Université de Neuchâtel

Faculté des Sciences
Institut de mathématiques
Rue Emile-Argand 11, 2000 Neuchâtel

On the topology of Lagrangian submanifolds in toric symplectic manifolds

Thèse

présentée à la Faculté des Sciences
pour l'obtention du grade de docteur ès Sciences en mathématiques

par

Joé Brendel

Acceptée sur proposition du jury :

Prof. Felix Schlenk, Université de Neuchâtel, directeur de thèse, rapporteur
Prof. Bruno Colbois, Université de Neuchâtel, expert interne
Prof. Grigory Mikhalkin, Université de Genève, rapporteur
Prof. Joontae Kim, Université de Sogang, rapporteur

Soutenue le 4 juillet 2022

UNIVERSITÉ DE
NEUCHÂTEL

Faculté des Sciences
Secrétariat-décanat de Faculté
Rue Emile-Argand 11
2000 Neuchâtel – Suisse
Tél : + 41 (0)32 718 21 00
E-mail : secretariat.sciences@unine.ch

IMPRIMATUR POUR THESE DE DOCTORAT

**La Faculté des sciences de l'Université de Neuchâtel
autorise l'impression de la présente thèse soutenue par**

Monsieur Joé BRENDEL

Titre:

“On the topology of Lagrangian submanifolds in toric symplectic manifolds”

sur le rapport des membres du jury composé comme suit:

- Prof. Felix Schlenk, directeur de thèse, Université de Neuchâtel, Suisse
- Prof. Bruno Colbois, Université de Neuchâtel, Suisse
- Prof. Grigory Mikhalkin, Université de Genève, Suisse
- Prof. ass. Joontae Kim, Sogang University, Corée du Sud

Neuchâtel, le 8 août 2022

Le Doyen, Prof. R. Bshary

Said the straight man
to the late man
Where have you been?
I've been here and
I've been there and
I've been in between.

KING CRIMSON

Acknowledgements

It is my pleasure to thank my advisor Felix Schlenk for his support and enthousiasm while guiding me through the five years I have spent writing this thesis. Thanks, Felix, for always being available to listen to my latest flimsy idea, no matter how dumb or false (most of the time both) it turned out to be. Thanks for the hikes, the fondues, the endless discussions and for tolerating my stubborn refusal to learn you-know-what. Thanks for your patience in waiting until *the candle was lit* - I know it took some time. Thanks for granting me the freedom to pursue my own questions and for showing me that the number of people interested in what I have to say is always a strictly positive integer.

Another person I can barely thank enough is Joontae Kim. Thanks, Joontae, for your generosity, your patience and your kindness. The list of things I have learned from you exceeds this meagre paragraph by some and many things on that list are beyond mathematics. Your influence on my trajectory has been crucial. I wish you and Jiyeon the best and thank both of you whole-heartedly for the warm welcome in Seoul and for the great times we have spent together in Switzerland and Korea. I am happy to count you among my collaborators and, more importantly, my friends.

I am indebted to Grisha Mikhalkin for his continuing support, for patiently explaining his ideas and for agreeing to serve as a member of my jury. Thanks also, Grisha, for organizing the Geneva half of GeNeSys and spoiling us with excellent tea and wine.

Thanks to Bruno Colbois for completing the jury of my thesis.

Thanks to Jonny Evans, Yael Karshon, Leonid Polterovich, Jack Smith and Umut Varolgunes for sharing some of their precious time with me to listen to my questions and ideas and, much more importantly, generously share their own.

Thanks to all of my various math teachers and professors throughout my life who went through the painstaking task of breathing life into theorems and equations. They are too many to name, but I would like to point out Anand Dessai, Patrick Ghanaat, Ivan Izmestiev and Patrick Krier who each had a lasting influence on me.

Si une thèse n'était que des maths, ça en ferait une activité ennuyeuse et frustrante à un point insupportable. C'est quelque chose que j'ai tendance à oublier de temps en temps et c'est avec grand plaisir que je remercie mes collègues et amis de Neuchâtel (et d'ailleurs) de me le rappeller. Vous me manquez déjà! Alex "pause caf?" pour les nombreuses pauses, les explications sur Baum-Connes que je faisais semblant de comprendre et les discussions homologiques qui était toutes fausses. Alex "chaud pour la grimpe?", le bloc est moins cool depuis que t'es à Heidelberg - l'institut aussi. Antoine pour l'hydromel, de m'avoir appris le Jass (environ cinq fois...) et d'organiser autant de trucs que tout le monde a envie de faire mais personne n'a envie d'organiser. Bernard, fir den Kimchi an d'Wanderungen. Carsten, mon (très) grand frère de thèse qui s'est toujours si bien occupé de moi. Cengiz, ich habe schon wieder vergessen, wie das Dreikörperproblem geht. Charel fir d'Wanderungen and d'Diskussiounen. Corentin pour les discussions à midi. Eda, Giulia and Oliver for the two hour lunch breaks. Édouard, la personne la plus sympa de l'institut. Eva, pour les moments à la coloc. Farshid, for the endless conversations about this, that and all the rest. Flavio for the contagious enthousiasm - you almost convinced me that algebra has some appeal. Joé, *wann mer dech net hätten, missten mer dech kréien.* Joel, für die unzähligen "Einstein würfelt nicht"-Erniedrigungen, Morse-Homologie in der Sonne und Informatik-Support. Heidi et Jonathan, je vous dois environ trente-cinq repas chez moi. Katie, it was a real pleasure working together! Laura de m'avoir convaincu de "juste peut-être essayer une seule fois de venir à la grimpe". C'est un des souvenirs les plus vifs que je garderai de Neuchâtel. Léo, pour le meilleur sourire de l'institut. Luc, mon "ancien combattant" préféré. Lucas, mein grosser Bruder auf den ich öfter hätte hören sollen. Luis, du flotten Borscht. Nicholas, *Wien Kaffi wëll muss Kaffi maachen. Mee wanns de Kaffi mëss, dann muss de en zéien loossen.* Merci fir deng Frëndschaft. Ouni déi wier ech eng langweileg Persoun déi langweileg Musék lauschtert. Oliver, für deine Perspektive. Patrick, fir déi flott Wanderungen (an nach aner trips) an deng dausend Interessen déis mat deenen ronderem dech deels. Philippe, mäin lieblings "Staadter Jengi". Radu pour les "questions bêtes" sur lesquelles j'ai toujours bien aimé réfléchir sans jamais trouver de réponse. Rémi pour les trois conversations qu'on a eu et que j'ai toutes retenues. Robin, *wer Glück hat, kann seine Freunde auf einer Hand zählen.* Danke für die Zeit in Fribourg und alles darüber hinaus. Stefano, for your smile and sharing your point of view on toric stuff. Thomas d'être "généralement sympa", pour les crêpes et les mille trucs tropicaux que tu m'as expliqué mille fois. Tom d'avoir été le papa des doctorants quand j'ai commencé ma thèse. Tu m'as manqué après ton départ! Merci aussi à Aïsha, Ana, Christina, Clayton, David, Elisa, Hélène, Jérémy, Laurent, Matt, Paul-Henry, Sanaz, Thiebout, Valentin pour les moments qu'on a passés ensemble. Merci à Florine et Rumo, merci à Séverine; quelles bonnes excuses de revenir à Fribourg de temps en temps! Merci à Afrim, le saloon est bordélique quand tu prends des vacances. Merci à Christine de faire marcher l'institut. Merci aux étudiants de Neuchâtel. Vous êtes paresseux, mais je vous aime bien quand-même.

Merci aux Ziegler pour "les mercredis". Merci à Nathalie et Yvan de me faire me sentir chez moi dès la première minute. Vous ne correspondez à aucun des clichés sur les beaux-parents et je ne sais pas comment vous faites.

Merci Schwesterhäerz, leider konnt ech dech an läschter Zäit net sou dacks stëppelen! Merci Mama and Papa fir d'Ënnerstëtzung déi ganz Joëren. Merci dass der iech d'Méi gemaach hutt mech sou dacks "am Exil" besichen ze kommen an mer bei all Plënner nees gehollef hutt. Ouni iech wier dat heiten alles net méiglech.

Merci à Morgane de tenir le soleil.

Abstract

This thesis consists of four main chapters. Chapter 2, *An introduction to the Chekanov torus*, is an expository article introducing all the concepts necessary to define and understand the Chekanov torus. We give a detailed introduction to Hamiltonian circle actions, symplectic reduction, lifting techniques and versal deformations and show how those can be used to show that the Chekanov torus is exotic. Chapter 3, *On the topology of real Lagrangians in toric symplectic manifolds*, is a joint article with Joontae Kim and Jiyeon Moon focusing on constructing examples of real Lagrangian submanifolds in toric manifolds by lifting symmetries from the moment polytope. We also prove convexity and tightness for the examples we construct and give an analogue of the Delzant construction. Chapter 4, *Real Lagrangian tori and versal deformations*, is an article focusing on obstructions for a given Lagrangian submanifold to be real and is, in some sense, complementary to Chapter 3. We develop a general obstruction in terms of versal deformations and displacement energy and apply this to toric fibres and Chekanov tori in toric manifolds. Chapter 5, *Squeezing via degenerations of the complex projective plane*, is an appendix to the paper *On certain quantifications of Gromov's non-squeezing theorem* by Kevin Sackel, Antoine Song, Umut Varolgunes and Jonathan J. Zhu. We prove that the symplectic four-ball can be squeezed after removing a subset of Minkowski dimension two.

Keywords: Symplectic topology, Lagrangian submanifolds, Toric geometry, Displacement energy, Versal deformations, Exotic tori

Résumé

Cette thèse consiste en quatre chapitres. Chapitre 2, *An introduction to the Chekanov torus*, est un article d'exposition introduisant tous les concepts nécessaires pour définir le tore de Chekanov. On donne une introduction détailée aux actions Hamiltoniennes d'un cercle, à la réduction symplectique et aux déformations versales tout en montrant comment combiner ces outils pour montrer que le tore de Chekanov est exotique. Chapitre 3, *On the topology of real Lagrangians in toric symplectic manifolds*, est un article avec Joontae Kim and Jiyeon Moon qui donne une construction de sous-variétés Lagrangiennes réelles dans les variétés toriques par relèvement de symétries du polytope moment. Nous discutons également certaines propriétés des exemples construits et donnons un analogue de la construction de Delzant. Chapitre 4, *Real Lagrangian tori and versal deformations*, est un article centré autour d'obstructions à une Lagrangienne d'être réelle et, dans ce sens, Chapitre 4 est complémentaire à Chapitre 3. On donne un critère général que l'on applique par la suite aux fibres toriques et aux tores de Chekanov dans les variétés toriques. Chapitre 5, *Squeezing via degenerations of the complex projective plane*, est un appendice à l'article *On certain quantifications of Gromov's non-squeezing theorem* par Kevin Sackel, Antoine Song, Umut Varolgunes et Jonathan J. Zhu. On montre que la boule symplectique de dimension quatre privée d'un sous-ensemble de dimension de Minkowski deux peut être plongé dans un cylindre de rayon inférieur à celui de la boule.

Mots-clé: Topologie symplectique, Sous-variétés Lagrangiennes, Géométrie torique, Énergie de déplacement, Déformations versales, Tores exotiques

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1 Introduction

The main chapters of this PhD thesis consist of four works completed during my PhD at Université de Neuchâtel under the guidance of Prof. Felix Schlenk.

Chapter 2, *An introduction to the Chekanov torus*, is an introductory article accompanying a set of lectures given in the *GGTI online lectures* organized by the *Gökova Geometry and Topology Institute* in August 2021. This introductory article has been accepted for publication in *Proceedings of the 27th Gökova Geometry-Topology Conference*.

Chapter 3, *On the topology of real Lagrangians in toric symplectic manifolds*, is a joint article with Joontae Kim and Jiyeon Moon focusing on constructing examples of real Lagrangian submanifolds. It has been accepted for publication in *Israel Journal of Mathematics*.

Chapter 4, *Real Lagrangian tori and versal deformations*, is an article about real Lagrangian submanifolds as well, but it focuses on obstructions to such embeddings instead of constructions. Hence it uses quite different methods. This article has been conditionally accepted for publictation in *Journal of Symplectic Geometry*.

Chapter 5, *Squeezing via degenerations of the complex projective plane*, is an article about certain quantitative questions surrounding Gromov's non-squeezing theorem. It will appear as an appendix to the paper *On certain quantifications of Gromov's non-squeezing theorem* by Kevin Sackel, Antoine Song, Umut Varolgunes and Jonathan J. Zhu which has been conditionally accepted for publication in *Geometry & Topology*.

A detailed introduction to each of these works can be found at the beginning of the corresponding chapter. Here we will give a broad introduction to these research questions with special focus on their context inside the larger world of symplectic geometry and on how the research questions are linked to each other.

1.1 Some classical mechanics

Symplectic geometry is the study of geometric and topological features of a manifold M equipped with a closed non-degenerate differential two-form ω. The study of these manifolds is historically motivated by classical mechanics. Let us briefly explain this. A (holonomic) mechanical system consists of the data (Q, V), where Q is a smooth manifold called *configuration space* and $V\colon Q \to \mathbb{R}$ is a smooth function giving the potential on that configuration space. Examples of configuration spaces are:

- A particle in space, $Q = \mathbb{R}^3$;
- Several particles in space, $Q = \mathbb{R}^3 \times \ldots \times \mathbb{R}^3$. Note that if we want to prevent these particles from colliding, then we need to remove subspaces of the form $x_i = x_j$,

where $x_i, x_j \in \mathbb{R}^3$ are the position vectors of the corresponding particles;

- the pendulum constrained to a plane, $Q = S^1$ or the pendulum freely moving in three-space $Q = S^2$;
- the double pendulum, $Q = S^1 \times S^1$.

The potential function $V\colon Q \to \mathbb{R}$ depends on the problem under consideration and it is the piece of data encoding the dynamics of the system. We can think for example of a homogeneous gravitational force field induced by the potential $V(q_1, q_2, q_3) = q_3$ acting on a point particle ($Q = \mathbb{R}^3$). Applying Newton's equation of motion (assuming our particle has unit mass),

$$\ddot{q} = -\nabla V(q), \tag{1.1.1}$$

we know from high-school that this particle will trace out a parabola contained in a plane. However, we quickly run into trouble when moving on to more complicated examples, such as the pendulum. It is the same problem as in $\mathbb{R}^3$, but restricted to $S^2 \subset \mathbb{R}^3$. We see that the force field isn not everywhere tangent to the sphere to which our particle is constrained. These constraints are quite tricky to handle when using Newton's equation (1.1.1). Fortunately, there is a trick allowing us to set up the problem more intrinsically. The idea is essentially to view the velocity (or momenta, to be more precise) as additional variables subjected to a differential equation of their own. More precisely, we replace Newton's equation by a system of equations

$$\begin{cases} \dot{q} = p \\ \dot{p} = -\nabla V(q), \end{cases} \tag{1.1.2}$$

where (in the above example) q is the coordinate on the sphere S^2. Setting

$$H(q,p) = \frac{1}{2}\|p\|^2 + V(q), \tag{1.1.3}$$

the equations of motion are given by

$$\begin{cases} \dot{q} = \partial_p H \\ \dot{p} = -\partial_q H. \end{cases} \tag{1.1.4}$$

This is the *Hamiltonian formalism* in classical mechanics. The function H defined by (1.1.3) is called the *Hamiltonian (function)* and it measures the total energy of the system.

Let us take this even further and reformulate the equations of motion using differential forms. Formally, our process of adding the variables p given by the momenta corresponds to passing to the cotangent bundle T^*Q. This is also called the *phase space* of the system. The phase space carries a canonical differential two-form given by

$$\omega = \sum_i dq_i \wedge dp_i, \tag{1.1.5}$$

where $q = (q_1, \dots, q_n)$ is a chart on Q and $p = (p_1, \dots, p_n)$ is the associated fibre chart for T^*Q. Indeed, one can check that this form is independent of the choice of coordinates.

It is called the *canonical two-form* on T^*Q and it is an example of a symplectic form. The equations of motion can now be written very compactly as,

$$dH = \iota(X)\omega, \tag{1.1.6}$$

where $X \in \Gamma(TQ)$ is a vector field whose flow reproduces the dynamics of the classical system at hand. Note that (1.1.6) defines a unique vector field since ω is *non-degenerate*, meaning that the map $Y \mapsto \iota(Y)\omega$ induces a bundle isomorphism $TQ \to T^*Q$. Note furthermore that ω is an *exact* differential form, meaning that it can be written as $\omega = d\lambda$, where λ is a differential one-form. In the general setting, as it turns out, it is enough to ask the symplectic form to be *locally* the differential of a one-form, i.e. to be *closed* instead of exact.

Definition 1.1.1. *A* symplectic manifold *is a smooth manifold M equipped with a closed, non-degenerate two-form ω.*

The non-degeneracy of the symplectic form implies that symplectic manifolds are even-dimensional. In the above construction this was automatic, since for each spacial coordinate q_i we added precisely one momentum coordinate p_i. The group of diffeomorphisms preserving the symplectic form under pull-back is called the *group of symplectomorphisms* and we denote it by $\mathrm{Symp}(M, \omega)$. Note that the equation (1.1.6) makes sense on any symplectic manifold and for any smooth function $H\colon M \to \mathbb{R}$. The flow of a vector field obtained by (1.1.6) is called *autonomous Hamiltonian flow*. More generally, we can consider flows of time-dependent vector fields defined by time-dependent families of Hamiltonian functions H_t (by abuse of terminology, this is again called a Hamiltonian function). The set of time-one maps of time-dependent Hamiltonian flows is a group, called the group of *Hamiltonian diffeomorphisms*. We will denote it by $\mathrm{Ham}(M, \omega)$. Hamiltonian diffeomorphisms preserve the symplectic form and hence there is an inclusion $\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}(M, \omega)$.

1.2 What is symplectic geometry?

Liouville's Theorem from classical mechanics states that Hamiltonian flows $t \mapsto \phi_t$ preserve the canonical volume form on phase space,

$$\phi_t^* \,\mathrm{vol} = \mathrm{vol}, \quad \mathrm{vol} = dq_1 \wedge dp_1 \wedge \ldots \wedge dq_n \wedge dp_n. \tag{1.2.7}$$

This theorem immediately follows from the formulation of Hamiltonian mechanics in terms of symplectic forms. Indeed, the volume form can be written as

$$\mathrm{vol} = \frac{1}{n!}(\omega \wedge \ldots \wedge \omega), \tag{1.2.8}$$

and therefore Liouville's theorem follows from the fact that Hamiltonian flows preserve the symplectic form. In other words, the symplectomorphisms are contained in the group of volume-preserving diffeomorphisms,

$$\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}(M, \omega) \subset \mathrm{Diff}_{\omega^{\wedge n}}(M). \tag{1.2.9}$$

This raises the following question.

How big a difference is there between symplectic and volume-preserving diffeomorphisms?

One answer to this question was given by Gromov in his celebrated non-squeezing theorem. We equip the vector space $\mathbb{R}^{2n} = \{(x_1, y_1, \dots, x_n, y_n)\}$ with the standard symplectic form $\omega_0 = \sum_i dx_i \wedge dy_i$. Note that this is the same as taking $T^*\mathbb{R}^n$ with the symplectic form (1.1.5). By $B^{2n}(R)$ we denote the ball of radius $R > 0$ and by $Z^{2n} = \{x_1^2 + y_1^2 < 1\}$ we denote the cylinder obtained by taking the coordinate pair (x_1, y_1) to lie in the unit disk.

Theorem 1.2.1 (Gromov non-squeezing)**.** *If $R > 1$, then there is no symplectomorphism $\phi \in \mathrm{Symp}(\mathbb{R}^{2n}, \omega_0)$ such that $\phi(B^{2n}(R)) \subset Z^{2n}$.*

Note that this is in stark contrast with the volume preserving case. Indeed, there is even a *linear* volume preserving map which squeezes a ball of arbitrarily large radius into Z^{2n}, obtained for example by scaling x_1, y_1 by some small enough $0 < \alpha < 1$ at the cost of scaling one of the remaining coordinates by α^{-2}.

This is the first occurrence of the phenomenon of *symplectic rigidity.* We (quite vaguley) define *symplectic rigidity* as

A phenomenon which occurs in the smooth world, but not in the symplectic world.

Gromov's theorem tells us that the ball can be squeezed in the smooth world but not in the symplectic one. We will encounter different flavours of symplectic rigidity throughout this introduction, most of which are related to a certain type of submanifolds of symplectic manifolds, called *Lagrangians.* Symplectic rigidity results are interesting due to the fact that the group of symplectic transformations is somewhat ill-understood[1] and rigidity results tell us something about the *size* of the group of $\mathrm{Symp}(M, \omega)$, namely that it is smaller than $\mathrm{Diff}(M)$ in some geometrically meaningful way.

1.3 Squeezing after removing a subset

Before moving on to Lagrangian submanifolds, let us stick to Gromov's non-squeezing theorem and discuss a related question, to which Chapter 5 of this thesis is dedicated. The following discussion is mainly based on [SSVZ21]. It is very natural to ask

What subset do we need to remove from the ball such that it can be squeezed?

Formally, we want to know something about sets $\Sigma \subset B^{2n}(R)$ such that there is a symplectomorphism ϕ for which

$$\phi(B^{2n}(R) \setminus \Sigma) \subset Z^{2n}. \tag{1.3.10}$$

Obviously, this question is interesting only if $R > 1$. A naive guess may be to try to quantify the volume one has to remove in order for squeezing to take place. However, Katok has shown the following.

---

[1] Its topology is explicitely determined only in a handful of examples, see for example the discussion in McDuff–Salamon [MS17, Section 10.4]

Theorem 1.3.1 (Katok [Kat73]). *Let $R > 0$. For every $\varepsilon > 0$, there is a symplectomorphism ϕ such that $\phi(B^{2n}(R)) \setminus Z^{2n}$ has volume smaller than ε.*

Again, we realize that volume considerations do not lead us to the heart of the matter. Strikingly, it suffices to remove a set of half the dimension of the ball in order to make squeezing possible. Let P be the (x_1, x_2)-plane in $\mathbb{R}^4 = \{(x_1, y_1, x_2, y_2)\}$.

Theorem 1.3.2 (Sackel–Song–Varolgunes–Zhu [SSVZ21]). *There is a symplectomorphism ϕ of $(\mathbb{R}^4, \omega_0)$ such that $\phi(B^4(2) \setminus P) \subset Z^4$.*

Furthermore, the following remarkable theorem appears in the same paper.

Theorem 1.3.3 (Sackel–Song–Varolgunes–Zhu [SSVZ21]). *Let $R > 1$ and let $\Sigma \subset B^4(R)$ such that there is a symplectomorphism ϕ with $\phi(B^4(R) \setminus \Sigma) \subset Z^4$. Then Σ has Minkowski dimension at least two.*

Note that Theorem 1.3.2 shows that the bound on dimension from Theorem 1.3.3 is optimal. A natural quantitative question imposes itself.

By how much can we squeeze if we allow ourselves to remove a subset of Minkowski dimension two?

More precisely, we would like to know for which $R > 1$ there is a subset $\Sigma \subset B^4(R)$ of Minkowski dimension two and a symplectomorphism ϕ such that $\phi(B^4(R) \setminus \Sigma) \subset Z^4$. In the first version of their preprint, Sackel–Song–Varolgunes–Zhu conjectured that one cannot do better than $R = 2$. However, we could prove the following.

Theorem 1.3.4 (B. Theorem 5.1.1). *For every $R < 3$, there is a set $\Sigma \subset B^4(R)$ of Minkowski dimension two such that $B^4(R) \setminus \Sigma$ symplectically embedds into Z^4.*

This theorem is the topic of Chapter 5 of this thesis. We construct a sequence of subsets Σ and corresponding embeddings which achieve an increasing sequence of squeezing factors $\{R_n\}_{n\in\mathbb{N}}$ with $R_n \to 3$. The first non-trivial squeezing of this sequenze reproduces Theorem 1.3.2. Furthermore, the members of the sequence of squeezings are in bijection with the so-called *Markov triples*, i.e. natural numbers $a, b, c \in \mathbb{N}$ solving the *Markov equation*,

$$a^2 + b^2 + c^2 = 3abc. \tag{1.3.11}$$

This is not the first occurence of Markov triples in symplectic geometry around related questions, see for example Galkin–Usnich [GA10], Hacking–Prokhorov [HP10], Vianna [Via16, Via17] and Evans–Smith [ES18]. See Section 5.1 for a more detailed account. Let us finish this discussion with the following natural question which is, as of now, wide open.

Question 1.3.5. What is the supremum over all $R > 1$ for which there is a subset $\Sigma \subset B^4(R)$ of Minkowski dimension two and a symplectomorphism ϕ such that $\phi(B^4(R) \setminus \Sigma) \subset Z^4$? Is it finite?

1.4 Lagrangian submanifolds

Gromov's theorem 1.2.1 is the first result exhibiting symplectic rigidity. By now, symplectic rigidity has been detected in many different contexts. One of the most prominent examples is *intersection rigidity* for Lagrangian submanifolds.

Definition 1.4.1. *A submanifold $L \subset (M, \omega)$ is called* Lagrangian *if it has half the dimension of M and ω vanishes on TL.*

By *intersection rigidity*, we mean that $L \subset M$ has self-intersections which cannot be removed by Hamiltonian isotopies,

$$L \cap \phi(L) \neq \varnothing, \quad \text{for all } \phi \in \mathrm{Ham}(M, \omega). \tag{1.4.12}$$

The striking fact here is that L has only half the dimension of the ambient space and still (1.4.12) holds. A Lagrangian submanifold for which (1.4.12) holds is called *non-displaceable*. Let us emphasize that we are interested in cases where L is *smoothly* displaceable (this is for example the case when its normal bundle is trivial) since then Hamiltonian non-displaceability is an honest symplectic rigidity phenomenon.

As it turns out, we can attach symplectic information to a Lagrangian submanifold L, even if it is displaceable. The idea is to quantify how much energy is needed to displace L by a Hamiltonian isotopy. This quantification is given in terms of the so-called *Hofer geometry* on the group of Hamiltonian diffeomorphisms. Let $A \subset M$ be any subset of M. Then we can define the *displacement energy* by

$$e(M, A) = \inf \left\{ \|H\| \;\middle|\; \phi_1^H(A) \cap A = \varnothing \right\}, \tag{1.4.13}$$

and set $e(M, A) = \infty$ if A is not displaceable, i.e. if the infimum is taken over the empty set. By ϕ_1^H we have denoted the time-one map of the Hamiltonian flow associated to the (time-dependent) Hamiltonian H and by $\| \cdot \|$ the Hofer norm, see for example (2.5.100). We discuss some details surrounding displacement energy in Section 2.5 and in Section 4.2. For a detailed account, see also the classic book by Polterovich [Pol01]. Displacement energy is a symplectic invariant and hence it can be used to distiguish subsets up to symplectomorphism. This leads us to another symplectic rigidity phenomenon for Lagrangian submanifolds.

Definition 1.4.2. *Let $L, L' \subset M$ be Lagrangian submanifolds. We say that L, L' are* symplectically equivalent *if there is a symplectomorphism mapping L to L'.*

Let us look at a family of examples in the symplectic vector space $(\mathbb{R}^4, \omega_0)$, called *product tori*,

$$T(a_1, a_2) = \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \,|\, \pi(x_1^2 + y_1^2) = a_1, \pi(x_2^2 + y_2^2) = a_2\}. \tag{1.4.14}$$

For every pair $(a_1, a_2) \in \mathbb{R}^2_{>0}$, this defines a Lagranian torus in $\mathbb{R}^4$. Product tori have been classified by Chekanov [Che96]. Two tori $T(a_1, a_2)$ and $T(b_1, b_2)$ are symplectically equivalent if and only if $(a_1, a_2) = (b_1, b_2)$ up to permutation. On the other hand, all $T(a_1, a_2)$ are smoothly isotopic. This means that this is yet another symplectic rigidity phenomenon. Let us push this line of inquiry a bit further and ask, *Is there any Lagrangian torus in $\mathbb{R}^4$ which is not symplectically equivalent to any of the product tori?*

This question was open for some time until Chekanov constructed the first example of such a torus in [Che96]. A torus which is not symplectically equivalent to a member of a set of *standard tori* (e.g. product tori) is called *exotic*. The construction and study of exotic tori is a very active field of research, see for example Chekanov–Schlenk [CS10, CS16], Galkin–Usnich [GA10], Auroux [Aur15], Vianna [Via16, Via17]. In terms of full classification of such Lagrangian submanifolds, very little is known. For example, it is not known whether there is another exotic torus in $\mathbb{R}^4$ distinct from Chekanov's example.

Let us briefly discuss *versal deformations*. This technique was used by Chekanov [Che96] in order to distiguish the product tori and show that the example he constructed is exotic. In Chapter 2 and Chapter 4, we heavily rely on this technique. The main idea of versal deformations is to strengthen an invariant of Lagrangian submanifolds by evaluating it on neighbouring Lagrangians. For example, say we wanted to distinguish the product torus $T(1,1)$ from Chekanov's exotic torus. Unfortunately, these tori have the same displacement energy. However, displacement energy *as a function on the space of Lagrangians* behaves locally differently on the Chekanov torus than it does on $T(1,1)$ and thus distiguishes the two tori. The crucial ingredient for versal deformations is Weinstein's neighbourhood theorem, which states, roughly speaking, that *if two Lagrangian submanifolds are diffeomorphic, then they have symplectomorphic neighbourhoods.* This fact is striking. Not only does it mean that the topology (i.e. the normal bundle) of a Lagrangian embedding is determined by the intrinsic topology of the Lagrangian, it also means that there are no local symplectic invariants attached to Lagrangian submanifolds beyond its diffeomorphism type. Since any smooth manifold L sits as a Lagrangian submanifold given by the zero section in its cotangent bundle T^*L (equipped with the symplectic form (1.1.5)), Weinstein's theorem implies in particular that locally we can always think about cotangent bundles. With this in mind, we can give a good description of the Lagrangian neighbours of a fixed Lagrangian submanifold L. Indeed, C^1-small Lagrangian perturbations of L correspond to the graphs of closed one-forms in T^*L and two such graphs are Hamiltonian isotopic if and only if the difference of their one-forms is exact. In other words, it follows from de Rham cohomology that the space of local Lagrangian perturbations up to locally supported Hamiltonian isotopies is isomorphic to the first cohomology $H^1(L;\mathbb{R})$. Since any symplectic invariant (such as dispacement energy) is in particular invariant under Hamiltonian isotopies, this yields a well-defined function defined on a neighbourhood U of $0 \in H^1(L;\mathbb{R})$,

$$H^1(L;\mathbb{R}) \supset U \to \mathbb{R} \cup \{\infty\} \tag{1.4.15}$$

Chapter 2 deals with these matters in depth. To finish this discussion and pivot to the topic of the next sections, let us state the following result which we prove in Chapter 4 using versal deformations and displacement energy.

Theorem 1.4.3 (B. Proposition 4.5.6)**.** *Let (M,ω) be a monotone toric symplectic manifold satisfying property FS. Then M contains an exotic monotone Lagrangian torus.*

Before discussing toric manifolds in 1.6, we introduce Hamiltonian group actions in the next section.

1.5 Hamiltonian group actions

Let us return to classical mechanics for a bit. A very important method to understand a dynamical system in mechanics is by finding *integrals* of the system. By *integrals*, we just mean preserved quantities. Having a preserved quantity is obviously a very useful feature for a dynamical system to have; indeed, let $F\colon M \to \mathbb{R}$ be a function which is constant along the solutions of a certain mechanical system. Instead of looking for solutions of the system in all of M, we can restrict our attention to the level sets $F^{-1}(c) \subset M$ and try to understand the system on these level sets.

Combining the fact that systems from classical mechanics are Hamiltonian (i.e. of the form (1.1.6)) with some symplectic geometry, this casual remark can be turned into a poweful technique. Let (M, ω) be a symplecic manifold (for example a phase space $M = T^*Q$) and $H\colon M \to \mathbb{R}$ a smooth function (for example the total energy of a mechanical system as in (1.1.3)). We obtain the associated Hamiltonian system by integrating the vector field X_H defined by (1.1.6). Now suppose that we have an integral, i.e. another function F which is constant along the solutions of the Hamiltonian system. This is equivalent to $dF(X_H) = 0$. Here something very interesting happens, namely we can rewrite this as

$$0 = dF(X_H) = (\iota(X_F)\omega)(X_H) = \omega(X_F, X_H) \tag{1.5.16}$$

Here we have used (1.1.6) for the function F, which defines the Hamiltonian vector field X_F associated to F. The point is that the right hand side of (1.5.16) puts F and H on equal footing. In other words, using the anti-symmetry of ω and doing the inverse development, we find that *F is preserved under the Hamiltonian flow of H if and only if H is preserved under the Hamiltonian flow of F.* The expression $\omega(X_F, X_H)$ is called the *Poisson bracket* of F and H and denoted by $\{F, H\}$. From (1.5.16), we can for example deduce conservation of energy for time-independent Hamiltonians H. Indeed, the Poisson bracket $\{H, H\}$ vanishes by anti-symmetry and from this we deduce that H is preserved under its own Hamiltonian flow.

