

ISOMETRIC GROUP ACTIONS ON BANACH SPACES AND REPRESENTATIONS VANISHING AT INFINITY

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Abstract. Our main result is that the simple Lie group $G = \mathrm{Sp}(n, 1)$ acts metrically properly isometrically on $L^p(G)$ if $p > 4n + 2$. To prove this, we introduce Property (BP_0^V) , with V being a Banach space: a locally compact group G has Property (BP_0^V) if every affine isometric action of G on V , such that the linear part is a C_0 -representation of G , either has a fixed point or is metrically proper. We prove that solvable groups, connected Lie groups, and linear algebraic groups over a local field of characteristic zero, have Property (BP_0^V) . As a consequence, for unitary representations, we characterize those groups in the latter classes for which the first cohomology with respect to the left regular representation on $L^2(G)$ is nonzero; and we characterize uniform lattices in those groups for which the first L^2 -Betti number is nonzero.

1. Introduction

The study of affine isometric actions on Hilbert spaces has, in recent years, found applications ranging from geometric group theory, to rigidity theory and to K -theory of C^* -algebras. For instance, if G is σ -compact, G has Kazhdan's Property (T) if and only if every affine isometric action on a Hilbert space has a fixed point (see [BHV, Chap. 2]). This is known to have strong group-theoretic consequences on G : for instance, this implies that G is compactly generated and has compact abelianization (see [BHV, Chap. 2] for a direct proof). On the other

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hand, a group is said to be a-T-menable if it admits a metrically proper affine action on a Hilbert space, and a deep result of Higson and Kasparov [HiKa] asserts that an a-T-menable group satisfies the Baum–Connes conjecture.

These notions naturally extend to affine isometric actions on Banach spaces. Generalizations of Property (T) to the context of uniformly convex Banach spaces have proven to be fruitful, producing new rigidity results [BFGM], [FM]; on the other hand, a result of Kasparov and Yu [KY] asserts that the Novikov conjecture holds for finitely generated groups embedding uniformly in superreflexive Banach spaces (in particular for those admitting a metrically proper isometric action on such a space).

Yu also recently proved [Yu] that every word hyperbolic group admits a metrically proper isometric action on the uniformly convex space $\ell^p(\Gamma \times \Gamma)$ for $p \gg 0$. This result contrasts with the existence of word hyperbolic groups with Property (T).

Our main result is the following theorem.

Theorem 1.1. *Let k be a local field. Let G be a simple algebraic group of rank 1 over k . Let p_0 be the Hausdorff dimension of the visual boundary of G . Then, for every $p > \max\{1, p_0\}$, there exists a metrically proper affine action of G on $L^p(G)$ with linear part $\lambda_{G,p}$.*

Note that this result cannot be extended to simple groups of rank ≥ 2 (see [BFGM, Theorem B]). We also prove (see Proposition 3.1) that every countable a-T-menable group admits a metrically proper affine action on some L^p -space for every $p \geq 1$.

A key ingredient of the proof of Theorem 1.1 is a result of Pansu [Pa1], [Pa2], who computes the first L^p -cohomology for semisimple Lie groups for $1 < p < \infty$. The first L^p -cohomology actually coincides (see Section 3.2) with the first cohomology of the group with values in the right regular representation on $L^p(G)$. The proof of Theorem 1.1 then consists in showing that nontrivial 1-cocycles on such representations are automatically proper. This latter fact is part of a more general phenomenon: the properness of nontrivial 1-cocycles on an isometric L^p -representation π of a group G is actually true under very general assumptions on G and π .

Our approach was initially motivated by the following example. The cyclic group \mathbf{Z} acts naturally on $\ell^2(\mathbf{Z})$; the corresponding operator T , given by the action of the positive generator of \mathbf{Z} , is usually called the bilateral shift. Now take $f \in \ell^2(\mathbf{Z})$, and let us consider the affine isometry T_f of $\ell^2(\mathbf{Z})$ given by $T_f(v) = Tv + f$. It is immediately checked that this isometry has a fixed point if and only if $f \in \text{Im}(T - 1)$. We show that otherwise the corresponding action is metrically proper, that is, for every/some $v \in \ell^2(\mathbf{Z})$, $\|T_f^n(v)\| \rightarrow \infty$ when $|n| \rightarrow \infty$. Our aim is to make this observation systematic.

One essential feature in the above context is that the representation of \mathbf{Z} on $\ell^2(\mathbf{Z})$ is C_0 . In a general context, let V be a Banach space. An isometric linear representation π of a locally compact group G is C_0 if for every $L \in V^*$ (the topological dual), and every $v \in V$, we have $L(\pi(g)v) \rightarrow 0$ when g tends to infinity. In other words, $\pi(g)v$ weakly tends to zero for every $v \in V$.

Definition 1. Let V be a Banach space. A locally compact group G has *Property* (BP_0^V) if, for every C_0 isometric linear representation π of G on V , any affine isometric action of G with linear part π either has a bounded orbit or is metrically proper. We say that G has *Property* (BP_0) if it has (BP_0^V) for every Hilbert space V .

The acronym (BP_0) stands for “Bounded or Proper with respect to C_0 -representations”.

Thus the observation above amounts to proving that \mathbf{Z} has *Property* (BP_0) . This is part of the following result.

Proposition 1.2 (see Proposition 2.8). *Let G be a locally compact group, and V a Banach space.*

- (1) *Suppose that G has two noncompact normal subgroups centralizing each other. Then G has *Property* (BP_0^V) .*
- (2) *Suppose that G has a noncompact normal subgroup with *Property* (BP_0^V) . Then G has *Property* (BP_0^V) .*

Corollary 1.3 (see Corollary 2.10, Proposition 2.12). *Let V be a Banach space.*

- (1) *Every solvable locally compact group has *Property* (BP_0^V) .*
- (2) *Every connected Lie group or linear algebraic group over a p -adic field has *Property* (BP_0^V) .*

As an application of Corollary 1.3, we characterize in Proposition 4.11 those connected Lie groups and linear algebraic groups over a p -adic field, such that the first cohomology of G with coefficients in the left regular representation λ_G on $L^2(G)$ is nonzero; this generalizes a result of Guichardet (Proposition 8.5 in Chapter III of [Gu2]) for simple Lie groups.

Proposition 1.4 (see Proposition 4.11). *Let G be a connected Lie group or $G = \mathbf{G}(\mathbf{K})$, the group of K -points of a linear algebraic group \mathbf{G} over a local field \mathbf{K} of characteristic zero. Assume G noncompact. Then the following are equivalent:*

- (i) $H^1(G, \lambda_G) \neq 0$.
- (ii) *Either G is amenable, or there exists a compact normal subgroup $K \subset G$ such that G/K is isomorphic to $\mathrm{PSL}_2(\mathbf{R})$ (case of Lie groups), or a simple algebraic group of rank 1 (case of an algebraic group over a p -adic field).*

We also characterize those uniform lattices Γ in a group as above, whose first L^2 -Betti number $\beta_{(2)}^1(\Gamma)$ is nonzero. For uniform lattices in connected Lie groups, this gives a new proof of Theorem 4.1 in [Eck].

Corollary 1.5 (see Corollary 4.12). *Let G be a connected Lie group or $G = \mathbf{G}(\mathbf{K})$ where \mathbf{K} is a local field of characteristic zero; let Γ be a uniform lattice in G . If the first L^2 -Betti number $\beta_{(2)}^1(\Gamma)$ is nonzero, then Γ is commensurable either to a non-abelian free group or to a surface group.*

In contrast with *Property* (BP_0) , we have

Proposition 1.6 (see Proposition 5.1). *There exists an affine isometric action of \mathbf{Z} on a complex Hilbert space, that is neither metrically proper nor bounded, and whose linear part has no finite-dimensional subrepresentation.*

This result can be extended to \mathbf{R} in view of the following result.

Proposition 1.7 (see Proposition 5.2). *Every isometric action of \mathbf{Z} on a complex Hilbert space can be extended to a continuous action of \mathbf{R} .*

While Property (BP_0) is a rule for connected Lie groups or linear algebraic groups over local fields of characteristic zero, this is certainly not the case for discrete groups:

Proposition 1.8 (see Proposition 4.3 and Corollary 4.7). *Non-abelian free groups and surface groups do not have Property (BP_0) .*

The proof of Proposition 1.8 follows from a direct and simple construction for free groups. However, for surface groups, our proof relies on von Neumann algebra arguments as well as the analytical zero divisor conjecture.

