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Mixed Equilibria and Dynamical Systems Arising from Fictitious Play in Perturbed Games

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Fictitious play in infinitely repeated, randomly perturbed games is investigated. Dynamical systems theory is used to study the Markov process $\{x_k\}$, whose state vector x_k lists the empirical frequencies of player's actions in the first k games. For 2×2 games with countably many Nash distribution equilibria, we prove that sample paths converge almost surely. But for Jordan's 3×2 matching game, there are robust parameter values giving probability 0 of convergence. Applications are made to coordination and anticoordination games and to general theory. Proofs rely on results in stochastic approximation and dynamical systems. *Journal of Economic Literature* Classification Numbers: C72, C73.

INTRODUCTION

We consider an infinitely repeated game that is *perturbed* in the sense of Harsanyi (1973); i.e., payoff functions vary randomly around a mean that defines the *unperturbed* (or *classical*) game, and *adaptive* in that players' actions are governed by experience. Following Fudenberg and Kreps (1993), we postulate that long-term behavior evolves according to *fictitious play*: At each game, each player knows her own payoff function for the next game, and chooses an action (pure strategy) calculated to optimize her payoff under the assumption that other players use the mixed strate-

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gies given by the empirical frequencies of past actions. In this article we present a mathematical framework for investigating such games for arbitrary finite numbers of players and actions. Applications are made to specific games and general theory.

Background. Fictitious play was introduced many years ago as an iterative method of solving an iterated unperturbed game [Brown (1951)]. Convergence was proved for 2×2 games [Robinson (1951), Miyasawa (1961), and Metrick and Polak (1994)], and for $n \times m$ games (n players, m actions each) in which players have identical payoff functions [Monderer and Shapley (1996)]; but Shapley (1964) exhibited 2×3 games without convergence. Cowan (1992) rigorously demonstrated chaotic behavior in 2×4 games.

Fictitious play has been examined as a learning rule for perturbed games [Fudenberg and Kreps (1993), Kaniovski and Young (1995), Benaïm and Hirsch (1994), and Fudenberg and Levine (1995, 1998)]. Books on adaptive games from a variety of perspectives include Weibull (1995), Vega-Redondo (1996), Samuelson (1997), Young (1998), Fudenberg and Levine (1998), and Hofbauer and Sigmund (1998); see also the special issue of *Games and Economic Behavior* [Sigmund and Young (1995)].

From now on we assume given an adaptive perturbed game in which players use fictitious play to choose their actions.

Let a_k denote the kth *action profile*, a vector listing the actions played in game k. The kth *state vector* x_k lists the empirical frequencies of each player's actions in the first k games. In fictitious play, the conditional probability distributions of a_{k+1} and x_{k+1} are functions of x_k and the random payoff function for game $k+1$. These vectors form two stochastic processes: the *action sequence* $\{a_k\}$ and the *state sequence* $\{x_k\}$. The latter is a Markov process with countably many states, essentially a generalized urn process [Arthur *et al*. (1987)]. Our results concern the probable behavior of state sequences, especially the location and shape of their limit sets, and the long-term averages of functions of action profiles, such as correlations of actions.

The following results are known for fictitious play for 2×2 games: If the unperturbed game has a unique Nash equilibrium (possibly mixed), then almost surely the state sequence converges to it [Fudenberg and Kreps (1993)]. Under generic conditions, when the payoffs are subject to a certain types of sufficiently small independent random perturbations, the state sequence converges almost surely to an equilibrium of the perturbed game which is close to a Nash equilibrium e of the unperturbed game; and e is a pure equilibrium of the unperturbed game, if any exists [Kaniovski and Young (1995) and Benaïm and Hirsch (1994)].

Up to now very little has been known, beyond these convergence theorems, about fictitious play for perturbed games with more than two players. A basic mathematical task is to devise methods for distinguishing between convergence and nonconvergence. Where convergence is certain in games with multiple equilibria, it is a challenging problem to estimate the probability distribution of the limiting equilibria. Where convergence is not certain, knowledge of the probable location and form of limit sets of sample paths is desirable.

New Results. Our main tool for investigating the Markov process $\{x_k\}$ is the dynamical system generated by the deterministic *game vector field* F in the compact convex polyhedron S comprising all possible state vectors: $F(x)$ points in the direction of the conditional expected change in the state vector, given that the current state is x [Eqs. (16), (17), (18)]. Equilibria (zeroes) of F are Nash distribution equilibria of the perturbed game (Definition 1.3), and solution curves to the *game differential equation* $dx/dt = F(x)$ may be viewed as caricatures of state sequences $\{x_k\}$. Our fundamental result, the limit set theorem 3.3, characterizes limit sets of sample paths in terms of the dynamical system Φ generated by F. Special properties of Φ for various classes of games, derived from results in dynamical systems theory, lead to conclusions about probable behavior of state sequences.

For a broad class of 2×2 games, including those with countably many Nash distribution equilibria, we prove almost sure convergence of the state sequence (Theorem 2.2); this extends the results of Fudenberg and Kreps (1993) for unique equilibrium games, and Kaniovski and Young (1995) for small noise. Further convergence theorems are obtained for several classes of $n \times 2$ coordination and anticoordination games (Theorems 6.10 and 6.11).

There is, however, no reason to expect convergence in general: For broad parameter ranges and not too much noise, nonconvergence is almost certain in an adaptive perturbed version of the 3-player matching game of Jordan (1993), even though there is a unique Nash distribution equilibrium (Theorem 6.9).

For fictitious play in general perturbed games, under mild restrictions there is positive probability that the limit set of the state sequence lies in any given attractor of the flow of F (Theorem 5.4). In particular, every asymptotically stable equilibrium has positive probability of being selected. On the other hand, there is zero probability of convergence to a linearly unstable equilibrium (Theorem 5.1). If $x_k \to x_*$ with positive probability, and the noise is diffuse near x_*, then almost surely all action profiles—even highly inefficient ones—are played infinitely often (Proposition 5.6). Theorem 4.1 gives information on long-term averages of functions of action profiles.

Outline of Contents. Section 1 gives the basic mathematical setup of perturbed games, fictitious play, Markov processes and the game vector field. Convergence theorems for 2×2 games are stated in Section 2. Section 3 analyzes dynamics of the game vector field and presents the limit set theorem. In Section 4 convergence for 2×2 games is proved, and also a result on limit sets of cumulative averages of general functions of action profiles. Section 5 contains results on equilibrium selection and action sequences for games with arbitrary numbers of players and pure strategies. In Section 6 we prove results on convergence and nonconvergence in $n \times 2$ coordination and anticoordination games, and constraints on sample paths are obtained.

1. GAMES WITH RANDOMLY PERTURBED PAYOFFS

In this section we introduce the adaptive model considered by Fudenberg and Kreps (1993) in which players determine pure by privately observed noisy payoffs, in the spirit of Harsanyi's 1973 purification theorem.

We denote the set $\{0, 1, 2, \dots\}$ of natural numbers by $\mathbf{N}$, the set of positive natural numbers by $\mathbf{N}_+$, and the Euclidean norm of $z \in \mathbf{R}^n$ by $\|z\| = \sqrt{\sum_{j=1}^n z_j^2}$.

The basic model is an infinitely repeated game played by $\mu \in \mathbf{N}_+$ players labeled by $i \in \{1, \dots, \mu\}$ at times $k = 1, 2 \dots$. It is convenient to use superscript $-i$ to refer to the set of players $\neq i$. Thus in a game with two players, $-i$ refers to the opponent of player i.

We assume that player i has a fixed, finite *action set* $A^i = \{1, 2, \dots, d_i\}$. The Cartesian product of the other players' action sets is denoted by A^{-i}; it has cardinality $d_{-i} = \prod_{j \neq i} d_j$. The set of *action profiles* is $A = A^1 \times \cdots \times A^{\mu}$, often identified with $A^i \times A^{-i}$.

The set S^i of mixed strategies for player i is the unit simplex $\Delta^{d_i - 1} \subset \mathbf{R}^{d_i}$ of dimension $d_i - 1$ spanned by the unit coordinate vectors:

$$S^i = \left\{ z \in \mathbf{R}^{d_i} \colon \sum_{l=1}^{d_i} z_l = 1,\ z_l \geq 0 \right\}.$$

We identify S^i with the set of probability measures on A^i. If $l \in A^i$, the corresponding vertex of S^i is denoted by $\mathbf{a}^l \in S^i$: the lth component of $\mathbf{a}^l$ is 1 for $l = a^i$ and 0 otherwise.

We set $S^{-i} = \prod_{j \neq i} S^j$. An element $r \in S^{-i}$ is a list $\{r^j\}_{j \neq i}$ of mixed strategies for player i's opponents.

The game's *state space* is the compact convex polyhedron,

$$S = S^1 \times \cdots \times S^\mu \subset \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_\mu} = \mathbf{R}^d,$$

where $d = \sum_{i=1}^{\mu} d_i$. There are natural identifications of S with $S^i \times S^{-i}$.

A *state* $s = (s^1, \ldots, s^\mu) \in S$ of the game is a list of mixed strategies, often interpreted as empirical frequencies of past actions. Given s, the mixed strategy profile of player i's opponents is obtained by deleting s^i from s, and denoted by $s^{-i} \in S^{-i}$; thus s is identified with (s^i, s^{-i}).

The payoffs to player i are determined by her *payoff function*

$$V^i \colon A^i \times A^{-i} \to \mathbf{R}.$$

When she plays $l \in A^i$ and her opponents play $r \in A^{-i}$, her payoff is $V^i[l,r]$. We identify V^i with a matrix of shape $d_i \times d_{-i}$.

Any function $\Psi \colon A \to \mathbf{R}$ extends to a unique d-linear function $\mathbf{R}^d \to \mathbf{R}$. This map, and its restriction to S, will also be denoted by Ψ. In this way the payoff function V^i has a *canonical extension* to S. If $p = (p^1, \ldots, p^\mu) \in S$ is interpreted as a list of mixed strategies, then $V^i(p)$ is the corresponding expected payoff to player i.

Nash Distribution Equilibria

Before analyzing fictitious play, we review one-shot games in which payoffs are randomly perturbed; here we do not consider repeated plays or adaptation.

For clarify, if W is a matrix we may denote W_{ij} by $W[i, j]$.

