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## On Reasoning about ‘Generally’ and ‘Rarely’ with Filter-like Family of Sets<sup>1</sup>

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### Abstract

We examine some issues in reasoning about ‘generally’ and ‘rarely’. The primary motivation is a precise qualitative approach to assertions and arguments involving such vague notions, which occur often in ordinary language and in some branches of science. We focus mainly on the intended meanings of such assertions and analyse some basic intuitions and their underlying presuppositions. This leads to distinguishing various versions according to their behaviour, which can be explained by means of filter-like families of sets. Such families provide bases for precise qualitative reasoning about some vague notions.

*Keywords.* Vague notions, generally, rarely, precise reasoning, families of sets, filter, ideal, logics for vague notions, several, many, most, few

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## 1 Introduction

In this paper we discuss, trying to explain, some fundamental issues in the precise treatment of assertions involving 'generally' and 'rarely'. We hope to clarify the role played by families of subsets in this context.<sup>3</sup>

Some vague notions occur often in ordinary language and in some branches of science and it would be desirable to reason about assertions involving them in a precise manner. The overall aim is having logics for (some versions of) these vague notions.<sup>4</sup> We shall focus mainly on the intended meanings of such assertions. By analysing some basic intuitions and their underlying presuppositions, we are led to distinguishing various versions according to their behaviour, which can be explained by means of families of sets. Such families, in turn, provide bases for precise qualitative reasoning about assertions involving some vague notions like 'generally' and 'rarely'.

Assertions and arguments involving some vague notions, such as 'generally', 'rarely', 'several', 'few', 'many', 'most', etc., occur often, both in ordinary language and in some branches of science. For instance, one often encounters assertions such as "Many bodies expand when heated", "Most birds fly" and "Few metals are liquid under ordinary conditions".<sup>5</sup> The assertions "Whoever likes sports watches Sports-TV" and "Boys generally like sports" appear to lead to "Boys generally watch Sports-TV". Such

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<sup>3</sup>A preliminary version of part of this exposition was presented at the II Simpósio Internacional Principia, held at Florianópolis in August 2002 (cf. [Vel'02]).

<sup>4</sup>We would like to have logics for some vague notions, such as one has logics embodying some mathematical notions ([B+F'85], p. 3).

<sup>5</sup>Such notions may also be useful in reporting experimental set-ups and results. More elaborate expressions are also used: a physician may say that a patient's background indicates a certain propensity, making him or her prone to some ailments.

qualitative arguments involving these vague notions appear frequently in several walks of life.<sup>6</sup>

Ideas concerning these notions have appeared in the literature. Some traditional square-of-opposition relations among 'few', 'many', and 'most' have been analysed [Pet'79] and a quantifier for 'most' in the sense of majority has been suggested [Res'62, Sla'88]. Systems with various generalised quantifiers, for notions such as 'many', 'few', 'most', etc., have been considered to be appropriate to treat quantified sentences in natural language (cf. [B+C'81], [Gra'03]). These works are also related to the tradition of analysis and formalisation of language [Fre'79, Tar'36, Chu'56, Mon'74].

We wish to reason about assertions involving vague notions in a precise manner. Here one may feel a certain tension. On the one hand, one needs a clear understanding of 'generally' and 'rarely' for precise reasoning; on the other hand, these notions appear to be quite vague.<sup>7</sup>

Our approach here will be as follows. We will first examine some intuitions behind 'generally' and 'rarely', which will indicate that we actually have various distinct versions of these vague notions. We will then consider abstract versions of these intuitions aiming at a unified treatment, which will in turn suggest how one can handle (some versions of) these vague notions by means of families of subsets. We will finally indicate how these ideas can be used to provide bases for logical systems.

## 2 Some accounts for 'generally' and 'rarely'

Various possible interpretations seem to be associated with vague notions 'generally' and 'rarely'. We shall consider some reasonable ones and examine some intuitions underlying them.

Consider assertions of the form "objects generally have a given property" and "objects rarely have a given property". How is one to understand these assertions? What would be the possible grounds for accepting them? We shall now examine some answers to these questions stemming from possible accounts for versions of 'generally' and 'rarely'.

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<sup>6</sup>A medical doctor usually prescribes a treatment considering it appropriate to a typical patient with such symptoms.

<sup>7</sup>Arguments involving such notions have been considered to be "unruly" to logical methods (cf., e. g., [Tou'58], p. 149). Vagueness is a source of controversies (cf., e. g., [Fin'75], [Wri'75], [Eva'78], [Pea'81]).

## 2.1 Numerical accounts for ‘generally’

Some accounts for ‘generally’ try to explain it in terms of relative frequency or size.

For instance, consider the assertion “Brazilians generally like soccer”. A relative-frequency account for it may be “The Brazilians that like soccer form a ‘likely’ portion”, with more than, say, 75 % of the population.

Now, consider the assertion “Viennese generally like music”. A size-based account for it might be “The Viennese that like music form a ‘sizeable’ set”, in the sense that their number is above, say, 1 million.<sup>8</sup>

These two accounts of ‘generally’ are quite similar.<sup>9</sup> These two accounts may be termed “metric”, as try to reduce it to a measurable aspect, so to speak. They seek to explicate “people generally have a property  $\varphi$ ” as “the people having  $\varphi$  form a ‘likely’ (or ‘sizeable’) set”, i. e. a set having “high” relative frequency (or cardinality), where ‘high’ is understood as above a given threshold.

These two metric accounts, however, differ in one important aspect, related to invariance. This can be seen by considering the relation of having the same size. On the one hand, the size accounts - cardinality above a given threshold - clearly fail to distinguish sets with the same cardinality: they are all either above or below the threshold. We may say that we have a non-local notion. In contrast, sets with the same size may very well have distinct probabilities.<sup>10</sup> Thus, in a probabilistic account of ‘generally’, the family of ‘likely’ sets, is not necessarily invariant under having the same size. It may be said to correspond to a local notion.

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<sup>8</sup>Notice that this threshold may depend on the person. Other persons may be inclined to use other thresholds and accept that a set of Viennese is ‘sizeable’ when its size exceeds 1.2 million or, say, 0.8 million.

<sup>9</sup>A size-based account for “Brazilians generally like soccer” might be “The Brazilians that like soccer form a ‘sizeable’ set”: their number is above, say, 80 million (Brazil has about 170 million inhabitants). Also, a relative-frequency account for “Viennese generally like music” may be “The Viennese that like music form a ‘likely’ portion”, with more than, say, 70 % of the population. Here, we use ‘Brazilian’ and ‘Viennese’ as inhabitant of Brazil and of Vienna, respectively.

<sup>10</sup>For instance, consider the even and odd naturals with probability 1/2; these sets have the same cardinality as the whole set of naturals (which has probability 1). Indeed, any infinite universe  $V$  can be partitioned as the union of two sets  $X$  and  $Y$ , both with the same cardinality as  $V$ ; so  $V$ ,  $X$  and  $Y$  cannot have all the same probability, even though they have the same size. For a finite universe, it suffices to consider a non-uniform distribution.

## 2.2 Relaxed accounts for ‘rarely’ and ‘generally’

The preceding accounts hinge on assigning a threshold, which may seem somewhat arbitrary. Even though they may suffice for some situations, such approaches do not appear to be appropriate for other cases, where they may fail to clarify the underlying intuitions.

We shall now examine some more relaxed accounts. For instance, consider the assertion “Natural numbers rarely divide sixty”. One may interpret, and explain, it by regarding it as asserting that “the divisors of sixty form a ‘small’ set”, where ‘small’ is understood as finite. Similarly, one would understand the assertion “Real numbers rarely are rational” as “the rational reals form a ‘small’ set”, with ‘small’ now taken as (at most) denumerable.

This account of ‘rarely’ is still quantitative and resorts to a threshold, but it is more relaxed. It tries to explicate “objects rarely have property  $\varphi$ ” as “the objects having  $\varphi$  form a ‘small’ set”, under a given sense of ‘small’ (capturing some idea of “having ‘very few’ elements”).

The intended meaning of “objects generally have property  $\varphi$ ” can also be given by means of the set of exceptions, i. e. those objects failing to have property  $\varphi$ . One may understand “Eagles generally fly” as “The non-flying eagles form a ‘small’ set”, which suggests paraphrasing it as “Eagles rarely fail to fly” or “Eagles rarely are non-flying (birds)”.

