Symplectic Embeddings in Dimension 4

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Abstract

Symplectic geometry is the underlying geometry of Hamiltonian dynamics. Since the proof of Gromov's non-squeezing theorem in 1985, symplectic embeddings have been at the heart of symplectic geometry. This thesis studies some symplectic embedding problems in dimension 4. We start by completely solving the problem of embedding an ellipsoid into a cube. This result is a refinement of the theorem proved by Gromov, McDuff-Polterovich and Biran about embeddings of a disjoint union of equal balls into a cube. In the second part of the thesis, we construct explicit embeddings of a disjoint union of balls into certain (non-disjoint) unions of an ellipsoid and a cylinder. It follows from Hutchings' ECH capacities that these embeddings are optimal.

Keywords: Symplectic embeddings; ECH capacities; Pell numbers

Résumé

La géométrie symplectique est la géométrie sous-jacente à la dynamique hamiltonienne. Depuis la démonstration du théorème de non-tassement de Gromov en 1985, les plongements symplectiques se trouvent au coeur de la géométrie symplectique. Cette thèse étudie certains problèmes de plongements symplectiques en dimension 4. Nous commençons par résoudre complètement le problème des plongements d'ellipsoïdes dans des cubes. Ce résultat est un raffinement du théorème de Gromov, McDuff-Polterovich et Biran sur les plongements d'une union disjointe de boules égales dans un cube. Dans la deuxième partie de la thèse, nous construisons des plongements explicites d'une union disjointe de boules dans certaines unions (non-disjointes) d'ellipsoïdes et de cylindres. Il découle des capacités ECH de Hutchings que ces plongements sont optimaux.

Mots clés: Plongements symplectiques; Capacités ECH; Nombres de Pell

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Contents

Chapter 1

Introduction

In the first section of this introduction, we give a motivating example for symplectic geometry: the *N-body problem*. Symplectic geometry arose from celestial mechanics, and some of its characteristic phenomena can be easily observed in this example. In Section 1.2, we generalize this example to Hamiltonian dynamics and introduce the mathematical framework in which the thesis will be placed. We reprove the observations of Section 1.1 in this general setting and introduce the notion of *symplectic structure*. In Section 1.3, we introduce *symplectic embedding problems*, the main topic of this thesis, by making a brief survey of some main results in the field. Finally, in Section 1.4 we present the results of the thesis and explain some of its consequences. This introduction has been mainly inspired by the books [A], [S2] and [Z]. Readers who are interested in a more detailed introduction are referred to these books.

1.1 The N-body problem

The *N-body problem* of celestial mechanics consists in studying the motion in the 3-dimensional space $\mathbb{R}^3$ of N particles with masses $m_1, \ldots, m_N > 0$ which are subject to gravitational forces. Denote the *position* of the particles at time $t \in \mathbb{R}$ by $x_1(t), \ldots, x_N(t) \in \mathbb{R}^3$. By Newton's second law of motion, the force $F_i\left(x_1, \ldots, x_N\right)$ acting on the particle x_i satisfies the equation

$$F_i\left(x_1, \ldots, x_N\right) = \sum_{j \neq i} m_i m_j \frac{x_j - x_i}{\left|x_j - x_i\right|^3} = m_i \ddot{x}_i \tag{1.1.1}$$

where $i = 1, \ldots, N$. These equations are not defined at *collisions*, that is, at the points

$$\Delta := \left\{ \left(x_1, \ldots, x_N\right) \in \mathbb{R}^{3N} \;\middle|\; \exists i \neq j \text{ with } x_i = x_j \right\}.$$

The space $\mathcal{C} := \mathbb{R}^{3N} \setminus \Delta$ where the equations (1.1.1) are defined is called the *configuration space* of the N-body problem. The *gravitational field*

$$F := (F_1, \ldots, F_N) : \mathcal{C} \longrightarrow \mathbb{R}^{3N}$$

has the important property to *derive from a potential*, which means that there exists a function $U \colon \mathcal{C} \longrightarrow \mathbb{R}$ called the *potential energy*, such that

$$F = -\nabla U.$$

The potential energy depends only on the positions of the particles and is explicitly given in the N-body problem by

$$U\left(x_1, \ldots, x_N\right) := -\sum_{i=1}^{N} \sum_{j>i} \frac{m_i m_j}{|x_j - x_i|}.$$

The *momentum* of the system at time $t \in \mathbb{R}$ is the vector $y \in \mathbb{R}^{3N}$ given by

$$y := (y_1, \ldots, y_N) = \left(m_1 \dot{x}_1, \ldots, m_N \dot{x}_N\right). \tag{1.1.2}$$

The *phase space* of the N-body problem is then defined as the set

$$\mathcal{P} := \mathcal{C} \times \mathbb{R}^{3N} = \left(\mathbb{R}^{3N} \setminus \triangle\right) \times \mathbb{R}^{3N} \subset \mathbb{R}^{6N}$$

consisting of all possible pairs (x, y) of positions $x \in \mathbb{R}^{3N} \setminus \Delta$ and momenta $y \in \mathbb{R}^{3N}$ of the particles.

We define the *kinetic energy* as the function $K \colon \mathbb{R}^{3N} \longrightarrow \mathbb{R}$ depending only on the momenta of the particles and given by

$$K\left(y_1, \ldots, y_N\right) := \sum_{i=1}^{N} \frac{1}{2m_i} |y_i|^2$$

and the *mechanical energy* as the function $H \colon \mathcal{P} \longrightarrow \mathbb{R}$ defined on the phase space and given by

$$H\left(x, y\right) = U(x) + K\left(y\right).$$

By differentiating H with respect to x and y, we obtain

$$\begin{cases} \dot{x} = & \dfrac{\partial H}{\partial y}(x, y), \\ \dot{y} = & -\dfrac{\partial H}{\partial x}(x, y). \end{cases}$$

These equations are called the *Hamiltonian equations.* Notice that the first equation is just a rephrasing of (1.1.2) while the second equation is a rephrasing of (1.1.1). The physical content of these equations is thus strictly the same as Newton's law. It is however a fruitful point of view to classical mechanics, as we will make apparent in the sequel. A first remark is that we have transformed the second-order ordinary differential equation (1.1.1) in a first-order ordinary differential equation.

The vector field

$$X_H := \begin{pmatrix} \dfrac{\partial H}{\partial y} \\ -\dfrac{\partial H}{\partial x} \end{pmatrix}$$

defined on the phase space $\mathcal{P}$ is called the *Hamiltonian vector field* associated to H. We want to study the *flow* φ_H^t associated to X_H, that is, the map $\varphi_H^t : I \times \mathcal{P} \longrightarrow \mathcal{P}$ defined by the equations

$$\begin{cases} \dfrac{d}{dt}\varphi_H^t(z) = & X_H\left(\varphi_H^t(z)\right), \\ \varphi_H^0(z) = & z, \end{cases} \tag{1.1.3}$$

for all $z := (x, y) \in \mathcal{P}$, where $I \subset \mathbb{R}$ is the maximal interval of existence of the flow.

The first important property of this flow is the *conservation of mechanical energy.*

Fact 1.1.1. *The flow φ_H^t preserves the mechanical energy H, that is, for any initial condition $z \in \mathcal{P}$ and for all $t \in \mathbb{R}$,*

$$H\left(\varphi_H^t(z)\right) = H(z).$$

Proof. It suffices to prove that $\frac{d}{dt}\left(H \circ \varphi_H^t\right) = 0$. In view of (1.1.3), we have

$$\begin{aligned} \frac{d}{dt}\left(H \circ \varphi_H^t\right) &= dH\left(\varphi_H^t\right) X_H\left(\varphi_H^t\right) \\ &= \frac{\partial H}{\partial x}\left(\varphi_H^t\right)\frac{\partial H}{\partial y}\left(\varphi_H^t\right) - \frac{\partial H}{\partial y}\left(\varphi_H^t\right)\frac{\partial H}{\partial x}\left(\varphi_H^t\right) \\ &= 0. \end{aligned}$$

□

The second property of the flow is the *conservation of volume.*

Fact 1.1.2. (Liouville's theorem) *The flow φ_H^t preserves the volume, that is, for any domain $D \subset \mathcal{P}$ and for all $t \in \mathbb{R}$,*

$$\mathrm{Vol}\varphi_H^t(D) = \mathrm{Vol}D.$$

Proof. If suffices to prove that the divergence of X_H vanishes. We have

$$\mathrm{div}X_H = \frac{\partial}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial}{\partial y}\frac{\partial H}{\partial x} = 0.$$

□

Another important property of the flow φ_H^t is the *conservation of the symplectic structure*. This property is a refinement of the conservation of volume, in the sense that the conservation of the symplectic structure implies the conservation of volume. To prove this, it is however convenient to generalize the N-body problem and to use the formalism of differential forms. In this framework, we will then prove the properties of the Hamiltonian flow φ_H^t stated in this section. This is the object of the next section.

1.2 Hamiltonian dynamics

In order to generalize the N-body problem, consider on $\mathcal{P} \subset \mathbb{R}^{6N}$ the *standard symplectic form*

$$\omega_0 = \sum_{i=1}^{3N} dx_i \wedge dy_i.$$

It is clear that $d\omega_0 = 0$, i.e. ω_0 is closed. Moreover, ω_0 is non degenerate, since its $3N$-th power

$$\omega_0^{3N} = (3N)!\, dx_1 \wedge dy_1 \wedge \ldots \wedge dx_{3N} \wedge dy_{3N}$$

is a multiple of the standard volume form. These two properties of the standard symplectic form ω_0 give rise to the following.

Definition 1.2.1. A *symplectic manifold* (M, ω) is a smooth manifold M equipped with a closed non-degenerate differential 2-form ω. The form ω is called a *symplectic form*.

The non-degeneracy of ω implies that the dimension of M is even: $\dim M = 2n$. Moreover, it implies that $\frac{1}{n!}\omega^n$ is a volume form on M.

In order to find a general definition for the Hamiltonian vector field X_H, remark that X_H satisfies the equation

$$\omega_0 (X_H, \cdot) = dH.$$

Moreover, due to the non-degeneracy of ω_0, it is the unique vector field with this property. This leads to the following definition.

Definition 1.2.2. Let $H \colon M \longrightarrow \mathbb{R}$ be a smooth function. Due to the non-degeneracy of ω, there exists a unique vector field X_H on M such that

$$\omega (X_H, \cdot) = dH. \tag{1.2.1}$$

The function H is then called a *Hamiltonian function* while the vector field X_H is called the *Hamiltonian vector field* associated to H. The flow φ_H^t associated to H is called the *Hamiltonian flow* associated to H.

We are now in position to prove in the general situation the properties of the Hamiltonian flow φ_H^t stated in the previous section. We start with Fact 1.1.1: the *conservation of mechanical energy.*

Proposition 1.2.3. *The flow φ_H^t preserves the Hamiltonian function H, that is, for any initial condition $z \in M$ and for all $t \in \mathbb{R}$,*

$$H\left(\varphi_H^t(z)\right) = H(z).$$

Proof. We have

$$\begin{aligned}\frac{d}{dt}\left(H \circ \varphi_H^t\right) &= dH\left(\varphi_H^t\right) X_H\left(\varphi_H^t\right) \\ &= \omega\left(X_H\left(\varphi_H^t\right), X_H\left(\varphi_H^t\right)\right) = 0.\end{aligned}$$

The second equality follows from (1.2.1) while the last equality follows from the non-degeneracy of ω. □

We now prove the *conservation of the symplectic structure.*

Proposition 1.2.4. *The flow φ_H^t preserves the symplectic form ω, that is, $\left(\varphi_H^t\right)^* \omega = \omega$ for all $t \in \mathbb{R}$.*

Proof. We have

$$\mathcal{L}_{X_H}\omega = d\iota_{X_H}\omega + \iota_{X_H}\underbrace{d\omega}_{=0} = ddH = 0.$$

The second equality follows from (1.2.1). □

Definition 1.2.5. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. A map $\varphi\colon M_1 \longrightarrow M_2$ is called *symplectic* if $\varphi^*\omega_2 = \omega_1$.

In dimension 2, symplectic maps agree with volume preserving maps that preserve the orientation since a symplectic form is a volume form. In higher dimension, it is still true that symplectic maps preserve the volume. Consider the volume forms $\Omega_1 := \frac{1}{n!}\omega_1^n$ on M_1 and $\Omega_2 := \frac{1}{n!}\omega_2^n$ on M_2.

Proposition 1.2.6. *A symplectic map φ preserves the volume, that is, $\varphi^*\Omega_2 = \Omega_1$.*

Proof. We have

$$\varphi^*\Omega_2 = \varphi^*\left(\frac{1}{n!}\omega_2^n\right) = \frac{1}{n!}\left(\varphi^*\omega_2\right)^n = \frac{1}{n!}\omega_1^n = \Omega_1.$$

□

In particular, we deduce Fact 1.1.2 from Propositions 1.2.4 and 1.2.6.

Corollary 1.2.7. *The flow φ_H^t preserves the volume.*

1.3 Symplectic embeddings

In the previous section, we have seen that diffeomorphisms which arise as time t maps of Hamiltonian flows preserve both the volume and the symplectic structure. Symplectic geometry studies among other things the properties of symplectic maps. They lie somewhere between the *rigid* Euclidean isometries and the *flexible* volume preserving maps. One goal of symplectic geometry is to determine in which contexts symplectic maps are rather flexible, and when they are rather rigid. One way to observe both phenomena is via symplectic embeddings.

Definition 1.3.1. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. A symplectic map $\varphi\colon M_1 \longrightarrow M_2$ is called a *symplectic embedding* if φ is a homeomorphism on its image. We denote it by $\varphi\colon M_1 \overset{s}{\hookrightarrow} M_2$.

Notice that due to the non-degeneracy of symplectic forms, symplectic maps are always immersions. It is thus enough to require the symplectic map to be a homeomorphism on its image in order to get an embedding.

The first important embedding result is undoubtedly *Gromov's nonsqueezing theorem* (see [G]). We consider the Euclidean $2n$-dimensional space $(\mathbb{R}^{2n}, \omega_0)$ endowed with the canonical symplectic form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Denote by

$$B(a) := \left\{ (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n} \;\mid\; \pi \left(\sum_{i=1}^{n} x_i^2 + y_i^2 \right) < a \right\}$$

the *ball* of radius $\sqrt{\frac{a}{\pi}} > 0$ and by

$$Z(A) := \left\{ (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n} \;\mid\; \pi \left(x_1^2 + y_1^2 \right) < A \right\}$$

the *symplectic cylinder*, that is, the product $D^2(A) \times \mathbb{R}^{2n-2}$ of the disc of area $A > 0$ with $\mathbb{R}^{2n-2}$.

Theorem 1.3.2. (Gromov's nonsqueezing theorem) *There exists a symplectic embedding $B(a) \overset{s}{\hookrightarrow} Z(A)$ if and only if $a \leqslant A$.*

The theorem says thus that the problem of symplectically embedding a ball into a symplectic cylinder is rigid: the best possible symplectic embedding is the inclusion.

In contrast with Gromov's nonsqueezing theorem, if we define the *isotropic cylinder* by

$$Z^{\text{iso}}(A) := \left\{ (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n} \;\mid\; \pi \left(x_1^2 + x_2^2 \right) < A \right\},$$

then $B(a) \overset{s}{\hookrightarrow} Z^{\text{iso}}(A)$ for all $a > 0$. Indeed, the linear map $\varphi_\varepsilon \colon \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ given by

$$\varphi_\varepsilon(x, y) = (\varepsilon x, \frac{1}{\varepsilon} y)$$

is symplectic for all $\varepsilon > 0$, and for any $a > 0$ there exists $\varepsilon > 0$ such that $\varphi_\varepsilon\left(B(a)\right) \subset Z^{\text{iso}}(A)$. This shows that the problem of symplectically embedding a ball into an isotropic cylinder is flexible.

Important obstructions to symplectic embeddings are given by symplectic capacities which have been introduced by Ekeland and Hofer in [EH]. Denote by $\mathrm{SM}(2n)$ the set of symplectic manifolds of dimension $2n$.

Definition 1.3.3. Assume $n \geqslant 2$. A *symplectic capacity* is a map

$$c \colon \mathrm{SM}(2n) \longrightarrow [0, +\infty]$$

with the following properties.

(Monotonicity) If there exists a symplectic embedding

$$(M_1, \omega_1) \overset{s}{\hookrightarrow} (M_2, \omega_2)\,,$$

then

$$c\left(M_1, \omega_1\right) \leqslant c\left(M_2, \omega_2\right).$$

(Conformality) If $\alpha > 0$, then

$$c\left(M, \alpha\omega\right) = \alpha c\left(M, \omega\right).$$

(Nontriviality) $c\left(B(1)\right) > 0$ and $c\left(Z(1)\right) < \infty$.

The nontriviality axiom excludes the volume to be a symplectic capacity. In view of the monotonicity axiom, symplectic capacities are symplectic invariants and can thus be used to find obstructions to symplectic embeddings. An example of a symplectic capacity is the *Gromov width* $G \colon \mathrm{SM}(2n) \longrightarrow [0, +\infty]$ which is defined by

$$G(M, \omega) := \sup \left\{ a \;\mid\; \exists\, \varphi \colon\; (B(a), \omega_0) \overset{s}{\hookrightarrow} (M, \omega) \right\}$$

where $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$ denotes the canonical symplectic form on $\mathbb{R}^{2n}$. Gromov's width measures the size of the biggest ball that symplectically embeds into the manifold. In fact, $G\left(B(a)\right) = G\left(Z(a)\right) = a$, so that Gromov's nonsqueezing theorem follows by monotonicity from the fact that Gromov's width is a capacity.

The next step is to refine this capacity. Indeed, Gromov's width can for example not distinguish a ball from a cylinder. This problem is solved by considering the following *ball-packing problem*. We define for all $k \geqslant 1$ a capacity $G_k \colon \mathrm{SM}(2n) \longrightarrow [0, +\infty]$ by

$$G_k(M, \omega) := \sup \left\{ a \mid \exists\, \varphi \colon \coprod_{i=1}^{k} (B(a), \omega_0) \overset{s}{\hookrightarrow} (M, \omega) \right\}, \tag{1.3.1}$$

that is, G_k measures the size of the biggest disjoint union of equal balls which embeds into the manifold. Of course, G_1 is the Gromov width. Now, G_2 distinguishes a ball from a cylinder. Indeed, $G_2(Z(a)) = a$ while we deduce from volume considerations that $G_2(B(a)) < a$.

A way to measure the flexibility (or rigidity) of this ball-packing problem is the following. We define for each $k \in \mathbb{N}$ the *k-th packing number* of the symplectic manifold (M, ω) by

$$p_k(M, \omega) := \sup \left\{ \frac{k \operatorname{Vol}(B(a))}{\operatorname{Vol}(M)} : \coprod_k (B(a), \omega_0) \overset{s}{\hookrightarrow} (M, \omega) \right\}. \tag{1.3.2}$$

The k-th packing number of M thus describes the supremum of the volume ratio which can be filled by symplectic embeddings of k disjoint equal balls $B(a)$ into M. Thus, if $p_k(M) = 1$, the ball-packing problem by k balls is flexible. On the other hand, if $p_k(B(1)) < 1$, there is some rigidity: we say that there is a *packing obstruction*.

These symplectic invariants are still not very well understood in arbitrary dimensions. However, there are some results in dimension 4. For example, all the capacities G_k and the packing numbers p_k have been computed for the 4-dimensional ball $B^4(1)$ by Gromov [G], McDuff-Polterovich [MP] and Biran [B1]. They found the following table.

k	1	2	3	4	5	6	7	8	$\geqslant 9$
G_k	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{6}{17}$	$1/\sqrt{k}$
p_k	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{20}{25}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1

This result shows that, while there is symplectic rigidity for many small k, the problem is flexible for large k.

We can get one step further in the refinement of these results. Define a *symplectic ellipsoid* by

$$E(a_1, a_2) = \left\{ (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 : \frac{\pi\,(x_1^2 + y_1^2)}{a_1} + \frac{\pi\,(x_2^2 + y_2^2)}{a_2} < 1 \right\}.$$

McDuff showed in [M2] that the ellipsoid $E(a_1, a_2)$ symplectically embeds into the ball $B(A)$ if and only if a certain finite disjoint union of balls $\coprod_i B(w_i)$ embeds into $B(A)$. The sizes w_i of the balls $B(w_i)$ are related to the continued fraction expansion of $\frac{a_2}{a_1}$. In particular, if $k \geqslant 1$ is a positive integer, then

$$E(a, ka) \overset{s}{\hookrightarrow} B(A) \text{ if and only if } \coprod_{i=1}^{k} B(a) \overset{s}{\hookrightarrow} B(A). \tag{1.3.3}$$

In [MS], McDuff and Schlenk used this result to solve the problem of embedding a 4-dimensional ellipsoid into a ball. They determined the function $c_{EB} \colon [1, +\infty[\longrightarrow [1, +\infty[$ defined by

$$c_{EB}(a) := \inf \left\{ A \,:\, E(1, a) \overset{s}{\hookrightarrow} B(A) \right\}. \tag{1.3.4}$$

Since $E(1, a) \overset{s}{\hookrightarrow} B(A)$ if and only if $E(\lambda, \lambda a) \overset{s}{\hookrightarrow} B(\lambda A)$, this result indeed completely solves the problem of embedding an ellipsoid into a ball. Moreover, by using (1.3.3), one can recover the capacities $G_k(B(1))$ and the packing numbers $p_k(B(1))$ by looking at embeddings of $E(1, k) \overset{s}{\hookrightarrow} B(A)$ since for all integers $k \geqslant 1$,

$$c_{EB}(k) = \frac{1}{G_k(B(1))}.$$

The structure of the graph of c_{EB} turns out to be very rich. For $1 \leqslant a \leqslant \tau^4$, where $\tau := \frac{1+\sqrt{5}}{2}$ is the golden ratio, the graph consists of an infinite piecewise linear staircase oscillating between flexible parts and more rigid parts. For $\tau^4 \leqslant a \leqslant 8 + \frac{1}{36}$, the graph is equal to the volume constraint except on a finite number of intervals. For $a \geqslant 8 + \frac{1}{36}$, the problem is flexible since $c_{EB}(a)$ coincides with the volume constraint.

There is another way to characterize the function c_{EB}. If (M, ω) is a 4-dimensional symplectic manifold, then there exists a whole sequence of symplectic capacities associated to M

$$0 = c_0(M, \omega) \leqslant c_1(M, \omega) \leqslant c_2(M, \omega) \leqslant \ldots \leqslant \infty$$

called *ECH capacities,* which have been introduced by Hutchings in [H1] using his embedded contact homology. ECH capacities give sharp obstructions to certain symplectic embeddings. In particular, Hutchings and McDuff showed in [H1] and [M3] that $E(a, b) \overset{s}{\hookrightarrow} E(c, d)$ if and only if $c_k(E(a, b)) \leqslant c_k(E(c, d))$ for all $k \geqslant 0$. This gives another characterization of the function c_{EB}, although it seems very difficult to deduce the shape of the graph of c_{EB} directly from the ECH capacities.

1.4 Presentation of the results

1.4.1 Symplectic embeddings of 4-dimensional ellipsoids into cubes

In Chapter 2, we solve the problem of embedding a 4-dimensional symplectic ellipsoid into a 4-dimensional cube. This is joint work with Dorothee Müller and has been accepted for publication in the *Journal of Symplectic Geometry* ([FM]).

Recall that McDuff and Schlenk determined in [MS] the function c_{EB} defined by (1.3.4) whose value at a is the infimum of the size of a 4-ball into which the ellipsoid $E(1,a)$ symplectically embeds (here, $a \geqslant 1$ is the ratio of the area of the large axis to that of the smaller axis of the ellipsoid). In Chapter 2, we look at embeddings into 4-dimensional cubes instead, and determine the function

$$c_{EC}(a) \,:=\, \inf \Big\{ A \,:\, E(1,a) \,\stackrel{s}{\hookrightarrow}\, C(A) \Big\}$$

whose value at a is the infimum of the size of a 4-cube $C(A) = D^2(A) \times D^2(A)$ into which the ellipsoid $E(1,a)$ symplectically embeds (where $D^2(A)$ denotes the disc in $\mathbb{R}^2$ of area A).

As for the function c_{EB}, this function refines the result of Gromov [G], McDuff-Polterovich [MP] and Biran [B1], where they computed for the 4-cube $C(1)$ all the capacities G_k and the packing numbers p_k defined by (1.3.1) and (1.3.2). They found the following table.

k	1	2	3	4	5	6	7	$\geqslant 8$
$G_k\,(C(1))$	1	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{4}{7}$	$\frac{8}{15}$	$\sqrt{2/k}$
$p_k\,(C(1))$	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{48}{49}$	$\frac{224}{225}$	1

Now since McDuff showed in [M2] that

$$E(a,ka) \stackrel{s}{\hookrightarrow} C(A) \text{ if and only if } \coprod_{i=1}^{k} B(a) \stackrel{s}{\hookrightarrow} C(A),$$

we see that here also the capacities $G_k\,(C(1))$ and the packing numbers $p_k\,(C(1))$ can be recovered from the function c_{EC} since for all integers $k \geqslant 1$

$$c_{EC}(k) = \frac{1}{G_k\,(C(1))}.$$

The structure of the graph of c_{EC} turns out to be very similar to the one of c_{EB}. For a less than the square σ^2 of the silver ratio $\sigma := 1 + \sqrt{2}$, the function c_{EC} is piecewise linear, with an infinite staircase converging to $(\sigma^2, \sqrt{\sigma^2/2})$ (see Figure 2.1.1). This staircase is determined by Pell numbers. On the interval $\left[\sigma^2, 7\frac{1}{32}\right]$, the function c_{EC} coincides with the volume constraint $\sqrt{\frac{a}{2}}$ except on seven disjoint intervals, where c_{EC} is piecewise linear (see Figure 2.1.2). Finally, for $a \geqslant 7\frac{1}{32}$ the functions c_{EC} and $\sqrt{\frac{a}{2}}$ are equal.

For the proof, we first translate the embedding problem $E(1,a) \hookrightarrow C(A)$ to a certain ball packing problem of the ball $B(2A)$. This embedding problem is then solved by adapting the method from [MS], which finds all exceptional spheres in blow-ups of the complex projective plane that provide an embedding obstruction.

We also prove in Chapter 2 that for any rational numbers $a, b > 0$, there exists a symplectic embedding $E(a,b) \hookrightarrow P(c,d) := D^2(c) \times D^2(d)$ if and only if there exists a symplectic embedding

$$B(a,b) \coprod B(c) \coprod B(d) \hookrightarrow B(c+d),$$

where $B(a,b)$ denotes a certain disjoint union of balls $\coprod_i B(w_i)$. The sequence of sizes w_i of the balls $B(w_i)$ are determined by the continued fraction of the rational number $\frac{b}{a}$.

A corollary of this result is that ECH-capacities are sharp for the problem of symplectically embedding an ellipsoid into a polydisc, that is,

$$E(a,b) \overset{s}{\hookrightarrow} P(c,d) \text{ if and only if } c_k(E(a,b)) \leqslant c_k(P(c,d))$$

for all $k \geqslant 0$. From this, we deduce that the ellipsoid $E(1,a)$ symplectically embeds into the cube $C(A)$ if and only if $E(1,a)$ symplectically embeds into the ellipsoid $E(A, 2A)$. Our embedding function c_{EC} thus also describes the smallest dilate of $E(1,2)$ into which $E(1,a)$ symplectically embeds.

1.4.2 Symplectic embeddings into the union of an ellipsoid and a cylinder

In Chapter 3 we study symplectic embeddings into the union of an ellipsoid and a cylinder. The results of Section 3.1 are joint work with Keon Choi, Daniel Cristofaro-Gardiner, Michael Hutchings and Vinicius G. B. Ramos and are part of [CGFHR] that has been accepted for publication in the *Journal of Topology*.

Let $a, b, c > 0$ be positive real numbers. Denote by

$$Z(a,b,c) := Z(a) \cup E(b,c)$$

the (non-disjoint) union of the symplectic cylinder $Z(a) = D^2(a) \times \mathbb{R}^2$ and the ellipsoid $E(b,c)$.

In Section 3.1 we solve some ball packing problems of $Z(a,b,c)$. More precisely we determine for certain values $b, c > 0$ the embedding capacity function

$$c\left(w_1,\ldots,w_n;b,c\right) := \inf\left\{\lambda \ \mid \ \coprod_{i=1}^{n} B\left(w_i\right) \overset{s}{\hookrightarrow} Z(\lambda,\lambda b,\lambda c)\right\}$$

for all positive real numbers $w_1, \ldots, w_n > 0$. We first compute the obstruction for these ball packings given by ECH capacities, leading to a lower bound on $c\left(w_1, \ldots, w_n; b, c\right)$. We then show that this lower bound is also an upper bound, by an explicit packing that is obtained by an embedding construction called symplectic shearing.

In Section 3.2 we construct an explicit embedding of $E\left(36, \frac{6}{5}\right)$ into $Z(1,6,6)$, using the symplectic folding method. This construction has the following consequences.

It shows that the problem of embedding an ellipsoid $E(a,b)$ into the union of an ellipsoid and a cylinder $Z(c,d,e) = Z(c) \cup E(d,e)$ is not rigid in general. Indeed, $E(36, \frac{6}{5}) \not\subset Z\left(1,6,6\right)$ which excludes trivial embeddings, and $\mathrm{vol}\left(E(36, \frac{6}{5})\right) > \mathrm{vol}\left(E\left(6,6\right)\right)$ which excludes embeddings of $E(36, \frac{6}{5})$ that take values in the ball $B(6) = E\left(6,6\right)$ alone (cf. [MS]).

Moreover, this embedding shows that the ball packing construction of Section 3.1 is not always optimal. Indeed, since $\coprod_{i=1}^{30} B\left(\frac{6}{5}\right) \overset{s}{\hookrightarrow} E\left(36, \frac{6}{5}\right)$, the embedding $E\left(36, \frac{6}{5}\right) \overset{s}{\hookrightarrow} Z(1,6,6)$ implies the existence of an embedding

$$\coprod_{i=1}^{30} B\left(\frac{6}{5}\right) \overset{s}{\hookrightarrow} Z(1,6,6)$$

while the shearing method of Section 3.1 would only give

$$\coprod_{i=1}^{30} B\left(\frac{6}{5}\right) \overset{s}{\hookrightarrow} \frac{36}{35} Z(1,6,6).$$

Chapter 2

Symplectic embeddings of 4-dimensional ellipsoids into cubes

2.1 Introduction

2.1.1 Statement of the result

Let $(\mathbb{R}^4, \omega)$ be the Euclidean 4-dimensional space endowed with the canonical symplectic form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Any open subset of $\mathbb{R}^4$ is also endowed with ω. Simple examples are the symplectic cylinders $Z(a) := D^2(a) \times \mathbb{R}^2$ (where $D^2(a)$ is the open disc of area a), the open symplectic ellipsoids

$$E(a_1, a_2) = \left\{ (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \, : \, \frac{\pi \left(x_1^2 + y_1^2\right)}{a_1} + \frac{\pi \left(x_2^2 + y_2^2\right)}{a_2} < 1 \right\},$$

and the open polydiscs $P\left(a_1, a_2\right) := D^2\left(a_1\right) \times D^2\left(a_2\right)$. We denote the open ball $E(a, a)$ (of radius $\sqrt{a/\pi}$) by $B(a)$ and the open cube $P(a, a)$ by $C(a)$. Since $D^2(a)$ is symplectomorphic to an open square, $D^2(a) \times D^2(a)$ is indeed symplectomorphic to a cube.

Given two open subsets U and V, we say that a smooth embedding $\varphi \colon U \hookrightarrow V$ is a *symplectic embedding* if φ preserves ω, that is, if $\varphi^* \omega = \omega$. In the sequel, we will write $\varphi \colon U \overset{s}{\hookrightarrow} V$ for such an embedding. Since symplectic embeddings are volume preserving, a necessary condition for the existence of a symplectic embedding $U \overset{s}{\hookrightarrow} V$ is, of course, $\mathrm{Vol}(U) \leqslant \mathrm{Vol}(V)$, where $\mathrm{Vol}(U) := \frac{1}{2} \int_U \omega \wedge \omega$. For volume preserving embeddings, this is the only condition (see e.g. [S1]). For symplectic embeddings, however, the situation is very different, as was detected by Gromov in [G]. Among many other things, he proved the following.

Example 2.1.1. (Gromov's nonsqueezing Theorem) There exists a symplectic embedding of the ball $B(a)$ into the cylinder $Z(A)$ if and only if $a \leqslant A$.

Notice that the volume of the cylinder $Z(A)$ is infinite, and that for any a the ball $B(a)$ embeds by a linear volume preserving embedding into $Z(A)$. Similarly, we also have

Example 2.1.2. There exists a symplectic embedding of the ball $B(a)$ into the cube $C(A)$ if and only if $a \leqslant A$.

The above results show that symplectic embeddings are much more special and in some sense "more rigid" than volume preserving embeddings. A next step was to understand this rigidity better. One way of doing this is to fix a domain $V \subset \mathbb{R}^4$ of finite volume, and to try to determine for each $k \in \mathbb{N}$ the *k-th packing number*

$$p_k(V) := \sup \left\{ \frac{k \operatorname{Vol}(B(a))}{\operatorname{Vol}(V)} \, : \, \bigsqcup_k B(a) \overset{s}{\hookrightarrow} V \right\}.$$

Here, $\bigsqcup_k B(a)$ is the disjoint union of k equal balls $B(a)$. It follows from Darboux's Theorem that always $p_k(V) > 0$. If $p_k(V) = 1$, one says that V admits a *full packing* by k balls, and if $p_k(V) < 1$, one says that there is a *packing obstruction*. Again, it is known that if we would consider volume preserving embeddings instead, then all packing numbers would always be 1.

In important work by Gromov [G], McDuff-Polterovich [MP] and Biran [B1] all the packing numbers of the 4-ball B and the 4-cube C were determined. The result for C is

k	1	2	3	4	5	6	7	$\geqslant 8$
p_k	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{48}{49}$	$\frac{224}{225}$	1

This result shows that, while there is symplectic rigidity for many small k, there is no rigidity at all for large k.

In order to better understand these numbers, we look at a problem that interpolates the above problem of packing by k equal balls. For $0 < a_1 \leqslant a_2$, consider the ellipsoid $E(a_1, a_2)$ defined above, and look for the smallest cube $C(A)$ into which $E(a_1, a_2)$ symplectically embeds. Since $E(a_1, a_2) \overset{s}{\hookrightarrow} C(A)$ if and only if $E(\lambda a_1, \lambda a_2) \overset{s}{\hookrightarrow} C(\lambda A)$, we can always assume that $a_1 = 1$, and therefore study the *embedding capacity function*

$$c_{EC}(a) := \inf \left\{ A \, : \, E(1, a) \overset{s}{\hookrightarrow} C(A) \right\}$$

on the interval $[1, \infty[$. It is clear that c_{EC} is a continuous and nondecreasing function. Since symplectic embeddings are volume preserving and the volumes of $E(1,a)$ and $C(A)$ are $\frac{1}{2}a$ and A^2 respectively, we must have the lower bound

$$\sqrt{\frac{a}{2}} \leqslant c_{EC}(a).$$

It is not hard to see that $\bigsqcup_k B(1) \overset{s}{\hookrightarrow} E(1,k)$. Therefore, $\bigsqcup_k B(1) \overset{s}{\hookrightarrow} C(A)$ whenever $E(1,k) \overset{s}{\hookrightarrow} C(A)$. In [M2], McDuff has shown that the converse is also true! Our ellipsoid embedding problem therefore indeed interpolates the problem of packing by k equal balls, and we get

$$p_k(C) = \frac{\operatorname{Vol}(E(1,k))}{\operatorname{Vol}(C(c_{EC}(k)))} = \frac{\frac{k}{2}}{(c_{EC}(k))^2}.$$

First upper estimates for the function $c_{EC}(a)$ were obtained in Chapter 4.4 of [S2] by explicit embeddings of ellipsoids into a cube. These upper estimates also suggested that symplectic rigidity for the problem $E(1,a) \overset{s}{\hookrightarrow} C(A)$ should disappear for large a.