Let us assume something stronger than the existence of an integral F, namely assume that the Hamiltonian flow of F is one-periodic. This means that we get an induced $S^1 = \mathbb{R}/\mathbb{Z}$-action on (M, ω). As mentioned above, the dynamics of H is constrained to level sets $F^{-1}(c)$, but furthermore we have found out that H is constant along the (S^1-) orbits of F and hence H descends to a function H_c on the quotient

$$M_c = F^{-1}(c)/S^1, \quad H_c\colon M_c \to \mathbb{R}. \tag{1.5.17}$$

Under some mild conditions, the space M_c is a manifold. Actually, one can prove that whenever M_c is a manifold, then it is a *symplectic manifold* with a symplectic form ω_c naturally induced by ω. This procedure is called *symplectic reduction.* To summarize, we have started with a Hamiltonian system (M, ω, H) on a symplectic $2n$-manifold which has an S^1-symmetry and we have produced a *reduced* Hamiltonian system (M_c, ω_c, H_c) on a symplectic manifold of dimension $2n-2$. In many situations, this reduced system is much easier to understand than the original one and solutions of the reduced system can be lifted back to the original space. Note also that a certain version of *Noether's theorem* is built into this formalism; *if F generates a symmetry of the system, then F is a preserved quantity of the system.* We refer to Section 2.2.7 for more details and to Section 2.2.8 for a detailed example.

Symplectic reduction can be carried out in a much more general setting, namely for Hamiltonian group actions by compact Lie groups. Throughout this thesis, we crucially use the interplay of the geometry of reduced spaces with the geometry of the total space. See for example Section 2.3 and Section 4.3. Roughly speaking, there is usually some kind of correspondence between symplectic objects (Lagrangian submanifolds, Hamiltonian flows,...) in the reduced space and S^1-invariant/equivariant objects in the total space.

Let us illustrate the usefulness of the above discussions by a classical example.

Remark 1.5.1 (Clairaut's relation)**.** *Clairaut's relation* states that there is a preserved quantity along the geodesics on a surface of revolution. In general, the geodesic flow on a Riemannian manifold (Q, g) corresponds to the Hamiltonian flow on T^*Q generated by the Hamiltonian of a free particle (meaning vanishing potential $V(q) = 0$),

$$H(q,p) = \frac{1}{2}\|p\|_g^2. \tag{1.5.18}$$

In other words, the dynamics of geodesic flows is a special case of Hamiltonian dynamics. Now let $Q \subset \mathbb{R}^3$ be a surface of revolution equipped with the restriction of the ambient metric. The geodesic system arising in this way has an obvious S^1-symmetry inherited from the S^1-symmetry of Q, namely the one by rotation around the axis that has swept out the surface of revolution. This rotation turns out to be a Hamiltonian S^1-symmetry, generated by the Hamiltonian function $G(q,p) = p(\partial_\vartheta)$, where $\partial_\vartheta \in \Gamma(TQ)$ is the vector field generating the rotation. Since $dH(X_G) = 0$, we deduce from (1.5.16) that G is preserved under the Hamiltonian flow of H. The function G is Clairaut's integral. Not only have we gotten the insight that Clairaut's relation just comes from the obvious symmetry of the setup, we also have a recipe for finding a formula for the integral: It is just the Hamiltonian generating the symmetry.

1.6 Toric manifolds

Most of the results presented in this thesis are statements about a certain type of symplectic manifolds, called *toric manifolds*. There are many different approaches to toric manifolds, see for example [CdS03], [Aud04] or [Eva21]. We will give a definition in terms of Hamiltonian group actions which is motivated by integrable systems in classical mechanics.

Let (M, ω, H) be a Hamiltonian system on a $2n$-dimensional manifold. Let $F_1, \dots, F_n$ be a set of pairwise commuting Hamiltonians, all of which commute with H. In the language of 1.5, this means that H has lots of integrals. For this to be meaningful, we need to assume that the functions F_i are independent in some way. Otherwise, we may choose $F_1 = \ldots = F_n = H$, which would be silly. Let us assume for example that the map $F = (F_1, \dots, F_n)\colon M \to \mathbb{R}^n$ has a regular value $c \in \mathbb{R}^n$. Then $F^{-1}(c)$ is a manifold and any integral curve of X_H which intersects $F^{-1}(c)$ is contained in $F^{-1}(c)$. This follows from the fact that F_i is constant along the flow of H. In fact, one can show that H can be written as a function of the F_i, meaning that n is the maximal number of independent commuting functions on a $2n$-dimensional manifold. If we assume furthermore that $F^{-1}(c)$ is compact and connected, then the *Arnold–Liouville theorem* completely describes the geometry and the dynamics close to $F^{-1}(c)$. It claims that the preimage $F^{-1}(c)$ is diffeomorphic to the torus T^n and that it has a neighbourhood which is symplectomorphic to $T^n \times V = (\mathbb{R}/\mathbb{Z})^n \times V = \{(\alpha_i, x_i)\}$, where V is an open subset of $\mathbb{R}^n$, equipped with the symplectic form $\sum_i d\alpha_i \wedge dx_i$. Furthermore, this diffeomorphism preserves the fibres of the map F, meaning that the space M is fibred by invariant tori in a neighbourhood of $F^{-1}(c)$. The x_i are called *action coordinates* and the α_i are called *angle coordinates*. Summarizing the above, we get a complete understanding of the Hamiltonian flow of H. We can write $H = h(x_1, \dots, x_n)$ as a function of the action coordinates and, using the normal form of the symplectic form, we find that its Hamiltonian flow is given by

$$\phi_t(\alpha_1, \dots, \alpha_n, x_1, \dots, x_n) = \left(\alpha_1 + t\frac{\partial h}{\partial x_1}, \dots, \alpha_n + t\frac{\partial h}{\partial x_n}, x_1, \dots, x_n\right). \tag{1.6.19}$$

Note that this is a linear translation in the torus component by a translation vector which depends on the action coordinates $x_1, \dots, x_n$. Let $\mu_i \colon U \to \mathbb{R}$ be the projection to the i-th action coordinate, where U denotes a neighbourhood of $F^{-1}(c)$. Then every μ_i generates a Hamiltonian S^1-action and these circle actions commute, which implies that they fit together to yield an action of the torus $T^n = S^1 \times \dots \times S^1$ on U. If the system μ_i of commuting Hamiltonian circle actions extends globally, then we call the resulting torus action by composition of Hamiltonian flows a *Hamiltonian torus action*.

Definition 1.6.1. *A compact symplectic $2n$-manifold (M, ω) is called* toric *if it admits an effective Hamiltonian T^n-action.*

In other words, toric symplectic manifolds are compact symplectic manifolds which admit *global* action-angle coordinates. Locally, action-angle coordinates always exist, but the condition that they extend globally imposes severe topological restrictions, as we shall see below. See Duistermaat [Dui80] for an excellent discussion of action-angle coordinates and the extension question.

The map $\mu = (\mu_1, \dots, \mu_n) \colon M \to \mathbb{R}^n$ whose components are the Hamiltonian functions generating the circle actions is called *moment map*. Moment maps of Hamiltonian torus actions have surprising geometric features. The Atyiah/Guillemin–Sternberg theorem shows that their image is always a rational polytope $\Delta \subset \mathbb{R}^n$ spanned by the image of the fixed points of the torus action. In the toric case, i.e. the case where the torus which acts has half the dimension of the ambient manifold, Delzant [Del88] showed that $\Delta \subset \mathbb{R}^n$ is a very special kind of polytope (called Delzant polytope) and showed that (M, ω) is actually classified by its Delzant polytope up to T^n-equivariant symplectomorphisms, see Theorem 2.6.3 for a precise statement. Note however that we have some freedom in choosing how to get to the Delzant polytope $\Delta \subset \mathbb{R}^n$. For instance, we can add constants to Hamiltonian functions without changing their flow, meaning Δ is determined only up to translations. We may also change the basis we choose in the torus T^n acting on the toric manifold. As it turns out, this is the only freedom which is allowed. Therefore the group of *integral affine transformations*

$$\operatorname{Aff}_n \mathbb{Z} = \mathrm{GL}(n; \mathbb{Z}) \ltimes \mathbb{R}^n \tag{1.6.20}$$

naturally acts on the base of $\mu \colon M \to \mathbb{R}^n$. Delzant's classification result gives rise to the following correspondence

$$\textit{Symplectic geometry of } (M,\omega) \leftrightarrow \textit{Integral affine geometry of } \Delta \subset \mathbb{R}^n.$$

This opens up a playground on which one can play around with symplectic invariants and see to which properties of the polytope they correspond. The latter are usually much easier to handle and read off than the former. In the next section we discuss an instance of this for displacement energy of toric fibres.

1.7 Toric fibres and their displacement energy

The orbits of toric T^n-actions are tori on which the symplectic form vanishes (isotropic submanifolds). There is an open and dense set in M – corresponding to the interior of the moment polytope – in which the orbits are of maximal dimension n meaning that they are Lagrangian submanifolds.

Definition 1.7.1. *Let* (M,ω) *be a toric symplectic manifold with moment map* $\mu\colon M \to \Delta$. *A Lagrangian torus of the form* $T(x) = \mu^{-1}(x)$ *for* $x \in \operatorname{int}\Delta$ *is called a* toric fibre.

Recalling the discussion in 1.4, we note that toric fibres in toric manifolds are the analogue of product tori in $\mathbb{R}^{2n}$. We now understand the word *exotic* in Theorem 1.4.3. It means that there is a Lagrangian torus which is not symplectically equivalent to a toric fibre. The symplectic topology of toric fibres is a well-studied topic, see for example the monumental work by Fukaya–Oh–Ohta–Ono [FOOO16] or Woodward [Woo11]. For us, there are two main facts which make the study of toric fibres via the method combining versal deformations with displacement energy (see (1.4.15)) quite accessible.

1. The versal deformation of a toric fibre is obvious. Indeed, we get a natural n-dimensional family of nearby Lagrangians by varying the base point of the toric fibration, and this family indeed yields a versal deformation. See for example 4.4.
2. The displacement energy of toric fibres can be computed in many cases.

Let us elaborate on the second point, which is somewhat subtle. We are interested in understanding the function

$$e_\Delta\colon \operatorname{int}\Delta \to \mathbb{R} \cup \{\infty\}, \quad x \mapsto e(M, T(x)), \tag{1.7.21}$$

which associates to every point in $\operatorname{int}\Delta$ the displacement energy of the corresponding toric fibre. In 4.3.2, we find that e_Δ is bounded from below by the piece-wise linear function $x \mapsto d_\Delta(x)$ which measures the integral affine distance of a point $x \in \operatorname{int}\Delta$ to the boundary of Δ, see Proposition 4.3.2. This is proved by using the Delzant construction and the computation of the displacement energy of product tori in $\mathbb{C}^N$. Let us emphasize that this bound holds for all toric manifolds.

However, this bound fails to be optimal in many cases. For example, in [FOOO16] it is shown that every toric manifold contains a non-displaceable toric fibre, i.e. there is $x_0 \in \operatorname{int}\Delta$ such that $e_\Delta(x_0) = \infty$. See also the detailed discussion in [McD11]. In the same paper (see also [CS10]), McDuff gives a highly effective method of producing upper bounds on displacement energy of toric fibres. This is called the method by *probes*. In many important special cases, combining these two bounds gives the answer.

Theorem 1.7.2 (Lemma 4.3.6)**.** *Let* (M, ω) *be a monotone toric manifold satisfying property* FS*. Then the functions* e_Δ *and* d_Δ *agree on an open and dense subset of* $\operatorname{int}\Delta$.

Property FS was introduced by Chekanov–Schlenk [CS] to make sure that an open and dense subset of toric fibres in (M, ω) are accessible by the method of probes. Imposing this condition should be ultimately obsolete in the sense that it is conjectured (and checked by computation in dimensions $\leqslant 18$) that monotonicity implies property FS. We refer to 4.4 for a detailed discussion. Let us also emphasize that in all examples (monotone or not) known to us, we have $e_\Delta = d_\Delta$ on an open and dense subset. Furthermore, knowing the function e_Δ on an open and dense subset is sufficient to extract information from versal deformations. These methods were used for example in the proof of Theorem 1.4.3.

1.8 Real Lagrangian submanifolds

An involution on a set X is a map $\sigma\colon X \to X$ which squares to the identity, $\sigma \circ \sigma = \mathrm{id}$. Involutions appear all thoughout mathematics, and their fixed point sets

$$\operatorname{Fix}\sigma = \{x \in X \mid \sigma(x) = x\} \subset X \qquad (1.8.22)$$

usually have very good properties; fixed point sets of group morphisms are subgroups, fixed point sets of smooth maps are submanifolds, fixed point sets of Riemannian isometries are totally geodesic submanifolds, etc. This is no different in symplectic geometry. Fixed point sets of symplectic involutions, i.e. maps $\sigma \in \operatorname{Symp}(M, \omega)$ satsisfying $\sigma^2 = \mathrm{id}$ are symplectic submanifolds. However, we are interested in *antisymplectic* involutions, diffeomorphisms $\sigma \in \operatorname{Diff}(M, \omega)$ satisfying

$$\sigma^2 = \mathrm{id}, \quad \sigma^*\omega = -\omega. \qquad (1.8.23)$$

The fixed point set of an antisymplectic involution is either empty or it is a Lagrangian submanifold.

Definition 1.8.1. *A Lagrangian* $L \subset (M, \omega)$ *is called* real *if it can be realized as the fixed point set of an antisymplectic involution.*

The name *real* is taken from algebraic geometry, where the *real locus* of a projective variety designates its fixed point set under complex conjugation. The study of real algebraic geometry is a very classical topic with a rich history, driven for example by Hilbert's sixteenth problem. *Real symplectic geometry* is the analogous study of real Lagrangian submanifolds, and we refer to the introductory sections 3.1 and 4.1 for a discussion on the existing literature and results in symplectic geometry.

Chapter 3 analyzes the topology of real Lagrangian submanifolds in toric symplectic manifolds. Its starting point was a question posed to me by Joontae Kim:

Can an involution of the Delzant polytope Δ *be lifted to an antisymplectic involution of the corresponding toric manifold?*

The answer turns out to be positive.

Theorem 1.8.2 (B.–Kim–Moon, Theorem 3.1.1)**.** *Let (M, ω) be a toric symplectic manifold with Delzant polytope Δ. Every involution of $\Delta \subset \mathbb{R}^n$ which preserves $\mathbb{Z}^n$ can be lifted to an antisymplectic involution of (M, ω).*

The Lagrangian fixed point set of these lifted involutions is never empty and thus it gives a large class of real Lagrangian submanifolds of toric manifolds. In fact, restricting our attention to *monotone* toric symplectic manifolds of dimension four (the so-called del Pezzo surfaces), we gave a complete classification of the topological type of real Lagranians (see 3.1) and showed that all the possible types are indeed realized by our construction. This is Theorem 3.1.4. Furthermore, we give a Delzant-type construction of the real Lagrangian manifolds arising in this way in Theorem 3.1.2 and generalize Duistermaat's convexity and tightness result [Dui83] in Theorem 3.1.3.

Chapter 4 is about the following question:

Which Lagrangian submanifolds are real?

Note that this question naturally splits into two parts - a smooth part and symplectic part. Given a Lagrangian submanifold $L \subset M$, we first need to know whether L can be realized as the fixed point set of a *smooth* involution. This is actually already a non-trivial question, and classical results by Smith give obstructions, see [Bor60] or 4.1 for details. If L can be realized as fixed point set of a smooth involution, we can meaningfully ask if it is *real*. If the answer to the smooth question is positive and the answer to the symplectic question is negative then this is an honest occurence of *symplectic rigidity* as discussed in Section 1.2.

Our main result is an obstruction in terms of the versal deformations of the Lagrangian submanifold. We proved that if $L \subset (M, \omega)$ is real, then its displacement energy germ (1.4.15) has to be invariant under central symmetry, see Theorem 4.2.10. Combining this with some knowledge of displacement energy of toric fibres (as discussed in 1.7) and with Theorem 1.8.2, we obtain the following result.

Theorem 1.8.3 (B. Theorem 4.1.2)**.** *Let (M, ω) be a monotone toric manifold with Delzant polytope Δ satisfying property FS. Then the monotone toric fibre $T(x_0) \subset M$ is real if and only if Δ is centrally symmetric, meaning $\Delta = -\Delta$.*

Note that a real Lagrangian in a monotone symplectic manifold is automatically monotone and hence this theorem settles the question which toric fibres in monotone toric manifolds (satisfying FS) are real. The main point in the proof of this theorem is the fact that one can recover Δ from the displacement energy germ of the monotone fibre $T(x_0)$. For any non-monotone fibre, this is false. We also prove that the Chekanov tori from Theorem 1.4.3 are not real, generalizing an earlier result by Kim [Kim21a] for the Chekanov torus in $S^2 \times S^2$, which was obtained by different methods.

Theorem 1.8.4 (B. Theorem 4.1.4)**.** *The Chekanov tori from Theorem 1.4.3 are not real.*

In case Δ is centrally symmetric, this is an honest occurence of symplectic rigidity. Indeed, the Chekanov tori we constructed are smoothly isotopic to the monotone toric fibre and hence they can be realized as fixed point sets of a smooth involution, see Proposition 4.5.9.

2 An introduction to the Chekanov torus

This is an introductory article which arose from expanded notes for a set of talks given in the GGTI Online Seminars organized by the Gökova Geometry and Topology Institute.

We give a mostly self-contained and elementary introduction to the Chekanov torus in $\mathbb{C}^2$, i.e. the first example of a Lagrangian torus in $(\mathbb{R}^4, \omega_0)$ which cannot be mapped to a product torus by a symplectomorphism. Our exposition emphasizes the relationship with Hamiltonian torus actions and symplectic reduction, and our goal is to explain how the Chekanov torus can be constructed in more general toric symplectic manifolds. Furthermore, we give a detailed introduction to the technique of versal deformations, which we use, together with the displacement energy of product tori, to prove that the Chekanov torus is not symplectomorphic to a product torus.

2.1 Introduction

2.1.1 A classification question in dimension two

Let $\mathcal{C}$ be a closed embedded smooth curve in the plane $\mathbb{R}^2$, i.e. a submanifold diffeomorphic to the circle. We are interested in classifying such curves up to applying area-preserving diffeomorphisms of $\mathbb{R}^2$. One obvious invariant of this classification question is the area contained by $\mathcal{C}$. More precisely, we know that the complement of $\mathcal{C}$ has one bounded component, say $U \subset \mathbb{R}^2$, and we can measure its Euclidean area,

$$\operatorname{area} \mathcal{C} = \operatorname{area} U = \int_U dx \wedge dy = \frac{1}{2} \int_{\mathcal{C}} x dy - y dx. \tag{2.1.1}$$

In fact, two curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^2$ are equivalent up to area-preserving diffeomorphism if and only if they contain the same area. Thus our first classification question is settled.

Let us formalize this question in terms of symplectic geometry. We replace area-preserving diffeomorphisms of $\mathbb{R}^2$ by diffeomorphisms preserving the two-form $\omega_0 = dx \wedge dy$. These are the diffeomorphisms which preserve area *and orientation.* They are the *symplectomorphisms* of $(\mathbb{R}^2, \omega_0)$,

$$\operatorname{Symp}(\mathbb{R}^2, \omega_0) = \{\phi \in \operatorname{Diff}(\mathbb{R}^2) \mid \phi^* \omega_0 = \omega_0\}. \tag{2.1.2}$$

In terms of symplectic geometry, curves in the plane are *Lagrangian submanifolds.* This means that they have half the dimension of the ambient manifold and that the symplectic form vanishes on their tangent space. By the same token as above, *Lagrangian circles in the plane are classified by the area they contain.* In other words, we can take the round circles in the plane as models, meaning that every Lagrangian circle $\mathcal{C}$ in the plane is of the form

$$S^1(a) = \{\pi(x^2 + y^2) = a\}, \quad a = \operatorname{area} \mathcal{C}, \tag{2.1.3}$$

after applying a symplectomorphism.

2.1.2 A classification question in dimension four

Let us now ask the same question in dimension four. The symplectic manifold we consider is

$$\mathbb{R}^4 = \{(x_1, y_1, x_2, y_2)\}, \quad \omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2. \tag{2.1.4}$$

The symplectomorphisms are again defined as the group of diffeomorphisms which preserve ω_0. The symplectic manifold $(\mathbb{R}^4, \omega_0)$ can be seen as the product of $(\mathbb{R}^2, \omega_0)$ with itself, where the symplectic form on the product of two symplectic manifolds is the sum of symplectic forms. This means that we obtain a natural family of Lagrangian submanifolds by taking the product $\mathcal{C}_1 \times \mathcal{C}_2$ of smooth embedded closed curves $\mathcal{C}_1 \subset \{(x_1, y_1)\}, \mathcal{C}_2 \subset \{(x_2, y_2)\}$ in the factors. These submanifolds are diffeomorphic to tori, and hence we will refer to them as *Lagrangian tori* from now on. For convenience, we can again put these Lagrangian submanifolds into the normal form given by products of round circles,

$$T(a_1, a_2) = \{\pi(x_1^2 + y_1^2) = a_1, \, \pi(x_2^2 + y_2^2) = a_2\}, \quad a_i = \operatorname{area} \mathcal{C}_i. \tag{2.1.5}$$

Indeed, we can achieve this normal form by applying symplectomorphisms in the factors of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. Members of the two-parameter family of tori $T(a_1, a_2)$ are called *product tori*. In light of the classifiaction in dimension two, it is natural to ask

Question 2.1.1. Is every Lagrangian torus $L \subset \mathbb{R}^4$ a product torus, up to applying a symplectomorphism?

This question was answered in the negative by Chekanov, who constructed the first counter-example, now called *Chekanov torus*, in [Che96]. See also Eliashberg–Polterovich [EP97]. Our goal is to give a detailed description of the Chekanov torus and discuss a symplectic invariant of Lagrangian submanifolds one can use to distiguish it from the product tori.

2.1.3 Hamiltonian torus actions

Note that the circles (2.1.3) are the orbits of the *harmonic oscillator*. By this, we mean the orbits of the vector field X_H defined by *Hamilton's equation*

$$dH = -\omega_0(X_H, \cdot) \tag{2.1.6}$$

for the Hamiltonian $H(x, y) = \pi(x^2 + y^2)$. Indeed, this implicit equation defines the rotational vector field $X_H = 2\pi(x\partial_y - y\partial_x)$ and hence its orbits are concentric circles. In terms of classical mechanics, this vector field defines the flow in the phase space of a point particle in one dimension submitted to a (suitably normalized) square potential. Hence the name *harmonic oscillator*. Note that the flow of X_H is periodic of unit period and hence it induces a so-called *Hamiltonian group action* by the circle $S^1 = \mathbb{R}/\mathbb{Z}$. In higher dimensions, one can make a similar construction. Indeed, let us define the pair of Hamiltonians

$$H_1(x, y) = \pi(x_1^2 + y_1^2), \quad H_2(x, y) = \pi(x_2^2 + y_2^2) \tag{2.1.7}$$

on $(\mathbb{R}^4, \omega_0)$. The Hamiltonian circle actions associated to H_1, H_2 are given by rotation in the coordinate planes $\{(x_1, y_1)\}$ and $\{(x_2, y_2)\}$, respectively. Furthermore, the

Hamiltonians H_1, H_2 *(Poisson-)commute*, meaning their respective Hamiltonian vector fields X_{H_1}, X_{H_2} satisfy $\omega(X_{H_1}, X_{H_2}) = 0$. Since they commute, the two circle actions can be combined to a Hamiltonian action of the two-torus on $\mathbb{R}^4$, see also Definition 2.2.27. The point here is that the orbits of the group action are precisely the product tori $T(a_1, a_2)$. Thinking slightly more geometrically, we obtain a singular fibration $(H_1, H_2) \colon \mathbb{R}^4 \to \mathbb{R}^2_{\geqslant 0}$ over the non-negative quadrant with product tori as fibres. Since the product tori are Lagrangian submanifolds, this is an example of a (singular) *Lagrangian torus fibration*. Readers who are more familiar with the language of integrable systems may think of (H_1, H_2) as an integrable system and of the torus orbits as Liouville tori. This kinship between torus actions and Lagrangian tori will play a central role in our exposition of the Chekanov torus. In fact, we will view the Chekanov torus as fibre of an *exotic* Lagrangian torus fibration on the complement $\mathbb{R}^4 \setminus P$ of the Lagrangian plane

$$P = \{(x, y, x, -y) \in \mathbb{R}^4 \mid (x, y) \in \mathbb{R}^2\}. \tag{2.1.8}$$

We construct this exotic Lagrangian torus fibration by an explicit deformation of the standard fibration in certain symplectic quotients. Since we carry out this construction in $\mathbb{R}^4$, we do everything by hand and refrain from making reference to the general theory of toric manifolds except in Section 2.6.

2.1.4 Outline

The main goal of this expository paper is to give an elementary description of the Chekanov torus, highlighting its relationship with symplectic reduction and Hamiltonian torus actions. Very little a priori knowledge of symplectic geometry is required since we discuss all relevant concepts before using them. Our exposition is largely self-contained except for

1. Some references in §2.2.3 to concepts from Morse theory which we do not introduce;
2. The discussion of relations to almost toric geometry in §2.3.4. See §2.1.5 for a discussion of references on this topic;
3. The computation of the displacement energy of product tori in $\mathbb{R}^{2n}$, see Example 2.5.5;
4. Section 2.6 on toric symplectic manifolds, where we only give an overview.

This paper roughly splits into two parts.

The first part consists of Section 2.2 giving an introduction to Hamiltonian circle actions and symplectic reduction, and of Section 2.3, where we define the Chekanov torus using the techniques from Section 2.2. The main theme of the first part is to exhibit the relationship between Lagrangian tori on the one hand and Hamiltonian torus actions on the other. In fact, we will modify the standard toric structure (2.1.7) on $\mathbb{R}^4$ to get a Lagrangian torus fibration on the complement of a Lagrangian plane $P \subset \mathbb{R}^4$ (see (2.1.8)) having Chekanov tori as some of its fibres. For one, this approach is naturally compatible with the use of versal deformations. Secondly, the same method readily generalizes to toric symplectic manifolds of any dimension and yields an exotic copy of the Chekanov torus in many cases. The latter generalization will be outlined in Section 2.6.

The second part of the paper consists of an introduction to versal deformations in Section 2.4 and a short introduction to displacement energy in Section 2.5. Combining these two ingredients yields a proof of the fact that the Chekanov torus is exotic. The last section is an outlook on the case of general toric symplectic manifolds and as such is less detailed than the rest of the paper. However, we will see that the construction of the Chekanov torus, as well as the invariant used to prove its exoticity, can be carried over almost verbatim to toric symplectic manifolds.

As an expository text, this paper contains little to no original research. The only thing that is somewhat new is our approach to the construction of the exotic Lagrangian torus fibration on $\mathbb{R}^4 \setminus P$, although very similar ideas have appeared in a recent preprint by Groman–Varolgunes [GV21, Chapter 7]. What is interesting here is that the construction we give is related (an probably equivalent) to certain tools from *almost toric geometry*. As we have tried to outline in §2.3.4, the construction of the exotic Lagrangian torus fibration we give in §2.3.3 should be the same as performing a *nodal trade* on $\mathbb{C}^2$ followed by a *nodal slide* which sends the node to infinity.

2.1.5 Further references

For a general introduction to all topics related to Lagrangian torus fibrations and integrable systems we heartily recommend the recent book by Evans [Eva21]. The classical references for Hamiltonian torus actions and toric structures are too many to name. Among them are Audin [Aud04], Canas da Silva [CdS03], Guillemin [Gui94] and the original article by Delzant [Del88]. For an introduction to integrable systems, offering a complementary point of view on the material we discuss, we recommend the foundational article by Duistermaat [Dui80]. For the construction of the Chekanov torus, see Eliashberg–Polterovich [EP97] and the articles [Che96], [CS10] and [CS16]. The latter three texts also use and discuss versal deformations. For a slightly different point of view, see Auroux [Aur07]. The construction of the Chekanov torus in toric manifolds given in Section 2.6 is based on [Bre20]. For more details on almost toric geometry, we again recommend the excellent exposition in [Eva21] as well as the original articles by Symington [Sym03] and Vianna [Via16], [Via17].

2.1.6 Acknowledgements

We thank Selman Akbulut, Emily Windes, Ustun Yildirim and Eylem Zeliha Yıldız for the invitation to the GGTI Online Seminars they have organized and for their patience in waiting for the accompanying notes. Many thanks also to the participants of the online seminars for lively discussions. We thank Felix Schlenk for many useful comments along the way and Umut Varolgunes for clarifying discussions concerning §2.3.3 and §2.3.4 and for pointing out [GV21].

2.2 Hamiltonian circle actions

We introduce basic notions from symplectic geometry, Hamiltonian circle actions and symplectic reduction. We especially focus on Hamiltonian circle actions on two-dimensional symplectic manifolds, since this is a special case of toric manifolds. In particular, as a toy case of the Delzant theorem, we classify all Hamiltonian circle actions on surfaces.

2.2.1 Background: Symplectic manifolds and Hamilton's equation

In §2.1.1 and §2.1.2, we have seen the standard symplectic structures on $\mathbb{R}^2$ and $\mathbb{R}^4$. In general, a *symplectic manifold* is a smooth manifold M equipped with a closed differential two-form ω which is *non-degenerate*. Non-degenerate here means that the map

$$TM \to T^*M, \quad X \mapsto \iota(X)\omega = \omega(X, \cdot) \tag{2.2.9}$$

is a bundle isomorphism, i.e. an isomorphism of vector spaces $T_pM \to T_p^*M$ in every point $p \in M$. Note that we have already made use of the non-degeneracy in equation (2.1.6). If ω_0 was degenerate, the vector field X_H would not be well-defined. In general, a vector field X_H defined by *Hamilton's equation*

$$dH = -\iota(X_H)\omega \tag{2.2.10}$$

is called *Hamiltonian vector field* and the smooth function H is called the *Hamiltonian (function)*. By integrating a Hamiltonian vector field, we obtain a *Hamiltonian flow*, denoted by ϕ_t^H. We say that it is *generated by the Hamiltonian H*. In general, we allow for time-dependent Hamiltonians in the definition of Hamiltonian flows, see Defintion 2.5.1. For now, however, it is enough to restrict our attention to *autonomous* Hamiltonian flows, i.e. flows coming from a time-independent Hamiltonian function.

Proposition 2.2.1. *Hamiltonian flows preserve the symplectic form. Furthermore, autonomous Hamiltonian flows preserve the Hamiltonian function,*

$$(\phi_t^H)^*\omega = \omega, \quad H \circ \phi_t^H = H. \tag{2.2.11}$$

The second statement follows directly from Hamilton's equation (2.2.10) and the antisymmetry of the symplectic form, $dH(X_H) = \omega(X_H, X_H) = 0$. The first statement follows from *Cartan's magic formula*, which expresses the Lie derivative of a differential form $\alpha \in \Omega^k(M)$ in terms of the exterior derivative and contraction with the vector field,

$$\mathcal{L}_X\alpha = d\iota(X)\alpha + \iota(X)d\alpha, \tag{2.2.12}$$

by which

$$\frac{d}{dt}(\phi_t^H)^*\omega = (\phi_t^H)^*\mathcal{L}_{X_H}\omega = (\phi_t^H)^*(d\iota(X_H)\omega + 0) = (\phi_t^H)^*ddH = 0. \tag{2.2.13}$$

2.2.2 Hamiltonian circle actions

Definition 2.2.2. *A Hamiltonian system (M, ω, H) is called* Hamiltonian circle action *if its flow is periodic of unit period, i.e. $\phi_1^H = \mathrm{id}$.*

The name comes from the fact that such a Hamiltonian system induces a smooth action of $S^1 = \mathbb{R}/\mathbb{Z}$ which preserves the symplectic form,

$$S^1 \times M \to M, \quad (\vartheta, p) \mapsto \phi_\vartheta^H(p). \tag{2.2.14}$$

Note that we are a bit sloppy with terminology here, since *Hamiltonian circle action* can mean two things: The Hamiltonian system (M, ω, H) or the circle action itself.

