal{H}_c(2n)\}. \tag{1.11}$$

In the framework of Hofer geometry, the energy of a Hamiltonian diffeomorphism is its distance from the identity mapping, see [5, 6, 9]. Notice that

$$\begin{aligned} \sigma_H(a) &= \inf_\phi \left\{ \sup_x \mu\big(p\big(\phi\big(B^{2n}(a)\big) \cap D_x\big)\big) + E(\phi) \right\}, \\ \widehat{\sigma}_H(a) &= \inf_\phi \left\{ \sup_x \widehat{\mu}\big(p\big(\phi\big(B^{2n}(a)\big) \cap D_x\big)\big) + E(\phi) \right\}, \end{aligned} \tag{1.12}$$

where ϕ varies over $\mathrm{Ham}_c(Z^{2n}(\pi))$. Theorem 1.2, therefore, says that the vanishing phenomenon, described by Theorem 1.1, persists if we restrict ourselves to the Hamiltonian diffeomorphism of $Z^{2n}(\pi)$ whose Hofer distance to the identity mapping is arbitrarily small.

The proof of Theorem 1.1 is easy: we just have to slice the ball by planes $x_1 =$ const, and then translate the ith slice by i vertically in the y_2-direction by a symplectomorphism ϕ_i. The proof of Theorem 1.2 is analogous but much trickier since we need to control the behaviour of the maps ϕ_i on those parts of their compact supports that are not translated vertically.

The symplectic embedding methods developed in this paper will be further applied in [11]. Being elementary and precise, we hope that they will prove useful in other symplectic embedding problems as well.

2 Results

We start with stating a generalization of Theorem 1.1. We denote by $\overline{\mu}$ the outer Lebesgue measure on $\mathbb{R}^2$, and by $\widehat{\mu}(T) = \mu(\widehat{T})$ the Lebesgue measure of the simply connected hull of the subset T of $\mathbb{R}^2$. For each subset S of the cylinder $Z^{2n}(\pi)$, we define

$$\begin{aligned} \sigma(S) &= \inf_{\varphi} \sup_{x} \overline{\mu}\big(p\big(\varphi(S) \cap D_x\big)\big), \\ \widehat{\sigma}(S) &= \inf_{\varphi} \sup_{x} \widehat{\mu}\big(p\big(\varphi(S) \cap D_x\big)\big), \end{aligned} \tag{2.1}$$

where φ varies over all symplectic embeddings of S into $Z^{2n}(\pi)$. We denote by $Z^{2n}(a)$ the closed cylinder $B^2(a) \times \mathbb{R}^{2n}$.

Theorem 2.1. Consider a subset S of $Z^{2n}(\pi)$. Then

(i) $\sigma(S) = 0$;

(ii) $\widehat{\sigma}(S) = 0$ if $S \subset Z^{2n}(a)$ for some $a < \pi$. □

In view of identity (1.6) we have $\widehat{\sigma}(S) = \pi$ whenever S contains the ball $B^{2n}(\pi)$.

Question 2.2. Is it true that $\widehat{\sigma}(\mathrm{Int}\, B^{2n}(\pi)) = \pi$?

A slightly weaker version of Theorem 2.1 has been proved in [10] by using a symplectic folding method. The method used here is more elementary and can also be used to prove a generalization of Theorem 1.2. We denote by $\mathcal{H}(2n)$ the set of smooth and bounded functions $H : \mathbb{R}^{2n} \to \mathbb{R}$ whose support is contained in $Z^{2n}(\pi)$ and whose Hamiltonian vector field X_H generates a flow on $\mathbb{R}^{2n}$. Again, the time-1-map of this flow is then denoted by ϕ_H. Using notation (1.8) we define, for each subset S of $Z^{2n}(\pi)$,

$$\begin{aligned} \sigma_H(S) &= \inf_H \Big\{ \sup_x \overline{\mu}\big(p\big(\phi_H(S) \cap D_x\big)\big) + \|H\| \Big\}, \\ \widehat{\sigma}_H(S) &= \inf_H \Big\{ \sup_x \widehat{\mu}\big(p\big(\phi_H(S) \cap D_x\big)\big) + \|H\| \Big\}, \end{aligned} \tag{2.2}$$

where H varies over $\mathcal{H}_c(2n)$ if S is bounded, and over $\mathcal{H}(2n)$ if S is unbounded. In order to state the main result of this paper we need yet another definition.

Definition 2.3. A subset S of $Z^{2n}(\pi)$ is *partially bounded* if at least one of the coordinate functions $x_2, \dots, x_n, y_2, \dots, y_n$ is bounded on S.

Theorem 2.4. Consider a partially bounded subset S of $Z^{2n}(\pi)$. Then

(i) $\sigma_H(S) = 0$;

(ii) $\widehat{\sigma}_H(S) = 0$ if $S \subset Z^{2n}(a)$ for some $a < \pi$. □

Of course,

$$\sigma_H\big(Z^{2n}(\pi)\big) = \widehat{\sigma}_H\big(Z^{2n}(\pi)\big) = \sigma_H\big(\operatorname{Int} Z^{2n}(\pi)\big) = \widehat{\sigma}_H\big(\operatorname{Int} Z^{2n}(\pi)\big) = \pi. \tag{2.3}$$

Question 2.5. Is it true that $\sigma_H(Z^{2n}(a)) = \widehat{\sigma}_H(Z^{2n}(a)) = 0$ for all $a \in]0, \pi[$?

Theorems 2.1 and 2.4 are proved in Sections 3 and 4. In Section 5, we reformulate these theorems in the language of symplectic capacities.

3 Proof of Theorem 2.1

The main ingredient in the proof of Theorem 2.1 is a special embedding result in dimension 4. We will use coordinates $z = (u, v, x, y)$ on $(\mathbb{R}^4, du \wedge dv + dx \wedge dy)$. We denote by $E_{(x,y)} \subset \mathbb{R}^4$ the affine plane

$$E_{(x,y)} = \mathbb{R}^2 \times \big\{(x, y)\big\}, \tag{3.1}$$

and given any subset S of $\mathbb{R}^4$, we set

$$\begin{aligned} \overline{\mu}\big(S \cap E_{(x,y)}\big) &= \overline{\mu}\big(p\big(S \cap E_{(x,y)}\big)\big), \\ \widehat{\mu}\big(S \cap E_{(x,y)}\big) &= \widehat{\mu}\big(p\big(S \cap E_{(x,y)}\big)\big). \end{aligned} \tag{3.2}$$

Fix an integer $k \geq 2$. We set

$$\epsilon = \frac{\pi}{k}, \qquad \delta = \frac{\epsilon}{4k}, \tag{3.3}$$

and we define closed rectangles P, P', and Q in $\mathbb{R}^2(u,v)$ as

$$\begin{aligned} P &= [0,\pi] \times [0,1], \\ P' &= [\delta, \pi-\delta] \times [\delta, 1-\delta], \\ Q &= [3\delta, \pi-3\delta] \times [3\delta, 1-3\delta]. \end{aligned} \tag{3.4}$$

The support of a map $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$ is defined as

$$\operatorname{supp}\varphi = \overline{\{z \in \mathbb{R}^4 \mid \varphi(z) \neq z\}}. \tag{3.5}$$

Proposition 3.1. There exists a symplectomorphism φ of $\mathbb{R}^4$ such that $\operatorname{supp}\varphi \subset P' \times \mathbb{R}^2$ and such that for each $(x,y) \in \mathbb{R}^2$,

$$\mu\big(\varphi\big(P' \times \mathbb{R} \times [0,1]\big) \cap E_{(x,y)}\big) \le 2\epsilon, \tag{3.6}$$

$$\widehat{\mu}\big(\varphi\big(Q \times \mathbb{R} \times [0,1]\big) \cap E_{(x,y)}\big) \le 2\epsilon. \tag{3.7}$$

□

Proof. We define closed rectangles R, R', and R'' in $\mathbb{R}^2(u,v)$ as

$$\begin{aligned} R &= [0,\epsilon] \times [0,1], \\ R' &= [\delta, \epsilon-\delta] \times [\delta, 1-\delta], \\ R'' &= [2\delta, \epsilon-2\delta] \times [2\delta, 1-2\delta], \end{aligned} \tag{3.8}$$

and we define closed rectangular annuli A and A' in $\mathbb{R}^2(u,v)$ as

$$A = \overline{R \setminus R'}, \qquad A' = \overline{R' \setminus R''}. \tag{3.9}$$

Then, $R = A \cup A' \cup R''$, see Figure 3.1.

We choose smooth cut-off functions $f_1, f_2 : \mathbb{R} \to [0,1]$ such that

$$\begin{aligned} f_1(t) &= \begin{cases} 0, & t \notin [\delta, \epsilon-\delta], \\ 1, & t \in [2\delta, \epsilon-2\delta], \end{cases} \\ f_2(t) &= \begin{cases} 0, & t \notin [\delta, 1-\delta], \\ 1, & t \in [2\delta, 1-2\delta], \end{cases} \end{aligned} \tag{3.10}$$

and we define the smooth function $H : \mathbb{R}^4 \to \mathbb{R}$ by

$$H(u,v,x,y) = -f_1(u)f_2(v)(1+\epsilon)x. \tag{3.11}$$

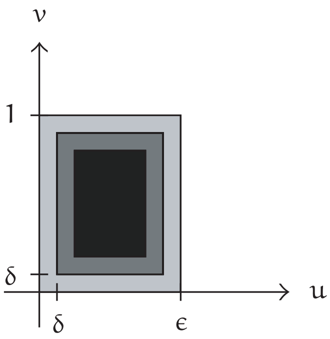

Figure 3.1 The decomposition $R = A \cup A' \cup R''$.

The Hamiltonian vector field X_H of H is given by

$$X_H(u, v, x, y) = (1+\epsilon)\begin{pmatrix} -f_1(u)f_2'(v)x \\ f_1'(u)f_2(v)x \\ 0 \\ f_1(u)f_2(v) \end{pmatrix}. \tag{3.12}$$

The time-1-map ϕ_H has the following properties:

(P1) $\operatorname{supp}\phi_H \subset R' \times \mathbb{R}^2$,
(P2) ϕ_H fixes $A \times \mathbb{R}^2$,
(P3) ϕ_H embeds $A' \times \mathbb{R}^2$ into $A' \times \mathbb{R}^2$,
(P4) ϕ_H translates $R'' \times \mathbb{R}^2$ by $(1+\epsilon)1_y$,

where we set $1_y = (0, 0, 0, 1)$.