Consider a classical μ-player game in which V^i denotes the payoff matrix for player i. A corresponding *perturbed* (or *augmented*) game [Fudenberg and Kreps (1993)] is specified by random $d_i \times d_{-i}$ matrices,

$$\mathbf{U}^i = V^i + \mathbf{E}^i, \qquad i = 1, \ldots, \mu; \tag{1}$$

where each $\mathbf{E}^i$ is a matrix-valued random variable with zero mean. There is no assumption here that $\mathbf{E}^i$ and $\mathbf{E}^j$ are independent for $i \neq j$. We refer to this as a *perturbed game*.

When playing the game, each player knows only her own payoff matrix. Properly speaking, the data (1) specify not a single game but a random game.

Suppose $r \in S^{-i}$ lists a set of mixed strategies for player i's opponents. An action $l \in A^i$ is called a *best response* of player i to r if it maximizes the expected payoff to player i, assuming that she plays l and her opponents plays r. Thus

$$l = \mathrm{Argmax}_{m \in \{1, \dots, d_i\}} \mathbf{U}^i[m, r].$$

To ensure uniqueness of the best response, we assume from now on that the random variables $\mathbf{U}^i$ satisfy the following *ad hoc* condition:

HYPOTHESIS 1.1. *For every mixed strategy $z \in S^{-i}$ and every $m, l \in A^i$ with $m \neq l$,*

$$\mathsf{P}\{\mathbf{U}^i[m, z] = \mathbf{U}^i[l, z]\} = 0.$$

For each $z \in S^{-i}$ the set of $d_i \times d_{-i}$ matrices U such that $U[m, z] = U[l, z]$ for some $l \neq m$ has zero Lebesgue measure. Therefore Hypothesis 1.1 is easily seen to be valid for a large class of random matrices.

The *best response map* of player i is the deterministic map

$$\beta^i \colon S^{-i} \to S^i$$

defined as follows: Given $s^{-i} \in S^{-i}$ and $l \in \{1, \dots, d_i\}$, let $\beta^i(s^{-i})_l$ denote the probability that action l is the best response of player i when the opponent uses the mixed strategy s^{-i}. Thus for each $l \in \{1, \dots, d_i\}$:

$$\beta^i(s^{-i})_l = \mathsf{P}\left\{l = \mathrm{Argmax}_{m \in \{1, \dots, d_i\}} \mathbf{U}^i[m, s^{-i}]\right\}. \tag{2}$$

It is easy to see that β^i is at least as smooth as the random variable $\mathbf{E}^i$.

DEFINITION 1.2. The *Nash map*

$$\nu \colon S \to S$$

is defined by

$$\nu(s^1, \dots, s^\mu) = \left(\beta^1(s^{-1}), \dots, \beta^\mu(s^{-\mu})\right). \tag{3}$$

To a joint mixed strategy s, the Nash map ν assigns the list comprising each player's best response to the opponents' strategies listed in s. Notice that ν is Lipschitz, C^r or analytic provided the probability distribution functions of the matrices $\mathbf{E}^i$ have the corresponding property.

DEFINITION 1.3. A *Nash distribution equilibrium* of the augmented game is a fixed point $s_* \in S$ of the Nash map.

Remark 1.4. If ν is continuous, the *existence* of Nash distribution equilibria follows from the Brouwer fixed point theorem. But the *calculation* of these equilibria by the players would require that the marginal probability laws of the random payoffs $\mathbf{U}^i$ are common knowledge. In this situation a Nash distribution equilibrium is the probability distribution of a

Nash equilibrium for the augmented game [Harsanyi (1973) and Fudenberg and Kreps (1993)]. Although we do not assume players have this knowledge, we show that in many cases the adaptive process of fictitious play causes empirical frequencies to converge to some Nash distribution equilibrium.

The set of probability measures on the finite set A is denoted by $\mathscr{P}(A)$: it is naturally identified with the simplex $\Delta^{d-1} \subset \mathbf{R}^d$. Closely related to the Nash map is the *joint best response map*

$$\hat{\nu}: S \to \mathscr{P}(A) = \Delta^{d-1}$$

defined for $a = (a^1 \cdots a^\mu) \in A$ by

$$\hat{\nu}(s)_a = \mathsf{P}\Big\{a^i = \operatorname{Argmax}_{j \in \{1,\dots,d_i\}} \mathbf{U}^i\big[j, s^{-i}\big] : i = 1,\dots,\mu\Big\}. \tag{4}$$

The number $\hat{\nu}(s)_a$ gives the probability that $a \in A$ constitutes the joint best responses to the mixed strategy profile $s \in S$. When the matrices $\{\mathbf{E}^i\}$ are independent,

$$\hat{\nu}(s)_a = \prod_{i=1}^{\mu} \beta^i(s^{-i})_{a^i}. \tag{5}$$

Fictitious Play and the Game Vector Field

We consider the state space,

$$S = \Delta^{d_1 - 1} \times \cdots \times \Delta^{d_\mu - 1} \subset \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_\mu}, \tag{6}$$

to be a submanifold (with corners) in $\mathbf{R}^d$, $d = d_1 + \cdots + d_\mu$. Its tangent space at every point is identified with the linear subspace

$$TS = \left\{ \big(y^1,\dots,y^\mu\big) \in \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_\mu} : \sum_{j=1}^{d_i} y^i_j = 0, i = 1,\dots,\mu \right\}. \tag{7}$$

A *vector field* on S is a map from S to TS.

The action of player i in game $k \in \mathbf{N}_+$ is denoted by $a^i_k \in A^i$, and $\mathbf{a}^i_k$ denotes the corresponding vertex of the simplex $S^i = \Delta^{d_i - 1}$. We may consider $\mathbf{a}^i_k$ either as a pure strategy in the simplex of all strategies, or as a Dirac measure in the simplex of measures on the finite action set A^i.

The μ-tuple

$$a_k = \big(a^1_k,\dots,a^\mu_k\big) \in A$$

is the *action profile* at time k. Corresponding to a_k is the *action vertex*

$$\mathbf{a}_k = \left(\mathbf{a}_k^1, \dots, \mathbf{a}_k^\mu\right) \in S,$$

which is an extreme point of the convex polyhedron S. Thus a sequence of games produces the sequence $\{a_k\}$ of action profiles, and the equivalent sequence $\{\mathbf{a}_k\}$ of vertices of S.

The *empirical frequency vector* of player i after the first k games is

$$x_k^i = \frac{1}{k} \sum_{j=1}^{k} \mathbf{a}_j^i \in S^i. \tag{8}$$

The mth component of x_k^i is the proportion of times in the first k games that player i has played action $m \in A^i$.

The *state of the game at time* k is the vector $x_k \in S$ listing the players' empirical frequencies at time k:

$$x_k = \left(x_k^1, \dots, x_m^\mu\right) \in S^1 \times \cdots \times S^\mu.$$

Note the recursion

$$x_{k+1} = \frac{k}{k+1} x_k + \frac{1}{k+1} \mathbf{a}_{k+1}. \tag{9}$$

We call $\{x_k\}_{k \in \mathbf{N}}$ the *state sequence* of the infinitely repeated game.

The *empirical joint frequency* tensor at time k is the map $C_k \colon A \to \mathbf{R}$ which assigns to an action profile the frequency with which it was played in the first k games. Formally,

$$C_k = \frac{1}{k} \sum_{j=1}^{k} \mathbf{I}_{a_j}, \tag{10}$$

where $\mathbf{I}_a$ denotes the indicator function of an action profile $a \in A$. For 2×2 games C_k is identified with a 2×2 matrix.

An *adaptive perturbed game* is a sequence of independent, identically distributed (IID) perturbed games. It is specified by data $(A^i, \{\mathbf{U}_k^i\}_{k \in \mathbf{N}_+})$ where A^i is player i's action set and $\mathbf{U}_k^i$ is a random $d_i \times d_{-i}$ matrix of the form

$$\mathbf{U}_k^i = V^i + \mathbf{E}_k^i, \qquad k = 1, 2, \dots, \tag{11}$$

with V_i a fixed matrix, and $\{\mathbf{E}_k^i\}_{k \in \mathbf{N}_+}$ a sequence of IID random matrices having mean zero; $\mathbf{E}_k^i$ and $\mathbf{E}_k^j$ need not be independent. The matrices V^i define the unperturbed game.

After the first k games and the random selection of noise matrices $\mathbf{E}^j_{k+1}$, the players play the augmented game defined by the payoff functions $\mathbf{U}^i_{k+1}$. They use the following adaptive procedure, *fictitious play*, for determining their actions a^i_{k+1} in game $k+1$. Player i knows her own payoff matrix $\mathbf{U}^i_{k+1}$ for round $k+1$ and her opponents' empirical frequency vector $x_k^{-i} \in S^{-i}$. She assumes the opponents use the mixed strategy x_k^{-i}, and she computes and plays her best response action $a^i_{k+1} \in A^i$ to x_k^{-i}. Thus

$$a^i_{k+1} = \operatorname{Argmax}_{m \in \{1, \dots, d_i\}} \mathbf{U}^i_{k+1}\left[m, x_k^{-i}\right]. \tag{12}$$

Therefore *the state sequence* $\{x_k\}$ *is a nonstationary discrete-time Markov process, with values in the compact, convex set* S.

Using the best response maps β^i [Eq. (2)] and the Nash map ν [Eq. (3)] we obtain the formulas,

$$\mathsf{P}\left\{a^i_{k+1} = l \mid x_k = x\right\} = \left(\beta^i(x^{-i})\right)_l, \tag{13}$$

$$\mathsf{E}\left(\mathbf{a}_{k+1} \mid x_k\right) = \nu(x^k). \tag{14}$$

From (13) and (14) we derive:

$$\mathsf{E}\left(x_{k+1} - x_k \mid x_k\right) = \frac{1}{k+1}\left(-x_k + \nu(x^k)\right). \tag{15}$$

The *game vector field* F: $S \to TS$ is defined as

$$\begin{aligned} F&: S \to TS \subset \mathbf{R}^{d_1} \times \cdots \times \mathbf{R}^{d_\mu}, \\ F(x) &= -x + \nu(x). \end{aligned} \tag{16}$$

$F(x)$ measures the extent to which x is not a Nash distribution equilibrium (Definition 1.3). Equations (14) and (15) give further meaning to F,

$$F(x) = \mathsf{E}\left(\mathbf{a}_{k+1} - x_k \mid x_k = x\right) \tag{17}$$

$$= (k+1)\mathsf{E}\left(x_{k+1} - x_k \mid x_k = x\right). \tag{18}$$

In other words: If the state at time k is x, then $F(x)$ is $k+1$ times the expected change in the state.