In this paper, we completely determine the function $c(a) := c_{EC}(a)$. In order to state our main theorem, we introduce two sequences of integers: the *Pell numbers* P_n and the *half companion Pell numbers* H_n, which are defined by the recurrence relations

$$\begin{aligned} P_0 = 0, P_1 = 1, &\quad P_n = 2P_{n-1} + P_{n-2}, \\ H_0 = 1, H_1 = 1, &\quad H_n = 2H_{n-1} + H_{n-2}. \end{aligned}$$

Thus, $P_2 = 2$, $P_3 = 5$, $P_4 = 12$, $P_5 = 29, \dots$ and $H_2 = 3$, $H_3 = 7$, $H_4 = 17$, $H_5 = 41, \dots$. The two sequences $(\alpha_n)_{n\geqslant 0}$ and $(\beta_n)_{n\geqslant 0}$ are then defined by

$$\alpha_n := \begin{cases} \dfrac{2P_{n+1}^2}{H_n^2} & \text{if } n \text{ is even,} \\ \dfrac{H_{n+1}^2}{2P_n^2} & \text{if } n \text{ is odd;} \end{cases} \qquad \beta_n := \begin{cases} \dfrac{H_{n+2}}{H_n} & \text{if } n \text{ is even,} \\ \dfrac{P_{n+2}}{P_n} & \text{if } n \text{ is odd.} \end{cases}$$

The first terms in these sequences are

$$\alpha_0 = 2 < \beta_0 = 3 < \alpha_1 = \frac{9}{2} < \beta_1 = 5 < \alpha_2 = \frac{50}{9} < \beta_2 = \frac{17}{3} < \dots .$$

More generally, for all $n \geqslant 0$,

$$\dots < \alpha_n < \beta_n < \alpha_{n+1} < \beta_{n+1} < \dots,$$

and both sequences converge to $\sigma^2 = 3 + 2\sqrt{2} \cong 5.83$, which is the square of the silver ration $\sigma := 1 + \sqrt{2}$.

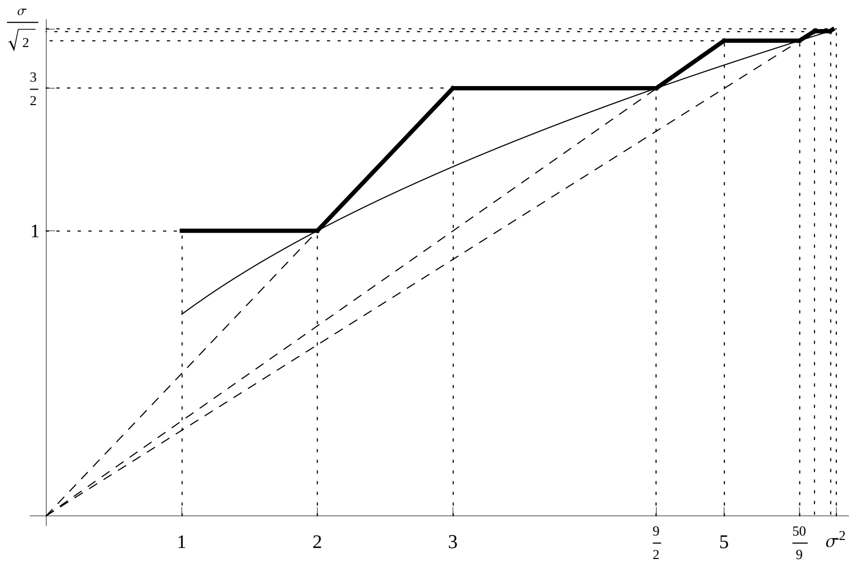

FIGURE 2.1.1 – The graph of c on the interval $[1, \sigma^2]$

Theorem 2.1.3. *(i) On the interval* $[1, \sigma^2]$,

$$c(a) = \begin{cases} 1 & \text{if } a \in [1,2]\,, \\ \dfrac{1}{\sqrt{2\alpha_n}}\, a & \text{if } a \in [\alpha_n, \beta_n]\,, \\ \sqrt{\dfrac{\alpha_{n+1}}{2}} & \text{if } a \in [\beta_n, \alpha_{n+1}]\,, \end{cases}$$

for all $n \geqslant 0$ *(see Figure 2.1.1).*

(ii) On the interval $\left[\sigma^2, 7\frac{1}{32}\right]$ *we have* $c(a) = \sqrt{\frac{a}{2}}$ *except on seven disjoint intervals, where* c *is piecewise linear (see Figure 2.1.2).*

(iii) For $a \geqslant 7\frac{1}{32}$ *we have* $c(a) = \sqrt{\frac{a}{2}}$.

The proof of (i) is given in Corollary 2.5.2, a more detailed statement as well as the proof of (ii) are given in Theorem 2.7.2, while the proof of (iii) is given in Lemma 2.4.1 and Proposition 2.7.7.

A similar result has been previously obtained by McDuff-Schlenk in [MS] for the embedding problem $E(1,a) \overset{s}{\hookrightarrow} B(A)$. These two results show that the structure of symplectic rigidity can be very rich.

For further results on packings of various symplectic manifolds by balls and ellipsoids we refer to [B2], [BuH1], [BuH2], [BuP], [LMS] and [O].

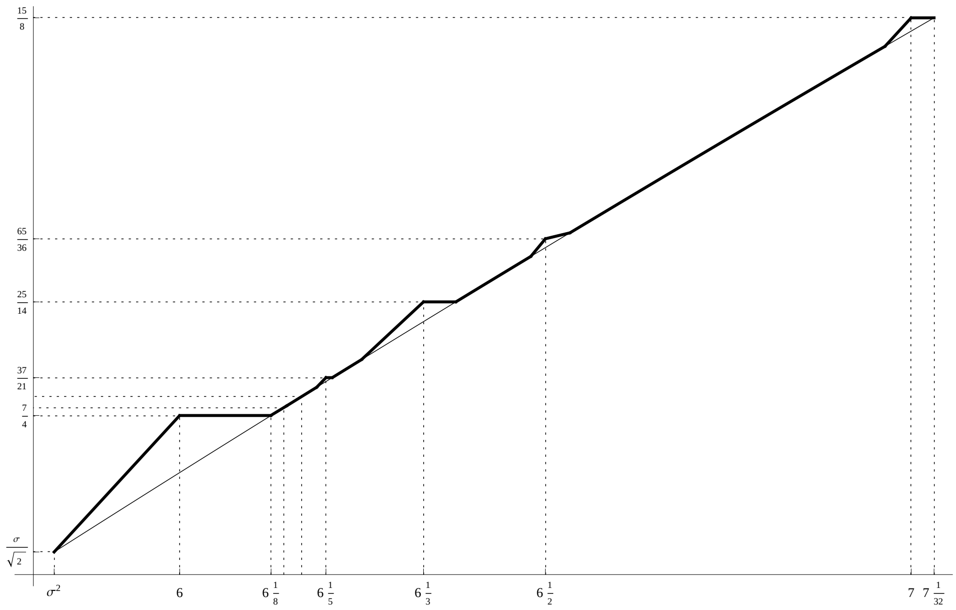

FIGURE 2.1.2 – The graph of c on the interval $\left[\sigma^2, 7\frac{1}{32}\right]$

2.1.2 Relations to ECH-capacities

There is a more combinatorial (but non-explicit) way of describing the embedding function $c_{EC}(a)$. Indeed, in [H1], Hutchings used his embedded contact homology to construct for each domain $U \subset \mathbb{R}^4$ a sequence of symplectic capacities $c^k_{ECH}(U)$, which for the ellipsoid $E(a,b)$ and the polydisc $P(a,b)$ are as follows.

Form the sequence $N_E(a,b)$ by arranging all numbers of the form $ma+nb$ with $m,n \geqslant 0$, in nondecreasing order (with multiplicities). Then for $k \geqslant 0$, the k-th ECH-capacity $c^k_{ECH}(E(a,b))$ is the $(k+1)$-th entry of $N_E(a,b)$. For instance, $c_{ECH}(E(1,1)) = \{0,1,1,2,2,2,3,3,3,3,4,\dots\}$.

Moreover, for polydiscs,

$$c^k_{ECH}(P(a,b)) \;=\; \min\left\{am+bn \;:\; m,n \in \mathbb{N};\; (m+1)(n+1) \geqslant k+1\right\}.$$

There exists a canonical way to decompose an ellipsoid $E(a,b)$ with $\frac{a}{b}$ rational into a finite disjoint union of balls $B(a,b) := \sqcup_i B(w_i)$ with weights w_i related to the continued fraction expansion of $\frac{a}{b}$. We shall explain this decomposition in more detail and prove the following proposition in the next section.

Proposition 2.1.4. *Let $a,b,c,d > 0$ with $\frac{a}{b}$ rational. Then there exists a symplectic embedding $E(a,b) \hookrightarrow P(c,d)$ if and only if there exists a symplectic embedding*

$$B(a,b) \bigsqcup B(c) \bigsqcup B(d) \;\hookrightarrow\; B(c+d).$$

Hutchings showed in Corollary 11 of [H2] how Proposition 2.1.4 implies that ECH-capacities form a complete set of invariants for the problem of symplectically embedding an ellipsoid into a polydisc:

Corollary 2.1.5. *There exists a symplectic embedding* $E(a,b) \hookrightarrow P(c,d)$ *if and only if* $c^k_{ECH}(E(a,b)) \leqslant c^k_{ECH}(P(c,d))$ *for all* $k \geqslant 0$.

It seems to be hard to derive Theorem 2.1.3 from Corollary 2.1.5 or vice-versa.

As a further corollary we obtain

Corollary 2.1.6. *The ellipsoid* $E(1,a)$ *symplectically embeds into the cube* $C(A)$ *if and only if* $E(1,a)$ *symplectically embeds into the ellipsoid* $E(A,2A)$.

Proof. By Corollary 2.1.5, $E(1,a)$ symplectically embeds into $C(A)$ if and only if $c^k_{ECH}(E(1,a)) \leqslant c^k_{ECH}(C(A))$ for all $k \geqslant 0$. By McDuff's proof of the Hofer Conjecture [M3], $E(1,a)$ symplectically embeds into $E(A,2A)$ if and only if $c^k_{ECH}(E(1,a)) \leqslant c^k_{ECH}(E(A,2A))$ for all $k \geqslant 0$. The corollary now follows from the remark on page 8098 in [H2], that says that for all $k \geqslant 0$

$$c^k_{ECH}(E(1,2)) = c^k_{ECH}(C(1)). \tag{2.1.1}$$

For the easy proof, we refer to Section 2.2. □

Remark 2.1.7. Recall that the ECH-capacities of $B(1)$ and $C(1)$ (or $E(1,2)$) are

$$c_{ECH}(B(1)) = \left(0^{\times 1}, 1^{\times 2}, 2^{\times 3}, 3^{\times 4}, 4^{\times 5}, 5^{\times 6}, 6^{\times 7}, 7^{\times 8}, 8^{\times 9}, 9^{\times 10}, \dots\right),$$
$$c_{ECH}(C(1)) = \left(0^{\times 1}, 1^{\times 1}, 2^{\times 2}, 3^{\times 2}, 4^{\times 3}, 5^{\times 3}, 6^{\times 4}, 7^{\times 4}, 8^{\times 5}, 9^{\times 5}, \dots\right).$$

One sees that the sequence $c_{ECH}(C(1))$ is obtained from $c_{ECH}(B(1))$ by some sort of doubling. This is reminiscent to the doubling in the definition of the Pell numbers: The Fibonacci and Pell numbers are defined recursively by

$$F_{n+1} = F_n + F_{n-1}, \qquad P_{n+1} = 2P_n + P_{n-1},$$

and while the Fibonacci numbers determine the infinite stairs of the function $c_{EB}(a)$ for $a \leqslant \tau^4$ (with τ the golden ratio, see [MS]), the Pell numbers determine the infinite stairs of the function $c_{EC}(a)$ for $a \leqslant \sigma^2$. This reminiscence may, however, be a coincidence. Indeed, for the ellipsoid $E(1,3)$ the sequence

$$c_{ECH}(E(1,3) = \left(0^{\times 1}, 1^{\times 1}, 2^{\times 1}, 3^{\times 2}, 4^{\times 2}, 5^{\times 2}, 6^{\times 3}, 7^{\times 3}, 8^{\times 3}, 9^{\times 4}, \dots\right)$$

is obtained from $c_{ECH}(B(1))$ by some sort of tripling, but the beginning of the function describing the embedding problem $E(1,a) \overset{s}{\hookrightarrow} E(A,3A)$ seems not to be given in terms of numbers defined by $G_{n+1} = 3G_n + G_{n-1}$.

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2.2 Proof of Proposition 2.1.4 and equalities (2.1.1)

In Section 2.2.1, we explain the canonical decomposition of $E(1,a)$ with $a \in \mathbb{Q}$ into a disjoint union of balls. We then prove Proposition 2.1.4 in Sections 2.2.2 and 2.2.3, and in Section 2.2.4 we prove equalities (2.1.1).

2.2.1 Decomposing an ellipsoid into a disjoint union of balls

In [M2], McDuff showed the following theorem.

Theorem 2.2.1. (McDuff [M2]) *Let $a, b > 0$ be two rational numbers. Then, there exists a finite sequence $(w_1, \dots, w_M)$ of rational numbers such that the closed ellipsoid $\overline{E}(a,b)$ symplectically embeds into the ball $B(A)$ if and only if the disjoint union of balls $\sqcup_i \overline{B}(w_i)$ symplectically embed into $B(A)$.*

The disjoint union $\sqcup_i \overline{B}(w_i)$ is then denoted by $\overline{B}(a,b)$. Following [MS], we will now explain one way to compute the weights $w_1, \dots, w_M$ in this decomposition. Notice that in [M2], the weights of the balls $B(w_i)$ are defined in a slightly different way. The proof that these weights agree with the weight expansion of a defined now can be found in the Appendix of [MS].

Definition 2.2.2. Let $a = \frac{p}{q} \geqslant 1$ be a rational number written in lowest terms. The *weight expansion* of a is the finite sequence $w(a) := (w_1, \dots, w_M)$ defined recursively by

- $w_1 = 1$, and $w_n \geqslant w_{n+1} > 0$ for all n;
- if $w_i > w_{i+1} = \ldots = w_n$ (where we set $w_0 := a$), then

$$w_{n+1} = \begin{cases} w_n & \text{if } w_{i+1} + \ldots + w_{n+1} = (n-i+1)w_{i+1} \\ & \qquad\qquad \leqslant w_i, \\ w_i - (n-i)w_{i+1} & \text{otherwise;} \end{cases}$$

- the sequence stops at w_n if the above formula gives $w_{n+1} = 0$.

Remark 2.2.3. If we regard this weight expansion as consisting of $N+1$ blocks on which the w_i are constant, that is

$$w(a) = (\underbrace{1, \dots, 1}_{l_0}, \underbrace{x_1, \dots, x_1}_{l_1}, \dots, \underbrace{x_N, \dots, x_N}_{l_N}) = \left(1^{\times l_0}, x_1^{\times l_1}, \dots, x_N^{\times l_N}\right),$$

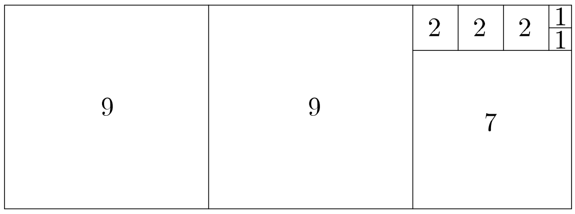

FIGURE 2.2.1 – The weight expansion of $\frac{25}{9}$.

then $x_1 = a - l_0 < 1$, and if we set $x_0 = 1$, then for all $2 \leqslant i \leqslant N$, $x_i = x_{i-2} - l_{i-1}x_{i-1}$. Moreover, the lengths of the blocks give the continued fraction of a since

$$a = l_0 + \cfrac{1}{l_1 + \cfrac{1}{l_2 + \cfrac{1}{\ddots + \cfrac{1}{l_N}}}} =: [l_0; l_1, \ldots, l_N] \, .$$

Example 2.2.4. The weight expansion of $\frac{25}{9}$ is $\left(1^{\times 2}, \frac{7}{9}, \frac{2}{9}^{\times 3}, \frac{1}{9}^{\times 2}\right)$. The continued fraction expansion of $\frac{25}{9}$ is thus $[2; 1, 3, 2]$. Notice that we also have

$$\frac{25}{9} = 2 \cdot 1^2 + \left(\frac{7}{9}\right)^2 + 3 \cdot \left(\frac{2}{9}\right)^2 + 2 \cdot \left(\frac{1}{9}\right)^2 .$$

This is no accident and is best explained geometrically as in Figure 2.2.1. The general result is stated in the next lemma.

Lemma 2.2.5. (McDuff-Schlenk [MS], Lemma 1.2.6) *Let $a = \frac{p}{q} \geqslant 1$ be a rational number with p, q relatively prime, and let $w := w(a) = (w_1, ..., w_M)$ be its weight expansion. Then*

(i) $w_M = \frac{1}{q}$;

(ii) $\sum w_i^2 = \langle w, w \rangle = a$;

(iii) $\sum w_i = a + 1 - \frac{1}{q}$.

2.2.2 Representations of balls and polydiscs

In the proof of Proposition 2.1.4, we shall use certain ways of representing open and closed balls and open polydiscs. Recall that $B(a)$ is the open ball in $\mathbb{R}^4$ of capacity $a = \pi r^2$, and that $P(a, b) = D^2(a) \times D^2(b)$, where $D^2(a)$ is the open disc in $\mathbb{R}^2$ of area a.

2.2.2.1 Representations as products

Denote by $\square(a,b)$ the open square $]0,a[\, \times\,]0,b[$ in $\mathbb{R}^2$. Since $D^2(a)$ is symplectomorphic to the open square $]0,a[\, \times\,]0,1[$, the polydisc $P(a,b)$ is symplectomorphic to

$$\square(a,b) \times \square(1,1) \, \subset \, \mathbb{R}^2(x) \times \mathbb{R}^2(y).$$

Next, consider the simplex

$$\triangle(a) \, := \, \Big\{ (x_1,x_2) \in \mathbb{R}^2(x) \, : \, 0 < x_1, x_2 \, ; \, x_1 + x_2 < a \Big\}.$$

Then $B(a)$ is symplectomorphic to the product

$$\triangle(a) \times \square(1,1) \, \subset \, \mathbb{R}^2(x) \times \mathbb{R}^2(y),$$

see [T] and Remark 9.3.1 of [S2].

2.2.2.2 Representations by the Delzant polytope

Denote by ω_{SF} the Study-Fubini form on the complex projective plane $\mathbb{C}P^2$, normalized by $\int_{\mathbb{C}P^1} \omega_{SF} = 1$. We write $\mathbb{C}P^2(a)$ for $(\mathbb{C}P^2, a\, \omega_{SF})$. Its affine part $\mathbb{C}P^2 \setminus \mathbb{C}P^1$ is symplectomorphic to the open ball $B(a)$. (Indeed, for $a = \pi$, the embedding

$$z = (z_1,z_2) \, \mapsto \, \left[z_1 : z_2 : \sqrt{1-|z|^2} \right]$$

is symplectic.)

The image of the moment map of the usual T^2-action on $\mathbb{C}P^2(a)$ is the closed triangle $\overline{\triangle(a)}$. For $b < a$, the preimage of $\overline{\triangle(b)} \subset \overline{\triangle(a)}$ is symplectomorphic to $\overline{B}(b)$. By precomposing the torus action with suitable linear torus automorphisms, one sees that also the closed triangles based at the other two corners of $\overline{\triangle(a)}$ correspond to closed balls in $\mathbb{C}P^2(a)$. We refer to [K] for details.

The image of the moment map of the usual T^2-action on $\mathbb{C}^2$ maps the polydisc $P(c,d)$ to the rectangle $[0,c[\, \times\, [0,d[\, \subset \mathbb{R}^2(x)$.

2.2.3 Proof of Proposition 2.1.4

Let now $a,b,c,d > 0$ with $\frac{a}{b}$ rational. We need to show that

$$E(a,b) \overset{s}{\hookrightarrow} P(c,d) \iff B(a,b) \sqcup B(c) \sqcup B(d) \overset{s}{\hookrightarrow} B(c+d).$$

"$\Longrightarrow$": By decomposing $E(a,b)$ into balls as before, we find that $B(a,b) \overset{s}{\hookrightarrow} P(c,d)$ (see also [M2]). Fix $\varepsilon > 0$. Then we have also $(1-\varepsilon)\overline{B}(a,b) \overset{s}{\hookrightarrow} P(c,d)$. Now represent the open balls $B(c), B(d), B(c+d)$

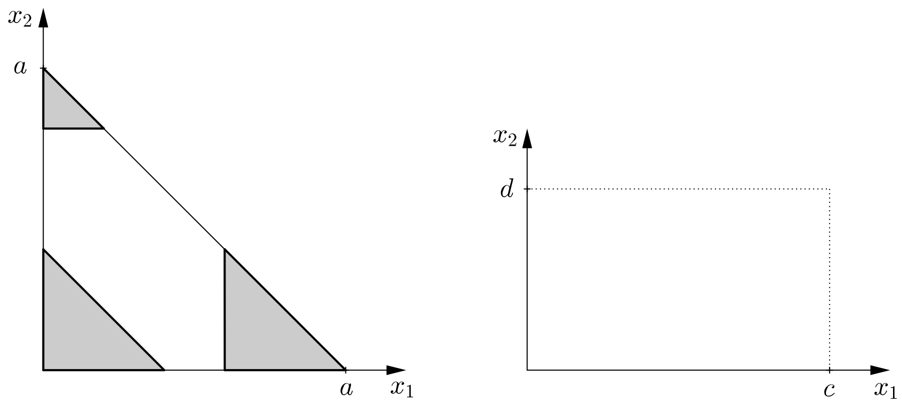

FIGURE 2.2.2 – Closed balls in $\mathbb{C}P^2(a)$ and the moment image of $P(c,d)$

and the polydisc as in Section 2.2.2.1 above. We then read off from Figure 2.2.3 that

$$(1-\varepsilon)\overline{B}(a,b) \sqcup B(c) \sqcup B(d) \overset{s}{\hookrightarrow} B(c+d).$$

This holds for every $\varepsilon > 0$. In view of [M1] we then also find a symplectic embedding $B(a,b) \sqcup B(c) \sqcup B(d) \overset{s}{\hookrightarrow} B(c+d)$.

"$\Longleftarrow$" : Assume now that $B(a,b) \sqcup B(c) \sqcup B(d) \overset{s}{\hookrightarrow} B(c+d)$. Fix $\varepsilon > 0$. Then

$$(1-\varepsilon)\overline{B}(a,b) \sqcup \overline{B}(c-\varepsilon) \sqcup \overline{B}(d-\varepsilon) \overset{s}{\hookrightarrow} \mathbb{C}P^2(c+d).$$

According to [M1], the space of symplectic embeddings of $\overline{B}(c-\varepsilon) \sqcup \overline{B}(d-\varepsilon)$ into $\mathbb{C}P^2(c+d)$ is connected. Any such isotopy extends to an ambient symplectic isotopy of $\mathbb{C}P^2(c+d)$. In view of this and by Section 2.2.2.2 we can thus assume that the balls $\overline{B}(c-\varepsilon)$ and $\overline{B}(d-\varepsilon)$ lie in $\mathbb{C}P^2(c+d)$ as shown in Figure 2.2.4.

The image of the balls $(1-\varepsilon)\overline{B}(a,b)$ must then lie over the gray shaded closed region. However, since the balls $\overline{B}(c-\varepsilon)$ and $\overline{B}(d-\varepsilon)$ are closed, the image of $(1-\varepsilon)\overline{B}(a,b)$ cannot touch the upper horizontal or the right vertical boundary of the gray shaded region. Moreover, according to Remark 2.1.E of [MP] we can assume that this image lies in the affine part of $\mathbb{C}P^2(c+d)$, i.e., the image of the balls $(1-\varepsilon)\overline{B}(a,b)$ lies over the gray shaded region deprived from the dark segment, and hence, by Section 2.2.2.2, in $P(c+\varepsilon, d+\varepsilon)$. We may suppose from the start that $c, d \geqslant 1$. Then $P(c+\varepsilon, d+\varepsilon) \subset (1+\varepsilon)P(c,d)$. We have thus found a symplectic embedding $(1-\varepsilon)\overline{B}(a,b) \overset{s}{\hookrightarrow} (1+\varepsilon)P(c,d)$. It is shown in Theorem 1.5 of [M2] that then also $(1-\varepsilon)\overline{E}(a,b) \overset{s}{\hookrightarrow} (1+\varepsilon)P(c,d)$. Hence

$$\frac{1-\varepsilon}{1+\varepsilon}\overline{E}(a,b) \overset{s}{\hookrightarrow} P(c,d).$$

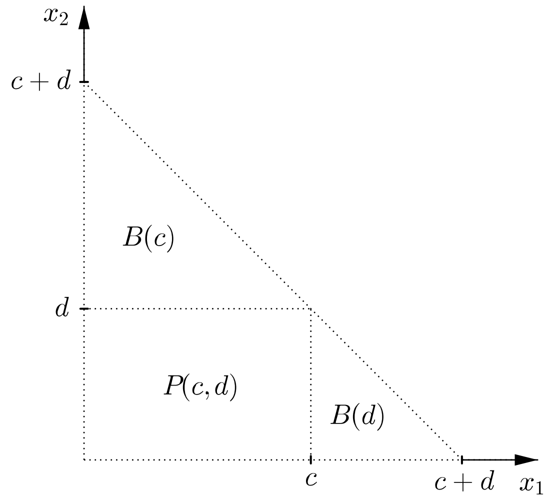

FIGURE 2.2.3 – $(1-\varepsilon)\overline{B}(a,b) \sqcup B(c) \sqcup B(d) \overset{s}{\hookrightarrow} B(c+d)$

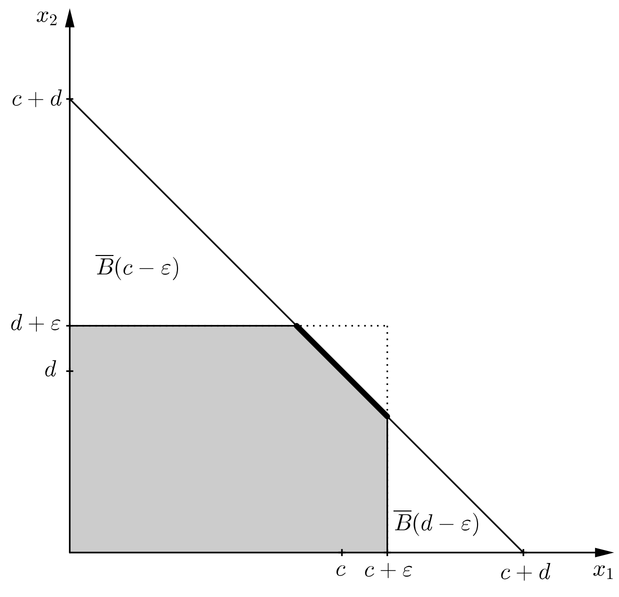

FIGURE 2.2.4 – How $\overline{B}(c-\varepsilon)$ and $\overline{B}(d-\varepsilon)$ lie in $\mathbb{C}P^2(c+d)$

It now follows again from [M2] that $E(a,b) \overset{s}{\hookrightarrow} P(c,d)$. (To be precise, [M2] considers embeddings of ellipsoids into open balls; however, the same arguments work for embeddings of ellipsoids into polydiscs.) $\square$

2.2.4 Proof of equalities (2.1.1)

Lemma 2.2.6. *For all $k \geqslant 0$,*

$$c^k_{ECH}\left(E(1,2)\right) = c^k_{ECH}\left(C(1)\right).$$

Proof. We will prove that $c^k_{ECH}\left(E(1,2)\right)$ and $c^k_{ECH}\left(C(1)\right)$ are both equal to the unique integer d such that

$$\left\lfloor \frac{d+1}{2} \right\rfloor \left\lceil \frac{d+1}{2} \right\rceil \leqslant k < \left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil.$$

For $c^k_{ECH}\left(E(1,2)\right)$, this follows from the fact that the number

$$\sharp\left\{(m,n) \in \mathbb{N}_0^2 : m+2n \leqslant d\right\}$$

of pairs of nonnegative integers (m,n) such that $m+2n \leqslant d$ is equal to $\left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil$. This, in turn, can easily be deduced from the identities

$$\sharp\left\{(m,n) \in \mathbb{N}_0^2 : m+2n = 2l\right\} = \sharp\left\{(m,n) \in \mathbb{N}_0^2 : m+2n = 2l+1\right\} = l+1.$$

On the other hand, we have

$$c^k_{ECH}\left(C(1)\right) = c^k_{ECH}\left(P(1,1)\right) = \min\left\{m+n : (m+1)(n+1) \geqslant k+1\right\}.$$

Fix a nonnegative integer k. Let $m_0, n_0 \in \mathbb{N}_0$ be two nonnegative integers such that

$$m_0 + n_0 = \min\left\{m+n : (m+1)(n+1) \geqslant k+1\right\}.$$

Without loss of generality, $m_0 \geqslant n_0$. Moreover, we can always take m_0, n_0 such that $m_0 - n_0 \in \{0,1\}$. Indeed, assume that $m_0 = n_0 + c$ with $c \geqslant 2$. Then for $m_0' = m_0 - 1$ and $n_0' = n_0 + 1$, we get

$$\begin{aligned}
\left(m_0'+1\right)\left(n_0'+1\right) &= m_0\left(n_0+2\right) = \left(n_0+c\right)\left(n_0+2\right) = n_0^2 + (c+2)n_0 + 2c \\
&> n_0^2 + (c+2)n_0 + c + 1 = \left(n_0+c+1\right)\left(n_0+1\right) \\
&= \left(m_0+1\right)\left(n_0+1\right) \geqslant k+1.
\end{aligned}$$

Thus (m_0', n_0') also realizes the minimum. Now, if m_0+n_0 is even, then $m_0 = n_0$ and we have to show that

$$\begin{aligned}
\left\lfloor \frac{2m_0+1}{2} \right\rfloor \left\lceil \frac{2m_0+1}{2} \right\rceil &= m_0\left(m_0+1\right) \leqslant k \\
&< \left(m_0+1\right)^2 = \left\lfloor \frac{2m_0+2}{2} \right\rfloor \left\lceil \frac{2m_0+2}{2} \right\rceil.
\end{aligned}$$

The first inequality follows from the minimality of m_0+n_0 while the second one follows from the fact that $\left(m_0+1\right)\left(n_0+1\right) \geqslant k+1$. The case m_0+n_0 odd is treated similarly. $\square$

2.3 Reduction to a constraint function given by exceptional spheres

In this section we explain how the function $c(a)$ can be described by the volume constraint $\sqrt{\frac{a}{2}}$ and the constraints coming from certain exceptional spheres in blow-ups of $\mathbb{C}P^2$. Since the function c is continuous, it suffices to determine c for each rational $a \geqslant 1$. The starting point is the following lemma, which is a special case of Proposition 2.1.4.

Lemma 2.3.1. *Let $a \geqslant 1$ be a rational number with weight expansion $w(a) = (w_1, \ldots, w_M)$ and $A > 0$. Then the ellipsoid $E(1,a)$ embeds symplectically into the cube $C(A)$ if and only if there is a symplectic embedding*

$$B(A) \sqcup B(A) \sqcup_i B(w_i) \overset{s}{\hookrightarrow} B(2A).$$

With this lemma, we have converted the problem of embedding an ellipsoid into a cube to the problem of embedding a disjoint union of balls into a ball. In [MP], the problem of embedding k disjoint balls into a ball was reduced to the question of understanding the symplectic cone of the k-fold blow-up X_k of $\mathbb{C}P^2$. Let $L := [\mathbb{C}P^1] \in H_2(X_k, \mathbb{Z})$ be the class of a line, let $E_1, \ldots, E_k \in H_2(X_k, \mathbb{Z})$ be the homology classes of the exceptional divisors, and denote by $l, e_1, \ldots, e_k \in H^2(X_k, \mathbb{R})$ their Poincaré duals. Let $-K := 3L - \sum E_i$ be the anti-canonical divisor of X_k, and define the corresponding *symplectic cone* $\mathcal{C}_K(X_k) \subset H^2(X_k, \mathbb{R})$ as the set of classes represented by symplectic forms ω with first Chern class $c_1(X_k, \omega)$ Poincaré dual to $-K$.

Theorem 2.3.2. (McDuff-Polterovich [MP]) *The union $\sqcup_{i=1}^k \overline{B}(w_i)$ embeds into the ball $B(\mu)$ or into $\mathbb{C}P^2(\mu)$ if and only if $\mu l - \sum w_i e_i \in \mathcal{C}_K(X_k)$.*

To understand $\mathcal{C}_K(X_k)$, we define as in [MS] the following set $\mathcal{E}_k \subset H_2(X_k)$.

Definition 2.3.3. $\mathcal{E}_k$ is the set consisting of $(0; -1, 0, \ldots, 0)$ and of all tuples $(d; m) := (d; m_1, \ldots, m_k)$ with $d \geqslant 0$ and $m_1 \geqslant \ldots \geqslant m_k \geqslant 0$ such that the class $E_{(d;m)} := dL - \sum m_i E_i \in H_2(X_k)$ is represented in X_k by a symplectically embedded sphere of self-intersection -1.

We will often write $\mathcal{E}$ instead of $\mathcal{E}_k$ if there is no danger of confusion. We then have the following description of $\mathcal{C}_K(X_k)$.

Proposition 2.3.4. (B [B1], Li-Li [LiLi], Li-Liu [LiLiu])

$$\mathcal{C}_K(X_k) = \left\{ \alpha \in H^2(X_k) : \alpha^2 > 0,\ \alpha(E) > 0, \forall E \in \mathcal{E}_k \right\}.$$

In order to give a characterization of the set $\mathcal{E}_k$, we need the following definition as in [MS].

Definition 2.3.5. A tuple $(d;m) := (d;m_1,\dots m_k)$ is said to be *ordered* if the m_i are in non-increasing order. The *Cremona transform* of an ordered tuple $(d;m)$ is

$$(2d - m_1 - m_2 - m_3; d - m_2 - m_3, d - m_1 - m_3, d - m_1 - m_2, m_4, \dots, m_k)\,.$$

A *Cremona move* of a tuple $(d;m)$ is the composition of the Cremona transform of $(d;m)$ with any permutation of the new obtained vector m.

Proposition 2.3.6. (McDuff-Schlenk [MS], Proposition 1.2.12 and Remark 3.3.1)

(i) All $(d;m) \in \mathcal{E}_k$ satisfy the two Diophantine equations

$$\sum m_i = 3d - 1,$$
$$\sum m_i^2 = d^2 + 1.$$

(ii) For all distinct $(d;m)\,,(d';m') \in \mathcal{E}_k$ we have

$$\sum m_i m_i' \leqslant dd'.$$

(iii) A tuple $(d;m)$ belongs to $\mathcal{E}_k$ if and only if $(d;m)$ satisfies the Diophantine equations in (i) and $(d;m)$ can be reduced to $(0;-1,0,\dots,0)$ by repeated Cremona moves.

Remark 2.3.7. Working directly with Lemma 2.3.1, Theorem 2.3.2 and Proposition 2.3.4 we find, as in [MS], that the only constraints for an embedding $E(1,a) \overset{s}{\hookrightarrow} C(A)$ are $A \geqslant \sqrt{\frac{a}{2}}$ and, for each class $(d;m) \in \mathcal{E}_k$,

$$2Ad \;\geqslant\; (m_1 + m_2)\; A + \langle (m_3,\dots,m_k)\,,\, w(a)\rangle\,. \qquad (2.3.1)$$

One can start from here and use Proposition 2.3.6 to prove Theorem 2.1.3. The analysis becomes, however, rather awkward, since the unknown A appears on both sides of (2.3.1).

To improve the situation, we shall apply a base change of $H_2(X_k)$, and express the elements of $\mathcal{E}$ in a new basis. Consider the product $S^2 \times S^2$ (whose affine part is a cube), and form the M-fold (topological) blow-up $X_M(S^2 \times S^2)$. A basis of $H_2(X_M(S^2 \times S^2))$ is given by $S_1, S_2, F_1, \dots, F_M$, where $S_1 := [S^2 \times \{\text{point}\}]$, $S_2 := [\{\text{point}\} \times S^2]$ and $F_1,\dots,F_M$ are the classes of the exceptional divisors.

Notice that there is a diffeomorphism $\varphi\colon X_M(S^2 \times S^2) \to X_{M+1}(\mathbb{C}P^2)$ such that the induced map in homology is

$$\begin{array}{rccl}
\varphi_*\colon & H_2(X_M(S^2\times S^2)) & \longrightarrow & H_2(X_{M+1}(\mathbb{C}P^2)) \\
& S_1 & \longmapsto & L - E_1 \\
& S_2 & \longmapsto & L \qquad - E_2 \\
& F_1 & \longmapsto & L - E_1 - E_2 \\
& F_i & \longmapsto & \qquad\qquad E_{i+1}.
\end{array}$$

The existence of such a φ is clear from a moment map picture such as Figure 2.2.4 above. With respect to the new basis $S_1, S_2, F_1, \ldots, F_M$ we write an element of $H_2\left(X_M\left(S^2 \times S^2\right)\right)$ as $(d, e; m_1, \ldots, m_M)$. Then

$$\varphi_*(d, e; m) \;=\; (d + e - m_1; d - m_1, e - m_1, m_2, \ldots, m_M)\,.$$

In the new basis, the constraint given by a class in $\mathcal{E}$ can be written in a more useful form:

Proposition 2.3.8. *(i) All $(d, e; m) \in \mathcal{E}_M$ satisfy the two Diophantine equations*

$$\sum m_i = 2(d+e) - 1,$$
$$\sum m_i^2 = 2de + 1.$$

(ii) For all distinct $(d, e; m), (d', e'; m') \in \mathcal{E}_M$, we have

$$\sum m_i m_i' \leqslant de' + d'e.$$

(iii) A tuple $(d, e; m)$ belongs to $\mathcal{E}_M$ if and only if $(d, e; m)$ satisfies the Diophantine equations of (i) and its image under φ_* can be reduced to $(0; -1, 0, \ldots, 0)$ by repeated Cremona moves.