For each subset T of $\mathbb{R}^2(u, v)$ and each $i \in \{1, \dots, k\}$, we define the translate T_i of T by

$$T_i = \big\{\big(u + (i-1)\epsilon, v\big) \mid (u, v) \in T\big\}. \tag{3.13}$$

With this notation we have

$$P = \bigcup_{i=1}^{k} R_i = \bigcup_{i=1}^{k} A_i \cup A_i' \cup R_i'', \tag{3.14}$$

see Figure 3.2. Set $H_i(u, v, x, y) = iH(u - (i-1)\epsilon, v, x, y)$. We define the smooth function

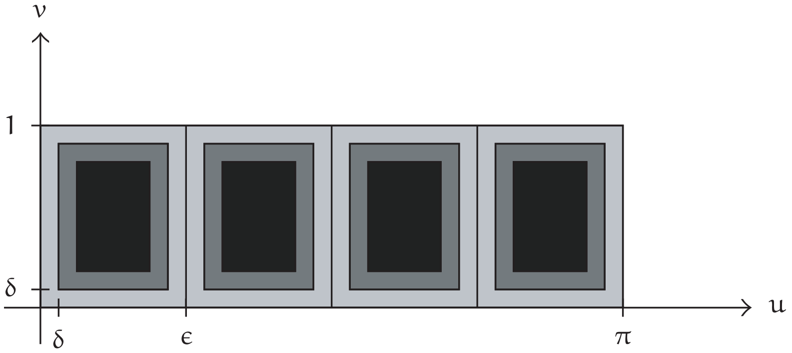

Figure 3.2 The decomposition $P = \bigcup_{i=1}^{k} R_i = \bigcup_{i=1}^{k} A_i \cup A_i' \cup R_i''$ for $k = 4$.

$\widetilde{H} : \mathbb{R}^4 \to \mathbb{R}$ by

$$\widetilde{H}(z) = \sum_{i=1}^{k} H_i(z), \tag{3.15}$$

and we define the symplectomorphism φ of $\mathbb{R}^4$ by $\varphi = \phi_{\widetilde{H}}$. In view of identity (3.12), we see that φ is of the form

$$\varphi(u, v, x, y) = (u', v', x, y'), \tag{3.16}$$

and in view of properties (P1)–(P4), we find that

$(\widetilde{\text{P1}})$ $\operatorname{supp} \varphi \subset P' \times \mathbb{R}^2$,
$(\widetilde{\text{P2}})$ φ fixes $\bigcup_{i=1}^{k} A_i \times \mathbb{R}^2$,
$(\widetilde{\text{P3}})$ φ embeds $A_i' \times \mathbb{R}^2$ into $A_i' \times \mathbb{R}^2$, $i = 1, \dots, k$,
$(\widetilde{\text{P4}})$ φ translates $R_i'' \times \mathbb{R}^2$ by $i(1+\epsilon)1_y$, $i = 1, \dots, k$,

see Figure 3.3.

3.1 Verification of estimates (3.6) and (3.7)

Fix $(x, y) \in \mathbb{R}^2$. We set

$$\begin{aligned} \mathcal{P}' &= p\big(\varphi\big(P' \times \mathbb{R} \times [0,1]\big) \cap E_{(x,y)}\big), \\ \mathcal{Q} &= p\big(\varphi\big(Q \times \mathbb{R} \times [0,1]\big) \cap E_{(x,y)}\big). \end{aligned} \tag{3.17}$$

Lemma 3.2. The estimate $\mu(\mathcal{P}') \leq 2\epsilon$ holds. □

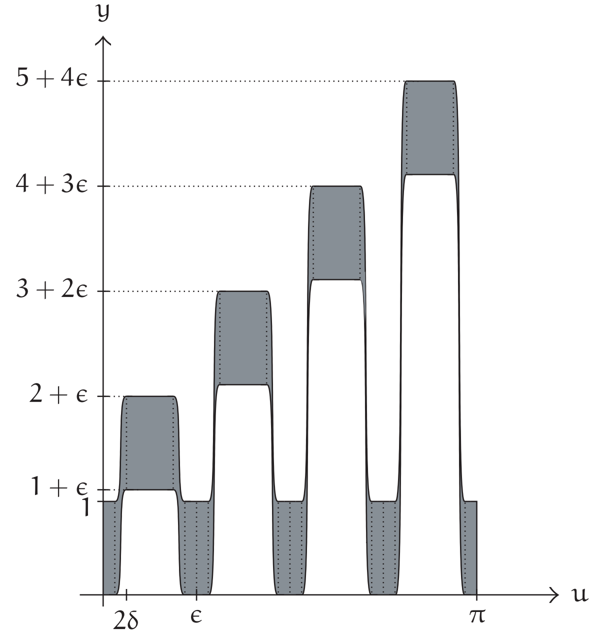

Figure 3.3 The intersection of $\varphi(P \times \mathbb{R} \times [0,1])$ with a plane $\{(u,v,x,y) \mid v, x \text{ constant}\}$ for $v \in [2\delta, 1-2\delta]$.

Proof. Using the definitions $\epsilon = \pi/k$ and $\delta = \epsilon/4k$ we estimate

$$\mu\big(A_i \cup A_i'\big) = \epsilon - (\epsilon - 4\delta)(1 - 4\delta) \le \frac{\epsilon}{k}, \quad i = 1, \dots, k. \tag{3.18}$$

Case 1 ($y \in [i^*(1+\epsilon), i^*(1+\epsilon)+1]$). According to properties $(\widetilde{P2})$–$(\widetilde{P4})$ we have $\mathcal{P}' \cap R_i'' = \varnothing$ if $i \neq i^*$, and so

$$\mathcal{P}' \subset R_{i^*} \cup \bigcup_{i=1}^{k} A_i \cup A_i'. \tag{3.19}$$

In view of estimate (3.18) we, therefore, find that

$$\mu(\mathcal{P}') \le \epsilon + k\frac{\epsilon}{k} = 2\epsilon. \tag{3.20}$$

Case 2 ($y \notin \bigcup_{i=1}^{k} [i(1+\epsilon), i(1+\epsilon)+1]$). According to properties $(\widetilde{P2})$–$(\widetilde{P4})$ we have $\mathcal{P}' \cap R_i'' = \varnothing$ for all i, and so

$$\mathcal{P}' \subset \bigcup_{i=1}^{k} A_i \cup A_i'. \tag{3.21}$$

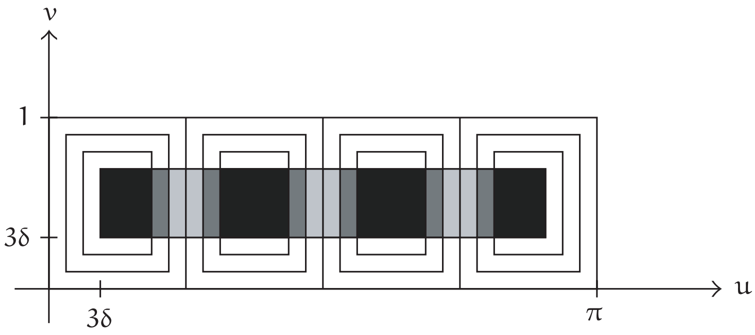

Figure 3.4 The subsets $\mathcal{A}_i, \mathcal{A}'_i$, and $\mathcal{R}''_i$ of Q, $i = 1, \ldots, 4$.

Therefore,

$$\mu(\mathcal{P}') \le \epsilon. \tag{3.22}$$

Estimates (3.20) and (3.22) yield that $\mu(\mathcal{P}') \le 2\epsilon$. ■

Lemma 3.3. The estimate $\widehat{\mu}(\mathcal{Q}) \le 2\epsilon$ holds. □

Proof. In view of the special form (3.16) of the map φ, we have

$$\mathcal{Q} = p\big(\varphi\big(Q \times \{x\} \times [0,1]\big) \cap E_{(x,y)}\big). \tag{3.23}$$

For $i = 1, \ldots, k$ we consider the intersections

$$\mathcal{A}_i = Q \cap A_i, \qquad \mathcal{A}'_i = Q \cap A'_i, \qquad \mathcal{R}''_i = Q \cap R''_i. \tag{3.24}$$

Each of the sets $\mathcal{A}_i$ and $\mathcal{A}'_i$ consists of one closed rectangle if $i \in \{1, k\}$, and of two closed rectangles if $i \in \{2, \ldots, k-1\}$, see Figure 3.4. The crucial observation in the proof is that for each i the simply connected hull of the part

$$p\big(\varphi\big(\mathcal{A}'_i \times \{x\} \times [0,1]\big) \cap E_{(x,y)}\big) \tag{3.25}$$

of $\mathcal{Q}$ is a simply connected subset of A'_i. Indeed, according to property $(\widetilde{\text{P3}})$, the closed and simply connected set $\varphi(\mathcal{A}'_i \times \{x\} \times [0,1])$ is contained in $A'_i \times \{x\} \times \mathbb{R}$, and so, the simply connected hull of $\varphi(\mathcal{A}'_i \times \{x\} \times [0,1]) \cap E_{(x,y)}$ is a simply connected subset of $A'_i \times \{(x,y)\}$.

We denote by $\widehat{\mathcal{Q}}$ the simply connected hull of $\mathcal{Q}$.