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2. Actions on Banach spaces

We define a *Banach pair* as a pair (V, \mathcal{L}) where V is a Banach space and \mathcal{L} is a linear subspace of V^* . We call it a Banach–Steinhaus pair if it satisfies the Banach–Steinhaus Property: any subset $X \subset V$ is bounded if and only if $L(X)$ is bounded for every $L \in \mathcal{L}$. For instance, the Banach–Steinhaus Theorem states that (V^*, V) is a Banach–Steinhaus pair for every Banach space V , and in particular (V, V^*) is a Banach–Steinhaus pair. If (V, \mathcal{L}) is a Banach–Steinhaus pair, and if W is a closed subspace of V , then $(W, \mathcal{L}|_W)$ is clearly a Banach–Steinhaus pair, where $\mathcal{L}|_W$ is the set of restrictions of $L \in \mathcal{L}$ to W .

If (V, \mathcal{L}) is a Banach pair, we say that an isometric linear action π of a locally compact group G on V is $C_0^{\mathcal{L}}$ if $L(\pi(g)v)$ tends to zero when $g \rightarrow \infty$, for every $v \in V$ and $L \in \mathcal{L}$. Note that being $C_0^{\mathcal{L}}$ definitely depends on \mathcal{L} (see the example below); however when the context is clear we write it C_0 .

Example 1. Let G be a discrete, infinite group. Consider its regular ℓ^1 -representation. Then it is $C_0^{c_0(G)}$ but not $C_0^{\ell^\infty(G)}$. Note that both are Banach–Steinhaus pairs. This example motivates the introduction of Banach–Steinhaus pairs different from (V, V^*) .

If π is a C_0 representation as above, and if W is a closed invariant subspace, defining a subrepresentation $\pi|_W$, then $\pi|_W$ is $C_0^{\mathcal{L}|_W}$.

Definition 2. Let (V, \mathcal{L}) be a Banach pair. A locally compact group G has *Property* $(FH_0^{(V, \mathcal{L})})$ (resp., $(BP_0^{(V, \mathcal{L})})$) if, for every C_0 -representation π of G , any affine isometric action of G on V with linear part π has a bounded orbit (resp. either has a bounded orbit or is metrically proper).

We say that G has *Property* $(\text{FH}_0^{([V], \mathcal{L})})$ if it has *Property* $(\text{FH}_0^{(W, \mathcal{L}|_W)})$ for every closed subspace W of V . We define analogously *Property* $(\text{BP}_0^{([V], \mathcal{L})})$.

Similarly, we say that G has *Property* (FH_0^V) (resp., (BP_0^V)) if it has *Property* $(\text{FH}_0^{(V, \mathcal{L})})$ (resp., $(\text{BP}_0^{(V, \mathcal{L})})$) for $\mathcal{L} = V^*$.

When the space V is superreflexive, i.e. isomorphic to a uniformly convex space, it is known that every nonempty bounded subset has a unique circumcentre (also called Chebyshev centre, see page 27 in [BL]). As a consequence, every isometric action with a bounded orbit on V has a globally fixed point.

Lemma 2.1. *Let a compact group K act by affine isometries on a Banach space. Then it fixes a point.*

Proof. Let Ω be an orbit. As Ω is compact, its closed convex hull X is also compact (see for instance [Rud, Theorem 3.25]). As K is amenable and acts on X by affine transformations, it has a fixed point. \square

Lemma 2.2. *Let (V, \mathcal{L}) be a Banach pair. Let K be a compact, normal subgroup of G . The following are equivalent:*

- (i) G has *Property* $(\text{FH}_0^{([V], \mathcal{L})})$ (resp., $(\text{BP}_0^{([V], \mathcal{L})})$);
- (ii) G/K has *Property* $(\text{FH}_0^{([V], \mathcal{L})})$ (resp., $(\text{BP}_0^{([V], \mathcal{L})})$).

Proof. The implication (i) \Rightarrow (ii) is clear. Suppose that the group G/K has *Property* $(\text{BP}_0^{([V], \mathcal{L})})$. By Lemma 2.1, the set W of K -fixed points is a nonempty closed affine subspace; moreover it is G -invariant. As G has *Property* $(\text{BP}_0^{(W, \mathcal{L}|_W)})$, its action on W , and therefore on V , is either bounded or metrically proper. The case of *Property* $(\text{FH}_0^{([V], \mathcal{L})})$ is similar. \square

Lemma 2.3. *Let (V, \mathcal{L}) be a Banach–Steinhaus pair. Let H, K be closed, noncompact subgroups of the locally compact group G which centralize each other. Let α be an affine isometric action of G on V , whose linear part π is a C_0 -representation. Then either $\alpha|_H$ and $\alpha|_K$ are both bounded, or they are both metrically proper.*

Proof. Set $b(g) = \alpha(g)(0)$. We assume that $\alpha|_H$ is not metrically proper, i.e.

$$M =: \liminf_{h \rightarrow \infty, h \in H} \|b(h)\| < \infty.$$

For $k \in K$, $h \in H$, the 1-cocycle relation gives

$$\pi(k)b(h) + b(k) = b(kh) = b(hk) = \pi(h)b(k) + b(h),$$

which we will use in the following form:

$$b(k) = \pi(h)b(k) + (1 - \pi(k))b(h).$$

Then, for every $L \in \mathcal{L}$ we have

$$L(b(k)) = L(\pi(h)b(k)) + L((1 - \pi(k))b(h)),$$

and thus

$$|L(b(k))| \leq |L(\pi(h)b(k))| + |L((1 - \pi(k))b(h))| \leq |L(\pi(h)b(k))| + 2\|L\|\|b(h)\|.$$

Taking the inferior limit when $h \rightarrow \infty$ in H , we obtain

$$|L(b(k))| \leq 2\|L\|M.$$

Thus $L(b(K))$ is bounded for every L ; as (V, \mathcal{L}) is a Banach–Steinhaus pair this means that $b(K)$ is bounded.

Inverting the roles of H and K , we can easily conclude. \square

The following proposition is an immediate consequence of Lemma 2.3, by taking $H = G$ and $K = Z(G)$.

Proposition 2.4. *Let G be a locally compact group with noncompact centre (e.g. a noncompact, locally compact abelian group). Then G has Property $(BP_0^{(V, \mathcal{L})})$ for every Banach–Steinhaus pair (V, \mathcal{L}) . \square*

In order to enlarge the class of groups for which we are able to prove Property (BP_0) , we need the following lemma.

Lemma 2.5. *Let (V, \mathcal{L}) be a Banach–Steinhaus pair. Let α be an affine isometric action of G on V , with linear part a C_0 -representation π . Set $b(g) = \alpha(g)(0)$. Let H be a closed, noncompact subgroup of G . Assume that there exists a sequence $(g_k)_{k \geq 1}$ in G , going to infinity, such that:*

- *the sequence $(b(g_k))$ is bounded in V ;*
- *for every $h \in H$, the set $\{g_k^{-1}hg_k \mid k \geq 1\}$ is relatively compact in G .*

Then $\alpha|_H$ is bounded.

Proof. Fix $M > 0$ such that $\|b(g_k)\| \leq M$ for every $k \geq 1$, and, for $h \in H$ define K_h as the closure of the set $\{g_k^{-1}hg_k \mid k \geq 1\}$, which is compact by assumption. Let us show that $\|b(h)\| \leq 2M$ for every $h \in H$. Noting that $hg_k = g_k h_k$, where $h_k = g_k^{-1}hg_k$, we expand $b(hg_k) - b(g_k) - b(h)$ in two ways, and we obtain

$$\pi(g_k)b(h_k) - b(h) = (\pi(h) - 1)b(g_k).$$

So, for every $L \in \mathcal{L}$,

$$\begin{aligned} |L(b(h))| &\leq |L(\pi(g_k)b(h_k))| + |L((\pi(h) - 1)b(g_k))| \\ &\leq |L(\pi(g_k)b(h_k))| + 2\|L\|\|b(g_k)\|. \end{aligned}$$

Using the assumption that $h_k \in K_h$ for every k , and the fact that for a C_0 -representation the decay of coefficients to 0 is uniform on compact subsets of the ambient Banach space we get, for $k \rightarrow \infty$,

$$|L(b(h))| \leq 2M\|L\|.$$

As (V, \mathcal{L}) is a Banach–Steinhaus pair, this implies that $b(H)$ is bounded. \square

The following lemma is a kind of a geometric Hahn–Banach statement.