Proof. Let $E \in \mathcal{E}$. The two identities in Proposition 2.3.6 (i) correspond to $c_1(E) = 1$ and $E \cdot E = -1$. For $E = d\,S_1 + e\,S_2 - \sum m_i F_i$ these identities become

$$c_1(E) = 2d + 2e - \sum m_i = 1,$$
$$E \cdot E = -\sum m_i^2 + 2de = -1,$$

proving (i). Assertion (ii) of Proposition 2.3.6 corresponds to positivity of intersection of J-holomorphic spheres representing $E, E' \in \mathcal{E}$. For distinct elements $E = (d, e; m)$ and $E' = (d', e'; m')$ in $\mathcal{E}$ we thus have

$$E \cdot E' = de' + ed' - \sum m_i m_i' \geqslant 0,$$

proving (ii). Assertion (iii) holds since φ_* is a base change. $\square$

In the sequel, given two vectors m and w of length M, we will denote by $\langle m, w \rangle = \sum_{i=1}^M m_i w_i$ the Euclidean scalar product in $\mathbb{R}^M$. Notice that we will also use this notation for vectors m and w of different lengths, meaning the Euclidean scalar product of the two vectors after adding enough zeros at the end of the shorter one.

Proposition 2.3.9. *Let $a \geqslant 1$ be a rational number with weight expansion $w(a) = (w_1, \ldots, w_M)$. For $(d, e; m) \in \mathcal{E}$, define the constraint*

$$\mu(d, e; m)(a) := \frac{\langle m, w(a) \rangle}{d + e}.$$

Then

$$c(a) = \sup_{(d,e;m)\in\mathcal{E}} \left\{ \sqrt{\frac{a}{2}}, \mu(d,e;m)(a) \right\}.$$

Proof. By Lemma 2.3.1, $E(1,a) \stackrel{s}{\hookrightarrow} C(A)$ if and only if

$$B(A) \sqcup B(A) \sqcup_i B\left(w_i\right) \stackrel{s}{\hookrightarrow} B(2A).$$

By Theorem 2.3.2, this is true if and only if

$$(2A)\, l - A\, e_1 - A\, e_2 - \sum_{i=1}^{M} w_i\, e_{i+2} \ \in\ \mathcal{C}_K. \tag{2.3.2}$$

Denote by $s_1, s_2, f_1, \dots, f_M$ the Poincaré duals of $S_1, S_2, F_1, \dots, F_M$. The base change in cohomology is then

$$\begin{array}{rccc} \varphi^*\colon & H^2\left(X_{M+1}\left(\mathbb{C}P^2\right)\right) & \longrightarrow & H^2\left(X_M\left(S^2\times S^2\right)\right) \\ & l & \longmapsto & s_1+s_2-f_1 \\ & e_1 & \longmapsto & s_2-f_1 \\ & e_2 & \longmapsto & s_1 \quad -f_1 \\ & e_i & \longmapsto & f_{i-1}. \end{array}$$

In this new basis of $H^2\left(X_M\left(S^2\times S^2\right)\right)$, (2.3.2) therefore becomes

$$A s_1 + A s_2 - \sum_{i=1}^{M} w_i\, f_{i+1} \ \in\ \mathcal{C}_K. \tag{2.3.3}$$

In view of Proposition 2.3.4, (2.3.3) translates to the conditions that for all $E := (d,e;m) \in \mathcal{E}$, we have $2A^2 - \sum w_i^2 > 0$ and

$$A s_1(E) + A s_2(E) - \sum w_i\, f_i(E) \ =\ (d+e)A - \sum m_i w_i \ >\ 0.$$

Recall from Lemma 2.2.5 (ii) that $\sum w_i^2 = a$. We conclude that $E(1,a) \stackrel{s}{\hookrightarrow} C(A)$ if and only if $A > \sqrt{\frac{a}{2}}$ and for all $(d,e;m) \in \mathcal{E}$

$$A \ >\ \frac{\sum m_i w_i}{d+e} = \frac{\langle m, w(a)\rangle}{d+e}.$$

This proves the proposition. ☐

Remark 2.3.10. By the symmetry between d and e in the formula for $\mu(d,e;m)(a)$, we can assume that all elements $(d,e;m) \in \mathcal{E}$ have $d \geqslant e$. We will use this convention throughout the paper.

The rest of this paper is devoted to the analysis of the constraints given in Proposition 2.3.9. This analysis follows the one in [MS]. However, several modifications are necessary.

2.4 Basic observations

Lemma 2.4.1. *For all $a \geqslant 8$, $c(a) = \sqrt{\frac{a}{2}}$.*

Proof. For all $(d, e; m) \in \mathcal{E}$, we have by definition of μ and Proposition 2.3.8 (i)

$$\mu(d, e; m)(a) := \frac{\langle m, w(a) \rangle}{d+e} \leqslant \frac{\sum m_i}{d+e} = \frac{2(d+e)-1}{d+e} < 2 \leqslant \sqrt{\frac{a}{2}}$$

for all $a \geqslant 8$. Therefore, by Proposition 2.3.9, $c(a) = \sqrt{\frac{a}{2}}$ for all $a \geqslant 8$. □

Lemma 2.4.2. *The function c has the following scaling property: for all $\lambda \geqslant 1$,*

$$\frac{c(\lambda a)}{\lambda a} \leqslant \frac{c(a)}{a}.$$

Proof. By definition of c, $E(1, a) \overset{s}{\hookrightarrow} C\left(c(a) + \varepsilon\right)$ for all $\varepsilon > 0$. Since $E(1, a)$ symplectically embeds into $C(A)$ if and only if $E(\lambda, \lambda a)$ symplectically embeds into $C(\lambda A)$, this is equivalent to $E(\lambda, \lambda a) \overset{s}{\hookrightarrow} C\left(\lambda c(a) + \varepsilon\right)$ for all $\varepsilon > 0$. Since $E(1, \lambda a) \subset E(\lambda, \lambda a)$ when $\lambda > 1$, this implies that

$$E(1, \lambda a) \overset{s}{\hookrightarrow} C\left(\lambda c(a) + \varepsilon\right)$$

for all $\varepsilon > 0$. Thus

$$c(\lambda a) := \inf\left\{A : E(1, \lambda a) \overset{s}{\hookrightarrow} C(A)\right\} \leqslant \lambda c(a) = \lambda a \frac{c(a)}{a}$$

as claimed. □

Lemma 2.4.3. *For $M \leqslant 7$, the sets $\mathcal{E}_M$ are finite and the only elements are*

$$\begin{array}{ccccc} (0,0;-1), & (1,0;1), & (1,1;1^{\times 3}), & (2,1;1^{\times 5}), & (2,2;2,1^{\times 5}), \\ (3,1;1^{\times 7}), & (3,2;2^{\times 2},1^{\times 5}), & (3,3;2^{\times 4},1^{\times 3}), & (4,3;2^{\times 6},1), & (4,4;3,2^{\times 6}). \end{array}$$

Proof. By Proposition 2.3.8 (i), we have for a class $(d, e; m) \in \mathcal{E}$ with $m = (m_1, \ldots, m_k)$,

$$(2(d+e)-1)^2 = \left(\sum_{i=1}^{k} m_i\right)^2 \leqslant k \sum_{i=1}^{k} m_i^2 = k(2de+1),$$

which is equivalent to

$$4\left(d^2+e^2\right) + 8de - 4(d+e) + 1 \leqslant 2kde + k,$$

and to

$$d^2 + e^2 \leqslant \frac{k-4}{2} de + d + e + \frac{k-1}{4}.$$

Now, if $k \leqslant 7$, then

$$d^2 + e^2 \leqslant \frac{3}{2}de + d + e + \frac{3}{2} \leqslant \frac{3}{4}\left(d^2 + e^2\right) + d + e + \frac{3}{2},$$

using the fact that $2de \leqslant d^2 + e^2$. This last inequality is equivalent to

$$d^2 - 4d + e^2 - 4e \leqslant 6$$

and finally to

$$(d-2)^2 + (e-2)^2 \leqslant 14,$$

which shows that $d, e \leqslant 5$. This shows that the sets $\mathcal{E}_M$ are finite for $M \leqslant 7$. To find the list of classes given above, it suffices to compute the solutions to the Diophantine equations of Proposition 2.3.8 (i) having $l(m) \leqslant 7$ and $d, e \leqslant 5$, and to show that they reduce to $(0, -1)$ by Cremona moves, which is the case. □

Definition 2.4.4. A class $(d, e; m) \in \mathcal{E}$ is said to be *obstructive* if there exists a rational number $a \geqslant 1$ such that $\mu(d, e; m)(a) > \sqrt{\frac{a}{2}}$.

Lemma 2.4.5. *Let $(d, e; m) \in \mathcal{E}$ be an obstructive class. Then either $e = d$ or $e = d - 1$.*

Proof. Suppose by contradiction that there exists a class $(d, e; m) \in \mathcal{E}$ obstructive at some point $a > 1$ such that $d = e + k$ with $k \geqslant 2$. Then, using Proposition 2.3.8 (i) and Lemma 2.2.5 (ii), we obtain

$$\begin{aligned}\sqrt{\frac{a}{2}} &< \mu(d, e; m)(a) = \frac{\langle m, w(a)\rangle}{d+e} \leqslant \frac{\|m\|\,\|w(a)\|}{d+e} = \frac{\sqrt{2de+1}\sqrt{a}}{d+e} \\ &= \frac{\sqrt{2(e+k)e+1}\sqrt{a}}{2e+k} = \frac{\sqrt{4e^2+4ke+2}\sqrt{a}}{\sqrt{2}(2e+k)} < \frac{\sqrt{4e^2+4ke+k^2}\sqrt{a}}{\sqrt{2}(2e+k)} \\ &= \sqrt{\frac{a}{2}},\end{aligned}$$

which is a contradiction. □

Remark 2.4.6. This lemma will be very useful in the sequel, because whenever we will have to prove some properties of obstructive classes, it will be sufficient to prove them for classes of the form $(d, d; m)$ or $(d + \frac{1}{2}, d - \frac{1}{2}; m)$ only. Since this will happen many times, we will not explicitly refer to this lemma each time.

Definition 2.4.7. We define the *error vector* of a class $(d, e; m)$ at a point a as the vector $\varepsilon := \varepsilon\left((d, e; m), a\right)$ defined by the equation

$$m = \frac{d+e}{\sqrt{2a}}\, w(a) + \varepsilon.$$

Lemma 2.4.8. *Let $a = \frac{p}{q} \geqslant 1$ be a rational number with weight expansion $w(a)$ and let $(d, e; m) \in \mathcal{E}$. Then*

(i) $\mu(d, e; m)(a) \leqslant \frac{\sqrt{2de+1}\sqrt{a}}{d+e}$. *In particular,*

$$\mu(d, d; m)(a) \leqslant \sqrt{1 + \tfrac{1}{2d^2}}\sqrt{\tfrac{a}{2}} \text{ and } \mu(d+\tfrac{1}{2}, d-\tfrac{1}{2}; m)(a) \leqslant \sqrt{1 + \tfrac{1}{4d^2}}\sqrt{\tfrac{a}{2}},$$

(ii) $\mu(d, e; m)(a) > \sqrt{\frac{a}{2}}$ *if and only if* $\langle \varepsilon, w(a) \rangle > 0$,

(iii) If $\mu(d, d; m)(a) > \sqrt{\frac{a}{2}}$ *(resp.* $\mu(d + \frac{1}{2}, d - \frac{1}{2}; m)(a) > \sqrt{\frac{a}{2}}$*), then* $\langle \varepsilon, \varepsilon \rangle < 1$ *(resp.* $\langle \varepsilon, \varepsilon \rangle < \frac{1}{2}$*),*

(iv) $-\sum_{i=1}^{M} \varepsilon_i = \frac{d+e}{\sqrt{2a}}\left(y(a) - \frac{1}{q}\right) + 1$, *where* $y(a) := a + 1 - 2\sqrt{2a}$.

Proof. (i) By the Cauchy-Schwarz inequality, Proposition 2.3.8 (i) and Lemma 2.2.5, we have

$$(d + e)\mu(d, e; m)(a) = \langle m, w(a) \rangle \leqslant \|m\| \, \|w(a)\| = \sqrt{2de + 1}\sqrt{a}.$$

In the case of a class $(d, d; m)$ we find that

$$\mu(d, d; m)(a) \leqslant \frac{\sqrt{2d^2 + 1}\sqrt{a}}{2d} = \sqrt{\frac{2d^2 + 1}{2d^2}}\sqrt{\frac{a}{2}} = \sqrt{1 + \frac{1}{2d^2}}\sqrt{\frac{a}{2}},$$

and in the case of a class $(d + \frac{1}{2}, d - \frac{1}{2}; m)$ that

$$\mu(d + \frac{1}{2}, d - \frac{1}{2}; m)(a) \leqslant \frac{\sqrt{2d^2 + \frac{1}{2}}\sqrt{a}}{2d} = \sqrt{1 + \frac{1}{4d^2}}\sqrt{\frac{a}{2}}.$$

(ii) Since

$$\begin{aligned}\langle \varepsilon, w(a) \rangle &= \left\langle m - \frac{d+e}{\sqrt{2a}} w(a), w(a) \right\rangle = \langle m, w(a) \rangle - \frac{d+e}{\sqrt{2a}} \|w(a)\|^2 \\ &= \langle m, w(a) \rangle - (d + e)\sqrt{\frac{a}{2}},\end{aligned}$$

we see that $\langle \varepsilon, w(a) \rangle > 0$ if and only if $\mu(d, e; m)(a) = \frac{\langle m, w(a) \rangle}{d+e} > \sqrt{\frac{a}{2}}$.

(iii) For a class $(d, d; m)$,

$$\begin{aligned}2d^2 + 1 = \langle m, m \rangle &= \left\langle \sqrt{\frac{2}{a}}\, dw(a) + \varepsilon, \sqrt{\frac{2}{a}}\, dw(a) + \varepsilon \right\rangle \\ &= 2d^2 + \frac{2\sqrt{2}}{\sqrt{a}}\, d \langle w(a), \varepsilon \rangle + \langle \varepsilon, \varepsilon \rangle\end{aligned}$$

shows that if $\mu(d,d;m)(a) > \sqrt{\frac{a}{2}}$, then by (ii) $\langle \varepsilon, \varepsilon \rangle < 1$. Similarly, for a class $(d+\frac{1}{2}, d-\frac{1}{2}; m)$,

$$2d^2 + \frac{1}{2} = 2d^2 + \frac{2\sqrt{2}}{\sqrt{a}} d \langle w(a), \varepsilon \rangle + \langle \varepsilon, \varepsilon \rangle$$

shows that if $\mu(d+\frac{1}{2}, d-\frac{1}{2}; m)(a) > \sqrt{\frac{a}{2}}$, then $\langle \varepsilon, \varepsilon \rangle < \frac{1}{2}$.

(iv) By Proposition 2.3.8 (i) and Lemma 2.2.5, we see that

$$2(d+e) - 1 = \sum_{i=1}^{M} m_i = \frac{d+e}{\sqrt{2a}} \left(a + 1 - \frac{1}{q} \right) + \sum_{i=1}^{M} \varepsilon_i.$$

Thus

$$-\sum_{i=1}^{M} \varepsilon_i = \frac{d+e}{\sqrt{2a}} \left(a + 1 - 2\sqrt{2a} \right) - \frac{d+e}{q\sqrt{2a}} + 1,$$

from which the result follows. □

Corollary 2.4.9. *Suppose that $c(a) > \sqrt{\frac{a}{2}}$ for some rational $a \geqslant 1$. Then*

(i) There exist classes $(d,e;m), (d',e';m') \in \mathcal{E}$ (possibly equal) and $\varepsilon > 0$ such that

$$c(z) = \begin{cases} \mu(d,e;m) & \text{if } z \in \left]a-\varepsilon, a\right], \\ \mu(d',e';m') & \text{if } z \in \left[a, a+\varepsilon\right[. \end{cases}$$

(ii) The set of classes $(d,e;m) \in \mathcal{E}$ such that $\mu(d,e;m)(a) = c(a)$ is finite.

(iii) For each of the intervals of (i), there exist rational coefficients $\alpha, \beta \geqslant 0$ such that $c(z) = \alpha + \beta z$.

Proof. Since $c(a) > \sqrt{\frac{a}{2}}$, there exists $D \in \mathbb{N}$ such that $c(a) > \sqrt{1+\frac{1}{D^2}}\sqrt{\frac{a}{2}}$. Since c is continuous, there exists $\varepsilon > 0$ such that $c(z) > \sqrt{1+\frac{1}{D^2}}\sqrt{\frac{z}{2}}$ for all $z \in \left]a-\varepsilon, a+\varepsilon\right[$. Now, if $\mu(d,e;m)(z) > \sqrt{1+\frac{1}{D^2}}\sqrt{\frac{z}{2}}$, the inequalities of Lemma 2.4.8 (i) imply that $d \leqslant D$. There are thus only finitely many classes with $\mu(d,e;m)(z) > \sqrt{1+\frac{1}{D^2}}\sqrt{\frac{z}{2}}$, and for all $z \in \left]a-\varepsilon, a+\varepsilon\right[$, $c(z)$ is the supremum of $\mu(d,e;m)(z)$ taken over finitely many classes. This proves (i) and (ii). To prove (iii), notice first that the constraints $\mu(d,e;m)$ are piecewise linear functions. Indeed, let $w(a) = (w_1(a), w_2(a), \dots)$ be the weight expansion of a, where the w_i are seen as functions of a. Then the w_i are piecewise linear functions and so is $\mu(d,e;m) = \frac{\langle m,w \rangle}{d+e}$. We can thus write $c(z) = \alpha + \beta z$ for z belonging to one of the intervals of (i). Now, since c is nondecreasing, $\beta \geqslant 0$, and by the scaling property of Lemma 2.4.2, $\alpha \geqslant 0$. □

Definition 2.4.10. A class $(d, e; m) \in \mathcal{E}$ is called perfect if there exists $b > 1$ and $\kappa > 0$ such that $m = \kappa\, w(b)$, that is, such that the vector m is a multiple of the weight expansion of b.

Lemma 2.4.11. *Let $(d, e; m) \in \mathcal{E}$ be a perfect class for some $b > 1$ with $d = e$ or $d = e + 1$. Then $c(b) = \mu(d, e; m)(b) > \sqrt{\frac{b}{2}}$ and $(d, e; m)$ is the only class such that $\mu(d, e; m)(b) = c(b)$.*

Proof. We first treat the case $d = e$. Let $(d, d; m) \in \mathcal{E}$ be a perfect class: $m = \kappa\, w(b)$ for some $b > 1$. By Proposition 2.3.8 (i),

$$2d^2 < 2d^2 + 1 = \langle m, m \rangle = \kappa^2 \langle w(b), w(b) \rangle = \kappa^2 b,$$

from which we deduce that $d < \kappa\sqrt{\frac{b}{2}}$. Then

$$\mu(d, d; m)(b) = \frac{\langle m, w(b) \rangle}{2d} = \frac{\kappa b}{2d} > \sqrt{\frac{b}{2}}.$$

This shows that $(d, d; m)$ is obstructive at b. But then $(d, d; m)$ is the only obstructive class at b. Indeed, if $(d', e'; m') \in \mathcal{E}$ is a class different of $(d, d; m)$, by positivity of intersections (Proposition 2.3.8 (ii)),

$$\kappa \langle m', w(b) \rangle = \sum m_i m_i' \leqslant d(d' + e').$$

Thus

$$\mu(d', e'; m')(b) = \frac{\langle m', w(b) \rangle}{d' + e'} \leqslant \frac{d \langle m', w(b) \rangle}{\kappa \langle m', w(b) \rangle} = \frac{d}{\kappa} < \sqrt{\frac{b}{2}}.$$

Consider now a class of the form $\left(d + \frac{1}{2}, d - \frac{1}{2}; m\right)$. By Proposition 2.3.8 (i), we have

$$2d^2 < 2d^2 + \frac{1}{2} = \langle m, m \rangle = \kappa^2 \langle w(b), w(b) \rangle = \kappa^2 b,$$

and thus $d < \kappa\sqrt{\frac{b}{2}}$ as in the case of a class $(d, d; m)$. The rest of the proof is then identical. □

Definition 2.4.12. Define the *length of a vector* m, denoted by $l(m)$, as the number of positive entries in m, and denote by $l(a)$ the *length of the weight expansion* $w(a)$ of a.

Lemma 2.4.13. *Let $(d, e; m) \in \mathcal{E}$ be an obstructive class. Let I be a maximal nonempty open interval on which $\mu(d, e; m)(a) > \sqrt{\frac{a}{2}}$. Then there exists a unique $a_0 \in I$ such that $l(a_0) = l(m)$. Moreover for all $a \in I$, $l(a) \geqslant l(a_0)$.*

Proof. Let us first prove that for all $a \in I$, $l(a) \geqslant l(m)$. If $l(a) < l(m)$, then, by Proposition 2.3.8 (i),

$$\sum_{i=1}^{l(a)} m_i^2 < 2de + 1.$$

Thus

$$\mu(d,e;m)(a) = \frac{\langle m, w(a)\rangle}{d+e} \leqslant \frac{\sqrt{\sum_{i=1}^{l(a)} m_i^2}\,\|w(a)\|}{d+e} \leqslant \frac{\sqrt{4de}}{d+e}\sqrt{\frac{a}{2}} \leqslant \sqrt{\frac{a}{2}},$$

and so $a \notin I$. Let us now prove the existence of an a_0 with $l(a_0) = l(m)$. Let $w(a) = (w_1(a), w_2(a), \ldots)$ be the weight expansion of a, where the w_i are again seen as functions of a. The w_i are piecewise linear functions and are linear on intervals that do not contain elements a' with $l(a') \leqslant i$. Hence if all $a \in I$ would have $l(a) > l(m)$, then the $l(m)$ first w_i would be linear, and $\mu(d,e;m)$ also. But this is impossible since $\sqrt{\frac{a}{2}}$ is concave. Thus there exists $a_0 \in I$ with $l(a_0) = l(m)$. The proof of uniqueness of a_0 follows from the fact that if $a < b$ and $l(a) = l(b)$, then there exists $c \in \,]a, b[$ such that $l(c) < l(a)$. □

Lemma 2.4.14. *Let $(d,e;m) \in \mathcal{E}$ be such that $\mu(d,e;m)(a) > \sqrt{\frac{a}{2}}$. Let $J = \{k, ..., k+s-1\}$ be a block of $s \geqslant 2$ consecutive integers such that the $w_i(a)$ are equal for all $i \in J$. Then we have the three following possibilities*

1. $m_k = \ldots = m_{k+s-1}$
2. $m_k - 1 = m_{k+1} = \ldots = m_{k+s-1}$
3. $m_k = \ldots = m_{k+s-2} = m_{k+s-1} + 1.$

Moreover, there is at most one block of length $s \geqslant 2$ where the m_i are not all equal, and if such a block J exists, then $\sum_{i\in J} \varepsilon_i^2 \geqslant \frac{s-1}{s}$.

The proof is similar to the proof of Lemma 2.1.7 in [MS].

Corollary 2.4.15. *If a class of the form $(d+\frac{1}{2}, d-\frac{1}{2}; m)$ is obstructive, then the m_i are constant on each block.*

Proof. Suppose there exists a block J of length $s \geqslant 2$ on which the $w_i(a)$ are not all equal. Then by Lemma 2.4.14, $\sum_{i\in J} \varepsilon_i^2 \geqslant \frac{s-1}{s} \geqslant \frac{1}{2}$, which contradicts Lemma 2.4.8 (iii), which states that $\sum \varepsilon_i^2 < \frac{1}{2}$. □

Lemma 2.4.16. *Let $(d,e;m) \in \mathcal{E}$ be an obstructive class at some rational $a \geqslant 1$ with $l(a) = l(m)$. Let $w_{k+1}, \ldots, w_{k+s}$ be a block which is not the first block of $w(a)$.*

(i) If the block is not the last one, then

$$|m_k - (m_{k+1} + \ldots + m_{k+s+1})| < \sqrt{s+2}.$$

If the block is the last one, then

$$|m_k - (m_{k+1} + \ldots + m_{k+s})| < \sqrt{s+1}.$$

(ii) It is always the case that

$$m_k - \sum_{i=k+1}^{M} m_i < \sqrt{M-k+1}.$$

The proof is similar to the one of Lemma 2.1.8 in [MS].

Proposition 2.4.17. *Let $(d, e; m) \in \mathcal{E}$ be an obstructive class at a point $a =: \frac{p}{q} \in \mathbb{Q}$ written in lowest terms with $l(a) = l(m)$. Let m_M be the last nonzero entry of the vector m and let I be the maximal open interval containing a such that $\mu(d, e; m)(a) > \sqrt{\frac{a}{2}}$. Then there exist integers $A < p$ and $B < (m_M + 1)q$ such that*

$$(d+e)\mu(d,e;m)(z) = \begin{cases} A + Bz & \text{if } z \leqslant a, z \in I, \\ (A + m_M p) + (B - m_M q)z & \text{if } z \geqslant a, z \in I. \end{cases}$$

Again, the proof is similar to the one of Proposition 2.3.2 in [MS].

2.5 The interval $[1, \sigma^2]$

The goal of this section is to prove part (i) of Theorem 2.1.3.

2.5.1 Preliminaries

Let us first recall that the *Pell numbers* P_n and the *half companion Pell numbers* H_n are defined by the recurrence relations

$$\begin{aligned} P_0 = 0,\, P_1 = 1, \quad P_n &= 2P_{n-1} + P_{n-2}, \\ H_0 = 1,\, H_1 = 1, \quad H_n &= 2H_{n-1} + H_{n-2}, \end{aligned}$$

respectively. It is then easy to see that

$$H_n = P_n + P_{n-1}.$$

Using this, we define the sequence $(\alpha_n)_{n \geqslant 0}$ by

$$\alpha_n := \begin{cases} \dfrac{2P_{n+1}^2}{H_n^2} =: \dfrac{p_n}{q_n} & \text{if } n \text{ is even,} \\ \dfrac{H_{n+1}^2}{2P_n^2} =: \dfrac{p_n}{q_n} & \text{if } n \text{ is odd.} \end{cases}$$

Set $W(\alpha_n) = q_n w(\alpha_n)$. Then, define $W'(\alpha_n)$ as the tuple obtained from $W(\alpha_n)$ by adding an extra 1 at the end. Define the classes $E(\alpha_n)$ by

$$E(\alpha_n) := \begin{cases} (P_{n+1}H_n, P_{n+1}H_n; W'(\alpha_n)) & \text{if } n \text{ is even,} \\ (P_n H_{n+1}, P_n H_{n+1}; W'(\alpha_n)) & \text{if } n \text{ is odd.} \end{cases}$$

For instance,

$$\begin{aligned} E(\alpha_0) &= \left(1, 1; 1^{\times 3}\right), \\ E(\alpha_1) &= \left(3, 3; 2^{\times 4}, 1^{\times 3}\right), \\ E(\alpha_2) &= \left(15, 15; 9^{\times 5}, 5, 4, 1^{\times 5}\right), \\ E(\alpha_3) &= \left(85, 85; 50^{\times 5}, 39, 11^{\times 3}, 6, 5, 1^{\times 6}\right). \end{aligned}$$

Moreover, we define the sequence $(\beta_n)_{n \geqslant 0}$ by

$$\beta_n := \begin{cases} \dfrac{H_{n+2}}{H_n} =: \dfrac{p_n}{q_n} & \text{if } n \text{ is even,} \\ \dfrac{P_{n+2}}{P_n} =: \dfrac{p_n}{q_n} & \text{if } n \text{ is odd.} \end{cases}$$

Set $W(\beta_n) = q_n w(\beta_n)$. Then the classes $E(\beta_n)$ are defined by

$$E(\beta_n) := \begin{cases} \left(\frac{1}{4}(H_n + H_{n+2}), \frac{1}{4}(H_n + H_{n+2}); W(\beta_n)\right) & \text{if } n \text{ is even,} \\ \left(\frac{1}{4}(P_n + P_{n+2}) + \frac{1}{2}, \frac{1}{4}(P_n + P_{n+2}) - \frac{1}{2}; W(\beta_n)\right) & \text{if } n \text{ is odd.} \end{cases}$$

For instance,

$$\begin{aligned} E(\beta_0) &= \left(1, 1; 1^{\times 3}\right), \\ E(\beta_1) &= \left(2, 1; 1^{\times 5}\right), \\ E(\beta_2) &= \left(5, 5; 3^{\times 5}, 2, 1^{\times 2}\right), \\ E(\beta_3) &= \left(9, 8; 5^{\times 5}, 4, 1^{\times 4}\right). \end{aligned}$$

Theorem 2.5.1. *For all $n \geqslant 0$, $E(\alpha_n), E(\beta_n) \in \mathcal{E}$.*

The proof that $E(\alpha_n) \in \mathcal{E}$ is given in the next subsection, while the proof that $E(\beta_n) \in \mathcal{E}$ is given in Corollary 2.6.19. Theorem 2.5.1 implies part (i) of Theorem 2.1.3:

Corollary 2.5.2. *On the interval $[1, \sigma^2]$,*

$$c(a) = \begin{cases} 1 & \text{if } a \in [1, 2], \\ \dfrac{1}{\sqrt{2\alpha_n}}\, a & \text{if } a \in [\alpha_n, \beta_n], \\ \sqrt{\dfrac{\alpha_{n+1}}{2}} & \text{if } a \in [\beta_n, \alpha_{n+1}], \end{cases}$$

for all $n \geqslant 0$.

Proof. Since for all $n \geqslant 0$, $E(\beta_n)$ is a perfect class, we know by Lemma 2.4.11 that $c(\beta_n) = \mu(E(\beta_n))(\beta_n)$. Hence

$$c(\beta_n) = \sqrt{\frac{\alpha_{n+1}}{2}}$$

for all $n \geqslant 0$. Indeed, for n even, we compute

$$\begin{aligned} c(\beta_n) &= \frac{2H_n \langle w(\beta_n), w(\beta_n)\rangle}{H_{n+2}+H_n} = \frac{2H_n\beta_n}{H_{n+2}+H_n} = \frac{2H_{n+2}}{H_{n+2}+H_n} \\ &= \frac{2(P_{n+2}+P_{n+1})}{P_{n+2}+P_{n+1}+P_n+P_{n-1}} = \frac{2(P_{n+2}+P_{n+1})}{4P_{n+1}} = \frac{H_{n+2}}{2P_{n+1}} = \sqrt{\frac{\alpha_{n+1}}{2}}, \end{aligned}$$

and for n odd,

$$\begin{aligned} c(\beta_n) &= \frac{2P_n \langle w(\beta_n), w(\beta_n)\rangle}{P_{n+2}+P_n} = \frac{2P_n\beta_n}{P_{n+2}+P_n} = \frac{2P_{n+2}}{P_{n+2}+P_n} \\ &= \frac{P_{n+2}}{\frac{1}{2}(P_{n+2}-P_n)+P_n} = \frac{P_{n+2}}{P_{n+1}+P_n} = \frac{P_{n+2}}{H_n} = \sqrt{\frac{\alpha_{n+1}}{2}}. \end{aligned}$$

Furthermore, $c(\alpha_n) = \sqrt{\frac{\alpha_n}{2}}$ for all $n \geqslant 0$. Indeed, for n even, we have

$$\mu(E(\alpha_n))(\alpha_n) = \frac{H_n^2\alpha_n}{2P_{n+1}H_n} = \frac{H_n\alpha_n}{2P_{n+1}} = \frac{P_{n+1}}{H_n} = \sqrt{\frac{\alpha_n}{2}}.$$

Thus for all $(d,e;m) \in \mathcal{E}$ distinct from $E(\alpha_n)$, we get by Proposition 2.3.8 (ii) that

$$P_{n+1}H_n(d+e) \geqslant \langle m, W'(\alpha_n)\rangle \geqslant H_n^2 \langle m, w(\alpha_n)\rangle,$$

and hence

$$\mu(d,e;m)(\alpha_n) = \frac{\langle m, w(\alpha_n)\rangle}{d+e} \leqslant \frac{P_{n+1}H_n}{H_n^2} = \frac{P_{n+1}}{H_n} = \sqrt{\frac{\alpha_n}{2}}.$$

Next, for n odd, we have

$$\mu(E(\alpha_n))(\alpha_n) = \frac{2P_n^2\alpha_n}{2P_nH_{n+1}} = \frac{P_n\alpha_n}{H_{n+1}} = \frac{H_{n+1}}{2P_n} = \sqrt{\frac{\alpha_n}{2}}.$$

Thus for all $(d,e;m) \in \mathcal{E}$ distinct from $E(\alpha_n)$, we get

$$P_nH_{n+1}(d+e) \geqslant \langle m, W'(\alpha_n)\rangle \geqslant 2P_n^2 \langle m, w(\alpha_n)\rangle,$$

and hence

$$\mu(d,e;m)(\alpha_n) = \frac{\langle m, w(\alpha_n)\rangle}{d+e} \leqslant \frac{P_nH_{n+1}}{2P_n^2} = \frac{H_{n+1}}{2P_n} = \sqrt{\frac{\alpha_n}{2}}.$$

Thus, by Proposition 2.3.9, we have $c(\alpha_n) = \sqrt{\frac{\alpha_n}{2}}$ for all $n \geqslant 0$ as required. Since c is nondecreasing, we get that $c(a) = \sqrt{\frac{\alpha_{n+1}}{2}}$ for $a \in [\beta_n, \alpha_{n+1}]$. Moreover, we have that for all $n \geqslant 0$

$$\frac{c(\beta_n)}{\beta_n} = \frac{c(\alpha_n)}{\alpha_n}.$$

Indeed, for n even,

$$\frac{c(\beta_n)}{\beta_n} = \frac{\sqrt{\frac{\alpha_{n+1}}{2}}}{\beta_n} = \frac{H_n}{2P_{n+1}} = \frac{\sqrt{\frac{\alpha_n}{2}}}{\alpha_n} = \frac{c(\alpha_n)}{\alpha_n},$$

and for n odd,

$$\frac{c(\beta_n)}{\beta_n} = \frac{\sqrt{\frac{\alpha_{n+1}}{2}}}{\beta_n} = \frac{P_n}{H_{n+1}} = \frac{\sqrt{\frac{\alpha_n}{2}}}{\alpha_n} = \frac{c(\alpha_n)}{\alpha_n}.$$

Hence, by the scaling property of Lemma 2.4.2, the function c has to be linear on $[\alpha_n, \beta_n]$ and thus $c(a) = \frac{1}{\sqrt{2\alpha_n}} a$ for $a \in [\alpha_n, \beta_n]$. □

2.5.2 The classes $E(\alpha_n)$ belong to $\mathcal{E}$

Lemma 2.5.3. *The classes $E(\alpha_n)$ satisfy the Diophantine conditions of Proposition 2.3.8 (i).*

Proof. We will prove this separately for n even and odd. In both cases, we will use Lemma 2.2.5 and the relation

$$-P_{2m}^2 + 2P_{2m}P_{2m-1} + P_{2m-1}^2 = 1$$

which can be easily deduced from the following identity

$$P_{2m-k} = (-1)^{k+1}(P_k H_{2m} - H_k P_{2m}),$$

given in Corollary 2.6.8 (v). For $n = 2m$, we obtain

$$\begin{aligned}
\sum m_i &= H_{2m}^2 \sum w_i + 1 = H_{2m}^2 \left(\frac{2P_{2m+1}^2}{H_{2m}^2} + 1 - \frac{1}{H_{2m}^2} \right) + 1 \\
&= 2(2P_{2m} + P_{2m-1})^2 + (P_{2m} + P_{2m-1})^2 \\
&= 9P_{2m}^2 + 10P_{2m}P_{2m-1} + 3P_{2m-1}^2 \\
&= 8P_{2m}^2 + 12P_{2m}P_{2m-1} + 4P_{2m-1}^2 - \left(-P_{2m}^2 + 2P_{2m}P_{2m-1} + P_{2m-1}^2\right) \\
&= 4P_{2m+1}H_{2m} - 1 = 2(d+e) - 1;
\end{aligned}$$

$$\begin{aligned}
\sum m_i^2 &= H_{2m}^4 \sum w_i^2 + 1 = H_{2m}^4 \frac{2P_{2m+1}^2}{H_{2m}^2} + 1 \\
&= 2P_{2m+1}^2 H_{2m}^2 + 1 = 2de + 1.
\end{aligned}$$

Moreover, for $n = 2m-1$,

$$\begin{aligned}\sum m_i &= 2P_{2m-1}^2 \sum w_i + 1 = 2P_{2m-1}^2 \left(\frac{H_{2m}^2}{2P_{2m-1}^2} + 1 - \frac{1}{2P_{2m-1}^2} \right) + 1 \\ &= (P_{2m} + P_{2m-1})^2 + 2P_{2m-1}^2 \\ &= P_{2m}^2 + 2P_{2m}P_{2m-1} + 3P_{2m-1}^2 \\ &= 4P_{2m}P_{2m-1} + 4P_{2m-1}^2 - \left(-P_{2m}^2 + 2P_{2m}P_{2m-1} + P_{2m-1}^2 \right) \\ &= 4P_{2m-1}H_{2m} - 1 = 2(d+e) - 1;\end{aligned}$$

$$\begin{aligned}\sum m_i^2 &= 4P_{2m-1}^4 \sum w_i^2 + 1 = 4P_{2m-1}^4 \frac{H_{2m}^2}{2P_{2m-1}^2} + 1 \\ &= 2P_{2m-1}^2 H_{2m}^2 + 1 = 2de + 1.\end{aligned}$$

This proves the lemma. □

We will now prove separately for n even and n odd that the classes $E(\alpha_n)$ reduce to $(0;-1)$ by standard Cremona moves.