Remark 2.2.3. (Periods of circle actions) Requiring Hamiltonian circle actions to have unit period is a choice which fixes the normalization of the Hamiltonian function. Indeed, if the Hamiltonian H has period τ, then we can replace it by τH, which has unit period. This explains why we have chosen $H = \pi(x^2 + y^2)$ in §2.1.1.

Let us now turn to some examples of symplectic manifolds and Hamiltonian circle actions on them.

Example 2.2.4. (The harmonic oscillator) Let $\mathbb{R}^2$ be equipped with the coordinates $(x, y) \in \mathbb{R}^2$. Throughout the text, we identify $\mathbb{C} = \mathbb{R}^2$ via the coordinate $z = x + iy$. The two form $\omega_0 = dx \wedge dy$ is symplectic. It is closed and, in fact, even exact. For example, one can pick any of the one-forms xdy, $-ydx$ or $\frac{1}{2}(xdy - ydx)$ as primitive. The latter primitive has the distinction of being invariant under the circle action $z \mapsto e^{2\pi i\vartheta}z$. The form ω_0 is non-degenerate. In fact, it induces the map $\partial_x \mapsto dy, \partial_y \mapsto -dx$ on $T\mathbb{C} \to T^*\mathbb{C}$ in the standard trivializations. The Hamiltonian $F = \pi(x^2 + y^2) = \pi|z|^2$ defines the Hamiltonian circle action $\phi^F_\vartheta(z) = e^{2\pi i\vartheta}z$ on $\mathbb{C}$. The image of the Hamiltonian is the non-negative reals, $F(\mathbb{C}) = \mathbb{R}_{\geqslant 0}$. See also Figure 2.1.

Example 2.2.5. (Rotation of the sphere) Let $S^2 \subset \mathbb{R}^3$ be the unit two-sphere equipped with the symplectic form ω_{S^2}, which measure Euclidean area. Let $G(x, y, z) = 2\pi z$ be the Hamiltonian given by a rescaling of the projection to the z-axis. The corresponding Hamiltonian flow ϕ^G_t is rotation of unit period around the z-axis. Indeed, one can write the symplectic form out in coordinates,

$$\omega_{S^2} = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy, \tag{2.2.15}$$

and check that the Hamiltonian vector field is $X_G = 2\pi(y\partial_x - x\partial_y)$. The image of the Hamiltonian is the closed segment $G(S^2) = [-2\pi, 2\pi]$. See also Figure 2.1.

Example 2.2.6. (Rotation of the cylinder) Let $Z = T^*S^1 \cong \mathbb{R} \times S^1$ be the cylinder equipped with the symplectic form $\omega_Z = dx \wedge d\vartheta$, where $x \in \mathbb{R}$ and $\vartheta \in S^1 = \mathbb{R}/\mathbb{Z}$. Then the projection $K(\vartheta, x) = x$ generates the rotation $\phi^K_t(\vartheta, x) = (\vartheta + t, x)$ of unit period. The image of the Hamiltonian is the real line $K(Z) = \mathbb{R}$. See also Figure 2.1.

Remark 2.2.7. Note that in the Examples 2.2.4, 2.2.5 and 2.2.6, the traces of the orbits of the Hamiltonian flow can be read off from the Hamiltonian function without any computation since the orbits are contained in level sets of the Hamiltonian and the ambient space has dimension two. However, this does not fix their parametrization. Indeed, take the example of $(\mathbb{C}, \omega_0)$ and note that all Hamiltonian functions of the form $H_f = f(\pi|z|^2)$ for a smooth function f have orbits contained in the circles $\{|z| = c\}$. Since the Hamiltonian vector field is given by

$$X_{H_f} = f' X_F, \tag{2.2.16}$$

for $F = \pi|z|^2$, the only H_f which yield circle actions are given by $H_f = kF + c$, where $k \in \mathbb{Z}$ and $c \in \mathbb{R}$ is a constant. See also Remark 2.2.15.

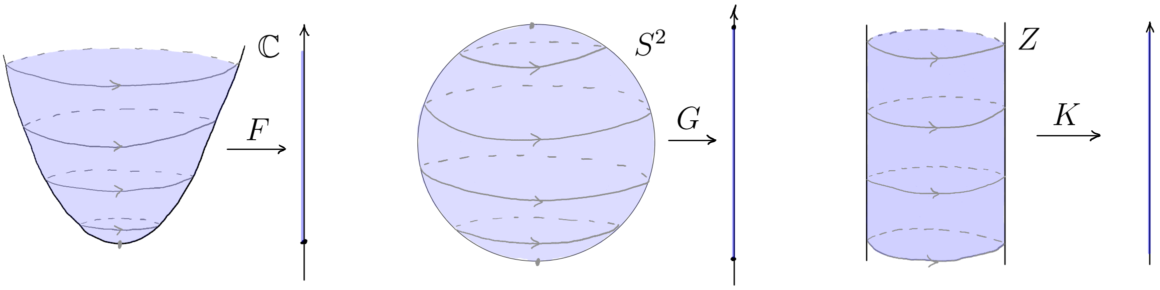


Figure 2.1: The circle actions from Examples 2.2.4, 2.2.5 and 2.2.6

Example 2.2.8. (Linear combinations of harmonic oscillators) Note that all of the above circle actions are defined on two-dimensional symplectic manifolds. Let us now consider $\mathbb{C}^n$ equipped with the standard symplectic form $\omega_0 = \sum_i dx_i \wedge dy_i$. We view this as the product of n copies of $(\mathbb{C}, \omega_0 = dx \wedge dy)$ and thus there are circle actions $F_i = \pi|z_i|^2$ which rotate the i-th coordinate plane and leave the others invariant. We can construct more complicated Hamiltonian systems from the F_i by taking linear combinations

$$F_\xi = \xi_1 F_1 + \ldots + \xi_n F_n, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n. \tag{2.2.17}$$

Note that Hamilton's equation (2.2.10) defines an $\mathbb{R}$-linear mapping $H \mapsto X_H$ and hence the corresponding Hamiltonian vector field is $X_{F_\xi} = \sum \xi_i X_{F_i}$. The corresponing Hamiltonian flow is given by

$$\phi_t^{F_\xi}(z_1, \ldots, z_n) = (e^{2\pi i \xi_1 t} z_1, \ldots, e^{2\pi i \xi_n t} z_n). \tag{2.2.18}$$

This yields a circle action of unit period if and only if $\xi \in \mathbb{Z}^n$. In §2.2.6, we will further generalize and geometrize this class of examples. Indeed, we will see in Example 2.2.28 that there is a *Hamiltonian group action* of the n-torus on $\mathbb{C}^n$. Hamiltonian flows of the type (2.2.18) arise by picking one-parameter subgroups of the torus.

2.2.3 Some Morse theory

We will see that Hamiltonian circle actions are in some sense very rigid. For example, the only compact surface admitting a Hamiltonian circle action is the sphere. In order to prove this, we will make use of Morse theory. Here, we assume that the reader is familiar with the basic ideas of Morse theory. In particular, we will make reference to the *Morse lemma* and *Morse indices* without introducing these concepts. The link between Hamiltonian circle actions and Morse theory comes from a local model of fixed points of the action. Let us first recall Darboux's theorem.

Proposition 2.2.9. *Let $p \in M$ be a point in a symplectic manifold (M^{2n}, ω). Then there is a chart $\varphi\colon \mathbb{C}^n \supset U \to V$ around p such that $\varphi^*\omega = \omega_0$.*

Sketch of proof. The proof has two steps. First one proves that the symplectic vector space $(T_pM, \Omega = \omega_p)$ obtained by restricting the symplectic form to the tangent space is equivalent to $(\mathbb{C}^n, \omega_0)$ by a symplectic isomorphism of vector spaces. In the second step, one picks a local diffeomorphism $\psi\colon T_pM \supset U' \to V$ for a contractible neighbourhood V of $p \in M$, for example by the exponential map associated to a Riemannian metric. Now there are two different symplectic forms on V, namely $\omega|_V$ and $\psi_*\Omega$. The discrepancy between these two symplectic forms can be corrected by applying the so-called *Moser trick*. The latter can be applied because V is contractible and hence the difference between the symplectic forms is exact, $\omega|_V - \psi_*\Omega = d\alpha$. ☐

As it turns out, Darboux's theorem can be made equivariant in some cases. Let us return to Hamiltonian circle actions. Let $H\colon M \to \mathbb{R}$ be a Hamiltoniann generating a circle action $\vartheta \mapsto \psi_\vartheta$ on (M, ω). Furthermore, let $p \in M$ be a fixed point of ψ_ϑ. By Hamilton's equation (2.2.10) this is equivalent to $p \in M$ being a critical point of H.

Proposition 2.2.10. *Let $p \in M$ be a fixed point of a Hamiltonian circle action generated by $H\colon M \to \mathbb{R}$. Then there is an S^1-invariant neighbourhood V of p and a chart $\varphi\colon \mathbb{C}^n \supset U \to V$ such that*

$$H_\xi = H \circ \varphi = \sum_{i=1}^{n} \xi_i \pi |z_i|^2, \quad \varphi^*\omega = \omega_0, \tag{2.2.19}$$

for some $\xi_i \in \mathbb{Z}^n$.

Sketch of proof. The strategy of proof is similar to that used for the Darboux theorem. The circle action can be linearized at the fixed point to yield a circle action on $(T_pM, \Omega = \omega_p)$ and the local diffeomorphism $\psi\colon T_pM \supset U' \to V$ can be chosen to be equivariant, by taking the exponential map of an averaged Riemannian metric for example. As it turns out, the Moser trick can be carried out equivariantly and hence one obtains an S^1-equivariant symplectomorphism $\varphi\colon \mathbb{C}^n \supset U \to V$. In order to see that $H \circ \varphi$ has the desired form (2.2.19) all that is left to do is to classify linear Hamiltonian circle actions on symplectic vector spaces. These are always of the form (2.2.18) for some $\xi_i \in \mathbb{Z}$ and since φ is a symplectomorphism, we obtain $H \circ \varphi = H_\xi$. ☐

Let us make two comments about this local normal form. First note that we have encountered Hamiltonians of the type (2.2.19) in Example 2.2.8. Proposition 2.2.10 thus shows that every Hamiltonian circle action is locally a linear combination of harmonic oscillators around its fixed points. Secondly, the proposition can be interpreted in terms of Morse theory. Indeed note that (2.2.19) is a variation on the *Morse Lemma*. However, note that in general H is not Morse properly speaking, since some ξ_i may be zero. However it is always *Morse-Bott*, meaning that it is non-degenerate on the directions transverse to the submanifold of fixed points.

Corollary 2.2.11. *Hamiltonians generating Hamiltonian circle actions are Morse-Bott and their Morse-Bott indices are even.*

Evenness of the Morse-Bott index follows immediately from the formula (2.2.19), since the coordinates $x_i = \mathrm{Re}(z_i)$ and $y_i = \mathrm{Im}(z_i)$ come in pairs which both have the same contribution to the Morse index. Thus Hamiltonians which generate circle actions are *perfect*, meaning that the Morse inequalities are equalities. In terms of Morse homology, the boundary operator vanishes and the Morse chain complex is isomorphic to Morse homology. In particular, every perfect Morse function on a compact connected orientable manifold has a unique local maximum and minimum, since the top and bottom homologies of such spaces have exactly one generator.

Corollary 2.2.12. *The only compact symplectic surface admitting a non-trivial Hamiltonian circle action is the sphere.*

Proof. First note that a surface admitting a symplectic structure is orientable. Let (Σ, ω_Σ) be a compact symplectic surface with a Hamiltonian circle action generated by H. We may assume that H is Morse, since otherwise it vanishes on a neighbourhood of its degenerate fixed points and hence vanishes on all of Σ. If Σ has genus > 0, then it has non-vanishing first homology. This is in contradiction with the fact that H is Morse and has only even indices. ☐

2.2.4 Hamiltonian circle actions on surfaces

Let (Σ, ω) be a connected symplectic surface equipped with a Hamiltonian circle action generated by H. Let us assume that the circle action is *effective*. Recall that a group action is called *effective* if there is no element of the group which acts trivially, except for the neutral element. For us, this just means that we normalize H such that $\phi_1^H = \mathrm{id}$, but $\phi_t^H \neq \mathrm{id}$ for all $0 < t < 1$. Hence assuming effectiveness is not really a restriction. The goal of this subsection is to give a classification of effective Hamiltonian circle actions on connected symplectic surfaces, see Theorem 2.2.17. This is interesting for us in two respects. First, it serves as a toy case for the general classification of toric symplectic manifolds by Delzant, see Theorem 2.6.3. Secondly, we heavily rely on this classification for the construction of the Chekanov torus in Section 2.3. Before going into details about this, let us gain some intuition for Hamiltonian circle actions on surfaces by looking at the symplectic area contained between two orbits. This is closely related to a very old theorem by Archimedes.

Remark 2.2.13. (Archimedes' theorem) Let $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the round sphere of unit radius and $Z = \{x^2 + y^2 = 1\} \subset \mathbb{R}^3$ the cylinder obtained as product of the unit circle and the real line. Archimedes observed that the area of slices of equal height

$$S^2 \cap \{a \leqslant z \leqslant b\}, \quad Z \cap \{a \leqslant z \leqslant b\} \tag{2.2.20}$$

is equal for all $-1 \leqslant a \leqslant b \leqslant 1$. We will see that this is equivalent to the fact that the Hamiltonian $H(x, y, z) = 2\pi z$ defines a Hamiltonian S^1-action on both surfaces.

Proposition 2.2.14. *Let (Σ, ω) be a connected symplectic surface and $H \colon \Sigma \to \mathbb{R}$ a proper Hamiltonian. Then (Σ, ω, H) is an effective Hamiltonian circle action if and only if H has only critical points of even index and for all $a, b \in H(\Sigma)$ we have*

$$\int_{H^{-1}(a,b)} \omega = b - a. \tag{2.2.21}$$

Proof. Let us first assume that (Σ, ω, H) is an effective circle action. By Corollary 2.2.11, the function H is Morse and all its critical points have even index. In particular, since Σ is connected, there are no critical values in (a, b). Thus the level set $H^{-1}(x)$ for $x \in$

(a,b) is the circle run through by an orbit of the Hamiltonian system. In particular, the set $H^{-1}(a,b)$ is a cylinder and our goal is to prove that this cylinder has area $b-a$. We start by picking a smooth curve $\gamma\colon (a,b)\to\Sigma$ such that

$$H(\gamma(x))=x. \tag{2.2.22}$$

We can think of γ as a local section of the system. By (2.2.22), γ intersects the orbits of the S^1-action ϕ_t^H transversely. Together with the circle action, the curve γ allows us to define a parametrization of the cylinder $H^{-1}(a,b)$,

$$h\colon (a,b)\times S^1\to H^{-1}(a,b)\subset\Sigma,\quad (x,t)\mapsto\phi_t^H(\gamma(x)). \tag{2.2.23}$$

Note that we have used the fact that the action is effective here. Otherwise, h may not be a diffeomorphism. We claim that h is a symplectomorphism when we equip $(a,b)\times S^1$ with the symplectic form $dx\wedge dt$. The claim then follows, since $(a,b)\times S^1$ has area $b-a$. To show that h is a symplectomorphism, it suffices to compute

$$h^*\omega(\partial_t,\partial_t)=h^*\omega(\partial_x,\partial_x)=0,\quad h^*\omega(\partial_x,\partial_t)=1, \tag{2.2.24}$$

since ∂_x,∂_t form a basis at each tangent space of the cylinder $(a,b)\times S^1$. The first two equalities in (2.2.24) follow directly from anti-symmetry of ω. For the third, compute

$$h_*\partial_x=\frac{\partial h}{\partial x}=(\phi_t^H)_*\dot\gamma,\quad h_*\partial_t=\frac{\partial h}{\partial t}=X_H\circ h. \tag{2.2.25}$$

From this, we deduce

$$h^*\omega(\partial_x,\partial_t)=\omega((\phi_t^H)_*\dot\gamma,X_H\circ h)=dH((\phi_t^H)_*\dot\gamma)=dH(\dot\gamma)=1 \tag{2.2.26}$$

For the second identity, we have used Hamilton's equation, for the third the invariance of H under its Hamiltonian flow and the fourth follows from the choice of γ by differentiating (2.2.22) with respect to x.

Let us now prove the converse. The condition that H only has critical points of even Morse index (here this means that the indices are equal to 0 or 2) implies that the level sets $H^{-1}(x)$ are connected. This follows from the connectedness of Σ and basic Morse theory, see for example [Aud04, Theorem IV.3.1]. In particular, every level $H^{-1}(x)$ is diffeomorphic to a circle by the properness of H. To prove that H induces a circle action, let $a,b\in H(\Sigma)$ such that $a<x<b$. By the above, we know that $H^{-1}(a,b)$ is an open cylinder. Again, we pick a section $\gamma\colon(a,b)\to\Sigma$ with $H(\gamma(x))=x$ and define the corresponding map

$$h\colon (a,b)\times\mathbb{R}\to H^{-1}(a,b)\subset\Sigma,\quad (x,t)\mapsto\phi_t^H(\gamma(x)). \tag{2.2.27}$$

By the same arguments as above, this map satisfies $h^*\omega=dx\wedge dt$, but as opposed to (2.2.23), this is not a diffeomorphism but rather a covering. Indeed, since H is proper and has no critical values in (a,b), there is a well-defined function $P\colon(a,b)\to\mathbb{R}$ such that $P(x)$ is the period of the Hamiltonian system at the level $x\in(a,b)$. This function is smooth since it is defined by the implicit equation $\phi^H_{P(x)}(\gamma(x))=\gamma(x)$. Our goal is

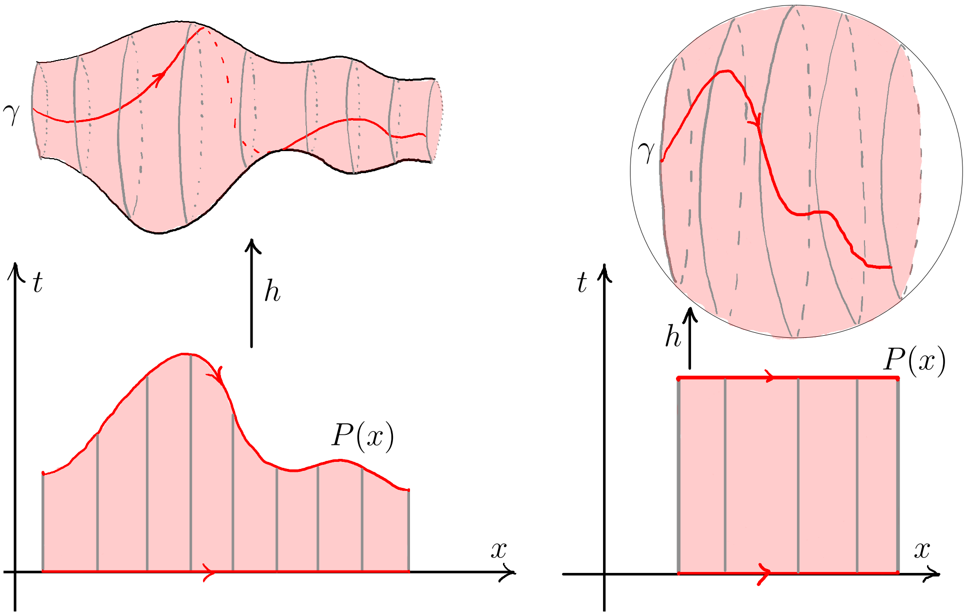


Figure 2.2: The map $h(x,t) = \phi_t^H(\gamma(x))$ induced by a Hamiltonian H as in (2.2.23) and (2.2.27). The Hamiltonian system is a circle action if and only if the period function $x \mapsto P(x)$ is constant.

to show that $P(x) = 1$ for all $x \in (a,b)$. Note that h induces a symplectomorphism from $((a,b)\times\mathbb{R})/(x,t)\sim(x,t+P(x))$ to $H^{-1}(a,b)$. But this implies that the symplectic area of slices $H^{-1}(a,y)$ for $a<y<b$ can be measured by integrating the period,

$$\int_a^y P(x)dx = \int_{H^{-1}(a,y)} \omega. \tag{2.2.28}$$

Differentiation with respect to y together with (2.2.21) yields $P(x)=1$. □

Note that the symplectomorphism (2.2.23) appears in the proof of both directions of the equivalence. To recapitulate, the map h defined in this way satisfies

$$h^*\omega = dx\wedge dt, \quad H\circ h = K, \tag{2.2.29}$$

where $K(x,t)=x$ is the Hamiltonian which induces the rotation on the cylinder. See also Example 2.2.6. In particular, h is equivariant, $h\circ\phi_t^K = \phi_t^H\circ h$. This means that it yields a *local equivalence* of the systems (Σ,ω,H) and the model space $(Z, dx\wedge dt, K)$. Local models of this kind are ubiquitous in the study of Hamiltonian group actions and integrable systems. In favourable situations, the local models can be pieced together to yield a global classification of the spaces at hand. This happens for example for toric symplectic manifolds, see Theorem 2.6.3 or the original paper [Del88]. The goal of this section is to illustrate this in the most basic case, namely the one of surfaces equipped with a Hamiltonian circle action. Before moving to this classification, let us make some more remarks concerning Proposition 2.2.14.

Remark 2.2.15. (Periods and area) Let $H\colon \Sigma\to\mathbb{R}$ be a proper Hamiltonian with only

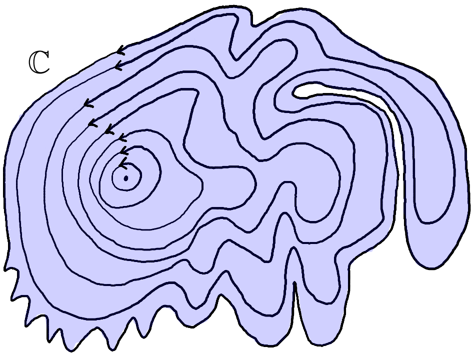


Figure 2.3: Any smooth family of circles in the plane is the orbit set of a Hamiltonian circle action.

critical points of even Morse index. Even if H itself does not generate a Hamiltonian S^1-action, we can come up with a Hamiltonian $\tilde{H}$ which generates a circle action and has the same orbits as H. This follows directly from Proposition 2.2.14 and one can given an explicit formula for $\tilde{H}$ by setting

$$\tilde{H}\colon \Sigma \to \mathbb{R}, \quad \tilde{H}(p) = \int_{H^{-1}(a,H(p))} \omega, \tag{2.2.30}$$

where $a \in \mathbb{R}$ is any real number. The function $\tilde{H}$ is well defined and smooth since all the level sets of H are smooth by the hypothesis that it has only critical points of even index. Note that changing a only changes $\tilde{H}$ by an additive constant. Furthermore, the function $\tilde{H}$ is the unique Hamiltonian generating a circle action with the same orbits as H. This sheds some more light on Remark 2.2.7. Using (2.2.30), it is easy to define a Hamiltonian circle action that has a given smooth family of circles in $\mathbb{C}$ as orbits, see Figure 2.3. Indeed, we can assign to any point the symplectic area bounded by the curve the point lies on.

Remark 2.2.16. (Hypotheses of Proposition 2.2.14) The left side of Figure 2.4 illustrates why the properness of H is important. One can define a Hamiltonian which has only critical points of even index and has the set of curves given in this figure as orbits, however one cannot find a corresponding circle action. One of the orbits is non-compact. In terms of Remark 2.2.15, this is reflected in the fact that the area contained by the orbits diverges as one approaches the non-compact orbit. After removal of the non-compact orbit, one can however define two distinct circle actions on the remaining halves. The right side of Figure 2.4 illustrates why one needs to exclude critical points of odd index. Again, one of the supposed orbits of the circle action is not a circle. This time the function assigning to each point the symplectic area of the sublevel set is well-defined but there is a jump in the period when one crosses the figure eight. One can define circle actions in the three regions arising after the removal of the figure eight.

The following theorem illustrates, both in the statement and in the strategy of proof,

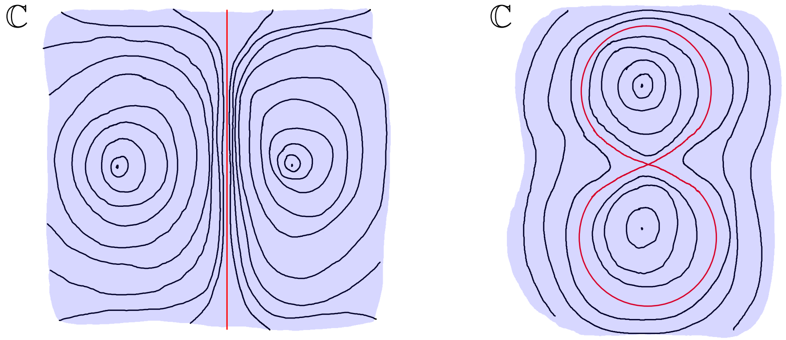


Figure 2.4: On the left: level sets of a non-proper function. On the right: level sets of a function with a critical point of odd Morse index.

the classical Delzant theorem 2.6.3 on the classification of toric symplectic manifolds. We will heavily rely on the classification of Hamiltonian circle actions on surfaces in Section 2.3.

Theorem 2.2.17. *(toy Delzant theorem) Connected symplectic surfaces carrying an effective Hamiltonian circle action are classified by the image of their Hamiltonian function. More precisely, two Hamiltonian circle actions (Σ, ω, H) and (Σ', ω', H') are S^1-equivariantly symplectomorphic if and only if the images $H(\Sigma), H'(\Sigma') \subset \mathbb{R}$ agree up to translation.*

Proof. The *only if* direction is obvious. For the other direction, we will see that most of the work is already contained in Proposition 2.2.14 and the normal form at fixed points which we have discussed in Proposition 2.2.10. Let (Σ, ω, H) be a Hamiltonian circle action on a connected surfaces with image $I = H(\Sigma)$. We will show that (Σ, ω, H) is equivariantly symplectomorphic to a model space determined by I up to translation. By connectedness, the images I is a (possibly infinite) segment. The set $H^{-1}(\operatorname{int} I)$ fibering over the interior of the image is equivariantly symplectomorphic to the standard Hamiltonian circle action on the open cylinder $\operatorname{int} I \times S^1$. This follows from the same argument as in the first half of the proof of Proposition 2.2.14. We denote this symplectomorphism by $\phi\colon \operatorname{int} I \times S^1 \to H^{-1}(\operatorname{int} I)$. If $I = \operatorname{int} I$, then we are done. If not, note that the set $I \setminus \operatorname{int} I$ consists of at most two boundary points, which are critical values of H and correspond to fixed points on Σ. This means that we need to extend the symplectomorphism ϕ over at most two fixed points. Assume that I has a boundary point which is a minimum of H (maxima can be treated similarly) and, up to translation, we can furthermore assume that this minimum is $0 \in \mathbb{R}$. Denote the corresponding fixed point on the surface by p. The normal form from Proposition 2.2.10 yields

$$\psi\colon \mathbb{C} \supset D(a) \to V, \quad H \circ \psi = |z|^2, \quad \psi^*\omega = \omega_0, \quad \varphi(0) = p, \quad i \in \{1, 2\}. \tag{2.2.31}$$

Here $D(a)$ denotes a disk of area a and V is an S^1-invariant neighbourhood of $p \in \Sigma$. In the normal form $H \circ \psi = \xi |z|^2$ given by Proposition 2.2.10 we have $\xi = 1$, since the action would not be effective for $|\xi| \neq 1$ and since we are dealing with a minimum (for a maximum, we have $\xi = -1$). Note that the normal form ψ is S^1-equivariant with respect to the harmonic oscillator system restricted to $D(a)$. Thus we now have a suitable model for a neighbourhood of p and this can be glued S^1-equivariantly to the existing model ϕ to obtain an extension over the fixed point p. This can be repeated in case there is another fixed point. $\square$

Note that the classification of connected symplectic surfaces carrying an effective Hamiltonian circle action boils down to classifying segments in $\mathbb{R}$ up to translation. Thus the list given by Examples 2.2.4, 2.2.5 and 2.2.6 is complete up to changing the total area for S^2, removing some of the fixed points (to get the open segments) and flipping the orientation of the circle action by passing from H to $-H$.

2.2.5 Background: Lagrangian submanifolds

Before continuing our discussion, let us introduce some more basic notions from symplectic geometry. Note that a submanifold $N \subset M$ of a symplectic manifold (M, ω) is not necessarily symplectic itself. Indeed, the restriction of the symplectic form $\omega|_{TN}$ to the subbundle TN may be degenerate.

Definition 2.2.18. *A submanifold $L^n \subset (M^{2n}, \omega)$ of half the dimension of M such that $\omega|_{TL} = 0$ is called* Lagrangian*.*

More generally, a submanifold N on which ω vanishes is called *isotropic*. An elementary argument shows that isotropic submanifolds have at most half the dimension of the ambient space. This means that *Lagrangians are maximal among the submanifolds on which ω is maximally degenerate.*

Definition 2.2.19. *Two Lagrangian submanifolds $L, L' \subset (M, \omega)$ are called* symplectomorphic *if there is a symplectomorphism $\phi \in \mathrm{Symp}(M, \omega)$ mapping L to L'. Similarly, they are called* Hamiltonian isotopic *if there is a Hamiltonian diffeomorphism mapping L to L'.*

The second equivalence relation is stronger than the first one, since Hamiltonian diffeomorphisms are in particular symplectomorphisms. However, in all of the examples we discuss in detail, there will be no distinction between these two equivalence relations. The Chekanov torus for example is exotic with respect to both relations. There is another, considerably weaker, equivalence relation on Lagrangian submanifolds. Two Lagrangian submanifolds are called *Lagrangian isotopic* if there is a smooth isotopy of submanifolds *all of which are Lagrangian* having as endpoints the two given Lagrangians. We will see that the Chekanov torus *is not* exotic with respect to this equivalence relation. There are too many interesting examples of Lagrangians in symplectic geometry to even start mentioning them all and so we will restrict our attention to the strict minimum of examples that we will need to proceed in our discussion. In fact, we have already encountered an important set of Lagrangians in the Introduction, see §2.1.2.

Example 2.2.20. (Product tori) Let $(\mathbb{C}^n, \omega_0)$ be the standard symplectic vector space and let $a_1, \ldots, a_n \in \mathbb{R}_{>0}$ be positive numbers. Then the *product tori*,

$$T(a_1, \ldots, a_n) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \pi|z_i|^2 = a_i\} \tag{2.2.32}$$

are Lagrangian submanifolds. This follows from the fact that the product of Lagrangian submanifolds is again Lagrangian and curves in symplectic surfaces are automatically

Lagrangian. Note the the parameters a_i are the area of the disk bounded by the respective circles (not their radius). We will sometimes denote the product tori by $T(a)$ for $a \in \mathbb{R}^n_{>0}$ for short.

Definition 2.2.21. *A Lagrangian torus $L \subset \mathbb{C}^n$ is called* exotic *if it cannot be mapped to a product torus by a symplectomorphism/Hamiltonian diffeomorphism.*

Remark 2.2.22. (Product tori in general manifolds) In conjunction with Darboux's theorem (see Proposition 2.2.9), this example shows that Lagrangian tori are abundant in every symplectic manifold. Indeed, let $\varphi\colon U \to V \subset M$ be a Darboux chart. Then for small enough $a \in \mathbb{R}^n_{>0}$, the product torus $T(a)$ is contained in U and hence its image $\varphi(T(a)) \subset M$ is Lagrangian. These tori were studied for example by Chekanov–Schlenk [CS16].

Let $L \subset (M, \omega)$ be a Lagrangian. Then to every disk with boundary on L, one can associate its symplectic area. Due to the Lagrangian condition and the closedness of the symplectic form, this yields a well-defined morphism on the homotopy classes of such disks,

$$\mathrm{Area}_L\colon \pi_2(M, L) \to \mathbb{R}, \quad D \mapsto \int_D \omega. \tag{2.2.33}$$

Let us call this the *area class* of L. The area class is a symplectic invariant in the sense that if there is another Lagrangian L' which is the image of L under a symplectomorphism, then

$$\mathrm{Area}_{L'} = \mathrm{Area}_L \circ (\phi|_L)_*, \quad (\phi|_L)_*\colon \pi_2(M, L) \to \pi_2(M, L'). \tag{2.2.34}$$

There is another morphism of this type, called the *Maslov class* of L which takes integer values and has the same symplectic invariance propery as the area class. We will not go into details about the definition of the Maslov class here, see for example the standard reference [MS17].