Lemma 2.6. *Let (V, \mathcal{L}) be a Banach–Steinhaus pair. Then there exists $\eta > 0$ with the following property: for every $v \in V$, there exists $L \in \mathcal{L}$ such that $\|L\| \leq 1$ and $L(v) \geq \eta\|v\|$.*

Proof. Suppose the contrary. For every n , there exists $v_n \in V$ of norm one such that for every $L \in \mathcal{L}$, we have $L(v_n) < 2^{-n}\|L\|$. Set $X = \{2^n v_n \mid n \geq 0\}$. Then $L(X)$ is bounded for every $L \in \mathcal{L}$; by the Banach–Steinhaus Property, X is bounded; this is a contradiction. \square

Lemma 2.7. *Let (V, \mathcal{L}) be a Banach–Steinhaus pair. Let G be a locally compact group, and N a noncompact, closed normal subgroup. Let α be an affine isometric action of G on V whose linear part is C_0 .*

- (1) *Suppose that $\alpha|_N$ is bounded. Then α is also bounded.*
- (2) *Suppose that $\alpha|_N$ is metrically proper. Then α is also metrically proper.*

Proof. (1) For $M \geq 0$, define A_M as the set of all $x \in V$ whose N -orbit has diameter at most M . Clearly A_M is G -invariant. By assumption, for some M (which we fix now), the set A_M is nonempty. We claim that it is bounded, allowing us to conclude.

Consider $x, y \in A_M$, and set $v = x - y$. Then, for $h \in N$,

$$\pi(h)v - v = \alpha(h)x - x - \alpha(h)y + y.$$

So

$$\|\pi(h)v - v\| \leq \|\alpha(h)x - x\| + \|y - \alpha(h)y\| \leq 2M.$$

Fix η and L as in Lemma 2.6. Then

$$\eta\|v\| \leq L(v) \leq |L(v) - L(\pi(h)v)| + |L(\pi(h)v)| \leq 2M + |L(\pi(h)v)|.$$

As N is noncompact, letting $h \rightarrow \infty$, we obtain $\|v\| \leq 2M/\eta$. Thus the diameter of A_M is bounded by $2M/\eta$.

(2) Suppose by contradiction that $\alpha|_N$ is metrically proper and α is not metrically proper. Then there exists a sequence (g_k) in G , tending to infinity, such that $(b(g_k))$ is bounded. As $\alpha|_N$ is unbounded, there exists, by Lemma 2.5, an element $n \in N$ such that the sequence $(g_k^{-1}ng_k)_{k \geq 1}$ is not relatively compact in N ; extracting if necessary we can suppose that it tends to infinity. Therefore, as $\alpha|_N$ is metrically proper, $\|b(g_k^{-1}ng_k)\|$ tends to infinity. But it is bounded by $2\|b(g_k)\| + \|b(n)\|$, which is bounded, a contradiction. \square

From this we deduce the following.

Proposition 2.8. *Let (V, \mathcal{L}) be a Banach–Steinhaus pair, and let G be a locally compact group. Let N be a noncompact, closed, normal subgroup of G . If either the centralizer $C_G(N)$ of N is noncompact, or N has Property $(BP_0^{(V, \mathcal{L})})$, then G also has Property $(BP_0^{(V, \mathcal{L})})$.*

Proof. Let α be an affine isometric action of G , with linear part a C_0 -representation π . If $\alpha|_N$ is bounded, then α is bounded by Lemma 2.7(1). Assume then that $\alpha|_N$ is unbounded. Then $\alpha|_N$ is metrically proper (in case $C_G(N)$ is noncompact, this follows from Lemma 2.3). Accordingly, by Lemma 2.7(2), α is metrically proper.

\square

Corollary 2.9. *For every Banach–Steinhaus pair (V, \mathcal{L}) , Properties $(\text{BP}_0^{([V], \mathcal{L})})$ and $(\text{FH}_0^{([V], \mathcal{L})})$ are preserved by extensions.*

Proof. Let $1 \rightarrow N \rightarrow G \rightarrow H \rightarrow 1$ be an extension of locally compact groups, and suppose that N and H have Property $(\text{BP}_0^{([V], \mathcal{L})})$. If N is compact, then, by Lemma 2.2, since H has Property $(\text{BP}_0^{([V], \mathcal{L})})$, so does G . If N is not compact, then, since it has Property $(\text{BP}_0^{([V], \mathcal{L})})$, by Proposition 2.8, G has Property $(\text{BP}_0^{([V], \mathcal{L})})$. The case of Property $(\text{FH}_0^{([V], \mathcal{L})})$ is similar (and easier). \square

Corollary 2.10. *Locally compact, solvable groups have Property $(\text{BP}_0^{(V, \mathcal{L})})$ for every Banach–Steinhaus pair (V, \mathcal{L}) .*

Proof. Since, using Proposition 2.4, locally compact abelian groups have Property $(\text{BP}_0^{(V, \mathcal{L})})$, this follows from Corollary 2.9. \square

Lemma 2.11. *Let (V, \mathcal{L}) be a Banach pair. Property $(\text{BP}_0^{(V, \mathcal{L})})$ is inherited from cocompact subgroups.*

Proof. The proof is straightforward. \square

Proposition 2.12. *Connected Lie groups, and linear algebraic groups over p -adic fields, have Property $(\text{BP}_0^{(V, \mathcal{L})})$ for every Banach–Steinhaus pair (V, \mathcal{L}) .*

Proof. This follows from Lemma 2.11 and Corollary 2.10, since G contains a solvable cocompact subgroup H : for linear algebraic groups over local fields of characteristic zero, this follows from [BT, Theorem 8.2]; for Lie groups, taking the quotient by the maximal solvable normal subgroup, we can also use [BT, Theorem 8.2]. \square

3. Metrically proper affine actions of rank 1 groups on L^p -spaces

3.1. Spaces with measured walls, and the non-Archimedean case

Recall that a locally compact σ -compact group is a-T-menable if it acts metrically properly isometrically on some Hilbert space.

Proposition 3.1. *Let Γ be a countable, discrete group. The following are equivalent:*

- (i) Γ is a-T-menable;
- (ii) for every $p \geq 1$, the group Γ acts metrically properly isometrically on some L^p -space.

Proof. We prove the nontrivial implication (i) \Rightarrow (ii). We recall from [CMV] that a space with measured walls is a 4-tuple $(X, \mathcal{W}, \mathcal{B}, \mu)$ where X is a set, \mathcal{W} is a set of partitions of X into two classes (called walls), \mathcal{B} is a σ -algebra of sets on \mathcal{W} , and μ is a measure on \mathcal{B} such that, for every pair x, y of distinct points in X , the set $\omega(x, y)$ of walls separating x from y belongs to \mathcal{B} and satisfies $w(x, y) =: \mu(\omega(x, y)) < \infty$.

It was proved in Proposition 1 of [CMV] that a countable group is a-T-menable if and only if it admits a metrically proper action on some space with measured walls (by this we mean that Γ preserves the measured wall space structure, and that the function $g \mapsto w(gx, x)$ is proper on Γ).

A half-space in a space with measured walls X is a class of the partition defined by some wall in \mathcal{W} . Let Ω be the set of half-spaces, $p : \Omega \rightarrow \mathcal{W}$ the canonical map (associating to any half-space the corresponding wall), $\mathcal{A} := p^{-1}(\mathcal{B})$ the pulled-back σ -algebra, and ν the pulled-back measure defined by

$$\nu(A) = \frac{1}{2} \int_{\mathcal{W}} \text{card}(A \cap p^{-1}(x)) d\mu(x)$$

for $A \in \mathcal{A}$. Let χ_x be the characteristic function of the set of half-spaces through x . For $x, y \in X$, we define a function $c(x, y) \in L^p(\Omega, \nu)$ by

$$c(x, y) = \chi_x - \chi_y.$$

Suppose that Γ acts properly on $(X, \mathcal{W}, \mathcal{B}, \mu)$. For $p \geq 1$, let π_p denote the quasi-regular representation of Γ on $L^p(\Omega, \nu)$. Observe that:

- $c(x, y) + c(y, z) = c(x, z)$;
- $c(gx, gy) = \pi_p(g)c(x, y)$;
- $\|c(x, y)\|_p^p = w(x, y)$;

for every $x, y, z \in X, g \in \Gamma$. Fixing a base-point $x_0 \in X$, the map

$$b : \Gamma \rightarrow L^p(\Omega, \nu) : g \mapsto c(gx_0, x_0)$$

defines a 1-cocycle in $Z^1(\Gamma, \pi_p)$. Since $\|b(g)\|_p = w(gx_0, x_0)^{1/p}$, this cocycle is proper, so that the corresponding affine isometric action is metrically proper. \square

Remark 1. What the above proof really shows is that every locally compact group acting properly on a space with measured walls, admits a metrically proper action on some L^p -space for every $p \geq 1$. Several nondiscrete examples appear in [CMV].