2.5.2.1 The classes $E(\alpha_{2m})$ reduce to $(0;-1)$

One readily checks that the classes $E(\alpha_{2m})$ reduce to $(0;-1)$ for $m = 0, 1, 2$. In the following, we reduce the classes $E(\alpha_{2m})$ for $m \geqslant 3$.

Lemma 2.5.4. *The continued fraction expansion of α_{2m} is*

$$\left[5; \{1,4\}^{\times(m-1)}, 1, 1, 3, 1, \{4,1\}^{\times(m-1)}\right].$$

Moreover, with $u_j := (2H_j - P_j)H_{2m} + H_jP_{2m}$,

$$\begin{aligned}E(\alpha_{2m}) = (&P_{2m+1}H_{2m}, P_{2m+1}H_{2m}; \\ &\left(\frac{1}{2}u_{2m}\right)^{\times 5}, u_{2m-1}, \left(\frac{1}{2}u_{2m-2}\right)^{\times 4}, \dots, u_3, \left(\frac{1}{2}u_2\right)^{\times 4}, \\ &u_1, H_{2m} + \frac{1}{2}P_{2m}, \left(\frac{1}{2}P_{2m}\right)^{\times 3}, P_{2m-1}, \\ &\left(\frac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, \dots, \left(\frac{1}{2}P_2\right)^{\times 4}, P_1, 1\Big).\end{aligned}$$

Proof. Since (α_n) is an increasing sequence converging to $\sigma^2 < 6$ and $\alpha_2 = \frac{50}{9} > 5$, the first term of $W'(\alpha_{2m})$ is

$$(q_{2m})^{\times 5} = \left(H_{2m}^2\right)^{\times 5} = \left(\frac{1}{2}u_{2m}\right)^{\times 5}.$$

To determine the next terms, we will first prove that for all $k \geqslant 1$, $u_{2k+1} > \frac{1}{2}u_{2k} > u_{2k-1}$. Indeed

$$\begin{aligned}
u_{2k+1} &= \left(2H_{2k+1} - P_{2k+1}\right) H_{2m} + H_{2k+1}P_{2m} \\
&= \left(4P_{2k} + P_{2k-1}\right) H_{2m} + \left(2H_{2k} + H_{2k-1}\right) P_{2m} \\
&> \left(\frac{1}{2}P_{2k} + P_{2k-1}\right) H_{2m} + \frac{1}{2}H_{2k}P_{2m} \\
&= \frac{1}{2}\left(2H_{2k} - P_{2k}\right) H_{2m} + \frac{1}{2}H_{2k}P_{2m} \\
&= \frac{1}{2}u_{2k} \\
&= \left(P_{2k-1} + \frac{5}{2}P_{2k-2} + P_{2k-3}\right) H_{2m} + \left(H_{2k-1} + \frac{1}{2}H_{2k-2}\right) P_{2m} \\
&> \left(P_{2k-1} + 2P_{2k-2}\right) H_{2m} + H_{2k-1}P_{2m} \\
&= \left(2H_{2k-1} - P_{2k-1}\right) H_{2m} + H_{2k-1}P_{2m} \\
&= u_{2k-1}.
\end{aligned}$$

The second term of $W'\left(\alpha_{2m}\right)$ is

$$\begin{aligned}
2P_{2m+1}^2 - 5H_{2m}^2 &= 3P_{2m}^2 - 2P_{2m}P_{2m-1} - 3P_{2m-1}^2 \\
&= \left(2H_{2m-1} - P_{2m-1}\right) H_{2m} + H_{2m-1}P_{2m} \\
&= u_{2m-1} < \frac{1}{2}u_{2m}.
\end{aligned}$$

Now for all $k \geqslant 1$, we have

$$\begin{aligned}
u_{2k+1} - 4\left(\frac{1}{2}u_{2k}\right) &= \left(2H_{2k+1} - P_{2k+1} - 4H_{2k} + 2P_{2k}\right) H_{2m} \\
&\quad + \left(H_{2k+1} - 2H_{2k}\right) P_{2m} \\
&= \left(2H_{2k-1} - P_{2k-1}\right) H_{2m} + H_{2k-1}P_{2m} = u_{2k-1} < \frac{1}{2}u_{2k};
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}u_{2k} - u_{2k-1} &= \left(H_{2k} - \frac{1}{2}P_{2k} - 2H_{2k-1} + P_{2k-1}\right) H_{2m} \\
&\quad + \left(\frac{1}{2}H_{2k} - H_{2k-1}\right) P_{2m} \\
&= \frac{1}{2}\left(2H_{2k-2} - P_{2k-2}\right) H_{2m} + \frac{1}{2}H_{2k-2}P_{2m} = \frac{1}{2}u_{2k-2} \\
&< u_{2k-1}.
\end{aligned}$$

Thus the first terms of $W'\left(\alpha_{2m}\right)$ are

$$\left(\frac{1}{2}u_{2m}\right)^{\times 5}, u_{2m-1}, \left(\frac{1}{2}u_{2m-2}\right)^{\times 4}, \ldots, u_3, \left(\frac{1}{2}u_2\right)^{\times 4}, u_1.$$

The next terms are $H_{2m}+\frac{1}{2}P_{2m}, \left(\frac{1}{2}P_{2m}\right)^{\times 3}, P_{2m-1}$. Indeed

$$\frac{1}{2}u_2 - u_1 = H_{2m} + \frac{1}{2}P_{2m} < H_{2m} + P_{2m} = u_1;$$

$$u_1 - \left(H_{2m} + \frac{1}{2}P_{2m}\right) = \frac{1}{2}P_{2m} < H_{2m} + \frac{1}{2}P_{2m};$$

$$H_{2m} + \frac{1}{2}P_{2m} - 3\left(\frac{1}{2}P_{2m}\right) = P_{2m-1} < \frac{1}{2}P_{2m}.$$

Notice that for all $k \geqslant 1$,

$$P_{k+1} > \frac{1}{2}P_k > P_{k-1},$$

and thus

$$P_{2k+1} - 4\left(\frac{1}{2}P_{2k}\right) = P_{2k-1} < \frac{1}{2}P_{2k};$$

$$\frac{1}{2}P_{2k} - P_{2k-1} = \frac{1}{2}P_{2k-2} < P_{2k-1}.$$

This proves that the last terms of $W'(\alpha_{2m})$ are

$$\left(\frac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, \ldots, \left(\frac{1}{2}P_2\right)^{\times 4}, P_1, 1$$

with the last 1 added by definition of $W'(\alpha_{2m})$. □

Let us introduce now some notations in order to simplify the expressions of the classes.

Definition 2.5.5. Set

$$A_k^m := \left(\left(\frac{1}{2}u_{2k}\right)^{\times 4}, u_{2k-1}, (P_{2k}H_{2m})^{\times 2}, \left(\frac{1}{2}u_{2k-2}\right)^{\times 4}, u_{2k-3}, \ldots, \left(\frac{1}{2}u_2\right)^{\times 4},\right.$$
$$\left. u_1, H_{2m} + \frac{1}{2}P_{2m}, \left(\frac{1}{2}P_{2m}\right)^{\times 3}, P_{2m-1}\right),$$
$$B_k^m := \left(\left(\frac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, \ldots, \left(\frac{1}{2}P_{2m-2k+2}\right)^{\times 4}, P_{2m-2k+1},\right.$$
$$\left.\left(\frac{1}{2}P_{2m-2k}\right)^{\times 8}, (P_{2m-2k-1})^{\times 2}, \ldots, \left(\frac{1}{2}P_2\right)^{\times 8}, (P_1)^{\times 2}, 1\right),$$
$$V_k^m := (P_{2k+1}H_{2m} + H_{2k}P_{2m}; A_k^m, B_k^m).$$

Thus A_k^m has the structure $\left[4,1,2,\{4,1\}^{\times(k-1)},1,3,1\right]$ and B_k^m has the structure $\left[\{4,1\}^{\times(k-1)},\{8,2\}^{\times(m-k)},1\right]$. We use here the convention that if $k=m$, B_m^m has the structure $\left[\{4,1\}^{\times(m-1)},1\right]$ and that if $k=1$, B_1^m has the structure $\left[\{8,2\}^{\times(m-1)},1\right]$.

The structure of the reduction process of a class $E(\alpha_{2m})$ will be the following. First, we compute in Lemma 2.5.7 that the image of $E(\alpha_{2m})$ under φ_* is $V_m^m = (P_{2m+1}H_{2m}+H_{2m}P_{2m}; A_m^m, B_m^m)$. Then, we reduce V_m^m in Lemma 2.5.8 and Lemma 2.5.9 to

$$V^m := \left(H_{2m}; H_{2m}-\frac{1}{2}P_{2m}, \left(\frac{1}{2}P_{2m}\right)^{\times 3}, (P_{2m-1})^{\times 2}, B_1^m\right)$$

in $4(m-2)+8$ Cremona moves. Finally, we show in Lemma 2.5.10 and Lemma 2.5.11 that V^m reduces to $(0;-1)$ in $5(m-2)+8$ moves.

Remark 2.5.6. Since the Cremona transform of a class $(d;m)$ only modifies the first 3 entries of the vector m, we will write some of the first entries of the classes and will abbreviate the other terms by $(*)$. It is important to notice that the terms denoted by $(*)$ will always be left invariant during the reduction process. Then, each time after applying the Cremona transform, we will reorder the entries of m. We will not always reorder the entries in decreasing order, but this will have no consequence on the reduction because we will reorder them in a way such that the first 3 entries of the vector m will always be the 3 biggest entries in decreasing order. We will write down each step of the reduction, that is the class obtained after applying the Cremona transform and reordering. But sometimes when the reordering will not be obvious, we will write the intermediate step before reordering and denote by $\rightarrow$ the Cremona transform of a class and by $\rightsquigarrow$ the reordering of a class. We will freely use the three relations

$$P_n = 2P_{n-1}+P_{n-2}, \quad H_n = P_n + P_{n-1},$$
$$P_{2m-k} = (-1)^{k+1}\left(P_k H_{2m} - H_k P_{2m}\right).$$

Lemma 2.5.7. *The image of $E(\alpha_{2m})$ by φ_* is the class*

$$\varphi_*\left(E\left(\alpha_{2m}\right)\right) = (P_{2m+1}H_{2m}+H_{2m}P_{2m}; A_m^m, B_m^m) = V_m^m.$$

Proof. The first terms of $E(\alpha_{2m})$ are $\left(P_{2m+1}H_{2m}, P_{2m+1}H_{2m}; \frac{1}{2}u_{2m}, (*)\right)$. Since $\frac{1}{2}u_{2m} = H_{2m}^2$, we get

$$\begin{aligned}\varphi_*\left(E\left(\alpha_{2m}\right)\right) &= \left(2P_{2m+1}H_{2m}-H_{2m}^2; \left(P_{2m+1}H_{2m}-H_{2m}^2\right)^{\times 2}, (*)\right)\\ &= \left(P_{2m+1}H_{2m}+H_{2m}P_{2m}; \left(P_{2m}H_{2m}\right)^{\times 2}, (*)\right).\end{aligned}$$

After reordering this class, we see that we indeed obtain V_m^m as required. ☐

Lemma 2.5.8. *For all $3 \leqslant k \leqslant m$, V_k^m reduces to V_{k-1}^m in 4 Cremona moves.*

Proof. We have

$$\begin{aligned}
V_k^m = \Big(& P_{2k+1}H_{2m} + H_{2k}P_{2m}; \frac{1}{2}\left(\left(2H_{2k} - P_{2k}\right)H_{2m} + H_{2k}P_{2m}\right)^{\times 4}, \\
& \left(2H_{2k-1} - P_{2k-1}\right)H_{2m} + H_{2k-1}P_{2m}, \left(P_{2k}H_{2m}\right)^{\times 2}, (*)\Big) \quad \to
\end{aligned}$$

$$\begin{aligned}
\Big(\Big(&-H_{2k} + \tfrac{7}{2}P_{2k}\Big)H_{2m} + \tfrac{1}{2}H_{2k}P_{2m}; \left(H_{2k-1}H_{2m}\right)^{\times 3}, \\
& \tfrac{1}{2}\left(\left(2H_{2k} - P_{2k}\right)H_{2m} + H_{2k}P_{2m}\right), \left(2H_{2k-1} - P_{2k-1}\right)H_{2m} + \\
& + H_{2k-1}P_{2m}, \left(P_{2k}H_{2m}\right)^{\times 2}, (*)\Big) \quad \rightsquigarrow
\end{aligned}$$

$$\begin{aligned}
\Big(\Big(&-H_{2k} + \tfrac{7}{2}P_{2k}\Big)H_{2m} + \tfrac{1}{2}H_{2k}P_{2m}; \tfrac{1}{2}\left(\left(2H_{2k} - P_{2k}\right)H_{2m} + H_{2k}P_{2m}\right), \\
& \left(2H_{2k-1} - P_{2k-1}\right)H_{2m} + H_{2k-1}P_{2m}, \left(P_{2k}H_{2m}\right)^{\times 2}, \\
& \left(H_{2k-1}H_{2m}\right)^{\times 3}, (*)\Big) \quad \to
\end{aligned}$$

$$\begin{aligned}
\Big(&\tfrac{3}{2}P_{2k}H_{2m} + \tfrac{1}{2}H_{2k-2}P_{2m}; \Big(2H_{2k} - \tfrac{5}{2}P_{2k}\Big)H_{2m} + \tfrac{1}{2}H_{2k-2}P_{2m}, P_{2k-2}H_{2m}, \\
& P_{2k-1}H_{2m} - H_{2k-1}P_{2m}, P_{2k}H_{2m}, \left(H_{2k-1}H_{2m}\right)^{\times 3}, (*)\Big) \quad \rightsquigarrow
\end{aligned}$$

$$\begin{aligned}
\Big(&\tfrac{3}{2}P_{2k}H_{2m} + \tfrac{1}{2}H_{2k-2}P_{2m}; P_{2k}H_{2m}, \left(H_{2k-1}H_{2m}\right)^{\times 3}, \\
& \Big(2H_{2k} - \tfrac{5}{2}P_{2k}\Big)H_{2m} + \tfrac{1}{2}H_{2k-2}P_{2m}, P_{2k-2}H_{2m}, \\
& P_{2m-(2k-1)}, (*)\Big) \quad \to
\end{aligned}$$

$$\begin{aligned}
\Big(&2P_{2k-1}H_{2m} + H_{2k-2}P_{2m}; \Big(2H_{2k} - \tfrac{5}{2}P_{2k}\Big)H_{2m} + \tfrac{1}{2}H_{2k-2}P_{2m}, \\
& \Big(-\tfrac{1}{2}P_{2k-2}H_{2m} + \tfrac{1}{2}H_{2k-2}P_{2m}\Big)^{\times 2}, H_{2k-1}H_{2m}, \\
& \Big(2H_{2k} - \tfrac{5}{2}P_{2k}\Big)H_{2m} + \tfrac{1}{2}H_{2k-2}P_{2m}, P_{2k-2}H_{2m}, \\
& P_{2m-(2k-1)}, (*)\Big) \quad \rightsquigarrow
\end{aligned}$$

$$\begin{aligned}
(&2P_{2k-1}H_{2m} + H_{2k-2}P_{2m}; H_{2k-1}H_{2m}, \\
& \Big(\Big(2H_{2k} - \tfrac{5}{2}P_{2k}\Big)H_{2m} + \tfrac{1}{2}H_{2k-2}P_{2m}\Big)^{\times 2}, P_{2k-2}H_{2m}, \\
& \Big(\tfrac{1}{2}P_{2m-(2k-2)}\Big)^{\times 2}, P_{2m-(2k-1)}, (*)\Big) \quad \to
\end{aligned}$$

$$\begin{aligned}
\Big(&P_{2k-1}H_{2m} + H_{2k-2}P_{2m}; P_{2k-2}H_{2m}, \Big(-\tfrac{1}{2}P_{2k-2}H_{2m} + \tfrac{1}{2}H_{2k-2}P_{2m}\Big)^{\times 2}, \\
& P_{2k-2}H_{2m}, \Big(\tfrac{1}{2}P_{2m-(2k-2)}\Big)^{\times 2}, P_{2m-(2k-1)}, (*)\Big) \quad \rightsquigarrow
\end{aligned}$$

$$\begin{aligned}
\Big(&P_{2k-1}H_{2m} + H_{2k-2}P_{2m}; \left(P_{2k-2}H_{2m}\right)^{\times 2}, \Big(\tfrac{1}{2}P_{2m-(2k-2)}\Big)^{\times 4}, \\
& P_{2m-(2k-1)}, (*)\Big).
\end{aligned}$$

Now, after reordering this last class, we obtain V_{k-1}^m as required. □

Lemma 2.5.9. *V_2^m reduces in 8 Cremona moves to the class*

$$V^m := \left(H_{2m}; H_{2m} - \frac{1}{2}P_{2m}, \left(\frac{1}{2}P_{2m}\right)^{\times 3}, (P_{2m-1})^{\times 2}, B_1^m\right).$$

Proof. We have

$$V_2^m = \Bigg(29H_{2m} + 17P_{2m}; \left(11H_{2m} + \frac{17}{2}P_{2m}\right)^{\times 4}, 9H_{2m} + 7P_{2m},$$
$$(12H_{2m})^{\times 2}, \left(2H_{2m} + \frac{3}{2}P_{2m}\right)^{\times 4}, H_{2m} + P_{2m}, H_{2m} + \frac{1}{2}P_{2m}, (*)\Bigg);$$

$$\Big(25H_{2m} + \tfrac{17}{2}P_{2m}; 11H_{2m} + \tfrac{17}{2}P_{2m}, 9H_{2m} + 7P_{2m}, (12H_{2m})^{\times 2},$$
$$(7H_{2m})^{\times 3}, \left(2H_{2m} + \tfrac{3}{2}P_{2m}\right)^{\times 4}, H_{2m} + P_{2m}, H_{2m} + \tfrac{1}{2}P_{2m}, (*)\Big);$$

$$\Big(18H_{2m} + \tfrac{3}{2}P_{2m}; 12H_{2m}, (7H_{2m})^{\times 3}, 4H_{2m} + \tfrac{3}{2}P_{2m}, \left(2H_{2m} + \tfrac{3}{2}P_{2m}\right)^{\times 4},$$
$$2H_{2m}, H_{2m} + P_{2m}, H_{2m} + \tfrac{1}{2}P_{2m}, P_{2m-3}, (*)\Big);$$

$$\Big(10H_{2m} + 3P_{2m}; 7H_{2m}, \left(4H_{2m} + \tfrac{3}{2}P_{2m}\right)^{\times 2}, \left(2H_{2m} + \tfrac{3}{2}P_{2m}\right)^{\times 4},$$
$$2H_{2m}, H_{2m} + P_{2m}, H_{2m} + \tfrac{1}{2}P_{2m}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 2}, P_{2m-3}, (*)\Big);$$

$$\Big(5H_{2m} + 3P_{2m}; \left(2H_{2m} + \tfrac{3}{2}P_{2m}\right)^{\times 4}, (2H_{2m})^{\times 2}, H_{2m} + P_{2m},$$
$$H_{2m} + \tfrac{1}{2}P_{2m}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 2}, P_{2m-3}, (*)\Big);$$

$$\Big(4H_{2m} + \tfrac{3}{2}P_{2m}; 2H_{2m} + \tfrac{3}{2}P_{2m}, (2H_{2m})^{\times 2}, H_{2m} + P_{2m}, H_{2m} + \tfrac{1}{2}P_{2m},$$
$$(H_{2m})^{\times 3}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, (*)\Big);$$

$$\Big(2H_{2m} + \tfrac{3}{2}P_{2m}; H_{2m} + P_{2m}, H_{2m} + \tfrac{1}{2}P_{2m}, \tfrac{3}{2}P_{2m}, (H_{2m})^{\times 3},$$
$$\left(\tfrac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, (*)\Big);$$

$$\Big(2H_{2m}; (H_{2m})^{\times 3}, H_{2m} - \tfrac{1}{2}P_{2m}, P_{2m-1}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, (*)\Big);$$

$$\Big(H_{2m}; H_{2m} - \tfrac{1}{2}P_{2m}, P_{2m-1}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, (*)\Big).$$

After reordering this last class, we obtain V^m as required. □

Lemma 2.5.10. *For all $m \geqslant 3$, V^m reduces in 5 Cremona moves to V^{m-1}.*

Proof. We have

$$V^m = \left(H_{2m}; H_{2m} - \tfrac{1}{2}P_{2m}, \left(\tfrac{1}{2}P_{2m}\right)^{\times 3}, (P_{2m-1})^{\times 2}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 8}, (*)\right);$$

$$\left(H_{2m} - \tfrac{1}{2}P_{2m}; \tfrac{1}{2}P_{2m}, (P_{2m-1})^{\times 3}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 8}, (*)\right);$$

$$\left(\tfrac{1}{2}P_{2m}; P_{2m-1}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 9}, (*)\right);$$

$$\left(P_{2m-1}; P_{2m-1} - \tfrac{1}{2}P_{2m-2}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 7}, (*)\right);$$

$$\left(P_{2m-1} - \tfrac{1}{2}P_{2m-2}; H_{2m-2}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 5}, (*)\right);$$

$$\left(H_{2m-2}; H_{2m-2} - \tfrac{1}{2}P_{2m-2}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 3}, (*)\right).$$

After reordering this class, we obtain V^{m-1} as required. □

Lemma 2.5.11. *V^2 reduces in 8 Cremona moves to $(0; -1)$.*

Proof. We have

$$V^2 = (17; 11, 6^{\times 3}, 5^{\times 2}, 1^{\times 11});$$
$$(11; 6, 5^{\times 3}, 1^{\times 11});$$
$$(6; 5, 1^{\times 12});$$
$$(5; 4, 1^{\times 10});$$
$$(4; 3, 1^{\times 8});$$
$$(3; 2, 1^{\times 6});$$
$$(2; 1^{\times 5});$$
$$(1; 1^{\times 2});$$
$$(0; -1).$$
□

2.5.2.2 The classes $E(\alpha_{2m-1})$ reduce to $(0; -1)$

One readily checks that the classes $E(\alpha_{2m-1})$ reduce to $(0; -1)$ for $m = 1, 2$. In the following, we reduce the classes $E(\alpha_{2m-1})$ for $m \geqslant 3$.

Lemma 2.5.12. *The continued fraction expansion of α_{2m-1} is*

$$\left[5; \{1, 4\}^{\times(m-2)}, 1, 3, 1, 1, \{4, 1\}^{\times(m-1)}\right].$$

Moreover, if $v_j := (2H_j + P_j)\, H_{2m} - H_j P_{2m}$*, then*

$$\begin{aligned} E\left(\alpha_{2m-1}\right) = \Bigg(& P_{2m-1}H_{2m}, P_{2m-1}H_{2m}; \\ & \left(\frac{1}{2}v_{2m-2}\right)^{\times 5}, v_{2m-3}, \left(\frac{1}{2}v_{2m-4}\right)^{\times 4}, \dots, v_3, \left(\frac{1}{2}v_2\right)^{\times 4}, \\ & v_1, \left(H_{2m} - \frac{1}{2}P_{2m}\right)^{\times 3}, \frac{1}{2}P_{2m}, P_{2m-1}, \\ & \left(\frac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, \dots, \left(\frac{1}{2}P_2\right)^{\times 4}, P_1, 1 \Bigg). \end{aligned}$$

Proof. The first terms of $W'\left(\alpha_{2m-1}\right)$ are $\left(\frac{1}{2}v_{2m-2}\right)^{\times 5}$ since $\frac{1}{2}v_{2m-2} = 2P_{2m-1}^2$ and $5 < \alpha_{2m-1} < 6$ for $m \geqslant 2$. Before determining the next terms, we prove that for all $k \geqslant 1$, $v_{2k+1} > \frac{1}{2}v_{2k} > v_{2k-1}$. Indeed

$$\begin{aligned} v_{2k+1} &= \left(H_{2k+1} + P_{2k+1}\right) P_{2m} + \left(2H_{2k+1} + P_{2k+1}\right) P_{2m-1} \\ &= \left(5P_{2k} + 2P_{2k-1}\right) P_{2m} + \left(8P_{2k} + 3P_{2k-1}\right) P_{2m-1} \\ &> \left(P_{2k} + \frac{1}{2}P_{2k-1}\right) P_{2m} + \left(\frac{3}{2}P_{2k} + P_{2k-1}\right) P_{2m-1} \\ &= \frac{1}{2}\left(H_{2k} + P_{2k}\right) P_{2m} + \frac{1}{2}\left(2H_{2k} + P_{2k}\right) P_{2m-1} \\ &= \frac{1}{2}v_{2k} \\ &= \left(\frac{5}{2}P_{2k-1} + P_{2k-2}\right) P_{2m} + \left(3P_{2k-1} + \frac{7}{2}P_{2k-2} + P_{2k-3}\right) P_{2m-1} \\ &> \left(2P_{2k-1} + P_{2k-2}\right) P_{2m} + \left(3P_{2k-1} + 2P_{2k-2}\right) P_{2m-1} \\ &= \left(H_{2k-1} + P_{2k-1}\right) P_{2m} + \left(2H_{2k-1} + P_{2k-1}\right) P_{2m-1} \\ &= v_{2k-1}. \end{aligned}$$

So the next term of $W'\left(\alpha_{2m-1}\right)$ is $H_{2m}^2 - 5\left(2P_{2m-1}^2\right) = v_{2m-3} < \frac{1}{2}v_{2m-2}$. Moreover, for all $k \geqslant 1$,

$$\begin{aligned} v_{2k+1} - 4\left(\frac{1}{2}v_{2k}\right) &= \left(2H_{2k+1} + P_{2k+1} - 4H_{2k} - 2P_{2k}\right) H_{2m} \\ &\quad - \left(H_{2k+1} - 2H_{2k}\right) P_{2m} \\ &= \left(2H_{2k-1} + P_{2k-1}\right) H_{2m} - H_{2k-1}P_{2m} = v_{2k-1} < \frac{1}{2}v_{2k}; \end{aligned}$$

$$\begin{aligned} \frac{1}{2}v_{2k} - v_{2k-1} &= \left(H_{2k} + \frac{1}{2}P_{2k} - 2H_{2k-1} - P_{2k-1}\right) H_{2m} \\ &\quad - \left(\frac{1}{2}H_{2k} - H_{2k-1}\right) P_{2m} \\ &= \frac{1}{2}\left(2H_{2k-2} + P_{2k-2}\right) H_{2m} - \frac{1}{2}H_{2k-2}P_{2m} = \frac{1}{2}v_{2k-2} \\ &< v_{2k-1}. \end{aligned}$$

This proves that the first terms of $W'(\alpha_{2m-1})$ are

$$\left(\frac{1}{2}v_{2m-2}\right)^{\times 5}, v_{2m-3}, \left(\frac{1}{2}v_{2m-4}\right)^{\times 4}, \ldots, v_3, \left(\frac{1}{2}v_2\right)^{\times 4}, v_1.$$

The next three terms are $\left(H_{2m} - \frac{1}{2}P_{2m}\right)^{\times 3}, \frac{1}{2}P_{2m}, P_{2m-1}$ since

$$\frac{1}{2}v_2 - v_1 = H_{2m} - \frac{1}{2}P_{2m} < 3H_{2m} - P_{2m} = v_1;$$

$$v_1 - 3\left(H_{2m} - \frac{1}{2}P_{2m}\right) = \frac{1}{2}P_{2m} < H_{2m} - \frac{1}{2}P_{2m};$$

$$H_{2m} - \frac{1}{2}P_{2m} - \frac{1}{2}P_{2m} = P_{2m-1} < \frac{1}{2}P_{2m}.$$

Since the last terms are the same as those of $W'(\alpha_{2m})$, the lemma is proved. □

Let us introduce again some notations.

Definition 2.5.13. Set

$$\begin{aligned}
\hat{A}_k^m := \Bigg(& \left(\frac{1}{2}v_{2k-2}\right)^{\times 4}, v_{2k-3}, (H_{2k-1}H_{2m} - H_{2k-1}P_{2m})^{\times 2}, \left(\frac{1}{2}v_{2k-4}\right)^{\times 4}, \\
& v_{2k-5}, \ldots, \left(\frac{1}{2}v_2\right)^{\times 4}, v_1, \left(H_{2m} - \frac{1}{2}P_{2m}\right)^{\times 3}, \frac{1}{2}P_{2m}, P_{2m-1}\Bigg), \\
\hat{B}_k^m := \Bigg(& \left(\frac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, \ldots, \left(\frac{1}{2}P_{2m-2k+2}\right)^{\times 4}, P_{2m-2k+1}, \\
& \left(\frac{1}{2}P_{2m-2k}\right)^{\times 8}, (P_{2m-2k-1})^{\times 2}, \ldots, \left(\frac{1}{2}P_2\right)^{\times 8}, (P_1)^{\times 2}, 1\Bigg), \\
\hat{V}_k^m := \Big(& P_{2k}H_{2m} + H_{2k-1}P_{2m}; \hat{A}_k^m, \hat{B}_k^m\Big).
\end{aligned}$$

Note that $\hat{B}_k^m$ is actually equal to the vector B_k^m that we used in the reduction of the classes $E(\alpha_{2m})$. Here, $\hat{A}_k^m$ has the structure

$$\left[4, 1, 2, \{4, 1\}^{\times(k-2)}, 3, 1, 1\right]$$

and $\hat{B}_k^m$ has the structure

$$\left[\{4, 1\}^{\times(k-1)}, \{8, 2\}^{\times(m-k)}, 1\right].$$

We use again the convention that if $k = m$, $\hat{B}_m^m$ has the structure $\left[\{4, 1\}^{\times(m-1)}, 1\right]$ and that if $k = 1$, $\hat{B}_1^m$ has the structure $\left[\{8, 2\}^{\times(m-1)}, 1\right]$.

Lemma 2.5.14. *The image of $E(\alpha_{2m-1})$ by φ_* is the class*

$$\varphi_*(E(\alpha_{2m-1})) = \left(P_{2m}H_{2m} - H_{2m-1}P_{2m}; \hat{A}^m_m, \hat{B}^m_m\right) = \hat{V}^m_m.$$

Proof. The first terms of $E(\alpha_{2m-1})$ are

$$\left(P_{2m-1}H_{2m}, P_{2m-1}H_{2m}; \frac{1}{2}v_{2m-2}, (*)\right).$$

Since $\frac{1}{2}v_{2m-2} = 2P^2_{2m-1}$, we get

$$\begin{aligned}\varphi_*(E(\alpha_{2m-1})) &= \left(2P_{2m-1}H_{2m} - 2P^2_{2m-1}; \left(P_{2m-1}H_{2m} - 2P^2_{2m-1}\right)^{\times 2}, (*)\right)\\ &= (P_{2m}H_{2m} - H_{2m-1}P_{2m};\\ &\qquad (H_{2m-1}H_{2m} - H_{2m-1}P_{2m})^{\times 2}, (*)\Big).\end{aligned}$$

After reordering, this last class is $\left(P_{2m}H_{2m} - H_{2m-1}P_{2m}; \hat{A}^m_m, \hat{B}^m_m\right) = \hat{V}^m_m$ as required. □

Lemma 2.5.15. *For all $3 \leqslant k \leqslant m$, $\hat{V}^m_k$ reduces to $\hat{V}^m_{k-1}$ in 4 Cremona moves.*

Proof. We have

$$\begin{aligned}\hat{V}^m_k = \Big(&P_{2k}H_{2m} - H_{2k-1}P_{2m}; \tfrac{1}{2}\left((2H_{2k-2} + P_{2k-2})H_{2m} - H_{2k-2}P_{2m}\right)^{\times 4},\\ &(2H_{2k-3} + P_{2k-3})H_{2m} - H_{2k-3}P_{2m},\\ &(H_{2k-1}H_{2m} - H_{2k-1}P_{2m})^{\times 2}, (*) \quad \to\end{aligned}$$

$$\begin{aligned}\Big(&\left(H_{2k-2} + \tfrac{9}{2}P_{2k-2}\right)H_{2m} - \left(\tfrac{1}{2}H_{2k-2} + 4P_{2k-2}\right)P_{2m};\\ &(2P_{2k-2}H_{2m} - 2P_{2k-2}P_{2m})^{\times 3},\\ &\tfrac{1}{2}\left((2H_{2k-2} + P_{2k-2})H_{2m} - H_{2k-2}P_{2m}\right),\\ &(2H_{2k-3} + P_{2k-3})H_{2m} - H_{2k-3}P_{2m},\\ &(H_{2k-1}H_{2m} - H_{2k-1}P_{2m})^{\times 2}, (*)\Big) \quad \rightsquigarrow\end{aligned}$$

$$\begin{aligned}\Big(&\left(H_{2k-2} + \tfrac{9}{2}P_{2k-2}\right)H_{2m} - \left(\tfrac{1}{2}H_{2k-2} + 4P_{2k-2}\right)P_{2m};\\ &\tfrac{1}{2}\left((2H_{2k-2} + P_{2k-2})H_{2m} - H_{2k-2}P_{2m}\right),\\ &(2H_{2k-3} + P_{2k-3})H_{2m} - H_{2k-3}P_{2m}, (H_{2k-1}H_{2m} - H_{2k-1}P_{2m})^{\times 2},\\ &(2P_{2k-2}H_{2m} - 2P_{2k-2}P_{2m})^{\times 3}, (*)\Big) \quad \to\end{aligned}$$

$$\begin{aligned}\Big(&\left(H_{2k-2} + \tfrac{7}{2}P_{2k-2}\right)H_{2m} - \left(\tfrac{1}{2}H_{2k-2} + 4P_{2k-2}\right)P_{2m};\\ &\left(H_{2k-2} - \tfrac{1}{2}P_{2k-2}\right)H_{2m} - \tfrac{1}{2}H_{2k-2}P_{2m}, H_{2k-3}H_{2m} - H_{2k-3}P_{2m},\\ &P_{2k-1}H_{2m} - H_{2k-1}P_{2m}, H_{2k-1}H_{2m} - H_{2k-1}P_{2m},\\ &(2P_{2k-2}H_{2m} - 2P_{2k-2}P_{2m})^{\times 3}, (*)\Big) \quad \rightsquigarrow\end{aligned}$$

$$\Big(\Big(H_{2k-2} + \tfrac{7}{2}P_{2k-2}\Big) H_{2m} - \Big(\tfrac{1}{2}H_{2k-2} + 4P_{2k-2}\Big) P_{2m};$$
$$H_{2k-1}H_{2m} - H_{2k-1}P_{2m}, (2P_{2k-2}H_{2m} - 2P_{2k-2}P_{2m})^{\times 3},$$
$$\Big(H_{2k-2} - \tfrac{1}{2}P_{2k-2}\Big) H_{2m} - \tfrac{1}{2}H_{2k-2}P_{2m}, H_{2k-3}H_{2m} - H_{2k-3}P_{2m},$$
$$P_{2m-(2k-1)}, (*)\Big) \quad \to$$

$$\Big(P_{2k-1}H_{2m} - 2P_{2k-2}P_{2m}; \Big(H_{2k-2} - \tfrac{1}{2}P_{2k-2}\Big) H_{2m} - \tfrac{1}{2}H_{2k-2}P_{2m},$$
$$\Big(-\tfrac{1}{2}P_{2k-2}H_{2m} + \tfrac{1}{2}H_{2k-2}P_{2m}\Big)^{\times 2}, 2P_{2k-2}H_{2m} - 2P_{2k-2}P_{2m},$$
$$\Big(H_{2k-2} - \tfrac{1}{2}P_{2k-2}\Big) H_{2m} - \tfrac{1}{2}H_{2k-2}P_{2m}, H_{2k-3}H_{2m} - H_{2k-3}P_{2m},$$
$$P_{2m-(2k-1)}, (*)\Big) \quad \rightsquigarrow$$

$$(P_{2k-1}H_{2m} - 2P_{2k-2}P_{2m}; 2P_{2k-2}H_{2m} - 2P_{2k-2}P_{2m},$$
$$\Big(\Big(H_{2k-2} - \tfrac{1}{2}P_{2k-2}\Big) H_{2m} - \tfrac{1}{2}H_{2k-2}P_{2m}\Big)^{\times 2}, H_{2k-3}H_{2m} - H_{2k-3}P_{2m},$$
$$\Big(\tfrac{1}{2}P_{2m-(2k-2)}\Big)^{\times 2}, P_{2m-(2k-1)}, (*)\Big) \quad \to$$

$$(P_{2k-2}H_{2m} - H_{2k-3}P_{2m}; H_{2k-3}H_{2m} - H_{2k-3}P_{2m},$$
$$\Big(-\tfrac{1}{2}P_{2k-2}H_{2m} + \tfrac{1}{2}H_{2k-2}P_{2m}\Big)^{\times 2}, H_{2k-3}H_{2m} - H_{2k-3}P_{2m},$$
$$\Big(\tfrac{1}{2}P_{2m-(2k-2)}\Big)^{\times 2}, P_{2m-(2k-1)}, (*)\Big) \quad \rightsquigarrow$$

$$\Big(P_{2k-2}H_{2m} - H_{2k-3}P_{2m}; (H_{2k-3}H_{2m} - H_{2k-3}P_{2m})^{\times 2},$$
$$\Big(\tfrac{1}{2}P_{2m-(2k-2)}\Big)^{\times 4}, P_{2m-(2k-1)}, (*)\Big).$$