Definition 2.2.23. *A Lagrangian submanifold $L \subset M$ is called* monotone *if the area and Maslov classes are positively proportional.*

Monotonicity is a symplectic invariant in the sense that it is preserved under symplectomorphisms. This follows immediately from (2.2.34) and the analogous property of the Maslov class.

Example 2.2.24. (Monotonicity of product tori) Let $M = \mathbb{C}^n$ and $L = T(a)$ be a product torus. Then $\pi_2(\mathbb{C}^n, T(a)) \cong \pi_1(T(a)) \cong \mathbb{Z}^n$ with generators given for example by the disks D_i bounding the circles $S^1(a_i)$ in the factors of $\mathbb{C}^n$ equipped with the natural orientation. The Maslov index of these disks is 2, independently of a. For the area class, we compute $\int_{D_i} \omega_0 = a_i$. We conclude that a product torus is monotone if and only if all a_i are equal. Monotone product tori are also sometimes called *Clifford tori.*

2.2.6 Background: Commuting Hamiltonians

Recall from Proposition 2.2.1 that an (autonomous) Hamiltonian function F is preserved by the Hamiltonian flow it generates. Infinitesimally, this can be expressed as $dF(X_F) = 0$. Two Hamiltonians are said to commute if one is preserved by the flow of the other.

Definition 2.2.25. *Two Hamiltonians F, G on a symplectic manifold (M, ω)* Poisson-commute *if $dF(X_G) = 0$. The Hamiltonians $F_1, \ldots, F_n$* commute *if they commute pairwise.*

First note that this relation is symmetric, meaning that F is preserved under the flow of G if and only if G is preserved under the flow of F. This follows immediately from Hamilton's equation and anti-symmetry of the symplectic form,

$$dF(X_G) = \omega(X_G, X_F) = -dG(X_F). \tag{2.2.35}$$

Remark 2.2.26. (Poisson brackets) Setting $\{F, G\} = \omega(X_F, X_G)$ induces a structure on the space of smooth functions on the symplectic manifold. This structure is called a *Poisson bracket* and it can be given by a set of axioms the study of which is of independent interest. Hence the term *Poisson-commute* for Hamiltonians satisfying $\{F, G\} = 0$.

If two Hamiltonians commute, then their corresponding Hamiltonian flows commute,

$$\phi_t^F \circ \phi_s^G = \phi_s^G \circ \phi_t^F, \quad \text{if } \omega(X_F, X_G) = 0. \tag{2.2.36}$$

This follows from the fact that $\omega(X_G, X_F)$ agrees with the *Lie bracket of vector fields* $[X_F, X_G]$ and the latter vanishes if and only if the corresponding flows commute. The converse is not true, but almost. If the flows commute, then $\omega(X_F, X_G)$ is locally constant as a function on the symplectic manifold.

Let us now consider the special case where the two commuting Hamiltonians F, G generate Hamiltonian circle actions. By commutativity of the flows (2.2.36), we can define an induced action of the two-torus $T^2 = (\mathbb{R}/\mathbb{Z})^2$,

$$\psi \colon T^2 \times M \to M, \quad (\vartheta = (\vartheta_1, \vartheta_2), p) \mapsto \psi_\vartheta(p) = (\phi_{\vartheta_1}^F \circ \phi_{\vartheta_2}^G)(p). \tag{2.2.37}$$

This leads us to the following definition.

Definition 2.2.27. *An action ψ of a smooth torus $T^k = (\mathbb{R}/\mathbb{Z})^k$ on a symplectic manifold (M, ω) is called* Hamiltonian *if there is a commuting set of Hamiltonains $F_1, \ldots, F_k$ each of which generates a Hamiltonian circle action such that $\psi_\vartheta = \phi_{\vartheta_1}^{F_1} \circ \ldots \circ \phi_{\vartheta_k}^{F_k}$ for all $\vartheta = (\vartheta_1, \ldots, \vartheta_k) \in T^k$.*

A map $\mu = (F_1, \ldots, F_k) \colon M \to \mathbb{R}^k$ generating a Hamiltonian T^k-action is called *moment map*. Moment maps have very interesting geometric properties. For example note that the orbits of the torus action are contained in the level sets of $\mu \colon M \to \mathbb{R}^k$ since the Hamiltonians are preserved by the Hamiltonian flows. Hence we often think of moment maps as *singular fibrations* which are compatible with the group action. Note furthermore that the moment map is not unique. For one, we can add constant vectors to it. Furthermore, we can replace for example F_1 by an integral linear combination $F_1 + k_2F_2 + \ldots + k_nF_n$ without changing the torus action. More generally, one can post-compose it with elements in $\mathrm{GL}(k; \mathbb{Z})$, which corresponds to changing the basis on the torus T^k which acts on the space. Therefore, it is useful to formulate the theory of Hamiltonian group actions and moment maps in a more invariant way. However, we will not go into details here, since we only need the following example of torus actions for the definition of the Chekanov torus.

Example 2.2.28. (Standard torus action on $\mathbb{C}^n$) The main example for us is the standard T^n-action on $\mathbb{C}^n$, which we have already mentioned in §2.1.3. It is generated by $F_i = \pi|z_i|^2$ and given by rotation in each of the factors of $\mathbb{C}^n$,

$$\psi_\vartheta(z_1, \dots, z_n) = (e^{2\pi i\vartheta_1} z_1, \dots, e^{2\pi i\vartheta_n} z_n), \quad \vartheta \in T^n. \tag{2.2.38}$$

The orbits of this action are the product tori from Example 2.2.20 as well as lower-dimensional tori on which some of the coordinates z_i vanish. This yields another proof of the fact that the product tori are Lagrangian. Indeed, their tangent spaces are spanned by the Hamiltonian vector fields $X_{F_1}, \dots, X_{F_n}$ and since F_i, F_j commute, we deduce $\omega(X_{F_i}, X_{F_j}) = 0$.

We can obtain other fairly obvious examples by taking products of the two-dimensional Examples 2.2.4, 2.2.5 and 2.2.6. All examples obtained in this way are of a very special type, since the torus that acts has half the dimension of the ambient space. As it turns out, this is the maximal case, meaning that there cannot be more than n independent commuting Hamiltonians on a $2n$-dimensional symplectic manifold. Such maximal Hamiltonian torus actions are called *toric* and they have very special properties, see Section 2.6.

2.2.7 Symplectic reduction of Hamiltonian circle actions

In the setting of smooth manifolds, one can divide out a free smooth action of a compact Lie group to obtain a quotient manifold. *Symplectic reduction* is the analogue of this operation for Hamiltonian group actions where the quotient manifold inherits a symplectic structure. Although symplectic reduction can be carried out in a much more general setting, we restrict our attention to Hamiltonian circle actions. Note that *a priori* it is not at all obvious how to get a symplectic structure on the quotient space of group actions. For one, the orbit space M^{2n}/S^1 of a Hamiltonian circle action (M, ω, H) has odd dimension and hence cannot carry a symplectic structure. This problem is resolved by not taking a global quotient, but rather restricting to a level set of the S^1-Hamiltonian H. We denote level sets by

$$Z_c = H^{-1}(c), \quad c \in \mathbb{R}. \tag{2.2.39}$$

Theorem 2.2.29. *(Marsden–Weinstein) Let (M, ω, H) be a Hamiltonian circle action and $c \in \mathbb{R}$ such that S^1 acts freely on the level set $Z_c = H^{-1}(c)$. Then c is a regular value and the quotient $M_c = Z_c/S^1$ carries a natural symplectic form ω_c such that $\pi_c^*\omega_c = \omega|_{Z_c}$. By $\pi_c \colon Z_c \to M_c$, we denote the quotient map.*

First note that the circle action generated by H automatically preserves the level sets Z_c of H. For the theorem to apply, we only need this action to be free. Furthermore, the fact that $c \in \mathbb{R}$ is a regular value follows directly from Hamilton's equation (2.2.10). Indeed, if the circle action on Z_c is free, then the Hamiltonian vector field does not vanish on Z_c and so neither does dH. This proves that Z_c is a submanifold. Symplectic reduction is best summarized by a *reduction diagram*,

$$\begin{tikzcd} Z_c = H^{-1}(c) \arrow[r, hook] \arrow[d, "\pi_c"] & (M, \omega) \\ (M_c, \omega_c). & \end{tikzcd} \tag{2.2.40}$$

Sketch of proof. By the hypotheses of the theorem, the pair $(Z_c, \omega|_{Z_c})$ is an $2n-1$-dimensional manifold equipped with a degenerate closed two-form and a free circle action. Note that the circle action preserves $\omega|_{Z_c}$ since Hamiltonian flows preserve symplectic forms. Since the quotient map is a submersion, this proves that there is a closed two-form ω_c on $M_c = Z_c/S^1$ such that $\pi_c^*\omega_c = \omega|_{Z_c}$. The hard part is to prove that ω_c is non-degenerate and hence symplectic. For this, note that Z_c is a *coisotropic submanifold*, meaning that the symplectic complement $T_pZ_c^\omega$ is contained in T_pZ_c for every $p \in Z_c$. In fact, we will prove that the symplectic complement is spanned by the Hamiltonian vector field X_H, which is tangent to Z_c. This is the *miracle of symplectic reduction*, since it shows that we can get rid of the degeneracy of the form $\omega|_{Z_c}$ by dividing out the group action and hence the form ω_c on the quotient is non-degenerate. To conclude, we prove that

$$TZ_c^\omega = \operatorname{span}\{X_H\}. \tag{2.2.41}$$

Since $Z_c = H^{-1}(c)$, the tangent bundle can be written as $TZ_c = \ker dH$. Therefore, $\omega(X_H, Y) = dH(Y) = 0$ for all $Y \in TZ_c$ and thus $X_H \in TZ_c^\omega$. On the other hand TZ_c^ω is one-dimensional and hence there is equality. □

There is a very natural relationship between the symplectic geometry of the reduced space (M_c, ω_c) and the S^1-equivariant symplectic geometry of (M, ω). We crucially exploit this relationship in our discussion of the Chekanov torus in Section 2.3. For the remainder of this subsection, let (M, ω, H) be a Hamiltonian circle action which admits symplectic reduction at the level $c \in \mathbb{R}$ and take the notation from (2.2.40).

Proposition 2.2.30. *There is a one-to-one correspondence between Lagrangian submanifolds $L_c \subset M_c$ and Lagrangian submanifolds $L \subset H^{-1}(c) \subset M$.*

Proof. Given $L_c \subset M_c$, set $L = \pi_c^{-1}(L_c)$. It follows from $\pi_c^*\omega_c = \omega$ that L is Lagrangian. On the other hand, given $L \subset H^{-1}(c)$, it follows from the Lagrangian condition that L is invariant under the circle action. This is actually a general fact: Lagrangian submanifolds contained in a level set of a Hamiltonian are invariant under the flow of that Hamiltonian. Indeed, due to the Lagrangian condition, it suffices to show that $\omega(Y, X_H) = 0$ for all $Y \in TL$ in order to conclude that $X_H \in TL$. Note however that $\omega(Y, X_H) = dH(Y) = 0$ due to that fact that $TH^{-1}(c) = \ker dH$. We obtain a Lagrangian in the symplectic quotient by setting $L_c = \pi_c(L) = L/S^1$. □

Now let (M, ω, F) be another (not necessarily periodic) Hamiltonian system on the same space as the circle action H. We say that F is *H-equivariant* if the Hamiltonians H, F commute. The commutativity of flows in (2.2.36) explains this terminology.

Proposition 2.2.31. *Every H-equivariant Hamiltonian system (M, ω, F) projects to a unique Hamiltonian system (M_c, ω_c, F_c), meaning that*

$$\pi_c \circ \phi_t^F = \phi_t^{F_c} \circ \pi_c, \quad t \in \mathbb{R}. \tag{2.2.42}$$

Conversely, a given Hamiltonian system (M_c, ω_c, F_c) can be lifted to a (non-unique) Hamiltonian system (M, ω, F) for which (2.2.42) holds.

Proof. For the first statement note that the H-equivariance $\omega(X_H, X_F) = dF(X_H) = 0$ implies that F is constant on the orbits of the circle action H and hence that the restriction $F|_{Z_c}$ descends to a well-defined function F_c in the quotient. Using $\pi_c^*\omega_c = \omega$ and $F_c \circ \pi_c = F|_{Z_c}$, it is straightforward to check that F_c generates the desired Hamiltonian system. For the converse, lift the function F_c given on M_c to the level set Z_c and extend it to M, for example by a cut-off in a tubular neighbourhood. The Hamiltonian system we obtain depends on the choice of this extension and is hence not unique. □

2.2.8 Example: Complex projective plane

As an example of the utility of symplectic reduction, we describe the complex projective plane $\mathbb{C}P^2$ as a symplectic quotient of $\mathbb{C}^3$. This can be seen as a symplectic version of the Hopf fibration which naturally equips $\mathbb{C}P^2$ with the Fubini–Study form $\omega_{\mathbb{C}P^2}$. The construction can be carried over to $\mathbb{C}P^n$ verbatim. Let $\mathbb{C}^3$ be equipped with the symplectic form ω_0 and let

$$H(z_1, z_2, z_3) = \pi(|z_1|^2 + |z_2|^2 + |z_3|^2). \tag{2.2.43}$$

This Hamiltonian defines the diagonal circle action,

$$\phi_t^H(z_1, z_2, z_3) = (e^{2\pi it} z_1, e^{2\pi it} z_2, e^{2\pi it} z_3). \tag{2.2.44}$$

The circle action is free on $H^{-1}(c)$ for all $c > 0$ and hence we can perform symplectic reduction, for example for $c = 3$,

$$\begin{tikzcd} S^5(3) = H^{-1}(3) \arrow[r, hook] \arrow[d, "\pi"] & (\mathbb{C}^3, \omega_0) \\ (\mathbb{C}P^2, \omega_{\mathbb{C}P^2}). & \end{tikzcd} \tag{2.2.45}$$

Obviously, the choice $c = 3$ is quite arbitrary. Any other convention yields the same symplectic form on $\mathbb{C}P^2$ up to scaling. Let us illustrate the lifting/projection constructions of §2.2.7 in this example. Proposition 2.2.30 tells us that there is a one-to-one correspondence between Lagrangian submanifolds of $\mathbb{C}P^2$ and Lagrangian submanifolds of $\mathbb{C}^3$ contained in $S^5(3)$. Take for example the Lagrangian torus $T(1,1,1) = \{\pi|z_i|^2 = 1\} \subset \mathbb{C}^3$. It is contained in $H^{-1}(3)$ and hence automatically invariant under the circle action. Therefore it projects to a Lagrangian torus in $\pi(T(1,1,1)) = T_{\mathrm{Cl}} \subset \mathbb{C}P^2$, known as the *Clifford torus*. The same can be done for all product tori (see Example 2.5.5),

$$T(a_1, a_2, a_3) \subset \mathbb{C}^3, \quad a_1 + a_2 + a_3 = 3. \tag{2.2.46}$$

Let's illustrate the inverse process of lifting Lagrangians from the reduced space by taking real projective space $\mathbb{R}P^2 \subset \mathbb{C}P^2$ obtained as the real locus in homogeneous coordinates. Lifting $\mathbb{R}P^2$ to $S^5 \subset \mathbb{C}^3$ yields an interesting Lagrangian $L = \pi^{-1}(\mathbb{R}P^2)$, which has the structure of a (non-trivial) principal S^1-bundle over $\mathbb{R}P^2$. In order to identify the topological type of L, note that it can be written as

$$L = \{(e^{2\pi it}x_1, e^{2\pi it}x_2, e^{2\pi it}x_3) \subset \mathbb{C}^3 \,|\, e^{2\pi it} \in S^1,\, x_i \in \mathbb{R},\, x_1^2 + x_2^2 + x_3^2 = 3\}. \tag{2.2.47}$$

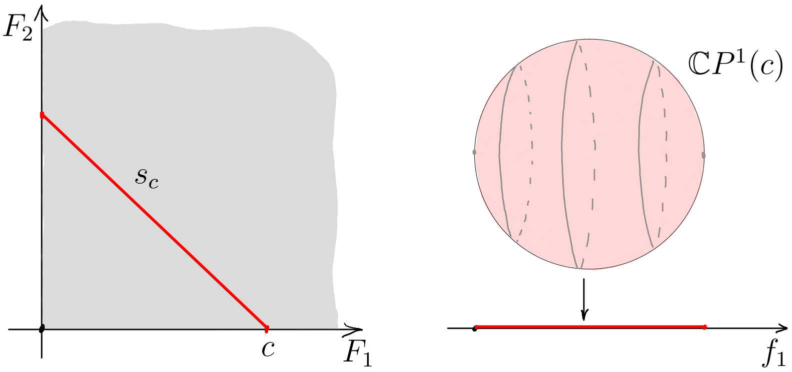


Figure 2.5: The symplectic reduction described in Remark 2.2.33 at the level $c > 0$.

This shows that L can be identified with $S^2 \times_{\mathbb{Z}_2} S^1$, where $\mathbb{Z}_2$ acts on S^2 and S^1 by the antipodal map. Using the projection $S^2 \times_{\mathbb{Z}_2} S^1 \to \mathbb{R}P^1 \cong S^1$, we see that L is the mapping torus of the antipodal map $x \mapsto -x$ on the two-sphere, meaning that L is a higher dimensional analogue of the Klein bottle.

As an illustration of Proposition 2.2.31, we construct a Hamiltonian T^2-action on $\mathbb{C}P^2$. In order to do so, let us first take a step back and note that (2.2.44) is in some sense induced by the standard torus action on $\mathbb{C}^3$ described in Example 2.2.28. Indeed, we obtain this circle action by restricting to the diagonal circle $S^1 \hookrightarrow T^3$. As it turns out, this is equivalent to the fact that we can write H in terms of the Hamiltonians F_i generating the torus action,

$$H = F_1 + F_2 + F_3, \quad F_i = \pi|z_i|^2. \tag{2.2.48}$$

This implies that H commutes with F_1, F_2, F_3, meaning that we obtain three reduced Hamiltonian systems on $\mathbb{C}P^2$ by Proposition 2.2.31 which we will denote by $f_1, f_2, f_3 \colon \mathbb{C}P^2 \to \mathbb{R}$. These reduced systems still generate circle actions and they still commute in the quotient. However, note that there is some redundancy. Indeed, we have restricted to the level set $H = 3$ in the reduction and hence we can use (2.2.48) to write $f_3 = 3 - f_1 - f_2$, meaning that the flow of f_3 can be written as a composition of the flows of f_1, f_2. Forgetting about f_3, we obtain a Hamiltonian T^2-action on $\mathbb{C}P^2$. This is just the *residual action* of the T^3-action in the sense that there is a natural action of the quotient $T^2 \cong T^3/S^1$ by the diagonal circle on $\mathbb{C}P^2$.

Remark 2.2.32. (Quotients of Hamiltonian torus actions) Note that the above discussion applies to a much more general setting. Whenever we have a Hamiltonian torus action of T^k and we can perform symplectic reduction with respect to a subtorus $K \subset T^k$, then the quotient carries a residual Hamiltonian action by T^k/K. This follows from the fact that the actions of T^k and K commute since T^k is abelian. One application of this is the *Delzant construction* which proves one direction of Theorem 2.6.3. Indeed, Delzant noticed that every toric manifold can be constructed as a quotient of some $(\mathbb{C}^N, \omega_0)$ equipped with the standard T^N-action such that the toric structure on the quotient comes as the residual torus action.

Remark 2.2.33. (The case of $\mathbb{C}P^1$) Let us take a closer look at the same construction for $\mathbb{C}P^1$, since we will use an analogous idea for the construction of the Chekanov torus in Section 2.3. In that case, reduction is performed on one of the sets

$$H = F_1 + F_2 = \pi(|z_1|^2 + |z_2|^2) = c, \quad c > 0, \tag{2.2.49}$$

to obtain $\mathbb{C}P^1(c)$. In the image $\mathbb{R}^2_{\geqslant 0}$ of the moment map $\mu = (F_1, F_2)$, these levels project to segments $s_c = \{F_1 + F_2 = c\} \cap \mathbb{R}^2_{\geqslant 0}$ as in Figure 2.5. As above, the circle actions generated by F_1, F_2 yield induced circle actions f_1, f_2 on $\mathbb{C}P^1$. The image of the map (f_1, f_2) is s_c. However, there is again some redudancy and we may pick f_1 as generator of the induced circle action on $\mathbb{C}P^1(c)$. This shows that the image of the Hamiltonian f_1 is the segment of length c coming from the projection of s_c onto the first coordinate axis. This allows us to apply Theorem 2.2.17 to conclude that $(\mathbb{C}P^1(c), \omega_c, f_1)$ is equivariantly symplectomorphic to the circle action $(S^2, \alpha\omega_{S^2}, \alpha G)$ from Example 2.2.5 with normalization $\alpha = \frac{c}{4\pi}$.

2.3 Constructing the Chekanov torus in $\mathbb{R}^4$

The construction of the Chekanov torus we give here is essentially due to Eliashberg–Polterovich [EP97]. The picture in terms of reduced spaces can also be found in [CS10], from which we have taken the depiction in Figure 2.7. Another elucidating account from a slightly different point of view is given by Auroux, for example in [Aur07].

2.3.1 A family of symplectic reductions

One of the key elements in this construction of the Chekanov torus is the Hamiltonian

$$F\colon \mathbb{C}^2 \to \mathbb{R}, \quad (z_1, z_2) \mapsto \pi(|z_1|^2 - |z_2|^2). \tag{2.3.50}$$

It induces the following Hamiltonian circle action

$$\psi_\vartheta(z_1, z_2) = (e^{2\pi i\vartheta} z_1, e^{-2\pi i\vartheta} z_2), \quad \vartheta \in S^1 = \mathbb{R}/\mathbb{Z}. \tag{2.3.51}$$

All of our constructions will be invariant or equivariant with respect to this circle action. Roughly speaking, we will perform a ψ-equivariant perturbation of the standard toric structure on $\mathbb{C}^2$. For this we first discuss the symplectic quotients obtained at different levels of F.

Let us denote the level sets of F by $Z_c = F^{-1}(c)$ for all $c \in \mathbb{R}$. First of all note that for all $c \neq 0$, this level set is smooth. In fact, it is diffeomorphic to $S^1 \times \mathbb{C}$. The only critical value of F is $0 \in \mathbb{R}$ and its only critical point is $0 \in \mathbb{C}^2$. Thus ψ has $0 \in \mathbb{C}^2$ as its only fixed point and thus we can perform symplectic reduction away from $c = 0$. We first treat the case $c \neq 0$ and then move on to the critical value $c = 0$.

Proposition 2.3.1. *Let $c \neq 0$. The symplectic quotient $M_c = F^{-1}(c)/S^1 = Z_c/S^1$ by the circle action is symplectomorphic to the standard symplectic plane $(\mathbb{C}, \omega_0)$.*

Before proving this claim, let us first take a step back and interpret the Hamiltonian S^1-action in terms of the toric structure on $\mathbb{C}^2$. Recall from §2.1.3 and Example 2.2.28 that

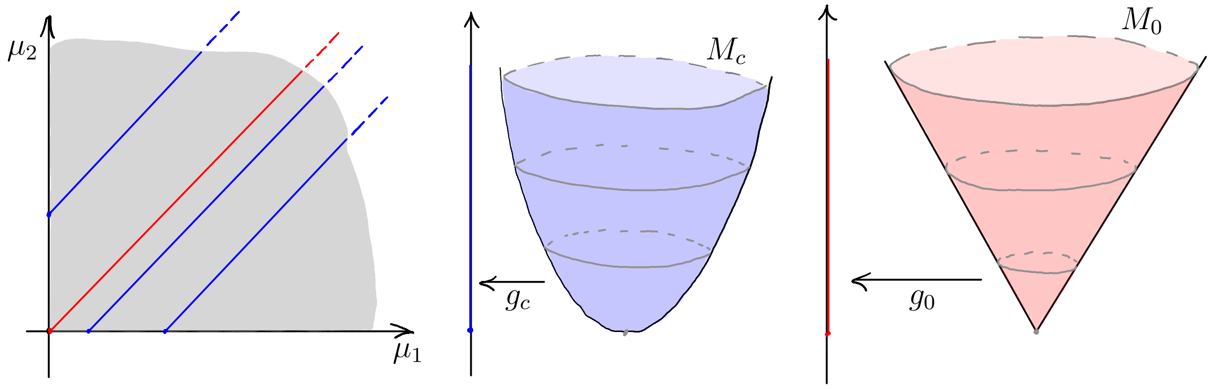


Figure 2.6: The family $\pi_c\colon Z_c \to M_c$ with the residual Hamiltonian circle actions on the quotients M_c.

the moment map of the standard toric structure on $\mathbb{C}^2$ is

$$\mu\colon \mathbb{C}^2 \to \mathbb{R}^2_{\geqslant 0}, \quad (z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2), \tag{2.3.52}$$

and that its action is by rotations in the factors of $\mathbb{C}^2$. Now note that our Hamiltonian F is compatible with this structure in the sense that we can write it as $F = \mu_1 - \mu_2$. In particular, this means that the level set Z_c projects to the half-line $\{\mu_1 - \mu_2 = c\} \subset \mathbb{R}^2_{\geqslant 0}$ in the base of the toric fibration. See Figure 2.6. The key observation for us is that after we quotient out the circle action generated by F, the quotient M_c still carries a residual circle action (coming from the initial toric T^2-action) which turns it into a toric manifold in its own right. Indeed, note that the pair of Hamiltonians F, μ_1 commutes and hence we may apply Proposition 2.2.31 to obtain a reduced Hamiltonian system $(M_c, \omega_c, (\mu_1)_c)$. For simplicity, let us call

$$g_c = (\mu_1)_c\colon M_c \to \mathbb{R}. \tag{2.3.53}$$

The reduced Hamiltonian g_c generates an effective Hamiltonian circle action on M_c. Recall that we have classified all effective Hamiltonian circle actions on symplectic surfaces in Theorem 2.2.17 in terms of the image of the Hamiltonian function. Here, the image of g_c is a half-line. Indeed, it is the projection of the half-line $\{\mu_1 - \mu_2 = c\} \subset \mathbb{R}^2_{\geqslant 0}$ to the first component. It follows that there is an equivariant symplectomorphism to the harmonic oscillator,

$$(M_c, \omega_c, g_c) \cong (\mathbb{C}, \omega_0, \pi|z|^2), \quad c \neq 0. \tag{2.3.54}$$

Note the similarity with Remark 2.2.33, where we have looked at the reduced spaces of the Hamiltonian $H = \pi(F_1 + F_2)$. In that case, the sets $H^{-1}(c) \cap \mathbb{R}^2_{\geqslant 0}$ are given by closed line segments, allowing us to equivariantly identify the quotients with the standard Hamiltonian circle action on the sphere via Theorem 2.2.17. Both these cases are instances of a more general principle which we briefly describe in Remark 2.6.8.

Let us now turn to the critical value $c = 0$. The only critical point $0 \in \mathbb{C}^2$ is the only fixed point of the circle action ψ restricted to Z_0. This means that we may perform symplectic reduction on its complement, $Z_0 \setminus \{0\}$. Note that there is again a residual circle action on the quotient $(Z_0 \setminus \{0\})/S^1$ and hence we can use this additional structure

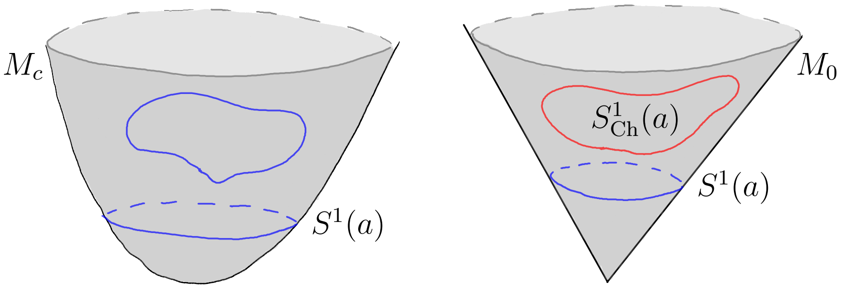


Figure 2.7: On the right: A standard circle $S^1(a)$ and a Chekanov circle $S^1_{\mathrm{Ch}}(a)$. On the left, both circles are Hamiltonian isotopic and so are the lifted tori. Such a Hamiltonian isotopy does not lift to the fibre over the origin in M_0.

to determine the quotient as above. This time, the image of the induced Hamiltonian is the open half-line and hence the quotient is symplectomorphic to $(\mathbb{C}^\times, \omega_0)$. Obviously, we can add a point to this space and extend the smooth and symplectic structures. Therefore we set $(M_0, \omega_0) = (\mathbb{C}, \omega_0)$. Let us emphasize that, as opposed to $M_{c\neq 0}$, this space is not defined by symplectic reduction, but it coincides with the symplectic quotient of $Z_0 \setminus \{0\}$ outside of $0 \in \mathbb{C}$. We will see that this distinction is crucial for the construction of the Chekanov torus. Let us briefly recapitulate. We have now constructed a one-parameter family of quotients

$$\begin{tikzcd}
Z_c = F^{-1}(c) \arrow[r, hook] \arrow[d, "\pi_c"] & (\mathbb{C}^2, \omega_0) \\
(M_c, \omega_c) \cong (\mathbb{C}, \omega_0), &
\end{tikzcd} \tag{2.3.55}$$

with the caveat that for $c = 0$ this is not an honest symplectic reduction. This is summarized by Figure 2.6. However, we obtain a map $\pi_0\colon Z_0 \to \mathbb{C}$ which restricts to a principal S^1-bundle coming from symplectic reduction over $\mathbb{C}^\times$ but which has a degenerate fibre $\{0\}$ over the origin $0 \in \mathbb{C}$.

Remark 2.3.2. For now, we will also be a little bit sloppy and identify $(M_c, \omega_c) = (\mathbb{C}, \omega_0)$, although this identification is not unique. In fact, for every smooth curve $\gamma\colon [0, +\infty) \to M_c$ with $\gamma(0)$ equal to the fixed point of the circle action g_c and transverse to its orbits, we obtain a symplectomorphism mapping the image of γ to the ray $\mathbb{R}_{\geqslant 0}$. See also the proof of Proposition 2.2.14.

2.3.2 The Chekanov torus

We can now turn to the definition of the Chekanov torus. The basic idea is to lift circles from the quotient spaces M_c to tori in $\mathbb{C}^2$ via (2.3.55). Since circles in the plane are Lagrangian, such tori are automatically Lagrangian by Proposition 2.2.30. Let $S^1_{\mathrm{Ch}}(a) \subset \mathbb{C}$ be a simple closed curve bounding area $a > 0$ in the quotient space $(M_0, \omega_0) = (\mathbb{C}, \omega_0)$ such that $S^1_{\mathrm{Ch}}(a)$ *does not enclose the origin*, see also Figure 2.7. From now on, we call such a curve *Chekanov circle*. Then we can define the following one-parameter family of tori.

Definition 2.3.3. *The* Chekanov torus of area parameter $a > 0$ *is defined as the lift*

$$T_{Ch}(a) = \pi_0^{-1}(S^1_{Ch}(a)) \subset \mathbb{C}^2. \tag{2.3.56}$$

This definition does not depend on the choice of the circle $S^1_{\mathrm{Ch}}(a)$ up to applying a Hamiltonian isotopy. Indeed, all circles enclosing area $a > 0$ in $\mathbb{C}^\times$ which do not enclose the origin are Hamiltonian isotopic by a Hamiltonian isotopy inside $\mathbb{C}^\times$, and such a Hamiltonian isotopy can be lifted by Proposition 2.2.31. Note that $S^1_{\mathrm{Ch}}(a)$ is Hamiltonian isotopic to the standard circle $S^1(a)$ (which does enclose the origin) through a Hamiltonian isotopy supported in $\mathbb{C}$ but *it is not* through a Hamiltonian isotopy supported in $\mathbb{C}^\times$. Now recall that the quotient map $\pi_0\colon Z_0 \to \mathbb{C}$ is singular at the origin and hence a Hamiltonian isotopy whose support contains $0 \in \mathbb{C}$ cannot be lifted. In fact, a smooth circle that passes through the origin lifts to a *pinched Lagrangian torus* in $\mathbb{C}^2$. See also Figure 2.8. We will see now that the standard circles $S^1(a)$ in the reduced spaces lift to product tori in $\mathbb{C}^2$. This motivates Definition 2.3.3 of the Chekanov torus, since it means that there are two kinds of circles in the reduced space $M_0 = \mathbb{C}$, those of standard type $S^1(a)$ (enclosing the origin) and those of Chekanov type $S^1_{\mathrm{Ch}}(a)$ (not enclosing the origin). Both types come in a one-parameter family indexed by the area they contain. Note that in all other reduced spaces with $c \neq 0$, this is not so, since the reduction $\pi_c\colon Z_c \to \mathbb{C}$ is defined everywhere.