To illustrate some features of this relaxed account in contrast to the metric accounts, consider the universe of natural numbers and imagine that one accepts the following assertions:

- $\theta$ : “Natural numbers rarely are below thirteen”,
- $\delta$ : “Natural numbers rarely divide twelve”.

In this case, one would probably accept also the assertions:

- $\gamma$ : “Naturals rarely are below thirteen and even”,
- $\eta$ : “Naturals rarely are below thirteen or divide twelve”.

The acceptance of the first two assertions, as well as inferring  $\gamma$  from them, might be explained by a metric account as above. This, however, does not seem to be the case with assertion  $\eta$ .<sup>11</sup> The more relaxed account can explain this situation, as we shall have occasion to see in section 4.

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<sup>11</sup>For instance, considering the universe of Brazilians, those liking basketball and those liking volleyball may have relative frequency below the threshold (of, say, 70 %), but those liking basketball or volleyball may happen to exceed the threshold.

### 2.3 Qualitative accounts for ‘generally’ and ‘rarely’

The accounts of ‘generally’ and ‘rarely’ mentioned so far may be termed “quantitative”. Even though they may suffice for various cases, such accounts do not seem to cover some situations, where these notions appear to present a qualitative character.

As an example, consider the assertion “Real numbers generally are rational”. How is one to understand this assertion? What would be the possible grounds for accepting it? The rationals do not seem to form a “likely”, “sizeable” or “large” set of reals in a quantitative sense: there are too few of them.<sup>12</sup> Yet, there seems to be a sense in which one may accept that ‘Real numbers generally are rational’. Indeed, one may say that “the rationals are ‘almost everywhere’ within the reals”, since near any real one finds a rational. In this sense, the rationals may be said to be “ubiquitous” within the reals [Gra’99, C+G’00]. More precisely, in any open neighbourhood of a real one finds a rational, thus the rational reals form a dense set of reals.

This example illustrates a local qualitative notion of ‘generally’. One explicates “objects generally have property  $\varphi$ ” by saying that “the set of objects having property  $\varphi$  is a dense set” in a given topology.<sup>13</sup>

We can perhaps distinguish the earlier quantitative accounts from the more flexible qualitative accounts in terms of the properties stressed. They are of a topological nature in the latter, rather than metrical as in the former. We can also see that the earlier quantitative versions can be subsumed under the more flexible qualitative notions.

### 2.4 Abstract versions: ‘important’ and ‘negligible’

We now consider some abstract versions of ‘generally’ and ‘rarely’.

We have seen various distinct notions of ‘generally’ and ‘rarely’. In the accounts examined, the intended meaning of “objects generally have a given property  $\varphi$ ” and of “objects rarely have property  $\varphi$ ” can be given in terms of sets of objects: of those objects having  $\varphi$  and of those objects failing to have  $\varphi$ , respectively.

<sup>12</sup>Indeed, it is the assertion “Real numbers generally are irrational” - in the sense “Real numbers rarely are rational” - that appears to be more reasonable, as explained above (in 2.2)

<sup>13</sup>A *dense* subset of the universe, in a given topology, is one having the universe as its closure (or equivalently, one intersecting every non-empty open set) [Kel’55].

We would like to give a unified treatment covering these various distinct notions of 'generally' and 'rarely'. For this purpose, we shall prefer to employ more neutral names encompassing these notions: we will use 'important' in lieu of 'sizeable', 'likely' or 'large' (corresponding to 'generally'), and, accordingly 'negligible' for 'non-sizeable', 'unlikely' or 'small' (corresponding to 'rarely').

The previous terms are somewhat vague, the more so with the new ones. Nevertheless, they present some advantages. First, the reliance on a - somewhat arbitrary - threshold is less stringent. Also, they have a wider range of applications, stemming from the somewhat liberal interpretation of 'important' as carrying considerable weight or importance. For instance, when saying "Unimportant meetings are those attended only by junior staff", one seems to be considering sets including only junior staff members as 'unimportant'.<sup>14</sup>

Notice that these notions of 'important' and 'negligible' are relative to the situation or person, as suggested by the examples.<sup>15</sup>

We have seen some ways of understanding, and explaining, these vague notions. We may summarise these various accounts of 'generally' and 'rarely' as follows.

1. Numeric accounts by 'likely' (high relative-frequency), e. g., "Brazilians generally like soccer" as "More than 75 % of the Brazilians like soccer".
2. Numeric accounts by 'sizeable' (large size), e. g., "Viennese generally like music" as "The Viennese liking music are more than 1 million".
3. Relaxed quantitative accounts by 'small' (cardinality), e. g., "Natural numbers rarely divide sixty" as "The divisors of sixty form a finite set".

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<sup>14</sup>Another example is "Important parties are those attended by the celebrities" (cf. 4.1 in section 4).

<sup>15</sup>Further examples illustrating these points are as follows [Vel'99]. First, consider two sets with the same size: one consisting of a horse and an ox, and another one consisting of a horse and a dog. These sets may be just as important to a conservationist. But, the former may be more important to a farmer, whereas the latter might be preferred by an English gentleman, keen on fox hunting. Now, consider two sets with distinct sizes: one consisting of thirty birds, and another one consisting of a couple of elephants. The Zoo director is likely to consider them equally important. But, an ornithologist might rank the former as more important, whereas a truck driver in charge of transporting them would probably give more attention to the latter. So, a smaller set may be more important than a larger set, or just as important.

4. Qualitative accounts by 'ubiquitous' (dense set), e. g., "Real numbers generally are rational" as "The rationals are almost everywhere within the reals".
5. Abstract accounts by 'important'/'negligible', e. g., "Unimportant meetings are those attended only by junior staff".

These accounts differ with respect to their reliance on a threshold as well as invariance (local or non-local notions). We may summarise these features in the following table.

Account	Reading	Threshold	Invariance
frequency	likely	+	N
size	sizeable	+	Y
relaxed	small	$\pm$	Y
qualitative	ubiquitous	-	N
abstract	important	-	N

### 3 Families for 'generally' and 'rarely'

We will now examine how one can handle (some versions of) vague notions like 'generally' and 'rarely' by means of families of subsets.

Various possible interpretations can be associated to the vague notions of 'generally' and 'rarely'. We would like to give a unified treatment for (some of) them. We might say that we really have a family of notions and we attempt to describe some of their common properties. Towards this goal, we shall try to explain these notions by relying on a relation comparing subsets of a given universe.

As a first candidate for such a comparison one might consider the relation  $\simeq$  of "having about the same size". It is tempting to consider that we have an equivalence relation. Indeed, reflexivity and symmetry seem reasonable; but, what about transitivity?<sup>16</sup> Actually, a notion such as "having about the same size" is not such a good starting point. This is so because one is naturally led to think that sets with the same size should have about

<sup>16</sup>Concerning transitivity: are we prepared to accept that the extremes  $X_0$  and  $X_n$  of a long chain  $X_0 \simeq X_1 \simeq \dots \simeq X_n$  still have about the same size? Even though adjacent sets may differ by a very small amount, the extremes may differ substantially. Transitivity of vague relations is connected to the so-called sorites paradoxes [Sai'89, Edw'72]: my age a second ago and now are practically the same, but I am definitely quite older than when I was born.

the same size. In other words, this is a non-local notion, whereas some of our notions are, in contrast, local ones.

In view of the preceding considerations, we shall use ‘roughly less important than’ for our comparison between subsets of a universe  $V$ , which we shall denote by  $\sqsubseteq$ .<sup>17</sup> We shall try to explicate our notions of ‘important’ and ‘negligible’ by relying on some reasonable properties of this relation  $\sqsubseteq$  between subsets of a given universe. Also, instead of assuming at the outset that we have an equivalence relation, we shall put forward some more basic - and hopefully more palatable - postulates. (This enterprise is somewhat reminiscent of that of “reverse mathematics”, with an important difference.<sup>18</sup>)

### 3.1 Basic ideas

We now have three vague notions, namely the properties ‘negligible’ and ‘important’, as well as the binary relation ‘roughly less important than’. We shall attempt to explain them by means of some properties of these notions, based mainly on common sense and ordinary understanding.