Case 1 ($y \in [0,1]$). According to properties $(\widetilde{P2})$–$(\widetilde{P4})$, we have $\mathcal{Q} \cap A_i = \mathcal{A}_i$ and $\mathcal{Q} \cap R_i'' = \varnothing$ for all i. In view of the above observation we conclude that

$$\widehat{\mathcal{Q}} \subset \bigcup_{i=1}^{k} \mathcal{A}_i \cup A_i'. \tag{3.26}$$

In view of estimate (3.18) we, therefore, find that

$$\mu(\widehat{\mathcal{Q}}) \leq k\frac{\epsilon}{k} = \epsilon. \tag{3.27}$$

Case 2 ($y \in [i^*(1+\epsilon), i^*(1+\epsilon)+1]$). According to properties $(\widetilde{P2})$–$(\widetilde{P4})$, we have $\mathcal{Q} \cap A_i = \varnothing$ for all i and $\mathcal{Q} \cap R_i'' = \varnothing$ if $i \neq i^*$. In view of the above observation we conclude that

$$\widehat{\mathcal{Q}} \subset R_{i^*} \cup \bigcup_{i=1}^{k} A_i'. \tag{3.28}$$

Therefore,

$$\mu(\widehat{\mathcal{Q}}) \leq \epsilon + \epsilon = 2\epsilon. \tag{3.29}$$

Case 3 ($y \notin [0,1] \cup \bigcup_{i=1}^{k} [i(1+\epsilon), i(1+\epsilon)+1]$). According to properties $(\widetilde{P2})$–$(\widetilde{P4})$, we have $\mathcal{Q} \cap A_i = \mathcal{Q} \cap R_i'' = \varnothing$ for all i. In view of the above observation, we conclude that

$$\widehat{\mathcal{Q}} \subset \bigcup_{i=1}^{k} A_i'. \tag{3.30}$$

Therefore,

$$\mu(\widehat{\mathcal{Q}}) \leq \epsilon. \tag{3.31}$$

Estimates (3.27), (3.29), and (3.31) yield that $\widehat{\mu}(\mathcal{Q}) = \mu(\widehat{\mathcal{Q}}) \leq 2\epsilon$. This completes the proof of Lemma 3.3. ■

In view of Lemmas 3.2 and 3.3, estimates (3.6) and (3.7) hold true. The proof of Proposition 3.1 is thus complete. ■

3.2 End of the proof of Theorem 2.1(i)

Fix $k \geq 2$ and set $\epsilon = \pi/k$. We choose a symplectomorphism α of $\mathbb{R}^2(u,v)$ such that $P' \subset \alpha(B^2(\pi))$. We refer to [10, Lemma 2.5] for an explicit construction. Choose an orientation preserving diffeomorphism $f : \mathbb{R} \to]0,1[$, and denote by f' its derivative. Then the map

$$\beta : \mathbb{R}^2 \longrightarrow \mathbb{R} \times]0,1[, \qquad (x,y) \longmapsto \left(\frac{x}{f'(y)}, f(y) \right) \tag{3.32}$$

is a symplectomorphism. We define the symplectic embedding $\Phi : \mathbb{R}^{2n} \hookrightarrow \mathbb{R}^{2n}$ by

$$\Phi = \big((\alpha^{-1} \times \mathrm{id}\,) \circ \varphi \circ (\alpha \times \beta) \big) \times \mathrm{id}_{2n-4}, \tag{3.33}$$

where φ is the map guaranteed by Proposition 3.1. Since

$$\operatorname{supp} \varphi \subset P' \times \mathbb{R}^2 \subset \alpha\big(B^2(\pi)\big) \times \mathbb{R}^2, \tag{3.34}$$

we have $\Phi(Z^{2n}(\pi)) \subset Z^{2n}(\pi)$. For each subset S of $Z^{2n}(\pi)$ and each point $z = (x, y, z_3, \dots, z_n) \in \mathbb{R}^{2n-2}$, we have

$$\begin{aligned} \Phi(S) \cap D_z &\subset \Phi\big(Z^{2n}(\pi)\big) \cap D_z \\ &= \big((\alpha^{-1} \times \mathrm{id}\,) \circ \varphi \circ (\alpha \times \beta) \big) \big(Z^4(\pi)\big) \cap D_{(x,y)} \\ &\subset \big((\alpha^{-1} \times \mathrm{id}\,) \circ \varphi \big) \big(\alpha\big(B^2(\pi)\big) \times \mathbb{R} \times [0,1] \big) \cap E_{(x,y)}. \end{aligned} \tag{3.35}$$

Using this, the facts that $\overline{\mu}$ is monotone and α^{-1} preserves μ, inclusions (3.34), and estimates (3.6) and (3.18), we can estimate

$$\begin{aligned} \overline{\mu}\big(\Phi(S) \cap D_z\big) &\le \mu\big(\varphi\big(\alpha\big(B^2(\pi)\big) \times \mathbb{R} \times [0,1] \big) \cap E_{(x,y)} \big) \\ &= \mu\big(\varphi\big(P' \times \mathbb{R} \times [0,1] \big) \cap E_{(x,y)} \big) + \mu\big(\alpha\big(B^2(\pi)\big) \setminus P' \big) \\ &\le 3\epsilon. \end{aligned} \tag{3.36}$$

Since this holds true for all $z \in \mathbb{R}^{2n-2}$, and since $k \ge 2$ was arbitrary, we conclude that $\sigma(S) = 0$.

3.3 End of the proof of Theorem 2.1(ii)

Choose $a < \pi$ so large that $S \subset Z^{2n}(a)$. We choose $k \ge 2$ so large that $a < \mu(Q)$. We then find a symplectomorphism α of $\mathbb{R}^2(u,v)$ such that

$$\alpha\big(B^2(a)\big) \subset Q, \qquad \alpha\big(B^2(\pi)\big) \supset P', \tag{3.37}$$

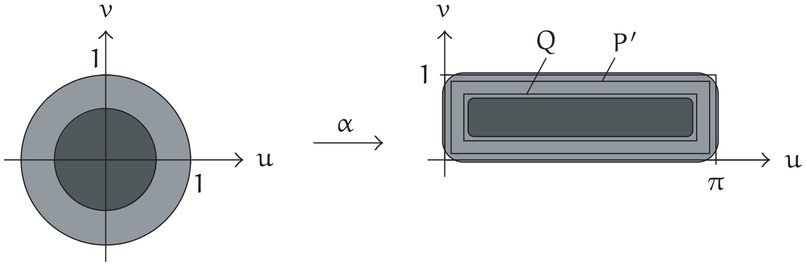

Figure 3.5 The symplectomorphism α.

see Figure 3.5. Again we refer to [10, Lemma 2.5] for an explicit construction. We choose a symplectomorphism $\beta : \mathbb{R}^2 \to \mathbb{R} \times]0,1[$ as above and define the symplectic embedding $\Phi : \mathbb{R}^{2n} \hookrightarrow \mathbb{R}^{2n}$ by

$$\Phi = \big((\alpha^{-1} \times \mathrm{id}) \circ \varphi \circ (\alpha \times \beta) \big) \times \mathrm{id}_{2n-4} \,. \tag{3.38}$$

Since $\operatorname{supp} \varphi \subset P' \times \mathbb{R}^2 \subset \alpha(B^2(\pi)) \times \mathbb{R}^2$ we have $\Phi(Z^{2n}(a)) \subset Z^{2n}(\pi)$. For each $z = (x, y, z_3, \dots, z_n) \in \mathbb{R}^{2n-2}$, we have

$$\begin{aligned} \Phi(S) \cap D_z &\subset \Phi\big(Z^{2n}(a)\big) \cap D_z \\ &= \big((\alpha^{-1} \times \mathrm{id}) \circ \varphi \circ (\alpha \times \beta) \big) \big(Z^4(a)\big) \cap D_{(x,y)} \\ &\subset \big((\alpha^{-1} \times \mathrm{id}) \circ \varphi \big) \big(Q \times \mathbb{R} \times [0,1] \big) \cap E_{(x,y)}. \end{aligned} \tag{3.39}$$

Using this, the facts that $\widehat{\mu}$ is monotone and α^{-1} preserves $\widehat{\mu}$, and estimate (3.7), we can estimate

$$\begin{aligned} \widehat{\mu}\big(\Phi(S) \cap D_z\big) &\le \widehat{\mu}\big(\varphi\big(Q \times \mathbb{R} \times [0,1]\big) \cap E_{(x,y)}\big) \\ &\le 2\epsilon. \end{aligned} \tag{3.40}$$

Since this holds true for all $z \in \mathbb{R}^{2n-2}$, and since we can choose k as large as we like, we conclude that $\widehat{\sigma}(S) = 0$. The proof of Theorem 2.1 is complete.

4 Proof of Theorem 2.4

As in the proof of Theorem 2.1, the main ingredient in the proof is a special embedding result in dimension 4. We denote by $\mathcal{H}(\mathbb{R}^4)$ the set of smooth and bounded functions $H : \mathbb{R}^4 \to \mathbb{R}$ whose Hamiltonian vector field X_H generates a flow on $\mathbb{R}^4$. Again, the time-1-map of this flow is then denoted by ϕ_H, and we set

$$\|H\| = \sup_{z\in\mathbb{R}^{2n}} H(z) - \inf_{z\in\mathbb{R}^{2n}} H(z). \tag{4.1}$$

Fix an integer $k \geq 2$, and set

$$\epsilon = \frac{\pi}{k}, \qquad \delta = \frac{\epsilon}{4k}, \qquad \nu = \frac{\delta}{4k}. \tag{4.2}$$

We use the notation of Section 3, and in addition, we define the closed rectangle P^ν in $\mathbb{R}^2(u,v)$ by

$$P^\nu = [\nu, \pi-\nu] \times [\nu, 1-\nu]. \tag{4.3}$$

The support of a function $H : \mathbb{R}^4 \to \mathbb{R}$ is defined as

$$\operatorname{supp} H = \overline{\left\{z \in \mathbb{R}^4 \mid H(z) \neq 0\right\}}. \tag{4.4}$$

Proposition 4.1. There exists $H \in \mathcal{H}(\mathbb{R}^4)$ such that $\operatorname{supp} H \subset P^\nu \times \mathbb{R}^2$ and $\|H\| \leq 2\epsilon$ and such that for each $(x,y) \in \mathbb{R}^2$,

$$\mu\left(\phi_H\left(P^\nu \times \mathbb{R} \times [0,1]\right) \cap E_{(x,y)}\right) \leq 3\epsilon, \tag{4.5}$$

$$\widehat{\mu}\left(\phi_H\left(Q \times \mathbb{R} \times [0,1]\right) \cap E_{(x,y)}\right) \leq 3\epsilon. \tag{4.6}$$