A tree $X = (V, E)$ is an example of a space with measured walls (with $\mathcal{W} = E, \mu =$ counting measure). The set Ω of half-spaces identifies with the set \mathbb{E} of oriented edges. Suppose that a locally compact group G acts properly cocompactly on a tree X . We choose a reference edge $e_0 \in \mathbb{E}$ and use it to lift the cocycle $b \in Z^1(G, \ell^p(\mathbb{E}))$ from the previous proof to a cocycle $\tilde{b} \in Z^1(G, L^p(G))$, by the formula $(\tilde{b}(g))(h) = (b(g))(he_0)$. Then

$$\|\tilde{b}(g)\|_p^p = \frac{m_0 d(gx_0, x_0)}{k},$$

where m_0 is the Haar measure of the stabilizer of e_0 in G , and k is the number of orbits of G in \mathbb{E} . This shows that \tilde{b} is a proper cocycle. We have proved:

Proposition 3.2. *Let G be a locally compact group. If G acts properly cocompactly on a locally finite tree (e.g., if G is a rank 1 simple algebraic group over a non-Archimedean local field), then for every $p \geq 1$, the group G admits a metrically proper isometric action on $L^p(G)$, with linear part the left regular representation $\lambda_{G,p}$. \square*

3.2. The Lie group case

Let M be a Riemannian manifold equipped with its Riemannian measure μ . Fix $p > 1$. Denote by $D_p(M)$ the vector space of differentiable functions whose gradient is in $L^p(TM)$. Equip $D_p(M)$ with a pseudo-norm $\|f\|_{D_p} = \|\nabla f\|_p$, which induces a norm on $D_p(M)$ modulo the constants. Denote by $\mathbf{D}_p(M)$ the completion of this normed vector space. We have $W^{1,p}(M) = L^p(M) \cap D_p(M)$. Hence, $W^{1,p}(M)$ canonically embeds in $\mathbf{D}_p(M)$ as a subspace.

The first L^p -cohomology of M is the quotient space

$$H_p^1(M) = \mathbf{D}_p(M)/W^{1,p}(M).$$

The first *reduced* L^p -cohomology of M is the quotient space

$$\overline{H}_p^1(M) = \mathbf{D}_p(M)/\overline{W^{1,p}(M)},$$

where $\overline{W^{1,p}(M)}$ is the closure of $W^{1,p}(M)$ in the Banach space $\mathbf{D}_p(M)$. Note that the two latter spaces coincide if and only if the norm $\|\cdot\|_{D_p}$ on the Sobolev space $W^{1,p}(M)$ is equivalent to the usual Sobolev norm $\|\cdot\|_p + \|\cdot\|_{D_p}$, that is, if M satisfies the strong Sobolev inequality in L^p : $\|f\|_p \leq C\|\nabla f\|_p$. If the group of isometries G of M acts cocompactly on M , the strong Sobolev inequality in L^p is satisfied if and only if G is either nonamenable or nonunimodular [Pit].

Assume now that $M = G$ is a connected, unimodular Lie group, endowed with a left-invariant Riemannian metric. Denote by $\rho_{G,p}$ the right regular representation on $D_p(G)$. Let $g \in G$ and $\gamma : [0, d(1, g)] \rightarrow G$ be a geodesic between 1 and g . For any $f \in D_p(G)$ and $x \in G$, we have

$$(f - \rho_{G,p}(g)f)(x) = f(x) - f(xg) = \int_0^{d(1,g)} \nabla f(\gamma_x(t)) \cdot \gamma'_x(t) dt,$$

where $\gamma_x(t) = x\gamma(t)$. Using Hölder's inequality, we deduce that

$$\|f - \rho_{G,p}(g)f\|_p \leq d(1, g)\|\nabla f\|_p.$$

Therefore, there is a well-defined map from $D_p(G)$ to $Z^1(G, \rho_{G,p})$

$$J : f \mapsto (b_f : g \mapsto f - \rho_{G,p}(g)f).$$

The map J induces an injective map from $\mathbf{D}_p(G)$ to $Z^1(G, \rho_{G,p})$. Moreover, b_f is a coboundary if and only if f is in $L^p(G) + \{\text{constants}\}$, i.e. if and only if the class of f is zero in $H_p^1(G)$. Hence, J induces an injective¹ linear map from $H_p^1(G)$ to $H^1(G, \rho_{G,p})$.

Let G be a simple Lie group of rank 1, with finite centre, equipped with a left-invariant Riemannian metric. Up to taking the quotient by the centre, G is $\text{PO}(n, 1)$, $\text{PU}(n, 1)$, $\text{PSp}(n, 1)$ or $\text{F}_{4(-20)}$ for some $n \geq 2$. Let ∂G be the sphere

¹Actually, J induces an isomorphism of topological vector spaces [T, Prop. 6.3] but this is more delicate and not needed here.

at infinity of the Riemannian symmetric space of G , and let p_0 be its Hausdorff dimension, so that

$$p_0 = \begin{cases} n - 1 & \text{if } G = \text{PO}(n, 1), \\ 2n & \text{if } G = \text{PU}(n, 1), \\ 4n + 2 & \text{if } G = \text{PSp}(n, 1), \\ 22 & \text{if } G = \text{F}_4(-20). \end{cases}$$

By a result of P. Pansu [Pa1], [Pa2], $H^1_{(p)}(G) \neq 0$ if and only if $p > \max\{1, p_0\}$. From the above discussion, we deduce that $H^1(G, \rho_{G,p}) \neq 0$ for those groups as soon as $p > \max\{1, p_0\}$. Together with the fact that connected Lie groups have Property $(\text{BP}_0^{L^p})$ for $1 < p < \infty$, this yields the following result.

Theorem 3.3. *Let G be a simple Lie group of rank 1 over k . Let p_0 be the Hausdorff dimension of the visual boundary of G . Then, for every $p > \max\{1, p_0\}$, there exists a metrically proper affine action of G on $L^p(G)$ with linear² part $\lambda_{G,p}$.*

4. Actions on Hilbert spaces

4.1. Property (BP_0)

Recall that a locally compact group G has Property (FH) if every affine isometric action of G on a Hilbert space has a fixed point. For G σ -compact, this is known to be equivalent to the celebrated Kazhdan's Property (T) (see [BHV, Chap. 2]).

When V is a Hilbert space (sufficiently large in comparison to G), we write (BP_0) and (FH_0) for (BP_0^V) and (FH_0^V) .

There is a simple characterization of groups with Property (FH_0) among groups with Property (BP_0) .

Proposition 4.1. *Let G be a locally compact group.*

- (1) *Suppose that G has Property (BP_0) . Then either G is a-T-menable or has Property (FH_0) .*
- (2) *If G is both a-T-menable and has Property (FH_0) , then it is compact.*

Proof. The first statement is clear. Suppose that G is a-T-menable and is not compact. Then G is σ -compact (take a metrically proper action α and consider $K_n = \{g \in G \mid \|\alpha(g)(0)\| \leq n\}$). Since G is a-T-menable, it is Haagerup, i.e. it has a C_0 -representation π with almost invariant vectors; since G is not compact, π has no invariant vector. By Proposition 2.5.3 in [BHV], $\infty\pi$ has nontrivial 1-cohomology, while it is C_0 . Hence G does not have Property (FH_0) . \square

Let us mention an application of Property (BP_0) in ergodic theory.

Proposition 4.2. *Let G be a locally compact group with Property (BP_0) and $\text{Hom}(G, \mathbf{R}) = 0$. Let G act (on the right), in a measure-preserving way, on a probability space (X, \mathcal{B}, μ) ; assume that the action is mixing. Let $F : X \times G \rightarrow \mathbf{C}$ be a measurable function such that:*

²Since G is unimodular, the representations $\lambda_{G,p}$ and $\rho_{G,p}$ are isomorphic.