In the sequel, we will examine properties connecting the vague notions ‘negligible’, ‘important’ and ‘roughly less important than’. Given a universe  $V$ , we consider the families  $\mathcal{N}$ , of negligible subsets, and  $\mathcal{W}$ , of important subsets, of universe  $V$ .<sup>19</sup> We shall postulate some reasonable properties connecting these families and the dominance relation  $\sqsubseteq$  of ‘roughly less important than’. The relative character of ‘negligible’ and ‘important’ is embodied in these families and in properties of the binary relation  $\sqsubseteq$  of ‘roughly less important than’, which may vary according to the situation. They, however, can be expected to share some general properties, if they are to be appropriate for capturing reasonable notions of ‘generally’ (and ‘rarely’), corresponding to ‘sizeable’, ‘likely’ or ‘large’ (and ‘non-sizeable’, ‘unlikely’ or ‘small’).

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<sup>17</sup>Previous presentations relied on a more symmetric relation (‘almost as important as’) for comparing subsets of the universe [Vel’99, Vel’02]. The present organisation is more modular and incorporates considerable improvements over the previous approaches (see also 3.5).

<sup>18</sup>“The fundamental question in reverse mathematics is to determine which set existence axioms are required to prove particular theorems of mathematics” ([Sol’99], p. 45). Here, instead of locating familiar axioms, we will be suggesting some new postulates, whence the need for justifying their acceptance on intuitive grounds.

<sup>19</sup>Notice that it is the properties ‘negligible’ and ‘important’ as well as the binary relation that are somewhat vague; the subsets of universe are usual (non-vague) sets (rather than, say, fuzzy sets).

We find convenient to divide our postulates into two groups, namely common postulates and specific postulates, depending on whether the properties they express may be expected to be fundamental to our notions or shared only by some special versions of them.

### 3.2 Common postulates

We shall first consider our basic postulates, expressing properties that may be expected to be common to our vague notions.

We shall be resorting to two kinds of arguments, namely intuitive arguments - based mainly on common sense and ordinary understanding - to try to justify the acceptance of the proposed postulates, as well as (simple) mathematical proofs, to derive - as consequences of our postulates - some properties (that seem to be intuitively expected).

A dictionary explanation for 'negligible' is: "Something that is negligible is so small or unimportant that is not worth considering or worrying about" [Col'87].<sup>20</sup> Also, one usually understands 'negligible' as "fit to be neglected or discarded" [Web'70].

Our idea of a set being 'roughly less important than' another set is "being practically within": except for a part that may be discarded, the former is within the latter. These explanations suggest that it appears reasonable to say that "a set is practically within another when the difference between the former and the latter is negligible".

We are thus led to formulate our first basic postulate, explicating dominance in terms of the family of negligible subsets.

#### 1. *Explicate dominance in terms of negligible*

For subsets  $S$  and  $T$  of the universe  $V$ :  $S$  is dominated by  $T$  iff the difference  $S - T$  is negligible.<sup>21</sup>

$$[\sqsubseteq \mathcal{N}]: S \sqsubseteq T \Leftrightarrow (S - T) \in \mathcal{N}$$

We can now see some immediate - and intuitively reasonable - consequences of this basic postulate  $[\sqsubseteq \mathcal{N}]$ .

First, complementation reverses dominance: if  $S \subseteq V$  is "practically within"  $T \subseteq V$ , then  $\overline{T} \subseteq V$  is "practically within"  $\overline{S} \subseteq V$ .<sup>22</sup>

<sup>20</sup>Notice that this explanation already suggests a connection between 'negligible' and 'important'.

<sup>21</sup>The difference (or relative complement) of two sets consists of the elements in one set but not in the other:  $S - T := \{x \in V : x \in S \& x \notin T\}$ .

<sup>22</sup>The (absolute) complement of a subset of the universe  $V$  consists of the elements (of the universe) outside the set:  $\overline{S} := \{x \in V : x \notin S\}$ .

1.a *Behaviour of dominance under complementation*

For subsets  $S$  and  $T$  of  $V$ :  $S$  is dominated by  $T$  iff the complement of  $S$  dominates the complement of  $T$ .

$$(\sqsubseteq^c): S \sqsubseteq T \Leftrightarrow \overline{T} \sqsubseteq \overline{S}^{23}$$

Second, the negligible subsets can be characterised as those “practically within” the empty set.

1.b *Characterisation of negligible subsets*

A subset  $S \subseteq V$  is negligible iff  $S$  is dominated by the empty set.

$$(\mathcal{N} \sqsubseteq \emptyset): S \in \mathcal{N} \Leftrightarrow S \sqsubseteq \emptyset^{24}$$

Our second postulate is suggested by the above consequence 1.b:  $(\mathcal{N} \sqsubseteq \emptyset)$ , which characterises the negligible subsets as those dominated by the empty set: those “practically within” the empty set. In a dual manner, we would expect the important subsets to be those with the universe “practically within”, i. e. those dominating the universe.

2. *Explicate important as dominating the universe*

A subset  $T \subseteq V$  is important iff  $T$  dominates the universe  $V$ .

$$[V \sqsubseteq \mathcal{W}]: T \in \mathcal{W} \Leftrightarrow V \sqsubseteq T$$

Our intuitive ideas about the notions of ‘negligible’ and ‘important’ suggest that an important subset has negligible complement. This duality is equivalent to 2:  $[V \sqsubseteq \mathcal{W}]$  in view of our first postulate  $[\sqsubseteq \mathcal{N}]$ .

2.a *Duality of negligible and important under complementation*

A subset  $S \subseteq V$  is negligible iff its complement  $\overline{S} \subseteq V$  is important.

$$(\mathcal{N}^c \mathcal{W}): S \in \mathcal{N} \Leftrightarrow \overline{S} \in \mathcal{W}^{25}$$

Our relation of dominance might be trivial in two respects: no dominance or dominance for any pair of subsets. This is not what one would expect and our next two postulates concern aspects of non-triviality of dominance.

We still do not know whether there is any dominance of subsets. Our intuitive ideas about dominance indicate that the empty set is (practically) within the universe. This is the content of our next postulate.

3. *Empty set dominated by universe*

The empty set  $\emptyset$  is dominated by the universe  $V$ .

$$[\emptyset V]: \emptyset \sqsubseteq V$$

<sup>23</sup>Consequence 1.a:  $(\sqsubseteq^c)$  follows from 1:  $[\sqsubseteq \mathcal{N}]$ , since  $\overline{\overline{T}} - \overline{\overline{S}} = S - T$ .

<sup>24</sup>Consequence 1.b:  $(\mathcal{N} \sqsubseteq \emptyset)$  follows from 1:  $[\sqsubseteq \mathcal{N}]$ , since  $S - \emptyset = S$ .

<sup>25</sup>Consequence 2.a:  $(\mathcal{N}^c \mathcal{W})$  is equivalent to 2:  $[V \sqsubseteq \mathcal{W}]$ , by 1.a:  $(\sqsubseteq^c)$  and 1.b:  $(\mathcal{N} \sqsubseteq \emptyset)$ , since  $S \in \mathcal{N}$  iff  $S \sqsubseteq \emptyset$  iff  $\overline{\emptyset} \sqsubseteq \overline{S}$  iff  $\overline{S} \in \mathcal{W}$ .

We can now see some immediate - and reasonable - consequences of our basic postulates so far.

Our intuitive ideas about the notion of 'negligible' suggest that the empty set is (most) negligible (and, dually, the universe is (most) important). Our basic postulates corroborate these ideas.

### 3.a *Empty set negligible*

The empty set  $\emptyset$  is negligible.

$$(\emptyset\mathcal{N}): \emptyset \in \mathcal{N}^{26}$$

Our intuitive ideas suggest that a subset of a set is (practically) within the set. This is the content of our next consequence.

### 3.b *Subset is dominated*

For subsets  $S$  and  $T$  of the universe  $V$ : if  $S$  is a subset of  $T$  then  $S$  is dominated by  $T$ .

$$(\subseteq\sqsubseteq): S \subseteq T \Rightarrow S \sqsubseteq T^{27}$$

We are also led to accept a set as (practically) within itself, as expressed by our next consequence.