□

Proof. As in the proof of Proposition 3.1, we start with describing a local model of our map. We define the closed rectangles R, R', and R'' and the closed rectangular annuli A and A' as in the proof of Proposition 3.1, and we define the closed rectangle R^ν in $\mathbb{R}^2(u,v)$ by

$$R^\nu = [\nu, \epsilon-\nu] \times [\nu, 1-\nu]. \tag{4.7}$$

We also define closed intervals I, I', and I'' in $\mathbb{R}(x)$ by

$$I = [0,\epsilon], \qquad I' = [\delta, \epsilon-\delta], \qquad I'' = [2\delta, \epsilon-2\delta], \tag{4.8}$$

and set

$$J = [0,\delta] \cup [\epsilon-\delta, \epsilon], \qquad J' = [\delta, 2\delta] \cup]\epsilon-2\delta, \epsilon-\delta[, \tag{4.9}$$

then $I = J \cup J' \cup I''$. We finally set

$$\check{y}_i = 1 + (2i - 1)\delta, \qquad \widehat{y}_i = 2i - \epsilon + 2i\delta. \tag{4.10}$$

Lemma 4.2. For each $i \in \{1, \dots, k\}$, there exists a smooth function $H_i : \mathbb{R}^4 \to \mathbb{R}$ with the following properties:

$(P1)_i$ $\operatorname{supp} H_i \subset R^{\nu} \times I \times \mathbb{R}$,

$(P2)_i$ ϕ_{H_i} fixes $A \times I \times [0, 1]$,

$(P3)_i$ ϕ_{H_i} embeds $A' \times I \times [0, 1]$ into $A' \times I \times \mathbb{R}$,

$(P4)_i$ ϕ_{H_i} fixes $R'' \times J \times [0, 1]$,

$(P5)_i$ ϕ_{H_i} embeds $R'' \times J' \times [0, 1]$ into

$$R'' \times (J \cup J') \times \mathbb{R} \coprod R'' \times I \times \left([\check{y}_i, \check{y}_i + \delta] \cup [\widehat{y}_i - \epsilon, \widehat{y}_i]\right), \tag{4.11}$$

$(P6)_i$ ϕ_{H_i} translates $R'' \times I'' \times [0, 1]$ by $2i1_y$,

$(P7)_i$ $\|H_i\| \leq 2\epsilon$,

see Figure 4.1. □

Proof. We will first construct a Hamiltonian diffeomorphism ϕ_F of small energy which disjoins $R'' \times I'' \times [0, 1]$ from itself, and will then construct a Hamiltonian diffeomorphism ϕ_{G_i} whose support is disjoint from $R \times I \times [0, 1]$, and which translates the image $\phi_F(R'' \times I'' \times [0, 1])$ far along the y-axis. The composition $\phi_{G_i} \circ \phi_F \circ \phi_{G_i}^{-1}$ will be the desired map ϕ_{H_i}. Both ϕ_F and ϕ_{G_i} are similar to the map ϕ_H constructed in Section 3, but now F and G_i have also an x-cut-off factor. In order to make the support of ϕ_{G_i} disjoint from $R \times I \times [0, 1]$, the function G_i must also have a y-cut-off factor. This will lead to technical complications.

Step 1 (construction of the map ϕ_F). We choose smooth cut-off functions $f_j : \mathbb{R} \to [0, 1]$, $j = 1, 2, 3$, such that

$$f_1(t) = \begin{cases} 0, & t \notin [\delta, \epsilon - \delta], \\ 1, & t \in [2\delta, \epsilon - 2\delta], \end{cases}$$
$$f_2(t) = \begin{cases} 0, & t \notin [\delta, 1 - \delta], \\ 1, & t \in [2\delta, 1 - 2\delta], \end{cases} \tag{4.12}$$
$$f_3(t) = \begin{cases} 0, & t \notin I', \\ 1, & t \in I'', \end{cases}$$

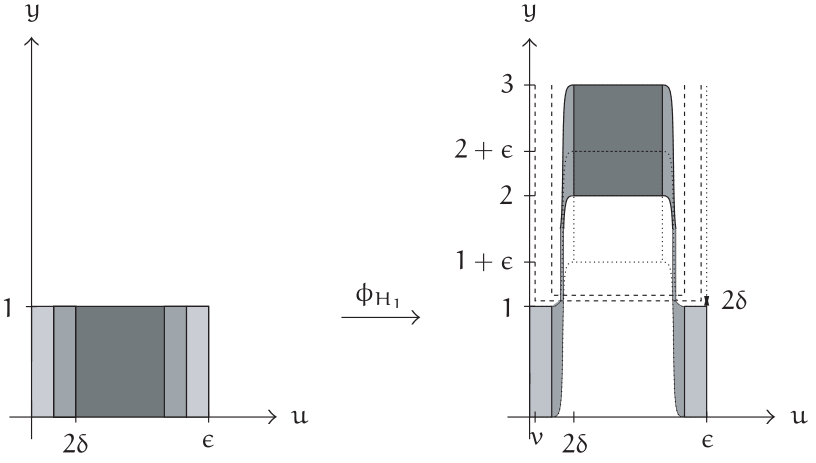

Figure 4.1 An impression of the map ϕ_{H_1}.

and we define the smooth function $F : \mathbb{R}^4 \to \mathbb{R}$ by

$$F(u, v, x, y) = -f_1(u)f_2(v)f_3(x)(1+\epsilon)x. \tag{4.13}$$

By the choice of the cut-off functions f_1, f_2, and f_3 we have

$$\operatorname{supp} F \subset R' \times I' \times \mathbb{R}, \tag{4.14}$$

and since $|f_3(x)x| \leq \epsilon - \delta$ for all x, we have

$$\|F\| \leq (1+\epsilon)(\epsilon - \delta) \leq 2\epsilon. \tag{4.15}$$

The Hamiltonian vector field X_F of F is given by

$$X_F(z) = (1+\epsilon)\begin{pmatrix} -f_1(u)f_2'(v)f_3(x)x \\ f_1'(u)f_2(v)f_3(x)x \\ 0 \\ f_1(u)f_2(v)\big(f_3'(x)x + f_3(x)\big) \end{pmatrix}. \tag{4.16}$$

Notice that

$$X_F(z) = (1+\epsilon)\big(f_3'(x)x + f_3(x)\big)1_y \quad \text{for all } z \in R'' \times I \times \mathbb{R}. \tag{4.17}$$

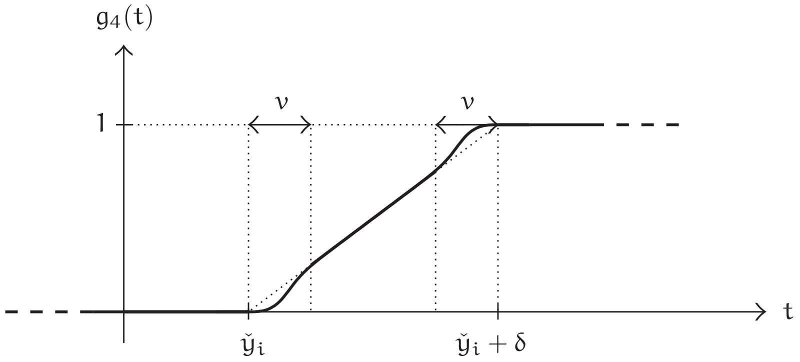

Figure 4.2 The cut-off function $g_4(t)$.

Step 2 (construction of the map ϕ_{G_i}). We choose smooth cut-off functions $g_j : \mathbb{R} \to [0,1]$, $j = 1,2,3,4$, such that

$$g_1(t) = \begin{cases} 0, & t \notin [\nu, \epsilon - \nu], \\ 1, & t \in [\delta, \epsilon - \delta], \end{cases}$$
$$g_2(t) = \begin{cases} 0, & t \notin [\nu, 1 - \nu], \\ 1, & t \in [\delta, 1 - \delta], \end{cases}$$
$$g_3(t) = \begin{cases} 0, & t \notin I, \\ 1, & t \in I', \end{cases} \tag{4.18}$$
$$g_4(t) = \begin{cases} 0, & t \le \check{y}_i, \\ 1, & t \ge \check{y}_i + \delta. \end{cases}$$

We can assume that $g_3'(t) \ge 0$ if $t \le \epsilon - \delta$, and that $g_4'(t) \ge 0$ for all $t \in \mathbb{R}$ and

$$g_4(t) = \frac{1}{\delta}\left(t - \check{y}_i\right) \quad \text{if } t \in \left[\check{y}_i + \nu, \check{y}_i + \delta - \nu\right], \tag{4.19}$$

see Figure 4.2. We define the smooth function $G_i : \mathbb{R}^4 \to \mathbb{R}$ by

$$G_i(u,v,x,y) = -g_1(u)g_2(v)g_3(x)g_4(y)(2i - 1 - \epsilon)x. \tag{4.20}$$

The Hamiltonian vector field X_{G_i} of G_i is given by

$$X_{G_i}(z) = (2i-1-\epsilon)\begin{pmatrix} -g_1(u)g_2'(v)g_3(x)g_4(y)x \\ g_1'(u)g_2(v)g_3(x)g_4(y)x \\ -g_1(u)g_2(v)g_3(x)g_4'(y)x \\ g_1(u)g_2(v)\big(g_3'(x)x+g_3(x)\big)g_4(y) \end{pmatrix}. \tag{4.21}$$

In view of the choice of the cut-off functions g_1 and g_2, we find that for all $z \in R' \times I \times \mathbb{R}$,

$$X_{G_i}(z) = (2i-1-\epsilon)\begin{pmatrix} 0 \\ 0 \\ -g_3(x)g_4'(y)x \\ \big(g_3'(x)x+g_3(x)\big)g_4(y) \end{pmatrix}. \tag{4.22}$$

Also, notice that

$$\operatorname{supp}\phi_{G_i} = \operatorname{supp}\phi_{G_i}^{-1} \subset R^{v} \times I \times \left[\check{y}_i, \infty\right[. \tag{4.23}$$

We define the smooth function $H_i : \mathbb{R}^4 \to \mathbb{R}$ by

$$H_i(z) = F\big(\phi_{G_i}^{-1}(z)\big). \tag{4.24}$$

According to the transformation law of Hamiltonian vector fields under symplectic transformations, we have

$$\phi_{H_i} = \phi_{G_i} \circ \phi_F \circ \phi_{G_i}^{-1}. \tag{4.25}$$

Step 3 (verification of properties $(P1)_i$–$(P7)_i$)

Property $(P1)_i$. This property follows from inclusions (4.14) and (4.23). In order to verify $(P2)_i$–$(P7)_i$ we observe that inclusion (4.23) and identity (4.25) imply that

$$\phi_{H_i}(z) = \big(\phi_{G_i} \circ \phi_F\big)(z) \quad \text{for all } z \in R \times I \times [0,1]. \tag{4.26}$$

Properties $(P2)_i$ and $(P4)_i$. Assume that $z \in A \times I \times [0,1]$, or that $z \in R'' \times J \times [0,1]$. Inclusion (4.14) implies that $\phi_F(z) = z$. Inclusion (4.23) and identity (4.26) now imply that $\phi_{H_i}(z) = z$.