- $\int_X |F(x, g)|^2 d\mu(x) < \infty$ for every $g \in G$;
- $F(x, gh) = F(x, g) + F(x.g, h)$ for every $g, h \in G$, almost everywhere in x .

Then the following alternative holds: either there exists $f \in L^2(X, \mu)$ such that $F(x, g) = f(x.g) - f(x)$ (for every $g \in G$, almost everywhere in x), or

$$\lim_{g \rightarrow \infty} \int_X |F(x, g)|^2 d\mu(x) = \infty.$$

Proof. Set $L_0^2(X, \mu) = \{f \in L^2(X, \mu) \mid \int_X f(x) d\mu(x) = 0\}$. As $\int_X F(x, gh) d\mu(x) = \int_X F(x, g) d\mu(x) + \int_X F(x, h) d\mu(x)$ for every $g, h \in G$, we have $F(\cdot, g) \in L_0^2(X, \mu)$ for every $g \in G$, in view of the assumption $\text{Hom}(G, \mathbf{R}) = 0$. Let π be the standard representation of G on $L_0^2(X, \mu)$. Then $F(\cdot, g)$ defines a 1-cocycle with respect to π . Since the G -action on X is mixing, the representation π is C_0 (see Theorem 2.9 in [BM]), so that the conclusion follows immediately from Property (BP₀). \square

4.2. Discrete groups without (BP₀)

Proposition 2.12 provides a wealth of groups with Property (BP₀). We now provide examples of groups *without* Property (BP₀) (in particular the free group F_n on n generators, $n \geq 2$).

Proposition 4.3. *Let H be an infinite group, K a nontrivial group, and F a common finite subgroup of H and K , which is distinct from K . Let $G = H *_F K$ be the amalgamated product. Then there exists a 1-cocycle with respect to the regular representation λ_G which is neither bounded nor proper. In particular, G does not have Property (BP₀).*

Proof. Let w be an ℓ^2 function on G which is left F -invariant, but not left K -invariant (in particular $w \neq 0$). Define $\alpha(k) = \lambda_G(k)$ for $k \in K$, and $\alpha(h) = t_w \circ \lambda_G(h) \circ t_{-w}$ for $h \in H$, where t_w denotes the translation by w on $\ell^2(G)$. Then α is well defined on $H *_F K$ (by the F -invariance assumption on w). The fixed point set of K is the set of all left K -invariant functions. The set of fixed points of H is reduced to $\{w\}$ (since H is infinite). Accordingly, the action has no fixed point. On the other hand, since H is infinite and has a fixed point, the action is not metrically proper. \square

To obtain other examples of groups without (BP₀), we first establish a connection with a classical conjecture on discrete groups. For a group Γ , we denote by $\mathbf{C}\Gamma$ the group algebra over \mathbf{C} , and by denote again by λ_Γ the left regular representation of $\mathbf{C}\Gamma$ on $\ell^2(\Gamma)$:

$$\lambda_\Gamma(f)\xi = f * \xi$$

($f \in \mathbf{C}\Gamma, \xi \in \ell^2(\Gamma)$). Here is the *analytical zero-divisor conjecture*:

Conjecture 1. *If Γ is a torsion-free group, then $\lambda_\Gamma(f)$ is injective, for every nonzero $f \in \mathbf{C}\Gamma$.*

The main result on Conjecture 1 is due to P. Linnell [Lin]: it holds for groups that are extensions of a right-orderable group by an elementary amenable group; in particular, we will use the fact that it holds for non-abelian free groups.

Lemma 4.4. *Let Γ be a torsion-free group satisfying Conjecture 1. Let $f_1, f_2 \in \mathbf{C}\Gamma$ be nonzero elements. There exists nonzero functions $\xi_1, \xi_2 \in \ell^2(\Gamma)$ such that $\lambda_\Gamma(f_1)\xi_1 + \lambda_\Gamma(f_2)\xi_2 = 0$.*

Proof. We start with a

Claim. *If $f \in \mathbf{C}\Gamma$ is a nonzero element, then $\lambda_\Gamma(f)$ has dense image.*

To see this, observe that the orthogonal of the image of $\lambda_\Gamma(f)$ is the kernel of $\lambda_\Gamma(f^*)$, which is $\{0\}$ as Γ satisfies Conjecture 1.

Let $L(\Gamma)$ be the von Neumann algebra of Γ , i.e. the bicommutant of $\lambda_\Gamma(\mathbf{C}\Gamma)$ in $\mathcal{B}(\ell^2(\Gamma))$. A (nonnecessarily closed) subspace of $\ell^2(\Gamma)$ is *affiliated with* $L(\Gamma)$ if it is invariant under the commutant $\lambda_\Gamma(\mathbf{C}\Gamma)'$ of $\lambda_\Gamma(\mathbf{C}\Gamma)$. For example, if $f \in \mathbf{C}\Gamma$, the image of $\lambda_\Gamma(f)$ is an affiliated subspace. A result of L. Aagaard [Aag] states that the intersection of two dense, affiliated subspaces is still dense. We apply this with the images of $\lambda_\Gamma(f_1)$ and of $\lambda_\Gamma(-f_2)$, so that there exist nonzero ξ_1, ξ_2 such that $\lambda_\Gamma(f_1)\xi_1 = \lambda_\Gamma(-f_2)\xi_2$. \square

Proposition 4.5. *Fix $k \geq 2$. Let w be a nontrivial reduced word in the free group F_k . There exists an unbounded 1-cocycle $b_w \in Z^1(F_k, \lambda_{F_k})$, such that $b_w(w) = 0$. In particular, b_w is not proper.*

Proof. We start with $k = 2$. Write F_2 as the free group on two generators s, t . Write w as a reduced word in $s^{\pm 1}, t^{\pm 1}$:

$$w = x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_n^{\epsilon_n}$$

($x_j \in \{s, t\}; \epsilon_j \in \{-1, 1\}$). If either s or t does not appear in w , then the existence of the desired cocycle follows from the proof of Proposition 4.3 (with $H = K = \mathbf{Z}$). So we may assume that both s and t appear in w . Set $\delta_j = (\epsilon_j - 1)/2$ and define two elements $f_{w,s}, f_{w,t} \in \mathbf{C}F_2$ by

$$\begin{aligned} f_{w,s} &= \sum_{j: x_j=s} \epsilon_j x_1^{\epsilon_1} \dots x_{j-1}^{\epsilon_{j-1}} x_j^{\delta_j}, \\ f_{w,t} &= \sum_{j: x_j=t} \epsilon_j x_1^{\epsilon_1} \dots x_{j-1}^{\epsilon_{j-1}} x_j^{\delta_j}. \end{aligned}$$

Note that $f_{w,s}$ and $f_{w,t}$ are nonzero, as s and t appear in w . Since F_2 satisfies Conjecture 1 (by Linnell's result already quoted [Lin]), we may appeal to Lemma 4.4 and find nonzero functions $\xi_s, \xi_t \in \ell^2(F_2)$ such that $\lambda_{F_2}(f_{w,s})\xi_s + \lambda_{F_2}(f_{w,t})\xi_t = 0$.

Then set $b_w(s) = \xi_s$, $b_w(t) = \xi_t$ and, using the freeness of F_2 , extend uniquely to a 1-cocycle $b_w \in Z^1(F_2, \lambda_{F_2})$. Using the relations

$$b(g_1 g_2 \dots g_m) = \sum_{j=1}^m \lambda_{F_2}(g_1 \dots g_{j-1}) b(g_j)$$

and $b(g^{-1}) = -\lambda_{F_2}(g^{-1})b(g)$ (for a cocycle b and $g_1, \dots, g_m, g \in F_2$), one checks that

$$b_w(w) = \lambda_{F_2}(f_{w,s})b_w(s) + \lambda_{F_2}(f_{w,t})b_w(t) = 0.$$

It remains to show that b_w is unbounded, i.e. that the corresponding affine action α_w has no fixed point. Let $H = \langle w \rangle$ be the cyclic subgroup generated by w . As the linear action is C_0 , the only fixed point of $\alpha_w|_H$ is 0. But 0 is clearly not fixed under $\alpha_w(s)$ or $\alpha_w(t)$, which completes the proof in case $k = 2$.