### 3.c *Reflexivity of dominance*

Each subset  $S \subseteq V$  dominates itself.

$$(\equiv\sqsubseteq): S \sqsubseteq S^{28}$$

As our next consequence, we have the empty set and the universe as the extremes of dominance.

### 3.d *Extremes of dominance*

Each subset  $S \subseteq V$  dominates the empty set  $\emptyset$  and is dominated by the universe  $V$ .

$$(\perp\top): \emptyset \sqsubseteq S \ \& \ S \sqsubseteq V^{29}$$

For all we know so far, our relation of dominance might collapse many subsets. Our intuitive ideas about dominance suggest that the (nonempty) universe is not (practically) within the empty set. This is the content of our fourth postulate.

### 4. *Universe not dominated by empty set*

The universe  $V$  is not dominated by the empty set  $\emptyset$ .

<sup>26</sup>Consequence 3.a:  $(\emptyset\mathcal{N})$  is equivalent to 3:  $[\emptyset V]$  under 1:  $[\sqsubseteq \mathcal{N}]$ , since  $\emptyset - V = \emptyset$ . Clearly,  $V \in \mathcal{W}$  is equivalent to  $\emptyset \in \mathcal{N}$  in view of the duality 2.a:  $(\mathcal{N}^c\mathcal{W})$ .

<sup>27</sup>Consequence 3.b:  $(\subseteq\sqsubseteq)$  follows from 1:  $[\sqsubseteq \mathcal{N}]$  and 3.a:  $(\emptyset\mathcal{N})$ , as  $S - T = \emptyset$ , when  $S \subseteq T$ .

<sup>28</sup>Consequence 3.c:  $(\equiv\sqsubseteq)$  follows immediately from 3.b:  $(\subseteq\sqsubseteq)$  (as  $S \subseteq S$ ).

<sup>29</sup>Consequence 3.d:  $(\perp\top)$  follows immediately from 3.b:  $(\subseteq\sqsubseteq)$  (as  $\emptyset \subseteq S \subseteq V$ ).

$[V\emptyset]: V \not\subseteq \emptyset$

Our intuitive ideas about the notion of ‘negligible’ suggest that the universe is not negligible (and, dually, the empty set is not important). Our basic postulates corroborate these ideas yielding non-triviality of our families: the existence of non-negligible (and non-important) subsets of the (nonempty) universe.

4.a *Universe not negligible*

The universe  $V$  is not negligible.

$(VN): V \notin \mathcal{N}^{30}$

Summarising, we have four basic postulates (cf. table 1).

1	$\sqsubseteq \mathcal{N}$	$S \sqsubseteq T \Leftrightarrow (S - T) \in \mathcal{N}$	characterise $\sqsubseteq$ via $\mathcal{N}$
2	$V \sqsubseteq \mathcal{W}$	$T \in \mathcal{W} \Leftrightarrow V \sqsubseteq T$	characterise $\mathcal{W}$ via $\sqsubseteq$
3	$\emptyset V$	$\emptyset \sqsubseteq V$	dominance
4	$V\emptyset$	$V \not\subseteq \emptyset$	non-dominance

Table 1: Basic postulates for dominance and families

These four basic postulates have some reasonably acceptable consequences (cf. table 2).

1.a	$\sqsubseteq^c$	$S \sqsubseteq T \Leftrightarrow \overline{T} \sqsubseteq \overline{S}$	$\sqsubseteq$ under complement
1.b	$\mathcal{N} \sqsubseteq \emptyset$	$S \in \mathcal{N} \Leftrightarrow S \sqsubseteq \emptyset$	$\mathcal{W}$ as dominated by $\emptyset$
2.a	$\mathcal{N}^c \mathcal{W}$	$S \in \mathcal{N} \Leftrightarrow \overline{S} \in \mathcal{W}$	$\mathcal{N}$ as complements of $\mathcal{W}$
3.a	$\emptyset \mathcal{N}$	$\emptyset \in \mathcal{N}$	void set is negligible
3.b	$\subseteq \sqsubseteq$	$S \subseteq T \Rightarrow S \sqsubseteq T$	subset dominated
3.c	$= \sqsubseteq$	$S \sqsubseteq S$	dominance is reflexive
3.d	$\perp \top$	$\emptyset \sqsubseteq S \sqsubseteq V$	$\emptyset$ & $V$ : extremes of $\sqsubseteq$
4.a	$V \mathcal{N}$	$V \notin \mathcal{N}$	universe not negligible

Table 2: Consequences of the basic postulates

Thus, these basic postulates, expressing reasonable assumptions about our three vague notions, lead to the some basic properties of the families  $\mathcal{N}$ , of negligible subsets, and  $\mathcal{W}$ , of important subsets, of a (nonempty) universe  $V$ . These families  $\mathcal{N}$  and  $\mathcal{W}$  are

<sup>30</sup>Consequence 4.a:  $(VN)$  is equivalent to 4:  $[V\emptyset]$  under 1.b:  $(\mathcal{N} \sqsubseteq \emptyset)$ . Clearly,  $\emptyset \notin \mathcal{W}$  is equivalent to  $V \notin \mathcal{N}$  in view of the duality 2.a:  $(\mathcal{N}^c \mathcal{W})$ .

- dual ( $S \in \mathcal{N}$  iff  $\bar{S} \in \mathcal{W}$ );
- proper ( $\emptyset \in \mathcal{N}$  &  $V \notin \mathcal{N}$ ;  $\emptyset \notin \mathcal{W}$  &  $V \in \mathcal{W}$ ).

### 3.3 Specific postulates

We shall now consider our specific postulates, expressing properties that may be expected to be shared only by some notions corresponding to generally' and 'rarely'. By duality, each property about a family has its dual version concerning the other family.

These specific postulates will be of a special nature. So, we will introduce them from a somewhat algebraic viewpoint. We shall present some more intuitive reasons for accepting them later on (in 3.5).

Our first specific postulate concerns the behaviour of dominance under union. If two subsets of the universe are practically within a common subset, it may be reasonable to expect that so is their union.

#### A. Dominance under union

If a subset  $R \subseteq V$  dominates subsets  $P$  and  $Q$  of the universe  $V$ ; then  $R$  also dominates their union  $P \cup Q$ .

$[\cup \sqsubseteq]$ :  $P \sqsubseteq R \& Q \sqsubseteq R \Rightarrow P \cup Q \sqsubseteq R$

A consequence of this specific postulate A:  $[\cup \sqsubseteq]$  (actually an equivalent formulation for it) is the following behaviour of the family of negligible subsets under union.

#### A'. Closure of negligible subsets under union

The union  $N' \cup N''$  of negligible subsets  $N'$  and  $N''$  is also negligible.

$(\mathcal{N}\cup)$ :  $N' \in \mathcal{N} \& N'' \in \mathcal{N} \Rightarrow N' \cup N'' \in \mathcal{N}$ <sup>31</sup>

Consequence A':  $(\mathcal{N}\cup)$  is equivalent by duality 2.a:  $(\mathcal{N}^c\mathcal{W})$  to the closure of the family  $\mathcal{W}$  of important subsets under intersection.

Our second specific postulate concerns the behaviour of dominance under intersection. Given two dominance relations, one may find reasonable to expect that the intersection of the lower subsets is dominated by the union of the upper subsets.

#### B. Dominance under intersection

For subsets  $P, Q, S$  and  $T$  of the universe  $V$ : if  $P$  is dominated by  $S$  and  $Q$  is dominated by  $T$ , then the intersection  $P \cap Q$  is dominated by the union  $S \cup T$ .

<sup>31</sup>Consequence A':  $(\mathcal{N}\cup)$  is equivalent to A:  $[\cup \sqsubseteq]$  under 1:  $[\sqsubseteq \mathcal{N}]$ , in view of the equality  $(P \cup Q) - R = (P - R) \cup (Q - R)$ .

$$[\cap \sqsubseteq]: P \subseteq S \& Q \subseteq T \Rightarrow P \cap Q \subseteq S \cup T$$

A consequence of this specific postulate  $[\cap \sqsubseteq]$  (actually equivalent to it) is the following behaviour of the family of negligible subsets under intersection.