Property $(P3)_i$. Assume that $z \in A' \times I \times [0,1]$. According to inclusion (4.14) and identity (4.17), we have $\phi_F(z) \in A' \times I \times \mathbb{R}$. Identities (4.22) and (4.26) now imply that $\phi_{H_i}(z) \in A' \times I \times \mathbb{R}$.

Property $(\mathrm{P5})_i$. Assume that $z \in R'' \times J' \times [0,1]$. Identity (4.17) yields

$$\phi_F(z) \in R'' \times J' \times \mathbb{R}. \tag{4.27}$$

Identity (4.21) implies that the restriction of ϕ_{G_i} to $R'' \times I \times \mathbb{R}$ is of the form

$$\phi_{G_i}(u,v,x,y) = \big(u,v,\varphi(x,y)\big), \tag{4.28}$$

where φ is a symplectomorphism of $I \times \mathbb{R}$. Let $\phi_F(z) = (u_0, v_0, x_0, y_0)$. According to identity (4.26), we need to show that

$$\varphi(x_0,y_0) \in (J \cup J') \times \mathbb{R} \coprod I \times \big([\check{y}_i, \check{y}_i + \delta] \cup [\widehat{y}_i - \epsilon, \widehat{y}_i]\big). \tag{4.29}$$

Assume first that $y_0 \le \check{y}_i$. Inclusion (4.23) implies that

$$\varphi(x_0,y_0) = (x_0,y_0) \in J' \times \mathbb{R}. \tag{4.30}$$

Assume now that $y_0 \ge \check{y}_i$. We let

$$\gamma(t) = \big(x(t),y(t)\big), \quad t \in [0,1], \tag{4.31}$$

be the segment of the solution of the system of ordinary differential equations

$$\begin{aligned} \dot{x}(t) &= (2i-1-\epsilon)\big(-g_3(x(t))g_4'(y(t))x(t)\big),\\ \dot{y}(t) &= (2i-1-\epsilon)\big(g_3'(x(t))x(t) + g_3(x(t))\big)g_4(y(t)) \end{aligned} \tag{4.32}$$

starting at $\gamma(0) = (x_0, y_0)$. Then $\gamma(1) = \varphi(x_0, y_0)$. Since $g_4'(y) \ge 0$ for all $y \in \mathbb{R}$, the first equation in (4.32) implies that $\dot{x}(t) \le 0$ for all $t \in [0,1]$, and so $x(t) \le x_0 \le \epsilon - \delta$ for all $t \in [0,1]$. Since $g_3'(x) \ge 0$ for all $x \le \epsilon - \delta$, the second equation in (4.32) implies that $\dot{y}(t) \ge 0$ for all $t \in [0,1]$.

Case 1 ($y_0 \ge \check{y}_i + \delta$). Since $g_4(y_0) = 1$ and $\dot{y}(t) \ge 0$, we have $g_4(y(t)) = 1$ for all $t \in [0,1]$, and so $\dot{x}(t) = 0$ for all $t \in [0,1]$. In particular, $\gamma(1) \in J' \times \mathbb{R}$.

Case 2 ($y_0 \in [\check{y}_i, \check{y}_i + \delta]$ and $x_0 \in]\delta, 2\delta[$). Since $\dot{x}(t) \le 0$ and $\dot{y}(t) \ge 0$ for all $t \in [0,1]$, we find that $x(1) \in [0, 2\delta[$, and so $\gamma(1) \in (J \cup J') \times \mathbb{R}$.

Case 3 ($y_0 \in [\check{y}_i, \check{y}_i + \delta]$ and $x_0 \in]\epsilon - 2\delta, \epsilon - \delta[$). We claim that

$$\gamma(1) \in [0,\delta] \times \mathbb{R} \cup [\delta, \epsilon - \delta] \times \big([\check{y}_i, \check{y}_i + \delta] \cup [\widehat{y}_i - \epsilon, \widehat{y}_i]\big). \tag{4.33}$$

We consider the closed rectangle

$$C = [\epsilon - 2\delta, \epsilon - \delta] \times \left[\check{y}_i, \check{y}_i + \delta\right], \tag{4.34}$$

and we denote the left, right, top, and bottom edges of C by L_l, L_r, L_t, and L_b. It is enough to prove claim (4.33) for $(x_0, y_0) = \gamma(0) \in L_l \cup L_r \cup L_t \cup L_b$, see Figure 4.3. Notice that, as long as $\gamma(t) \in [\delta, \epsilon - \delta] \times [\check{y}_i, \check{y}_i + \delta]$, system (4.32) reads

$$\begin{aligned} \dot{x}(t) &= (2i - 1 - \epsilon)\big(-g_4'\big(y(t)\big)x(t)\big), \\ \dot{y}(t) &= (2i - 1 - \epsilon)g_4\big(y(t)\big), \end{aligned} \tag{4.35}$$

and that

$$\begin{aligned} \dot{x}(t) &= 0, \\ \dot{y}(t) &= 2i - 1 - \epsilon, \end{aligned} \tag{4.36}$$

if $\gamma(t) \in [\delta, \epsilon - \delta] \times [\check{y}_i + \delta, \infty[$.

Assume $\gamma(0) \in L_b$. Then $g_4'(y_0) = g_4(y_0) = 0$, and so (4.35) implies that

$$\gamma(1) = \gamma(0) = \big(x_0, \check{y}_i\big) \in [\delta, \epsilon - \delta] \times \left[\check{y}_i, \check{y}_i + \delta\right]. \tag{4.37}$$

Assume $\gamma(0) \in L_t$. Then (4.36) implies that

$$\gamma(1) = \gamma(0) + (0, 2i - 1 - \epsilon) = \big(x_0, \widehat{y}_i\big) \in [\delta, \epsilon - \delta] \times \left[\widehat{y}_i - \epsilon, \widehat{y}_i\right]. \tag{4.38}$$

Assume $\gamma(0) \in L_r$. In order to understand the locus of $\varphi(L_r)$, we consider the horizontal and the vertical line

$$L_h = [\delta, \epsilon - \delta] \times \left\{\check{y}_i + \delta\right\}, \qquad L_v = \{\delta\} \times \left[\check{y}_i, \check{y}_i + \delta\right], \tag{4.39}$$

and first check that the trajectory γ_ν starting at $(\epsilon - \delta, \check{y}_i + \nu) \in L_r$ crosses L_v, see Figure 4.3. According to choice (4.19), system (4.35) reads

$$\begin{aligned} \dot{x}(t) &= (2i - 1 - \epsilon)\frac{1}{\delta}\big(-x(t)\big), \\ \dot{y}(t) &= (2i - 1 - \epsilon)\frac{1}{\delta}\big(y(t) - \check{y}_i\big) \end{aligned} \tag{4.40}$$

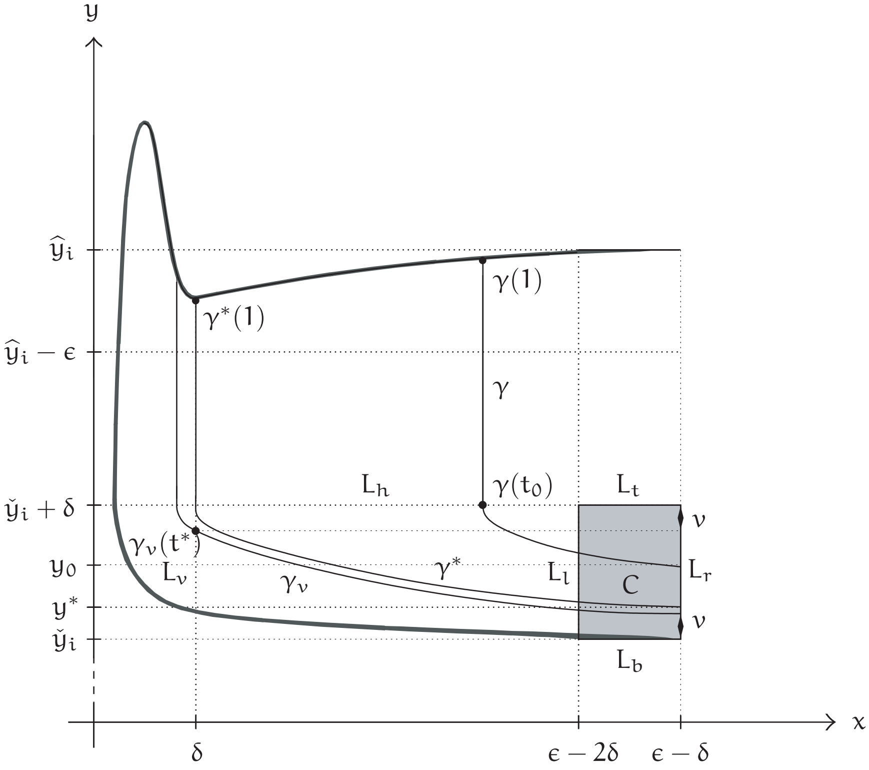

Figure 4.3 The image $\phi(C)$, the curves γ_ν and γ^* and a curve γ starting at $(\epsilon - \delta, y_0)$ with $y_0 > y^*$.

as long as $\gamma_\nu(t) \in [\delta, \epsilon - \delta] \times [\check{y}_i + \nu, \check{y}_i + \delta - \nu]$. We set

$$t^* = \frac{\delta}{2i - 1 - \epsilon} \log \frac{\epsilon - \delta}{\delta}. \tag{4.41}$$

Solving (4.40) for the initial condition $(x_0, y_0) = (\epsilon - \delta, \check{y}_i + \nu)$, and using $\nu((\epsilon - \delta)/\delta) = \delta - \nu$, we find that

$$\gamma_\nu(t^*) = \big(\delta, \check{y}_i + \delta - \nu\big) \in L_\nu. \tag{4.42}$$

It follows that there exists a unique $y^* \in \,]\check{y}_i + \nu, \check{y}_i + \delta[$ such that

(i) the curve $\gamma(t)$ does not cross L_h if $y_0 \in \,]\check{y}_i, y^*[$,

(ii) the curve $\gamma(t)$ does cross L_h if $y_0 \in [y^*, \check{y}_i + \delta[$.