Suppose now $k \geq 2$. View F_k as a subgroup of index $k - 1$ in F_2 . The restriction of λ_{F_2} to F_k is the direct sum of $k - 1$ copies of λ_{F_k} . Project the cocycle b_w , given by the first part of the proof, to each of these $k - 1$ summands. This way, we get $k - 1$ cocycles in $Z^1(F_k, \lambda_{F_k})$, each of them vanishing on w . At least one of them is unbounded, because $b|_{F_k}$ is unbounded. \square

Corollary 4.6. *For $k \geq 2$, let $\Gamma = F_k *_Z G$ be an amalgamated product over Z , an infinite cyclic subgroup. Then Γ does not have Property (BP_0) .*

Proof. Let $w \in F_k$ and $g \in G$ correspond to the positive generators of the copies of Z that are amalgamated. Choosing representatives for the left cosets of F_k in Γ , identify $\lambda_\Gamma|_{F_k}$ with $\infty \lambda_{F_k} =: \lambda_{F_k} \oplus \lambda_{F_k} \oplus \dots$. Let $b_w \in Z^1(F_k, \lambda_{F_k})$, as in Proposition 4.5. Define an affine action α of F_k , with linear part $\lambda_\Gamma|_{F_k}$, by

$$\alpha(x)(v_1, v_2, v_3, \dots) = (\lambda_{F_k}(x)v_1 + b_w(x), \lambda_{F_k}(x)v_2, \lambda_{F_k}(x)v_3, \dots)$$

($x \in F_k$). On the other hand, view $\lambda_\Gamma|_G$ as an affine action of G . Since

$$\alpha(w) = \lambda_\Gamma(w) = \lambda_\Gamma(g),$$

these two affine actions can be “glued together”, i.e. extend to an affine action $\tilde{\alpha}$ of Γ , with linear part λ_Γ . By the very construction, $\tilde{\alpha}$ has unbounded orbits and is not metrically proper. \square

Corollary 4.7. *The surface groups Γ_g ($g \geq 2$) do not have Property (BP_0) .*

Proof. The presentation

$$\Gamma_g = \left\langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1]^{-1} = \prod_{j=2}^g [a_j, b_j] \right\rangle$$

exhibits Γ_g as an amalgamated product $F_2 *_Z F_{2g-2}$ so Corollary 4.6 applies. \square

Here is an intriguing question, in view of the fact that $\mathrm{PSL}_2(\mathbf{Z})$ contains a free group of finite index.

Question 1. *Does $\mathrm{PSL}_2(\mathbf{Z}) \simeq C_2 * C_3$ have Property (BP_0) ?*

4.3. Application to the regular representation

Let us recall that Guichardet [Gu1, Théorème 1] proved that, if π is a representation without nonzero fixed vector of a locally compact, σ -compact group, the space $B^1(G, \pi)$ is closed in $Z^1(G, \pi)$ (endowed with the topology of uniform convergence on compact subsets) if and only if π does not almost have invariant vectors. In particular, $H^1(G, \pi) \neq 0$ if π almost has invariant vectors. This rests on a clever use of the open mapping theorem for Fréchet spaces. Using this, we can reprove the following result, first proved in [AW] (see also [BCV]).

Proposition 4.8. *Let G be a σ -compact, locally compact group. If G is Haagerup, then it is a - T -menable.*

Proof. Set $H = G \times \mathbf{Z}$; then H is σ -compact, locally compact, is Haagerup, and has noncompact centre. Hence, by Proposition 2.4, it has Property (BP_0) . Take a C_0 -representation π of H , almost having invariant vectors. By Guichardet's result recalled above, there exists an affine action α of H , with linear part π , and without fixed point. By Property (BP_0) , the action α is metrically proper. So the restriction $\alpha|_G$ is metrically proper too. \square

If G is σ -compact and amenable, the representation π in the above proof can be taken as the left regular representation of $G \times \mathbf{Z}$ on $L^2(G \times \mathbf{Z})$. (By way of contrast, if Γ is a discrete, nonamenable group, then $H^1(\Gamma \times \mathbf{Z}, \lambda_{\Gamma \times \mathbf{Z}}) = 0$ by Corollary 10 in [BV]).

Concerning affine actions on $L^2(G)$, we have the following:

Conjecture 2. *For an amenable group G , every affine action with linear part λ_G is either bounded or metrically proper.*

Evidence for this conjecture comes from the fact that Proposition 2.8, Corollary 2.10 and Proposition 2.12 establish it in numerous cases: amenable groups with infinite centre, solvable groups, amenable Lie groups, etc. More evidence comes from a result proved in [MV]: if Γ is a countable amenable group, and A is any infinite subgroup, then the restriction map $H^1(\Gamma, \lambda_\Gamma) \rightarrow H^1(A, \lambda_{\Gamma|_A})$ is injective. If true, our conjecture would provide a conceptual explanation of this fact.

Being more ambitious, one may even ask

Question 2. *Does every amenable group have Property (BP_0) ?*

We now turn to the study of some groups G for which $H^1(G, \lambda_G) \neq 0$.

Lemma 4.9. *Let G be a locally compact, second countable group. Suppose that, for some $k \geq 2$, the group G has closed normal subgroups N_1, \dots, N_k such that $[N_i, N_j] = 1$ whenever $i \neq j$ and $G = N_1 \dots N_k$. Let π be a unitary representation such that $\overline{H^1}(G, \pi) \neq 0$. Then at least one of the N_i has an invariant vector by π .*

Proof. There is an obvious map p of $N = \prod_{i=1}^k N_i$ onto G . Then $\overline{H^1}(N, \pi \circ p) \neq 0$. This uses the standard fact that every compact subset of G is the image of a compact subset of N (note that we use here σ -compactness).

Suppose that for some i , the group N_i has no invariant vector by $\pi \circ p$. Write $N = N_i \times \prod_{j \neq i} N_j$; by [Sha, Prop. 3.2] (which uses second countability), $\prod_{j \neq i} N_j$ has an invariant vector by $\pi \circ p$, so that for every $j \neq i$, N_j has an invariant vector by $\pi \circ p$. \square

Proposition 4.10. *Let G be a connected Lie group or $G = \mathbf{G}(\mathbf{K})$, the group of \mathbf{K} -points of a linear algebraic group \mathbf{G} over a local field \mathbf{K} of characteristic zero. Suppose that G has a C_0 -representation π such that $\overline{H^1}(G, \pi) \neq 0$. Then either G is amenable, or has a compact subgroup K such that G/K is a simple Lie group (resp., a simple linear algebraic group) with trivial centre.*

Proof. By Property (BP₀), G has the Haagerup property (as recalled in the proof of Proposition 4.1). If G is a connected Lie group, by [CCJJV, Chap. 4], $G = RS_1 \dots S_k$ where R, S_1, \dots, S_k centralize each other, R is a connected amenable Lie group, and each S_i is a simple, noncompact, connected Lie group with the Haagerup Property (with possibly infinite centre). In the case of an algebraic group, the same conclusion holds [Cor], except that the S_i 's are simple linear algebraic groups.

If G is not amenable, then $k \geq 1$, and in this case by Lemma 4.9 it follows that $k = 1$ and R is compact. By [Sha, Cor. 3.6], the centre $Z(G)$ has an invariant vector by π and thus is compact since π is C_0 ; since in our situation $Z(S_1) \subset Z(G)$, we see that S_1 has finite centre, so that $K = RZ(S_1)$ is compact and G/K is a simple group with trivial centre. \square

Proposition 4.11. *Let G be a connected Lie group or $G = \mathbf{G}(\mathbf{K})$ where \mathbf{K} is a local field of characteristic zero. Assume G noncompact. Then the following are equivalent:*

- (i) $H^1(G, \lambda_G) \neq 0$.
- (ii) *Either G is amenable, or there exists a compact normal subgroup $K \subset G$ such that G/K is isomorphic to $\mathrm{PSL}_2(\mathbf{R})$ (case of Lie groups), or a simple algebraic group of rank 1 (case of an algebraic group over a p -adic field).*

Proof. Suppose (i). If G is not amenable, then, by the result of Guichardet already mentioned [Gu1, Théorème 1], one has $\overline{H^1}(G, \lambda_G) = H^1(G, \lambda_G) \neq 0$. By Proposition 4.10, G has a compact normal subgroup K such that $S = G/K$ is simple with trivial centre. Moreover, G does not have Property (T), hence has rank 1 [DK]. This settles the non-Archimedean case. If G is a Lie group, then by [Mar, Theorem 6.4], λ_G contains an irreducible subrepresentation σ factoring through S , such that $\overline{H^1}(G, \sigma) = \overline{H^1}(S, \sigma) \neq 0$. Then $\sigma \leq \lambda_S$, as S is cocompact in G , so that $H^1(S, \lambda_S) \neq 0$. By a result of Guichardet (Proposition 8.5 in Chapter III of [Gu2]), this implies that $S \simeq \mathrm{PSL}_2(\mathbf{R})$.