Proposition 2.3.4. *Let $\pi_c\colon Z_c \to M_c$ be the symplectic reduction induced by $F = \pi(|z_1|^2 - |z_2|^2) = c$ on $\mathbb{C}^2$ for $c \neq 0$. Then the lifts of standard circles $S^1(a) = \{\pi|z|^2 = a\} \subset M_c = \mathbb{C}$ are product tori,*

$$\pi_c^{-1}(S^1(a)) = \begin{cases} T(a, a-c) & \text{if } c < 0, \\ T(a+c, a) & \text{if } c > 0. \end{cases} \tag{2.3.57}$$

Furthermore, the lift of any simple closed curve is Hamiltonian isotopic to a product torus.

Proof. Since $S^1(a)$ is the orbit of the residual Hamiltonian circle action (2.3.53), it lifts to an orbit of the T^2-action on $\mathbb{C}^2$, and this orbit is a product torus. The identity (2.3.57) follows from the identification of the toric moment map image of g_c with $\{\mu_1 - \mu_2 = c\} \cap \mathbb{R}^2_{\geqslant 0}$, see the discussion surrounding (2.3.54) and Figure 2.6. Let us now prove that lifts of simple closed curves from the spaces $M_{c\neq 0}$ yield product tori up to applying a Hamiltonian isotopy. This follows from the fact that every such curve in $(\mathbb{C}, \omega_0)$ is Hamiltonian isotopic to some $S^1(a)$ (see the discussion in §2.1.1) together with the fact that Hamiltonian isotopies can be lifted through symplectic reductions by Proposition 2.2.31. □

In particular, we have proven that we cannot come up with exotic tori by lifting circles from the reduced spaces M_c for $c \neq 0$. For $c = 0$, however, the situation is different, as we have seen in Definition 2.3.3. Similarly as above, we have

$$\pi_0^{-1}(S^1(a)) = T(a, a), \quad a > 0. \tag{2.3.58}$$

Remark 2.3.5. (Equivariant equivalence) Since the two types of circles $S^1(a), S^1_{\mathrm{Ch}}(a) \subset \mathbb{C}^\times$ are not Hamiltonian isotopic, we have shown that there is no ψ-equivariant Hamiltonian isotopy mapping $T_{\mathrm{Ch}}(a)$ to the product torus $T(a, a)$. In Section 2.5, we will see that there is no general such Hamiltonian isotopy, i.e. that the Chekanov torus is exotic.

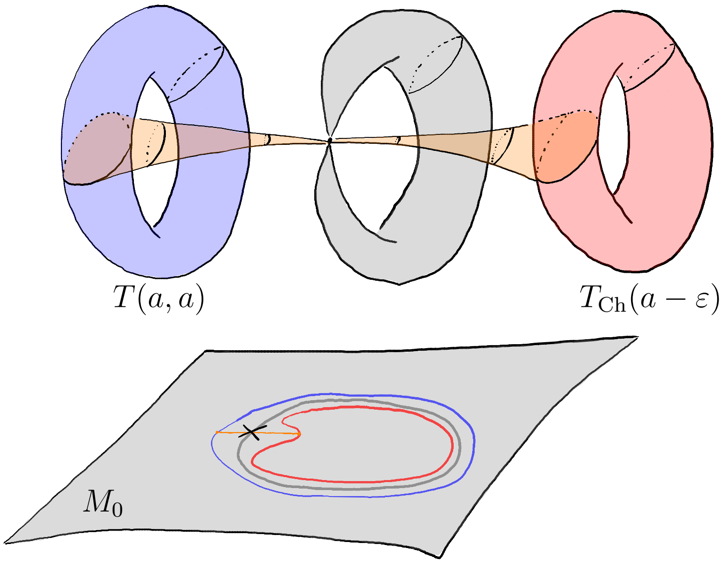


Figure 2.8: The outer circle lifts to a product torus. The inner circle lifts to a Chekanov torus. There is an intermediate circle lifting to a pinched torus, since the circle fibres of $\pi_0\colon Z_0 \to M_0$ degenerate to a point fibre over the point marked by a cross. The lift of the transverse segment is the union of two Lagrangian disks meeting at the singular point of the pinched torus.

Remark 2.3.6. (Monotonicity of the Chekanov torus) Proposition 2.3.4 allows us to compute the area and Maslov class of the Chekanov torus $T_{\mathrm{Ch}}(a)$. The area and Maslov classes depend continuously on the Lagrangian we choose (in the C^1-topology, see for example Remark 2.4.1). We can define the sequence of Lagrangian tori $L_n = \pi^{-1}_{1/n}(S^1_{\mathrm{Ch}}(a))$ which converges to the Chekanov torus $T_{\mathrm{Ch}}(a)$. However by the second statement of Proposition 2.3.4, we obtain $L_n \cong T(a+\frac{1}{n}, a)$ and hence the Chekanov torus has the same area and Maslov class as the product torus $T(a,a)$. Therefore, it is monotone. In particular, we cannot use these invariants to prove that the Chekanov torus is exotic.

2.3.3 An exotic Lagrangian torus fibration

Let us look once more at the quotient map $\pi_0\colon Z_0 \to \mathbb{C}$ coming from the reduction of the circle action generated by $F = \pi(|z_1|^2 - |z_2|^2)$. The standard Lagrangian torus fibration by product tori descends to the fibration of $\mathbb{C}^\times$ by standard circles as on the left hand side of Figure 2.9. Conversely, the lift of the fibration by circles lifts to a Lagrangian torus fibration of, at the very least, the set $Z_0 \setminus \{0\}$. Let us now try to obtain a fibration of $Z_0 \setminus \{0\}$ by Chekanov tori instead of product tori. The previous paragraph §2.3.2 suggests how to do this: We should lift a fibration of $\mathbb{C}^\times$ by circles which do not enclose the origin, as these lift to Chekanov tori. A moment's reflection shows that this is impossible. The best we can do is shown on the right hand side of Figure 2.9, which shows a circle fibration (with one singular point where the circles degenerate to a point) of the plane from which we removed a set which is diffeomorphic to a half-line with endpoint in the origin. Note that we are only interested in Hamiltonian isotopy classes and hence it does not matter what this set looks like precisely, nor what the circle fibration on its

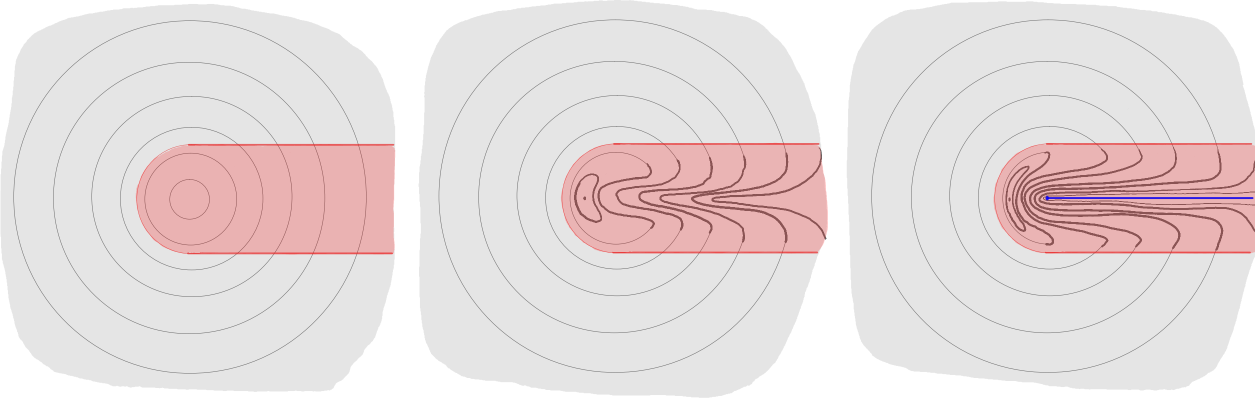


Figure 2.9: A deformation of the standard circle action on $\mathbb{C}$ (on the left) to one on the slit plane $\mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\geqslant 0}$ (on the right). The latter has only Chekanov circles as orbits, as none of them encloses the origin. Note that the orbits are standard outside an ε-neighbourhood of the slit.

complement looks like as long as it has only one singular fibre. Thus from now on, we assume that the set we remove is $\mathbb{R}_{\geqslant 0}$ and we denote the slit plane by $\mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\geqslant 0}$. Furthermore, we take a circle fibration which is standard outside of an ε-neighbourhood of the slit $\mathbb{R}_{\geqslant 0}$ as in Figure 2.9. We give a concrete such fibration in Lemma 2.3.8, but for now we consider it given. For simplicity, we encode this circle fibration as the level sets of a function $h \in C^\infty(\mathbb{C}^-)$. Note that we may even assume, by Remark 2.2.15, that h generates a Hamiltonian circle action; this then determines h up to an additive constant. Let us set

$$h \colon \mathbb{C}^- \to \mathbb{R}, \quad h(z) = \text{area contained by the circle fibre through } z. \tag{2.3.59}$$

We obtain the Chekanov tori as lifts,

$$T_{\mathrm{Ch}}(a) = \pi_0^{-1}(h^{-1}(a)), \quad a > 0. \tag{2.3.60}$$

Thus we obtain a fibration of $\pi_0^{-1}(\mathbb{C}^-)$ by Chekanov tori. Note that the lift $\pi_0^{-1}(\mathbb{R}_{\geqslant 0})$ is given (in a suitable identification of $(Z_0 \setminus \{0\})/S^1$ with $\mathbb{C}^\times$) by the Lagrangian plane

$$P = \{(z, \overline{z}) \in \mathbb{C}^2 \,|\, z \in \mathbb{C}\}. \tag{2.3.61}$$

We extend the Lagrangian torus fibration of $Z_0 \setminus P$ to a Lagrangian torus fibration of $\mathbb{C}^2 \setminus P$.

Theorem 2.3.7. *There is a smooth map $p \colon \mathbb{C}^2 \setminus P \to \mathbb{R} \times \mathbb{R}_{\geqslant 0}$ such that the fibres over $\mathbb{R} \times \mathbb{R}_{>0}$ are Lagrangian tori and*

(1) the fibres over the ray $\{0\} \times \mathbb{R}_{>0}$ are Chekanov tori,

$$p^{-1}(0, a) = T_{Ch}(0, a), \quad a > 0, \tag{2.3.62}$$

(2) all other fibres are Hamiltonian isotopic to product tori,

$$p^{-1}(c,a) \cong \begin{cases} T(a, a-c) & \text{if } c < 0, \\ T(a+c, a) & \text{if } c > 0. \end{cases} \tag{2.3.63}$$

In order to extend the Lagrangian torus fibration of $Z_0 \setminus P$, we need to work across different reduced spaces, since $\mathbb{C}^2$ is decomposed into slices $\mathbb{C}^2 = \cup_{c\in\mathbb{R}} Z_c$ and each of the slices reduces to one of the M_c. Recall that up until now, we were sloppy about how we identify $M_c = \mathbb{C}$ but now we need to make this precise. For this, we use an idea going back to Eliashberg–Polterovich [EP97] and featuring prominently in Auroux [Aur07], namely we consider the following holomorphic function

$$f\colon \mathbb{C}^2 \to \mathbb{C}, \quad (z_1, z_2) \mapsto z_1 z_2. \tag{2.3.64}$$

Note that this function is invariant under the circle action ψ generated by the Hamiltonian F. Therefore f factors through the reduced spaces,

$$\begin{tikzcd} Z_c = F^{-1}(c) \arrow[r, hookrightarrow] \arrow[d, "\pi_c"] & (\mathbb{C}^2, \omega_0) \arrow[r, "f"] & (\mathbb{C}, \omega_0) \\ (M_c, \omega_c). \arrow[rru] & & \end{tikzcd} \tag{2.3.65}$$

Recall that the quotients carry a residual circle action induced by rotation in the first factor, $\vartheta.(z_1, z_2) = (e^{2\pi i\vartheta} z_1, z_2)$. Since f is equivariant with respect to this action and the standard action on $\mathbb{C}$, we deduce that the identification $M_c = \mathbb{C}$ induced by f is equivariant. Following [GV21, Section 7], we note that this induces a map

$$\pi\colon \mathbb{C}^2 \to \mathbb{C} \times \mathbb{R}, \quad (z_1, z_2) \mapsto (z_1 z_2, F(z_1, z_2) = \pi(|z_1|^2 - |z_2|^2)), \tag{2.3.66}$$

which is ψ-invariant and such that $\pi|_{Z_c} = \pi_c$. In other words, π is a global quotient map for the Hamiltonian S^1-action which realizes symplectic reduction on the slices Z_c.

The main idea in order to extend the fibration on $Z_0 \setminus P$ is to define a one-parameter family $\{h^t\}_{t\in[0,1]}$ of Hamiltonian circle actions on $\mathbb{C}$ (or $\mathbb{C}^-$, respectively) which interpolate between the standard circle action and the one defined by (2.3.59) and take the $\mathbb{R}$-component in the image of (2.3.66) as a parameter to lift this family of systems. More precisely, let us construct this one-parameter family.

Lemma 2.3.8. *There is a family of functions $h^s \in C^\infty(\mathbb{C})$ for $s \in [0,1)$ and $h^1 \in C^\infty(\mathbb{C}^-)$ which generate Hamiltonian S^1-actions on the respective spaces with the following properties.*

(1) For $s = 0$, we have $h^0(z) = \pi|z|^2$;

(2) The family depends smoothly on s, meaning $s \mapsto h^s$ is smooth for all $s \in [0,1)$ and $s \mapsto h^s|_{\mathbb{C}^-}$ is smooth for all $s \in [0,1]$;

(3) For all $\varepsilon > 0$, we can choose h^s such that their orbits are circles centered in $0 \in \mathbb{C}$ outside of a ε-neighbourhood of $\mathbb{R}_{\geqslant 0} \subset \mathbb{C}$.

See Figure 2.9 for an illustration of what the orbits of this family of functions looks like. Actually, one way of finding such a family of functions is starting with the corresponding families of orbits as in the figure and defining h^s as in (2.3.59).

Proof of Lemma 2.3.8. Pick a smooth family of $\{\tilde{h}^s\}_{s\in[0,1]}$ with $\tilde{h}^s \in C^\infty(\mathbb{C})$ for $s \in [0,1)$ and $\tilde{h}^1 \in C^\infty(\mathbb{C}^-)$ satisfying properties (1)–(3), but which do not necessarily have periodic flows. This can be explicitely done as follows. For $\varepsilon > 0$, let U_ε be the ε-neighbourhood of the slit $\mathbb{R}_{\geqslant 0} \subset \mathbb{C}$, i.e. all points at Euclidean distance less than ε from $\mathbb{R}_{\geqslant 0}$. Let $\chi_\varepsilon \colon \mathbb{C} \to [0,1]$ be a smooth characteristic function of the slit $\mathbb{R}_{\geqslant 0}$ with

$$\operatorname{supp}(\chi_\varepsilon) \subset U_\varepsilon, \quad \chi_\varepsilon|_{\mathbb{R}_{\geqslant 0}} = 1, \quad \chi_\varepsilon|_{\mathbb{C}^-} < 1. \tag{2.3.67}$$

Define the family of functions

$$\tilde{h}^s(z) = \pi\,|z|^2 + \frac{s\chi_\varepsilon(z)}{1 - s\chi_\varepsilon(z)}. \tag{2.3.68}$$

Note that $\tilde{h}^1$ is not defined on the slit (as expected) and that for a suitable choice of χ_ε, any $\tilde{h}^s$ has a unique critical point in $(-\varepsilon, 0]$. This implies that all level sets are diffeomorphic to circles. This family satisfies properties (1)–(3) but it does not define circle actions in general. However, Remark 2.2.15 allows us to conclude by setting the value of h^s equal to the symplectic area contained by the level sets of $\tilde{h}^s$. ☐

Proof of Theorem 2.3.7. Let $\varepsilon > 0$ be a small parameter and let $b \colon \mathbb{R} \to [0,1]$ be a bump function with support in $(-\varepsilon, \varepsilon)$ and with $b(0) = 1$ as its unique maximum. Let $\{h^s\}_{s\in[0,1]}$ be the family of functions from Lemma 2.3.8 with ε the parameter used for (3) and set

$$\tilde{H} \colon \mathbb{C} \times \mathbb{R} \setminus (\mathbb{R}_{\geqslant 0} \times \{0\}) \to \mathbb{R}, \quad (z, r) \mapsto h^{b(r)}(z). \tag{2.3.69}$$

Let us define $H = \tilde{H} \circ \pi$ as the lift of $\tilde{H}$ under the map (2.3.66). We claim that $p = (F, H)$ is the desired Lagrangian torus fibration. First note that H is not defined on the preimage of the set $\mathbb{R}_{\geqslant 0} \times \{0\}$ and this preimage is precisely the Lagrangian plane P. Now let us look at a level set $p^{-1}(c, a) = \{F = c, H = a\}$. Such a level set is given as the lift of a circle $\tilde{H}^{-1}(a) \cap \{r = c\}$ under π. Recall that $\pi|_{Z_c} \colon Z_c \to \{r = c\}$ is the symplectic reduction at level $F = c$. Thus we can use Proposition 2.2.30 to conclude that the fibre $p^{-1}(c, a)$ is a Lagrangian torus. For $c = 0$, it is a Chekanov torus. This follows from the definition of Chekanov tori and our definition of $\tilde{H}$. Indeed, all circles $\tilde{H}^{-1}(a) \cap \{r = 0\}$ do not enclose the origin, since $\tilde{H}(z, 0) = h^{b(0)}(z) = h^1(z)$. The Hamiltonian isotopy (2.3.63) of all other fibres with product tori follows form Proposition 2.3.4. ☐

Remark 2.3.9. (Is it a moment map?) Although we suspect this to be true, the map $p = (F, H)$ we have constructed in §2.3.3 is a priori *not* a moment map of a Hamiltonian T^2-action on $\mathbb{C}^2 \setminus P$. The Hamiltonians commute by construction, $dF(X_H) = 0$, and F generates a circle action, but it is non-trivial to show that the second Hamiltonian H generates a circle action. Although the corresponding flows in the quotients M_c are circle actions, this does not imply that the lifted flow is periodic. Indeed, recall from the proof of Proposition 2.2.31 that lifted flows depend on the extension of the lifted Hamiltonian.

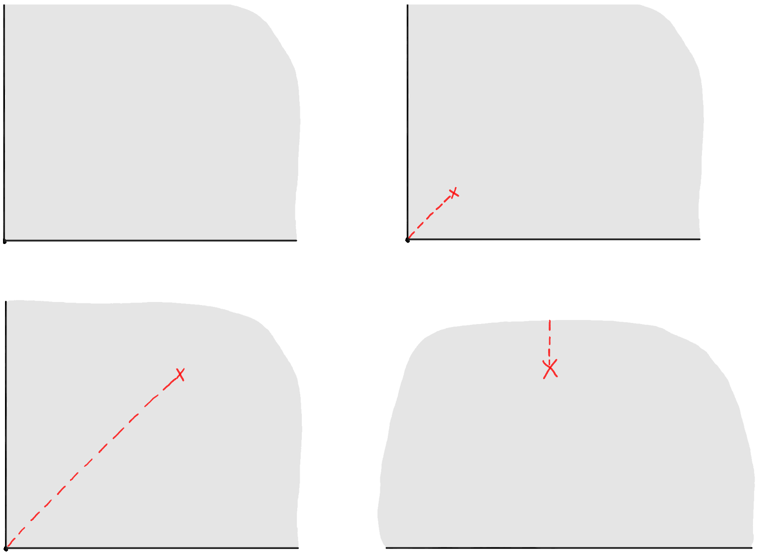


Figure 2.10: An illustration of the three operations in almost toric geometry for $\mathbb{C}^2$. The first diagram is the toric base $\mathbb{R}^2_{\geqslant 0}$ of $\mathbb{C}^2$. The second is obtained from the first by a *nodal trade*. The third is obtained from the second by a *nodal slide*. The fourth is obtained from the third by a *mutation*.

2.3.4 The Chekanov torus and almost toric fibrations

This section is somewhat speculative. It merely serves to pique the interest of the reader who is familiar with almost toric geometry. Its content does not appear anywhere else in the present manuscript (except for the discussion in 2.6.3) and hence it may be skipped by the reader who is unfamiliar with or uninterested in almost toric fibrations. In §2.3.3, we have passed from the standard toric structure

$$\mu\colon \mathbb{C}^2 \to \mathbb{R}^2_{\geqslant 0}, \quad (z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2), \tag{2.3.70}$$

defining a Lagrangian torus fibration by product tori to an *exotic* fibration, i.e. a fibration having Chekanov tori as some of its fibres,

$$p\colon \mathbb{C}^2 \setminus P \to \mathbb{R} \times \mathbb{R}_{\geqslant 0}, \quad (z_1, z_2) \mapsto (F(z_1, z_2) = \pi(|z_1|^2 - |z_2|^2), H(z_1, z_2)). \tag{2.3.71}$$

We briefly explain the relationship of this operation with certain operations from *almost toric geometry*. Almost toric geometry is a generalization (in dimension four) of toric geometry, where one allows for more complicated singularities in the fibration, namely so-called *focus-focus singularities*. We do not give details on almost toric geometry here, nor do we actually *prove* some kind of equivalence of the construction from §2.3.3 and almost toric operations, since this would be outside the scope of the present exposition.

The almost toric geometry package includes three basic operations: *nodal trades*, *nodal slides* and *mutations*. Here we use the terminology of [Eva21]. These operations *do not change the symplectic geometry of the manifold under consideration*, they merely give different Lagrangian fibrations of a fixed manifold. The nodal trade operation [Eva21, Section 8.2] is the most important one. In terms of singularities, it exchanges an *elliptic-*

elliptic singularity with a *focus-focus* singularity. The former is of toric type and simply corresponds to a vertex in a toric moment map image, and the latter is the only non-toric singularity of the almost-toric world. We illustrate the operations by the example of $\mathbb{C}^2$, see Figure 2.10. Passing from the first picture to the second in Figure 2.10 combinatorially encodes the nodal trade. The base point marked by a cross in Figure 2.10 is called a *node* and its corresponding fibre is a pinched torus. The main conceptual difference with the toric case is that the base of regular fibres of the fibration is not simply-connected anymore. Indeed, this is crucial since the torus fibration around a node has non-trivial *monodromy*. This monodromy is combinatorially encoded by the dashed line, we again refer to Sections 6, 7 and 8 in [Eva21] for details. One can slide the node along the direction of the dashed line, this is called a nodal slide [Eva21, Section 8.3]. Passing from the second to the third picture in Figure 2.10 represents a nodal slide. The mutation operation [Eva21, Section 8.4] is represented by passing from the third to the fourth picture, and geometrically it corresponds to choosing a different fundamental domain of the covering associated to the non simply-connected base.

Let us now describe an (equivalent?) procedure using the approach from §2.3.3. For this, we consider a parametric version $\{p_t\}_{t\in[0,1]}$ interpolating between the standard fibration (2.3.70) and the exotic one (2.3.71). Define

$$p_t = (F, H_t), \quad H_t = \tilde{H}_t \circ \pi, \quad \tilde{H}_t(z, r) = h^{tb(r)}(z), \tag{2.3.72}$$

where we use the same construction and conventions as in the proof of Theorem 2.3.7, see also the discussion around (2.3.69). Note that $p_1 = p$. Since the bump function b takes values in $[0,1]$ and the functions h^s from Lemma 2.3.8 are defined on $\mathbb{C}$ for all $s \in [0,1)$, we deduce that p_t is defined on all of $\mathbb{C}^2$ for each $t \in [0,1)$, in contrast to $p = p_1$. Another difference with the case $s = 1$ is the presence of a singular fibre in the interior of the base $\mathbb{R} \times \mathbb{R}_{\geqslant 0}$. Indeed, note that $\tilde{H}_t|_{\{r=0\}} = h^t$ and h^t has a level set passing through the origin $(0,0) \in \mathbb{C} \times \mathbb{R}$. Recall that the fibration (2.3.66) is singular at $(0,0)$ and thus the corresponding fibre is a pinched torus as shown in Figure 2.8. Thus the basis of this fibration is as depicted in Figure 2.11 for different values of $s \in [0,1]$. The cross here depicts a pinched torus fibre. This fibre gets pushed off to infinity in $t = 1$, at the cost of removing a Lagrangian plane from the total space, see also Figure 2.9. This suggests that the family p_t encodes a nodal trade when we pass from p_0 to p_t for small $t > 0$ and a nodal slide with increasing t.

Remark 2.3.10. (Nodal trade at infinity) We have described a process to pass from a Lagrangian torus fibration on $\mathbb{C}^2$ to one on $\mathbb{C}^2 \setminus P$. This suggests that there is an inverse procedure which should consist of adding a Lagrangian plane to the latter space and extending the Lagrangian torus fibration over it. This should maybe be viewed as a *nodal trade at infinity*, since it is inverse to pushing off a node to infinity. This should correspond to the *integrable Lagrangian surgery* appearing in [GV21, Section 7.4].

2.4 Versal deformations of Lagrangians

In this section, we describe small deformations of a given Lagrangian submanifold using Weinstein's theorem 2.4.2. Up to (locally supported) Hamiltonian isotopies, the deformation space of a given Lagrangian L is parametrized by a small neighbourhood in $H^1(L;\mathbb{R})$.

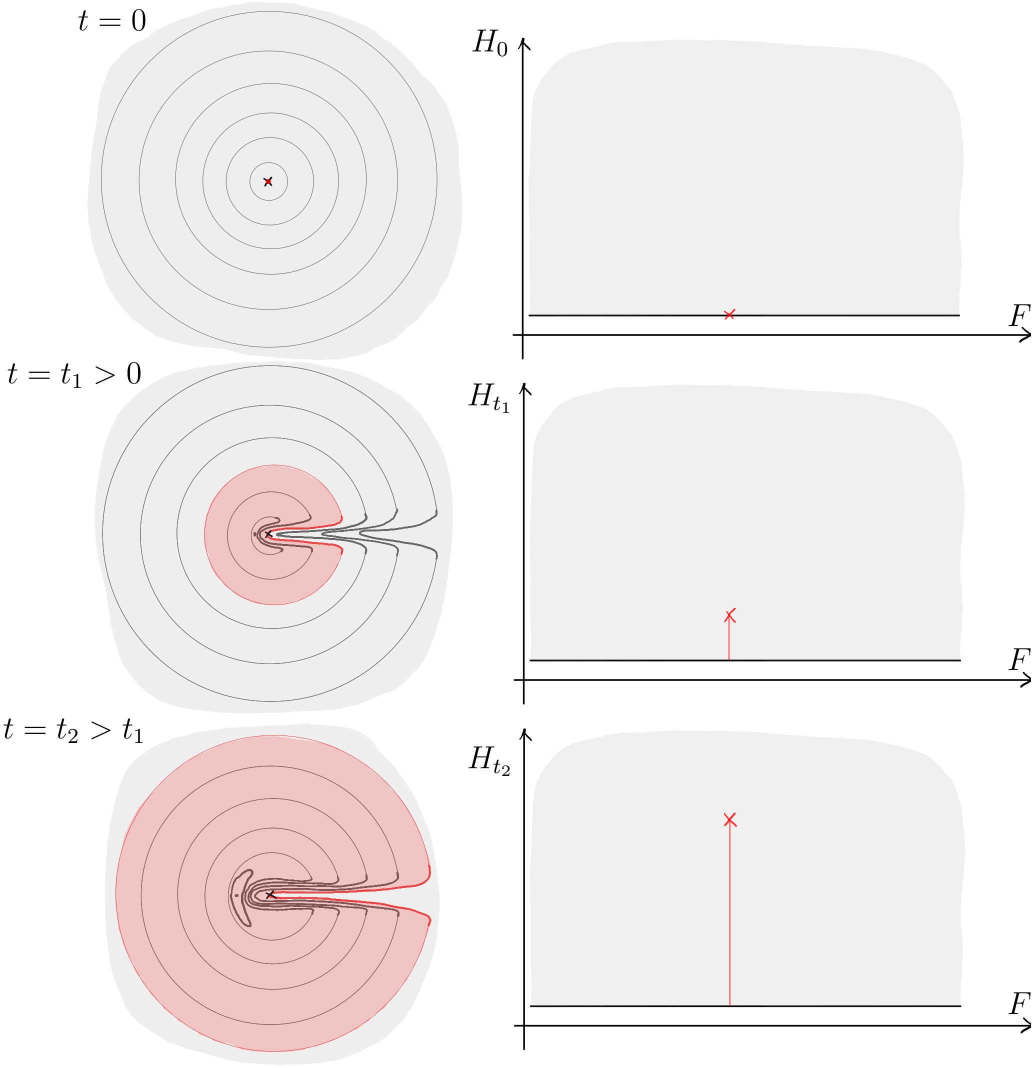


Figure 2.11: The image of the map $p_t = (F, H_t)$ for different values of the parameter t on the right. On the left, the fibrations induced on the reduced space M_0. The level set passing through the origin lifts to a pinched torus, as in Figure 2.8. The image of the pinched torus is marked by a cross on the right. The fibres over the vertical segment under the cross are Chekanov tori.

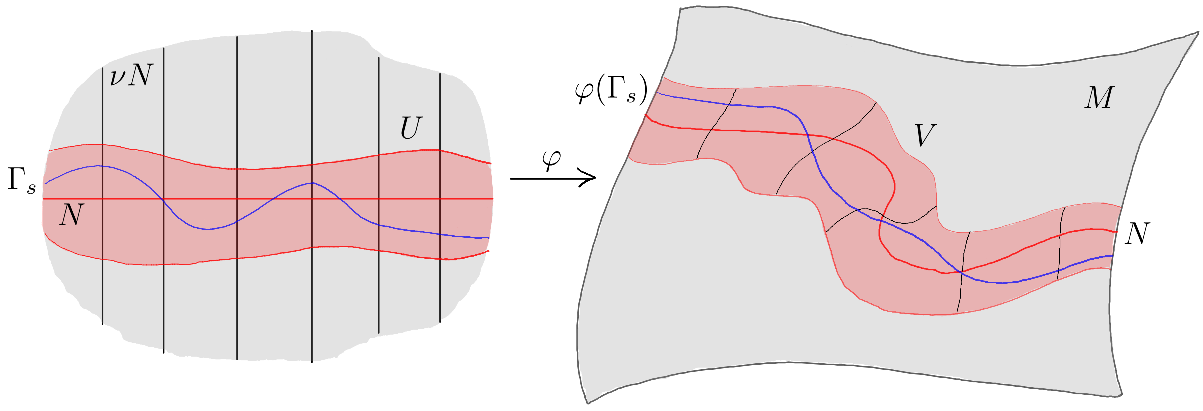


Figure 2.12: Deformations of submanifolds correspond to sections of the normal bundle via the tubular neighbourhood theorem.

This yields a method of strengthening symplectic invariants of Lagrangian manifolds, since one can evaluate a given symplectic invariant on the deformations of L instead of only on L. These deformations are known as *versal deformations* and the idea of using them to strengthen given invariants goes back to Chekanov [Che96], see also Chekanov–Schlenk [CS10], [CS16].

2.4.1 Deformations of smooth submanifolds

We start by recalling smooth deformations. Let $N \subset M$ be a smooth submanifold. We would like to describe the ways in which N can be smoothly deformed and for this we need the normal bundle. Recall that the *normal bundle* $\nu N \to N$ of N is defined by the short exact sequence of vector bundles over N

$$0 \to TN \to TM|_N \to \nu N \to 0. \tag{2.4.73}$$

In other words, the normal bundle is the collection of directions which are transversal to $TN \subset TM|_N$. Despite its somewhat misleading name, we do not need to pick a Riemannian metric (or something along those lines) to define the normal bundle. The *tubular neighbourhood theorem* states that a neighbourhood of N in M looks like a neighbourhood of the zero section in νN. Let $U \subset \nu N$ and $V \subset M$ be such neighbourhoods with a diffeomorphism $\varphi\colon U \to V$. We obtain a bijection

$$\Gamma(\nu N) \supset \mathcal{U} \to \mathcal{V} \subset \{\text{submanifolds of } M \text{ in } V\}, \quad s \mapsto \varphi(\Gamma_s). \tag{2.4.74}$$

By $\Gamma(\nu N)$, we denote the sections of the normal bundle, by $\Gamma_s \subset \nu N$ the graph of a section s and by $\mathcal{U}$ the set of sections which lie in U. The set $\mathcal{V}$ is defined as the submanifolds inside V *which are graphs* in the corresponding tubular neighbourhood. The map (2.4.74) is thus tautologically a bijection.