Our analysis of fictitious play relies on a close connection between limits of sample paths $\{x_k\}$, and the dynamics of the deterministic *game differential equation* $dx/dt = F(x)$. In a similar way, we will analyze the empirical joint frequencies in terms of the dynamics of the system of differential equations

$$\frac{dx}{dt} = F(x),$$
$$\frac{dC}{dt} = -C + \hat{\nu}(x).$$

In the next section we give some interesting consequences of the general theory which can be easily stated without the full mathematical formalism of Section 3.

We record for reference the obvious but useful fact alluded to above:

PROPOSITION 1.5. *The zeroes of the game vector field F are the Nash distribution equilibria.*

2. ASYMPTOTIC BEHAVIOR OF ADAPTIVE 2 × 2 PERTURBED GAMES

In this section we describe the asymptotic behavior of the state sequence for 2×2 games having arbitrarily many Nash distribution equilibria. Proofs are postponed to Sections 3 and 4.

Convergence of Empirical Frequencies

Let $(A^1, \{\mathbf{U}_k^1\}, A^2, \{\mathbf{U}_k^2\})$ be an adaptive 2-player perturbed game in which $A^1 = A^2 = \{1, 2\}$ (each player has two pure strategies), and the random 2×2 matrices $\mathbf{U}_k^i$ are as in (11). Each action set A^i, $i = 1, 2$ has two elements. We identify the one-dimensional simplex S^i with the closed unit interval $I = [0, 1]$ by the map $(s, 1 - s) \mapsto s$. In this way a game state in the original *simplicial coordinates*

$$\big((x^1, 1 - x^1), (x^2, 1 - x^2)\big)$$

is given the *interval coordinates*

$$(x^1, x^2) \in I \times I.$$

In addition to Hypothesis 1.1, we further assume:

HYPOTHESIS 2.1. *The Nash map ν: $I \times I \to I \times I$ (Eq. (3)) is Lipschitz continuous.*

This technical assumption is crucial to our proofs, as it validates the standard theorems of existence, uniqueness, and continuity of solutions to differential equations. In some cases, as, for example, in classical fictitious play for games with fixed payoff matrices, it is not satisfied; but for many perturbed games it holds.

Given a state sequence $\{x_k\}_{k \in \mathbf{N}}$ in $I \times I$ resulting from infinitely repeated fictitious play—thus a sample path of a stochastic process satisfying (5)—we say that a point $x_* \in I \times I$ is a *limit point* of $\{x_k\}_{k \in \mathbf{N}}$ if $\lim_{i \to \infty} x_{k_i} = x_*$ for some sequence $k_i \to \infty$. The set of such limit points is the *state limit set* $L\{x_k\}$. Because the sequence $\{x_k\}$ is a random variable, the state limit set is a set-valued random variable, that is, a function assigning a set to each element of a probability space. By extension of the usual probabilistic convention we refer to it as a set, just as a numerical random variable is treated as a number.

The following theorem extends Proposition 8.1 of Fudenberg and Kreps (1993) to arbitrary 2×2 games; it also generalizes a result of Kaniovski and Young (1995) for small noise.

Recall that $\hat{\nu}$ is the joint best response map [Eq. (4)]. Let $\mathscr{E} \subset I \times I$ denote the set of Nash distribution equilibria.

THEOREM 2.2. *Consider a* 2×2 *adaptive perturbed game satisfying Hypotheses* 1.1 *and* 2.1. *Let* $\{x_k\}$ *denote the sequence of empirical frequency vectors, and* $\{C_k\}$ *the sequence of empirical joint frequency matrices. Then*:

(a) *With probability* 1, *the limit set* $L\{x_k\}$ *is a point or a compact arc in* $\mathscr{E}$ *that is the graph of a strictly increasing or decreasing function.*

(b) *If* $\mathscr{E}$ *is finite or countably infinite then almost surely* $\{x_k\}$ *converges to a Nash distribution equilibrium.*

(c) *Let* x_* *be a Nash distribution equilibrium such that* $\mathsf{P}\{x_k \to x_*\} > 0$. *Then*

$$\mathsf{P}\{C_k \to \hat{\nu}(x_*) | x_k \to x_*\} = 1.$$

This result shows that when there are only countably many Nash distribution equilibria, players behave in the long run as though they have computed the equilibria of the game. (Players who know the probability laws of the stochastic matrices $\mathbf{U}^1$ and $\mathbf{U}^2$, and who read Remark 1.4, can do that.)

It is a generic condition for a vector field to have finite equilibrium set, that is, it holds for a dense open set of C^1 (continuously differentiable) vector fields on any compact manifold. In this sense, (b) and (c) imply that for most games of the type considered, sequences of sample paths and joint frequency matrices converge almost surely. The following corollary of (a) gives an easily verified criterion for finiteness of equilibria, hence for convergence. The hypothesis holds, for example, if entries in the payoff matrices (11) are independent with analytic density functions:

COROLLARY 2.3. *In addition to the assumptions of Theorem* 2.2, *assume that the Nash map extends to a real analytic map* $\mathbf{R}^2 \to I \times I$. *Then* $\mathscr{E}$ *is finite, and sample paths converge almost surely, as do joint frequency matrices.*

Theorem 2.2(c) and Eq. (5) imply:

COROLLARY 2.4. *Let* $x_* \in I \times I$ *be as in Theorem* 2.2(c). *Suppose that for each* k, *the two payoff perturbation matrices* $\mathbf{E}_k^1, \mathbf{E}_k^2$ *are independent random variables. Then almost surely the sequence* $\{C_k\}$ *of empirical joint frequency matrices converges to the* 2×2 *matrix whose* (i, j)*th entry is* $x_*^{\,i} x_*^{\,j}$.

The proof of Theorem 2.2 makes use of the material in Sections 3 and 4. Part (a) is a consequence of the limit set theorem 3.3; part (b) derives from Lemma 4.3, and (c) from Theorem 4.1. Details are given in Section 4.

Remark 2.5. Our assumptions on the noise matrices in Theorem 2.2 and its corollaries are less restrictive than those of Fudenberg and Kreps (1993), who require that the uncertainty in the payoff to a player depends only on that player's action [Hypothesis 6.1(ii)]. Corollary 2.3 implies almost sure convergence of state sequences for noise matrices of this form having analytic probability distribution functions.

3. CONTINUOUS TIME DYNAMICS AND FICTITIOUS PLAY

This section introduces the mathematical basis for our analysis of adaptive perturbed games, in terms of game vector field $F(x) = -x + \nu(x)$ [Eq. (16)] and the corresponding game differential equation in S:

$$\frac{dx}{dt} = F(x), \tag{19}$$

whose solution flow is denoted by Φ.

Invariant Sets and Attractors

Throughout the remainder of this section we assume:

HYPOTHESIS 3.1. *The game vector field* F *is locally Lipschitz.*

Next we introduce tools enabling us to analyze game asymptotics in terms of the dynamics of F. Denoting the dimension of S by

$$n = \sum_1^{\mu} (d_i - 1) = d - \mu,$$

we identify S with a compact convex subset of $\mathbf{R}^n$ having nonempty interior, and its tangent space TS with $\mathbf{R}^n$; for convenience we assume the origin belongs to the interior of S. Under this identification, the game vector field is a Lipschitz map

$$F: S \to \mathbf{R}^n.$$

It is convenient to extend F to a Lipschitz map defined on all of $\mathbf{R}^n$ as follows: for x outside S, set $F(x) = F(\rho(x))$ where ρ: $\mathbf{R}^n \to S$ is the retraction along rays emanating from some point of S. This makes F: $\mathbf{R}^n \to \mathbf{R}^n$ a bounded Lipschitz map.

It follows that F is *completely integrable*, meaning that its trajectories are defined for all values of t. Therefore F generates a *flow*

$$\Phi: \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n,$$

where for each $y \in \mathbf{R}^n$, the function $t \mapsto \Phi_t(y)$ is the solution to the initial value problem

$$\frac{dx}{dt} = F(x), \tag{20}$$

$$x(0) = y. \tag{21}$$

The parameterized curve $t \mapsto \Phi_t(y)$ is the *trajectory* of y; the image of this curve is the *orbit* of y.

For each fixed $t \in \mathbf{R}$, the map $y \mapsto \Phi_t y$ is a homeomorphism of $\mathbf{R}^n$. We view the flow as the collection of maps $\{\Phi_t: \mathbf{R}^n \to \mathbf{R}^n\}_{t \in \mathbf{R}}$, with Φ_0 denoting the identity map of $\mathbf{R}^n$. Uniqueness of solutions to Eq. (??) implies the composition law $\Phi_s \circ \Phi_t = \Phi_{s+t}$.

An *equilibrium* (or stationary point) p is a zero of F; this is equivalent, by uniqueness of solutions, to $\Phi_t(p) = p$ for all t. A point y is *periodic* if $\Phi_T(y) = y$ for some $T > 0$. The *limit set* (more properly, the omega limit set) of x (and of its orbit and trajectory) is the set of points of the form $\lim_{k \to \infty} \Phi_{t_k}(x)$ for some sequence $t_k \to \infty$.

A set Q is *invariant* if $\Phi_t(Q) = Q$ for all t; it is *forward invariant* if this holds for all $t \geq 0$. It can be shown that S is forward invariant, using convexity of S and the fact that $F(x) - x \in S$ for all S [see Definition 1.2 and Eq. (16)]. Equilibria, periodic orbits, and omega limit sets are also invariant.

$\Phi|Q$ denotes the restriction of Φ to an invariant set Q, that is, the collection of maps $\Phi_t|Q$: $Q \to Q$. When Q is forward invariant but not invariant, by a slight abuse of language $\Phi|Q$ denotes the *semiflow* $\{\Phi_t|Q\}_{t \geq 0}$. If $Q = \mathbf{R}^n$ we identify $\Phi|Q$ with Φ.

We fix attention on $\Phi|Q$ where Q denotes a forward invariant set. A subset K of Q is an *attractor* (for $\Phi|Q$) provided K is nonempty, compact, and invariant, and there is a neighborhood $U \subset Q$ of K with the property that $\lim_{t \to \infty} \operatorname{dist}(\Phi_t x, K) = 0$ uniformly for $x \in U$. Here $\operatorname{dist}(a, K)$ means the distance from a to the nearest point of K. Speaking loosely, we say that an attractor captures the trajectories of all nearby points. An asymptotically stable limit cycle or equilibrium is an example of an attractor. If Q is compact, it is trivially an attractor; any other attractor is a *proper* attractor.