Now, after reordering this last class, we obtain $\hat{V}^m_{k-1}$ as required. □

Lemma 2.5.16. *$\hat{V}^m_2$ reduces in 5 Cremona moves to the class*

$$\hat{V}^m := \left(H_{2m} - \frac{1}{2}P_{2m}; \frac{1}{2}P_{2m}, (P_{2m-1})^{\times 3}, \hat{B}^m_1\right).$$

Proof. We have

$$\hat{V}^m_2 = \left(12H_{2m} - 7P_{2m}; \left(4H_{2m} - \frac{3}{2}P_{2m}\right)^{\times 4}, 3H_{2m} - P_{2m},\right.$$
$$\left.(7H_{2m} - 7P_{2m})^{\times 2}, \left(H_{2m} - \frac{1}{2}P_{2m}\right)^{\times 3}, (*)\right);$$

$$\Big(12H_{2m} - \tfrac{19}{2}P_{2m}; 4H_{2m} - \tfrac{3}{2}P_{2m}, 3H_{2m} - P_{2m}, (7H_{2m} - 7P_{2m})^{\times 2},$$
$$(4H_{2m} - 4P_{2m})^{\times 3}, \Big(H_{2m} - \tfrac{1}{2}P_{2m}\Big)^{\times 3}, (*)\Big);$$

$$\Big(10H_{2m} - \tfrac{19}{2}P_{2m}; 7H_{2m} - 7P_{2m}, (4H_{2m} - 4P_{2m})^{\times 3}, 2H_{2m} - \tfrac{3}{2}P_{2m},$$
$$\Big(H_{2m} - \tfrac{1}{2}P_{2m}\Big)^{\times 3}, P_{2m-1}, P_{2m-3}, (*)\Big);$$

$$\Big(5H_{2m} - 4P_{2m}; 4H_{2m} - 4P_{2m}, \left(2H_{2m} - \tfrac{3}{2}P_{2m}\right)^{\times 2}, \left(H_{2m} - \tfrac{1}{2}P_{2m}\right)^{\times 3},$$
$$P_{2m-1}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 2}, P_{2m-3}, (*)\Big);$$
$$\left(2H_{2m} - P_{2m}; \left(H_{2m} - \tfrac{1}{2}P_{2m}\right)^{\times 3}, (P_{2m-1})^{\times 2}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, (*)\right);$$
$$\left(H_{2m} - \tfrac{1}{2}P_{2m}; \left(H_{2m} - \tfrac{1}{2}P_{2m}\right)^{\times 3}, (P_{2m-1})^{\times 2}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, (*)\right).$$

After reordering this last class, we obtain $\hat{V}^m$ as required. □

Lemma 2.5.17. *For all $m \geqslant 3$, $\hat{V}^m$ reduces in 5 Cremona moves to $\hat{V}^{m-1}$.*

Proof. We have

$$\hat{V}^m = \left(H_{2m} - \tfrac{1}{2}P_{2m}; \tfrac{1}{2}P_{2m}, (P_{2m-1})^{\times 3}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 8}, (*)\right);$$
$$\left(\tfrac{1}{2}P_{2m}; P_{2m-1}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 9}, (*)\right);$$
$$\left(P_{2m-1}; P_{2m-1} - \tfrac{1}{2}P_{2m-2}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 7}, (*)\right);$$
$$\left(P_{2m-1} - \tfrac{1}{2}P_{2m-2}; H_{2m-2}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 5}, (*)\right);$$
$$\left(H_{2m-2}; H_{2m-2} - \tfrac{1}{2}P_{2m-2}, \left(\tfrac{1}{2}P_{2m-2}\right)^{\times 3}, (*)\right);$$
$$\left(H_{2m-2} - \tfrac{1}{2}P_{2m-2}; \tfrac{1}{2}P_{2m-2}, P_{2m-3}, (*)\right).$$

After reordering this class, we obtain $\hat{V}^{m-1}$ as required. □

Lemma 2.5.18. *$\hat{V}^2$ reduces in 7 Cremona moves to $(0; -1)$.*

Proof. We have

$$\hat{V}^2 = \left(11; 6, 5^{\times 3}, 1^{\times 11}\right);$$
$$\left(6; 5, 1^{\times 12}\right);$$
$$\left(5; 4, 1^{\times 10}\right);$$
$$\left(4; 3, 1^{\times 8}\right);$$
$$\left(3; 2, 1^{\times 6}\right);$$
$$\left(2; 1^{\times 5}\right);$$
$$\left(1; 1^{\times 2}\right);$$
$$(0; -1).$$

□

2.6 The interval $[\sigma^2, 6]$

In this section we prove that $c(a) = \frac{a+1}{4}$ on the interval $[\sigma^2, 6]$, which is a part of Theorem 2.7.2. Notice that the class $(2, 2; 2, 1^{\times 5})$ gives the constraint $\frac{a+1}{4}$ on $[\sigma^2, 6]$. If thus suffices to show that no class gives a stronger constraint. We begin by showing this on the interval $\left[5\frac{12}{13}, 6\right]$, and will then spend some efforts to extend it to $[\sigma^2, 6]$.

Proposition 2.6.1. *For $a \in \left[5\frac{12}{13}, 6\right]$, we have $c(a) = \frac{a+1}{4}$.*

Proof. Write $a = 5 + x \in \left[5\frac{12}{13}, 6\right[$. Then $w(a) = (1^{\times 5}, x, w_7, \ldots, w_M)$. Since $x \geqslant \frac{12}{13}$, $1 - x \leqslant \frac{x}{9}$, thus at least the first nine of the weights $w_7, \ldots, w_M$ are equal. Then, by Corollary 1.2.4 in [MS], the $M - 6$ balls of weights $w_7, \ldots, w_M$ fully fill the ball of weight λ, where $a = 5 + x^2 + \lambda^2$. Thus to prove that $c(a) = \frac{a+1}{4}$ on $[\sigma^2, 6]$, it suffices to check that the seven balls of weights $1^{\times 5}, x, \lambda$ embed into the cube $C\left(\frac{(5+x)+1}{4}\right)$, or in other words, that the finite number of classes of $\mathcal{E}_M$ with $M \leqslant 7$ don't give any embedding constraint stronger than $\frac{a+1}{4}$ for these seven balls.

This is clear for the classes belonging to $\mathcal{E}_M$ with $M \leqslant 6$. The strongest constraint of $\mathcal{E}_7$ comes from the class $(4, 3; 2^{\times 6}, 1)$ for which

$$\mu\left(4, 3; 2^{\times 6}, 1\right)(5 + x) = \frac{10 + 2x + \lambda}{7}.$$

Notice that $\frac{10+2x+\lambda}{7} \leqslant \frac{(5+x)+1}{4}$ if and only if $\lambda \leqslant \frac{2-x}{4}$. Recall that $a = 5 + x = 5 + x^2 + \lambda^2$, whence $\lambda^2 = x - x^2$. Since $x - x^2 \leqslant \left(\frac{2-x}{4}\right)^2$ for $x \in \left[\frac{12}{13}, 1\right]$. Thus the class $(4, 3; 2^{\times 6}, 1)$ indeed gives no stronger constraint than $\frac{a+1}{4}$. Finally, by continuity of c, $c(6) = \frac{6+1}{4} = \frac{7}{4}$. □

Proposition 2.6.2. *Let $(d, e; m) \in \mathcal{E}$ be a class such that $\mu(d, e; m)(a) > \frac{a+1}{4} \geqslant \sqrt{\frac{a}{2}}$ for some $a \in [\sigma^2, 6]$. Then*

(i) $d < \frac{2\sqrt{a}}{\sqrt{a^2-6a+1}}$ for a class of the form $(d, d; m)$ and $d < \frac{\sqrt{2a}}{\sqrt{a^2-6a+1}}$ for a class of the form $(d + \frac{1}{2}, d - \frac{1}{2}; m)$.

(ii) Moreover, if we denote $\lambda^2 := 1 - \sum_{i=1}^{M} \varepsilon_i^2$ (respectively $\lambda^2 := \frac{1}{2} - \sum_{i=1}^{M} \varepsilon_i^2$) for a class of the form $(d, d; m)$ (respectively for a class of the form $(d + \frac{1}{2}, d - \frac{1}{2}; m)$), then $\lambda^2 > d^2 \sqrt{\frac{2}{a}}\, y(a)$.

Proof. (i) Let us first prove this for a class of the form $(d, d; m)$. By Lemma 2.4.8 (i), we have

$$\frac{a+1}{4} < \mu(d, d; m)(a) \leqslant \sqrt{1 + \frac{1}{2d^2}}\sqrt{\frac{a}{2}}.$$

Thus
$$\frac{(a+1)^2}{8a} - 1 < \frac{1}{2d^2}$$
and so
$$d < \frac{2\sqrt{a}}{\sqrt{a^2-6a+1}}.$$
Similarly, for a class of the form $(d+\frac{1}{2}, d-\frac{1}{2}; m)$, we have
$$\frac{a+1}{4} < \mu(d+\frac{1}{2}, d-\frac{1}{2}; m)(a) \leqslant \sqrt{1+\frac{1}{4d^2}}\sqrt{\frac{a}{2}}$$
and thus
$$d < \frac{\sqrt{2a}}{\sqrt{a^2-6a+1}}.$$
(ii) For a class of the form $(d,d;m)$, we have by Proposition 2.3.8 (i),
$$\begin{aligned} 2d^2+1 = \langle m,m\rangle &= \left\langle \frac{\sqrt{2}d}{\sqrt{a}}w(a)+\varepsilon, \frac{\sqrt{2}d}{\sqrt{a}}w(a)+\varepsilon \right\rangle \\ &= 2d^2 + \frac{2\sqrt{2}d}{\sqrt{a}}\langle w(a),\varepsilon\rangle + \langle \varepsilon,\varepsilon\rangle . \end{aligned}$$
Thus
$$\langle w(a),\varepsilon\rangle = \underbrace{(1-\langle\varepsilon,\varepsilon\rangle)}_{=\lambda^2}\frac{1}{2d}\sqrt{\frac{a}{2}}. \tag{2.6.1}$$
On the other hand
$$\begin{aligned} \frac{a+1}{4} < \mu(d,d;m)(a) = \frac{\langle m, w(a)\rangle}{2d} &= \frac{1}{2d}\left\langle \frac{\sqrt{2}d}{\sqrt{a}}w(a)+\varepsilon, w(a)\right\rangle \\ = \sqrt{\frac{a}{2}} + \frac{1}{2d}\langle \varepsilon, w(a)\rangle . \end{aligned}$$
Thus
$$\langle \varepsilon, w(a)\rangle > 2d\left(\frac{a+1}{4} - \sqrt{\frac{a}{2}}\right) = \frac{1}{2}d\underbrace{\left(a+1-2\sqrt{2a}\right)}_{=y(a)}.$$
Inserting (2.6.1) in this inequality, we get
$$\frac{\lambda^2}{2d}\sqrt{\frac{a}{2}} > \frac{d}{2}y(a),$$
and finally
$$\lambda^2 > d^2\sqrt{\frac{2}{a}}\,y(a).$$

Similarly, for a class $(d+\frac{1}{2}, d-\frac{1}{2}; m)$, we have

$$2d^2+\frac{1}{2}=2d^2+\frac{2\sqrt{2}d}{\sqrt{a}}\left\langle w(a),\varepsilon\right\rangle+\left\langle \varepsilon,\varepsilon\right\rangle,$$

so

$$\left\langle w(a),\varepsilon\right\rangle=\underbrace{\left(\frac{1}{2}-\left\langle \varepsilon,\varepsilon\right\rangle\right)}_{=\lambda^2}\frac{1}{2d}\sqrt{\frac{a}{2}}.$$

The rest of the proof is then identical to the case of a class $(d,d;m)$. ☐

Notice that the continued fraction of $\sigma^2=3+2\sqrt{2}$ is $[5;1,4,1,4,\ldots]$. We will now define the so called *convergents* c_k of σ^2 and some other numbers $u_k(j)$ and $v_k(j)$ which will play a crucial role in the proof of Theorem 2.6.21.

Definition 2.6.3. For all $k,j\geqslant 1$, set

$$\begin{aligned}
c_{2k-1} &:= \left[5;\{1,4\}^{\times(k-1)},1\right]=\left[5;\{1,4\}^{\times(k-2)},1,5\right],\\
c_{2k} &:= \left[5;\{1,4\}^{\times k}\right],\\
u_k(j) &:= \left[5;\{1,4\}^{\times(k-1)},1,5,j\right],\\
v_k(j) &:= \left[5;\{1,4\}^{\times(k-1)},1,j\right].
\end{aligned}$$

Lemma 2.6.4. *For all $k,j\geqslant 1$, we have the following relations written in lowest terms*

(i) $c_{2k-1}=\dfrac{\frac{1}{2}P_{2k+2}}{\frac{1}{2}P_{2k}},\quad c_{2k}=\dfrac{P_{2k+3}}{P_{2k+1}},$

(ii) $u_k(j)=\dfrac{\frac{1}{2}\left(jP_{2k+4}+P_{2k+2}\right)}{\frac{1}{2}\left(jP_{2k+2}+P_{2k}\right)},$

(iii) $v_k(j)=\dfrac{\frac{1}{2}jP_{2k+2}+P_{2k+1}}{\frac{1}{2}jP_{2k}+P_{2k-1}}.$

Proof. We use the fact that if $[a_0;a_1,\ldots,a_M]$ is a continued fraction and $\frac{p_k}{q_k}:=[a_0;a_1,\ldots,a_k]$ is its k-th convergent written in lowest terms, then for any real number x,

$$[a_0;a_1,\ldots,a_k,x]=\frac{xp_k+p_{k-1}}{xq_k+q_{k-1}},$$

written in lowest terms.

(i) We argue by induction on k. Assertion (i) is clear for $k=1$. Assume it holds for $k-1$. Then

$$c_{2k-1}=\left[5;\{1,4\}^{\times(k-1)},1\right]=\frac{\frac{1}{2}P_{2k}+P_{2k+1}}{\frac{1}{2}P_{2k-2}+P_{2k-1}}=\frac{\frac{1}{2}P_{2k+2}}{\frac{1}{2}P_{2k}},$$

and

$$c_{2k} = \left[5; \{1,4\}^{\times k}\right] = \frac{P_{2k+1} + 2P_{2k+2}}{P_{2k-1} + 2P_{2k}} = \frac{P_{2k+3}}{P_{2k+1}}.$$

The proofs of (ii) and (iii) are then straightforward. □

Corollary 2.6.5. *We have*

(i) $c_2 < c_4 < \ldots < c_{2k} < \ldots < \sigma^2 < \ldots < c_{2k+1} < \ldots < c_3 < c_1$*,*

(ii) $c_{2k+1} < \ldots < u_k(2) < u_k(1) = v_k(6) < v_k(7) < \ldots < c_{2k-1}$*.*

Proof. (i) It is a property of convergents that the even convergents (respectively odd convergents) of a real number a form an increasing sequence (respectively decreasing sequence) converging to a.

(ii) This follows from Lemma 2.6.4 and the identity $P_{2k+1}P_{2k-1} - P_{2k}^2 = 1$ for all $k \geqslant 1$. □

Lemma 2.6.6. *To prove that a quadratic identity of the form*

$$Q(s) := \sum_{i,j \geqslant 0} a_{ij} P_{s+i} P_{s+j} + \sum_{j \geqslant 0} b_j P_{2s+j} + (-1)^s c = 0$$

holds for all $s \geqslant 0$*, it suffices to check it for three distinct values of* s*. Moreover, if* Q *is homogeneous and linear (that is,* $c = 0$ *and* $a_{ij} = 0$ *for all* i, j*), it suffices to check it for two distinct values of* s*.*

Proof. The proof is similar to the proof of Proposition 3.2.3 in [MS]. The only part of the proof which is slightly different is the proof of their Lemma 3.2.2, which we have adapted in the following Lemma 2.6.7. □

Lemma 2.6.7. *For all* $i \geqslant 0$*, there exists a finite number of rational coefficients* a_{ij}, c_i *such that the identity*

$$P_{s+i} P_s = \sum_{j \geqslant 0} a_{ij} P_{2s+j} + (-1)^s c_i$$

holds for all $s \geqslant 0$*. Moreover,* $c_i = -\sum_{j \geqslant 0} a_{ij} P_j$*.*

Proof. In view of the relation

$$P_k = 2P_{k-1} + P_{k-2} \tag{2.6.2}$$

it suffices to prove it for two distinct values of i. We claim that for $i = 0$ and $i = 2$ we have the following relations for all $s \geqslant 0$:

$$4P_s^2 = P_{2s+1} - P_{2s} - (-1)^s, \tag{2.6.3}$$
$$8P_{s+2}P_s = P_{2s+1} + P_{2s+3} - 6(-1)^s, \tag{2.6.4}$$

To prove this, we use the two well-known identities for Pell numbers,

$$P_k^2 = P_{k+1}P_{k-1} - (-1)^k, \tag{2.6.5}$$
$$P_{2k-1} = P_k^2 + P_{k-1}^2. \tag{2.6.6}$$

We start with $i = 0$. The relation is true for $s = 0$. Now, if $s \geqslant 1$, applying (2.6.2) to (2.6.3) gives

$$8P_s^2 = P_{s+1}^2 + 2P_s^2 + P_{s-1}^2 - 2(-1)^s.$$

Using again (2.6.2), we get

$$6P_s^2 = P_{s+1}^2 + P_{s-1}^2 - 2(-1)^s = 4P_s^2 + 4P_sP_{s-1} + 2P_{s-1}^2 - 2(-1)^s. \tag{2.6.7}$$

Finally, applying once more (2.6.2), we obtain

$$2P_s^2 = 2P_{s-1}\left(2P_s + P_{s-1}\right) - 2(-1)^s = 2P_{s+1}P_{s-1} - 2(-1)^s,$$

which is true by (2.6.5).

For $i = 2$, applying (2.6.5) to the LHS of (2.6.4), and (2.6.6) to the RHS, we obtain

$$8P_{s+1}^2 + 8(-1)^{s+1} = P_{s+2}^2 + 2P_{s+1}^2 + P_s^2 - 6(-1)^s,$$

which is equivalent to

$$6P_{s+1}^2 = P_{s+2}^2 + P_s^2 - 2(-1)^{s+1},$$

which is true by (2.6.7).

Finally, we easily check that the formula for c_0 and c_2 holds. □

Corollary 2.6.8. *For all $k, j \geqslant 1$, we have:*

(i) If we abbreviate $u := u_k(j) =: \frac{p}{q}$, then $q^2\left(u^2 - 6u + 1\right) = j^2 + 6j + 1$,

(ii) If $v := v_k(j) = \frac{p}{q}$, then $q^2\left(v^2 - 6v + 1\right) = j^2 - 4j - 4$,

(iii) If we denote $\frac{p}{q} := u_k(2)$, then $p^2 - 6pq + q^2 - 16 = 1$, and if $\frac{p}{q} := u_k(3)$, then $p^2 - 6pq + q^2 - 12 = 16$,

(iv) If we denote $\frac{p}{q} := v_k(6)$, then $5p^2 - 30pq + 5q^2 - 32 = 8$, and if $\frac{p}{q} := v_k(7)$, then $3p^2 - 18pq + 3q^2 - 28 = 23$.

(v) $P_{2m-k} = (-1)^{k+1}\left(P_kH_{2m} - H_kP_{2m}\right)$ for all $m \geqslant 0$ and $k \leqslant 2m$.

Proof. By Lemma 2.6.6 we only have to check the first four identities for three values of k. It is easy to see that they are true for $k = 1, 2, 3$. Similarly, it suffices to check the last identity for three even and three odd values. □

Lemma 2.6.9. *Set* $\varphi(a) := \frac{a+1}{4}$ *and* $\psi(a) = \sqrt{\frac{a}{2}}$. *Then*

(i) $\varphi\left(u_k(j+1)\right) > \psi\left(u_k(j)\right)$ *for all* $k, j \geqslant 1$,

(ii) $\varphi\left(v_k(j)\right) > \psi\left(v_k(j+1)\right)$ *for all* $k \geqslant 1, j \geqslant 6$.

Proof. (i) Abbreviate $u := u_k(j+1)$ and $u' := u_k(j)$. Due to Corollary 2.6.8 (i) we have

$$u^2 = \frac{(j+1)^2 + 6(j+1) + 1}{q^2} + 6u - 1.$$

We have to prove that $\frac{u+1}{4} > \sqrt{\frac{u'}{2}}$ which is equivalent to

$$u^2 + 2u + 1 > 8u',$$

which becomes

$$\frac{(j+1)^2 + 6(j+1) + 1}{q^2} + 8u > 8u',$$

and finally

$$8(u' - u)q^2 < (j+1)^2 + 6(j+1) + 1. \tag{2.6.8}$$

By Lemma 2.6.4 (ii)

$$\begin{aligned} u' - u &= \frac{\frac{1}{2}\left(jP_{2k+4} + P_{2k+2}\right)}{\frac{1}{2}\left(jP_{2k+2} + P_{2k}\right)} - \frac{\frac{1}{2}\left((j+1)P_{2k+4} + P_{2k+2}\right)}{\frac{1}{2}\left((j+1)P_{2k+2} + P_{2k}\right)} \\ &= \frac{(j+1)P_{2k+2}^2 - (j+1)P_{2k}P_{2k+4} + jP_{2k+4}P_{2k} - jP_{2k+2}^2}{\left((j+1)P_{2k+2} + P_{2k}\right)\left(jP_{2k+2} + P_{2k}\right)} \\ &= \frac{P_{2k+2}^2 - P_{2k}P_{2k+4}}{\left((j+1)P_{2k+2} + P_{2k}\right)\left(jP_{2k+2} + P_{2k}\right)} \\ &= \frac{4}{\left((j+1)P_{2k+2} + P_{2k}\right)\left(jP_{2k+2} + P_{2k}\right)} \end{aligned}$$

since $P_{2k+2}^2 - P_{2k}P_{2k+4} = 4$ for all $k \geqslant 1$ by Lemma 2.6.6. Inserting this in (2.6.8) gives

$$\frac{8\left((j+1)P_{2k+2} + P_{2k}\right)}{jP_{2k+2} + P_{2k}} < (j+1)^2 + 6(j+1) + 1,$$

since $q^2 = \frac{1}{4}\left((j+1)P_{2k+2} + P_{2k}\right)^2$. This inequality is now true for all $k, j \geqslant 1$ since the left hand side is smaller than 16, and the right hand side is bigger than 17.

(ii) Abbreviate $v := v_k(j)$ and $v' := v_k(j+1)$. Due to Corollary 2.6.8 (ii), we have

$$v^2 = \frac{j^2 - 4j - 4}{q^2} + 6v - 1.$$

We have to prove that $\frac{v+1}{4} > \sqrt{\frac{v'}{2}}$ which is equivalent to

$$v^2 + 2v + 1 > 8v',$$

which becomes

$$\frac{j^2 - 4j - 4}{q^2} + 8v > 8v',$$

and finally

$$8(v' - v)q^2 < j^2 - 4j - 4. \tag{2.6.9}$$

By Lemma 2.6.4 (iii)

$$\begin{aligned}
v' - v &= \frac{\frac{1}{2}jP_{2k+2} + P_{2k+1}}{\frac{1}{2}jP_{2k} + P_{2k-1}} - \frac{\frac{1}{2}(j+1)P_{2k+2} + P_{2k+1}}{\frac{1}{2}(j+1)P_{2k} + P_{2k-1}} \\
&= \frac{jP_{2k-1}P_{2k+2} + (j+1)P_{2k}P_{2k+1} - jP_{2k}P_{2k+1} - (j+1)P_{2k-1}P_{2k+2}}{2\left(\frac{1}{2}jP_{2k} + P_{2k-1}\right)\left(\frac{1}{2}(j+1)P_{2k} + P_{2k-1}\right)} \\
&= \frac{P_{2k}P_{2k+1} - P_{2k-1}P_{2k+2}}{2\left(\frac{1}{2}jP_{2k} + P_{2k-1}\right)\left(\frac{1}{2}(j+1)P_{2k} + P_{2k-1}\right)} \\
&= \frac{-1}{\left(\frac{1}{2}jP_{2k} + P_{2k-1}\right)\left(\frac{1}{2}(j+1)P_{2k} + P_{2k-1}\right)}
\end{aligned}$$

since $P_{2k}P_{2k+1} - P_{2k-1}P_{2k+2} = -2$ for all $k \geqslant 1$ by Lemma 2.6.6. Inserting this into (2.6.9) gives

$$-\frac{8\left(\frac{1}{2}jP_{2k} + P_{2k-1}\right)}{\frac{1}{2}(j+1)P_{2k} + P_{2k-1}} < j^2 - 4j - 4,$$

since $q^2 = \left(\frac{1}{2}jP_{2k} + P_{2k-1}\right)^2$. This inequality is now true for all $k \geqslant 1, j \geqslant 6$ since the left hand side is negative, and the right hand side is positive. ☐

Definition 2.6.10. A point $a \in [\sigma^2, 6]$ is said to be *regular* if for all $(d, e; m) \in \mathcal{E}$ such that $l(m) = l(a)$, it holds that $\mu(d, e; m)(a) \leqslant \frac{a+1}{4}$.

Proposition 2.6.11. *Assume that the points*

c_{2k-1} *for all* $k \geqslant 1$,
$u_k(j)$ *for all* $k \geqslant 1, j \geqslant 2$,
$v_k(j)$ *for all* $k \geqslant 1, j \geqslant 6$

are regular. Then $c(a) = \frac{a+1}{4}$ *on* $[\sigma^2, 6]$.

Proof. Assume by contradiction that $c(a_0) > \frac{a_0+1}{4}$ for some $a_0 \in [\sigma^2, 6]$. Since for all $a \in\,]\sigma^2, 6]$, we have $c(a) \geqslant \frac{a+1}{4} > \sqrt{\frac{a}{2}}$, the function $c(a)$ is piecewise linear on $]\sigma^2, 6]$ by Corollary 2.4.9. Let $S \subset\,]\sigma^2, 6[$ be the set of non-smooth points of c on $]\sigma^2, 6[$. Decompose this set as $S = S_+ \cup S_-$, where S_+ (respectively S_-) consists of the points $s \in S$ near which c is concave (respectively convex). Since $c(s) > \frac{s+1}{4}$ for all for all $s \in S_+$, the biggest point of S is in S_- because $c(a) = \frac{a+1}{4}$ for $a \in \left[5\frac{12}{13}, 6\right]$ by Proposition 2.6.1. And since $c\left(\sigma^2\right) = \frac{\sigma^2+1}{4}$, it follows that the set S_+ is non-empty. Let $s_0 = \max S_+$. Then $s_0 \in \,]\sigma^2, 6[$. By Corollary 2.4.9 (i) there exists $(d, e; m) \in \mathcal{E}$ and $\varepsilon > 0$ such that

$$c(z) = \mu(d, e; m)(z) \tag{2.6.10}$$

on $[s_0, s_0 + \varepsilon[$. Abbreviate $\mu(z) := \mu(d, e; m)(z)$. Then, $\mu(s_0) = c(s_0) > \frac{s_0+1}{4} > \sqrt{\frac{s_0}{2}}$. Let I be the maximal open interval containing s_0 on which $\mu(z) > \sqrt{\frac{z}{2}}$ for all $z \in I$. By Lemma 2.4.13, there exists a unique $s' \in I$ with $l(m) = l(s')$, and $l(m) < l(z)$ for all other $z \in I$. Moreover, by Proposition 2.4.17, the constraint $\mu(z)$ is given by two linear functions on I:

$$\mu(z) = \begin{cases} \alpha + \beta z & \text{if } z \leqslant s', z \in I, \\ \alpha' + \beta' z & \text{if } z \geqslant s', z \in I. \end{cases}$$

Thus, s' is the only non-smooth point of μ on I. But since $s_0 \in S_+$ and $\mu \leqslant c$, s_0 is also a non-smooth point of μ, and so $s' = s_0$. Now, since c is nondecreasing and by (2.6.10), we see that $\beta' \geqslant 0$.

Let $k \geqslant 1$ be such that $s_0 \in [c_{2k+1}, c_{2k-1}]$. Since c_{2k+1} and c_{2k-1} are regular by assumption, we have $s_0 \in \,]c_{2k+1}, c_{2k-1}[$. Notice that $u_k(j) \to c_{2k+1}$ and $v_k(j) \to c_{2k-1}$ as $j \to \infty$. Let u_-, u_+ be the two points from the sequence

$$c_{2k+1} < \ldots < u_k(2) < u_k(1) = v_k(6) < v_k(7) < \ldots < c_{2k-1}$$

of Corollary 2.6.5 (ii) such that $s_0 \in [u_-, u_+]$. Since u_- and u_+ are regular by assumption, we must have $s_0 \in \,]u_-, u_+[$. But then Lemma 2.6.9 shows that $\mu(s_0) > \frac{s_0+1}{4} > \frac{u_-+1}{4} = \varphi\left(u_-\right) > \psi\left(u_+\right) = \sqrt{\frac{u_+}{2}}$. And since $\beta' \geqslant 0$, we find that $\mu\left(u_+\right) \geqslant \mu\left(s_0\right) > \sqrt{\frac{u_+}{2}}$, and thus $u_+ \in I$, and so $l\left(u_+\right) > l\left(s_0\right)$. But, for all $z \in \,]u_-, u_+[$ we have $l(z) > l\left(u_-\right)$ and $l(z) > l\left(u_+\right)$. In particular, $l\left(s_0\right) > l\left(u_+\right)$, which is a contradiction. □

Lemma 2.6.12. *The points $u_k(j)$ with $k \geqslant 1, j \geqslant 2$ are regular.*

Proof. Abbreviate $u := u_k(j) =: \frac{p}{q}$. Let us first prove that a class of the form $(d+\frac{1}{2}, d-\frac{1}{2}; m)$ with $l(m) = l(u)$ cannot give an obstruction bigger than $\frac{u+1}{4}$. Suppose by contradiction that there is such a class $(d + \frac{1}{2}, d - \frac{1}{2}; m)$. By Proposition 2.6.2 (i) and Corollary 2.6.8 (i),

$$\frac{\sqrt{2}d}{q\sqrt{u}} < \frac{2}{q\sqrt{u^2 - 6u + 1}} = \frac{2}{\sqrt{j^2 + 6j + 1}},$$

which is smaller than 1 for all $j \geqslant 2$. Since $l(m) = l(u)$, $m_i \geqslant 1$ for all i. Thus

$$\sum \varepsilon_i^2 \geqslant j\left(1 - \frac{\sqrt{2}d}{q\sqrt{u}}\right)^2 > j\left(1 - \frac{2}{\sqrt{j^2+6j+1}}\right)^2 =: s(j).$$

Now, since $s(j)$ is increasing for $j \geqslant 2$ and $s(2) > \frac{1}{2}$, we have $\mu(d+\frac{1}{2}, d-\frac{1}{2}; m)(u) \leqslant \sqrt{\frac{u}{2}} < \frac{u+1}{4}$ by Lemma 2.4.8 (iii). The lemma is thus proven for a class of the form $(d+\frac{1}{2}, d-\frac{1}{2}; m)$.

Let us now prove it for a class of the form $(d, d; m)$. Suppose that there exists a class $(d, d; m) \in \mathcal{E}$ with $l(m) = l(u)$ such that $\mu(d, d; m)(u) > \frac{u+1}{4}$. By Proposition 2.6.2 (i) and Corollary 2.6.8 (i),

$$\frac{\sqrt{2}d}{q\sqrt{u}} < \frac{2\sqrt{2}}{q\sqrt{u^2-6u+1}} = \frac{2\sqrt{2}}{\sqrt{j^2+6j+1}},$$

which is smaller than 1 for all $j \geqslant 2$. Since $l(m) = l(u)$, $m_i \geqslant 1$ for all i. Thus

$$\sum \varepsilon_i^2 \geqslant j\left(1 - \frac{\sqrt{2}d}{q\sqrt{u}}\right)^2 > j\left(1 - \frac{2\sqrt{2}}{\sqrt{j^2+6j+1}}\right)^2 =: s(j).$$

Now, since $s(j)$ is increasing for $j \geqslant 2$ and $s(4) > 1$, we have $\mu(d, d; m)(u) \leqslant \sqrt{\frac{u}{2}} < \frac{u+1}{4}$ by Lemma 2.4.8 (iii). So the lemma is proven for $j \geqslant 4$.

It remains to show the lemma for $j = 2, 3$. By Proposition 2.6.2 (ii),

$$\sum \varepsilon_i^2 = 1 - \lambda^2 < 1 - d^2\sqrt{\frac{2}{u}}y(u).$$

Thus

$$\begin{aligned} 0 = \sum \varepsilon_i^2 + \lambda^2 - 1 &> j\left(1 - \frac{\sqrt{2}d}{q\sqrt{u}}\right)^2 + d^2\sqrt{\frac{2}{u}}y(u) - 1 \\ &= \left(\frac{2j}{q^2u} + \sqrt{\frac{2}{u}}y(u)\right)d^2 - \frac{2\sqrt{2}j}{q\sqrt{u}}d + j - 1 =: f(d). \end{aligned}$$

To obtain a contradiction, we need to show that $f(d) \geqslant 0$ for all $d \geqslant 1$. Since $y(u) > 0$ for $u > \sigma^2$, it is sufficient to show that the discriminant of $f(d)$ is negative, that is:

$$\frac{8j^2}{q^2u} - 4(j-1)\left(\frac{2j}{q^2u} + \sqrt{\frac{2}{u}}y(u)\right) \leqslant 0,$$

which is equivalent to

$$\frac{j}{q^2} + 2(j-1)u \leqslant (j-1)\sqrt{\frac{u}{2}}(u+1).$$

Taking squares and using $u = \frac{p}{q}$, we get

$$0 \leqslant -2j^2 + pq\left((j-1)^2p^2 - 6(j-1)^2pq + (j-1)^2q^2 - 8j(j-1)\right). \tag{2.6.11}$$

For $j = 2$, this gives

$$0 \leqslant -8 + pq\left(p^2 - 6pq + q^2 - 16\right),$$

and this inequality is true since $pq \geqslant 26$ and $p^2 - 6pq + q^2 - 16 = 1$ by Corollary 2.6.8 (iii). For $j = 3$, we get

$$0 \leqslant -9 + pq\left(2p^2 - 12pq + 2q^2 - 24\right),$$

and this inequality is also true since $pq \geqslant 57$ and $2p^2 - 12pq + 2q^2 - 24 = 32$ by Corollary 2.6.8 (iii). This concludes the proof. □

Lemma 2.6.13. *The points $v_k(j)$ with $k \geqslant 1, j \geqslant 6$ are regular.*

Proof. Abbreviate $v := v_k(j) =: \frac{p}{q}$. Let us first prove that a class of the form $(d+\frac{1}{2}, d-\frac{1}{2}; m)$ with $l(m) = l(v)$ cannot give an obstruction bigger than $\frac{v+1}{4}$. Suppose by contradiction that there is such a class $(d+\frac{1}{2}, d-\frac{1}{2}; m)$. By Proposition 2.6.2 (i) and Corollary 2.6.8 (ii),

$$\frac{\sqrt{2}d}{q\sqrt{u}} < \frac{2}{q\sqrt{v^2 - 6v + 1}} = \frac{2}{\sqrt{j^2 - 4j - 4}},$$

which is smaller than 1 for all $j \geqslant 6$. Since $l(m) = l(v)$, $m_i \geqslant 1$ for all i. Thus

$$\sum \varepsilon_i^2 \geqslant j\left(1 - \frac{\sqrt{2}d}{q\sqrt{u}}\right)^2 > j\left(1 - \frac{2}{\sqrt{j^2 - 4j - 4}}\right)^2 =: s(j).$$

Now, since $s(j)$ is increasing for $j \geqslant 6$ and $s(6) > \frac{1}{2}$, we have $\mu(d + \frac{1}{2}, d - \frac{1}{2}; m)(v) \leqslant \sqrt{\frac{v}{2}} < \frac{v+1}{4}$ by Lemma 2.4.8 (iii). The lemma is thus proven for a class of the form $(d + \frac{1}{2}, d - \frac{1}{2}; m)$.