Conversely, suppose (ii). If G is amenable, then $H^1(G, \lambda_G)$ is not Hausdorff, hence is nonzero. Otherwise, suppose G nonamenable, and consider K as in (ii). By Proposition 3.2 and Theorem 3.3 (noticing that $p_0 = 1$ for $\mathrm{PSL}_2(\mathbf{R})$), we have $H^1(G/K, \lambda_{G/K}) \neq 0$. Then $H^1(G, \lambda_G) \neq 0$ by the same elementary argument as used in the proof of Proposition 3.2. \square

Corollary 4.12. *Let G be a connected Lie group or $G = \mathbf{G}(\mathbf{K})$ where \mathbf{K} is a local field of characteristic zero; let Γ be a uniform lattice in G . If the first L^2 -Betti number $\beta_{(2)}^1(\Gamma)$ is nonzero, then Γ is commensurable either to a non-abelian free group or to a surface group (more precisely: Γ has a finite index subgroup Γ_0 with a finite normal subgroup N such that Γ_0/N is either a non-abelian free group or a surface group).*

Proof. From $\beta_{(2)}^1(\Gamma) > 0$, it follows that Γ (and also G) is nonamenable: see Theorem 0.2 in [CG]. On the other hand, it was proved in [BV] that, for Γ a finitely generated nonamenable group: $\beta_{(2)}^1(\Gamma) > 0$ if and only if $H^1(\Gamma, \lambda_\Gamma) \neq 0$. Since Γ is uniform in G , we have by Shapiro's Lemma (Proposition 4.6 in Chapter

III of [Gu2]):

$$0 \neq H^1(\Gamma, \lambda_\Gamma) = H^1(G, \text{Ind}_\Gamma^G \lambda_\Gamma) \simeq H^1(G, \lambda_G).$$

By Proposition 4.11, the group G admits a compact normal subgroup K such that G/K is isomorphic either to $\text{PSL}_2(\mathbf{R})$ or to a simple algebraic group of rank 1 over a p -adic group. Let $p : G \rightarrow G/K$ be the quotient map. Then $p(\Gamma)$ is a uniform lattice in G/K . By Selberg’s Lemma, find a finite-index torsion-free subgroup $\tilde{\Gamma}_0$ of $p(\Gamma)$: then $\tilde{\Gamma}_0$ is either a surface group (case of $\text{PSL}_2(\mathbf{R})$) or a non-abelian free group (non-Archimedean case). Set $\Gamma_0 = p^{-1}(\tilde{\Gamma}_0)$, a finite-index subgroup of Γ . Conclude by observing that the kernel $\text{Ker}(p|_{\Gamma_0})$ is contained in $\Gamma \cap K$, so is finite. \square

The preceding result overlaps a result of B. Eckmann (Theorem 4.1 in [Eck]), who characterized lattices Γ (not necessarily uniform) with $\beta_{(2)}^1(\Gamma) > 0$ in a connected Lie group.

4.4. Some non- σ -compact groups

Here is a curiosity. Start with the observation from the proof of Proposition 4.1 that a locally compact, non- σ -compact group cannot be a-T-menable. Accordingly, if it has Property (BP_0) , then it also has Property (FH_0) .

The above observation shows that the σ -compactness assumption is necessary in Guichardet’s result mentioned above. It also provides, in the non-Fréchet case, some explicit counterexamples to the statement of the open mapping theorem.

Proposition 4.13. *Let G be a non- σ -compact locally compact amenable group with Property (BP_0) . Endow $Z^1(G, \lambda_G)$ with the topology of uniform convergence on compact subsets. Then the map*

$$\partial : \begin{cases} L^2(G) \rightarrow Z^1(G, \lambda_G), \\ \xi \mapsto (g \mapsto \lambda_G(g)\xi - \xi), \end{cases}$$

is a continuous linear bijective homomorphism, whose inverse is not continuous.

Proof. The map ∂ is linear, injective (as G is not compact) and surjective (since G has Property (FH_0)). It is clearly continuous. By amenability of G , for every $\varepsilon > 0$ and every compact subset $K \subset G$, there exists a unit vector $\xi \in L^2(G)$ such that

$$\max_{g \in K} \|(\partial\xi)(g)\| < \varepsilon.$$

This clearly shows that ∂^{-1} is not continuous. \square

Example 2. Examples of non- σ -compact amenable groups with Property (BP_0) include:

- Uncountable solvable groups (by Corollary 2.10);
- Discrete groups of the form $G = F^I$, where F is a nontrivial finite group and I is any infinite set. Indeed G is amenable, as it is locally finite, and since G is isomorphic to $G \times G$ (as I is infinite), G contains an infinite normal subgroup with infinite centralizer, so Proposition 2.8 applies.

5. Actions of \mathbf{Z} and \mathbf{R}

5.1. Actions of \mathbf{Z}

We have shown that every action of \mathbf{Z} on a Hilbert space with C_0 linear part is either bounded or metrically proper.

An example of Edelstein [Ede] shows that the C_0 assumption cannot be dropped. Let us briefly recall his example. On \mathbf{C} , consider the rotation r_n with centre 1 and angle $2\pi/n!$. Consider the abstract product $\mathbf{C}^{\mathbf{N}}$, and, for $(z_n)_{n \in \mathbf{N}} \in \mathbf{C}^{\mathbf{N}}$, set $r((z_n)_{n \in \mathbf{N}}) = (r_n(z_n))_{n \in \mathbf{N}}$. This self-map is bijective and has the constant sequence 1 as unique fixed point. Moreover, it can be shown that $r(\ell^2(\mathbf{N})) = \ell^2(\mathbf{N})$. Thus r induces an affine isometry of $\ell^2(\mathbf{N})$, which has no fixed point since the constant 1 is not in $\ell^2(\mathbf{N})$. However, the action is not metrically proper; actually 0 is a recurrent point: an easy computation gives $\|r^{n!}(0)\|^2 \leq \sum_{k>n} (2\pi n!/k!)^2$, and this sum clearly tends to zero. Notice that r almost has fixed points: with v_m the characteristic function of $\{1, \dots, m\}$, one has $\lim_{m \rightarrow \infty} \|r(v_m) - v_m\| = 0$.

Observe that this isometry has diagonalizable linear part. Let us now provide another counterexample with further assumptions on the linear part.

Definition 3. A unitary or orthogonal representation of a group is *weakly C_0* if it has no nonzero finite-dimensional subrepresentation.³

Proposition 5.1. *There exists an affine isometric action of \mathbf{Z} on a complex Hilbert space, which is neither bounded nor metrically proper, and has a weakly C_0 linear part.*

Proof. Write σ for the affine action of \mathbf{Z} , and π for its linear part. Let μ be a probability measure on $[0, 1]$ and write $H = L^2([0, 1], \mu)$. Let $\pi(1)$ be the multiplication by the function $e(x) = \exp(2i\pi x)$. Write $\sigma(1) = \tau_1 \circ \pi(1)$ where τ_1 is the translation by the constant function 1. Note that π is weakly C^0 if and only if the spectrum of $\pi(1)$ has no atom, i.e. μ is nonatomic.

Let b be the corresponding cocycle and write $c(n) = \|b(n)\|^2$. An immediate computation shows that

$$c(n) = \int \phi_n(x) d\mu(x)$$

where $\phi_n(x) = |\sin(\pi nx) / \sin(\pi x)|^2$.

Let N_n be an increasing sequence of integers and let ε_n be a decreasing sequence in $]0, 1[$, such that $\varepsilon_n \rightarrow 0$. Moreover, let us assume that $N_n/N_{n+1} = o(\varepsilon_n)$.