Let $K \subset Q$ be an attractor. Its *basin* is the open, forward invariant set of all points of Q whose trajectories tend to K. If the basin is all of Q then K is the *global* attractor for $\Phi|Q$. It can be shown that S contains a global attractor for Φ (taking $Q = \mathbf{R}^d$).

We call Q *attractor-free* if Q is a nonempty compact invariant set and there is no proper attractor for $\Phi|Q$. Many types of compact invariant sets are known to be attractor-free, such as periodic orbits, omega limit sets, and supports of ergodic invariant Borel measures.

The interplay between attractors and attractor-free sets is very helpful in analyzing long-term dynamical behavior. The following simple but useful result says that an attractor-free compact invariant set is contained in every attractor whose basin it meets:

LEMMA 3.2. *Let Q be forward invariant for a flow Φ, let $\Lambda \subset Q$ be compact and invariant, and assume $\Phi|\Lambda$ is attractor-free. Suppose $K \subset Q$ is an attractor for $\Phi|Q$ and the basin B of K meets Λ. Then $K \supset \Lambda$.*

Proof. $\Lambda \cap K$ is nonempty: Given $x \in \Lambda \cap B$, there is a sequence $t_k \to \infty$ such that $\Phi_{t_k} x$ converges, necessarily to a point in $\Lambda \cap K$. It is now easy to see that $\Lambda \cap K$ is an attractor for $\Phi|\Lambda$. Therefore Λ, being attractor-free, coincides with $\Lambda \cap K$. QED

The Limit Set Theorem

The following theorem describes the state limit set $L\{x_k\}$ in terms of the dynamics of the game vector field F.

THEOREM 3.3 (Limit Set Theorem). *With probability* 1, *the state limit set $L\{x_k\}$ has the following properties*:

(a) *$L\{x_k\}$ is an invariant set for the flow of the game vector field F.*

(b) *$L\{x_k\}$ is compact, connected and attractor-free.*

Being a compact invariant set, $L\{x_k\}$ lies in the global attractor. From Lemma 3.2 we obtain a very useful result:

COROLLARY 3.4. *With probability* 1, *the state limit set is contained in every attractor whose basin it meets*.

The proof of Theorem 3.3 is based on the important recursion

$$x_{k+1} - x_k = \frac{1}{k+1}\left[F(x_k) + Z_{k+1}\right], \tag{22}$$

where the random variable $\{Z_{k+1}\}$ is defined by (22):

$$Z_{k+1} = (k+1)(x_{k+1} - x_k) - F(x_k).$$

LEMMA 3.5. (i) *The vector field F is locally Lipschitz*.
(ii) *There exists $R > 0$ such that $\|x_k\| < R$, $\|Z_{k+1}\| < R$ for all k*.
(iii) $\mathsf{E}(Z_{k+1}|x_k) = 0$.

Proof. Statement (i) is Hypothesis 3.1, and (ii) and (iii) follow from Eqs. (15) and (18). QED

Proof of Theorem 3.3. A recursion such as (22) is a particular form of a *stochastic approximation process*. Theorem 3.3 follows from a general result proved by Benaïm (1996) [see also Benaïm and Hirsch (1996)], concerning the asymptotic behavior of stochastic approximation processes satisfying Lemma 3.5. QED

4. APPLICATIONS OF THE LIMIT SET THEOREM

Correlated Strategies

The limit set theorem 3.3 characterizes limit sets of the vector of empirical frequencies of players' actions. Here we consider limit sets of long run averages of an arbitrary deterministic function evaluated on the sequence of action profiles. In particular, we estimate the location of the limit set of the sequence $\{C_k\}$ of joint frequency tensors.

Consider a function $H: A \to \mathbf{R}^m$. After each game, H is evaluated on the current action profile $a_k \in A$, producing a stochastic process $\{H(a_k)\}$. The kth empirical frequency vector of the process is the random vector

$$\langle H \rangle_k = \frac{1}{k}\sum_{j=1}^{k} H(a_j).$$

Denote by $L[H] \subset \mathbf{R}^m$ the limit set of the stochastic sequence $\{\langle H \rangle_k\}$.

The following result uses the machinery of the limit set theorem to estimate the location of $L[H]$ in terms of the deterministic function $\overline{H}$

giving the expected value of H after the next play, conditioned on the current state,

$$\overline{H}: S \to \mathbf{R}^m,$$
$$\overline{H}(x) = \mathsf{E}\big(H(a_{k+1})|x_k = x\big) = \sum_{a \in A} H(a)\hat{\nu}(x)_a,$$

where $\hat{\nu}(x)_a$ is the value at $a \in A$ of the joint best response map [Eq. (4)]. Hypothesis 3.1 implies $\overline{H}$ is Lipschitz continuous.

THEOREM 4.1. *For any H: $A \to \mathbf{R}^m$, the map $\overline{H}$: $S \to \mathbf{R}^m$ has the following properties*:

(a) *The limit set of the sequence $\{\langle H \rangle_k\}$ is almost surely a compact connected subset of the convex hull of $\overline{H}(L(\{x_k\}))$.*

(b) *If the state sequence converges almost surely to a unique equilibrium x_*, then $\langle H \rangle_k$ converges almost surely to $\overline{H}(x_*)$.*

When H assigns to action profile a its indicator function, $\langle H \rangle_k$ is just the empirical joint frequency tensor C_k [Eq. (10)]. Theorem 4.1 implies that the limit set of the sequence $\{C_k\}$ is almost surely a compact connected set contained in the convex hull of $\hat{\nu}(L\{x_k\})$, where $\hat{\nu}$ is the joint best response map [Eq. (4)]. Therefore in case $x_k \to x_*$, we conclude that $C_k \to \hat{\nu}(x_*)$ almost surely. In this way, part (c) of Theorem 2.2 is proved.

Proof of Theorem 4.1. Let $u_k = \langle H \rangle_k \in \mathbf{R}^m$. A computation shows that the process

$$\big\{(x_k, u_k) \in \mathbf{R}^n \times \mathbf{R}^m\big\}_{k \in \mathbf{N}}$$

is a stochastic approximation process (22) whose corresponding ordinary differential equation in $\mathbf{R}^n \times \mathbf{R}^m$ is given by the system,

$$\frac{dx}{dt} = F(x), \qquad \frac{du}{dt} = -u + \overline{H}(x), \tag{23}$$

where F denotes the game vector field. Notice that the evolution of $x(t)$ is independent of u.

Let $L \subset \mathbf{R}^n$ denote the limit set of $\{x_k\}$, and let $L' \subset L \times \mathbf{R}^m$ denote the limit set of the sequence $\{(x_k, u_k)\}$. Since L is invariant under F, it is clear that $L \times \mathbf{R}^m$ is invariant under System (23).

Let Ψ denote the flow in $\mathbf{R}^n \times \mathbf{R}^m$ of Eq. (23) and let $C \subset \mathbf{R}^m$ denote the closed convex hull of $\overline{H}(L)$. We claim $L \times C$ *contains global attractor for* $\Psi|(L \times \mathbf{R}^m)$. If this claim is true, then $L' \subset L \times C$ by Lemma 3.2, which implies the conclusion (a) of Theorem 4.1; and (a) implies (b).

To prove the claim, define $V\colon L \times \mathbf{R}^m \to \mathbf{R}_+$ by $V(x,u) = \operatorname{dist}(u, C)$. By convexity of C,

$$\begin{aligned}\operatorname{dist}\big((1-t)u + t\overline{H}(x), C\big) &\le (1-t)\operatorname{dist}(u,C) + t\operatorname{dist}\big(\overline{H}(x), C\big)\\ &= (1-t)\operatorname{dist}(u,C),\end{aligned}$$

where the last equality holds because $\overline{H})(x) \in C$; here $\operatorname{dist}(y, C)$ is the distance from y to the nearest point of C. Thus

$$\frac{\operatorname{dist}\big(u + t\big(-u + \overline{H}(x)\big), C\big) - \operatorname{dist}(u,C)}{t} \le -\operatorname{dist}(u,C).$$

Now V is a continuous convex function because C is convex, so V admits a right partial derivative with respect to u. Therefore, letting t go to zero in the last inequality gives

$$\frac{d}{dt}V\big(\Psi_t(x,u)\big) \le -V\big(\Psi_t(x,u)\big),$$

whence $V(\Psi_t(x,u)) \le e^{-t}V((x,u))$ by a standard theorem in differential inequalities. This implies that $L \times C$ contains the global attractor for $\Psi|(L \times \mathbf{R}^m)$. QED

Proof of Theorem 2.2

LEMMA 4.2. *The flow* Φ *of the game vector field* F *for an adaptive perturbed* 2×2 *game is area-decreasing in* $I \times I$. *When* F *is* C^1, *its divergence is* -2.

Proof. Here $n = 2$. As usual we identify the state space with $I \times I$. If the game vector field

$$F\colon I \times I \to \mathbf{R}^2, \qquad F(x) = -x + \nu(x)$$

is C^1, then F has negative divergence. For by definition of ν [the Nash map, Eq. (3)],

$$F^i(x^1, x^2) = -x^i + \beta^i(x^{-i}), \qquad i = 1, 2, \tag{24}$$

whence the divergence of F is $\partial F^1/\partial x^1 + \partial F^2/\partial x^2 = -2$. By Liouville's formula [Hartman (1964)], $\frac{d}{dt}\operatorname{Det}(D\Phi_t(x)) = -2$, showing that Φ_t decreases areas exponentially for $t > 0$. The case where F is merely Lipschitz follows by approximating each component β^i by a C^1 map $I \to I$ and applying standard continuity theorems in differential equations. QED

COROLLARY 4.3. *No nonempty open subset of* $I \times I$ *is invariant under* Φ, *and no compact invariant set* $M \subset I \times I$ *separates the plane*.

Proof. The first statement follows from Lemma 4.2. If M separates the plane, its complement has at least one connected component $A \subset I \times I$; but A is an open invariant set. QED

Lemma 4.2 and its proof extend to arbitrary games, giving the following general result:

PROPOSITION 4.4. *For an adaptive perturbed game satisfying Hypotheses* 1.1 *and* 2.1, *the flow* Φ *of the game vector field* F *decreases volume in the state space* S. *When* F *is* C^1, *its divergence is* $-d$ *where* d *is the dimension of* S. *If* $A \subset S$ *has volume* C *then the volume of* $\Phi^t A$ *is* $e^{-td}C$.