*B'. Closure of negligible subsets under intersection*

The intersection  $N' \cap N''$  of negligible subsets  $N'$  and  $N''$  is negligible.

$$(\mathcal{N}\cap): N' \in \mathcal{N} \& N'' \in \mathcal{N} \Rightarrow N' \cap N'' \in \mathcal{N}^{32}$$

Much as before, this consequence B':  $(\mathcal{N}\cap)$  is equivalent by duality 2.a:  $(\mathcal{N}^c\mathcal{W})$  to the closure of the family  $\mathcal{W}$  of the important subsets under union.

Our third specific postulate concerns the behaviour of dominance under inclusion. It may be reasonable to expect that, when a set is practically within another, the same will happen to each subset of the former.

*C. Dominance under inclusion*

For subsets  $Q$  and  $R$  of the universe  $V$ : if set  $R$  dominates set  $Q$ , then  $R$  will also dominate every subset  $P \subseteq Q$ .

$$[\subseteq \sqsubseteq]: P \subseteq Q \& Q \subseteq R \Rightarrow P \subseteq R$$

As a consequence of this specific postulate C:  $[\subseteq \sqsubseteq]$  (actually an equivalent formulation), we have the following behaviour of the family of negligible subsets under inclusion.

*C'. Closure of negligible subsets under subset*

Each subset of a negligible subset is also negligible.

$$(\mathcal{N}\subseteq): S \subseteq N \& N \in \mathcal{N} \Rightarrow S \in \mathcal{N}^{33}$$

We notice that postulate C:  $[\subseteq \sqsubseteq]$  is stronger than postulate B:  $[\subseteq \sqsubseteq]$  (as a family closed under subsets must be closed under intersection). Also, as before by duality 2.a:  $(\mathcal{N}^c\mathcal{W})$ , this consequence C':  $(\mathcal{N}\subseteq)$  is equivalent to the closure of the family  $\mathcal{W}$  of important subsets under supersets.

Our fourth specific postulate concerns viewing negligible and important as alternatives. One may be willing to accept that a subset of the universe must be negligible or important.

*D. Alternatives negligible or important*

If a subset  $S \subseteq V$  is not negligible, then it must be important.

$$[\mathcal{N}|\mathcal{W}]: S \notin \mathcal{N} \Rightarrow S \in \mathcal{W}$$

<sup>32</sup>Consequence B':  $(\mathcal{N}\cap)$  is equivalent to B:  $[\subseteq \sqsubseteq]$  under 1: 1:  $[\sqsubseteq \mathcal{N}]$ , as  $\emptyset \cap \emptyset = \emptyset$  and  $(P \cap Q) - (S \cup T) = (P - S) \cap (Q - T)$ .

<sup>33</sup>Consequence C':  $(\mathcal{N}\subseteq)$  is equivalent to C:  $[\subseteq \sqsubseteq]$  under 1:  $[\sqsubseteq \mathcal{N}]$ , since  $P - R \subseteq Q - R$ , when  $P \subseteq Q$ .

A consequence of this specific postulate  $[\mathcal{N}|\mathcal{W}]$  (actually equivalent to it) by duality 2.a:  $(\mathcal{N}^c\mathcal{W})$  ) is the following behaviour of the family of negligible subsets under complementation.

D'. *Negligible subsets under complementation*

If a subset  $S \subseteq V$  is not negligible, then its complement  $\bar{S} \subseteq V$  is negligible.  $(\mathcal{N}^c)$ :  $S \notin \mathcal{N} \Rightarrow \bar{S} \in \mathcal{N}$

Much as before, this consequence D':  $(\mathcal{N}^c)$  is equivalent to the family  $\mathcal{W}$  of important subsets being prime: if a subset  $T \subseteq V$  is not important, then its complement  $\bar{T} \subseteq V$  is important.

Summarising, we have four specific postulates, with equivalent formulations (cf. table 3).

A	$\cup \sqsubseteq$	$P \sqsubseteq R \& Q \sqsubseteq R \Rightarrow P \cup Q \sqsubseteq R$	$\sqsubseteq$ under union $\cup$
A'	$\mathcal{N} \cup$	$N', N'' \in \mathcal{N} \Rightarrow N' \cup N'' \in \mathcal{N}$	$\mathcal{N}$ $\cup$ -closed
B	$\cap \sqsubseteq$	$P \sqsubseteq S \& Q \sqsubseteq T \Rightarrow P \cap Q \sqsubseteq S \cup T$	$\sqsubseteq$ under intersection $\cap$
B'	$\mathcal{N} \cap$	$N', N'' \in \mathcal{N} \Rightarrow N' \cap N'' \in \mathcal{N}$	$\mathcal{N}$ $\cap$ -closed
C	$\sqsubseteq \sqsubseteq$	$P \sqsubseteq Q \& Q \sqsubseteq R \Rightarrow P \sqsubseteq R$	$\sqsubseteq$ under inclusion $\sqsubseteq$
C'	$\mathcal{N} \sqsubseteq$	$S \sqsubseteq N \in \mathcal{N} \Rightarrow S \in \mathcal{N}$	$\mathcal{N}$ down-closed
D	$\mathcal{N} \mathcal{W}$	$S \notin \mathcal{N} \Rightarrow S \in \mathcal{W}$	Set in $\mathcal{N}$ or in $\mathcal{W}$
D'	$\mathcal{N}^c$	$S \notin \mathcal{N} \Rightarrow \bar{S} \in \mathcal{N}$	$S$ or $\bar{S}$ must be negligible

Table 3: Specific postulates for dominance and families

Each specific postulate on its own may appear somewhat reasonable, but some combinations of them can lead to consequences that are perhaps less palatable. For instance, postulates A:  $[\cup \sqsubseteq]$  and C:  $[\sqsubseteq \sqsubseteq]$  imply the transitivity of dominance:  $P \sqsubseteq Q \& Q \sqsubseteq R \Rightarrow P \sqsubseteq R$ .<sup>34</sup> Also, our postulates can be used to characterise some interesting classes of families of subsets, as we will see in the sequel.

### 3.4 Postulates and families

We will now examine how our postulates can be used to characterise some classes of families of subsets, corresponding to interesting versions of ‘generally’ and ‘rarely’.

<sup>34</sup>Postulates A:  $[\cup \sqsubseteq]$  and C:  $[\sqsubseteq \sqsubseteq]$ , under 1:  $[\sqsubseteq \mathcal{N}]$ , yield the transitivity of dominance  $\sqsubseteq$ , because of the inclusion  $P - R \sqsubseteq (P - Q) \cup (Q - R)$ .

We will consider the following classes of families of (negligible) subsets (corresponding to versions of 'rarely').

- Lattices: closed under union and intersection.
- Down-closed: closed under subsets.
- Ideals: closed under union and subsets.
- Prime ideals: maximal proper ideals.

The dual families of (important) subsets (corresponding to versions of 'generally') form the following classes.

- Lattices: closed under intersection and union.
- Up-closed: closed under supersets.
- Filters: closed under intersection and supersets.
- Ultrafilters: maximal proper filters.

These families have many applications in Logic and in areas of Mathematics. In the sequel, we shall see how to use them for reasoning about (some versions of) 'generally' and 'rarely'.<sup>35</sup>

We now wish to see how our postulates can be used to characterise these classes of families of subsets, corresponding to versions of 'generally' and 'rarely'.

We have already seen (in 3.2) that, by our four basic postulates, the families  $\mathcal{N}$  (of negligible subsets) and  $\mathcal{W}$  (of important subsets) of a (nonempty) universe  $V$  are proper and as dual.<sup>36</sup>

We also know that our specific postulates correspond to closure properties of the families  $\mathcal{N}$  and  $\mathcal{W}$  (cf. table 3 in 3.3). Thus, our specific postulates lead to families of subsets as above.

We now wish to see the converse: how such families of subsets lead to models of our basic and specific postulates. Given such a family, we can

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<sup>35</sup>Some examples are as follows. The sets having more than, say, 70 % of the elements form an up-closed family (corresponding to a notion of 'several'). Both the finite unions of intervals of the reals and the cofinite open subsets of an infinite topological space form lattices (corresponding to notions of 'many'). The subsets including a given nonempty set as well as the cofinite subsets of an infinite universe form filters (corresponding to notions of 'most'). The subsets having a given element of an universe form an ultrafilter.