The trajectory γ^* starting at $(\epsilon - \delta, y^*)$ is shown in Figure 4.3.

In case (i), either $\gamma(t)$ does not cross the line L_ν, in which case $\gamma(1) \in [\delta, \epsilon - \delta] \times [\check{y}_i, \check{y}_i + \delta]$, or $\gamma(t)$ crosses L_ν, in which case $\dot{x}(t) \leq 0$ implies $\gamma(1) \in [0, \delta] \times \mathbb{R}$.

Assume now that we are in case (ii), and that $\gamma(t) = (x(t), y(t))$ is the trajectory starting at $(y_0, \epsilon - \delta)$. We define t_0 through the identity $\gamma(t_0) \in L_h$, see Figure 4.3. In order to estimate t_0 from above, we first notice that identity (4.42) and $\check{y}_i + \nu < y_0$ imply that

$$y\big(t^*\big) > \check{y}_i + \delta - \nu. \tag{4.43}$$

In view of the second equation in (4.35), the fact that $y(t)$ and $g_4(y)$ are increasing, and estimate (4.43), we have

$$\dot{y}(t) \geq (2i - 1 - \epsilon)\left(1 - \frac{1}{\delta}\nu\right) \quad \text{for all } t \geq t^*. \tag{4.44}$$

Using estimate (4.44), identity (4.41), that $\nu = \delta/4k$, and $\delta = \epsilon/4k$, we can estimate

$$\begin{aligned} t_0 &< t^* + \frac{\nu}{(2i-1-\epsilon)\left(1 - \frac{1}{\delta}\nu\right)} \\ &< \frac{1}{2i-1-\epsilon}\big(\delta \log(4k-1) + \delta\big) \\ &< \frac{1}{2i-1-\epsilon}\epsilon. \end{aligned} \tag{4.45}$$

In view of the second equation in (4.36), and the relation $\widehat{y}_i = (\check{y}_i + \delta) + (2i - 1 - \epsilon)$, we thus finally find

$$\widehat{y}_i \geq y(1) = \check{y}_i + \delta + \big(1 - t_0\big)(2i - 1 - \epsilon) > \widehat{y}_i - \epsilon \tag{4.46}$$

and so $\gamma(1) \in [\delta, \epsilon - \delta] \times [\widehat{y}_i - \epsilon, \widehat{y}_i]$.

Assume finally that $\gamma(0) \in L_l$. Since $\dot{x}(t) \leq 0$ and $\dot{y}(t) \geq 0$ for all $t \in [0, 1]$, we have $\gamma(1) \in [0, \epsilon - 2\delta] \times [\check{y}_i, \infty[$. The part of $\varphi(L_l)$ contained in $[\delta, \epsilon - 2\delta] \times [\check{y}_i + \delta, \infty[$ lies above the corresponding part of $\varphi(L_r)$ and below the line $\{(x, y) \mid y = \widehat{y}_i\}$. Our result for $\varphi(L_r)$ therefore implies that this part of $\varphi(L_l)$ is contained in $[\delta, \epsilon - 2\delta] \times [\widehat{y}_i - \epsilon, \widehat{y}_i]$, see Figure 4.3. We conclude that (4.33) also holds true for all points $\gamma(0) \in L_l$. This completes the verification of property $(\mathrm{P5})_i$.

Property $(\mathrm{P6})_i$. Assume that $z \in R'' \times I'' \times [0, 1]$. Identity (4.17) yields

$$\phi_F(z) = z + (1 + \epsilon)1_y \in R'' \times I'' \times [1 + \epsilon, \infty[. \tag{4.47}$$

In view of identities (4.26) and (4.22), and the choice of the cut-off functions g_3 and g_4, we therefore find

$$\phi_{H_i}(z) = \phi_{G_i}\big(z + (1 + \epsilon)1_y\big) = z + (1 + \epsilon)1_y + (2i - 1 - \epsilon)1_y = z + 2i1_y. \tag{4.48}$$

Property $(P7)_i$. Using definition (4.24) of H_i, and estimate (4.15), we finally estimate $\|H_i\| = \|F\| \le 2\epsilon$.

The proof of Lemma 4.2 is complete. ■

We proceed with the proof of Proposition 4.1. As in Section 3 we define for each subset T of $\mathbb{R}^2(u, v)$ and each $i \in \{1, \dots, k\}$ the translate T_i of T by

$$T_i = \{(u + (i-1)\epsilon, v) \mid (u, v) \in T\}, \tag{4.49}$$

and for each subset X of $\mathbb{R}(x)$, each $i \in \{1, \dots, k\}$, and $j \in \mathbb{Z}$, we define the translate X_{ij} of X by

$$X_{ij} = \{x + 4(i-1)\delta + j\epsilon \mid x \in X\}. \tag{4.50}$$

Let H_i be the functions guaranteed by Lemma 4.2, and define for each $i \in \{1, \dots, k\}$ and $j \in \mathbb{Z}$ the smooth function $H_{ij} : \mathbb{R}^4 \to \mathbb{R}$ by

$$H_{ij}(z) = H_i(u - (i-1)\epsilon, v, x - 4(i-1)\delta - j\epsilon, y). \tag{4.51}$$

In view of Lemma 4.2, we have the following properties:

$(P1)_{ij}$ $\operatorname{supp} H_{ij} \subset R_i^\gamma \times I_{ij} \times \mathbb{R}$,
$(P2)_{ij}$ $\phi_{H_{ij}}$ fixes $A_i \times I_{ij} \times [0, 1]$,
$(P3)_{ij}$ $\phi_{H_{ij}}$ embeds $A_i' \times I_{ij} \times [0, 1]$ into $A_i' \times I_{ij} \times \mathbb{R}$,
$(P4)_{ij}$ $\phi_{H_{ij}}$ fixes $R_i'' \times J_{ij} \times [0, 1]$,
$(P5)_{ij}$ $\phi_{H_{ij}}$ embeds $R_i'' \times J_{ij}' \times [0, 1]$ into

$$R_i'' \times (J_{ij} \cup J_{ij}') \times \mathbb{R} \coprod R_i'' \times I_{ij} \times \left([\check{y}_i, \check{y}_i + \delta] \cup [\widehat{y}_i - \epsilon, \widehat{y}_i]\right), \tag{4.52}$$

$(P6)_{ij}$ $\phi_{H_{ij}}$ translates $R_i'' \times I_{ij}'' \times [0, 1]$ by $2i1_y$,
$(P7)_{ij}$ $\|H_{ij}\| \le 2\epsilon$.

Since the sets $R_i^\gamma \times I_{ij} \times \mathbb{R}$ are mutually disjoint, properties $(P1)_{ij}$ guarantee that the function

$$H(z) = \sum_{i=1}^{k} \sum_{j \in \mathbb{Z}} H_{ij}(z) \tag{4.53}$$

belongs to $\mathcal{H}(4)$. Properties $(\mathrm{P1})_{ij}$ also imply that $\operatorname{supp} H \subset P^{\nu} \times \mathbb{R}^2$. Properties $(\mathrm{P1})_{ij}$ and $(\mathrm{P7})_{ij}$ imply that

$$\|H\| \le \sup_{i,j} \|H_{ij}\| \le 2\epsilon. \tag{4.54}$$

4.1 Verification of estimates (4.5) and (4.6)

Fix $(x_0, y_0) \in \mathbb{R}^2$. We set

$$\begin{aligned} \mathcal{P}^{\nu} &= p\big(\phi_H\big(P^{\nu} \times \mathbb{R} \times [0,1]\big) \cap E_{(x_0,y_0)}\big), \\ \mathcal{Q} &= p\big(\phi_H\big(Q \times \mathbb{R} \times [0,1]\big) \cap E_{(x_0,y_0)}\big). \end{aligned} \tag{4.55}$$

Since

$$\mathbb{R}(x) = \coprod_{i=1}^{k} \coprod_{j \in \mathbb{Z}} [-2\delta, 2\delta[_{ij}, \tag{4.56}$$

there exists a unique pair $(i_0, j_0) \in \{1, \dots, k\} \times \mathbb{Z}$ such that $x_0 \in [-2\delta, 2\delta[_{i_0 j_0}$. For $i \in \{1, \dots, k\}$ we define j_i by

$$j_i = \begin{cases} j_0 & \text{if } i \le i_0, \\ j_0 - 1 & \text{if } i > i_0. \end{cases} \tag{4.57}$$

According to properties $(\mathrm{P1})_{ij}$, we have

$$\mathcal{P}^{\nu} \cap R_i = p\big(\phi_{H_{ij_i}}\big(\big(P^{\nu} \cap R_i\big) \times I_{ij_i} \times [0,1]\big) \cap E_{(x_0,y_0)}\big), \tag{4.58}$$

$$\mathcal{Q} \cap R_i = p\big(\phi_{H_{ij_i}}\big(\big(Q \cap R_i\big) \times I_{ij_i} \times [0,1]\big) \cap E_{(x_0,y_0)}\big), \tag{4.59}$$

see Figure 4.4.