Remark 2.4.1. (C^1-topology) A section of a vector bundle is small in the C^1-topology if its values *and* its derivatives are small. Although one needs to choose a C^1-norm in the process, the corresponding topology is well-defined. Via the tubular neighbourhood theorem and the bijection (2.4.74), this induces a topology on the set of submanifolds which are smoothly isotopic to $N \subset M$. We call this the C^1-topology, as well. By definition, one can choose deformations C^1-small enough so that they correspond to a graph in a tubular neighbourhood.

2.4.2 Cotangent bundles and Weinstein's theorem

Now let $L \subset (M, \omega)$ be a Lagrangian submanifold of a symplectic manifold. Then there is a natural identification (i.e. a vector bundle isomorphism) of the normal bundle νL with the cotangent bundle T^*L. Indeed, the symplectic form yields a surjective homomorphism

$$TM|_L \to T^*L, \quad X \mapsto \iota(X)\omega|_{TL}. \tag{2.4.75}$$

By the definition of Lagrangian submanifolds, the kernel of this homomorphism is given by $TL \subset TM|_L$ and hence a comparison with (2.4.73) proves that $\nu L \cong T^*L$ by the map

$$\nu L = TM|_L / TL \to T^*L, \quad [X] \mapsto \iota(X)\omega|_{TL}. \tag{2.4.76}$$

This is a well-defined explicit identification. The tubular neighbourhood theorem applied to Lagrangian submanifolds thus states that a neighbourhood of L looks like a neighbourhood of the zero-section $0_L \subset T^*L$. *Weinstein's (Lagrangian neighbourhood) theorem* upgrades the diffeomorphism in this claim to a symplectomorphism.

Theorem 2.4.2. *(Weinstein's theorem) Let $L \subset (M, \omega)$ be a Lagrangian submanifold. Then there is a neighbourhood $U \subset T^*L$ of the zero–section, a neighbourhood $V \subset M$ of L and a symplectomorphism*

$$\varphi\colon (T^*L, \omega_{can}) \supset U \to V \subset (M, \omega), \quad \varphi^*\omega = \omega_{can} \tag{2.4.77}$$

mapping the zero–section to L.

Note that we tolerate a small abuse of notation here, since L refers to an abstract manifold on one hand and to a Lagrangian submanifold of M on the other. We call such a symplectomorphism $\varphi\colon U \to V$ a *Weinstein chart*. The symplectic form ω_{can} is the canonical symplectic form on cotangent bundles. It is an exact form, $\omega_{\mathrm{can}} = -d\lambda$, where λ is defined as $\sum p_i dq_i$ in a bundle chart of the cotangent bundle coming from a chart of the underlying manifold. See any textbook on symplectic geometry for more details on λ and ω_{can} and for a proof of Weinstein's theorem, for example [MS17, §3].

2.4.3 Versal deformations

We have seen that *smooth* deformations of a Lagrangian submanifold are sections of the cotangent bundle, i.e. one-forms. We will see that *Lagrangian* deformations of a Lagrangian submanifold correspond to closed one-forms. We first restrict our attention to the case of the cotangent bundle T^*L of a compact manifold L. This will serve as model, via Weinstein's theorem, for a general Lagrangian. Recall the following classical fact about graphs of one-forms.

Proposition 2.4.3. *Let T^*L be the cotangent bundle of a compact smooth manifold L equipped with its canonical symplectic form $\omega_{can} = -d\lambda$. Then we have*

(1) The graph $\Gamma_\alpha \subset T^*L$ of a one-form $\alpha \in \Omega^1(L)$ is Lagrangian if and only if the one-form is closed;

(2) Two graphs $\Gamma_\alpha, \Gamma_\beta \subset T^*L$ are Hamiltonian isotopic if and only if the forms are cohomologous, i.e. there is a smooth function $f \in C^\infty(L)$ for which $\alpha - \beta = df$.

It follows that the set of C^1-small Lagrangian perturbations of the zero-section (viewed a Lagrangian in T^*L) corresponds to a neighbourhood of the one-form $0 \in \Omega^1(L)$ in the space of closed one-forms. Furthermore, we obtain a bijection between such perturbed Lagrangians up to Hamiltonian isotopy and small elements in the first cohomology $H^1(L;\mathbb{R})$. This readily follows from the definition of de Rham-cohomology as the space of closed forms quotiented by the space of exact forms.

Now let $L \subset M$ be a compact Lagrangian in a symplectic manifold (M, ω) and take a Weinstein chart, $\varphi\colon U \to V$ with conventions as in (2.4.77). By Proposition 2.4.3 this yields a homeomorphism (in the respective C^1-topologies),

$$\Omega^1_{\mathrm{cl}}(L) \supset \widehat{\mathcal{U}} \to \widehat{\mathcal{V}} \subset \{\text{Lagrangians in } M\}, \quad \alpha \mapsto \varphi(\Gamma_\alpha). \tag{2.4.78}$$

By $\widehat{\mathcal{U}}$ we have denoted the set of closed sections contained in U and by $\widehat{\mathcal{V}}$ the set of Lagrangians in V which are graphs in the Weinstein chart. Furthermore, we may quotient out by exact one-forms on the left-hand side and by Hamiltonian isotopies supported in V on the right-hand side to obtain an induced homeomorphism. Note that the submanifolds in the image of (2.4.78) are all Lagrangian isotopic to L within the Weinstein neighbourhood V.

Proposition 2.4.4. *Let $L \subset M$ be a compact Lagrangian submanifold. Every Weinstein chart $\varphi\colon U \to V$ yields a C^1-homeomorphism*

$$w_L\colon H^1(L;\mathbb{R}) \supset \mathcal{U} \to \mathcal{V} \subset \{\textit{Lagrangians in } M\}/\!\sim, \quad [\alpha] \mapsto [\varphi(\Gamma_\alpha)]. \tag{2.4.79}$$

The equivalence relation on Lagrangian submanifolds is given by Hamiltonian isotopies which are supported in V. Furthermore, this homeomorphism is independent of the choice of Weinstein chart.

The first claim of the proposition follows from the above discussion. We postpone the proof of the independence of the Weinstein chart to Remark 2.4.9, where we construct an inverse to w_L that does not depend on the choice of the Weinstein chart.

Definition 2.4.5. *Let $L \subset (M, \omega)$ be a compact Lagrangian. Then the associated map w_L from* (2.4.79) *is called the* versal deformation *of L.*

Example 2.4.6. Let $L \subset (M, \omega)$ be a Lagrangian sphere. Since $H^1(L;\mathbb{R}) = 0$, the versal deformation is trivial. Indeed, any Lagrangian isotopy of a sphere is automatically Hamiltonian.

Example 2.4.7. Let (S^2, ω_{S^2}) be the two-sphere and as Lagrangian submanifold we choose the equator $L = G^{-1}(0)$, which we have viewed as orbit of the Hamiltonian system described in Example 2.2.5. We identify $(T^*S^1, -d\lambda_{can})$ with the cylinder $(\mathbb{R} \times S^1, dx \wedge dt)$ and define a Weinstein chart omitting the North and South Poles $N, S \in S^2$,

$$\varphi\colon (-2\pi, 2\pi) \times S^1 \to S^2 \setminus \{N, S\}, \quad (x, t) \mapsto \phi^G_t(\gamma(x)). \tag{2.4.80}$$

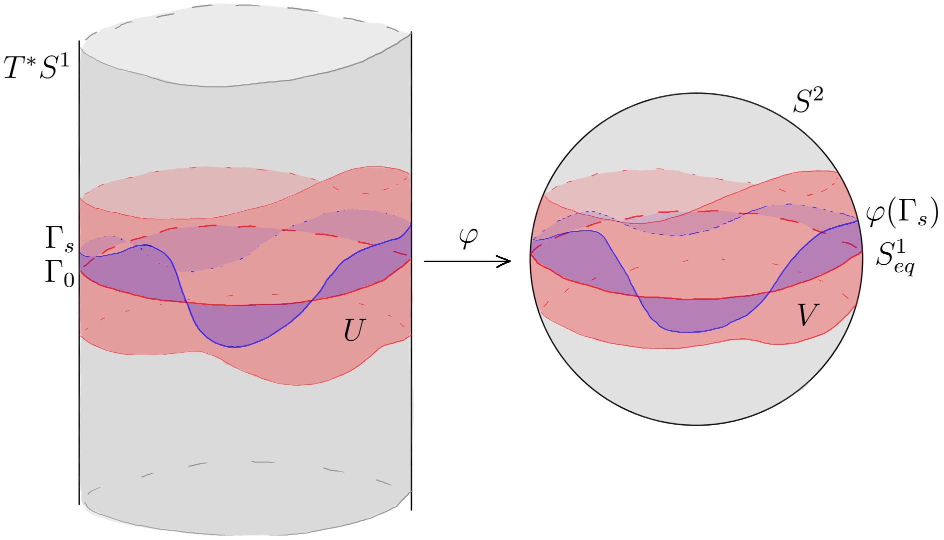


Figure 2.13: A Weinstein chart $\varphi\colon U \to V$ of the equator $S^1_{eq} \subset S^2$. The area bounded by the zero-section $\Gamma_0 \subset T^*S^1$ and the graph $\Gamma_s \subset T^*S^1$ is equal to the area bounded by the equator $S^1_{eq} \subset S^2$ and $\varphi(\Gamma_s) \subset S^2$.

Here we have chosen a smooth curve $\gamma\colon [-2\pi, 2\pi] \to S^2$ such that $\gamma(-2\pi) = S$ and $\gamma(2\pi) = N$ and $G(\gamma(x)) = x$. It follows that φ is a symplectomorphism as in the proof of Proposition 2.2.14. Since L is one-dimensional, every one-form is closed and the homomorphism

$$\Omega^1(S^1) \to \mathbb{R}, \quad \alpha \mapsto \int_{[S^1]} \alpha \tag{2.4.81}$$

descends to an isomorphism $H^1(L;\mathbb{R}) \cong \mathbb{R}$. The generator $[S^1] \in H_1(S^1;\mathbb{Z})$ here carries the orientation coming from the circle action generated by G. The integral (2.4.81) corresponds to the area contained between the zero-section and the graph of α. By applying the Weinstein chart (which preserves area), we obtain that the versal deformation of L by a small deformation parameter $a \in H^1(L;\mathbb{R})$ can be represented by any Lagrangian circle $L(a)$ in S^2 such that the cylinder bounded by L and $L(a)$ contains symplectic area a. For convenience, we can choose constant one-forms. Under our Weinstein chart, these correspond to orbits of the Hamiltonian circle action G, meaning that the map

$$H^1(L;\mathbb{R}) \cong \mathbb{R} \supset (-2\pi, 2\pi) \to \{\text{Lagrangians in } S^2\}, \quad a \mapsto G^{-1}(a) \tag{2.4.82}$$

induces a versal deformation. We will see that it is not a coincidence that one can choose nearby orbits of a Hamiltonian circle action as Lagrangian neighbours of the orbit $G^{-1}(0)$. See for example §2.4.4.

We would usually like to think of versal deformations as explicit families of Lagrangian submanifolds,

$$H^1(L;\mathbb{R}) \supset \mathcal{U} \to \{\text{Lagrangians in } M\}, \quad a \mapsto L(a). \tag{2.4.83}$$

However, this way of thinking is a little dangerous, since by far not all families of Lagrangian submanifolds are versal deformations, i.e. come from Weinstein neighbourhoods. For one, the family needs to be continuous and satisfy $L(0) = L$, where L is the original Lagrangian we want to deform. But there are more subtle conditions, which we discuss now. For clarity, we illustrate these by Example 2.4.7.

Our Lagrangian in Example 2.4.7 was the equatorial circle $L \subset S^2$. We realized its versal deformation by the set of circles of constant height $G^{-1}(a)$. This choice is obviously not unique, since there are many Lagrangians which represent the same Hamiltonian isotopy class. However not every one-parameter family of Lagrangians yields a versal deformation. For one, a family $a \mapsto L(a)$ has to be surjective onto Hamiltonian isotopy classes. In Example 2.4.7, one can easily imagine a one-parameter family of circles for which the areas of the two disks they bound are constant. All such circles are Hamiltonian isotopic, meaning that curve $a \mapsto [L(a)]$ is constant on the level of Hamiltonian isotopy classes. This is clearly not a versal deformation. But it gets more subtle, since versal deformations encode certain quantitative properties of the symplectic form. Note, again in Example 2.4.7, that the deformation parameter encodes the symplectic area contained between $G^{-1}(a)$ and its deformation. For example the family of circles of constant height given by

$$H^1(S^1;\mathbb{R}) \supset (-2\pi, 2\pi) \to \{\text{Lagrangians in } S^2\}, \quad a \mapsto G^{-1}(a^3) \tag{2.4.84}$$

exhausts all Hamiltonian classes of neighbourhing Lagrangians, but it is not a versal deformation of L. Indeed the symplectic area between L and $G^{-1}(a^3)$ is a^3 instead of a. We have the following general fact about versal deformations.

Proposition 2.4.8. *Let $L \subset (M,\omega)$ be a compact Lagrangian submanifold and let $a \mapsto L(a)$ be a versal deformation of L, meaning that every Lagrangian $L(a)$ for $a \in \mathcal{U} \subset H^1(L;\mathbb{R})$ is of the form $L(a) = \varphi(\Gamma_\alpha)$ for a Weinstein chart φ and a closed one-form α with $[\alpha] = a$. Then the Lagrangians $L(a)$ have the following property*

$$\int_{Z_\xi(L(a))} \omega = \langle a, \xi \rangle, \qquad \textit{for all } \xi \in H_1(L). \tag{2.4.85}$$

Before discussing the proof, we need to explain the notation in (2.4.85), see also Figure 2.14. Let $a \in \mathcal{U}$ and $\xi \in H_1(L)$, then by $Z_\xi(L(a))$ we denote a cylinder with one boundary component on L and one boundary component on $L(a)$ such that the boundary component on L is in the homology class of $\xi \in H_1(L)$. Note that (2.4.85) implicitly states that the value of the integral on the left-hand side is independent of all the choices we made to get a cylinder $Z_\xi(L(a))$. This will become apparent in the proof.

Proof of Proposition 2.4.8. The proof is a computation. The first crucial ingredient is the fact that we work in a Weinstein chart $\varphi\colon U \to V$ and hence the symplectic form ω is exact when restricted to V. Indeed, by (2.4.77), we have $\varphi^*\omega|_V = -d\lambda$, where λ is the Liouville one-form. For simplicity, we will make the computation in the cotangent bundle $U \subset T^*L$. The second crucial ingredient is the following property of the Liouville one-form

$$\alpha^*\lambda = \alpha, \quad \text{for all } \alpha \in \Omega^1(L). \tag{2.4.86}$$

This is the property which bridges the gap between purely cohomological data on the one hand and quantitative data about the symplectic form ω on the other. Now let $L(a) =$

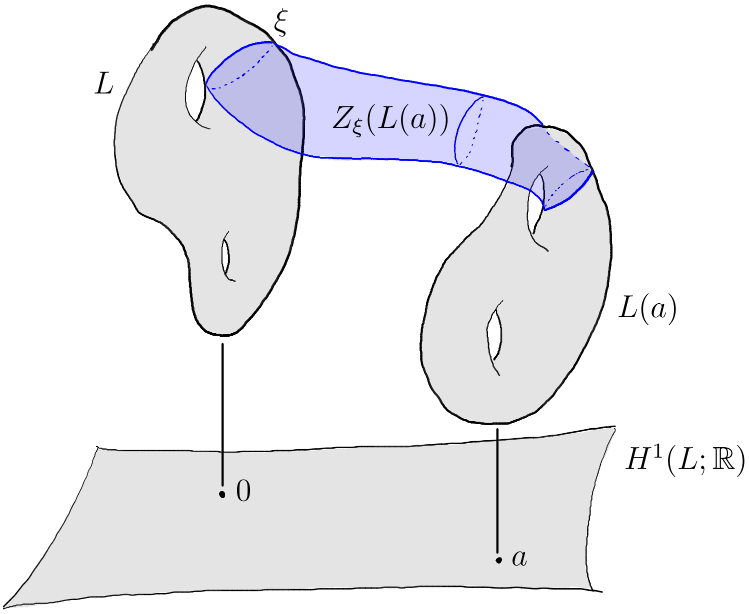


Figure 2.14: An illustration of Proposition 2.4.8. If $L(a)$ is a versal deformation of L for $a \in H^1(L;\mathbb{R})$, then the area of a cylinder over a cycle representing $\xi \in H_1(L)$ is equal to $\langle a, \xi\rangle$.

$\varphi(\Gamma_\alpha)$ for $a = [\alpha]$. Let $\xi \in H_1(L)$ and let $c\colon S^1 \to L \subset T^*L$ be a curve representing ξ. Then

$$\int_{Z_\xi(L(a))} \omega = \int_{\varphi^{-1}(Z_\xi(L(a)))} \varphi^*\omega = \int_{\alpha(c)} \lambda - \int_c \lambda = \int_c \alpha^*\lambda = \int_c \alpha = \langle a, \xi\rangle. \tag{2.4.87}$$

For the second identity, we have used Stokes and the fact that $L(a) = \varphi(\Gamma_\alpha)$. For the third identity, we have used that λ vanishes on the zero-section. This follows from the expression in coordinates $\lambda = \sum_i p_i dq_i$ or also from property (2.4.86). □

Remark 2.4.9. The identity (2.4.85) can be used to construct an inverse to the map w_L from (2.4.79). Indeed, to a Lagrangian L' which is Lagrangian isotopic to L inside a tubular neighbourhood V of L, one can associate the corresponding cohomology class $a_{L'} \in H^1(L;\mathbb{R})$ by measuring the symplectic areas of cylinders between L and L' as in (2.4.85),

$$\mathcal{V} \to \mathcal{U} \subset H^1(L;\mathbb{R}), \quad [L'] \mapsto a_{L'}, \quad \int_{Z_\xi(L')} \omega = \langle a_{L'}, \xi\rangle, \quad \forall \xi \in H_1(L). \tag{2.4.88}$$

This map is well-defined and it depends only on the Hamiltonian isotopy class of L'. The computation in the proof of Proposition 2.4.8 shows that it is an inverse to (2.4.79). Since this map does not depend on a Weinstein chart, this proves that (2.4.79) is independent of the choice of Weinstein chart.

2.4.4 Versal deformations of product tori

Note the similarity between the area condition (2.4.85) for versal deformations and the one in (2.2.21) for Hamiltonian circle actions. Indeed, an effective Hamiltonian circle action (Σ, ω, H) yields an associated versal deformation of a circle $H^{-1}(c)$ by setting

$$(-\varepsilon, \varepsilon) \to \{\text{Lagrangians in } \Sigma\}, \quad a \mapsto H^{-1}(c+a), \tag{2.4.89}$$

where $\varepsilon > 0$ is chosen small enough so that there are no critical values in $(c-\varepsilon, c+\varepsilon)$. Indeed, we can use exactly the same argument as in Example 2.4.7. In fact, the same argument can be used for all *toric* Hamiltonian torus actions. Since we have not yet introduced toric structures, we restrict our attention to the case of product tori in $\mathbb{C}^n$, see also Example 2.2.20. The proof we give can be carried over *verbatim* to get a versal deformation of general toric fibres[1]. Indeed, we will make use of the commuting Hamiltonians

$$F_i(z) = \pi |z_i|^2, \quad \omega(X_{F_i}, X_{F_j}) = 0, \quad i, j \in \{1, \ldots, n\}. \tag{2.4.90}$$

These fit together to a moment map $\mu = (F_1, \ldots, F_n)\colon \mathbb{C}^n \to \mathbb{R}^n_{\geqslant 0}$ the fibres of which are precisely the product tori,

$$\mu^{-1}(x) = T(x), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n_{>0}. \tag{2.4.91}$$

See also Example 2.2.28. In the general toric case, there is a similar moment map and the same arguments apply to yield a versal deformation of its Lagrangian fibres. Note that $H^1(T(x), \mathbb{R}) \cong \mathbb{R}^n$ and there is a canonical identification $H^1(T(x), \mathbb{R}) = \mathbb{R}^n$ by choosing a dual basis to the circle orbits of the F_i. The versal deformation is simply given by varying the base point of $\mu\colon \mathbb{C}^n \to \mathbb{R}^n_{\geqslant 0}$.

Proposition 2.4.10. *Let $c \in \mathbb{R}^n_{>0}$, then we obtain a versal deformation of the product torus $T(c) \subset \mathbb{C}^n$ by setting*

$$H^1(T(c), \mathbb{R}) \supset (\mathbb{R}^n_{>0} - c) \to \{\textit{Lagrangians in } \mathbb{C}^n\}, \quad a \mapsto T(c+a). \tag{2.4.92}$$

By $\mathbb{R}^n_{>0} - c$ we denote the positive orthant shifted by $-c \in \mathbb{R}^n$.

Proof. We denote the abstract n-torus by $T^n = (\mathbb{R}/\mathbb{Z})^n$. Its cotangent bundle $(T^*T^n, \omega_{can} = d\lambda)$ can be naturally identified with

$$\mathbb{R}^n \times T^n = \{(x, \vartheta) = (x_1, \ldots, x_n, \vartheta_1, \ldots, \vartheta_n) \,|\, x_i \in \mathbb{R},\, \vartheta_i \in S^1\}, \quad \omega = \sum_{i=1}^n dx_i \wedge d\vartheta_i. \tag{2.4.93}$$

This is the n-fold product of Example 2.2.6 and hence there is a natural T^n-action on the space (2.4.93) given by rotation in the T^n-coordinate. It is generated by the commuting set of Hamiltonians $G_i(x, \vartheta) = x_i$. Define

$$\varphi\colon (\mathbb{R}^n_{>0} - c) \times T^n \to \mathbb{C}^n, \quad (x, \vartheta) \mapsto (\phi^{F_1}_{\vartheta_1} \circ \ldots \circ \phi^{F_n}_{\vartheta_n})(f(c+x)), \tag{2.4.94}$$

---

[1] Note that for a product torus $T(c) \subset \mathbb{C}^n$, everything splits into factors and so one could just use the product of versal deformations of the factors $S^1(c_i) \subset \mathbb{C}$. In the general toric setting this is no longer true and we choose an approach, which may seem too complicated for $\mathbb{C}^n$, but which generalizes to the toric setting.

where $f\colon \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0} \subset \mathbb{C}^n$ is a diffeomorphism. We claim that, for a suitable choice of f, this map is a T^n-equivariant symplectomorphism onto its image intertwining the action by rotation on the cotangent bundle with the standard action on $\mathbb{C}^n$. This claim finishes the proof, since it shows that φ is a Weinstein chart for $T(c)$ such that the graphs of constant one-forms, $\{a\} \times T^n$ are mapped to $T(c+a)$. The T^n-equivariance of (2.4.94) is obvious by definition and does not depend on the choice of f. We now need to choose an f for which φ is symplectic. Note the similarity of φ with the symplectomorphism h constructed in the proof of Proposition 2.2.14. The role of the curve γ as *transverse coordinate* to the orbits of the Hamiltonian system is played here by f. The crucial property we need in order for φ to be symplectic is the analogue of (2.2.22),

$$F_i(f(x)) = x_i, \quad i \in \{1, \dots, n\}. \tag{2.4.95}$$

Equivalently, f is a section if we view the moment map $\mu\colon \mathbb{C}^n \to \mathbb{R}^n_{\geqslant 0}$ as a (singular) fibration. From (2.4.95) we deduce that the suitable definition for f is $f_i(x) = \frac{\sqrt{x_i}}{\pi}$. In order for φ to be symplectic, it suffices to check that

$$\omega_0(\varphi_*\partial_{x_i}, \varphi_*\partial_{x_j}) = 0, \quad \omega_0(\varphi_*\partial_{\vartheta_i}, \varphi_*\partial_{\vartheta_j}) = 0, \quad \omega_0(\varphi_*\partial_{x_i}, \varphi_*\partial_{\vartheta_j}) = \delta_{ij}. \tag{2.4.96}$$

Note that $\varphi_*\partial_{x_i}$ is contained in the Lagrangian subset $\mathbb{R}^n \subset \mathbb{C}^n$, which proves the first set of equations. For the second set of equations, note that $\varphi_*\partial_{\vartheta_i} = X_{F_i}$ and use the commutativity of the Hamiltonians F_i, F_j. Apply Hamilton's equation to the third set of equations to find

$$\omega_0(\varphi_*\partial_{x_i}, X_{F_j}) = dF_j(\varphi_*\partial_{x_i}) = \delta_{ij}. \tag{2.4.97}$$

In the last equality we have used the fact that the Hamiltonian flows $\phi_t^{F_k}$ preserve the functions F_j along with (2.4.95). □

2.5 Displacement energy and exoticity

We use a symplectic invariant called *displacement energy* to prove that the Chekanov torus is exotic. The displacement energy $e(M, A) \in \mathbb{R} \cup \{\infty\}$ is a numerical invariant attached to any compact subset $A \subset M$. It essentially measures how much *Hofer energy* is needed for the set A to be displaced from itself by a Hamiltonian isotopy. The displacement energy by itself is not sufficient to distinguish the Chekanov torus T^2_{Ch} from the product torus $T(1,1)$. Indeed, we will see that

$$e(\mathbb{C}^2, T_{\mathrm{Ch}}) = 1 = e(\mathbb{C}^2, T(1,1)). \tag{2.5.98}$$

Chekanov's crucial insight was that the displacement energy invariant may be strenghtened by evaluating it on the versal deformation of the Lagrangians in question. This indeed yields a means of distinguishing the Lagrangian tori in question.

2.5.1 The group of Hamiltonian diffeomorphisms

Let (M, ω) be a symplectic manifold. Recall from §2.2.1 that to any smooth function H on M, we can associate a Hamiltonian vector field X_H via Hamilton's equation $dH = -\iota(X_H)\omega$. By integrating X_H, we obtain a Hamiltonian flow ϕ_t^H. For the previous sections, it was sufficient to think about *autonomous* Hamiltonians flows, i.e. those flows which are generated by a vector field which is *constant in time*. However, in order to define the full group of Hamiltonian diffeomorphisms, we need to allow for time-dependent Hamiltonian vector fields.

Definition 2.5.1. *A smooth family of diffeomorphisms $t \mapsto \phi_t$ with $\phi_0 = \mathrm{id}$ is called* Hamiltonian flow *if there is a smooth family of Hamiltonian functions $t \mapsto H_t$ such that the vector field tangent to ϕ_t at time t is the Hamiltonian vector field associated to H_t via Hamilton's equation* (2.2.10).

Furthermore, we define the *group of Hamiltonian diffeomorphisms* (or simply *Hamiltonian group*) $\mathrm{Ham}(M, \omega)$ as the set of diffeomorphisms of M that can be realized as the time-one map of a Hamiltonian flow. One can verify that this is indeed a group. It is useful to think of this group as an infinite-dimensional Lie group with Lie algebra given by the Hamiltonian vector fields. Hamiltonian vector fields are in bijection with smooth functions up to adding a constant to the function, therefore we may think of the Lie algebra of Hamiltonian diffeomorphisms as $C^\infty(M)/\mathbb{R}$. A big difference between autonomous and time-dependent Hamiltonian flows is that the latter do not preserve the Hamiltonian function generating them. However, time-dependent Hamiltonian flows still preserve the symplectic form, meaning that we obtain an inclusion

$$\mathrm{Ham}(M, \omega) \subseteq \mathrm{Symp}(M, \omega). \tag{2.5.99}$$

In fact, the Hamiltonian group is even contained in the identity component of the symplectic group, $\mathrm{Symp}_0(M, \omega)$. In case the first homology $H^1(M; \mathbb{R})$ vanishes, this inclusion is an equality. Indeed, a vector field X generates a symplectic isotopy if and only if $\mathfrak{L}_X \omega = d\iota(X)\omega = 0$. In case the first cohomology vanishes, the closed form $\iota(X)\omega$ is automatically exact meaning that there is a function H with $dH = -\iota(X)\omega$. On more general manifolds, the difference between elements in $\mathrm{Symp}_0(M, \omega)$ and Hamiltonian diffeomorphisms is measured by the *symplectic flux*. We refer to [Pol01, Chapter 14] for more details.

Remark 2.5.2. (Compact support) Note that we have not assumed that the Hamiltonian functions (and hence the Hamiltonian flows) have compact support, as is the case for example in [Pol01]. For example, the harmonic oscillator $F(z) = \pi|z|^2$ does not have compact support. This means that we may run into trouble sometimes, since some Hamiltonian flows may not be defined for all times if we do not assume compact support.

2.5.2 Displacement energy

We will only give the key definitions and examples here. For a detailed introduction to the Hamiltonian group and the geometry induced on it by the Hofer norm, we again refer to Polterovich [Pol01]. Let $H = \{H_t\}_{t \in [0,1]}$ be a time-dependent Hamiltonian function on a symplectic manifold (M, ω). For the following discussion we either need to assume that M is compact or that H_t is compactly supported. The *Hofer norm* of H_t is defined as

$$\|H\| = \int_0^1 \left(\max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) \right) dt. \tag{2.5.100}$$

Definition 2.5.3. *Let* $A \subset M$ *be a compact subset. Its* displacement energy *is defined as*

$$e(M,A) = \inf\left\{ \|H\| \;\middle|\; \phi_1^H(A) \cap A = \varnothing \right\}. \tag{2.5.101}$$

If the set in (2.5.101) *is empty, we set* $e(M,A) = \infty$ *by convention. In that case* A *is called* non-displaceable*.*

Recall that by ϕ_t^H we denote the Hamiltonian flow at time t generated by H. The displacement energy is a symplectic invariant in the sense that

$$e(M,\phi(A)) = e(M,A), \quad \text{for all } \phi \in \mathrm{Symp}(M,\omega). \tag{2.5.102}$$

Although the displacement energy is defined for general subsets, we will be exclusively interested in the displacement energy of Lagrangians. We thus obtain an invariant to distinguish Lagrangian submanifolds. For all of our applications, it is enough to know the displacement energy of product tori in $\mathbb{C}^n$, see also Example 2.2.20.

Example 2.5.4. (Circles in the plane) The circle $S^1(x) \subset \mathbb{C}$ containing area $x > 0$ has displacement energy

$$e(\mathbb{C}, S^1(x)) = x. \tag{2.5.103}$$

The upper bound $e(\mathbb{C}, S^1(x)) \leqslant x$ can be proven by explicitely writing down a Hamiltonian which displaces $S^1(x)$ using energy $x + \varepsilon$ for all $\varepsilon > 0$. The lower bound can be obtained either by the calculus of variations of the action functional or by J-holomorphic curve techniques, see Theorem 2.5.6. This illustrates that, despite its simplicity, this result is non-trivial.

Example 2.5.5. (Product tori) By taking products of the previous example we obtain product tori $T(x_1,\dots,x_n) \subset \mathbb{C}^n$. Note that a Hamiltonian which displaces a factor $S^1(x_i) \subset \mathbb{C}$ of the product torus displaces the product torus when completed with the identity in the remaining components and when suitably cut off. Therefore, we obtain $\min\{x_1,\dots,x_n\}$ as an upper bound. As it turns out, this bound is sharp,

$$e(\mathbb{C}^n, T(x_1,\dots,x_n)) = \min\{x_1,\dots,x_n\}. \tag{2.5.104}$$

Again, the lower bound is non-trivial and follows for example from Theorem 2.5.6.

A classical result by Chekanov [Che98], see also Oh [Oh97], gives a lower bound on the displacement energy in terms of the minimal area of non-constant J-holomorphic curves. Let $\mathcal{J}(M,\omega)$ be the space of ω-tame almost complex structures and define

$$\sigma(M,L,J) = \inf\left\{ \int_D u^*\omega \;\middle|\; u\colon (D,\partial D) \to (M,L)\ J\text{-holomorphic, non-constant} \right\}. \tag{2.5.105}$$

By tameness and Gromov compactness, this value is attained and thus strictly positive. Now set

$$\sigma(M,L) = \sup_{J \in \mathcal{J}(M,\omega)} \sigma(M,L,J). \tag{2.5.106}$$

Theorem 2.5.6. *(Chekanov)*

$$\sigma(M,L) \leqslant e(M,L). \tag{2.5.107}$$

In practice, we can thus *choose* a suitable tame almost complex structure J and compute $\sigma(M,L,J)$ to obtain a lower bound. For $S^1(a) \subset \mathbb{C}$, see Example 2.5.4, we can pick the standard complex structure on $\mathbb{C}$ and see that the area-minimal non-constant disk has area a. A similar argument works in Example 2.5.5.