<sup>36</sup>We have  $\emptyset \in \mathcal{N}$  &  $V \notin \mathcal{N}$ ,  $\emptyset \notin \mathcal{W}$  &  $V \in \mathcal{W}$ , and  $S \in \mathcal{N}$  iff  $\overline{S} \in \mathcal{W}$

construct a structure that will satisfy our four basic postulates, as well as some specific postulates.

This construction can be done as follows. Consider a family  $\mathcal{W} \subseteq \wp(V)$  of (important) subsets of a nonempty universe  $V$ , and define

- family  $\mathcal{N}$  of (negligible) subsets by  $\mathcal{N} := \{S \subseteq V : \bar{S} \in \mathcal{W}\}$ ;
- relation  $\sqsubseteq$  of (dominance) by  $S \sqsubseteq T \Leftrightarrow (S - T) \in \mathcal{N}$ .

We now have a structure  $\mathcal{M} := \langle V, \sqsubseteq, \mathcal{N}, \mathcal{W} \rangle$ . This structure  $\mathcal{M}$  will satisfy our four basic postulates whenever family  $\mathcal{W} \subseteq \wp(V)$  is proper:  $\emptyset \notin \mathcal{W}$  and  $V \in \mathcal{W}$ .<sup>37</sup>

We can also see that each specific postulate corresponds to a closure property of the family  $\mathcal{W} \subseteq \wp(V)$ .<sup>38</sup>

Thus, each one of the above families of subsets corresponds to a set of our specific postulates. This is summarised in the following table.<sup>39</sup>

Postulates	$\Leftrightarrow$	Family $\mathcal{N}$	Family $\mathcal{W}$
A, B		lattice	lattice
C		downward	upward
A, (B,) C		ideal	filter
A, (B,) C, D		prime ideal	ultrafilter

We thus see that each one of the above families  $\mathcal{N}$ , of negligible subsets, and  $\mathcal{W}$ , of important subsets, gives rise to a model of our four basic postulates and corresponding specific postulates.

In this sense, we can say that our postulates characterise these classes of families of subsets (as well as some other interesting classes).

<sup>37</sup>We can see that structure  $\mathcal{M}$  satisfies postulate 1:  $[\sqsubseteq \mathcal{N}]$ , by the definition of  $\sqsubseteq$ . Now, by the definition of  $\mathcal{N}$ ,  $\mathcal{M}$  satisfies 2:  $[V \sqsubseteq \mathcal{W}]$  (by the definition of  $\sqsubseteq$ ), 3.a:  $(\emptyset \mathcal{N})$  (whence 3:  $[\emptyset V]$ ), as  $V \in \mathcal{W}$ , and 4.a:  $(V \mathcal{N})$  (whence 4:  $[V \emptyset]$ ), as  $\emptyset \notin \mathcal{W}$ .

<sup>38</sup>Indeed, we can see that we have

$$\begin{aligned}
 \mathcal{M} \models A &\Leftrightarrow \mathcal{W} \cap\text{-closed} \\
 \mathcal{M} \models B &\Leftrightarrow \mathcal{W} \cup\text{-closed} \\
 \mathcal{M} \models C &\Leftrightarrow \mathcal{W} \text{ up-closed} \\
 \mathcal{M} \models D &\Leftrightarrow \mathcal{W} \text{ prime}
 \end{aligned}$$

This shows a correspondence between specific postulates and closure properties.

<sup>39</sup>As noted before, postulate C implies postulate B.

### 3.5 Specific postulates revisited

We have introduced our specific postulates from a somewhat algebraic viewpoint in 3.3. We now wish to reconsider them: we will present some more intuitive reasons for accepting these specific postulates.

For our present purpose, it is convenient to consider another vague notion, namely a binary relation  $\approx$  of similarity: ‘about as important as’. Our intuitive idea of two sets being “about as important” is that they are “practically the same”: except for a part that may be discarded, the sets are equal.

We are thus led to imagine that this notion of similarity can be explained in terms of the family of negligible subsets, much as before. We thus consider the following definition.

*Define similarity in terms of negligible*

Subsets  $S$  and  $T$  of the universe  $V$  are similar iff both differences  $S - T$  and  $T - S$  are negligible.

$$\approx \mathcal{N}: S \approx T \Leftrightarrow (S - T) \in \mathcal{N} \& (T - S) \in \mathcal{N}$$

We can now see some simple, and intuitively reasonable, consequences of this definition  $\approx \mathcal{N}$  and our basic postulate 1:  $[\sqsubseteq \mathcal{N}]$ .<sup>40</sup>

*Characterise similarity in terms of dominance*

Subsets  $S$  and  $T$  of the universe  $V$  are similar iff they dominate each other.

$$(\approx \sqsubseteq): S \approx T \Leftrightarrow S \sqsubseteq T \& S \sqsupseteq T$$

*Invariance of similarity under complementation*

Subsets  $S$  and  $T$  of the universe  $V$  are similar iff their complements are similar.

$$(\approx^c): S \approx T \Leftrightarrow \bar{S} \approx \bar{T}$$

We will now examine some connections among our notions that may be considered somewhat reasonable.

A basic connection concerns the behaviour of the negligible subsets under similarity. One may expect that subset of the universe that is “practically the same” as a negligible one is negligible as well.

0. *Behaviour of negligible subsets under similarity*

A subset  $S \subseteq V$  similar to a negligible subset  $N \subseteq V$  is also negligible.

$$< \mathcal{N} \approx >: S \approx N \& N \in \mathcal{N} \Rightarrow S \in \mathcal{N}$$

Our next connection concerns the behaviour of similarity under union with negligible sets. One may find reasonable to accept that the addition

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<sup>40</sup>We can also characterise the sets that are both negligible and important as those similar to their own complements:  $S \in \mathcal{N} \cap \mathcal{W} \Leftrightarrow S \approx \bar{S}$ .

of a negligible set should have negligible impact, leaving a set practically the same as before.

I. *Behaviour of similarity under union with negligible set*

The union of subset  $S \subseteq V$  with a negligible subset  $N \subseteq V$  is similar to  $S$ .  
 $\langle +\mathcal{N} \rangle: N \in \mathcal{N} \Rightarrow S \cup N \approx S$

This connection I:  $\langle +\mathcal{N} \rangle$  (in the presence of the basic connection 0:  $\langle \mathcal{N} \approx \rangle$ ) yields consequence A':  $(\mathcal{N} \cup)$ . Thus, these somewhat reasonable connections provide intuitive support for specific postulate A:  $[\cup \sqsubseteq]$ .

Connections 0:  $\langle \mathcal{N} \approx \rangle$  and I:  $\langle +\mathcal{N} \rangle$  yield consequence A':  $(\mathcal{N} \cup)$ .  
 $\langle \mathcal{N} \approx \rangle \ \& \ \langle +\mathcal{N} \rangle \vdash (\mathcal{N} \cup)$ <sup>41</sup>

A connection analogous to the preceding one deals with the behaviour of similarity under removal of a negligible set. As before, one may be willing to accept that the removal of a negligible set should have negligible impact, leaving a set about as important as before.

II. *Behaviour of similarity under removal of negligible set*

The result  $S - N$  of removing a negligible subset  $N \subseteq V$  from subset  $S \subseteq V$  is similar to  $S$ .

$\langle -\mathcal{N} \rangle: N \in \mathcal{N} \Rightarrow S - N \approx S$

This connection II:  $\langle -\mathcal{N} \rangle$  (in the presence of the basic connection 0:  $\langle \mathcal{N} \approx \rangle$ ) yields consequence B':  $(\mathcal{N} \cap)$ . So, these somewhat reasonable connections provide intuitive justification for specific postulate B:  $[\subseteq \sqsubseteq]$ .

Connections 0:  $\langle \mathcal{N} \approx \rangle$  and II:  $\langle -\mathcal{N} \rangle$  yield consequence B':  $(\mathcal{N} \cap)$ .  
 $\langle \mathcal{N} \approx \rangle \ \& \ \langle -\mathcal{N} \rangle \vdash (\mathcal{N} \cap)$ <sup>42</sup>

Another connection analogous to the preceding ones deals with the behaviour of the negligible subsets under dominance  $\sqsubseteq$ . Much as before, one may find reasonable to consider as negligible a subset of the universe that is practically within a negligible one.