Lemma 4.3. The estimate $\mu(\mathcal{P}^{\nu}) \le 3\epsilon$ holds. ☐

Proof. According to definition (4.10) of $\check{y}_i$ and $\widehat{y}_i$, the sets

$$[2i, 2i+1] \cup \big[\check{y}_i, \check{y}_i + \delta\big] \cup \big[\widehat{y}_i - \epsilon, \widehat{y}_i\big], \quad i = 1, \dots, k, \tag{4.60}$$

are mutually disjoint.

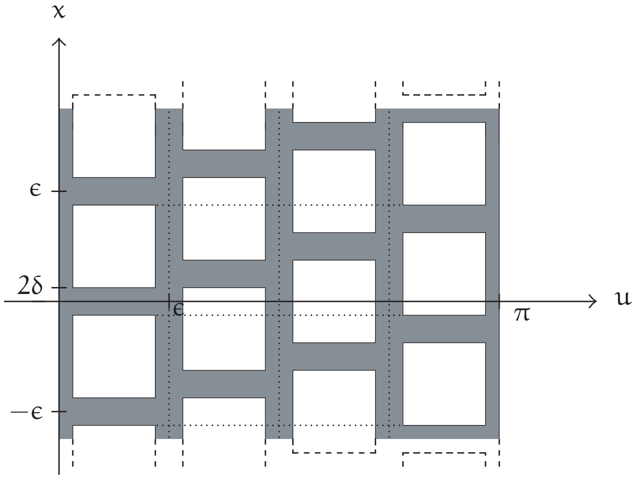

Figure 4.4 The (u, x)-cut-off region of the function H.

Case 1 $(y_0 \in [2i^*, 2i^*+1] \cup [\check{y}_{i^*}, \check{y}_{i^*}+\delta] \cup [\widehat{y}_{i^*}-\epsilon, \widehat{y}_{i^*}])$. According to identity (4.58) and properties $(\mathrm{P2})_{ij_i}$–$(\mathrm{P6})_{ij_i}$, we have $\mathcal{P}^\nu \cap R_i'' = \varnothing$ if $i \notin \{i_0, i^*\}$, and so

$$\mathcal{P}^\nu \subset R_{i_0} \cup R_{i^*} \cup \bigcup_{i=1}^{k} A_i \cup A_i'. \tag{4.61}$$

In view of estimate (3.18) we, therefore, find that

$$\mu(\mathcal{P}^\nu) \leq 2\epsilon + k\frac{\epsilon}{k} = 3\epsilon. \tag{4.62}$$

Case 2 $(y_0 \notin \bigcup_{i=1}^{k}[2i, 2i+1] \cup [\check{y}_i, \check{y}_i+\delta] \cup [\widehat{y}_i-\epsilon, \widehat{y}_i])$. According to properties $(\mathrm{P2})_{ij_i}$–$(\mathrm{P6})_{ij_i}$ we have $\mathcal{P}^\nu \cap R_i'' = \varnothing$ if $i \neq i_0$, and so

$$\mathcal{P}^\nu \subset R_{i_0} \cup \bigcup_{i=1}^{k} A_i \cup A_i'. \tag{4.63}$$

Therefore,

$$\mu(\mathcal{P}^\nu) \leq 2\epsilon. \tag{4.64}$$

Estimates (4.62) and (4.64) yield that $\mu(\mathcal{P}^\nu) \leq 3\epsilon$. ■

Lemma 4.4. The estimate $\widehat{\mu}(\mathcal{Q}) \leq 3\epsilon$ holds. □

Proof. For $i = 1, \dots, k$, we define $\mathcal{A}_i$, $\mathcal{A}_i'$, and $\mathcal{R}_i''$ as in (3.24). As in the proof of Lemma 3.3, the crucial observation in this proof is that for each i the simply connected hull of the part

$$p\big(\phi_{H_{ij_i}}(\mathcal{A}_i' \times I_{ij_i} \times [0,1]) \cap E_{(x_0,y_0)}\big) \tag{4.65}$$

of $\mathcal{Q}$ is a simply connected subset of A_i'. Indeed, according to property $(P3)_{ij_i}$, the closed and simply connected set $\phi_{H_{ij_i}}(\mathcal{A}_i' \times I_{ij_i} \times [0,1])$ is contained in $A_i' \times I_{ij_i} \times \mathbb{R}$, and so the simply connected hull of $\phi_{H_{ij_i}}(\mathcal{A}_i' \times I_{ij_i} \times [0,1]) \cap E_{(x_0,y_0)}$ is a simply connected subset of $A_i' \times \{(x_0, y_0)\}$.

Again we denote by $\widehat{\mathcal{Q}}$ the simply connected hull of $\mathcal{Q}$. According to definition (4.10) of $\check{y}_i$ and $\widehat{y}_i$, the $k+1$ sets

$$[0,1], \quad [2i, 2i+1] \cup \big[\check{y}_i, \check{y}_i + \delta\big] \cup \big[\widehat{y}_i - \epsilon, \widehat{y}_i\big], \quad i = 1, \dots, k, \tag{4.66}$$

are mutually disjoint.

Case 1 ($y_0 \in [0,1]$). According to identity (4.58) and properties $(P2)_{ij_i}$–$(P6)_{ij_i}$ we have $\mathcal{Q} \cap A_i = \mathcal{A}_i$ for all i and $\mathcal{Q} \cap R_i'' = \varnothing$ if $i \neq i_0$. In view of the above observation, we conclude that

$$\widehat{\mathcal{Q}} \subset R_{i_0} \cup \bigcup_{i=1}^{k} A_i \cup A_i'. \tag{4.67}$$

In view of estimate (3.18) we, therefore, find that

$$\mu(\widehat{\mathcal{Q}}) \le \epsilon + k\frac{\epsilon}{k} = 2\epsilon. \tag{4.68}$$

Case 2 ($y_0 \in [2i^*, 2i^*+1] \cup [\check{y}_{i^*}, \check{y}_{i^*} + \delta] \cup [\widehat{y}_{i^*} - \epsilon, \widehat{y}_{i^*}]$). According to properties $(P2)_{ij_i}$–$(P6)_{ij_i}$, we have $\mathcal{Q} \cap A_i = \varnothing$ for all i and $\mathcal{Q} \cap R_i'' = \varnothing$ if $i \notin \{i_0, i^*\}$. In view of the above observation we conclude that

$$\widehat{\mathcal{Q}} \subset R_{i_0} \cup R_{i^*} \cup \bigcup_{i=1}^{k} A_i'. \tag{4.69}$$

Therefore,

$$\mu(\widehat{\mathcal{Q}}) \le 2\epsilon + \epsilon = 3\epsilon. \tag{4.70}$$

Case 3 ($y_0 \notin [0,1] \cup \bigcup_{i=1}^{k}[2i,2i+1] \cup [\check{y}_i, \check{y}_i+\delta] \cup [\widehat{y}_i - \epsilon, \widehat{y}_i]$). According to properties $(P2)_{ij_i}$–$(P6)_{ij_i}$, we have $\mathcal{Q} \cap A_i = \varnothing$ for all i and $\mathcal{Q} \cap R_i'' = \varnothing$ if $i \neq i_0$. In view of the above observation, we conclude that

$$\widehat{\mathcal{Q}} \subset R_{i_0} \cup \bigcup_{i=1}^{k} A_i'. \tag{4.71}$$

Therefore,

$$\mu(\widehat{\mathcal{Q}}) \le \epsilon + \epsilon = 2\epsilon. \tag{4.72}$$

Estimates (4.68), (4.70), and (4.72) yield that $\widehat{\mu}(\mathcal{Q}) = \mu(\widehat{\mathcal{Q}}) \le 3\epsilon$. This completes the proof of Lemma 4.4. ■

In view of Lemmas 4.3 and 4.4, estimates (4.5) and (4.6) hold true. The proof of Proposition 4.1 is thus complete. ■

4.2 End of the proof of Theorem 2.4(i)

Consider a partially bounded subset S of $Z^{2n}(\pi)$. There exists $i \in \{2,\dots,n\}$ and $b > 0$, such that $x_i(S) \subset [-b,b]$, or $y_i(S) \subset [-b,b]$. We can assume, without loss of generality, that $i = 2$. If $x(S) \subset [-b,b]$, we define the symplectomorphism σ of $\mathbb{R}^2(x,y)$ by $\sigma(x,y) = (-y,x)$, and we let σ be the identity mapping otherwise. Define the symplectomorphism τ of $\mathbb{R}^2(x,y)$ by

$$\tau(x,y) = \left(2b\,x, \frac{1}{2b}y + \frac{1}{2}\right). \tag{4.73}$$

The composition $\mathrm{id}_2 \times (\tau \circ \sigma) \times \mathrm{id}_{2n-4}$ maps S into

$$B^2(\pi) \times \mathbb{R} \times [0,1] \times \mathbb{R}^{2n-4}. \tag{4.74}$$

Fix $k \ge 2$. We choose a symplectomorphism α of $\mathbb{R}^2(u,v)$, such that $P^\nu \subset \alpha(B^2(\pi))$. Let $H \in \mathcal{H}(\mathbb{R}^4)$ be the function guaranteed by Proposition 4.1. We define the smooth and bounded function $K : \mathbb{R}^{2n} \to \mathbb{R}$ by

$$K(z_1, z_2, z_3, \dots, z_n) = H\big(\alpha(z_1), (\tau \circ \sigma)(z_2)\big). \tag{4.75}$$

Since

$$\operatorname{supp} H \subset P^\nu \times \mathbb{R}^2 \subset \alpha\big(B^2(\pi)\big) \times \mathbb{R}^2, \tag{4.76}$$

the support of K is contained in $Z^{2n}(\pi)$, and since $\|H\| \le 2\epsilon$, we have

$$\|K\| = \|H\| \le 2\epsilon. \tag{4.77}$$

Moreover, the transformation law of Hamiltonian vector fields under symplectic transformations shows that $K \in \mathcal{H}(2n)$ and

$$\phi_K = \Big(\big(\alpha \times (\tau \circ \sigma)\big)^{-1} \circ \phi_H \circ \big(\alpha \times (\tau \circ \sigma)\big)\Big) \times \mathrm{id}_{2n-4}\,. \tag{4.78}$$