2.5.3 Displacement energy and symplectic reduction

As it turns out, symplectic reduction is a very useful tool for computing displacement energy. This is due to the fact that if one can displace a certain set A in a symplectic quotient (M_c, ω_c) of (M,ω) by a Hamiltonian isotopy, then one can displace its lifted set $\pi_c^{-1}(A)$ by the lifted Hamiltonian isotopy.

Proposition 2.5.7. *Let (M,ω,H) be a Hamiltonian circle action which admits symplectic reduction at the level $c \in \mathbb{R}$. Then, in the notation of* (2.2.40)*, we have*

$$e(M, \pi_c^{-1}(A)) \leqslant e(M_c, A), \quad A \subset M_c. \tag{2.5.108}$$

Sketch of proof. If the set A is non-displaceable, there is nothing to prove. Assume it is displaceable. Let $G_c = \{G_{c,t}\}_{t\in[0,1]}$ be a (time-dependent) Hamiltonian with Hofer norm $g > 0$ which displaces the set A. Then any lifted Hamiltonian $G = \{G_t\}_{t\in[0,1]}$ as in Proposition 2.2.31 dispaces the set $\pi^{-1}(A)$. Furthermore, we can choose a suitable cut-off to extend G_t to all of M so that it also has Hofer norm g. ☐

Remark 2.5.8. (General symplectic reduction) Although in §2.2.7 we have discussed symplectic reduction only for Hamiltonian circle actions, an analogous statement to Proposition 2.5.7 holds for symplectic reduction by any group.

The relationship between symplectic reduction and displacement energy was for example used in [AM13] and [Bre20]. Note also that the method of probes by McDuff [McD11] follows from Proposition 2.5.7. As in Remark 2.2.33 and §2.3.1, performing symplectic reduction on a segment yields a two-dimensional reduced space (in the case of probes it is a disk) in which invariant Lagrangians project to a curve. In a second step, displacements of a curve can be lifted to displacements of the original Lagrangian.

2.5.4 Displacement energy germs

We now combine versal deformations and displacement energy. Let $L \subset (M,\omega)$ be a compact Lagrangian and $w_L \colon \mathcal{U} \to \mathcal{V}$ a versal deformation of L as in Proposition 2.4.4. Recall that $\mathcal{V}$ is a set of Lagrangians up to locally supported Hamiltonian isotopies. Since the displacement energy is invariant under Hamiltonian isotopies by (2.5.102), we can consider the displacement energy of the image of versal deformations,

$$\mathcal{E}_L \colon H^1(L;\mathbb{R}) \supset \mathcal{U} \to \mathbb{R} \cup \{\infty\}, \quad a \mapsto e(M, w_L(a)). \tag{2.5.109}$$

Definition 2.5.9. *The function $\mathcal{E}_L$ is called* displacement energy germ *of the compact Lagrangian $L \subset (M,\omega)$.*

As the terminology suggests, we should think of $\mathcal{E}_L$ as a germ instead of a function (although we will be sloppy here concerning this point). Indeed, the domain of the function $\mathcal{E}_L$ we have defined depends on the choice of the Weinstein neighbourhood used to define the versal deformation. Hence the domain of definition may shrink when considering the intersection for two different choices. The germ of the function however is well-defined, by Proposition 2.4.4. In view of the symplectic invariance of displacement energy, we obtain the following invariance of $\mathcal{E}_L$ on the intersection of domains of definition,

$$\mathcal{E}_{\phi(L)} = \mathcal{E}_L \circ (\phi_L)^*, \quad \text{where } (\phi_L)_* \colon H_1(L;\mathbb{Z}) \to H_1(\phi(L),\mathbb{Z}). \tag{2.5.110}$$

From now on, denote by $L = T(1,1) \subset \mathbb{C}^2$ the Clifford torus and by $L' = T_{\mathrm{Ch}}(1) \subset \mathbb{C}^2$ the Chekanov torus. The arguments for general area parameter $a > 0$ are strictly analogous. We will show that

$$\mathcal{E}_L(a,b) = \min\{1+a, 1+b\}, \quad \mathcal{E}_{L'}(s,t) = 1+t \tag{2.5.111}$$

on some neighbourhoods of the origin. Here we have chosen suitable linear identifications $H^1(L;\mathbb{R}) \cong \mathbb{R}^2 \cong H^1(L';\mathbb{R})$. By the invariance (2.5.110), this shows that the Chekanov torus is exotic.

Theorem 2.5.10. *(Chekanov [Che96], Eliashberg–Polterovich [EP97]) The Chekanov torus $L' = T_{Ch}(1) \subset \mathbb{C}^2$ is exotic, i.e. there is no symplectomorphism of $\mathbb{C}^2$ mapping L' to a product torus.*

For the product torus L, recall that we have already determined an explicit versal deformation of product tori in Proposition 2.4.10,

$$(-1,\infty) \times (-1,\infty) \to \{\text{Lagrangians in } \mathbb{C}^2\}, \quad (a,b) \mapsto T(1+a, 1+b) \tag{2.5.112}$$

We have identified $H^1(L;\mathbb{R}) \cong \mathbb{R}^2$ by the identification dual to $H_1(L;\mathbb{Z}) = \mathbb{Z}\langle[\gamma_1],[\gamma_2]\rangle$, where γ_1 is the orbit of the Hamiltonian circle action F_1 and γ_2 the orbit of F_2. Combining this versal deformation with Example 2.5.5, we obtain the first equation from (2.5.111).

2.5.5 The Chekanov torus

Recall from Remark 2.3.6 that it is enough to distiguish L and L' in order to prove Theorem 2.5.10. Indeed, if L' is Hamiltonian isotopic to a product torus, then this product torus must be monotone of the same proportionality constant between area and Maslov class. Recall from the same remark that the area and Maslov classes are insufficient to distinguish L and L'. Before turning to versal deformations, let us first show that displacement energy by itself is also insufficent. Indeed, recall from Example 2.5.5 that $e(\mathbb{C}^2, L) = 1$.

Proposition 2.5.11. *The displacement energy of the Chekanov torus is $e(\mathbb{C}^2, L') = 1$.*

Proof. Let us first prove the upper bound $e(\mathbb{C}^2, L') \leqslant 1$. Recall that, by Definition 2.3.3, the Chekanov torus is a lift of a circle $S^1_{\mathrm{Ch}}(1)$ from the reduced space $\pi_0 \colon Z_0 \setminus \{0\} \to \mathbb{C}^\times$. By Proposition 2.5.7 we can estimate,

$$e(\mathbb{C}^2, L') \leqslant e(\mathbb{C}, S^1_{\mathrm{Ch}}(1)) \leqslant 1. \tag{2.5.113}$$

For the lower bound, we note that displacement energy is upper semi-continuous in the C^1-topology on the space of Lagrangian manifolds. This follows from the fact that if a Hamiltonian diffeomorphism displaces a given compact Lagrangian L, then it also displaces a small Weinstein neighbourhood of L and in particular all the Lagrangian neighbours contained in that neighbourhood. Now take the sequence L_n (with $a = 1$) as in Remark 2.3.6 which converges to L' and satisfies $L_n \cong T(1 + \frac{1}{n}, 1)$. This implies $e(L_n) = 1$ by Example 2.5.5, which allows us to conclude by upper semi-continuity. $\square$

The construction of the Chekanov torus as lift $\pi_0^{-1}(S^1_{\mathrm{Ch}}(1)) \subset \mathbb{C}^2$, see also Defintion 2.3.3, suggests to use the following family of Lagrangians to find a versal deformation,

$$(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to \{\text{Lagrangians in } \mathbb{C}^2\}, \quad (s,t) \mapsto L'(s,t) = \pi_s^{-1}(S^1_{\mathrm{Ch}}(1+t)). \tag{2.5.114}$$

This is indeed a versal deformation as can be shown by choosing an equivariant Weinstein chart with respect to the action $e^{2\pi i\vartheta}(z_1, z_2) = (e^{2\pi i\vartheta} z_1, e^{-2\pi i\vartheta} z_2)$ on $\mathbb{C}^2$. A deformation of the form $L'(0,t)$ is a Chekanov torus and, as in Proposition 2.5.11, its displacement energy is equal to $1 + t$. By Proposition 2.3.4, we see that any $L'(s,t)$ for $s \neq 0$ is Hamiltonian isotopic to a product torus,

$$L'(s,t) \cong \begin{cases} T(1+t, 1+t-s) & \text{if } s < 0, \\ T(1+t+s, 1+t) & \text{if } s > 0. \end{cases} \tag{2.5.115}$$

Note that in both cases of (2.5.115), the displacement energy is equal to $1 + t$ and hence the second identity from (2.5.111) follows. This finishes the proof of the exoticity of L'.

2.6 Outlook : Toric manifolds

2.6.1 Overview

Let us first pick up the discussion of Hamiltonian torus actions from §2.2.6. Recall that a Hamiltonian T^k-action is a smooth group action $\vartheta \mapsto \psi_\vartheta$ of the k-torus on a symplectic manifold (M, ω) which can be written as a composition of Hamiltonian flows,

$$\psi_\vartheta = \phi^{F_1}_{\vartheta_1} \circ \ldots \circ \phi^{F_k}_{\vartheta_k}, \quad \vartheta = (\vartheta_1, \ldots, \vartheta_k) \in T^k = (\mathbb{R}/\mathbb{Z})^k \tag{2.6.116}$$

for a set of k commuting Hamiltonians F_i, each of which generates a circle action. The map $\mu = (F_1, \ldots, F_k) \colon M \to \mathbb{R}^k$ is called a *moment map* of the action ψ. Note that the moment map determines the torus action, but the converse is not true. Indeed, one can add constants to the F_i and apply base changes in $\mathrm{GL}(k;\mathbb{Z})$ to μ without changing the action. We call elements in the group $\mathrm{Aff}_k(\mathbb{Z}) = \mathrm{GL}(k;\mathbb{Z}) \ltimes \mathbb{R}^k$ *affine integral transformations.* Atiyah [Ati82] and Guillemin–Sternberg [GS82] proved that the moment map of a Hamiltonian torus action has the following remarkable property.

Theorem 2.6.1. *(Atiyah/Guillemin–Sternberg) The image $\mu(M) \subset \mathbb{R}^k$ of the moment map μ of a Hamiltonian torus action on a compact symplectic manifold (M, ω) is a rational convex polytope given by the convex hull of the images of fixed points.*

The image $\Delta = \mu(M)$ is called the *moment polytope* of (M, ω, μ). The proof of the theorem uses Morse-theoretic arguments akin to the discussion from §2.2.3. In fact, one of the crucial ingredients is the fact that Hamiltonians generating circle actions are Morse-Bott functions all of whose critical submanifolds have even Morse index. Now one can ask the following classification question:

Can we reconstruct (M, ω, μ) *from the moment polytope* $\Delta \subset \mathbb{R}^n$*?*

In general, the answer is *no*. Note however that the question is reasonable, since we have proved in Theorem 2.2.17 that the answer is *yes* in the (very) special case of circle actions on symplectic surfaces. As it turns out, the appropriate subset of Hamiltonian torus actions to look at is the *toric* ones.

Definition 2.6.2. *A Hamiltonian torus action* ψ *on a symplectic manifold* (M, ω) *is called* toric *if it is effective and the torus has half the dimension of* M*.*

We obtain examples of toric symplectic manifolds by taking products of symplectic surfaces carrying a Hamiltonian circle action, such as the plane $\mathbb{C}$, the sphere S^2 and the cylinder T^*S^1, see Examples 2.2.4, 2.2.5 and 2.2.6. The respective moment polytopes are products of the images of the Hamiltonians generating the circle actions. From now on, we assume that M has dimension $2n$, meaning that the acting torus has dimension n. Delzant [Del88] proved that *compact* toric manifolds are classified by their moment polytopes.

Theorem 2.6.3. *(Delzant) There is a bijective correspondence between* Delzant polytopes *up to affine integral transformations and compact toric symplectic manifolds* (M, ω, μ) *up to equivariant symplectomorphisms.*

By *Delzant polytopes* we mean all convex polytopes that can be realized as moment polytopes of toric systems. They have the following nice characterization. A convex polytope $\Delta \subset \mathbb{R}^n$ is called *Delzant* if all edges have rational slope and at every vertex of Δ the set of *primitive vectors* v_i defined by the outgoing edges at that vertex form a basis of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. The *primitive vector* associated to a directed rational line is the first vector lying in the lattice $\mathbb{Z}^n$. In coordinates, this just means that all coordinates are integers and have no common divisors. In particular, all vertices of a Delzant polytope $\Delta \subset \mathbb{R}^n$ have exactly n outgoing edges. Note that n lattice vectors $v_1, \dots, v_n \in \mathbb{Z}^n$ are primitive and form a basis of $\mathbb{Z}^n$ if and only if $\det(v_1, \dots, v_n) = \pm 1$. Let us make some examples.

Example 2.6.4. (The Delzant condition) The first two vertices in Figure 2.15 are Delzant, the third one is not.

Remark 2.6.5. (Delzant construction) Note that Theorem 2.6.3 states in particular that for a given Delzant polytope $\Delta \subset \mathbb{R}^n$, one can always find a toric manifold (M, ω, μ) having Δ as moment polytope. In fact, one can give an explicit construction of the corresponding toric manifold as a symplectic quotient of some $\mathbb{C}^N$. The toric structure on the quotient M is induced by the standard toric structure (see Example 2.2.28) on $\mathbb{C}^N$ as a residual action. In fact, we have encountered an example of the Delzant construction in §2.2.8, where we have represented $\mathbb{C}P^2$ as symplectic quotient of $\mathbb{C}^3$ by the diagonal circle action. The pair (f_1, f_2) is the induced toric moment map and its image is the following Delzant polytope

$$\Delta_{\mathbb{C}P^2} = \{(f_1, f_2) \in \mathbb{R}^2 \mid f_1, f_2 \geqslant 0, f_1 + f_2 \leqslant 3\}. \tag{2.6.117}$$

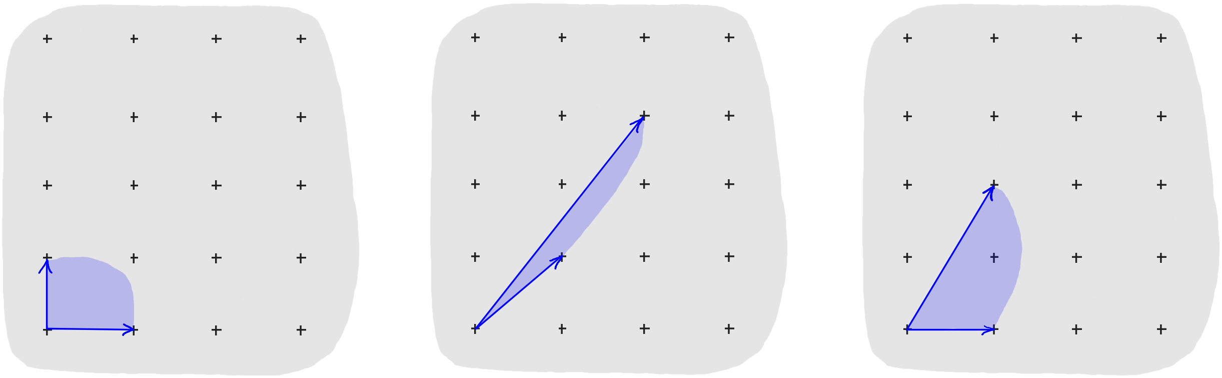


Figure 2.15: The first two vertices are Delzant, the last one is not.

Remark 2.6.6. (Integrable systems) From a dynamical point of view, toric manifolds are a special case of *integrable systems*. An integrable Hamiltonian system on a $2n$-dimensional symplectic manifold (M, ω) is a set of commuting Hamiltonians $F_1, \ldots, F_n$ such that the map $(F_1, \ldots, F_n)\colon M \to \mathbb{R}^n$ is proper and regular (meaning that $dF_1, \ldots, dF_n$ are linearly independent) on an open and dense subset of M. Note that here we do not suppose that the F_i generate circle actions. Still, the Arnold–Liouville theorem states that locally, after a change of variables, one can get a Hamiltonian torus action that has the same torus-fibres as the original Hamiltonian system. Such coordinates are called *action-angle coordinates*, the action coordinates being the values of the components of the local moment map and the angle coordinates the corresponding coordinates on the torus fibre. Toric symplectic manifolds can thus be seen as integrable systems for which the action-angle coordinates extend globally. This is not the case for all integrable systems. For example for the spherical pendulum, one needs at least two *action-angle charts*. See the classical reference [Dui80] for more details.

It is very useful to view the moment map $\mu\colon M \to \Delta$ of a toric manifold as a singular fibration. Indeed, recall that for general Hamiltonian torus actions, orbits of the action are included in a fibre of the moment map (since commuting Hamiltonians are preserved under their flows). In the toric case, *the orbits are precisely equal to the fibres of the moment map*. This follows purely from dimensional reasons. Furthermore, the orbits are always *isotropic submanifolds* of (M, ω), meaning that the symplectic form vanishes on their tangent space. In particular, the orbits of maximal dimension are Lagrangian. One can also show that the dimension of the orbit $\mu^{-1}(x) \subset M$ is equal to the dimension of the maximal facet containing $x \in \Delta$. See Figure 2.16 for a sketch of the fibration structure. This leads us to the following definition.

Definition 2.6.7. *Let (M, ω, μ) be a toric manifold and $x \in \operatorname{int}(\Delta)$ a point in the interior of the moment polytope. Then the (Lagrangian) fibre $T_x = \mu^{-1}(x) \subset M$ is called* toric fibre.

This is a generalization of product tori. Indeed, a product torus $T(x) \subset \mathbb{C}^n$ is the toric fibre $\mu^{-1}(x)$ of the standard toric structure from Example 2.2.28. In light of this, we call a Lagrangian torus in a toric manifold *exotic* if it is not symplectomorphic/Hamiltonian isotopic to a toric fibre.

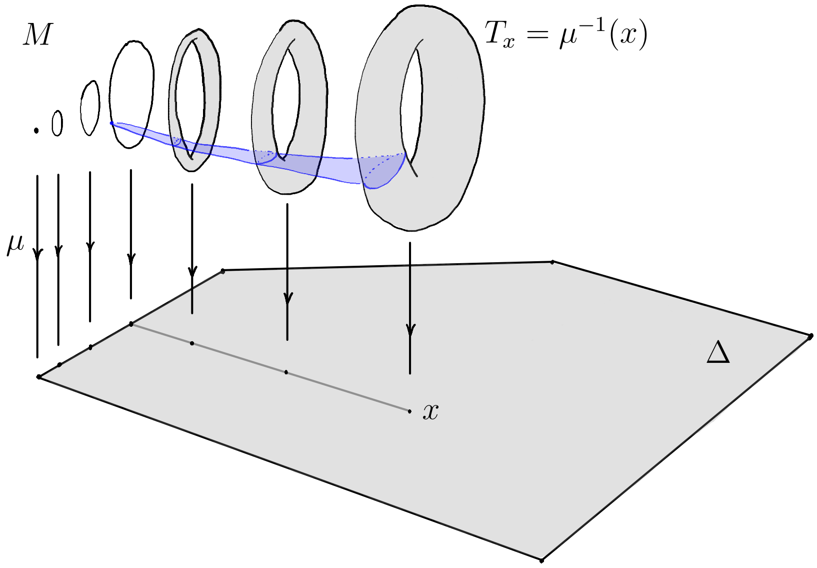


Figure 2.16: The singular fibration structure of the moment map $\mu\colon M \to \Delta$ of a toric manifold. The fibres over a k dimensional face of Δ are k-dimensional isotropic tori.

2.6.2 Chekanov tori in toric manifolds

The construction of the Chekanov torus we discussed in Section 2.3 can be easily adapted to toric manifolds. In favourable situations, we can distiguish it from toric fibres by the methods of Section 2.5 and thus we get an exotic torus in toric manifolds. This was first done by Chekanov–Schlenk [CS10] for $S^2 \times S^2$. We refer to [Bre20] for details of the content of this subsection, where we carry out these ideas in the general toric context.

Recall that the construction in Section 2.3 came from a one-parameter family of symplectic reductions with two-dimensional reduced spaces. To write down the Chekanov torus, it was sufficient to look at one particular (singular) reduced space, namely M_0. The family $\pi_c\colon Z_c \to M_c$ was needed to determine the versal deformation of the Chekanov torus and to distinguish it from the product torus. Here, it will be similar, except that we will need to consider an $(n-1)$-parametric family of two-dimensional reduced spaces.

First we pick a vertex $w \in \Delta$ at which we carry out the construction. Let $v_1, \dots, v_n \in \mathbb{Z}^n$ be the primitive vectors associated to the edges emanating from w. The family of two-dimensional reduced spaces will come from a family of lines with directional vector $v_1 + \ldots + v_n$. Note that the situation in Figure 2.6 is a special case of this. Without loss of generality, assume

$$w = 0 \in \mathbb{R}^n, \quad v_1 = e_1, \dots, v_n = e_n, \tag{2.6.118}$$

for the standard basis vectors $e_i \in \mathbb{Z}^n$. This is a normal form at the vertex v, which can be achieved by applying an element of the group of affine integral transformations $\mathrm{Aff}_n(\mathbb{Z}) = \mathrm{GL}(n;\mathbb{Z}) \ltimes \mathbb{R}^n$. Now let $U \subset \mathbb{R}^{n-1}$ be a neighbourhood of $0 \in \mathbb{R}^{n-1}$. The family of reductions will be indexed by elements $c \in U$ and we will view $U \subset \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$. Define the family of line segments and the corresponding pre-images by

$$l_c = (c + \mathbb{R}(e_1 + \ldots + e_n)) \cap \Delta, \quad Z_c = \mu^{-1}(l_c). \tag{2.6.119}$$

We want to perform symplectic reduction on the $\{Z_c\}_{c \in U}$ with respect to the following T^{n-1}-action,

$$T^{n-1} = \{(e^{2\pi i \vartheta_1}, \ldots, e^{2\pi i \vartheta_{n-1}}, e^{2\pi i(-\vartheta_1 - \ldots - \vartheta_{n-1})}) \mid \vartheta_1, \ldots, \vartheta_{n-1} \in S^1\} \subset T^n. \tag{2.6.120}$$

This torus action is not chosen at random. In fact, it is the *orthogonal complement* of the directional vector $e_1 + \ldots + e_n$ of the l_c. This is a special case of the following remark.

Remark 2.6.8. (Toric reduction) Note that reduction with respect to the full T^n-action on a toric manifold is quite uninteresting. Indeed, the reduced space will be a point. However, we may pick a subtorus $K \subset T^n$ and try to reduce with respect to the action of K. In that case, the quotient carries an induced action of T^n/K and one can show that this induced action is also Hamiltonian. There is a nice way to interpret this construction on the Delzant polytope of the toric manifold at hand. Indeed, picking a subtorus $K \subset T^n$ is equivalent to picking a rational affine vector subspace V_K in the image space $\mathbb{R}^n$ of the moment map $\mu \colon M \to \mathbb{R}^n$. In fact, under some natural identifications, the subspace V_K is the orthogonal complement of the Lie algebra $\mathrm{Lie}(K) \subset \mathrm{Lie}(T^n) = \mathbb{R}^n$. In the case that the symplectic reduction on the intersection $V_K \cap \Delta$ by K is regular, the quotient is automatically toric by the induced action of T^n/K, and its moment polytope is given by the intersection $V_K \cap \Delta$. Furthermore, one can determine from the pair (Δ, V_K) whether symplectic reduction is admissible. This means that (as is common in the toric world) everything is determined by the geometry of the moment polytope. See the forthcoming [Bre22] for more details. Note that we have already encountered several examples of toric reduction, namely in §2.2.8 and in §2.3.1. In fact, the Delzant construction is a special case of toric reduction, see Remark 2.6.5.

Note that the action of T^{n-1} is automatically free on $l_c \cap \operatorname{int} \Delta$ and hence we only need to worry about the endpoints of the segments l_c. In fact, let us restrict our attention to the half-open segments l_c' obtained by keeping the boundary point of l_c closest to the vertex $v = 0 \in \mathbb{R}^n$ and removing the other one. We remove this boundary point, since what happens away from the vertex v depends heavily on the geometry of Δ, see also Example 2.6.15. Close to the vertex v, the situation is however quite simple. Whenever l_c' intersects the boundary $\partial\Delta$ in a codimension one facet, the action is free and when it does not, then the action is not free. In the latter case we obtain a singular reduced space as in the discussion in §2.3.1. We denote the quotient maps by $\pi_c \colon Z_c' = \mu^{-1}(l_c') \to M_c$. Since the l_c are half open segments, the reduced spaces are symplectomorphic to symplectic disks (some of them having a singularity at the origin) equipped with the standard symplectic form and a natural circle action. This follows from the same arguments as in the case of $\mathbb{C}^2$ using Theorem 2.2.17. As for $\mathbb{C}^2$, the toric fibres project to the orbits of the residual circle action on the disks. Again, as for $\mathbb{C}^2$, Chekanov tori can be defined by making use of the points where symplectic reduction is not defined, by taking circles $S^1_{\mathrm{Ch}}(a)$ which are not Hamiltonian isotopic to the standard circles.

Definition 2.6.9. *The* Chekanov torus *of area parameter $a > 0$ in (M, ω) with respect to the vertex v is defined as the lift $\pi_0^{-1}(S^1_{Ch}(a))$.*

By our choice of directional vector $e_1 + \ldots + e_n$, we obtain the following.

Proposition 2.6.10. *For an open and dense subset $U_{reg} \subset U$, the reduction $\pi_c \colon Z'_c = \mu^{-1}(l'_c) \to M_c$ with $c \in U_{reg}$ is regular and the reduced space is symplectomorphic to a disk. In particular, the circles $S^1(a)$ and $S^1_{Ch}(a)$ are Hamiltonian isotopic in M_c for $c \in U_{reg}$ and so are the corresponding lifted tori.*

As in the case of $\mathbb{C}^2$, this will turn out to be crucial when trying to show that the Chekanov torus is exotic, since it proves that most neighbouring tori in the versal deformation are Hamiltonian isotopic to toric fibres and hence we can compute their displacement energy. In fact, we make the following assumption about the displacement energy of toric fibres of a toric manifold $\mu \colon M \to \Delta$.

Assumption 2.6.11. *Assume the displacement energy of a toric fibre $T_x = \mu^{-1}(x) \subset M$ is given by the affine distance of x to $\partial\Delta$,*

$$e(M, T_x) = d_{aff}(x, \partial\Delta) \tag{2.6.121}$$

for x in an open and dense subset of Δ.

See [Bre20, Section 3] for the definition of affine distance and a discussion of why this is a reasonable assumption. For example, one can prove that $e(M, T_x) \geqslant d_{aff}(x, \partial\Delta)$ for all toric fibres $T_x \subset M$ and without any additional assumptions on the toric symplectic manifold M. For our purposes, it is enough to have (2.6.121) on an open and dense subset. For example for $\mathbb{C}P^2$ and $S^2 \times S^2$, the level sets of the function $x \mapsto e(M, T_x)$ are given in Figure 2.17. Note that the level sets are just scalings of the boundary of the corresponding moment polytopes. Let us now restrict our attention to the case where M is *monotone*. We will not give the general definition of a monotone symplectic manifold here, since in the toric setting monotonicity can be easily read off from the moment polytope. A toric symplectic manifold M is *monotone* if there is a *central point* $x(\Delta) \in \Delta$, meaning a point which is equidistant (in terms of affine distance) to all facets of Δ. The toric fibre $T_{x(\Delta)}$ over the central point is a monotone Lagrangian torus. In both cases in Figure 2.17, the central point is the point where the level set degenerates to a point.

Let us now compute the displacement energy germ of the central fibre $T_{x(\Delta)}$. A versal deformation of $T_{x(\Delta)}$ is given by $b \mapsto T_{x(\Delta)+b}$ for small enough b. We have proved this in the special case of product tori in $\mathbb{C}^n$ in Proposition 2.4.10, and exactly the same argument goes through in the case of toric fibres. In conjunction with Assumption 2.6.11, this yields

$$\mathcal{E}_{T_{x(\Delta)}}(b) = d_{aff}(x(\Delta) + b, \partial\Delta), \tag{2.6.122}$$

on an open dense subset in a neighbourhood of the origin $0 \in H^1(T^n; \mathbb{R})$. The point here is that the level sets of $\mathcal{E}_{T_{x(\Delta)}}$ are given by scalings of Δ. From the symplectic invariance of $\mathcal{E}_{T_{x(\Delta)}}$ we deduce that the $\mathrm{GL}(n; \mathbb{Z})$-equivalence class of Δ is a symplectic invariant of $T_{x(\Delta)}$.

Remark 2.6.12. (The central fibre) In particular, the displacement energy germ of the central fibre $T_{x(\Delta)}$ in a monotone toric manifold *determines* its ambient space up to the

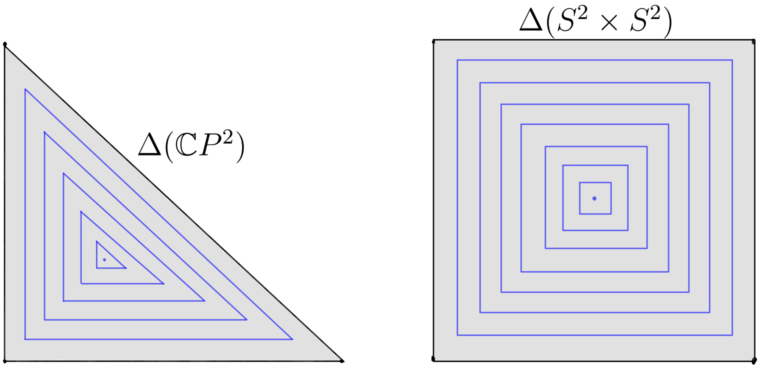


Figure 2.17: The level sets of the displacement energy function $x \mapsto e(M, T_x)$ of toric fibres for $M = \mathbb{C}P^2$ on the left and for $M = S^2 \times S^2$ on the right. The central point $x(\Delta)$ is at the central dot and the corresponding fibre is non-displaceable.

scaling of the symplectic form. Indeed, we have seen that $\mathcal{E}_{T_{x(\Delta)}}$ determines Δ up to scaling and that Δ determines (M, ω) by Delzant's theorem 2.6.3.

Let $T^v_{\text{Ch}} \subset M$ be the *monotone* Chekanov torus with respect to a given vertex $v \in \Delta$. This means that we lift the circle $S^1_{\text{Ch}}(a)$ in the reduced space M_0, where the area parameter $a > 0$ is chosen so that $\pi_0(T_{x(\Delta)}) = S^1(a)$, i.e. so that T^v_{Ch} is in the same area class as the central fibre $T_{x(\Delta)}$. We get a versal deformation of T^v_{Ch} by using the family of reduced spaces $\{M_c\}_{c \in U}$ as constructed above. Indeed, the assignment

$$(b_1, \dots, b_n) \mapsto \pi^{-1}_{(b_1, \dots, b_{n-1})}(S^1_{\text{Ch}}(a + b_n)), \quad b \in U \times (-\varepsilon, \varepsilon) \tag{2.6.123}$$

is a versal deformation of T^v_{Ch} for small enough $(-\varepsilon, \varepsilon)$. The tori in (2.6.123) are Hamiltonian isotopic to toric fibres for b in the open and dense subset $U_{reg} \times (-\varepsilon, \varepsilon)$ coming from Proposition 2.6.10. Using this, we can explicitely determine $\mathcal{E}_{T^v_{\text{Ch}}}$ on an open and dense subset. Its level sets are given by scalings of a polytope

$$\Delta^v_{\text{Ch}} = \Psi(\Delta). \tag{2.6.124}$$

The map Ψ is a piecewise linear homeomorphism of $\mathbb{R}^n$. In the normal form (2.6.118), it is given by

$$\Psi\colon \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_n \\ \vdots \\ x_{n-1} - x_n \\ \min\{x_1, \dots, x_n\} \end{pmatrix}. \tag{2.6.125}$$

We conclude that if Δ and Δ^v_{Ch} are not $\text{GL}(n; \mathbb{Z})$-equivalent, then the Chekanov torus T^v_{Ch} is exotic.