III. *Behaviour of negligible subsets under dominance*

A subset  $S \subseteq V$  dominated by a negligible subset  $N \subseteq V$  is negligible.

$\langle \mathcal{N} \sqsubseteq \rangle: S \sqsubseteq N \ \& \ N \in \mathcal{N} \Rightarrow S \in \mathcal{N}$

<sup>41</sup>Indeed, from  $N'' \in \mathcal{N}$ , connection I:  $\langle +\mathcal{N} \rangle$  yields  $N' \cup N'' \approx N'$ , whence, with  $N' \in \mathcal{N}$ , connection 0:  $\langle \mathcal{N} \approx \rangle$  yields  $N' \cup N'' \in \mathcal{N}$ . We thus have consequence A':  $(\mathcal{N} \cup)$ .

<sup>42</sup>Connections 0:  $\langle \mathcal{N} \approx \rangle$  and II:  $\langle -\mathcal{N} \rangle$  yield consequence B':  $(\mathcal{N} \cap)$ , in view of the equality  $N' - (N' - N'') = N' \cap N''$ . Indeed, from  $N'' \in \mathcal{N}$ , II:  $\langle -\mathcal{N} \rangle$  yields  $N' - N'' \approx N'$ , whence, with  $N' \in \mathcal{N}$ , 0:  $\langle \mathcal{N} \approx \rangle$  yields  $N' - (N' - N'') \in \mathcal{N}$ . So we have B':  $(\mathcal{N} \cap)$ .

This connection III:  $\langle \mathcal{N} \sqsubseteq \rangle$  (in the presence of the common postulates) yields consequence C’:  $(\mathcal{N} \subseteq)$ . Thus, this somewhat reasonable connection lends some intuitive justification for specific postulate C:  $[\subseteq \sqsubseteq]$ . Connection III:  $\langle \mathcal{N} \sqsubseteq \rangle$  (under consequence 3.b:  $(\subseteq \sqsubseteq)$ ) yields consequence C’:  $(\mathcal{N} \subseteq)$ .  
 $(\subseteq \sqsubseteq) \& \langle \mathcal{N} \sqsubseteq \rangle \vdash (\mathcal{N} \subseteq)$

We now come to our final connection, which is probably the least intuitively acceptable one (and with more profound impact). The underlying idea is that the universe is so important (i. e. carries so much weight) that any attempt to cover it by finitely many subsets must employ an important subset (one carrying considerable weight, or equivalently, almost as important as the entire universe).<sup>43</sup>

IV. *Finite cover of universe and important subsets*

A finite cover of universe  $V$  must have an important set.

$$\langle VW \rangle: V = T_1 \cup \dots \cup T_n \Rightarrow \exists k : T_k \in \mathcal{W}^{44}$$

This connection IV:  $\langle VW \rangle$  yields consequence D’:  $(\mathcal{N}^c)$ . Thus, this connection provides some intuitive support for specific postulate D:  $[\mathcal{N}|\mathcal{W}]$ .

Connection IV:  $\langle VW \rangle$  yields consequence D’:  $(\mathcal{N}^c)$ .

$$\langle \mathcal{N} \approx \rangle \& \langle -\mathcal{N} \rangle \vdash (\mathcal{N} \cap)$$

Summarising, we have seen some reasons (of varying intuitive appeal) providing some support for accepting our specific postulates, introduced from a somewhat algebraic viewpoint in 3.3.

## 4 Reasoning about ‘generally’ with families

The preceding ideas can be employed to provide bases for precise reasoning with assertions involving (some versions of) the vague notions ‘generally’ and ‘rarely’. In the sequel, we shall first illustrate how these ideas, giving some precise meanings to (versions of) of ‘generally’, also serve to reason and then indicate how one can set up logical systems on these bases.

The intended meaning of ‘generally’ (and ‘rarely’), at least in some cases, can be given by means of families of ‘important’ (and ‘negligible’)

<sup>43</sup>Over an infinite universe, one may regard the finite subsets as not carrying considerable weight. Another example where this connection holds is provided by considering as carrying considerable weight the subsets with elephants.

<sup>44</sup>An equivalent formulation of connection IV:  $\langle VW \rangle$  (by the duality 2.a:  $(\mathcal{N}^c \mathcal{W})$ ) is: an empty finite intersection must have a negligible set.

<sup>45</sup>Case  $n = 2$  of connection IV:  $\langle VW \rangle$  yields consequence D’:  $(\mathcal{N}^c)$ , since  $V = S \cup \bar{S}$ .

sets. Considering a given property  $\varphi$ , one can understand “objects generally have property  $\varphi$ ” as “the objects having  $\varphi$  form an important set”, in the sense of belonging to a given family of important subsets of the universe of discourse.

The preceding section shows that the families of important subsets corresponding to (some) notions of ‘generally’ can be characterised by postulates. Now, these postulates provide bases for analysing, and reasoning about, situations involving assertions with ‘generally’.

#### 4.1 Reasoning with families for ‘generally’

We will now illustrate how the postulates characterising (some versions of) ‘generally’ can be used in analysing, and reasoning about, situations involving assertions with ‘generally’.

We shall first examine a simple example. Consider the universe of Brazilians and imagine that one accepts the two assertions:

- $\varphi$ : “Brazilians generally shave their faces”;
- $\lambda$ : “Brazilians generally shave their legs”.

In this case, one would probably accept also the assertion

- $\mu$ : “Brazilians generally shave their faces or sport a moustache”.

This, however, does not seem to be the case with the assertion

- $\nu$ : “Brazilians generally shave their faces and shave their legs”.

The reason for accepting the assertion  $\mu$  should be clear (see also below). Assertion  $\nu$  can be seen not acceptable by considering males and females.<sup>46</sup>

For convenience, we will employ ‘several’ for the sense of ‘generally’ in this example. Thus, the explanation can be seen to hinge on the following ideas:

- if  $F$  has several elements  $F$  is a subset of  $M$  and  $F \subseteq M$ , then  $M$  also has several elements;

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<sup>46</sup>The “Brazilians that shave their faces” are generally males, whereas the “Brazilians that shave their legs” are generally females. So, the “Brazilians that shave their faces and shave their legs” form a rather small fraction of the population.

- even though both  $F$  and  $L$  have several elements, their intersection  $F \cap L$  may fail to have several elements.

So, the situation in this example can be explained by considering the family  $\mathcal{W}$  of important sets (corresponding ‘generally’) to be closed under supersets, but not under intersection.<sup>47</sup>

For another example, consider the universe of American males<sup>48</sup>. Imagine that one accepts the following three assertions:

- $\beta$ : “American males generally like beer”;
- $\sigma$ : “American males generally like sports”;
- $\varepsilon$ : “American males generally are Democrats or Republicans”.

In this case, one would probably accept also the two assertions:

- $\alpha$ : “American males generally like alcoholic beverages”;
- $\tau$ : “American males generally like beer and sports”.

Acceptance of assertion  $\alpha$  should be clear and an explanation for accepting assertion  $\tau$  can be given by means of the exceptions (see below). On the other hand, even though one accepts the assertion  $\varepsilon$ , neither one of the two assertions “American males generally are Democrats” and “American males generally are Republicans” seems to be equally acceptable.

For convenience, let us use ‘most’ for the sense of ‘generally’ in this example. Thus, the situation can be explained as follows:

- if  $B$  has most element and  $B \subseteq A$ , then  $A$  will have most elements as well;
- if both  $B$  and  $S$  have most elements, their complements  $\overline{B}$  and  $\overline{S}$  are small and so will be their union  $\overline{B \cup S}$  small, thus the intersection will have most elements;
- the union  $D \cup R$  may have most elements, without either  $D$  or  $R$  having most elements.

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<sup>47</sup>Thus, family  $\mathcal{W}$  is up-closed, but not a filter. An up-closed family  $\mathcal{W}$  with both  $F$  and  $L$  also has a superset  $M$  of  $F$ , but not necessarily  $F \cap L$ .

<sup>48</sup>This example is similar to that of natural numbers in 2.2.