Proof of Theorem 2.2. We first prove that the limit set Λ of a state sequence is almost surely a compact connected subset of $\mathscr{E} = F^{-1}(0) \subset S$, the set of Nash distribution equilibria. By Theorem 3.3, Λ is almost surely a compact, connected attractor-free invariant set. Let $\delta \in \{0, 1, 2\}$ denote the topological dimension of Λ [Hurewicz and Wallman (1948)]. If $d = 0$ then Λ, being connected, is a singleton (because a zero-dimensional set is totally disconnected), so Λ is an equilibrium by invariance. If $\delta = 2$, the interior of Λ is nonempty, contradicting Corollary 4.3, so this is impossible. If $\delta = 1$ we use a result of Hirsch and Pugh (1988): Any one-dimensional attractor-free set for a planar flow that does not separate the plane, consists entirely of stationary points. Thus in all cases $\Lambda \subset \mathscr{E}$.

Now $\mathscr{E}$ is the intersection of the graphs of $x^2 = \beta^2(x^1)$ and $x^1 = \beta^1(x^2)$ by Eq. (24). Therefore each compact connected subset of $\mathscr{E}$ maps injectively into I under each of the two coordinate projections $I \times I \to I$. It follows that every connected subset of $\mathscr{E}$ is either a point or a compact arc. This proves (a) and (b) of Theorem 2.2. Assertion (c) follows Theorem 4.1(b). QED

Proof of Corollary 4.3. Let $\tilde{\nu}$: $\mathbf{R}^2 \to I \times I$ be a real analytic extension the Nash map (Definition 1.2). Then $\mathscr{E}$ is the fixed point set of $\tilde{\nu}$, so $\mathscr{E}$ is a compact analytic variety. The preceding proof shows that every connected component of $\mathscr{E}$ is either a point or a compact arc. As a compact analytic variety has a finite number of connected components, and none of them can be an arc, $\mathscr{E}$ is finite. QED

5. EQUILIBRIUM SELECTION AND LOCAL STABILITY

We consider an adaptive perturbed game with any number μ of players and arbitrary finite action sets. Even if we know that state sequences converge almost surely, it is not obvious which equilibrium is selected. Here we address this important question of *equilibrium selection* and the related problem of *path dependence*.

We recall some standard terms from dynamical systems. A map is C^r, $r \geq 1$ if it is differentiable and its partial derivatives up to order r are continuous; C^0 means continuous. It is easy to see that the game vector field $F: S \to TS$ is C^r when the perturbation matrices $\mathbf{E}^i$ in Eq. (1) have C^{r-1} densities.

Assume F is C^1 and let $x_* \in \mathscr{E}$. If the Jacobian matrix $DF(x_*)$ is invertible, the equilibrium x_* is *simple*. If all eigenvalues of the Jacobian matrix $DF(x_*)$ have nonzero real parts, x_* is *hyperbolic*. If all eigenvalues have negative real parts, x_* is *linearly stable*; if some eigenvalue has positive real part, x_* is *linearly unstable*.

We call x_* *asymptotically stable* if there exists a neighborhood $U \subset S$ of x^* such that $\lim_{t\to\infty} \Phi_t(y) = x_*$ uniformly in $y \in U$, where Φ denotes the flow of F [see Eqs. (20), (21)]. It is a standard fact that linearly stable equilibria are asymptotically stable.

Nonconvergence to Unstable Equilibria

Let $x \in S$ be a game state for an adaptive perturbed game with any number of players. The game is *diffuse at* x if whenever the game state is x, every action profile a has a positive probability of being selected at the next play: $\hat{\nu}(x)_a > 0$ for all $a \in A$.

The following theorem shows that unstable equilibria of the game vector field are eliminated as outcomes of diffuse adaptive perturbed games with C^2 game vector fields:

THEOREM 5.1. *Let $x_* \in S$ be a linearly unstable equilibrium of F. If F is C^2 in a neighborhood of x_*, and the game is diffuse at x_*, then* $\mathsf{P}\{\lim_{k\to\infty} x_k = x_*\} = 0$.

Proof. We derive Theorem 5.1 from the following useful result due to Pemantle (1990):

THEOREM 5.2 (Pemantle). *Consider the stochastic approximation process* (22) *in* $\mathbf{R}^n$:

$$x_{k+1} - x_k = \frac{1}{k+1}\left[F(x_k) + Z_{k+1}\right],$$

where the sequence $\{Z_{k+1}\}$ of $\mathbf{R}^n$-valued random variables is a priori bounded with zero conditional expectations. Let x_* be a linearly unstable equilibrium of F. Assume F is C^2 in a neighborhood of x_*, and that there exists $c > 0$ and a neighborhood N of x_* such that for every unit vector $\Theta \in \mathbf{R}^n$, the following *condition holds*:

$$\mathsf{E}\big(\max(0, \langle Z_{k+1}, \Theta \rangle) | x_k \in N\big) > c. \tag{25}$$

Then $\mathsf{P}\{\lim_{k \to \infty} x_k = x_*\} = 0$.

Using the joint best response map, we express the game vector field as

$$F(x) = \sum_{a \in A} \hat{\nu}(x)_a \mathbf{a} - x,$$

by Eqs. (14) and (16). Since $F(x_*) = 0$ we have

$$x_* = \sum_{a \in A} \hat{\nu}(x_*)_a \mathbf{a}. \tag{26}$$

This exhibits x_* as a convex combination of all the extreme points a of S with strictly positive coefficients. Considering S as a convex body in $\mathbf{R}^n$, we have proved that *the diffuse equilibrium* x_* *is in the interior of* S.

We use this to verify Pemantle's hypothesis (25). The function of $x \in S$ defined as

$$\mathsf{E}\big(\max(0, \langle Z_{k+1}, \Theta \rangle) | x_k = x\big)$$

is independent of k and continuous in x. It therefore suffices to show:

$$\mathsf{E}\big(\max(0, \langle Z_{k+1}, \Theta \rangle) | x_k = x_*\big) > 0. \tag{27}$$

From Eq. (15) we have

$$\begin{aligned} Z_{k+1} &= \mathbf{a}_{k+1} - \mathsf{E}(\mathbf{a}_{k+1} | x_k) \\ &= \mathbf{a}_{k+1} - \sum_{a \in A} \hat{\nu}(x_k)_a \mathbf{a}. \end{aligned} \tag{28}$$

Fix a unit vector $\Theta \in TS$. Let A_+ denote the set of extreme points $a \in A$ for which $\langle \mathbf{a} - x_{*0}, \Theta \rangle > 0$. Then A_+ is nonempty: for from Eq. (26) and the identity $\sum_{a \in A} \hat{\nu}_a \equiv 1$ we get:

$$\sum_{a \in A} \langle \mathbf{a} - x_*, \Theta \rangle = \sum_{a \in A} \langle \hat{\nu}(x_*)_a (\mathbf{a} - \mathbf{a}), \Theta \rangle = 0.$$

From the definition (7) of TS there exists $a \in A$ such that $\langle a - x_*, \Theta \rangle \neq 0$, so the last equation implies A_+ is nonempty. We therefore have from (28):

$$
\begin{aligned}
&\mathsf{E}\big(\max(0, \langle Z_{k+1}, \Theta \rangle) | x_k = x_*\big) \\
&\quad = \mathsf{E}\left(\max\left(0, \left\langle \mathbf{a}_{k+1} - \sum_{a \in A} \hat{\nu}_a(x_*)\mathbf{a}, \Theta \right\rangle | x_k = x_* \right)\right) \\
&\quad = \mathsf{E}\big(\max(0, \langle \mathbf{a}_{k+1} - x_*, \Theta \rangle | x_k = x_*)\big) \text{ (by (26))}, \\
&\quad = \sum_{a \in A} \hat{\nu}(x_*)_a \max(0, \langle \mathbf{a} - x_*, \Theta \rangle) \\
&\quad = \sum_{a \in A_+} \hat{\nu}(x_*)_a \langle \mathbf{a} - x_*, \Theta \rangle > 0.
\end{aligned}
$$

This verifies (27), and Theorem 5.1 follows from Pemantle's theorem. QED

Remark 5.3.(*a*) Results similar to Theorem 5.1 can be found in Nevelson and Hasminskii (1973) (for totally unstable points), Arthur *et al.* (1988), Brandière and Duflo (1996) (for linearly unstable points), under various assumptions.

(b) In Theorem 5.1 the assumption that the game is diffuse at x_* is chosen for the sake of simplicity and can be weakened as follows. Let P denote the linear projection onto the unstable linear subspace E_u of x_* along the complementary invariant linear subspace; denote by $V(x_*)$ the bilinear form on E_u giving the covariance of $P(a_{k+1})$ conditioned on $x_k = x_*$. The conclusion of Theorem 5.1 holds provided $V(x_*)$ is nondegenerate; the proof is essentially the same.

Positive Probability of Every Attractor

The following theorem implies that for a diffuse adaptive perturbed game admitting several asymptotically stable equilibria, even if a particular equilibrium is more efficient than the others, at every stage there is positive probability that the state sequence converges toward a less efficient equilibrium.

THEOREM 5.4. *Consider a diffuse adaptive perturbed game with any number of players, having state space S. For every attractor $A \subset S$ of the game vector field,*

$$
\mathbf{P}\left\{ \lim_{k \to \infty} \operatorname{dist}(x_k, A) = 0 \right\} > 0.
$$

In particular, if x_* *denotes any asymptotically stable equilibrium,*

$$\mathbf{P}\left\{\lim_{k\to\infty} x_k = x_*\right\} > 0.$$

Proof. For asymptotically stable equilibria, the proof is a consequence of a well-known general result for urn processes that exploits the countability of the state space [Arthur *et al*. (1987) and Benaïm and Hirsch (1995)]. For general attractors this is proven in [Benaïm (1999), Sect. 7]. QED

Theorem 5.4 can be compared to a result of Arthur *et al*. (1988): if B is a connected component of the equilibrium set having nonempty interior, then even for discontinuous F, there is positive probability that $\lim_{k\to\infty} \operatorname{dist}(x_k, B) = 0$; note that B is not assumed to be an attractor.

The following consequence of Theorems 5.1 and 5.4 completely characterizes the long run qualitative behavior of fictitious play for a generic class of 2×2 games:

COROLLARY 5.5. *Consider a* 2×2 *adaptive perturbed game satisfying Hypotheses* 1.1 *and* 1.2, *with all Nash distribution equilibria simple*. *Then*:

(i) *The sequence* $\{x_k\}$ *of empirical frequency vectors converges with probability* 1 *to a Nash distribution equilibrium.*

(ii) *Let* x_* *be a linearly unstable equilibrium such that the Nash map is* C^2 *in a neighborhood of* x_*, *and the joint best response matrix* $\hat{\nu}(x_*)$ *has strictly positive entries* (*see Eq*. (4)). *Then* $\mathsf{P}\{\lim_{k\to\infty} x_k = x_*\} = 0$.