Let us now prove the lemma for a class of the form $(d, d; m)$. Suppose that there exists a class $(d, d; m) \in \mathcal{E}$ with $l(m) = l(v)$ such that $\mu(d, d; m)(v) > \frac{v+1}{4}$. By Proposition 2.6.2 (i) and Corollary 2.6.8 (ii),

$$\frac{\sqrt{2}d}{q\sqrt{u}} < \frac{2\sqrt{2}}{q\sqrt{v^2 - 6v + 1}} = \frac{2\sqrt{2}}{\sqrt{j^2 - 4j - 4}},$$

which is smaller than 1 for all $j \geqslant 6$. Since $l(m) = l(u)$, $m_i \geqslant 1$ for all i. Thus

$$\sum \varepsilon_i^2 \geqslant j\left(1 - \frac{\sqrt{2}d}{q\sqrt{u}}\right)^2 > j\left(1 - \frac{2\sqrt{2}}{\sqrt{j^2 - 4j - 4}}\right)^2 =: s(j).$$

Now, since $s(j)$ is increasing for $j \geqslant 6$ and $s(8) > 1$, we have $\mu(d,d;m)(v) \leqslant \sqrt{\frac{v}{2}} < \frac{v+1}{4}$ by Lemma 2.4.8 (iii). So the lemma is proven for $j \geqslant 8$.

It remains to show it for $j = 6, 7$. If $j = 6$, the same arguments as in the proof of Lemma 2.6.12 in the cases $j = 2, 3$ show that a point $v_k(6) =: \frac{p}{q}$ is regular if (2.6.11) is satisfied for $j = 6$, that is if and only if

$$0 \leqslant -72 + pq\left(25p^2 - 150pq + 25q^2 - 160\right).$$

This inequality is true since $pq \geqslant 287$ and $25p^2 - 150pq + 25q^2 - 160 = 40$ by Corollary 2.6.8 (iv). Similarly, a point $v_k(7) =: \frac{p}{q}$ is regular if (2.6.11) is satisfied for $j = 7$, that is if and only if

$$0 \leqslant -98 + pq\left(36p^2 - 216pq + 36q^2 - 336\right),$$

which is true since $pq \geqslant 376$ and $36p^2 - 216pq + 36q^2 - 336 = 276$ by Corollary 2.6.8 (iv). This completes the proof. □

Definition 2.6.14. Define for $k \geqslant 1$ and $i \geqslant 0$ the points

$$b_k(i) := v_k(2 + 2i) = \left[5; \{1, 4\}^{\times(k-1)}, 1, 2 + 2i\right].$$

In particular, for all $k \geqslant 1$, $u_k(1) = v_k(6) = b_k(2)$. Let $\frac{p}{q} := b_k(i)$ written in lowest terms. We will now associate to every $b_k(i)$ a class $E\left(b_k(i)\right) \in \mathcal{E}$ for which we will prove that it gives the constraint at $b_k(i)$. We distinguish the cases i even and i odd.

If $i = 2j$, set

$$m_k(2j) := qw\left(b_k(2j)\right)$$

but with the last block $\left(1^{\times(4j+2)}\right)$ being replaced by the block

$$\left(j + 1, j, 1^{\times(2j+1)}\right),$$

and $d_k(2j) := q\frac{1+b_k(2j)}{4} = \frac{1}{4}(p + q)$. Then define the class

$$E\left(b_k(2j)\right) := \left(d_k(2j), d_k(2j); m_k(2j)\right).$$

If $i = 2j + 1$, set

$$m_k(2j + 1) := qw\left(b_k(2j + 1)\right)$$

but with the last block $\left(1^{\times(4j+4)}\right)$ being replaced by the block

$$\left((j + 1)^{\times 2}, 1^{\times(2j+2)}\right),$$

and $d_k(2j + 1) := q\frac{1+b_k(2j+1)}{4} = \frac{1}{4}(p + q)$. Then define the class

$$E\left(b_k(2j + 1)\right) := \left(d_k(2j + 1) + \frac{1}{2}, d_k(2j + 1) - \frac{1}{2}; m_k(2j + 1)\right).$$

We will now prove that the classes $E(b_k(i))$ belong to $\mathcal{E}$.

Lemma 2.6.15. *For all $k \geqslant 1, i \geqslant 0$ the classes $E(b_k(i))$ satisfy the Diophantine conditions of Proposition 2.3.8 (i).*

Proof. We first treat the case $i = 2j$. Abbreviate $b := b_k(2j) =: \frac{p}{q}$, $m := m_k(2j)$, $w := w(b_k(2j))$ and $d := d_k(2j)$. By Lemma 2.2.5 (iii),

$$\sum_l m_l = q \sum_l w_l = q\left(b + 1 - \frac{1}{q}\right) = 4\left(q\frac{1+b}{4}\right) - 1 = 4d - 1,$$

which proves that the first equation holds. For the second equation, we have

$$\sum_l m_l^2 = q^2\left(\sum_l w_l^2\right) - 2j - 1 + (j+1)^2 + j^2 = q^2 b + 2j^2 = pq + 2j^2.$$

On the other hand,

$$2d^2 + 1 = 2\left(q^2\frac{(1+b)^2}{16}\right) + 1 = \frac{1}{8}(p+q)^2 + 1.$$

By Lemma 2.6.6, it suffices that this equals $pq + 2j^2$ for three small values of j, which is the case.

Similarly in the case $i = 2j+1$, abbreviate $b := b_k(2j+1) =: \frac{p}{q}$, $m := m_k(2j+1)$, $w := w(b_k(2j+1))$ and $d := d_k(2j+1)$. We then have

$$\sum_l m_l = q \sum_l w_l = q\left(b + 1 - \frac{1}{q}\right) = 4\left(q\frac{1+b}{4}\right) - 1 = 4d - 1.$$

For the second equation we have

$$\sum_l m_l^2 = q^2\left(\sum_l w_l^2\right) - 2j - 2 + 2(j+1)^2 = q^2 b + 2j^2 + 2j = pq + 2j^2 + 2j.$$

On the other hand,

$$2d^2 + \frac{1}{2} = 2\left(q^2\frac{(1+b)^2}{16}\right) + \frac{1}{2} = \frac{1}{8}(p+q)^2 + \frac{1}{2}.$$

Now use again Lemma 2.6.6 to check that this equals $pq + 2j^2 + 2j$ for all $j \geqslant 0$. □

Lemma 2.6.16. *The classes* $E(b_k(2j))$ *reduce to* $(0;-1)$ *for all* $k \geqslant 1, j \geqslant 0$.

Proof. The proof is by induction over k. For the initial step $k=1$, we have $b_1(2j) = \frac{24j+17}{4j+3} =: \frac{p}{q}$, $qw(b_1(2j)) = \left((4j+3)^{\times 5}, 4j+2, 1^{\times(4j+2)}\right)$, and $d_1(2j) = 7j+5$. Thus

$$E(b_1(2j)) = \left(7j+5, 7j+5; (4j+3)^{\times 5}, 4j+2, j+1, j, 1^{\times(2j+1)}\right).$$

We now show that this class reduces to $(0;-1)$.

$$\left(7j+5, 7j+5; (4j+3)^{\times 5}, 4j+2, j+1, j, 1^{\times(2j+1)}\right) \xmapsto{\varphi_*}$$
$$\left(10j+7; (4j+3)^{\times 4}, 4j+2, (3j+2)^{\times 2}, j+1, j, 1^{\times(2j+1)}\right);$$
$$\left(8j+5; 4j+3, 4j+2, (3j+2)^{\times 2}, (2j+1)^{\times 3}, j+1, j, 1^{\times(2j+1)}\right);$$
$$\left(5j+3; 3j+2, (2j+1)^{\times 3}, (j+1)^{\times 2}, j^{\times 2}, 1^{\times(2j+1)}\right);$$
$$\left(3j+2; 2j+1, (j+1)^{\times 3}, j^{\times 2}, 1^{\times(2j+1)}\right);$$
$$\left(2j+1; j+1, j^{\times 3}, 1^{\times(2j+1)}\right);$$
$$\left(j+1; j, 1^{\times(2j+2)}\right).$$

This class reduces to $(0;-1)$ in $j+1$ steps since a class of the type $\left(s+1; s, 1^{\times t}\right)$ for $s \geqslant 1, t \geqslant 2$ reduces to $\left(s; s-1, 1^{\times(t-2)}\right)$ by a standard Cremona move.

We turn now to the general case. We will freely use the definitions of the Pell numbers P_n and the Half companion Pell numbers H_n given in Definition 2.6.3 and the fact that for all $n \geqslant 0$, $H_n = P_n + P_{n-1}$. Suppose that the class $E(b_{k-1}(2j))$ reduces to $(0;-1)$ and let us show that the class $E(b_k(2j))$ also reduces to $(0;-1)$. We have

$$b_k(2j) = \frac{2jP_{2k+2} + H_{2k+2}}{2jP_{2k} + H_{2k}},$$
$$d_k(2j) = jH_{2k+1} + P_{2k+1}.$$

The first terms of the class $E(b_k(2j))$ are

$$\Big(jH_{2k+1} + P_{2k+1}, jH_{2k+1} + P_{2k+1}; (2jP_{2k} + H_{2k})^{\times 5}, 4jP_{2k-1} + H_{2k-1},$$
$$(2jP_{2k-2} + H_{2k-2})^{\times 4}, 4jP_{2k-3} + 2H_{2k-3}, (*)\Big),$$

where $(*)$ stands for all the next terms.

The image of $E\left(b_k(2j)\right)$ under φ_* is

$$\Big(2jP_{2k+1}+H_{2k+1};\left(2jP_{2k}+H_{2k}\right)^{\times 4},4jP_{2k-1}+2H_{2k-1},\left(jH_{2k}+P_{2k}\right)^{\times 2}\\ \left(2jP_{2k-2}+H_{2k-2}\right)^{\times 4},4jP_{2k-3}+2H_{2k-3},(*)\Big).$$

To finish the proof, we will show that $\varphi_*\left(E\left(b_k(2j)\right)\right)$ reduces to the class $\varphi_*\left(E\left(b_{k-1}(2j)\right)\right)$ in four steps.

$$\Big(2jP_{2k+1}+H_{2k+1};\left(2jP_{2k}+H_{2k}\right)^{\times 4},4jP_{2k-1}+2H_{2k-1},\\ \left(jH_{2k}+P_{2k}\right)^{\times 2},\left(2jP_{2k-2}+H_{2k-2}\right)^{\times 4},4jP_{2k-3}+2H_{2k-3},(*)\Big);$$

$$(2j\left(H_{2k}+P_{2k-1}\right)+3H_{2k-1}+2P_{2k-1};2jP_{2k}+H_{2k},4jP_{2k-1}+2H_{2k-1},\\ \left(jH_{2k}+P_{2k}\right)^{\times 2},\left(2jP_{2k-1}+H_{2k-1}\right)^{\times 3},\left(2jP_{2k-2}+H_{2k-2}\right)^{\times 4},\\ 4jP_{2k-3}+2H_{2k-3},(*));$$

$$\Big(j\left(3H_{2k}-2P_{2k}\right)+2H_{2k-1}+P_{2k-1};jH_{2k}+P_{2k},\left(2jP_{2k-1}+H_{2k-1}\right)^{\times 3},\\ jH_{2k-1}+P_{2k-1},\left(2jP_{2k-2}+H_{2k-2}\right)^{\times 4},4jP_{2k-3}+2H_{2k-3},\\ jH_{2k-2}+P_{2k-2},(*));$$

$$\Big(jH_{2k}+P_{2k};2jP_{2k-1}+H_{2k-1},\left(jH_{2k-1}+P_{2k-1}\right)^{\times 2},\\ \left(2jP_{2k-2}+H_{2k-2}\right)^{\times 4},4jP_{2k-3}+2H_{2k-3},jH_{2k-2}+P_{2k-2},(*)\Big);$$

$$\Big(2jP_{2k-1}+H_{2k-1};\left(2jP_{2k-2}+H_{2k-2}\right)^{\times 4},4jP_{2k-3}+2H_{2k-3},\\ \left(jH_{2k-2}+P_{2k-2}\right)^{\times 2},(*)\Big).$$

It is important to note that $(*)$ was left invariant during the whole reduction process. So the last class is precisely $\varphi_*\left(E\left(b_{k-1}(2j)\right)\right)$. □

Lemma 2.6.17. *The classes* $E\left(b_k(2j+1)\right)$ *reduce to* $(0;-1)$ *for all* $k\geqslant 1, j\geqslant 0$.

Proof. The proof is again by induction over k. For $k=1$, we have that $b_1(2j+1)=\frac{24j+29}{4j+5}=:\frac{p}{q}$, $qw\left(b_1(2j+1)\right)=\Big((4j+5)^{\times 5},4j+4,1^{\times(4j+4)}\Big)$, $d_1(2j+1)=7j+\frac{17}{2}$. Thus

$$E\left(b_1(2j+1)\right)=\Big(7j+9,7j+8;(4j+5)^{\times 5},4j+4,(j+1)^2,1^{\times(2j+2)}\Big).$$

We show now that this class reduces to $(0;-1)$.

$$\Big(7j+9,7j+8;(4j+5)^{\times 5},4j+4,(j+1)^2,1^{\times(2j+2)}\Big)\overset{\varphi_*}{\longmapsto}$$

$$\Big(10j+2;(4j+5)^{\times 4},4j+4,3j+4,3j+3,(j+1)^2,1^{\times(2j+2)}\Big);$$

$$\Big(8j+9;4j+5,4j+4,3j+4,3j+3,(2j+2)^{\times 3},(j+1)^2,1^{\times(2j+2)}\Big);$$

$$\Big(5j+5;3j+3,(2j+2)^{\times 3},(j+1)^{\times 3},j,1^{\times(2j+2)}\Big);$$

$$\left(3j+3; 2j+2, (j+1)^{\times 4}, j, 1^{\times(2j+2)}\right);$$
$$\left(2j+2; (j+1)^{\times 3}, j, 1^{\times(2j+2)}\right);$$
$$\left(j+1; j, 1^{\times(2j+2)}\right).$$

As seen before, this class reduces to $(0;-1)$ in $j+1$ steps.

We turn now to the general case. Suppose that the class $E\left(b_{k-1}(2j+1)\right)$ reduces to $(0;-1)$ and let us show that the class $E\left(b_k(2j+1)\right)$ also reduces to $(0;-1)$. We have

$$b_k(2j+1) = \frac{2jP_{2k+2}+P_{2k+3}}{2jP_{2k}+P_{2k+1}},$$

$$d_k(2j+1) = jH_{2k+1} + \frac{1}{2}H_{2k+2}.$$

The first terms of the class $E\left(b_k(2j+1)\right)$ are

$$\Big(jH_{2k+1}+\tfrac{1}{2}H_{2k+2}+\tfrac{1}{2}, jH_{2k+1}+\tfrac{1}{2}H_{2k+2}-\tfrac{1}{2}; \left(2jP_{2k}+P_{2k+1}\right)^{\times 5},$$
$$4jP_{2k-1}+2P_{2k}, \left(2jP_{2k-2}+P_{2k-1}\right)^{\times 4}, 4jP_{2k-3}+2P_{2k-2}, (*)\Big).$$

The image of $E\left(b_k(2j+1)\right)$ under φ_* is

$$\Big(2jP_{2k+1}+P_{2k+2}; \left(2jP_{2k}+P_{2k+1}\right)^{\times 4}, 4jP_{2k-1}+2P_{2k},$$
$$jH_{2k}+\tfrac{1}{2}H_{2k+1}+\tfrac{1}{2}, jH_{2k}+\tfrac{1}{2}H_{2k+1}-\tfrac{1}{2}, \left(2jP_{2k-2}+P_{2k-1}\right)^{\times 4},$$
$$4jP_{2k-3}+2P_{2k-2}, (*)),$$

To finish the proof, we will show that $\varphi_*\left(E\left(b_k(2j+1)\right)\right)$ reduces to the vector $\varphi_*\left(E\left(b_{k-1}(2j+1)\right)\right)$ in four steps.

$$\Big(2jP_{2k+1}+P_{2k+2}; \left(2jP_{2k}+P_{2k+1}\right)^{\times 4}, 4jP_{2k-1}+2P_{2k},$$
$$jH_{2k}+\tfrac{1}{2}H_{2k+1}+\tfrac{1}{2}, jH_{2k}+\tfrac{1}{2}H_{2k+1}-\tfrac{1}{2}, \left(2jP_{2k-2}+P_{2k-1}\right)^{\times 4},$$
$$4jP_{2k-3}+2P_{2k-2}, (*));$$

$$(j\left(2P_{2k}+4P_{2k-1}\right)+\left(P_{2k+1}+2P_{2k}\right); 2jP_{2k}+P_{2k+1}, 4jP_{2k-1}+2P_{2k},$$
$$jH_{2k}+\tfrac{1}{2}H_{2k+1}+\tfrac{1}{2}, jH_{2k}+\tfrac{1}{2}H_{2k+1}-\tfrac{1}{2}, \left(2jP_{2k-1}+P_{2k}\right)^{\times 3},$$
$$\left(2jP_{2k-2}+P_{2k-1}\right)^{\times 4}, 4jP_{2k-3}+2P_{2k-2}, (*)\Big);$$

$$\Big(j\left(P_{2k}+3P_{2k-1}\right)+\Big(\tfrac{1}{2}P_{2k+1}+\tfrac{3}{2}P_{2k}-\tfrac{1}{2}\Big); jH_{2k}+\tfrac{1}{2}H_{2k+1}-\tfrac{1}{2},$$
$$\left(2jP_{2k-1}+P_{2k}\right)^{\times 3}, jH_{2k-1}+\tfrac{1}{2}H_{2k}-\tfrac{1}{2}, \left(2jP_{2k-2}+P_{2k-1}\right)^{\times 4},$$
$$4jP_{2k-3}+2P_{2k-2}, jH_{2k-2}+\tfrac{1}{2}H_{2k-1}-\tfrac{1}{2}, (*)\Big);$$

$$\Big(jH_{2k}+\tfrac{1}{2}H_{2k+1}-\tfrac{1}{2}; 2jP_{2k-1}+P_{2k}, \Big(jH_{2k-1}+\tfrac{1}{2}H_{2k}-\tfrac{1}{2}\Big)^{\times 2},$$
$$\left(2jP_{2k-2}+P_{2k-1}\right)^{\times 4}, 4jP_{2k-3}+2P_{2k-2},$$
$$jH_{2k-2}+\tfrac{1}{2}H_{2k-1}-\tfrac{1}{2}, (*)\Big);$$

$$\Big(2jP_{2k-1} - P_{2k}; (2jP_{2k-2} + P_{2k-1})^{\times 4}, 4jP_{2k-3} + 2P_{2k-2},$$
$$jH_{2k-2} + \tfrac{1}{2}H_{2k-1} + \tfrac{1}{2}, jH_{2k-2} + \tfrac{1}{2}H_{2k-1} - \tfrac{1}{2}, (*)\Big).$$

Since $(*)$ was left invariant during the whole reduction process, the last class is precisely $\varphi_* \left(E\left(b_{k-1}(2j+1)\right)\right)$. ☐

Proposition 2.6.18. *For all $k \geqslant 1, i \geqslant 0$, we have $E\left(b_k(i)\right) \in \mathcal{E}$.*

Proof. We have to show that the classes $E\left(b_k(i)\right)$ satisfy the Diophantine conditions of Proposition 2.3.8, which we have done in Lemma 2.6.15, and that they reduce to $(0; -1)$ by Cremona moves, which we have done for i even in Lemma 2.6.16 and for i odd in Lemma 2.6.17. The proof is thus complete. ☐

Corollary 2.6.19. *For all $n \geqslant 0$, the classes $E\left(\beta_n\right)$ of Theorem 2.5.1 belong to $\mathcal{E}$.*

Proof. Notice that by Lemma 2.6.4, for all $k \geqslant 0$,

$$b_k(0) = v_k(2) = \frac{P_{2k+2} + P_{2k+1}}{P_{2k} + P_{2k-1}} = \frac{H_{2k+2}}{H_{2k}} = \beta_{2k},$$
$$b_k(1) = v_k(4) = \frac{2P_{2k+2} + P_{2k+1}}{2P_{2k} + P_{2k-1}} = \frac{P_{2k+3}}{P_{2k+1}} = \beta_{2k+1}.$$

Hence by Definition 2.6.14, we see that for all $k \geqslant 0$, $E\left(b_k(0)\right) = E\left(\beta_{2k}\right)$ and $E\left(b_k(1)\right) = E\left(\beta_{2k+1}\right)$. Thus all the classes $E\left(\beta_n\right)$ belong indeed to $\mathcal{E}$. ☐

Corollary 2.6.20. $c\left(b_k(i)\right) = \frac{b_k(i)+1}{4}$ *for all $k \geqslant 1$ and $i \geqslant 2$.*

Proof. Since $b_k(i) \in \left]\sigma^2, 6\right[$, we can write them as $b_k(i) = 5 + x$ where $x \in [0, 1[$. Now, $\left(2, 2; 2, 1^{\times 5}\right) \in \mathcal{E}$, thus

$$\mu\left(2, 2; 2, 1^{\times 5}\right)\left(b_k(i)\right) = \frac{6 + x}{4} = \frac{b_k(i) + 1}{4}.$$

So $c\left(b_k(i)\right) \geqslant \mu\left(2, 2; 2, 1^{\times 5}\right)\left(b_k(i)\right) = \frac{b_k(i)+1}{4}$.

Let us show the converse inequality. Abbreviate $b := b_k(i) =: \frac{p}{q}$ in lowest terms, $d := d_k(i)$ and $m := m_k(i)$. Then

$$\mu\left(E(b)\right)(b) = \frac{\langle m, w(b)\rangle}{2d} = \frac{\langle qw(b), w(b)\rangle}{2d} = \frac{2b}{1+b} < \sqrt{\frac{b}{2}} < \frac{b+1}{4}$$

since $b > \sigma^2$. Now if $(d', e'; m') \in \mathcal{E}$ is a class different from $E(b)$, we have by Proposition 2.3.8 (ii) that $\langle m, m'\rangle \leqslant d(d' + e')$. Using the definitions of d and m and the fact that m is written in decreasing order, we get

$$q\frac{1+b}{4}(d' + e') \geqslant q\left\langle w(b), m'\right\rangle,$$

thus

$$\mu(d', e'; m')(b) = \frac{\langle m', w(b) \rangle}{d' + e'} \leqslant \frac{b+1}{4}.$$

The proof is complete. ☐

Theorem 2.6.21. *$c(a) = \frac{a+1}{4}$ on $[\sigma^2, 6]$.*

Proof. By Proposition 2.6.11 it suffices to show that the points c_{2k-1} for all $k \geqslant 1$, $u_k(j)$ for all $k \geqslant 1, j \geqslant 2$, and $v_k(j)$ for all $k \geqslant 1, j \geqslant 6$ are regular. By Lemma 2.6.12 and Lemma 2.6.13, the points $u_k(j)$ and $v_k(j)$ are regular. Moreover, for all $k \geqslant 1$, $v_k(j) \underset{j \to \infty}{\longrightarrow} c_{2k-1}$. But by Corollary 2.6.20, $c\left(v_k(2+2i)\right) = c\left(b_k(i)\right) = \frac{b_k(i)+1}{4}$ for $i \geqslant 2$. So, by continuity of c, we get that $c\left(c_{2k-1}\right) = \frac{c_{2k-1}+1}{4}$, and the points c_{2k-1} are thus regular. This completes the proof. ☐

2.7 The interval $[6, 8]$

2.7.1 Preliminaries

We will use the fact that if $[l_0; l_1, ..., l_N]$ is a continued fraction of a rational number $\frac{p}{q}$ and $\frac{p_k}{q_k} := [l_0; l_1, ..., l_k]$ is its k-th convergent written in lowest terms, then for any real number x,

$$[l_0; l_1, ..., l_k, x] = \frac{x p_k + p_{k-1}}{x q_k + q_{k-1}},$$

written in lowest terms. In particular, $q_k = l_k q_{k-1} + q_{k-2}$. It is then easy to see that if $L := \sum_i l_i$, then

$$q = q_N \geqslant L. \tag{2.7.1}$$

Recall also that we defined the *error vector* of a class $(d, e; m)$ at a point a as the vector $\varepsilon := \varepsilon\left((d, e; m), a\right)$ defined by the equation

$$m = \frac{d+e}{\sqrt{2a}} w(a) + \varepsilon.$$

Now set then $M := l(a) = L + l_0$,

$$\sigma := \sum_{i > l_0} \varepsilon_i^2,$$

and

$$\sigma' := \sum_{i=l_0+1}^{M-l_N} \varepsilon_i^2 < \sigma.$$

Then by Lemma 2.4.8.3, $\sigma < 1$ for a class of the form $(d, d; m)$ and $\sigma < \frac{1}{2}$ for a class of the form $(d + \frac{1}{2}, d - \frac{1}{2}; m)$.

Lemma 2.7.1. *Let* $(d, e; m) \in \mathcal{E}$ *be a class such that there exists* $a =: \frac{p}{q} \in \,]\sigma^2, 8[$ *with* $l(a) = l(m)$ *and*

$$\mu(d, e; m)(a) > \sqrt{\frac{a}{2}}.$$

Assume that $y(a) := a + 1 - 2\sqrt{2a} > \frac{1}{q}$, *and set* $v_M := \frac{d+e}{q\sqrt{2a}}$. *Then*

(i) $|\sum \varepsilon_i| \leqslant \sqrt{\sigma L}$,

(ii) If $v_M < 1$, *then* $|\sum \varepsilon_i| \leqslant \sqrt{\sigma' L}$,

(iii) If $v_M \leqslant \frac{1}{2}$, *then* $v_M > \frac{1}{3}$ *and* $\sigma' \leqslant \frac{1}{2}$. *If* $v_M \leqslant \frac{3}{4}$, *then* $\sigma' \leqslant \frac{7}{8}$,

(iv) Set $\delta := y(a) - \frac{1}{q} > 0$. *Then for both types of classes* $(d, d; m)$ *and* $(d + \frac{1}{2}, d - \frac{1}{2}; m)$ *we have*

$$d \leqslant \frac{\sqrt{a}}{\sqrt{2}\delta}\left(\sqrt{\sigma L} - 1\right) \leqslant \frac{\sqrt{a}}{\sqrt{2}\delta}\left(\sqrt{\sigma q} - 1\right) < \frac{\sqrt{a}}{\sqrt{2}\delta}\left(\frac{\sigma}{\delta v_M} - 1\right).$$

If $v_M < 1$, σ *can be replaced by* σ'.

Proof. The proofs of (i), (ii) and (iii) are the same as in the proof of Lemma 5.1.2 in [MS]. To prove (iv), we notice first that $\sum \varepsilon_i < 0$. Indeed, by Lemma 2.4.8.4, $-\sum_{i=1}^{M} \varepsilon_i = \frac{d+e}{\sqrt{2a}}\left(y(a) - \frac{1}{q}\right) + 1$. Since $y(a) > \frac{1}{q}$ by assumption, we obtain the desired inequality.
Then, using (2.7.1) and (i), we find

$$\sqrt{\sigma q} \geqslant \sqrt{\sigma L} \geqslant \frac{d+e}{\sqrt{2a}}\left(y(a) - \frac{1}{q}\right) + 1 = \frac{d+e}{\sqrt{2a}}\delta + 1 = \delta q v_M + 1 > \delta q v_M.$$

Thus

$$\sqrt{q} < \frac{\sqrt{\sigma}}{\delta v_M}.$$

For both types of classes $(d, d; m)$ and $(d + \frac{1}{2}, d - \frac{1}{2}; m)$, we get

$$d \leqslant \frac{\sqrt{a}}{\sqrt{2}\delta}\left(\sqrt{\sigma L} - 1\right) \leqslant \frac{\sqrt{a}}{\sqrt{2}\delta}\left(\sqrt{\sigma q} - 1\right) < \frac{\sqrt{a}}{\sqrt{2}\delta}\left(\frac{\sigma}{\delta v_M} - 1\right).$$

If $v_M < 1$, the same arguments go through, when replacing σ by σ'. □

2.7.2 The interval $[6, 7]$

We start by stating a more precise version of part (ii) of Theorem 2.1.3.

Theorem 2.7.2. *On the interval* $\left[\sigma^2, 7\frac{1}{32}\right]$, $c(a) = \sqrt{\frac{a}{2}}$ *except on the seven disjoint intervals* $]u_x, v_x[$ *given in the following table. For each of these intervals, there exist a class* $(d, e; m) \in \mathcal{E}$ *and a rational number* $x \in]u_x, v_x[$ *with* $l(x) = l(m)$ *such that* $c(z) = \mu(d, e; m)(z) = \frac{1}{d+e}(A + Bz)$ *on* $[u_x, x]$, *and* $c(z) = \mu(d, e; m)(z) = \frac{1}{d+e}(A' + B'z)$ *on* $[x, v_x]$. *We list all these informations in the table below as well as the values of* $c(x)$ *and* $\sqrt{\frac{x}{2}}$.

x	$(d,e;m)$	(A,B)	(A',B')	$c(x)$	$c(x) \cong$	$\sqrt{\frac{x}{2}} \cong$
6	$(2,2;2,1^{\times 5})$	$(1,1)$	$(7,0)$	$\frac{7}{4}$	1.75	1.73
$6\frac{1}{7}$	$(28,28;16^{\times 6},3,2^{\times 6})$	$(6,15)$	$(92,1)$	$\frac{687}{392}$	1.752551	1.752549
$6\frac{1}{6}$	$(14,14;8^{\times 6},2,1^{\times 5})$	$(6,7)$	$(43,1)$	$\frac{295}{168}$	1.75595	1.75594
$6\frac{1}{5}$	$(11,10;6^{\times 6},1^{\times 5})$	$(6,5)$	$(37,0)$	$\frac{37}{21}$	1.762	1.761
$6\frac{1}{3}$	$(7,7;4^{\times 6},1^{\times 3})$	$(6,3)$	$(25,0)$	$\frac{25}{14}$	1.79	1.78
$6\frac{1}{2}$	$(9,9;5^{\times 6},3,2)$	$(0,5)$	$(26,1)$	$\frac{65}{36}$	1.81	1.80
7	$(4,4;3,2^{\times 6})$	$(1,2)$	$(15,0)$	$\frac{15}{8}$	1.88	1.87

x	u_x	$u_x \cong$	v_x	$v_x \cong$
6	$\sigma^2 = 3 + 2\sqrt{2}$	5.83	$6\frac{1}{8} = \frac{49}{8}$	6.13
$6\frac{1}{7}$	$\frac{2}{225}\left(347 + 28\sqrt{151}\right)$	6.142842	$4\left(173 - 70\sqrt{6}\right)$	6.142872
$6\frac{1}{6}$	$\frac{2}{7}\left(11 + 4\sqrt{7}\right)$	6.16657	$153 - 14\sqrt{110}$	6.16676
$6\frac{1}{5}$	$\frac{3}{100}\left(107 + 7\sqrt{201}\right)$	6.19	$\frac{2738}{441}$	6.21
$6\frac{1}{3}$	$\frac{1}{9}\left(31 + 7\sqrt{13}\right)$	6.2488	$\frac{625}{98}$	6.38
$6\frac{1}{2}$	$\frac{162}{25}$	6.48	$55 - 9\sqrt{29}$	6.53
7	$\frac{1}{2}\left(7 + 4\sqrt{3}\right)$	6.96	$7\frac{1}{32} = \frac{225}{32}$	7.03

Lemma 2.7.3. *The classes $(d,e;m)\in\mathcal{E}$ such that $\mu(d,e;m)\left(6\frac{1}{k}\right)>\sqrt{\frac{6\frac{1}{k}}{2}}$ and $l(m)=l\left(6\frac{1}{k}\right)$ for some $k=1,\dots,8$ are given in the following table.*

k	$(d,e;m)$
7	$(28,28;16^{\times 6},3,2^{\times 6})$
7	$(196,196;112^{\times 5},111,16^{\times 7})$
6	$(14,14;8^{\times 6},2,1^{\times 5})$
6	$(84,84;48^{\times 5},47,8^{\times 6})$
5	$(11,10;6^{\times 6},1^{\times 5})$
4	$(28,28;16^{\times 5},15,4^{\times 4})$
3	$(7,7;4^{\times 6},1^{\times 3})$
2	$(9,9;5^{\times 6},3,2)$
1	$(4,4;3,2^{\times 6})$

Proof. In the case $k=1$, since $6\frac{1}{1}=7$, we only have to check which elements of the finite set $\mathcal{E}_7$ are obstructive at 7. It turns out that the only obstructive one is $(4,4;3,2^{\times 6})$.

Let us now treat the cases $k=2,\dots,8$. Suppose that there is a class of the form $(d,d;m)$ which is obstructive at some $6\frac{1}{k}$. Since $l\left(6\frac{1}{k}\right)=6+k$, by Lemma 2.4.14 the vector m has to be of one of the five forms

$$\left(a^{\times 6},b^{\times k}\right),\qquad \left(a+1,a^{\times 5},b^{\times k}\right),\qquad \left(a^{\times 5},a-1,b^{\times k}\right),$$
$$\left(a^{\times 6},b+1,b^{\times (k-1)}\right),\quad \left(a^{\times 6},b^{\times (k-1)},b-1\right).$$

Define ε_a and ε_b by

$$a=\frac{\sqrt{2}d}{\sqrt{6\frac{1}{k}}}+\varepsilon_a\quad\text{and}\quad b=\frac{\sqrt{2}d}{k\sqrt{6\frac{1}{k}}}+\varepsilon_b.$$

If $m=\left(a^{\times 6},b^{\times k}\right)$, then $|a-kb|=|\varepsilon_a-k\varepsilon_b|\leqslant|\varepsilon_a|+k\,|\varepsilon_b|$. Since by Lemma 2.4.8 (iii), $\sum\varepsilon_i^2<1$, we find $|\varepsilon_a|+k\,|\varepsilon_b|<\sqrt{k+1}$, and thus $|a-kb|\leqslant\left\lceil\sqrt{k+1}-1\right\rceil$. Hence

$$s:=a-kb\in\begin{cases}\{0,\pm1\} & \text{if } k\in\{2,3\},\\ \{0,\pm1,\pm2\} & \text{if } k\in\{4,\dots,8\}.\end{cases}$$

The Diophantine equations of Proposition 2.3.8 (i) then become

$$\begin{aligned}4d&=\;6a+kb+1,\\ 2d^2&=\;6a^2+kb^2-1.\end{aligned}$$

Thus $(6a+kb+1)^2 = 8\left(6a^2+kb^2-1\right)$. Replacing a by $kb+s$, we can solve this equation in b for the values of k and s given above. We find three solutions to the equation with $b \geqslant 1$, namely when (k,s,b) is equal to $(3,0,3)$, $(3,1,1)$ or $(3,-1,5)$. This leads to the vectors $(16,16;9^{\times 6},3^{\times 3})$, $(7,7;4^{\times 6},1^{\times 3})$ and $(25,25;14^{\times 6},5^{\times 3})$, respectively. Since only $(7,7;4^{\times 6},1^{\times 3})$ reduces to $(0;-1,0,\ldots,0)$ by Cremona moves, this is the only class of the form $\left(d,d;a^{\times 6},b^{\times k}\right)$ potentially obstructive at some $6\frac{1}{k}$, and it indeed is obstructive at $6\frac{1}{3}$.

In the case where $m = \left(a+1,a^{\times 5},b^{\times k}\right)$, $\sigma = k\left|\varepsilon_b\right|^2 \leqslant \frac{1}{6}$. Thus, $|a-kb| \leqslant |\varepsilon_a| + k\left|\varepsilon_b\right| \leqslant 1+\sqrt{\frac{k}{6}}$, and thus

$$s := a-kb \in \begin{cases} \{0,\pm 1\} & \text{if } k \in \{2,\ldots,5\}\,, \\ \{0,\pm 1,\pm 2\} & \text{if } k \in \{6,\ldots,8\}\,. \end{cases}$$

From the Diophantine equations we obtain

$$(6a+kb+2)^2 = 8\left(6a^2+2a+kb^2\right).$$

Replacing a by $kb+s$, we obtain no solutions with $b \geqslant 1$ for the accepted values of k and s.