Thus, we can account for the situation in this example by considering the family  $\mathcal{W}$  of important sets (corresponding ‘generally’) to be to be closed under supersets and under intersection, but not prime.<sup>49</sup>

The detection of the appropriate notion of ‘generally’, and of the nature of the corresponding family of important sets, will hinge on non-logical information depending on the situation. To illustrate this issue, imagine that a socialite, eager to attend interesting parties, receives pieces of advice as follows:

1. “Important parties are those attended by the celebrities”;
2. “Important parties are those attended by Madonna”.

The former advisor considers a set of guests as important when it includes the celebrities, whereas the latter advisor understands as important sets of guests those where Madonna is. In both interpretations, the family  $\mathcal{W}$  of important sets is a filter, which is an ultrafilter in the Madonna interpretation, but not necessarily so in the celebrities interpretation.<sup>50</sup>

## 4.2 Logics for ‘generally’

We shall now briefly indicate how one can set up logical systems for expressing and reasoning about assertions involving (some versions of) ‘generally’ on the basis of their characteristic postulates. The goal is having logics for some vague notions, much as we have “logics embodying mathematical concepts” [B+F’85].

Our logics for ‘generally’ add to classical first-order logic a (non-standard) generalised quantifier  $\nabla$ , with intended interpretation “forming an important set of objects of the universe of discourse”.

The syntax of our logics is obtained by extending the usual first-order syntax by the new quantifier. We extend the usual first-order syntax by adding the new quantifier  $\nabla$  together with a new (variable-binding) formation rule giving generalised formulas, of the form  $\nabla z\varphi$ .<sup>51</sup>

<sup>49</sup>Thus, family  $\mathcal{W}$  is a filter, but not an ultrafilter. A filter  $\mathcal{W}$  with  $B$ ,  $S$  and  $D \cup R$  also has a superset  $A$  of  $B$  and  $B \cap S$  in  $\mathcal{W}$ , but not necessarily  $D$  or  $R$ .

<sup>50</sup>In both cases, the family  $\mathcal{W}$  is a principal filter: it consists of the sets including a generator. It is an ultrafilter when the generating set has a single element.

<sup>51</sup>With this new quantifier we can express assertions, such as “Birds generally fly” and “Metals generally are solid”, as well as properties like “people are generally taller than  $x$ ”.

The semantics for our logics is obtained by extending the usual first-order definition of satisfaction to the new quantifier. For this purpose, we resort to complex structures, a complex structure  $\mathcal{M}^{\mathcal{K}}$  being the expansion of a first-order structure  $\mathcal{M}$  by a family  $\mathcal{K}$  (of important) subsets of its universe. We then extend the usual Tarskian definition of satisfaction to generalised formulas, so as to capture the above interpretation: a generalised formula  $\nabla z\varphi$  is satisfied iff the extension of  $\varphi$  belongs to the given complex  $\mathcal{K}$ .<sup>52</sup>

So, the propositional connectives as well as the classical quantifiers  $\forall$  and  $\exists$  will keep their familiar interpretations.<sup>53</sup>

We can set up deductive systems for our logics by adding to a calculus for classical first-order logic some schemata: basic and specific schemata. The basic schemata code fundamental properties common to proper complexes.<sup>54</sup>

The specific schemata code closure properties characterising complexes: up-closed families, lattices, filters, or ultrafilters.<sup>55</sup>

These systems provide sound and complete deductive calculi for reasoning about assertions involving ‘generally’.<sup>56</sup> Our logics for ‘generally’ are (proper) conservative extensions of classical first-order logic<sup>57</sup>, with which

<sup>52</sup>More precisely, given a complex structure  $\mathcal{M}^{\mathcal{K}} = \langle \mathcal{M}, \mathcal{K} \rangle$ , for a formula  $\nabla z\varphi(\underline{u}, z)$ , we define  $\mathcal{M}^{\mathcal{K}} \models \nabla z\varphi(\underline{u}, v)[\underline{a}]$  iff  $\{b \in M : \mathcal{M}^{\mathcal{K}} \models \varphi(\underline{u}, z)[\underline{a}, b]\}$  is in  $\mathcal{K}$ .

<sup>53</sup>For a purely first-order formula  $\theta(\underline{u})$  (without  $\nabla$ ),  $\mathcal{M}^{\mathcal{K}} \models \theta(\underline{u})[\underline{a}]$  iff  $\mathcal{M} \models \theta(\underline{u})[\underline{a}]$ .

<sup>54</sup>The four basic schemata are of two kinds.

$$\begin{array}{ll} \nabla z\varphi \rightarrow \exists z\varphi & [\emptyset \notin \mathcal{K}] \\ \forall z\varphi \rightarrow \nabla z\varphi & [M \in \mathcal{K}] \end{array}$$

These two basic schemata code proper complexes.

$$\begin{array}{ll} \nabla z\varphi(z) \leftrightarrow \nabla v\varphi(v), \text{ for a new } v & [\text{alphabetic variant}] \\ \forall z(\psi \leftrightarrow \theta) \rightarrow (\nabla z\psi \leftrightarrow \nabla z\theta) & [\text{extensionality}] \end{array}$$

These two basic schemata code invariance under syntax.

<sup>55</sup>The specific schemata are as follows

$$\begin{array}{ll} \forall z(\psi \rightarrow \theta) \rightarrow (\nabla z\psi \rightarrow \nabla z\theta) & [\text{up-closed}] \\ (\nabla z\psi \wedge \nabla z\theta) \rightarrow \nabla z(\psi \wedge \theta) & [\cap\text{-closed}] \\ (\nabla z\psi \wedge \nabla z\theta) \rightarrow \nabla z(\psi \vee \theta) & [\cup\text{-closed}] \\ \neg \nabla z\varphi \rightarrow \nabla z\neg\varphi & [\text{prime}] \end{array}$$

These schemata code properties of specific complexes.

<sup>56</sup>A sentence  $\tau$  is derivable from a set  $\Gamma$  iff  $\tau$  holds in every complex model of  $\Gamma$  (in each case).

<sup>57</sup>For classical formulas (without  $\nabla$ ), our  $\nabla$ -axioms add no extra deductive power.

they share various metamathematical properties, such as compactness and Löwenheim-Skolem properties.<sup>58</sup>

## 5 Conclusion

We have examined some fundamental issues in the precise treatment of assertions involving ‘generally’ and ‘rarely’, trying to explain them and to clarify the role played by families of subsets in this context.

Assertions and arguments involving vague qualitative notions, such as ‘rarely’, ‘generally’, ‘most’, ‘many’, etc., occur often both in ordinary language and in some branches of science. This provides one of the motivations for undertaking such analyses.

We have examined some meanings for ‘generally’ and ‘rarely’. The analysis of some basic intuitions and their underlying presuppositions has led to distinguishing various versions according to their behaviour. These various versions - corresponding to notions such as ‘several’ (or ‘many’) and ‘most’ - can be rendered precise by resorting to families of subsets. The properties of these families can be used for reasoning about assertions with (some versions of) ‘generally’ and ‘rarely’.

By introducing generalised quantifiers over such families, one can obtain logical systems, which provide rigorous bases for qualitative reasoning about vague notions of ‘generally’. These logics are conservative extensions of classical first-order logic, with which they share various properties. These systems are undergoing further investigation [V+C’01, V+V’01, RHV’01, V+V’02].<sup>59</sup> They appear to have some interesting connections with fuzzy logic [Zad’75] as used in expert systems [Tur’84], natural language [B+C’81, Mon’74] and empirical reasoning<sup>60</sup>. Such connections suggest the possibility of other applications [C+V’77, Vel’98].

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These extensions are proper, because sentences, such as  $\exists u \forall z u \dot{=} z$ , cannot be expressed without  $\nabla$  [V+C’01, Vel’99].

<sup>58</sup>The apparent conflict with Lindström’s results ([Lin’66], [Bar’77]) is explained because we are using a non-standard notion of model (due to the complexes). This feature may confer to our logics for ‘generally’ some independent model-theoretic interest.

<sup>59</sup>These developments include proof methods and sorted versions (to express relative ‘generally’, since relativisation fails to express the intended meaning, due to properties of  $\nabla$  and  $\rightarrow$  [C+V’97, V+C’01]).

<sup>60</sup>Some questions motivating the introduction of sorts to express relative ‘generally’ appear to be connected to the so-called paradoxes of confirmation ([Hem’45], [Sai’89]).

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