(iii) *Suppose* $\hat{\nu}(x)$ *has strictly positive entries at each state* $x \in I \times I$. *Then* $\mathsf{P}\{\lim_{k\to\infty} x_k = x_*\} > 0$ *at every linearly stable equilibrium* x_*.

Proof. Part (i) follows from Theorem 2.2(b) because simple equilibria are isolated. Parts (ii) and (iii) follow from Theorems 5.1 and 5.4. QED

Paths to Equilibrium

The following result gives a further restriction on action sequences in diffuse games. It implies that even for convergence to a clearly optimal equilibrium, the players occasionally, but infinitely often, play the worst possible actions.

PROPOSITION 5.6. *Let* x_* *denote a diffuse equilibrium with* $\mathsf{P}\{\lim_{k\to\infty} x_k = x_*\} > 0$. *Then the conditional probability that every action profile* $a \in A$ *is played infinitely often, given that* $x_k \to x_*$, *is* 1.

Proof. Let $\hat{\nu}(x_*)_a = c > 0$. Let $\mathscr{F}_k$ denote the sigma field generated by $a^1, a^2, \ldots, a^k$. By the generalized Borel–Cantelli lemma [Doob (1953,

p. 324)], the following two sets of events coincide except for a set of measure zero:

$$\{a_k = a \text{ infinitely often}\},$$

and

$$\left\{ \sum_k \mathsf{P}\{a_{k+1} = a | \mathscr{F}_k\} = \infty \right\}.$$

Observe also that

$$\left\{ \sum_k \mathsf{P}\{a_{k+1} = a | \mathscr{F}_k\} = \infty \right\} = \left\{ \sum_k \hat{\nu}(x_k)_a = \infty \right\} \supset \{x_k \to x_*\},$$

where the last inclusion holds because $\Xi(x_n, a)$ converges to $c > 0$ on the set of events $\{x_k \to x_*\}$. Therefore, modulo sets of measure zero:

$$\{x_k \to x_*\} \subset \{a_k = a \text{ infinitely often}\}.$$

QED

6. BEYOND 2 × 2 GAMES

In this section we consider some adaptive perturbed games with more than two players. First we treat a version of Jordan's 3-player matching game. For sufficiently concentrated noise (e.g., low variance if the noise is Gaussian), almost surely sample paths do not converge to the unique Nash distribution equilibrium, and for $n = 3$ they cluster at a periodic orbit of the game vector field. For sufficiently diffuse noise, on the other hand, sample paths almost surely converge. Later we describe $n \times 2$ generalized coordination and anticoordination games, identify classes where convergence is guaranteed, and investigate the geometry of limit sets.

There are $n \geq 2$ players indexed modulo n by $i \in \{1, \dots, n\}$, each player having two pure strategies. The state space, formally $(\Delta^1)^n$ [Eq. (6)], is conveniently identified with the n-cube $I^n = [0,1]^n$, with $p = (p^1, \dots, p^n) \in I^n$ denoting the state in which player i's empirical frequency of playing action 1 is p^i. The game vector field is $F\colon I^n \to \mathbf{R}^n$; the set of Nash distribution equilibria is denoted by $\mathscr{E} = F^{-1}(0) \subset I^n$.

The payoff function for player i is given by a $2 \times 2^{n-1}$ matrix

$$\mathbf{U}_k^i = V^i + \mathbf{E}_k^i, \qquad i \in \{1,2\}, k \in \mathbf{N}_+;$$

the matrix entry indexed by action profile $(a, b) \in \{1,2\} \times \{1,2\}^{n-1}$ is denoted by $\mathbf{E}_k^i[a, b]$.

Throughout this section we make the following assumptions [used by Fudenberg and Kreps (1993)] on the noise matrices $\mathbf{E}_k^i$:

HYPOTHESIS 6.1. (i) $\{\mathbf{E}_k^i\}_{k \in \mathbf{N}_+}$ *is an IID sequence of zero mean random matrices having constant rows*: $\mathbf{E}_k^i[a, b] = \eta_{k,a}^i$ *for each action profile* (a, b)

(ii) *For each* i, *the random variables* $\eta_{k,2}^i - \eta_{k,1}^i$ *have a common probability density function* $f^i \colon \mathbf{R} \to [0, 1]$ *that is smooth and strictly positive*.

Note that $\eta_{k,2}^i - \eta_{k,1}^i$ also has zero mean. We denote its probability distribution function by

$$K^i \colon \mathbf{R} \to [0, 1], \qquad K^i(x) = \int_{-\infty}^{x} f^i(u)\, du. \tag{29}$$

Zero mean and positivity of f^i imply

$$0 < K^i < 1, \qquad K^i(0) = \tfrac{1}{2}. \tag{30}$$

Equation (31) below shows that the probabilities governing fictitious play are functions of K^i. When the K^i are analytic, we speak of *analytic noise*.

Nonconvergence in Jordan's Matching Game

Jordan (1993) exhibited a classical 3×2 matching game with a unique Nash equilibrium, for which fictitious play does not converge. We generalize this to $n \times 2$ perturbed games and obtain broad parameter ranges guaranteeing convergence and nonconvergence.

Payoffs in the $n \times 2$ unperturbed matching game as described in terms of the matrix

$$V = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The payoff to player $i \in \{1, \dots, n\}$ depends only on the actions of players i and $i + 1$, and is determined by the matrix $V^i = V$: If they, respectively, play $k, l \in \{1, 2\}$, the payoff to i is V_{kl}^i. Thus player i tries to match player $i + 1$. The payoff to player n, however, is determined by

$$V^n = -V = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix};$$

player n tries *not* to match player 1.

Consider now the adaptive perturbed game with payoffs in game k determined as above by noisy 2×2 payoff matrices

$$\mathbf{U}_k^i = V^i + \mathbf{E}_k^i, \qquad i = 1, \dots, n$$

subject to Hypothesis 6.1. In game $k+1$ player i plays the pure strategy that maximizes his expected payoff, under the assumption (usually mistaken) that player $i+1$ plays the mixed strategy given by empirical frequency vector x_k^{i+1}. Thus the probability, conditioned on x_k^{i+1}, that player i plays strategy 1 in game $k+1$, is the same as the conditional probability

$$\mathsf{P}\left\{\left(\mathbf{U}_{k+1}^i x_{k+1}^i\right)_1 \geq \left(\mathbf{U}_{k+1}^i x_{k+1}^i\right)_2 | x_{k+1}^i\right\}.$$

The form of V and Hypothesis 6.1 imply:

$$\mathsf{P}\left\{\left(\mathbf{U}_{k+1}^i x_{k+1}^i\right)_1 \geq \left(\mathbf{U}_{k+1}^i x_{k+1}^i\right)_2 | x_{k+1}^i\right\} = \begin{cases} K^i\left(4p_k^{i+1} - 2\right) & \text{if } i < n \\ 1 - K^n\left(4p_k^1 - 2\right) & \text{if } i = n. \end{cases} \tag{31}$$

We calculate the game vector field F in terms of the functions

$$h^i\colon [0,1] \to [0,1],$$
$$h^i(s) = K^i(4s - 2).$$

Note that $0 < h^i < 1$. From Eqs. (29) and (31):

$$h^{i\prime}(s) = K^{i\prime}(4s-2) = 4f^i(4s-2) > 0. \tag{32}$$

The game differential equation is

$$\frac{dp^i}{dt} = F^i(p) \equiv -p^i + h^i\left(p^{i+1}\right), \qquad i = 1, \dots, n-1; \tag{33}$$

$$\frac{dp^n}{dt} = F^n(p) \equiv -p^n + 1 - h^n\left(p^1\right). \tag{34}$$

While the state space for the extended game is I^n, the game vector field F is defined in all of $\mathbf{R}^n$ by (33), (34). It is easily seen that there is a global attractor in $\mathrm{Int}([0,1]^n)$.

LEMMA 6.2. *There is a unique Nash distribution equilibrium p_*, given by*

$$p_*^1 = p_*^2 = \cdots = p_*^n = \tfrac{1}{2}.$$

Proof. Equation (30) implies that the right-hand side of System (33), (34) vanishes if $p = p_*$. Any equilibrium q_*, of System (33), (34) of must satisfy

$$q_*^1 = h^1 \circ \cdots \circ h^{n-1}\left(1 - h^n\left(q_*^1\right)\right).$$

Because the h^i is strictly increasing, $q_*^1 = \frac{1}{2}$ is the unique solution to this fixed point equation. By induction it follows that $q_*^i = \frac{1}{2}$ for all i. QED

Notice that this p_* is also the unique Nash equilibrium of the unperturbed game.

The next two theorems illustrate how the noise densities f^i influence the game dynamics and the state limit set $L\{x_k\}$:

THEOREM 6.3. *Assume*

$$\prod_{i=1}^{n} f^i(0) > \sup_k \left\{ \frac{1}{4} \left| \cos\left(\frac{2k\pi}{n} \right) \right|^{-n}, k = 0, \dots, n-1 \right\}. \tag{35}$$

Then

(i) *There is zero probability that* $\lim_{k\to\infty} x_k = p_*$.

(ii) *Assume* $n \geq 3$ *is odd and the noise is analytic*. *Then the game vector field has an attracting nonequilibrium periodic orbit* $\Gamma \in [0,1]^n$, *and* $L\{x_k\} = \Gamma$ *with positive probability.*

(iii) *Assume* $n = 3$ *and the noise is analytic*. *Then there exists only a finite number of nonequilibrium periodic orbits, and almost surely* $L\{x_k\}$ *is one of them*.

Remark 6.4. When the perturbations have the form $\mathbf{E}_{ml}^i = \varepsilon \eta_m^i$, it can be shown that inequality (35) holds if the parameter $\varepsilon > 0$ is small enough.

Proof. (i) The characteristic polynomial $P(\lambda)$ of $DF(p_*)$ is

$$P(\lambda) = -(1+\lambda)^n - (4\rho)^n,$$

where $\rho = [\prod_{i=1}^n f^i(0)]^{1/n}$. Therefore the eigenvalues of $DF(p_*)$ are

$$\lambda_k = -1 + 4\rho \exp\left(\frac{2k\pi}{n} \right), \qquad k = 0, \dots, n-1; i = \sqrt{-1}.$$

Under the assumption of Theorem 6.1, all eigenvalues have nonzero real parts and some eigenvalues have positive real parts. Part (i) therefore follows from Theorem 5.1.