For each subset S of $Z^{2n}(\pi)$ and each point $z = (x, y, z_3, \dots, z_n) \in \mathbb{R}^{2n-2}$, we have

$$\begin{aligned} \phi_K(S) \cap D_z &\subset \Big(\big(\alpha \times (\tau \circ \sigma)\big)^{-1} \circ \phi_H\Big)\big(\alpha\big(B^2(\pi)\big) \times \mathbb{R} \times [0,1]\big) \cap E_{(x,y)} \\ &= \big((\alpha^{-1} \times \mathrm{id}\,) \circ \phi_H\big)\big(\alpha\big(B^2(\pi)\big) \times \mathbb{R} \times [0,1]\big) \cap E_{(x',y')}, \end{aligned} \tag{4.79}$$

where we set $(x', y') = (\tau \circ \sigma)(x, y)$. Using the facts that $\overline{\mu}$ is monotone and α^{-1} preserves μ, inclusions (4.76), and estimates (4.5) and $\pi - \mu(P^\nu) \le \epsilon$, we can estimate

$$\begin{aligned} \overline{\mu}\big(\phi_K(S) \cap D_z\big) &\le \mu\big(\phi_H\big(\alpha\big(B^2(\pi)\big) \times \mathbb{R} \times [0,1]\big) \cap E_{(x',y')}\big) \\ &= \mu\big(\phi_H\big(P^\nu \times \mathbb{R} \times [0,1]\big) \cap E_{(x',y')}\big) + \mu\big(\alpha\big(B^2(\pi)\big) \setminus P^\nu\big) \\ &\le 4\epsilon. \end{aligned} \tag{4.80}$$

Since this holds true for all $z \in \mathbb{R}^{2n-2}$, we conclude, together with estimate (4.77), that

$$\sup_z \overline{\mu}\big(\phi_K(S) \cap D_z\big) + \|K\| \le 6\epsilon. \tag{4.81}$$

Recall that $k \ge 2$ was arbitrary, and that $\epsilon = \pi/k$. If S is unbounded, we therefore conclude that $\sigma(S) = 0$. If S is bounded, we denote by ϕ_K^t, $t \in \mathbb{R}$, the Hamiltonian flow generated by K. Since K is supported in $Z^{2n}(\pi)$ and since S is bounded, we find a ball $B \subset \mathbb{R}^{2n-2}$ such that

$$\bigcup_{t \in [0,1]} \phi_K^t(S) \subset B^2(\pi) \times B. \tag{4.82}$$

Choose a smooth compactly supported function $f : \mathbb{R}^{2n-2} \to [0,1]$ such that $f|_B = 1$. The function $\widetilde{K} : \mathbb{R}^{2n} \to \mathbb{R}$ defined by

$$\widetilde{K}(z_1, z_2, \dots, z_n) = f(z_2, \dots, z_n) K(z_1, \dots, z_n) \tag{4.83}$$

belongs to $\mathcal{H}_c(2n)$. Moreover, $\|\widetilde{K}\| \le \|K\| \le 2\epsilon$ and $\phi_{\widetilde{K}}(S) = \phi_K(S)$. In view of estimate (4.81), we therefore find

$$\sup_z \overline{\mu}\big(\phi_{\widetilde{K}}(S) \cap D_z\big) + \big\|\widetilde{K}\big\| \le 6\epsilon. \tag{4.84}$$

Since $k \ge 2$ was arbitrary, we conclude that $\sigma(S) = 0$. The proof of Theorem 2.4(i) is complete.

4.3 End of the proof of Theorem 2.4(ii)

Consider a partially bounded subset S of $Z^{2n}(\pi)$ which is contained in $Z^{2n}(a)$, for some $a < \pi$. Proceeding as above, we find that the composition $\mathrm{id}_2 \times (\tau \circ \sigma) \times \mathrm{id}_{2n-4}$ maps S into

$$B^2(a) \times \mathbb{R} \times [0,1] \times \mathbb{R}^{2n-4}. \tag{4.85}$$

We choose $k \ge 2$ so large that $a < \mu(Q)$. We then find a symplectomorphism α of $\mathbb{R}^2(u,v)$ such that

$$\alpha\big(B^2(a)\big) \subset Q, \qquad \alpha\big(B^2(\pi)\big) \supset P^{\nu}. \tag{4.86}$$

We define $K : \mathbb{R}^{2n} \to \mathbb{R}$ by formula (4.75). Then $K \in \mathcal{H}(2n)$ and $\|K\| \le 2\epsilon$. For each $z = (x, y, z_3, \dots, z_n) \in \mathbb{R}^{2n-2}$, we have

$$\begin{aligned} \phi_K(S) \cap D_z &\subset \big((\alpha^{-1} \times \mathrm{id}) \circ \phi_H\big)\big(\alpha\big(B^2(a)\big) \times \mathbb{R} \times [0,1]\big) \cap E_{(x',y')} \\ &\subset \big((\alpha^{-1} \times \mathrm{id}) \circ \phi_H\big)\big(Q \times \mathbb{R} \times [0,1]\big) \cap E_{(x',y')}. \end{aligned} \tag{4.87}$$

Using the facts that $\widehat{\mu}$ is monotone and α^{-1} preserves $\widehat{\mu}$, and estimate (4.6), we can estimate

$$\begin{aligned} \widehat{\mu}\big(\phi_K(S) \cap D_z\big) &\le \widehat{\mu}\big(\phi_H\big(Q \times \mathbb{R} \times [0,1]\big) \cap E_{(x',y')}\big) \\ &\le 3\epsilon. \end{aligned} \tag{4.88}$$

Proceeding as above and recalling that we can choose k as large as we like, we conclude that $\widehat{\sigma}(S) = 0$. The proof of Theorem 2.4(ii) is complete.

5 Measuring intersections by symplectic capacities

Up till now we have measured the intersections $\varphi(S) \cap D_x$ by the outer Lebesgue measure $\overline{\mu}$ and by $\widehat{\mu}$. There are many other ways of measuring a subset of $\mathbb{R}^2$ in a symplectic way. We recall the following definition from [1, 5].

Definition 5.1. A *symplectic capacity on* $(\mathbb{R}^2, \omega_0)$ is a map c associating with each subset T of $\mathbb{R}^2$ a number $c(T) \in [0, \infty]$ in such a way that the following axioms are satisfied.

(A1) Monotonicity: $c(T) \le c(T')$ if there exists a symplectomorphism φ of $\mathbb{R}^2$ such that $\varphi(T) \subset T'$.

(A2) Conformality: $c(\lambda T) = \lambda^2 c(T)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.

(A3) Nontriviality: $c(B^2(\pi)) = \pi$.

A symplectic capacity c on $\mathbb{R}^2$ is called *intrinsic* if it satisfies the following stronger monotonicity axiom.

(A1′) Monotonicity: $c(T) \le c(T')$ if there exists a symplectic embedding $\varphi : T \hookrightarrow T'$.

Examples of intrinsic symplectic capacities on $\mathbb{R}^2$ are the outer Lebesgue measure $\overline{\mu}$, the Gromov width [3] and the Hofer-Zehnder capacity [5]. Examples of symplectic capacities on $\mathbb{R}^2$ which are not intrinsic are $\widehat{\mu}$, the first Ekeland-Hofer capacity [1] and the displacement energy [4]. Indeed, for each of these symplectic capacities we have $c(S^1) = \pi$, while $c(S^1) = 0$ for any intrinsic symplectic capacity. It is known that for any $a \in]0, \pi]$ there exists a symplectic capacity c on $\mathbb{R}^2$ such that $c(S^1) = a$, see [10, Proposition B.11].

For each subset S of $Z^{2n}(\pi)$ and each symplectic capacity c on $\mathbb{R}^2$, we define

$$\sigma(S; c) = \inf_{\varphi} \sup_{x} c\big(p\big(\varphi(S) \cap D_x\big)\big), \tag{5.1}$$

where φ varies over all symplectic embeddings of S into $Z^{2n}(\pi)$. With this notation we have $\sigma(S) = \sigma(S; \overline{\mu})$ and $\widehat{\sigma}(S) = \sigma(S, \widehat{\mu})$.

Corollary 5.2. Consider a subset S of $Z^{2n}(\pi)$ and a symplectic capacity c on $\mathbb{R}^2$. Then

(i) $\sigma(S; c) = 0$ if c is intrinsic;

(ii) $\sigma(S; c) = 0$ if $S \subset Z^{2n}(a)$ for some $a < \pi$. □

Proof. Consider a bounded subset T of $\mathbb{R}^2$. According to [10, Theorem B.7] we have $c(T) \le \overline{\mu}(T)$ for every intrinsic symplectic capacity c on $\mathbb{R}^2$, and $c(T) \le \widehat{\mu}(T)$ for every symplectic capacity c on $\mathbb{R}^2$. Corollary 5.2 thus follows from Theorem 2.1. ■

For each subset S of $Z^{2n}(\pi)$ and each symplectic capacity c on $\mathbb{R}^2$, we define

$$\sigma_H(S; c) = \inf_{H} \Big\{ \sup_{x} c\big(p\big(\phi_H(S) \cap D_x\big)\big) + \|H\| \Big\}, \tag{5.2}$$

where H varies over $\mathcal{H}_c(2n)$ if S is bounded, and over $\mathcal{H}(2n)$ if S is unbounded. With this notation we have $\sigma_H(S) = \sigma_H(S; \overline{\mu})$ and $\widehat{\sigma}_H(S) = \sigma_H(S, \widehat{\mu})$.

Corollary 5.3. Consider a partially bounded subset S of $Z^{2n}(\pi)$ and a symplectic capacity c on $\mathbb{R}^2$. Then

(i) $\sigma_H(S;c) = 0$ if c is intrinsic;

(ii) $\sigma_H(S;c) = 0$ if $S \subset Z^{2n}(a)$ for some $a < \pi$. □

Proof. Corollary 5.3 follows from Theorem 2.4 in the same way as Corollary 5.2 followed from Theorem 2.1. ■

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