For all positive integer n , write

$$I_n(k) = \left[\frac{k - \varepsilon_n}{N_n}, \frac{k + \varepsilon_n}{N_n} \right] \cap [0, 1]$$

and

$$K_n = K_{n-1} \cap \bigcup_{k=0}^{N_n} I_n(k).$$

³ C_0 (resp. weakly C_0) representations are often called mixing (resp. weakly mixing).

Finally, write

$$K = \bigcap_n K_n.$$

One can check easily that K is homeomorphic to a Cantor space.

Let μ be a probability measure on $[0, 1]$ such that:

- its support is contained in $K \cap [0, 1/2]$;
- there exists a subsequence ε_{k_n} such that

$$\mu([0, \sqrt{\varepsilon_{k_n}}]) = \sqrt{\varepsilon_{k_n}}. \tag{1}$$

We choose the sequence k_n such that each interval $I_n = [\sqrt{\varepsilon_{k_{n+1}}}, \sqrt{\varepsilon_{k_n}}]$ intersects K nontrivially. Take for μ_n any nonatomic measure supported by $K \cap I_n$ such that $\mu_n(I_n) = \sqrt{\varepsilon_{k_n}} - \sqrt{\varepsilon_{k_{n+1}}}$ and define $\mu = \sum_n \mu_n$: clearly, μ is nonatomic.

Claim 1. *The action σ has no fixed point (so it has unbounded orbits).*

If σ has a fixed point f , then $f(x) = (1 - \exp(2i\pi x))^{-1}$ μ -a.e. Let us show that f does not belong to $L^2([0, 1])$. Indeed, note that $|f|^2 = (1/\sin(\pi x))^2$. For all $x \in [0, \sqrt{\varepsilon_{k_n}}]$, we have

$$\sin(\pi x)^2 \leq \pi^2 x^2 \leq \pi^2 \varepsilon_{k_n}$$

and by (1)

$$\mu([0, \sqrt{\varepsilon_{k_n}}]) = \sqrt{\varepsilon_{k_n}}.$$

It follows that

$$\int |f|^2 d\mu \geq \frac{1}{\pi^2 \sqrt{\varepsilon_{k_n}}}$$

which proves Claim 1.

Claim 2. *If moreover $\varepsilon_{k_n} = o(N_{k_n}^{-4})$ (for instance, $N_n = 2^{n!}$ and $\varepsilon_n = (N_n)^{-5}$), then $c(N_{k_n})$ tends to 0, so that the action is not metrically proper.*

Indeed, let us show that $c(N_{k_n}) = o(1)$.

First, note that for all $x \in K$, the fractional part of $N_{k_n} \cdot x$ is less than ε_{k_n} . Thus, for every $x \geq \sqrt{\varepsilon_{k_n}}$ and every $x \in K$, it states

$$\phi_{N_{k_n}}(x) \leq \left(\frac{\sin(2\pi \varepsilon_{k_n})}{\sin(2\pi \sqrt{\varepsilon_{k_n}})} \right)^2 \leq \pi^2 \varepsilon_{k_n} / 4.$$

On the other hand, we have

$$\frac{\sin(2\pi N_{k_n} x)}{\sin(2\pi x)} \leq N_{k_n}$$

and by (1)

$$\mu([0, \sqrt{\varepsilon_{k_n}}]) = \sqrt{\varepsilon_{k_n}}.$$

It follows that

$$c(N_{k_n}) \leq \sqrt{\varepsilon_{k_n}} \cdot N_{k_n}^2 + \pi^2 \varepsilon_{k_n} / 4.$$

So we get $c(N_{k_n}) = o(1)$. \square

5.2. Actions of \mathbf{R}

Let us now show that the “pathological” actions of \mathbf{Z} described above can be extended to \mathbf{R} .

Recall that a group G is said to be *exponential* if, for every $g \in G$, there is a one-parameter subgroup through g (i.e. a continuous homomorphism $\beta : \mathbf{R} \rightarrow G$ such that $\beta(1) = g$). Clearly, an exponential group has to be arc-connected.

Endow the group of affine isometries of a complex Hilbert space, $\mathcal{H} \times \mathcal{U}(\mathcal{H})$, with the product topology, for the natural topology on \mathcal{H} and the norm operator topology on the unitary group $\mathcal{U}(\mathcal{H})$.

Proposition 5.2. *The group of affine isometries of a complex Hilbert space \mathcal{H} is exponential.*

Proof. Let $\alpha(v) = Uv + b$ be an affine isometry of \mathcal{H} .

By the spectral theorem, we find a projection-valued measure \mathbf{P} on $[-\pi, \pi[$ such that $U = \int_{-\pi}^{\pi} e^{ix} d\mathbf{P}(x)$, in the sense that, for every $\xi, \eta \in \mathcal{H}$,

$$\langle U\xi \mid \eta \rangle = \int_{-\pi}^{\pi} e^{ix} d\mu_{\xi, \eta}(x)$$

where $\mu_{\xi, \eta}(A) = \langle \mathbf{P}(A)\xi \mid \eta \rangle$ for any Borel subset $A \subset [-\pi, \pi[$. Consider the one-parameter group of unitary operators

$$v(s) = \int_{[-\pi, \pi[} e^{isx} d\mathbf{P}(x).$$

For every $\xi \in \mathcal{H}$ and $t \in \mathbf{R}$, define

$$b_{\xi}(t) = \int_0^t v(s)\xi ds.$$

It is straightforward that $b_{\xi} \in Z^1(\mathbf{R}, v)$. Let us consider the operator $A = \int_0^1 v(s) ds$. Then $b_{\xi}(1) = A\xi$ for every $\xi \in \mathcal{H}$. Thus, to show that there exists $\xi \in \mathcal{H}$ such that $b = b_{\xi}(1)$, it suffices to establish that A is invertible.

By Fubini’s Theorem

$$\int_0^1 v(s) ds = \int_0^1 \left(\int_{[-\pi, \pi[} e^{isx} d\mathbf{P}(x) \right) ds = \int_{[-\pi, \pi[} \frac{e^{ix} - 1}{ix} d\mathbf{P}(x).$$

Since the function $x \mapsto ix/(e^{ix} - 1)$ is bounded on $[-\pi, \pi[$, we obtain that

$$\int_{[-\pi, \pi[} \frac{ix}{e^{ix} - 1} d\mathbf{P}(x)$$

is a bounded operator on \mathcal{H} , and is the inverse of A ; so we may take $\xi = A^{-1}(b)$. \square

In view of Proposition 5.1, we obtain

Corollary 5.3. *There exists an affine isometric action of \mathbf{R} on a complex Hilbert space that is neither bounded nor metrically proper. Moreover, it can be chosen weakly C_0 .*

Remark 2. Proposition 5.2 is *false* for real Hilbert spaces. This follows from the fact that the orthogonal group of a real Hilbert space is *not* exponential. This is clear in finite dimension (the group $O(n)$ is not connected), and was observed by Putnam and Wintner [PW2] in infinite dimension (although the orthogonal group is then connected [PW1]). An example of an orthogonal transformation which is not in the image of the exponential map is a reflection

$$S = \text{diag}(-1, 1, 1, 1, \dots);$$

this can be seen by noticing that S is not a square in the orthogonal group: indeed, if $S = R^2$, since R commutes with S it stabilizes the -1 -eigenspace of S , which leads to a contradiction.

Added in proof. Let us point out a paper by J. Peterson and A. Thom, *Group cocycles and the ring of affiliated operators*, preprint, Aug. 2007, arXiv:0708.4327, which contains answers to some of the questions of the present paper. In Theorem 2.7 of that paper, it is proved that if Γ is a group with $\beta_{(2)}^1(\Gamma) > 0$ contains an infinite amenable subgroup, then Γ does not have Property (BP_0) : this vastly generalizes our Proposition 4.3 and Corollary 4.6, and it answers our Question 1 in the negative. Next, our Conjecture 2 is proved in Theorem 2.5 of the Peterson–Thom paper. Finally, in Theorem 2.9 of that paper, it is shown that the infinite symmetric group and some other locally finite groups, have Property (BP_0) : this gives more hope for a positive answer to our Question 2. Let us mention that our Proposition 3.1 was generalized to locally compact second countable groups by Chatterji, Drutu, and Haglund. See Corollary 1.5 in *Kazhdan and Haagerup properties from the median viewpoint*, preprint, Jan. 2008, arXiv:0704.3749.

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