(ii) Equations (33), (34) constitute a *monotone cyclic feedback system* in the sense of Mallet-Paret and Smith (1990), and it satisfies the hypothesis of their Theorem 4.3. By that result, there is an attracting periodic orbit Γ. The probability that $L\{x_k\} \subset \Gamma$ is positive (Theorem 5.4). Because $L\{x_k\}$ is invariant under the flow of F (Theorem 3.3), and Γ is an orbit, if

$L\{x_k\} \subset \Gamma$ then $L\{x_k\} = \Gamma$. Thus

$$\mathsf{P}\{L\{x_k\} = \Gamma\} > 0.$$

(iii) Assume $n = 3$. Under the change of variables

$$y_1 = p_1, \qquad y_2 = 1 - p_2, \qquad y_3 = p_3,$$

which amounts to relabeling the actions of player 2, and does not change the coordinates of the equilibrium, System (33), (34) becomes

$$\begin{aligned} \frac{dy^1}{dt} &= G^1(y) \equiv -y^1 + h^1(1 - y^2), \\ \frac{dy^2}{dt} &= G^2(y) \equiv -y^2 - h^2(y^3), \\ \frac{dy^3}{dt} &= G^3(y) \equiv -y^3 + 1 - h^3(y^1). \end{aligned} \tag{36}$$

This system is totally competitive and irreducible. Inequality (35) implies that $DF(p_*)$, and hence $DG(p_*)$, has one negative eigenvalue and two complex eigenvalues with positive real parts. Moreover, the negative eigenvalue has an eigenvector with all components positive, while the invariant linear subspace corresponding to the other eigenvalues is transverse to all positive vectors. [These are well-known implications of the Perron–Frobenius theorem applied to $-DF(p_*)$.]

A fundamental property of totally competitive irreducible systems in $\mathbf{R}^n$ having a global attractor, is the existence of a globally attracting, unordered invariant surface Σ [Theorems 1.1 and 1.7 of Hirsch (1988a); see also Takáč (1990) and Hirsch (1989)]. Any such surface is C^1 [Tereščák (1994)] and can be proven to be the graph of a C^1 map $\mathbf{R}^{n-1} \to \mathbf{R}^n$ with negative partial derivatives. In the present context, this shows that the two-dimensional unstable manifold of x_* is a neighborhood of x_* in Σ. Therefore x_* is a repellor for the flow in Σ.

All compact invariant sets lie in the global attractor, hence in Σ. Therefore the Poincaré–Bendixson theory of planar flows implies that every connected attractor-free set (e.g., $L\{x_k\}$ by the limit set theorem) is either a periodic orbit or a set of equilibria.

Real analyticity of F can be used to show that the nonstationary periodic orbits Γ_i are finite in number, by an implicit function theorem argument [compare Jiang (1991)]. As $L\{x_k\}$ cannot be $\{x_*\}$ by Pemantle's Theorem 5.2, $L\{x_k\}$ must be one of the Γ_i. QED

For $n = 3$, Theorem 6.3 precludes convergence when the densities f^i satisfy $f^1(0)f^2(0)f^3(0) > 2$. The following result, on the other hand, guarantees convergence provided the noise is sufficiently diffuse. This is not surprising, but it is interesting to obtain a concrete sufficient condition. Still unknown is what can occur for other densities. Are there cases where both convergence and nonconvergence have positive probability?

THEOREM 6.5. *Assume $n = 3$ and $f^i(s) < \frac{1}{2}$ for $|s| < 2$. Then almost surely $\{x_k\}$ converges to p_*.*

Proof. It is easy to see that all eigenvalues of $DG(p_*)$ have negative real parts, so p_* is an attractor for the flow $\Phi = \{\Phi_t\}_{t \in \mathbf{R}}$ induced by G.

We use the coordinates y^i, described above, that make G totally competitive. As in the proof of Theorem 6.3(iii), there is an invariant C^1 surface $\Sigma \subset \mathbf{R}^3$ that attracts all solutions.

Let $W \subset \Sigma$ denote the basin of attraction of p_* for the flow in Σ. We show by contradiction that

$$W \supset [0,1]^3 \cap \Sigma. \tag{37}$$

If not, then the Poincaré–Bendixson theorem implies that the boundary in Σ of the open set W is a periodic orbit

$$\Lambda \subset \operatorname{Int}\left([0,1]^3\right) \cap (\Sigma \setminus p_*).$$

But this implies that Λ is an attractor, which contradicts Λ being the boundary of W.

Fix $z \in \Lambda$ and a unit vector $u \in \mathbf{R}^3$. Recall that the curve

$$c\colon \mathbf{R} \to \mathbf{R}^3, \qquad c(t) = D\Phi_t(z)u$$

satisfies the variational equation $\frac{dc}{dt} = DG(\Phi_t z)c(t)$. Estimating $\frac{dc}{dt}$ we obtain:

$$\begin{aligned}
\|D\Phi_t(z)u\| &= \left\| D\Phi_0(z)u + t\frac{d}{dt}\Big|_{t=0} D\Phi_t(z)u \right\| + o(t) \\
&= \|(I + tDG(z))u\| + o(t) \\
&\le e^{t\|DG(z)\|} + o(t),
\end{aligned}$$

where I is the identity matrix and $\|A\|$ denotes the operator norm of matrix A. From System (36) and the hypothesis of Theorem 6.5, we derive the estimate

$$\|DG(z)\| < 1 + 4 \cdot \tfrac{1}{2} = 3,$$

whence

$$\|D\Phi_t(z)u\| < e^{3t} + o(t).$$

Since $D\Phi_{-t}(z) = (D\Phi_t(z))^{-1}$, we also have

$$\|D\Phi_t(z)u\| > e^{-3t} + o(t).$$

The chain rule and invariance of $\mathrm{Int}([0,1]^3)$ under $D\Phi_t$ for $t \geq 0$ imply

$$\|D\Phi_t(z)u\| > e^{-3t}. \tag{38}$$

It follows that *every real eigenvalue of* $D\Phi_t(z)$ *exceeds* e^{-3t} *if* $t > 0$.

Let Λ have period $T > 0$ and fix a point $q \in \Lambda$. Because the matrices $-DG(x)$ are nonnegative and irreducible, the matrix $D\Phi_{-T}(q)$ is strictly positive [Hirsch (1985b), Kunze and Siegel (1994), and Smith (1995)]. By the Perron–Frobenius theorem, $D\Phi_{-T}(q)$ has a simple eigenvalue $\rho > 0$ equal to the spectral radius of $D\Phi_{-T}(q)$, and corresponding to ρ there is unique positive unit eigenvector v for $D\Phi_{-T}(q)$. Since ρ^{-1} is an eigenvalue of $D\Phi_T(q)$ with eigenvector v, we have

$$\rho^{-1} > e^{-3T}. \tag{39}$$

Now $D\Phi_T(q)$ also has the eigenvector $F(q)$ for the eigenvalue 1. The two eigenvectors $F(q)$ and v are independent, because v is positive (all components are positive), while if $F(q)$ were positive or negative, then the forward orbit of q would converge [Selgrade (1979)].

Therefore there must be a third eigenvector w of $D\Phi_T(q)$ such that w, $F(q)$, and v are linearly independent. Because q lies in the invariant surface S, and $\Phi_T|S$ is a diffeomorphism, the tangent plane T_qS to S at q is mapped isomorphically to itself by $D\Phi_T(q)$. Since $D\Phi_T(q)|T_qS$ has the real eigenvector $F(q)$, it has a second real eigenvector; as v is transverse to T_q, it follows that w is tangent to Σ at q.

Let $\mu > 0$ be the third eigenvalue of $D\Phi_T(q)$, corresponding to w. Because the determinant is the product of the eigenvalues,

$$\mathrm{Det}\, D\Phi_T(q) = \rho^{-1}\mu > e^{-3T}\mu. \tag{40}$$

On the other hand, Liouville's formula [Hartman (1964)] shows that

$$\mathrm{Det}\, D\Phi_T(q) = \exp\left(\int_0^T \mathrm{Tr}\, DG(\Phi_t q)\, dt\right),$$

where Tr denotes the trace of a matrix. From Eq. (32),

$$\operatorname{Tr} DG(\Phi_t q) = -3.$$

Therefore from Eq. (40),

$$\mu < e^{3T} \operatorname{Det} D\Phi_T(q) = e^{3T}e^{-3T} = 1.$$

Thus the two eigenvalues for $D(\Phi_T|S)(q)$ are 1 and μ, $0 < \mu < 1$. This makes Λ an attractor for $\Phi|\Sigma$, leading to the desired contradiction. This proves (37).

It follows that $\{p_*\}$ is a global attractor for the flow in $[0,1]^3$. Therefore almost surely sample paths converge to p_*, by the limit set theorem 3.3.

QED

Coordination and Anticoordination Games

We now consider a broad class of $n \times 2$ games whose game vector fields turn out to have very convenient dynamical properties. For general n we show that for many coordination games there is at least a positive probability of convergence, while for $n = 3$ it is certain. This contrasts with Jordan's matching game discussed above, which is an anticoordination game having no chance of convergence in certain parameter regimes.

We keep the notation of Section 1, assuming additionally that each player has exactly two pure strategies.

A classical 2×2 game with payoff matrices V^i is sometimes informally termed a "coordination game" if the diagonal entries of V^i dominate columns, i.e.,

$$V^i_{11} > V^i_{12}, \qquad V^i_{22} > V^i_{21},$$

because players do better if they both play the same action rather than different actions. We call it a *generalized coordination game* if each payoff matrix has the weaker property that the sum of the diagonal entries is not less than the sum of the off-diagonal entries, that is,

$$V^i_{11} + V^i_{22} \geq V^i_{21} + V^i_{12}.$$

When the opposite weak inequality holds for both matrices, the game is a *generalized anticoordination game*. We use the term *strict* if the defining inequalities are strict.

An example of a 2×2 generalized strict coordination game is given by

$$V^1 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \qquad V^2 = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

There is a unique Nash equilibrium: It is clearly optimal for player 1 to play action 2 and player 2 to play action 1. Thus the players do not necessarily "coordinate" their actions. Remark 6.7 below gives a sense in which action choices tend to reinforce each other under fictitious play.