As in the previous case, when $m = \left(a^{\times 5},a-1,b^{\times k}\right)$, we have

$$s := a-kb \in \begin{cases} \{0,\pm 1\} & \text{if } k \in \{2,\ldots,5\}\,, \\ \{0,\pm 1,\pm 2\} & \text{if } k \in \{6,\ldots,8\}\,. \end{cases}$$

The Diophantine equations become $(6a+kb)^2 = 8\left(6a^2-2a+kb^2\right)$, which yields four solutions with $b \geqslant 1$, namely the tuples (k,s,b) equal to $(2,1,1)$, $(4,0,4)$, $(6,0,8)$ and $(7,0,16)$ which give the vectors $(5,5;3^{\times 5},2,1^{\times 2})$, $(28,28;16^{\times 5},15,4^{\times 4})$, $(84,84;48^{\times 5},47,8^{\times 6})$ and $(196,196;112^{\times 5},111,16^{\times 7})$, respectively. These vectors all reduce to $(0;-1,0,\ldots,0)$ by Cremona moves, but the first one is not obstructive at $6\frac{1}{2}$. So we add only the three last vectors to our table.

For the case $m = \left(a^{\times 6},b+1,b^{\times(k-1)}\right)$ notice that if $\varepsilon \in \mathbb{R}$ and $k \in \mathbb{N}$ are such that $(k-1)\varepsilon^2+(\varepsilon+1)^2 \leqslant 1$, then $\varepsilon \in \left[-\frac{2}{k},0\right]$. Thus

$$|(k-1)\varepsilon+(\varepsilon+1)| = |k\varepsilon+1| \leqslant 1.$$

Since $\sigma \geqslant \frac{k-1}{k}$, we get

$$\begin{aligned} |a-kb-1| &= |a-(b+1)-(k-1)b| = |\varepsilon_a-(\varepsilon_b+1)-(k-1)\varepsilon_b| \\ &\leqslant |\varepsilon_a| + |(k-1)\varepsilon_b+\varepsilon_b+1| \leqslant 1+1 = 2. \end{aligned}$$

Thus

$$s := a-kb-1 \in \{0,\pm 1,\pm 2\}\,.$$

The Diophantine equations become $(6a+kb+2)^2 = 8\left(6a^2+kb^2+2b\right)$, which when we replace a by $kb+s+1$ gives the three tuples of solutions (k,s,b) equal to $(2,0,2)$, $(6,1,6)$ and $(7,1,2)$, which yields the vectors $\left(9,9;5^{\times 6},3,2\right)$, $\left(14,14;8^{\times 6},2,1^{\times 5}\right)$ and, again, $\left(28,28;16^{\times 6},3,2^{\times 6}\right)$, respectively. All three vectors reduce to $(0;-1,0,\ldots,0)$ by Cremona moves, and they are obstructive at $6\frac{1}{k}$ for $k=2,6,7$ respectively.

For the case $m=\left(a^{\times 6},b^{\times(k-1)},b-1\right)$ we find similarly as in the previous case that

$$s := a-kb+1 \in \{0,\pm 1,\pm 2\}\,.$$

The Diophantine equations become $(6a+kb)^2 = 8\left(6a^2+kb^2-2b\right)$, which when we replace a by $kb+s-1$ gives as only solution with $b \geqslant 2$ the tuple $(k,s,b)=(2,0,3)$. This gives again the vector $\left(9,9;5^{\times 6},3,2\right)$.

The last case we have to treat is the case of an obstructive class of the form $(d+\frac{1}{2},d-\frac{1}{2};m)$. By Corollary 2.4.15, the only possibility for m is to be of the form $\left(a^{\times 6},b^{\times k}\right)$. We saw earlier that in this case we have

$$s := a-kb \in \begin{cases} \{0,\pm 1\} & \text{if } k\in\{2,3\}\,,\\ \{0,\pm 1,\pm 2\} & \text{if } k\in\{4,\ldots,8\}\,.\end{cases}$$

Now the Diophantine equations are

$$\begin{aligned} 4d &= 6a+kb+1,\\ 2d^2 &= 6a^2+kb^2-\tfrac{1}{2}.\end{aligned}$$

This leads to the equation $\frac{1}{8}\left(6a+kb+3\right)^2-\frac{1}{2}\left(6a+kb+3\right)+1=6a^2+kb^2$. When replacing a by $kb+s$, we obtain as only solution with $b\geqslant 1$ the tuple $(5,1,1)$ which gives the vector $\left(11,10;6^{\times 6},1^{\times 5}\right)$. This vector reduces to $(0;-1,0,\ldots,0)$ by Cremona moves and is obstructive at $6\frac{1}{5}$. ☐

Lemma 2.7.4. *The classes given in Lemma 2.7.3 are the only obstructive classes on the interval* $\left[6\frac{1}{8},7\right]$.

Proof. We claim that it suffices to prove that for all $a\in\left[6\frac{1}{8},7\right]$, there is no other class $(d,e;m)\in\mathcal{E}$ with $l(m)=l(a)$ that is obstructive at a. Indeed, suppose that $\mu(d,e;m)(a)>\sqrt{\frac{a}{2}}$ for some $a\in\left[6\frac{1}{8},7\right]$, and let I be the maximal nonempty interval containing a on which $\mu(d,e;m)(z)>\sqrt{\frac{z}{2}}$. Then, by Lemma 2.4.13, there exists a unique $a_0\in I$ such that $l\left(a_0\right)=l(m)$ and $l\left(a_0\right)\leqslant l(a)$ for all $a\in I$. Since for $M\leqslant 6$, $\mathcal{E}_M$ is finite, explicit calculations show that none of these classes is obstructive for $a\geqslant 6\frac{1}{8}$. Thus $l(m)>6$, and $a_0>6$. This implies that $a_0\geqslant 6\frac{1}{8}$. Indeed, $a_0<6\frac{1}{8}$ would contradict the fact that $l\left(a_0\right)\leqslant l(a)$ for all $a\in I$ since $l(a)>l\left(6\frac{1}{8}\right)$ for all $a\in\left]6,6\frac{1}{8}\right[$. A similar argument also shows that $a_0\leqslant 7$. Thus, $a_0\in\left[6\frac{1}{8},7\right]$ and this proves the claim.

We will thus prove that for each $a = 6\frac{p}{q} \in \left[6\frac{1}{8}, 7\right]$ there is no class $(d, e; m) \in \mathcal{E}$ with $l(m) = l(a)$ obstructive at a, and different from those given in Lemma 2.7.3. By Lemma 2.7.3, we only have to prove for $\frac{p}{q} \neq \frac{1}{k}$ with $k = 1, \dots, 8$. We will separate the proof in three cases: $3 \leqslant q \leqslant 8$, $9 \leqslant q \leqslant 39$, $q \geqslant 40$.

Case 1: $3 \leqslant q \leqslant 8$: In this case, $2 \leqslant p \leqslant q$. Notice that for all these values of p and q, $y\left(6\frac{p}{q}\right) > \frac{1}{q}$. We can thus apply Lemma 2.7.1 (iv). We get that if $(d, e; m) \in \mathcal{E}$ is obstructive at $6\frac{p}{q}$, then

$$d \leqslant \frac{\sqrt{6\frac{p}{q}}}{\sqrt{2}\left(y\left(6\frac{p}{q}\right) - \frac{1}{q}\right)}\left(\sqrt{q} - 1\right) \tag{2.7.2}$$

since $\sigma < 1$ for an obstructive class. We now use the computer program `SolLess[a,D]` given in the Appendix which computes for a rational number a and a natural number D all obstructive classes $(d, e; m)$ at a with $l(m) = l(a)$ and $d \leqslant D$. The code shows that there are no such classes for $3 \leqslant q \leqslant 8$.

Case 2: $9 \leqslant q \leqslant 39$: Since $y\left(6\frac{1}{8}\right) = \frac{1}{8}$ and y is increasing for $a > 2$, we have

$$y(a) - \frac{1}{q} \geqslant \frac{1}{8} - \frac{1}{9} > 0$$

for all $a \in \left[6\frac{1}{8}, 7\right]$. We can thus again apply Lemma 2.7.1 (iv) and obtain again (2.7.2), but this time for $1 \leqslant p \leqslant q$. Again, the code `SolLess[a,D]` shows that for $9 \leqslant q \leqslant 39$ there are no obstructive classes $(d, e; m)$ at $a = 6\frac{p}{q}$ with $l(m) = l(a)$.

Case 3: $q \geqslant 40$: For all $a = 6\frac{p}{q} \in \left[6\frac{1}{8}, 7\right]$, we have $\delta := y(a) - \frac{1}{q} \geqslant \frac{1}{8} - \frac{1}{40} = \frac{1}{10}$. Suppose that $(d, e; m) \in \mathcal{E}$ is obstructive at some $a = 6\frac{p}{q}$ with $q \geqslant 40$. We distinguish two cases: (i) $m_1 = m_6$, (ii) $m_1 \neq m_6$.

(i) Notice that by Lemma 2.7.1 (iii),

$$\begin{aligned}
&\text{if } v_M \in \left[\tfrac{1}{3}, \tfrac{1}{2}\right], && \text{then } \tfrac{\sigma'}{v_M} \leqslant \tfrac{1/2}{1/3} = \tfrac{3}{2},\\
&\text{if } v_M \in \left[\tfrac{1}{2}, \tfrac{2}{3}\right], && \text{then } \tfrac{\sigma'}{v_M} \leqslant \tfrac{7/8}{1/2} = \tfrac{7}{4},\\
&\text{if } v_M \geqslant \tfrac{2}{3}, && \text{then } \tfrac{\sigma}{v_M} \leqslant \tfrac{3}{2}.
\end{aligned}$$

By Lemma 2.7.1 (iv) we get that if $a = 6\frac{p}{q} \in \left]6\frac{1}{k+1}, 6\frac{1}{k}\right[$ for some $k = 1, \dots, 7$ and $q \geqslant 40$, then for all obstructive classes $(d, e; m)$ at a with $m_1 = m_6$

$$d \leqslant \frac{\sqrt{6\frac{1}{k}}}{\sqrt{2}\left(y\left(6\frac{1}{k+1}\right) - \frac{1}{40}\right)}\left(\frac{1}{y\left(6\frac{1}{k+1}\right) - \frac{1}{40}}\frac{7}{4} - 1\right).$$

Here we used the computer program $\texttt{InterSolLess1}[\texttt{k},\texttt{D}]$ given in the Appendix which gives for $k \in \{1,\dots,7\}$ and a natural number D a finite list of classes $(d,e;m)$ with $m_1 = m_6$ and $d \leqslant D$ which can potentially be obstructive at some $a = 6\frac{p}{q} \in \left]6\frac{1}{k+1}, 6\frac{1}{k}\right[$ with $q \geqslant 40$. Applied to our case, the code gives only one class that reduces to $(0;-1,0,\dots,0)$ by Cremona moves, namely $(d,e;m) = (99,99;56^{\times 6},14^{\times 4},1^{\times 3})$. By Lemma 2.4.14, the a in question can be $[6;3,1,3] = 6\frac{4}{15}$ or $[6,3,1,1,2] = 6\frac{5}{18}$, and the class turns out to give no obstruction at these two points.

(ii) Since $m_1 \neq m_6$, we know by Lemma 2.4.14 that $\sigma \leqslant \frac{1}{6}$. This implies that $v_M \geqslant 1 - \frac{1}{2\sqrt{3}}$ because the last two weights of $w\left(\frac{p}{q}\right)$ are always $\frac{1}{q}$. Then by Lemma 2.7.1 (iv) we get that if $a = 6\frac{p}{q} \in \left]6\frac{1}{k+1}, 6\frac{1}{k}\right[$ for some $k = 1,\dots,7$ and $q \geqslant 40$, then for all obstructive classes $(d,e;m)$ at a with $m_1 \neq m_6$ we have

$$d \leqslant \frac{\sqrt{6\frac{1}{k}}}{\sqrt{2}\left(y\left(6\frac{1}{k+1}\right) - \frac{1}{40}\right)} \left(\frac{1}{y\left(6\frac{1}{k+1}\right) - \frac{1}{40}} \frac{\frac{1}{6}}{\left(1 - \frac{1}{2\sqrt{3}}\right)} - 1\right).$$

Here we used the computer program $\texttt{InterSolLess2}[\texttt{k},\texttt{D}]$ which gives for $k \in \{1,\dots,7\}$ and a natural number D a finite list of classes $(d,e;m)$ with $m_1 \neq m_6$ and $d \leqslant D$ which can potentially be obstructive at some $a = 6\frac{p}{q} \in \left]6\frac{1}{k+1}, 6\frac{1}{k}\right[$ with $q \geqslant 40$. Applied to our case, the code gives no class that reduces to $(0;-1,0,\dots,0)$ by Cremona moves. □

Remark 2.7.5. The three programs $\texttt{SolLess}[\texttt{a},\texttt{D}]$, $\texttt{InterSolLess1}[\texttt{k},\texttt{D}]$ and $\texttt{InterSolLess2}[\texttt{k},\texttt{D}]$ give, for a natural number D, solutions $(d,e;m)$ with $d \leqslant D$. But, in the case of classes of the form $(d+\frac{1}{2}, d-\frac{1}{2}; m)$, we give estimates for d in Lemma 2.7.4. Thus for these classes, we have to add $\frac{1}{2}$ to our estimates when using the programs.

Proof of Theorem 2.7.2. We have already proven in Theorem 2.6.21 that the class $(2,2;2,1^{\times 5})$ gives the constraint $c(a) = \mu\left(2,2;2,1^{\times 5}\right)(a) = \frac{a+1}{4}$ on $[\sigma^2, 6]$. We postpone the proof that $c(a) = \mu\left(4,4;3,2^{\times 6}\right)(a) = \frac{15}{8}$ on $\left[7, 7\frac{1}{32}\right]$ to Corollary 2.7.8.

Since by Lemma 2.7.4, the only obstructive classes on the interval $\left[6\frac{1}{8}, 7\right]$ are those of Lemma 2.7.3, $c\left(6\frac{1}{8}\right) = \frac{7}{4} = \sqrt{\frac{6\frac{1}{8}}{2}}$ because an explicit computation shows that none of them is obstructive at $6\frac{1}{8}$. Hence, $c(a) = \frac{7}{4}$ for all $a \in \left[6, 6\frac{1}{8}\right]$ since c is nondecreasing.

In order to determine c on the interval $\left[6\frac{1}{8}, 7\right]$, Lemma 2.7.4 shows that we only have to work out the constraints given by the classes of Lemma 2.7.3. Notice that for $a \in \left]6\frac{1}{k+1}, 6\frac{1}{k}\right[$, the first terms of the weight expansion of a

are $w(a) = \left(1^{\times 6}, (a-6)^{\times k}, 1-k(a-6), \ldots\right)$. We can thus easily compute the constraints of all the classes. In the next table, we write the constraints given by the classes of Lemma 2.7.3 that do not appear in Theorem 2.7.2 and we then simply verify that they indeed do not give new obstructions.

x	$(d,e;m)$	(A,B)	(A',B')	$\mu(x)$
$6\frac{1}{7}$	$(196,196;112^{\times 5},111,16^{\times 7})$	$(-1,112)$	$(687,0)$	$\frac{687}{392}$
$6\frac{1}{6}$	$(84,84;48^{\times 5},47,8^{\times 6})$	$(-1,48)$	$(295,0)$	$\frac{295}{168}$
$6\frac{1}{4}$	$(28,28;16^{\times 5},15,4^{\times 4})$	$(-1,16)$	$(99,0)$	$\frac{99}{56}$

x	u_x	$u_x \cong$	v_x	$v_x \cong$
$6\frac{1}{7}$	$\frac{1}{112}\left(344+7\sqrt{2415}\right)$	6.142844	$\frac{471969}{76832}$	6.142870
$6\frac{1}{6}$	$\frac{1}{48}\left(148+7\sqrt{447}\right)$	6.16660	$\frac{87025}{14112}$	6.16674
$6\frac{1}{4}$	$\frac{1}{16}\left(50+7\sqrt{51}\right)$	6.2494	$\frac{9801}{1568}$	6.25

The proof of Theorem 2.7.2 (up to Corollary 2.7.8) is complete. □

2.7.3 The interval $[7,8]$

Lemma 2.7.6. *Assume that there exists a class $(d,e;m) \in \mathcal{E}$ such that $\mu(d,e;m)(a) > \sqrt{\frac{a}{2}}$ for some $a \in \left[7\frac{1}{32}, 8\right]$ with $l(a) = l(m)$. Then $m_1 = \ldots = m_7$ and $d \leqslant 13$.*

Proof. Notice first that

$$y(a) \geqslant y\left(7\frac{1}{32}\right) = \frac{17}{32} > \frac{1}{q}$$

for all $q \geqslant 2$. We distinguish two cases: $q \geqslant 12$ and $q \leqslant 11$.

If $q \geqslant 12$, then $\delta = y(a) - \frac{1}{q} \geqslant \frac{17}{32} - \frac{1}{12} = \frac{43}{96}$. Assume by contradiction that $m_1 \neq m_7$. Then by Lemma 2.4.14, $\sigma \leqslant \frac{1}{7}$ and so $v_M \geqslant \frac{1}{2}$. Thus,

$$\frac{\sigma}{v_M \delta} \leqslant \frac{192}{301} < 1.$$

But this contradicts Lemma 2.7.1 (iv).

To prove that $d \leqslant 13$, notice first that by Lemma 2.7.1 (iii),

$$\begin{aligned}
&\text{if } v_M \in \left[\tfrac{1}{3}, \tfrac{1}{2}\right], && \text{then } \tfrac{\sigma'}{v_M} \leqslant \tfrac{1/2}{1/3} = \tfrac{3}{2},\\
&\text{if } v_M \in \left[\tfrac{1}{2}, \tfrac{2}{3}\right], && \text{then } \tfrac{\sigma'}{v_M} \leqslant \tfrac{7/8}{1/2} = \tfrac{7}{4},\\
&\text{if } v_M \geqslant \tfrac{2}{3}, && \text{then } \tfrac{\sigma}{v_M} \leqslant \tfrac{3}{2}.
\end{aligned}$$

Then, since $\sqrt{a} \leqslant 2\sqrt{2}$, we get by Lemma 2.7.1 (iv) that

$$d \leqslant \frac{2\sqrt{2}}{43/96\sqrt{2}} \left(\frac{1}{43/96} \frac{7}{4} - 1 \right) + \frac{1}{2} < 14.$$

Thus $d \leqslant 13$.

Let now $q \leqslant 11$. Notice that $a \leqslant 7\frac{q-1}{q}$ and

$$\delta = y(a) - \frac{1}{q} \geqslant y\left(7\frac{1}{q}\right) - \frac{1}{q}.$$

By Lemma 2.7.1 (iv), we have

$$d \leqslant \frac{\sqrt{a}}{\sqrt{2}\delta} \left(\sqrt{\sigma q} - 1\right) + \frac{1}{2} \leqslant \frac{\sqrt{7\frac{q-1}{q}}}{\sqrt{2}\left(y\left(7\frac{1}{q}\right) - \frac{1}{q}\right)} \left(\sqrt{q} - 1\right) + \frac{1}{2}.$$

Since the RHS is strictly smaller than 11 for all $2 \leqslant q \leqslant 11$, we see that $d \leqslant 10$.

Assume now by contradiction that $m_1 \neq m_7$. Then $\sigma \leqslant \frac{1}{7}$. If $2 \leqslant q \leqslant 7$, then $\sqrt{\sigma q} - 1 \leqslant 0$ which contradicts Lemma 2.7.1 (iv). If $8 \leqslant q \leqslant 11$, then

$$v_M = \frac{d+e}{q\sqrt{2a}} \leqslant \frac{\sqrt{2}d}{q\sqrt{a}} \leqslant \frac{\sqrt{2}10}{8\sqrt{7}},$$

and so, by Lemma 2.4.14,

$$\langle \varepsilon, \varepsilon \rangle \geqslant \frac{6}{7} + 2\left(1 - v_M\right)^2 > 1$$

which contradicts Lemma 2.4.8 (iii). ☐

Proposition 2.7.7. $c(a) = \sqrt{\frac{a}{2}}$ *for all* $a \in \left[7\frac{1}{32}, 8\right]$.

Proof. Suppose by contradiction that there exists $a \geqslant 7\frac{1}{32}$ and $(d, e; m) \in \mathcal{E}$ with $\mu(d, e; m)(a) > \sqrt{\frac{a}{2}}$. Let I be the maximal open interval containing a on which $(d, e; m)$ is obstructive. Then, by Lemma 2.4.13, there exists $a_0 \in I$ with $l\left(a_0\right) = l(m)$ and $l(a) \geqslant l\left(a_0\right)$ for all $a \in I$.

Using the finite list of $\mathcal{E}_7$ in Lemma 2.4.3 we check by hand that no class in $\mathcal{E}_7$ is obstructive for $a \geqslant 7\frac{1}{32}$. Thus $l(m) > 7$ and so $a_0 > 7$. But then $a_0 \geqslant 7\frac{1}{32}$. Indeed, assume by contradiction that $a_0 < 7\frac{1}{32}$. Then since $a_0, a \in I$, $7\frac{1}{32}$ will also belong to I. But, for all $z \in \left]7, 7\frac{1}{32}\right[$, $l(z) > l\left(7\frac{1}{32}\right)$, and this contradicts the fact that $l(a) \geqslant l\left(a_0\right)$ for all $a \in I$.

Now by Lemma 2.7.6, we find that $d \leqslant 13$ and $m_1 = \ldots = m_7$. Since there are only finitely many classes satisfying these conditions, we can compute them explicitly. We find that there is only one class satisfying the conditions, namely $\left(8, 7; 4^{\times 7}, 1\right)$, but this class is not obstructive for $a \geqslant 7\frac{1}{32}$. ☐

Corollary 2.7.8. *$c(a) = \frac{15}{8}$ for all $a \in \left[7, 7\frac{1}{32}\right]$.*

Proof. Since the class $(d, e; m) = \left(4, 4; 3, 2^{\times 6}\right)$ gives the constraint

$$\mu(d, e; m)(a) = \frac{15}{8} = \sqrt{\frac{7\frac{1}{32}}{2}}$$

for all $a \geqslant 7$, we see that $c(a) = \frac{15}{8}$ on $\left[7, 7\frac{1}{32}\right]$ because c is nondecreasing by Lemma 2.4.2. □

Chapter 3

Symplectic embeddings into the union of an ellipsoid and a cylinder

3.1 Ball packings of the union of an ellipsoid and a cylinder

3.1.1 Statement of the result

Consider the Euclidean space $\mathbb{R}^4$ endowed with its canonical symplectic structure $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Define the open *symplectic cylinder* $Z(a) := D^2(a) \times \mathbb{R}^2$ as the symplectic product of the open disc $D^2(a)$ of area a with $\mathbb{R}^2$ and define the open *symplectic ellipsoid* by

$$E(b,c) := \left\{ (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \, : \, \frac{\pi\left(x_1^2 + y_1^2\right)}{b} + \frac{\pi\left(x_2^2 + y_2^2\right)}{c} < 1 \right\}.$$

Denote the open ball $E(a,a)$ (of radius $\sqrt{a/\pi}$) by $B(a)$. Let

$$Z(a,b,c) := Z(a) \cup E(b,c)$$

be the (non disjoint) union of the cylinder $Z(a)$ with the ellipsoid $E(b,c)$. The aim of this section is to realize some optimal symplectic embeddings of a disjoint union of balls $B(w_1) \amalg \ldots \amalg B(w_n)$ into the union $Z(a,b,c)$ of a cylinder and an ellipsoid for certain values of a, b, c. Since

$$\coprod_{i=1}^{n} B(w_i) \stackrel{s}{\hookrightarrow} Z(a,b,c)$$

if and only if

$$\coprod_{i=1}^{n} B(\lambda w_i) \stackrel{s}{\hookrightarrow} Z(\lambda a, \lambda b, \lambda c),$$

we can always assume that $a = 1$, and therefore study the *embedding capacity function* defined by

$$c\left(w_1, \ldots, w_n; b, c\right) := \inf\left\{\lambda \mid \coprod_{i=1}^{n} B\left(w_i\right) \overset{s}{\hookrightarrow} Z(\lambda, \lambda b, \lambda c)\right\}.$$

The main result of this section is the following theorem.

Theorem 3.1.1. *Let $w_1 \geqslant \ldots \geqslant w_n > 0$ and $b, c > 0$ be positive real numbers such that $c > 1$ and $1 \leqslant b \leqslant \frac{c}{c-1}$. Then*

$$c\left(w_1, \ldots, w_n; b, c\right) = \max\left\{\frac{w_1}{c}, \lambda_1, \ldots, \lambda_n\right\},$$

where

$$\lambda_k := \frac{\sum_{i=1}^{k} w_i}{k + \frac{b(c-1)}{c}}.$$

In particular, this theorem realizes an optimal embedding of a disjoint union of balls into the union $Z(a, b, b)$ of the cylinder $Z(a)$ and the ball $B(b)$ whenever $a \leqslant b \leqslant 2a$.

The outline of the proof of Theorem 3.1.1 is as follows. In Subsection 3.1.2, we use the ECH capacities of $\coprod_{i=1}^{n} B\left(w_i\right)$ and $Z(a, b, c)$, that have been computed in [CGFHR], in order to determine an obstruction to the ball packing problem, leading to a lower bound on $c\left(w_1, \ldots, w_n; b, c\right)$. In Subsection 3.1.3, we give an explicit embedding construction, using the symplectic shearing method, which realizes the obstruction. This will lead to the required upper bound on $c\left(w_1, \ldots, w_n; b, c\right)$. In particular, it follows from the proof that ECH capacities are sharp for embedding a disjoint union of balls into the union of a cylinder and an ellipsoid under the assumptions of Theorem 3.1.1.

3.1.2 The obstruction given by ECH capacities

3.1.2.1 ECH capacities

Let (X, ω) be a symplectic 4-dimensional manifold. The ECH capacities are a sequence of real numbers

$$0 = c_0\left(X, \omega\right) \leqslant c_1\left(X, \omega\right) \leqslant c_2\left(X, \omega\right) \leqslant \ldots \leqslant \infty$$

associated to the manifold X that have been introduced in [H1].

We give here the properties of ECH capacities that we will use.

1. (Monotonicity) If there exists a symplectic embedding

$$(X_1, \omega_1) \overset{s}{\hookrightarrow} (X_2, \omega_2),$$

then

$$c_k(X_1, \omega_1) \leqslant c_k(X_2, \omega_2)$$

for all $k \geqslant 0$.

2. (Conformality) If $\alpha > 0$, then

$$c_k(X, \alpha\omega) = \alpha c_k(X, \omega)$$

for all $k \geqslant 0$.

3. (Disjoint Union) Let $(X_1, \omega_1), \ldots, (X_n, \omega_n)$ be symplectic 4-dimensional manifolds. Then

$$c_k\left(\coprod_{i=1}^{n} (X_i, \omega_i)\right) = \max_{k_1+\ldots+k_n=k} \sum_{i=1}^{n} c_{k_i}(X_i, \omega_i) \tag{3.1.1}$$

for all $k \geqslant 0$.

4. (Ellipsoid) The ECH capacities of the ellipsoid $E(a_1, a_2)$ are given by

$$c_k(E(a_1, a_2)) = N(a_1, a_2)_k \tag{3.1.2}$$

for all $k \geqslant 0$, where

$$N(a_1, a_2) := \{ma_1 + na_2 \mid m, n \in \mathbb{N} \cup \{0\}\}$$

arranged in nondecreasing order, with repetitions and with the indices starting at $k = 0$.

5. (Union of a Cylinder and an Ellipsoid) Under the assumptions of Theorem 3.1.1, the ECH capacities of the union $Z(1, b, c)$ of the cylinder $Z(1)$ and the ellipsoid $E(b, c)$ are given by

$$c_k(Z(1, b, c)) = \begin{cases} k + \frac{b(c-1)}{c} & \text{if } k \geqslant \frac{b}{c} \\ kc & \text{if } k \leqslant \frac{b}{c} \end{cases} \tag{3.1.3}$$

for all $k = 1, \ldots, n$.

Properties 1,2,3 and 4 are proved in [H2] while Property 5 is proved in [CGFHR].

3.1.2.2 Proof of the lower bound

Lemma 3.1.2. *Let $w_1 \geqslant \ldots \geqslant w_n > 0$ and $b, c > 0$ be positive real numbers such that $c > 1$ and $a \leqslant b \leqslant \frac{c}{c-1}$. Then*

$$c\left(w_1, \ldots, w_n; b, c\right) \geqslant \max\left\{\frac{w_1}{c}, \lambda_1, \ldots, \lambda_n\right\},$$

where $\lambda_k := \frac{\sum_{i=1}^k w_i}{k + \frac{b(c-1)}{c}}$.

Proof. We need to show that if there exists an embedding

$$\coprod_{i=1}^n B\left(w_i\right) \overset{s}{\hookrightarrow} Z(\lambda, \lambda b, \lambda c),$$

then

$$\lambda \geqslant \max\left\{\frac{w_1}{c}, \lambda_1, \ldots, \lambda_n\right\}.$$

By Monotonicity and Conformality of the ECH capacities, we have that

$$\lambda \geqslant \frac{c_k\left(\coprod_{i=1}^n B\left(w_i\right)\right)}{c_k\left(Z(1, b, c)\right)}$$

for all $k \geqslant 0$. So it suffices to show that for $k = 1, \ldots, n$

$$\frac{c_k\left(\coprod_{i=1}^n B\left(w_i\right)\right)}{c_k\left(Z(1, b, c)\right)} \geqslant \max\left\{\frac{w_1}{c}, \lambda_1, \ldots, \lambda_n\right\}.$$

Notice first that by (3.1.2), $c_1\left(B(a)\right) = c_1\left(E(a, a)\right) = a$ for all $a > 0$. Thus, by (3.1.1), we have for $k = 1, \ldots, n$

$$\begin{aligned} c_k\left(\coprod_{i=1}^n B\left(w_i\right)\right) &= \max_{k_1 + \ldots + k_n = k}\left\{c_{k_1}\left(B\left(w_1\right)\right) + \ldots + c_{k_n}\left(B\left(w_n\right)\right)\right\} \\ &\geqslant c_1\left(B\left(w_1\right)\right) + \ldots + c_1\left(B\left(w_k\right)\right) = \sum_{i=1}^k w_i \end{aligned}$$

We distinguish two cases: $b \leqslant c$ and $b \geqslant c$.

If $b \leqslant c$, then $\lambda_1 \geqslant \frac{w_1}{c}$. Therefore, it is sufficient to show that $\lambda \geqslant \max\left\{\lambda_1, \ldots, \lambda_n\right\}$. Since $\frac{b}{c} \leqslant 1$, we have by (3.1.3), that for all $k = 1, \ldots, n$

$$\frac{c_k\left(\coprod_{i=1}^n B\left(w_i\right)\right)}{c_k\left(Z(1, b, c)\right)} \geqslant \frac{\sum_{i=1}^k w_i}{k + \frac{b(c-1)}{c}} = \lambda_k.$$

If $b \geqslant c$, then by (3.1.3),

$$\frac{c_1\left(\coprod_{i=1}^n B\left(w_i\right)\right)}{c_1\left(Z(1, b, c)\right)} = \frac{w_1}{c}.$$

Moreover, if $k \geqslant \frac{b}{c}$, we have

$$\frac{c_k\left(\coprod_{i=1}^n B\left(w_i\right)\right)}{c_k\left(Z(1,b,c)\right)} \geqslant \frac{\sum_{i=1}^k w_i}{k+\frac{b(c-1)}{c}} = \lambda_k,$$

and if $k \leqslant \frac{b}{c}$, we have

$$\frac{c_k\left(\coprod_{i=1}^n B\left(w_i\right)\right)}{c_k\left(Z(1,b,c)\right)} \geqslant \frac{\sum_{i=1}^k w_i}{kc} \geqslant \frac{\sum_{i=1}^k w_i}{k+\frac{b(c-1)}{c}} = \lambda_k.$$

□

3.1.3 The explicit embedding

3.1.3.1 Prismification of $B(a)$ and coprismification of $Z(a,b,c)$

We first introduce some notation. For $a, b > 0$ set

$$\begin{aligned} \triangle(a,b) &:= \left\{(u,v) \in \mathbb{R}^2 \;\middle|\; u,v>0,\ \frac{u}{a}+\frac{v}{b}<1\right\}, \\ \square(a,b) &:=]0,a[\, \times\,]0,b[\,, \\ \square(\infty,b) &:=]0,\infty[\, \times\,]0,b[\,, \end{aligned}$$

and abbreviate $\triangle(a) := \triangle(a,a)$ and $\square := \square(1,1)$. If A and B are subsets of $\mathbb{R}^2$, define the symplectic product

$$A \times B := \left\{(x_1,y_1,x_2,y_2) \in \mathbb{R}^4 \;\middle|\; (x_1,y_1) \in A, (x_2,y_2) \in B\right\}$$

and the Lagrangian product

$$A \times_L B := \left\{(x_1,y_1,x_2,y_2) \in \mathbb{R}^4 \;\middle|\; (x_1,x_2) \in A, (y_1,y_2) \in B\right\}.$$

It was observed by Traynor in [T], that for every $\varepsilon > 0$ there exists a symplectic embedding

$$(1-\varepsilon)B(a) \overset{s}{\hookrightarrow} \triangle(a) \times_L \square. \tag{3.1.4}$$

The set $\triangle(a) \times_L \square$ is then called a *prismification* of the ball $B(a)$ (see Figure 3.1.1).

Next, let us "coprismify" the union of an ellipsoid and a cylinder, $Z(a,b,c) = Z(a) \cup E(b.c)$. Define

$$C(a,b,c) := (\square(\infty,a) \cup \triangle(b,c)) \times_L T^2,$$

where T^2 is the torus $\mathbb{R}^2/\mathbb{Z}^2$ (see Figure 3.1.2). Then for every $\varepsilon > 0$ there exists a symplectic embedding

$$C(a,b,c) \overset{s}{\hookrightarrow} (1+\varepsilon)Z(a,b,c). \tag{3.1.5}$$

The set $C(a,b,c)$ is then called a *coprismification* of $Z(a,b,c)$. This is a consequence of the following more general fact:

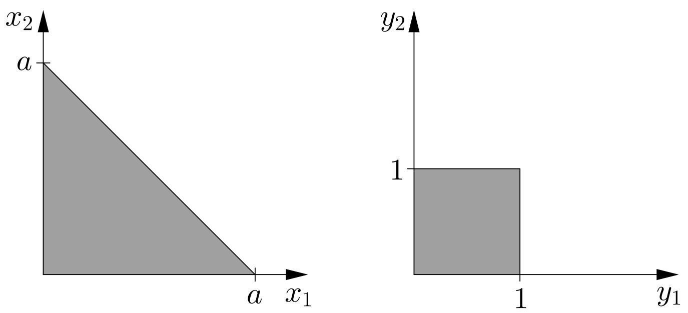

Figure 3.1.1: Prismification of $B(a)$

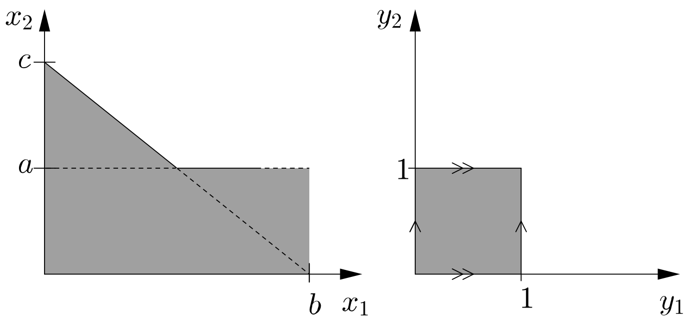

Figure 3.1.2: Coprismification of $Z(a, b, c)$

Lemma 3.1.3. *Let Ω be a domain in the open positive quadrant of the plane. Then there exists a symplectic embedding*

$$\mathrm{int}(\Omega) \times_L \mathbb{R}^2/\mathbb{Z}^2 \overset{s}{\hookrightarrow} X_\Omega,$$

where

$$X_\Omega := \left\{ (z_1, z_2) \in \mathbb{C}^2 = \mathbb{R}^4 \;\mid\; \left(|z_1|^2, |z_2|^2 \right) \in \Omega \right\}.$$

Proof. The map

$$\begin{aligned} \left\{ z \in \mathbb{C}^2 \;\mid\; z_1, z_2 \neq 0 \right\} &\overset{\simeq}{\longrightarrow} \left\{ x \in \mathbb{R}^2 \;\mid\; x_1, x_2 > 0 \right\} \times_L \frac{\mathbb{R}^2}{\mathbb{Z}^2} \\ \left(r_1 e^{i\theta_1}, r_2 e^{i\theta_2} \right) &\longmapsto \left(\pi \left(r_1^2, r_2^2 \right), \frac{1}{2\pi} (\theta_1, \theta_2) \right) \end{aligned}$$

is a symplectomorphism. The inverse of this map restricts to a symplectic embedding

$$\mathrm{int}(\Omega) \times_L \frac{\mathbb{R}^2}{\mathbb{Z}^2} \overset{s}{\hookrightarrow} X_\Omega.$$

□

3.1.3.2 Symplectic shearing

We now briefly recall the method of *symplectic shearing*. For more details on shearing see for example [LMS]. Let $k \geqslant 0$ be an integer and consider the matrix $A_k := \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$. The linear map

$$\hat{A}_k := \begin{pmatrix} A_k & 0 \\ 0 & \left(A_k^T \right)^{-1} \end{pmatrix}$$

is a symplectomorphism of $\mathbb{R}^4 = \mathbb{R}^2(x) \times_L \mathbb{R}^2(y)$.

If we apply $\hat{A}_k$ to the prismification $\triangle(a) \times_L \square$ of the ball $B(a)$, the effect in the x-plane is to distort $\triangle(a)$ to the triangle $A_k(\triangle(a))$ with vertices $(0,0)$, $(a,0)$ and $(-ka, a)$, while the effect in the y-plane is to distort $\square$ to the parallelogram $\left(A_k^T \right)^{-1}(\square)$ with vertices $(0,0)$, $(1,k)$, $(1,k+1)$ and $(0,1)$ (see Figure 3.1.3). Now, since $\left(A_k^T \right)^{-1} \in SL_2(\mathbb{Z})$, the parallelogram $\left(A_k^T \right)^{-1}(\square)$ injectively projects to the torus, so that the map $\hat{A}_k$ extends to an embedding

$$\varphi_k \colon \triangle(a) \times_L \square \overset{s}{\hookrightarrow} A_k(\triangle(a)) \times_L T^2.$$

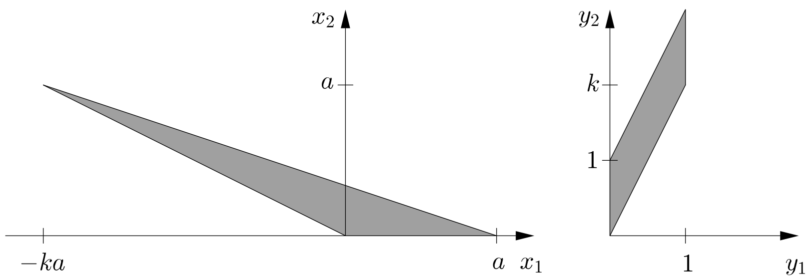

Figure 3.1.3: Shearing of $\triangle(a) \times_L \square$

3.1.3.3 Proof of the upper bound

Lemma 3.1.4. *Assume that* $w_1 \geqslant \ldots \geqslant w_n > 0$, $b > 0$ *and* $c > 1$. *Then,*

$$c\left(w_1, \ldots, w_n; b, c\right) \leqslant \max\left\{\frac{w_1}{c}, \lambda_1, \ldots, \lambda_n\right\},$$

where $\lambda_k := \frac{\sum_{i=1}^k w_i}{k + \frac{b(c-1)}{c}}$.

Proof. We need to show that if

$$\lambda > \max\left\{\frac{w_1}{c}, \lambda_1, \ldots, \lambda_n\right\},$$

then there exists an embedding

$$\coprod_{i=1}^n B\left(w_i\right) \overset{s}{\hookrightarrow} Z(\lambda, \lambda b, \lambda c).$$

Let $k \in \{1, \ldots, n\}$ be an index such that $\lambda_k = \max\left\{\lambda_1, \ldots, \lambda_n\right\}$. Assume that $\lambda > \max\left\{\frac{w_1}{c}, \lambda_k\right\}$. By the prismification (3.1.4) and the coprismification (3.1.5), it is enough to construct a symplectic embedding

$$\coprod_{i=1}^n \triangle\left(w_i\right) \times_L \square \overset{s}{\hookrightarrow} C(\lambda, \lambda b, \lambda c).$$

First note that $\lambda_k \leqslant w_i$ for all $i \leqslant k$, and $\lambda_k \geqslant w_i$ for all $i > k$. Indeed, since $\lambda_k := \frac{\sum_{i=1}^k w_i}{k + \frac{b(c-1)}{c}}$, we have

$$w_k - \lambda_k = \left(k + \frac{b(c-1)}{c} - 1\right)\left(\lambda_k - \lambda_{k-1}\right),$$
$$\lambda_k - w_{k+1} = \left(k + \frac{b(c-1)}{c} + 1\right)\left(\lambda_k - \lambda_{k+1}\right).$$

Since λ_k is maximal, we deduce that $w_k \geqslant \lambda_k \geqslant w_{k+1}$. The rest follows from $w_1 \geqslant \ldots \geqslant w_n$.

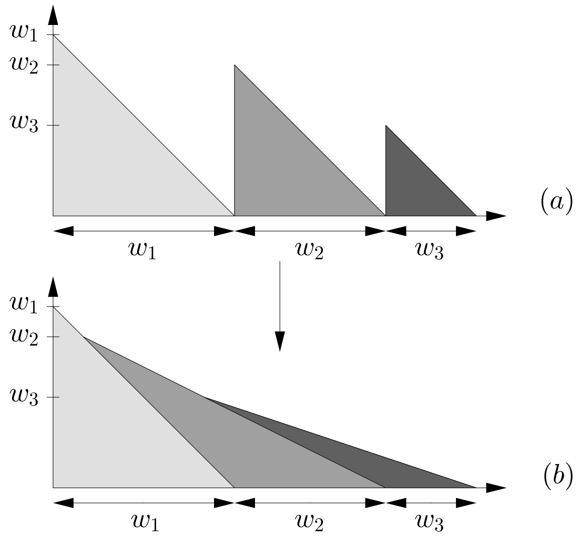

Figure 3.1.4: The shearing construction

Since $\lambda \geqslant w_i$ for all $i > k$, it follows that we just need to find a symplectic embedding

$$\varphi\colon \coprod_{i=1}^{k} \triangle(w_i) \times_L \square \overset{s}{\hookrightarrow} C(\lambda, \lambda b, \lambda c). \tag{3.1.6}$$

The sets $\triangle(w_i) \times_L \square$ with $i > k$ can then be embedded into $C(\lambda, \lambda b, \lambda c) \setminus \operatorname{Im} \varphi$ by appropriate translations along the x_1-axis.

We now construct the symplectic embedding φ in (3.1.6) by shearing. Place the sets $\triangle(w_\ell) \times_L \square$ with $\ell = 1, \dots, k$ side by side such that the vertices of each triangle $\triangle(w_\ell)$ are

$$\left(\sum_{i=1}^{\ell-1} w_i,\, 0\right),\ \left(\sum_{i=1}^{\ell} w_i,\, 0\right),\ \left(\sum_{i=1}^{\ell-1} w_i,\, w_\ell\right)$$

(see Figure 3.1.4(a)).

Apply then to each $\triangle(w_\ell) \times_L \square$ the shear $\varphi_{\ell-1}$, which embeds $\triangle(w_\ell) \times_L \square$ into $A_{\ell-1}(\triangle(w_\ell)) \times_L T^2$. In the x-plane, the effect is to embed the triangles $\triangle(w_\ell)$ into the triangles with vertices

$$\left(\sum_{i=1}^{\ell-1} w_i,\, 0\right),\ \left(\sum_{i=1}^{\ell} w_i,\, 0\right),\ \left(\sum_{i=1}^{\ell-1} w_i - (\ell-1)w_\ell,\, w_\ell\right)$$

(see Figure 3.1.4(b)). We denote the triangles $A_{\ell-1}(\triangle(w_\ell))$ by T_ℓ.

To finish the proof, we need to show that the disjoint union of triangles $\coprod_{\ell=1}^{k} T_\ell$ is contained in the region Ω bounded by the axes, the line segment

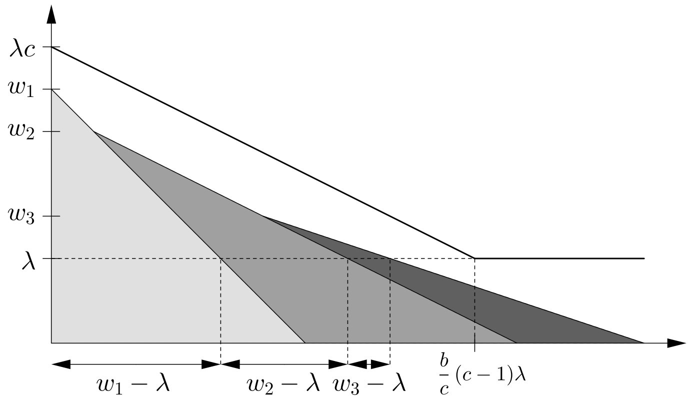

Figure 3.1.5: The triangles $\coprod_{\ell=1}^{k} T_\ell$ fit into the region Ω

from $(0,\lambda c)$ to $\left(\frac{b}{c}(c-1)\lambda,\lambda\right)$, and the horizontal ray extending to the right from the latter point (see Figure 3.1.5).

Observe that the right edge of T_ℓ has slope $-\frac{1}{\ell}$. And if $\ell > 1$ then the left edge of T_ℓ is a subset of the right edge of $T_{\ell-1}$. In particular, the upper boundary of the union of their closures (call this path Λ) is the graph of a convex function.

To verify that the triangles $T_1,\ldots,T_k$ are contained in Ω, we need to check that the path Λ does not go above the upper boundary of Ω (see Figure 3.1.5). The initial endpoint of Λ is $(0,w_1)$, which is not above the upper boundary of Ω by our assumption that $\lambda > \frac{w_1}{c}$. Next, Λ crosses the horizontal line of height λ at the point

$$\left(\sum_{\ell=1}^{k}(w_\ell-\lambda)\,,\,\lambda\right).$$

By convexity, it is enough to check that this point is not to the right of the corner $\left(\frac{b}{c}(c-1)\lambda,\lambda\right)$ of $\partial\Omega$. This holds because

$$\lambda > \lambda_k = \frac{\sum_{\ell=1}^{k} w_\ell}{k+\frac{b(c-1)}{c}}$$

implies that

$$\sum_{\ell=1}^{k}(w_\ell-\lambda) < \frac{b}{c}(c-1)\lambda.$$

This concludes the proof. □

3.2 The embedding of $E\left(36, \frac{6}{5}\right)$ into $Z\,(1, 6, 6)$

Consider the problem of finding, given $a, b \geqslant 1$, the infimum λ^* of those λ such that $E(a, 1) \overset{s}{\hookrightarrow} Z(\lambda, \lambda b, \lambda b)$, where $Z(\lambda, \lambda b, \lambda b) := Z(\lambda) \cup B(\lambda b)$. We ask for which $a, b \geqslant 1$ one has

1. $\operatorname{vol}(E(a, 1)) > \operatorname{vol}(B(\lambda^* b))$,

2. $E(a, 1) \not\subset Z(\lambda^*, \lambda^* b, \lambda^* b)$.

The first condition is imposed to exclude embeddings of $E(a, 1)$ that take values in the ball $B(\lambda^* b)$ alone (cf. [MS]), while the second condition excludes trivial embeddings. By Corollary 1.13 in [CGFHR], there are pairs $a, b \geqslant 1$ such that no embedding satisfying (1) and (2) exists.

In Proposition 3.2.1, we show that $E(36, \frac{6}{5}) \overset{s}{\hookrightarrow} Z\,(1, 6, 6)$ by using symplectic folding, an embedding technique invented in [LM] and refined in [S2]. The rescaled embedding $E(30, 1) \overset{s}{\hookrightarrow} Z\left(\frac{5}{6}, 5, 5\right)$ satisfies (1) and (2). Proposition 3.2.1 thus shows that the problem of embedding an ellipsoid into the union of a ball and a cylinder is not totally rigid. Actually, multiple symplectic folding gives similar results for all values $b > 4$, that is for all $b > 4$ there exist $a, \lambda > 0$ such that $E(a, 1) \overset{s}{\hookrightarrow} Z(\lambda, \lambda b, \lambda b)$ and such that this embedding satisfies (1) and (2).

Another consequence of this embedding is that although Lemma 3.1.1 holds for all values of $b > 0$ and $c > 1$, the ball packing construction described in the proof is not optimal for all these values of b, c. Indeed, since $\coprod_{i=1}^{30} B\left(\frac{6}{5}\right) \overset{s}{\hookrightarrow} E\left(36, \frac{6}{5}\right)$, the embedding $E\left(36, \frac{6}{5}\right) \overset{s}{\hookrightarrow} Z(1, 6, 6)$ implies the existence of an embedding

$$\coprod_{i=1}^{30} B\left(\frac{6}{5}\right) \overset{s}{\hookrightarrow} Z(1, 6, 6).$$

But the shearing construction of Lemma 3.1.1 only leads to an embedding

$$\coprod_{i=1}^{30} B\left(\frac{6}{5}\right) \overset{s}{\hookrightarrow} \frac{36}{35} Z(1, 6, 6).$$

Proposition 3.2.1. $E\left(36, \frac{6}{5}\right)$ *symplectically embeds into* $Z(1, 6, 6)$.

Proof. Consider the prismification $\triangle\left(36, \frac{6}{5}\right) \times_L \square$ of the ellipsoid $E\left(36, \frac{6}{5}\right)$ and the coprismification

$$C(1, 6, 6) := (\square(\infty, 1) \cup \triangle(6)) \times_L \square \;\subset\; (\square(\infty, 1) \cup \triangle(6)) \times_L T^2$$

of $Z(1, 6, 6)$. To prove the proposition, it suffices to show that $\triangle\left(36, \frac{6}{5}\right) \times_L \square$ symplectically embeds into $C(1, 6, 6)$.

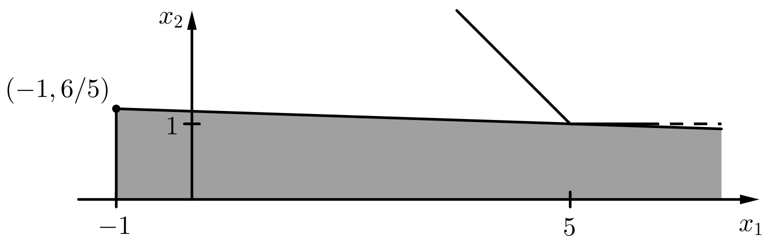

Figure 3.2.1: Disposing the triangle $\triangle\left(36, \frac{6}{5}\right)$ in the (x_1, y_2)-plane

We construct this embedding by using symplectic folding. For the details of symplectic folding, we refer to Sections 3 and 4 in [S2], which we will closely follow in the sequel.

Place the triangle $\triangle\left(36, \frac{6}{5}\right)$ in the (x_1, x_2)-plane such that its vertices are the points $(-1, 0)$, $\left(-1, \frac{6}{5}\right)$ and $(35, 0)$ (see Figure 3.2.1). Since the point $(5, 1)$ is on the boundary of the triangle, we only have to fold the left part of $\triangle\left(36, \frac{6}{5}\right)$ into $\triangle(6)$ in order to realize the required embedding.

Define ℓ as the unique height of the triangle such that the distance to the left edge is equal to $1 + \ell$ (see Figure 3.2.2 (a)). One can check that $\ell = 1 + \frac{4}{31}$. First separate the large fibers from the small ones and connect them by a tunnel of length ℓ using a map $\beta \times id$ constructed as in Step 1 of Section 3.2 in [S2] (see Figure 3.2.2 (b)).

Now lift the fibers analogously to the procedure described in Step 3 of Section 3.2 in [S2]. Specifically, define L as the projection of the tunnel to the x_1-axis. Take a cut-off function c over L and define the "lift" map

$$\varphi\left(x_1, y_1, x_2, y_2\right) := \left(x_1,\ y_1 - c\left(x_1\right) y_2,\ x_2 + \ell - \int_0^{x_1} c(t)dt,\ y_2\right).$$

The image of $\triangle\left(36, \frac{6}{5}\right) \times_L \square$ under φ is drawn in Figure 3.2.2 (c).

Now apply the folding map γ_2 defined in Step 4 of Section 4.2 in [S2] to the image of φ. By Lemma 4.2.1 in [S2], the "stairs" S connecting the two "floors" F_1 and F_2 in Figure 3.2.2 (d) are contained in the rectangle with horizontal edge of length ℓ, and vertical edge of length 2ℓ. One readily computes that by our choice of ℓ, this rectangle fits into $\triangle(6)$. The upper right corner of F_2 is the point $\left(3 + \frac{8}{31}, 2 + \frac{51}{155}\right)$, which lies in the interior of $\triangle(6)$. The floor F_2 thus also fits into $\triangle(6)$, which proves that the ellipsoid $E\left(36, \frac{6}{5}\right)$ can be folded into $Z(1, 6, 6)$ (see Figure 3.2.3).

$\square$

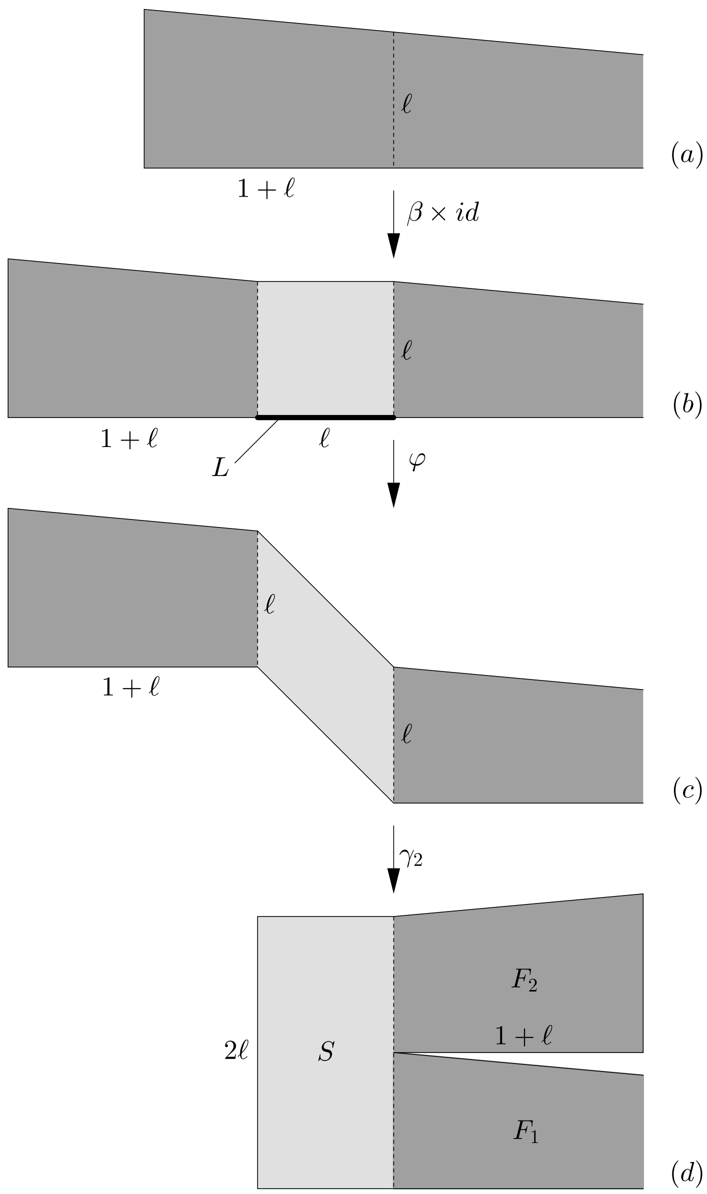

Figure 3.2.2: The folding construction

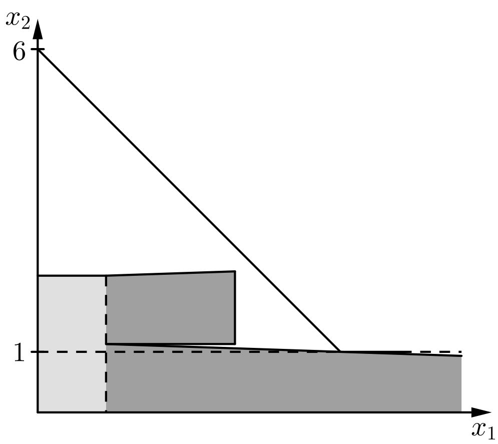

Figure 3.2.3: The ellipsoid $E\left(36, \frac{6}{5}\right)$ can be folded into $Z(1, 6, 6)$

Appendix A

Computer programs

A.1 Computing c at a point $a \in \left[6\frac{1}{8}, 7\right]$

We used the computer in Lemma 2.7.4 to compute c at points $\frac{p}{q} \in \left[6\frac{1}{8}, 7\right]$ with $q \leqslant 39$. In this section, we explain the code SolLess[a, D] which computes for a rational number a and a natural number D all classes $(d, e; m)$ obstructive at a, with $l(m) = l(a)$ and $d \leqslant D$. We have just adapted the program SolLess[a, D] given in the Appendix of [MS] to our case. The modules W[a], P[k], Difference[M] are exactly the same as in [MS].

```
W[a_] := Module[{aa = a, M, i = 2, L, u, v},
  M = ContinuedFraction[aa];
  L = Table[1, {j, M[[1]]}];
  {u, v} = {1, aa - Floor[aa]};
  While[i <= Length[M],
    L = Join[L, Table[v, {j, M[[i]]}]];
    {u, v} = {v, u - M[[i]] v};
    i++];
  Return[L]]
```

```
P[k_] := Module[{kk = k, PP, T0, i},
  T0 = Table[0, {u, 1, k}];
  T0p = ReplacePart[T0, 1, 1];
  T1 = Table[1, {u, 1, k}];
  T1m = ReplacePart[T1, 0, -1];
  PP = {T0, T0p, T1, T1m};
  Return[PP]]
```

```
Difference[M_] := Module[{V = M, vN, V1, l, L = {}, D, PP,
    i, j, N},
  l = Length[V];
  If[l == 1,
    L = P[V[[1]]]
    ];
  If[l > 1,
    vN = V[[-1]];
    V1 = Delete[V, -1];
    D = Difference[V1];
    PP = P[vN];
    i = 1;
    While[i <= Length[D],
      j = 1;
      While[j <= Length[PP],
        N = Join[D[[i]], PP[[j]]];
        L = Append[L, N];
        j++];
      i++]
    ];
  Return[L]]
```

The following module Sol0[a, d] gives for a rational number a all vectors of the form $(d, d; m)$ with $l(m) = l(a)$ which satisfy the Diophantine equations of Proposition 2.3.8 (i) and such that $\mu(d, d; m)(a) > \sqrt{\frac{a}{2}}$. The code Sol1[a, d] does the same thing for a class of the form $(d, d-1; m)$. Note that both modules do not verify whether the vectors reduce to $(0; -1)$ by repeated Cremona moves. We have just adapted the code Sol[a, d] of [MS], using that in our case, the volume constraint is $\sqrt{\frac{a}{2}}$ instead of $\sqrt{a}$ and that for a class of the form $(d, d; m)$ the Diophantine equations become

$$\sum m_i = 4d - 1, \qquad \sum m_i^2 = 2d^2 + 1,$$

and for a class of the form $(d, d-1; m)$, they become

$$\sum m_i = 4d - 3, \qquad \sum m_i^2 = 2d^2 - 2d + 1.$$

```
Sol0[a_, d_] := Module[{aa = a, dd = d, M, F, D, i, V,
    L = {}},
  M = ContinuedFraction[aa];
  F = Floor[((2*dd)/Sqrt[2*aa]) W[aa]];
  D = Difference[M];
  i = 1;
```

```
  While[i <= Length[D],
    V = Sort[F + D[[i]], Greater];
    SV = Sum[V[[j]], {j, 1, Length[V]}];
    If[{SV, V.V} == {4*dd - 1, 2*dd^2 + 1} && V[[-1]] > 0 &&
      W[aa].V/(2*dd) >= Sqrt[aa/2],
    L = Append[L, V]
    ];
  i++];
  Return[{{dd, dd}, Union[L]}]
```

```
Sol1[a_, d_] := Module[{aa = a, dd = d, M, F, D, i, V,
    L = {}},
  M = ContinuedFraction[aa];
  F = Floor[((2*dd - 1)/Sqrt[2*aa]) W[aa]];
  D = Difference[M];
  i = 1;
  While[i <= Length[D],
    V = Sort[F + D[[i]], Greater];
    SV = Sum[V[[j]], {j, 1, Length[V]}];
    If[{SV, V.V} == {4*dd - 3, 2*dd^2 - 2*d + 1}
      && V[[-1]] > 0 && W[aa].V/(2*dd - 1) > Sqrt[aa/2],
    L = Append[L, V]
    ];
  i++];
  Return[{{dd, dd - 1}, Union[L]}]]
```

Finally, we collect in the code SolLess[a, D] the vectors $(d, e; m)$ with $l(m) = l(a)$ that are obstructive at a and such that $d \leqslant D$.

```
SolLess[a_, D_] := Module[{aa = a, DD = D, d = 1, Ld,
    L = {}},
  While[d <= D,
    Ld = Sol0[aa, d];
    If[Length[Ld[[2]]] > 0,
      L = Append[L, Ld]
      ];
    Ld = Sol1[aa, d];
    If[Length[Ld[[2]]] > 0,
      L = Append[L, Ld]
      ];
    d++];
  Return[L]]
```

A.2 Computing c on an interval $\left]6\frac{1}{k+1}, 6\frac{1}{k}\right[$ with $k \in \{1, \ldots 7\}$

In Lemma 2.7.4 we used the codes `InterSolLess1[k, D]` and `InterSolLess2[k, D]` which give for $k \in \{1, \ldots, 7\}$ and a natural number D, a finite list of vectors $(d, e; m)$ with $d \leqslant D$ which can potentially be obstructive at some $a \in \left]6\frac{1}{k+1}, 6\frac{1}{k}\right[$. By Lemma 2.4.14, if a class $(d, e; m) \in \mathcal{E}$ is obstructive at some point $a \in \left[6\frac{1}{8}, 7\right]$, then we have three possibilities:

(i) $m_1 = \ldots = m_6$,

(ii) $m_1 - 1 = m_2 = \ldots = m_6$,

(iii) $m_1 = \ldots = m_5 = m_6 + 1$.

The code `InterSolLess1[k, D]` treats the case (i) while the cases (ii) and (iii) are covered by `InterSolLess2[k, D]`. We used the programs `Solutions[a, b]` and `sum[L]` exactly as they were in [MS]. `Solutions[a, b]` gives for $a, b \in \mathbb{N}$ all vectors m which are solution of the equations

$$\sum m_i = a, \qquad \sum m_i^2 = b,$$

and `sum[L]` computes the sum of the entries of a vector L.

```
Solutions[a_, b_]
  := Solutions[a, b, Min[a, Floor[Sqrt[b]]]]


Solutions[a_, b_, c_] :=
  Module[{A = a, B = b, C = c, i, m, K, j, V, L = {}},
    If[A^2 < B,
      L = {}
      ];
    If[A^2 == B,
      If[A > C,
        L = {},
        L = {{A}}
        ]
      ];
    If[A^2 > B,
      i = 1;
      m = Min[Floor[Sqrt[B]], C];
      While[i <= m,
        K = Solutions[A - i, B - i^2, i];
        j = 1;
```

```
        While[j <= Length[K],
          V = Prepend[K[[j]], i];
          L = Append[L, V];
          j++
          ];
        i++]
    ];
  Return[Union[L]]]

sum[L_] := Sum[L[[j]], {j, 1, Length[L]}]
```

A.2.1 Finding obstructive classes $(d, e; m)$ with $m_1 = \ldots = m_6$

We have adapted the modules `P[k]`, `Prelist[k,d]` from [MS] to the fact that the first six entries of m have to be equal instead of the first seven entries as it was the case in [MS]. The module `Prelist[k,d]` becomes `Prelist[k,d,c]` where $c = 0$ in the case of a class of the form $(d, d; m)$ and $c = 1$ when the class is of the form $(d, d-1; m)$. As before, we have adapted the code to take into account that we have another volume constraint and other Diophantine equations. Note that [MS] used their Lemma 2.1.7 and Lemma 2.1.8 which are also true in our case as stated in Lemma 2.4.14 and Lemma 2.4.16.

```
P[k_] := Module[{kk = k, PP, T0, i},
  T0 = Table[0, {i, 6 + kk}];
  Tm = ReplacePart[T0, -1, -1];
  Tp = ReplacePart[T0, 1, 7];
  PP = {Tm, T0, Tp};
  Return[PP]
  ]

Prelist[k_, d_, c_] :=
  Module[{kk = k, dd = d, case = c, u, v, m1, M1, mx, Mx, f,
    t, PP, M, MM,i = 0, j = 0, s = 1, S, T, K, l, L = {}},
  u = 1/(kk + 1);
  v = 1/kk;
  m1 = Round[(Sqrt[2]*dd)/Sqrt[6 + v]];
  M1 = Round[(Sqrt[2]*dd)/Sqrt[6 + u]];
  mx = Floor[(Sqrt[2]*dd)/Sqrt[6 + v] u] - 1;
  Mx = Ceiling[(Sqrt[2]*dd)/Sqrt[6 + u] v] + 1;
  f = Ceiling[Sqrt[kk + 2] - 1];
  t = -f;
  PP = P[kk];
```

```
While[i <= M1 - m1,
  While[j <= Mx - mx,
    While[s <= 3,
      While[t <= f,
      M = Join[Table[m1 + i, {u, 6}],
        Table[mx + j, {u, kk}]];
      M = M + PP[[s]];
      S = Sum[M[[u]], {u, 7, 7 + kk - 1}];
      M = Append[M, M[[6]] - S + t];
      T = 1;
      If[M == Sort[M, Greater] && M[[-1]] > 0,
        T = 1, T = 0];
      S = sum[M];
      If[case == 0,
        A = 4*dd - 1 - S;
        B = 2*dd^2 + 1 - M.M;
      ];
      If[case == 1,
        A = 4*dd - 3 - S;
        B = 2*dd*(dd - 1) + 1 - M.M;
      ];
      B = 2*dd^2 + 1 - M.M;
      If[Min[A, B] < 0,
        T = 0];
      If[T == 1,
        K = Solutions[A, B, M[[-1]]];
        l = 1;
        While[l <= Length[K],
          MM = Join[M, K[[l]]];
          While[MM[[-1]] == 0,
            MM = Drop[MM, -1]];
          L = Append[L, MM];
          l++]
        ];
      t++];
    t = -f;
    s++];
  s = 1;
  j++];
j = 0;
i++];
Return[{{dd, dd - case}, Union[L]}]]
```

As in [MS], the module `InterSol[k, d, c]` reduces the number of candidates given by the code `Prelist[k, d, c]`. As before, $c = 0$ in the case of a class of the form $(d, d; m)$ and $c = 1$ for a class of the form $(d, d - 1; m)$.

```
InterSol[k_, d_, c_] :=
  Module[{kk = k, dd = d, case = c, L, M, T, K = {}, i = 1,
    l, rest},
  L = Prelist[kk, dd, case][[2]];
  While[i <= Length[L],
    M = L[[i]];
    l = Length[M];
    T = 1;
    If[l <= 6 + kk + 2, T = 0];
    If[M[[-2]] - M[[-1]] > 1, T = 0];
    If[M[[-3]] > M[[-2]] + 1 && Abs[M[[-3]] - M[[-2]]
      - M[[-1]]] > 1,
      T = 0];
    If[kk == 1 && l >= 9,
      If[M[[8]] - M[[9]] > 1 && Abs[M[[7]] - (M[[8]]
      + M[[9]])] > 1,
        T = 0]];
    rest = Sum[M[[j]], {j, 7 + kk, l}];
    If[M[[6 + kk]] - rest >= Sqrt[l - kk - 5], T = 0];
    If[T == 1, K = Append[K, M]];
    i++];
  Return[{{dd, dd - case}, K}]]
```

Finally, we collect all the solutions for $d \leqslant D$ in `InterSolLess1[k, D]`.

```
InterSolLess1[k_, D_] := Module[{kk = k, DD = D, LL = {}, Q,
    d = 1},
  While[d <= DD,
    Q = InterSol[kk, d, 0];
    If[Length[Q[[2]]] > 0,
      LL = Append[LL, Q]];
    Q = InterSol[kk, d, 1];
    If[Length[Q[[2]]] > 0,
      LL = Append[LL, Q]];
    d++];
  Return[LL]]
```

A.2.2 Finding obstructive classes $(d, e; m)$ with $m_1 \neq m_6$

The code InterSolLess2[k, D] gives for $k \in \{1, \ldots, 7\}$ and a natural number D, a finite list of vectors $(d, e; m)$ with $d \leqslant D$ and $m_1 \neq m_6$ which can potentially be obstructive at some $a \in \left]6\frac{1}{k+1}, 6\frac{1}{k}\right[$. By Lemma 2.4.15, if a class $(d, e; m) \in \mathcal{E}$ with $m_1 \neq m_6$ is obstructive at some $a \in [6, 7[$, then necessarily $d = e$. Moreover, Lemma 2.4.14 shows that either $m_1 - 1 = m_2 = \ldots = m_6$ or $m_1 = \ldots = m_5 = m_6 + 1$. Notice that the first terms of the weight expansion of some $a \in \left]6\frac{1}{k+1}, 6\frac{1}{k}\right[$ are $\left(1^{\times 6}; (a-6)^{\times k}, \ldots\right)$. Thus the vector m is either of the form $\left(M+1, M^{\times 5}, m^{\times k}, \ldots\right)$ or of the form $\left(M, (M-1)^{\times 5}, m^{\times k}, \ldots\right)$. To find the vectors m of the form $\left(M+1, M^{\times 5}, m^{\times k}, \ldots\right)$, we vary M and $m \leqslant M$ as long as $(M+1) + 5M + km \leqslant 4d - 1$ and $(M+1)^2 + 5M^2 + km^2 \leqslant 2d^2 + 1$ and then use the code Solutions[a, b] from [MS] to find the solutions of the equations

$$\sum m_i = 4d - 1 - ((M+1) + 5M + km),$$
$$\sum m_i^2 = 2d^2 - 1 - \left(M+1)^2 + 5M^2 + km^2\right).$$

The case of a solution vector m of the form $\left(M, (M-1)^{\times 5}, m^{\times k}, \ldots\right)$ is then treated similarly.

```
InterSolLess2[kk_, DD_] := Module[{k = kk, D = DD, d, M, m,
    Sol,i, j},
  For[d = 1, d <= D, d++,
    M = 1;
    While[6*M + 1 <= 4*d - 1 && 6*M^2 + 2*M + 1
        <= 2*d^2 + 1,
      m = 1;
      While[
        6*M + 1 + k*m <= 4*d - 1 &&
          6*M^2 + 2*M + 1 + k*m^2 <= 2*d^2 + 1 && m <= M,
        Sol =
          Solutions[4*d - 1 - (6*M + 1 + k*m),
            2*d^2 + 1 - (6*M^2 + 2*M + 1 + k*m^2)];
        If [Length[Sol] > 0,
          For[i = 1, i <= Length[Sol], i++,
            If[Sol[[i]][[1]] <= m,
              For[j = 1, j <= k, j++,
                Sol[[i]] = Prepend[Sol[[i]], m];
                ];
```

```
            For[j = 1, j <= 5, j++,
              Sol[[i]] = Prepend[Sol[[i]], M];
              ];
            Sol[[i]] = Prepend[Sol[[i]], M + 1];
            Print[{{d, d}, Sol[[i]]}];
            ]
          ]
        ];
      m++;
      ];
    M++;
    ];
  M = 1;
  While[6*M - 1 <= 4*d - 1 && 6*M^2 - 2*M + 1
      <= 2*d^2 + 1,
    m = 1;
    While[
      6*M - 1 + k*m <= 4*d - 1 &&
        6*M^2 - 2*M + 1 + k*m^2 <= 2*d^2 + 1 && m <= M,
      Sol =
        Solutions[4*d - 1 - (6*M - 1 + k*m),
          2*d^2 + 1 - (6*M^2 - 2*M + 1 + k*m^2)];
      If [Length[Sol] > 0,
        For[i = 1, i <= Length[Sol], i++,
          If[Sol[[i]][[1]] <= m,
            For[j = 1, j <= k, j++,
              Sol[[i]] = Prepend[Sol[[i]], m];
              ];
            Sol[[i]] = Prepend[Sol[[i]], M - 1];
            For[j = 1, j <= 5, j++,
              Sol[[i]] = Prepend[Sol[[i]], M];
              ];
            Print[{{d, d}, Sol[[i]]}];
            ]
          ]
        ];
      m++;
      ];
    M++;
    ];
  ]
]
```

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