Consider an $n \times 2$ game Γ. To each ordered pair (i, j) of distinct players and each action profile $a \in A = \{1, 2\}^n$ there corresponds a *partial* 2×2 game $\Gamma^{ij}(a)$ obtained by freezing the actions of other players. The first player of $\Gamma^{ij}(a)$ is player i of Γ and the second player is j; we informally denote these players as i and j for convenience. For any action profile $b = (b^1, b^2)$ of the partial game, player i receives the same payoff he receives in Γ when i, j play b^1, b^2, respectively, and each other player $r \notin \{i, j\}$ plays a^r (the rth coordinate of a); the payoff to j in $\Gamma^{ij}(a)$ is defined similarly.

An $n \times 2$ game Γ is a *generalized coordination* (*respectively, anticoordination*) *game* if each partial 2×2 game $\Gamma^{ij}(a)$ has the corresponding property. Such a game is *irreducible* if for every $i \neq j$ in $\{1, \dots, n\}$ there exists $m \geq 1$ and a sequence $i_0, \dots, i_m$ such that: $i = i_0$, $j = i_m$, $i_r \neq i_{r+1}$, and the partial 2×2 games for players i_r and i_{r+1} are strict for $r = 0, \dots, m - 1$. We apply these definitions to an adaptive perturbed game to mean that the unperturbed game has the corresponding form.

These definitions depend on the labeling of the actions. It is sometimes possible to transform a given game into a generalized coordination or anticoordination game by relabeling actions. Jordan's matching game, for instance, can be transformed into a generalized anticoordination game, provided the number of players is odd.

The next proposition gives conditions on the game vector field F equivalent to generalized coordination and anticoordination. A vector field G in Euclidean space is *cooperative* if its Jacobian matrices have nonnegative off-diagonal entries, i.e., $\partial G_i / \partial x_j \geq 0$ for $i \neq j$. When the off-diagonal entries are nonpositive, G is called *competitive*, and *totally competitive* provided all entries are negative. If the Jacobian matrices are irreducible, G is called irreducible.

PROPOSITION 6.6. *Let F denote the game vector field of an $n \times 2$ adaptive perturbed game satisfying Hypothesis* 6.1. *F is cooperative* (*respectively, totally competitive*) *for a generalized coordination* (*respectively, generalized anticoordination*) *game. In either case, F is irreducible provided the game is irreducible.*

Remark 6.7. This proposition suggests an interpretation for a generalized coordination (respectively, anticoordination) game under fictitious play: At each round of play, player i's probability of playing action $a \in \{1, 2\}$ is a nondecreasing (respectively, nonincreasing) function of player j's empirical frequency of past plays of a, for $j \neq i$.

Proof of Proposition 6.6. Our definitions are conveniently rephrased as follows. Let $I = (i_1, \dots, i_m)$ be a sequence of $m \geq 1$ distinct players and $J = (b^1, \dots, b^m) \in \{1,2\}^m$ a sequence of m pure strategies. For any action profile $a \in A$, let $T_J^I(a)$ denote the action profile obtained from a by replacing a^{i_l} by b^l for $l = 1, \dots, m$; this defines a map $T_J^I\colon A \to A$. Composing this with player i's payoff function $V^i\colon A \to \mathbf{R}$ in Γ and forming the canonical extension, we obtain a smooth map $V^i \circ T_J^I\colon S \to \mathbf{R}$. In this notation, the payoff function to i in the partial game $\Gamma^{ij}(a)$ is the map

$$A^i \times A^j = \{1,2\} \times \{1,2\} \to \mathbf{R}, \qquad b \mapsto \left(V^i \circ T_b^{ij}\right)(a). \tag{41}$$

It is clear that Γ is a coordination game if and only if for every pair of distinct players $i \neq j$,

$$V^i \circ T_{1,1}^{ij} + V^i \circ T_{2,2}^{ij} \geq V^i \circ T_{1,2}^{ij} + V^i \circ T_{2,1}^{ij}, \tag{42}$$

and Γ is a generalized anticoordination game if and only if

$$V^i \circ T_{1,1}^{ij} + V^i \circ T_{2,2}^{ij} \leq V^i \circ T_{1,2}^{ij} + V^i \circ T_{2,1}^{ij}. \tag{43}$$

Hypothesis 6.1 implies that the Nash map ν [see (3)] is given by

$$\nu^i(p) = K^i\left(\left(V^i \circ T_1^i - V^i \circ T_2^i\right)(p)\right),$$

with K^i as in (29). It follows that the game vector field $F(x) = -x + \nu(x)$ has the formula,

$$F^i(p) = -p^i K^i\left(\left(V^i \circ T_1^i - V^i \circ T_2^i\right)(p)\right), \tag{44}$$

with $\partial F^i / \partial p^i = -1$, and for $i \neq j$:

$$\begin{aligned} \frac{\partial F^i}{\partial p^j} &= f^i\left(\left[V^i \circ T_1^i - V^i \circ T_2^i\right](p)\right) \\ &\quad \times\left(\left[V^i \circ T_{1,1}^{ij} + V^i \circ T_{2,2}^{ij} - V^i \circ T_{1,2}^{ij} - V^i \circ T_{2,1}^{ij}\right](p),\right). \end{aligned}$$

Because $f^i > 0$ (Hypothesis 6.1), these formulas and Eqs. (42), (43) imply the proposition. QED

From now on we assume given an adaptive perturbed $n \times 2$ game satisfying Hypothesis 6.1, *whose unperturbed game Γ is an irreducible generalized coordination or generalized anticoordination game*.

Remark 6.8. We conjecture that for fictitious play in any perturbed generalized coordination game, the state sequence converges almost surely.

Suggestive evidence for this is that almost every trajectory of the game vector field F converges, because F is cooperative and irreducible [Smith and Thieme (1991) and Smith (1995)]. Further motivation is proven by the next three theorems.

The following result extends convergence for 2×2 games with a unique equilibrium [Fudenberg and Kreps (1993)], to $n \times 2$ coordination games:

THEOREM 6.9. *The state sequence converges almost surely for any irreducible generalized coordination game having only one Nash distribution equilibrium.*

Proof. As F is cooperative and irreducible (Proposition 6.6) and has a unique equilibrium p_*, every trajectory converges to p_* and p_* is asymptotically stable [Hirsch (1985b, p. 431, Corollary)]. The conclusion from the classical stochastic approximation theorem of Robbins–Monro (1951); alternatively, the order-preserving property of the flow [Hirsch (1985b)] can be used to show that $\{p_*\}$ is the only compact invariant set, whence the limit set theorem concludes the proof. QED

THEOREM 6.10. *For a* 3×2 *irreducible generalized coordination game, almost surely* $L\{x^k\}$ *is a compact connected subset of* $\mathscr{E}$. *Thus* $\{x_k\}$ *converges almost surely provided* $\mathscr{E}$ *is countable. When the noise is analytic,* $\mathscr{E}$ *is finite.*

Proof. F is a cooperative irreducible vector field by Proposition 6.6, and we may assume $L\{x_k\}$ is connected, compact, and attractor-free by the limit set theorem 3.3. This implies that either $L\{x_k\}$ is an arc of equilibria, or else $L\{x_k\}$ is unordered for the vector ordering in $\mathbf{R}^3$ [Hirsch (1999)]. In the latter case $L\{x_k\}$ is unordered and lies in a connected invariant surface Σ homeomorphic to the plane [Hirsch (1988a)]. Recall that F has negative divergence by Proposition 4.4 [or Eq. (44)], which implies $L\{x_k\}$ does not separate Σ [Hirsch (1989)], and this entails $L\{x_k\} \subset \mathscr{E}$ [Hirsch and Pugh (1988)].

For analytic noise, F is an analytic map in $\mathbf{R}^3$. Therefore the equilibrium set $F^{-1}(0)$ is a compact, real analytic variety $Z \subset I^3$. We will prove Z is zero dimensional and thus finite.

Any connected component $X \subset Z$ is unordered for the vector ordering in $\mathbf{R}^3$ by Jiang (1991). This implies X lies in an invariant surface Σ homeomorphic to the plane [Hirsch (1988a)]; therefore X has dimension at most 2. The space X, being an analytic variety, can be triangulated; its dimension is ≤ 2 because it lies in a surface. If $\dim(X) = 1$, it is known that every vertex must belong to at least two 1-simplices. Therefore X contains a simple closed curve C, which separates Σ. The same holds if $\dim(X) = 2$: take C to be the boundary of a 2-simplex. But under assumption (b), this separation contradicts the earlier conclusion that the flow in S decreases area. QED

THEOREM 6.11. *Consider an* $n \times 2$ *irreducible generalized coordination game. If either*

(a) *the set* $\mathscr{E}$ *of Nash distribution equilibria is finite, or*

(b) *the noise is analytic,*

then there exists a Nash distribution equilibrium p_* *such that the sequence* $\{x_k\}$ *of empirical frequency vectors converges to* p_* *with positive probability.*

Proof. F is cooperative and irreducible by Proposition 6.6. Therefore if either (a) or (b) holds, there exists at least one asymptotically stable equilibrium for F [Hirsch (1985a) and Jiang (1991)]. The existence of p_* follows from Theorem 5.4. QED

Anticoordination games are more difficult to analyze. Aside from Jordan's game treated above, we have only the following general constraint on state limit sets. Denote the vector ordering of $\mathbf{R}^n$ by $\leq$, writing $x < y$ if $x^i \leq y^i$ for all i but $x \neq y$. A set is *unordered* if no two of its points are comparable under the vector ordering.

THEOREM 6.12. *For any* $n \times 2$ *irreducible generalized anticoordination game, there exists an invariant, unordered* C^1 *hypersurface* $\Sigma \subset \mathbf{R}^n = \mathbf{R}^{n-1} \times \mathbf{R}^n$, *depending only on the game vector field, such that almost surely the state limit set lies in* Σ; *and* Σ *is the graph of a* C^1 *map* h: $\mathbf{R}^{n-1} \to \mathbf{R}$ *having negative partial derivatives.*

Proof. The existence of Σ is proved as in the proof of Theorem 6.12. As $L\{x_k\}$ is almost surely a compact invariant set by the limit set theorem, it almost surely lies in K, hence in Σ. QED

The following form of Theorem 6.12 resembles a conservation law:

COROLLARY 6.13. *Given an irreducible generalized anticoordination game, there is a* C^1 *function* E: $\mathbf{R}^n \to \mathbf{R}$, *depending only on the game vector field, such that*

(i) $\lim_{k \to \infty} E(x_k) = 0$ *almost surely*

(ii) $\partial E / \partial x^i < 0$, $i = 1, \ldots, n$

Proof. Let h be as in Theorem 6.12, and set $E(x) = h(x^1, \ldots, x^{n-1}) - x^n$. QED

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