Theorem 2.6.13. *([Bre20]) Let (M, ω) be a monotone toric symplectic manifold satisfying Assumption 2.6.11. Then M contains an exotic Chekanov torus.*

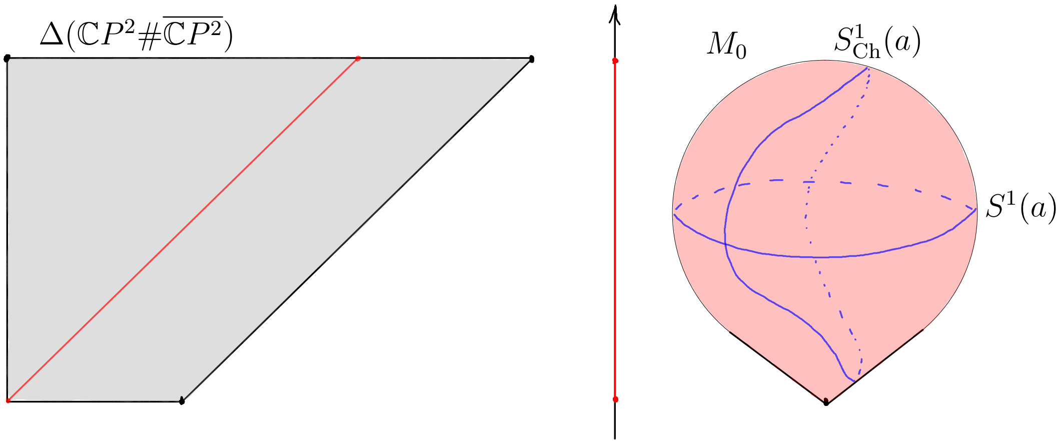


Figure 2.18: The polytope of the monotone blow-up of $\mathbb{C}P^2$ is shown on the left. Performing symplectic reduction on the indicated segment, we obtain the space sketched on the right. The Chekanov circle $S^1_{\mathrm{Ch}}(a)$ is Hamiltonian isotopic to the standard circle $S^1(a)$.

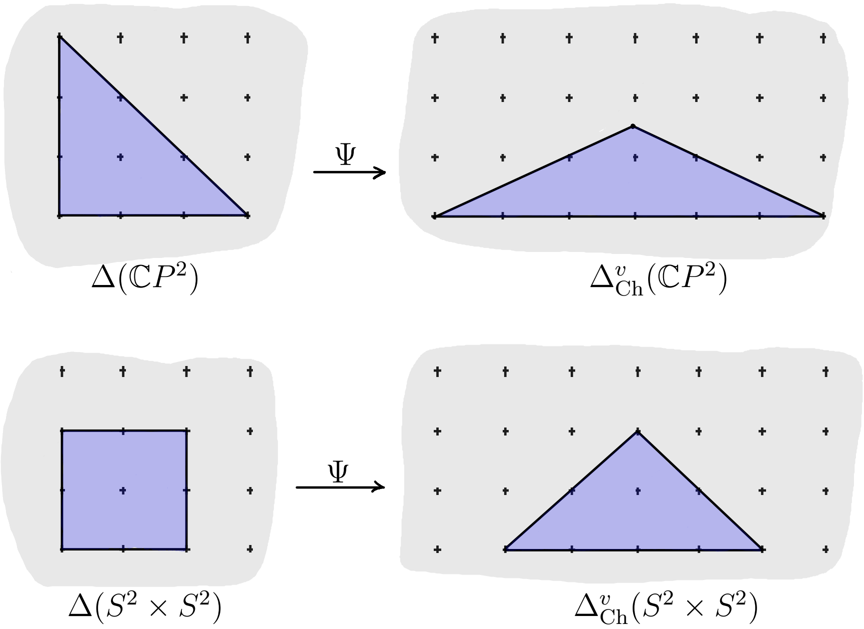
Remark 2.6.14. (Choice of the vertex) There are some vertices $v \in \Delta$ of Delzant polytopes for which the polytope Δ^v_{Ch} is $\mathrm{GL}(n;\mathbb{Z})$-equivalent to Δ, see Example 2.6.15. However, for a given Delzant polytope Δ, there is always a vertex v for which T^v_{Ch} is exotic. Indeed, let v be adjacent to the largest facet (in terms of integral affine $(n-1)$-volume) of Δ. Since Ψ merges all facets adjacent to v into one facet, the volume of the largest facet strictly increases when we pass from Δ to Δ^v_{Ch}.

Example 2.6.15. (A non-exotic Chekanov torus) Let us illustrate that the Chekanov torus T^v_{Ch} is not exotic for a certain choice of v and Δ. Let $\Delta(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})$ be the polytope given in Figure 2.18 and $v = 0$ the vertex at the origin. This is the Delzant polytope of the monotone blow-up of $\mathbb{C}P^2$. Then the reduced space M_0 is smooth at the endpoint opposite to 0. This means that we can isotope Chekanov circles $S^1_{\mathrm{Ch}}(a)$ to standard circles $S^1(a)$ and hence the lifted Chekanov torus is Hamiltonian isotopic to a toric fibre.

2.6.3 Discussion and an open question

Let us look at some examples of the polytopes Δ^v_{Ch} appearing as level sets of displacement energy germs. We again take the examples $\mathbb{C}P^2$ and $S^2 \times S^2$ and apply the map Ψ to the Delzant polytopes to obtain the corresponding Chekanov polytopes, see Figure 2.19. The reader who is familiar with almost toric geometry recognizes these as the almost-toric base polytopes of $\mathbb{C}P^2$ and $S^2 \times S^2$ obtained from their Delzant polytopes after one mutation, see [Via16], [Via17] or [Eva21, Section 8.4]. The idea to use mutations to find exotic Lagrangian tori first appeared as a conjecture in [GA10] and was then proved by Vianna. As was noticed by Chekanov–Schlenk, one can also use the approach of displacement energy germs to distiguish Vianna's tori and, as in the classical toric case, the level sets of the displacement energy germs coincide with the almost toric base polytopes. Hence distinguishing them boils down to distinguishing polytopes up to the action of $\mathrm{Aff}_2(\mathbb{Z})$.

Note that our approach from §2.6.2 starts at the other end: We know how to construct an exotic Lagrangian torus T^v_{Ch} in a toric manifold and find, via displacement energy



$\Delta(\mathbb{C}P^2)$ $\Delta^v_{\text{Ch}}(\mathbb{C}P^2)$

$\Delta(S^2 \times S^2)$ $\Delta^v_{\text{Ch}}(S^2 \times S^2)$

Figure 2.19: The Delzant polytope $\Delta(M)$ on the left and the associated Chekanov polytope $\Delta^v_{\text{Ch}}(M)$ on the right for $M = \mathbb{C}P^2$ and for $M = S^2 \times S^2$.

germs, an associated polytope $\Delta^v_{\text{Ch}} = \Psi(\Delta)$. This begs the following question about toric symplectic manifolds, see also Remark 2.6.12.

Question 2.6.16. Is there some kind of *almost toric fibration* on M having (a suitably scaled version of) the Chekanov polytope Δ^v_{Ch} as basis?

See Figure 2.20 for a depiction of the Delzant polytopes and the Chekanov polytopes Δ^v_{Ch} associated to $\mathbb{C}P^3$ and $S^2 \times S^2 \times S^2$. These are arguably the simplest 6-dimensional toric manifolds one should think about. The corresponding Chekanov polytopes are very far from Delzant, as the smoothness condition is not verified by any of their vertices and the Chekanov polytope for $S^2 \times S^2 \times S^2$ even has a vertex at which 6 edges meet. Note that we do not even know what *almost toric fibration* means in dimensions $\geqslant 6$, since almost toric fibrations are defined by the kinds of singularities we admit in the fibration and adding the so-called *focus-focus singularities* to the list of admissible ones seems to be a particularity of dimension four. This leads to the question: *What kind of singularities should we admit in higher dimensional almost toric fibrations?* Furthermore, one may wonder whether there is a meaningful notion of mutation which would allow us to produce more than one exotic fibration (and thus more than one exotic torus). Note that our approach *by hand* in Section 2.3 suggests where to start to construct this kind of mutation. Namely one can look at a family of symplectic reductions exhausting all of M and perturb the residual Hamiltonian circle action. In fact, this is what we have done locally when we have constructed the versal deformation of the Chekanov torus in §2.6.2. The almost toric fibration should appear as an extension of this local construction to all of M. Although the higher dimensional analogue of the construction by hand seems tricky, we will try to carry this out in future work.

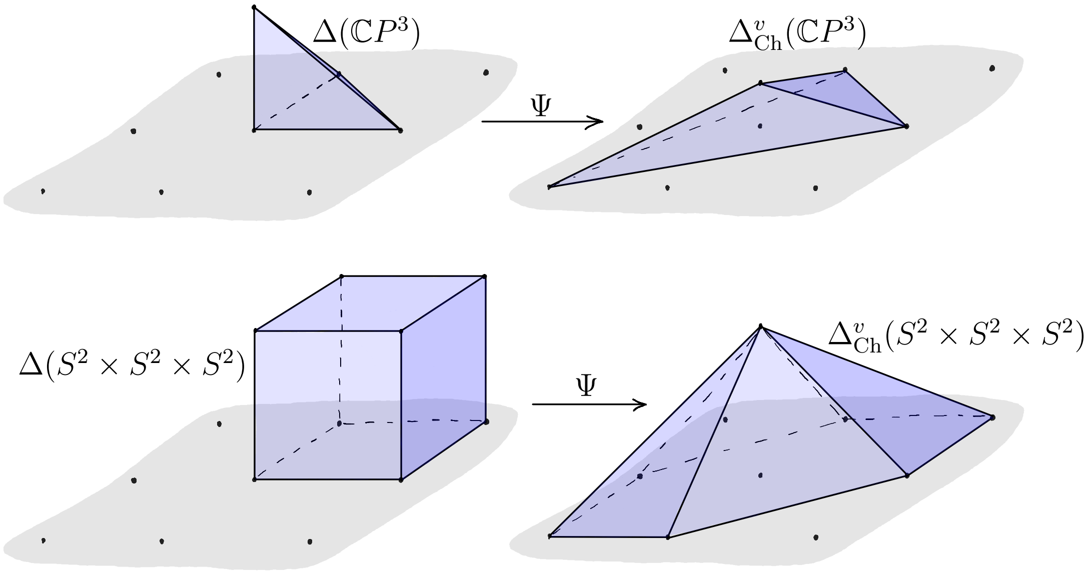


Figure 2.20: The Delzant polytope $\Delta(M)$ on the left and the associated Chekanov polytope $\Delta^v_{\mathrm{Ch}}(M)$ on the right for $M = \mathbb{C}P^3$ and for $M = S^2 \times S^2 \times S^2$.

3 On the topology of real Lagrangians in toric symplectic manifolds

This chapter consists of the article [BKM19] joint with Joontae Kim and Jiyeon Moon.

We explore the topology of real Lagrangian submanifolds in a toric symplectic manifold which come from involutive symmetries on its moment polytope. We establish a real analog of the Delzant construction for those real Lagrangians, which says that their diffeomorphism type is determined by combinatorial data. As an application, we realize all possible diffeomorphism types of connected real Lagrangians in toric symplectic del Pezzo surfaces.

3.1 Introduction

A diffeomorphism R on a symplectic manifold is called an *antisymplectic involution* if it is an involution, $R \circ R = \mathrm{id}$, and if it is antisymplectic, $R^*\omega = -\omega$. Fixed point sets of antisymplectic involutions are either empty or Lagrangian. A Lagrangian $L \subset M$ is called *real* if it is the fixed point set of an antisymplectic involution. We restrict ourselves to the study of real Lagrangians in *toric* symplectic manifolds.

A symplectic manifold (M, ω) of dimension $2n$ is called *toric* if it is equipped with an effective Hamiltonian action of the torus T^n. Complex projective space $\mathbb{C}P^n$ is a typical example. A classical result by Atiyah–Guillemin–Sternberg [Ati82, GS82] states that the image of the moment map μ of a Hamiltonian torus action is a convex polytope $\Delta \subset \mathrm{Lie}(T^n)^* = (\mathfrak{t}^n)^*$, called the *moment polytope*. In the case of $\mathbb{C}P^n$ the moment polytope is the n-simplex. Toric manifolds are classified up to equivariant symplectomorphisms by their moment polytope. This was proved by Delzant [Del88], who starts out with a given polytope satisfying certain properties (called *Delzant polytope*) and gives an explicit description of M as a symplectic quotient of a symplectic vector space. For details on the Delzant construction, see Section 3.3.2.

Let $\mathcal{S}_\Delta$ denote the group of lattice-preserving automorphisms of $(\mathfrak{t}^n)^*$ which leave Δ invariant. We construct antisymplectic involutions from symmetries of the moment polytope.

Theorem 3.1.1. *Let (M, ω) be a toric symplectic manifold with moment map μ and moment polytope Δ. Furthermore, let $\sigma \in \mathcal{S}(\Delta)$ be an involution of Δ. Then σ lifts to an antisymplectic involution R^σ of M,*

$$\mu \circ R^\sigma = \sigma \circ \mu. \tag{3.1.1}$$

The antisymplectic involution R^σ we construct is not unique with respect to the property (3.1.1). Henceforth, we will refer to it as the *standard antisymplectic lift of σ*. The most basic example for Theorem 3.1.1 is the following one. Let (S^2, ω) be the two-sphere equipped with its area form. The toric structure is given by rotation around a fixed axis and the corresponding moment map is given by projection onto that axis, see Figure 3.1. Therefore Δ can be identified with a segment in $\mathbb{R}$. Let σ be the only non-trivial involution on Δ given by the flip around the mid-point of the segment. The corresponding antisymplectic involution on S^2 is given by the flip fixing the equator.

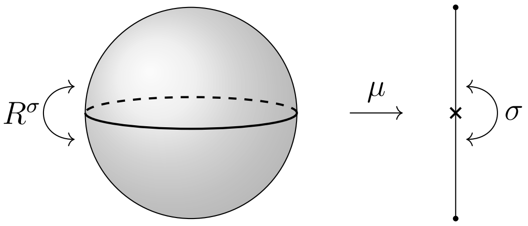


Figure 3.1: The flip σ on Δ and the corresponding antisymplectic involution R^σ on S^2.

Even for general toric symplectic manifolds M, there is a particularly simple way of understanding the involutions R^σ in Theorem 3.1.1 if we restrict our attention to the open and dense subset $\mu^{-1}(\mathring{\Delta})$ formed by the pre-image of the interior of Δ. In fact, $\mu^{-1}(\mathring{\Delta})$ is equivariantly symplectomorphic to $T^n \times \mathring{\Delta}$, when we equip the latter space with the natural T^n-action and the symplectic form coming from the inclusion $T^n \times \mathring{\Delta} \subset T^*T^n \cong T^n \times (\mathfrak{t}^n)^*$. Under this identification, the moment map corresponds to the natural projection $T^n \times \mathring{\Delta} \to \mathring{\Delta}$. We observe

1. For any lattice preserving involution σ on the moment polytope, the map $(\sigma^T)^{-1} \times \sigma$ defines a symplectomorphism on $T^n \times \mathring{\Delta}$. The transpose $\sigma^T \colon \mathfrak{t}^n \to \mathfrak{t}^n$ is well-defined on T^n, since it preserves the lattice. Hence we obtain a *symplectic* involution on $T^n \times \mathring{\Delta}$.

2. There is a natural antisymplectic involution R^0 on $T^n \times \mathring{\Delta}$ given by taking the group inverse on the T^n-component. The involution R^0 preserves the fibres of $T^n \times \mathring{\Delta} \to \mathring{\Delta}$.

The desired antisymplectic involution R^σ is obtained by composing the maps obtained in the two observations. Since they commute, the resulting diffeomorphism will indeed be an involution. The main problem with this heuristic argument is extending everything to the singular fibres over $\partial\Delta$. In Section 3.4 we thus stick to the more conventional approach via Delzant's point of view on toric manifolds. As we shall see, this approach also has the merit of providing a method to understand the fixed point set of R^σ.

In the special case where $\sigma = \mathrm{id}$, we obtain an antisymplectic involution R^0 which leaves the moment map invariant $\mu \circ R^0 = \mu$. This involution is widely known in toric geometry, where it corresponds to complex conjugation. Its fixed point set $\operatorname{Fix} R^0$ is the *real locus* of the toric variety, in the case of $\mathbb{C}P^n$ it corresponds to $\mathbb{R}P^n$. Duistermaat [Dui83] studied more general real Lagrangians $L = \operatorname{Fix} R$ in Hamiltonian T^k-spaces (M, ω, μ) for any $k \leqslant n$, which arise as the fixed point set of involutions leaving the moment map invariant, $\mu \circ R = \mu$. He proved that real Lagrangians of this type are tight and have a convex image under the moment map. Tightness of the real Lagrangian L means that for any $\xi \in \mathfrak{t}^n$ the restriction $H_\xi|_L$ of the Hamiltonian function $H_\xi = \langle \mu, \xi \rangle$ is *tight* in the sense that the sum of the Betti numbers of L and the one of the critical set of $H_\xi|_L$ coincide.

Another class of interesting Lagrangians are regular fibres $\mu^{-1}(x) \subset M$ of the toric moment maps. Entov–Poterovich [EP09] studied the rigidity of intersections of *Lagrangian fibres*, namely that the barycentric fibre in a closed monotone symplectic manifold cannot be displaced by a Hamiltonian isotopy. See also results of Fukaya–Oh–Ohta–Ono [FOOO10]. Theorem 3.1.1 shows that the toric fibre $\mu^{-1}(0)$ is real whenever the moment polytope is invariant under the central symmetry $\sigma = -\,\mathrm{id}$. Indeed, we will see that $\mathrm{Fix}(R^\sigma) \neq \varnothing$, and since $0 \in (\mathfrak{t}^n)^*$ is the only fixed point of $-\,\mathrm{id}$, the fixed point set of $R^{-\mathrm{id}}$ is the entire fibre $\mu^{-1}(0)$ for dimensional reasons. Under some additional assumptions on M, one can show that $\mu^{-1}(0)$ being real is a sufficient condition for Δ to be invariant under $-\,\mathrm{id}$. We refer to [Bre20] for details. In a sense, the two classical situations $\sigma = \mathrm{id}$ and $\sigma = -\,\mathrm{id}$ are opposite to each other and all other R^σ which we obtain from Theorem 3.1.1 are intermediate cases.

The remainder of the paper is dedicated to a topological study of the fixed point sets of the involutions R^σ. The main result in this direction is the so-called *real Delzant construction*, which states that the diffeomorphism type of $L = \mathrm{Fix}(R^\sigma)$ is completely determined by the moment polytope Δ and the involution $\sigma \in \mathcal{S}_\Delta$.

We briefly explain relevant notions in the classical Delzant construction, see Section 3.3.2 for details. The moment polytope Δ with k facets yields the moment map

$$\nu\colon \mathbb{C}^k \longrightarrow \mathfrak{k}^*,$$

where $\mathfrak{k} = \mathrm{Lie}(K)$ is the Lie algebra of the kernel K of the characteristic map $\pi\colon T^k \to T^n$. By the Marsden–Weinstein theorem, we can reconstruct the toric symplectic manifold,

$$M \cong \nu^{-1}(0)/K.$$

In a similar vein, we will define a real analog of the maps ν and π, namely

$$\pi_R\colon\ \mathrm{Fix}(\rho_{T^k}) \longrightarrow \mathrm{Fix}(R_{T^n})$$

the *real characteristic map*, and

$$\nu_R\colon\ \mathrm{Fix}(\rho) \longrightarrow (\mathfrak{k}/\mathfrak{k}_R)^*,$$

where $\mathfrak{k}_R \subset \mathfrak{k}$ is the Lie algebra of the kernel $K_R = \ker \pi_R$. Here ρ and ρ_{T^k} are involutions on $\mathbb{C}^k$ and T^k, respectively, determined by the involution $\sigma \in \mathcal{S}_\Delta$. See Section 3.5 for details.

The main result of the paper is the following *r*eal Delzant construction.

Theorem 3.1.2. *Let (M, ω, μ) be a toric symplectic manifold and let R^σ be the standard antisymplectic involution of M given by the lift of an involution $\sigma \in \mathcal{S}_\Delta$. Then the real Lagrangian $L = \mathrm{Fix}(R^\sigma)$ is diffeomorphic to $\nu_R^{-1}(0)/K_R$.*

As a partial generalization of Duistermaat's result, we prove convexity and tightness for the real Lagrangians $L = \mathrm{Fix}(R^\sigma)$.

Theorem 3.1.3. *Let (M, ω, μ) be a toric symplectic manifold with moment polytope Δ and let R^σ be the standard antisymplectic involution of M given by the lift of an involution*

$\sigma \in \mathcal{S}_\Delta$. *Then* $\mu(L) = \mathrm{Fix}(\sigma)$ *is convex, and for any* $\xi \in \mathfrak{t}^n$ *we have*

$$\dim H_*(L;\mathbb{Z}_2) = \dim H_*(\mathrm{Crit}(H_\xi|_L);\mathbb{Z}_2),$$

where H_ξ *is the smooth function* $\langle \mu, \xi \rangle$ *on* M *and* $\mathrm{Crit}(H_\xi|_L)$ *denotes the set of critical points of* $H_\xi|_L$.

In particular, both Theorem 3.1.2 and 3.1.3 imply that $L = \mathrm{Fix}(R^\sigma)$ is not empty. Example 3.6.3 shows that the tightness and the convexity, in general, fail if the real Lagrangian is not of the form $\mathrm{Fix}(R^\sigma)$.

As an application, we show that the class of real Lagrangians that come from involutive symmetries of moment polytopes provides a starting point for the classification of real Lagrangians in toric symplectic del Pezzo surfaces.

Recall that the symplectic del Pezzo surfaces $Q = S^2 \times S^2$ and $X_k = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ for $0 \leq k \leq 3$ are *monotone* and toric. Being monotone means that their first Chern class is positively proportional to the cohomology class of the symplectic form, see Section 3.7. Using Smith theory, the Arnold lemma, and homological obstructions for Lagrangians, we show that any real Lagrangian L in a toric symplectic del Pezzo surface M must be diffeomorphic to one of cases listed in Table 3.1. We then realize all of these possible cases as fixed point sets of lifted antisymplectic involutions R^σ. The real Delzant construction will be used to determine their diffeomorphism types. We refer to Section 3.7 for details.

Theorem 3.1.4. *Let* L *be a connected real Lagrangian submanifold of a toric symplectic del Pezzo surface* M. *Then* L *is diffeomorphic to one of the surfaces in Table 3.1, and each of these diffeomorphism types is realized as the fixed point set* $\mathrm{Fix}(R^\sigma)$ *of an antisymplectic involution from Theorem 3.1.1.*

X	$L = \mathrm{Fix}(R)$					
$S^2 \times S^2$	S^2	T^2				
X_0			$\mathbb{R}P^2$			
X_1				$\mathbb{R}P^2 \# \mathbb{R}P^2$		
X_2			$\mathbb{R}P^2$		$\#_3 \mathbb{R}P^2$	
X_3	S^2	T^2		$\mathbb{R}P^2 \# \mathbb{R}P^2$		$\#_4 \mathbb{R}P^2$

Table 3.1: The diffeomorphism types of connected real Lagrangians in toric symplectic del Pezzo surfaces.

Task. *Classify connected real Lagrangians in toric symplectic del Pezzo surfaces up to Hamiltonian isotopy.*

For fixed diffeomorphism type of the real Lagrangian, uniqueness up to Hamiltonian isotopy is known for S^2 in $S^2 \times S^2$ and X_3 by [Hin04] and [Eva10], and for $\mathbb{R}P^2$ in X_0 by [LW12]. Indeed, every real Lagrangian in a monotone symplectic manifold is monotone and there are no exotic monotone submanifolds in these cases.

Question 3.1.5. Is every connected real Lagrangian with fixed diffeomorphism type in toric del Pezzo surfaces unique up to Hamiltonian isotopy?

Since $S^2 \times S^2$ and X_3 admit infinitely many exotic monotone Lagrangian tori [Via16], a positive answer to this question would crucially depend on the submanifold being real. Recently, the uniqueness of real Lagrangian tori in $S^2 \times S^2$ was established in [Kim20].

3.2 Basic geometry

We refer to [Sja10], [Aud04] and [MS17, Chapter 5] for (real) symplectic geometry and Hamiltonian torus actions.

3.2.1 Basics

Let (M, ω) be a symplectic manifold and let the n-torus T^n act on M by symplectomorphisms. We denote this action by $(t, p) \mapsto t.p$ for $p \in M$ and $t \in T^n$ and the corresponding Lie algebra by $\mathfrak{t}^n = \mathrm{Lie}(T^n)$. The associated infinitesimal action $\mathfrak{t}^n \to \Gamma(TM)$ is defined by

$$\xi \longmapsto X_\xi, \quad (X_\xi)_p := \left.\frac{d}{ds}\right|_{s=0} \exp(s\xi).p, \quad p \in M. \tag{3.2.2}$$

A symplectic T^n-action on a symplectic manifold M is called *Hamiltonian* if there exists a smooth map $\mu : M \to (\mathfrak{t}^n)^*$ such that

1) for each $\xi \in \mathfrak{t}^n$ we have $d\langle \mu, \xi\rangle = \iota_{X_\xi}\omega$, where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{t}^n$ and $(\mathfrak{t}^n)^*$,

2) the map μ is invariant under the T^n-action, i.e. $\mu(t.p) = \mu(p)$ for all $t \in T^n$ and $p \in M$.

The map μ is called a *moment map* of the Hamiltonian T^n-action.

Definition 3.2.1. *A triple (M, ω, μ) is called a* Hamiltonian T^n-space *if (M, ω) is a symplectic manifold equipped with a Hamiltonian T^n-action and $\mu\colon M \to (\mathfrak{t}^n)^*$ is a moment map associated to the action.*

The equation $d\langle \mu, \xi\rangle = \iota_{X_\xi}\omega$ means that the Hamiltonian flow of $\langle \mu, \xi\rangle \in C^\infty(M)$ at time t corresponds to the action of $\exp(t\xi)$ on M. Furthermore, this equation can be used to prove the following geometric properties of the moment map

$$(\ker d\mu|_p)^\omega = T_p(T^n p), \tag{3.2.3}$$
$$\mathrm{Ann}(\mathrm{im}\, d\mu|_p) = \mathrm{Lie}(\mathrm{Stab}(p)). \tag{3.2.4}$$

A classical result by Atiyah–Guillemin–Sternberg states that the image of μ is a convex polytope in $(\mathfrak{t}^n)^*$, called the *moment polytope*.

Remark 3.2.2. As for Hamiltonians in general, adding a constant vector to the moment map does not change the group action it generates. We choose the normalization $\int_M \mu\omega^n = 0 \in (\mathfrak{t}^n)^*$ for compact M unless otherwise stated.

The *standard lattice* $\mathfrak{t}^n_{\mathbb{Z}}$ is defined as the kernel of the exponential map $\exp : \mathfrak{t}^n \to T^n$. Furthermore, the group formed by the automorphisms of $\mathfrak{t}^n$ which preserve the standard lattice will be denoted by $\operatorname{Aut}_{\mathbb{Z}} \mathfrak{t}^n$. The dual of the standard lattice is defined by

$$(\mathfrak{t}^n_{\mathbb{Z}})^* = \{\eta \in \mathfrak{t}^* \mid \langle \eta, x\rangle \in \mathbb{Z} \text{ for all } x \in \mathfrak{t}^n_{\mathbb{Z}}\}.$$

The corresponding group $\operatorname{Aut}_{\mathbb{Z}} (\mathfrak{t}^n)^*$ is defined similarly.

Remark 3.2.3. Since $T^n \cong \mathfrak{t}^n / \mathfrak{t}^n_{\mathbb{Z}}$, any element $\alpha \in \operatorname{Aut}_{\mathbb{Z}} \mathfrak{t}^n$ induces a group automorphism A of T^n. Conversely, for any group automorphism A of T^n, its differential A_* belongs to $\operatorname{Aut}_{\mathbb{Z}} \mathfrak{t}^n$.

Recall that for a given Hamiltonian T^n-space, one can perform symplectic reduction on certain level sets of the moment map in order to obtain a new symplectic manifold. See [CdS03] for details.

Proposition 3.2.4. *Let (M, ω, μ) be a Hamiltonian T^n-space and $0 \in (\mathfrak{t}^n)^*$ a regular value of μ such that T^n acts freely on the corresponding level set $\mu^{-1}(0)$. Then the quotient*

$$\widehat{M} = \mu^{-1}(0)/T^n$$

carries a unique symplectic structure $\widehat{\omega}$ such that

$$\iota^*\omega = p^*\widehat{\omega}$$

where $\iota : \mu^{-1}(0) \hookrightarrow M$ is the natural inclusion and $p : \mu^{-1}(0) \twoheadrightarrow \widehat{M}$ is the natural projection.

The space $(\widehat{M}, \widehat{\omega})$ is called *symplectic quotient* or *Marsden–Weinstein quotient* at the level 0. This construction is best summarized by the reduction diagram

$$\begin{tikzcd}
(\mu^{-1}(0), \iota^*\omega = p^*\widehat{\omega}) \arrow[r, hook, "\iota"] \arrow[d, two heads, "p"', "/T^n"] & (M, \omega) \\
(\widehat{M}, \widehat{\omega}). &
\end{tikzcd}$$

3.2.2 Compatible maps

Our constructions of symplectomorphisms and antisymplectic involutions in Section 3.4 rely heavily on the toric structure and hence they are *compatible* with the corresponding torus action in a suitable sense. As this compatibility is crucial for our applications, we discuss it here in detail. Let (M, ω, μ) be a Hamiltonian T^n-space. We will define a notion of compatibility between the torus action and a given diffeomorphism φ of M which either preserves or reverses the symplectic form, i.e. which is either symplectic or antisymplectic. In order to treat both cases simultaneously, we attach a sign $\varepsilon(\varphi) \in \{-1, 1\}$ to the diffeomorphism φ such that

$$\varphi^*\omega = \varepsilon(\varphi)\omega.$$

Proposition 3.2.5. *Let φ be a diffeomorphism of a Hamiltonian T^n-space (M, ω, μ) satisfying $\varphi^*\omega = \varepsilon(\varphi)\omega$ for $\varepsilon(\varphi) \in \{-1, 1\}$. Then the following are equivalent.*

1) There is a group automorphism $\beta : T^n \to T^n$ such that

$$\varphi(t.p) = \beta(t).\varphi(p), \quad p \in M,\, t \in T^n; \tag{3.2.5}$$

2) There is a map $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t}^n)^*$ such that

$$\mu \circ \varphi = \sigma \circ \mu; \tag{3.2.6}$$

3) There is a map $\alpha \in \mathrm{Aut}_{\mathbb{Z}}\, \mathfrak{t}^n$ such that

$$\varphi_*^{-1}(X_\xi \circ \varphi) = X_{\alpha(\xi)}, \quad \xi \in \mathfrak{t}^n. \tag{3.2.7}$$

Furthermore, if the statements are true, then the above maps are related by

$$\beta_*^{-1} = \varepsilon(\varphi)\sigma^* = \alpha. \tag{3.2.8}$$

Proof. First suppose that $\varepsilon(\varphi) = 1$.
We will show that both 1) and 2) are equivalent to the infinitesimal condition 3). Since the exponential map of T^n is surjective, 1) is equivalent to

$$\varphi(\exp s\xi.p) = \beta(\exp s\xi).\varphi(p), \quad p \in M,\, \xi \in \mathfrak{t}^n.$$

Differentiating with respect to s and rearranging terms, we obtain

$$\varphi_*^{-1}(X_\xi)_{\varphi(p)} = (X_{\beta_*^{-1}\xi})_p, \quad p \in M,\, \xi \in \mathfrak{t}^n.$$

The equivalence of 1) and 3) follows by defining, with the help of Remark 3.2.3, $\alpha := \beta_*^{-1} \in \mathrm{Aut}_{\mathbb{Z}}\, \mathfrak{t}^n$ and conversely by defining β as the automorphism obtained by lifting α^{-1} to T^n.

In order to prove the equivalence of 2) and 3), recall that given a Hamiltonian H, its vector field X_H transforms under a symplectomorphism φ to the Hamiltonian vector field $X_{H\circ\varphi} = \varphi_*^{-1}(X_H \circ \varphi)$. Since X_ξ is the vector field corresponding to the Hamiltonian function $H = \langle \mu, \xi \rangle$, identity (3.2.7) can be rewritten as

$$X_{\langle \mu\circ\varphi, \xi\rangle} = X_{\langle \mu, \alpha(\xi)\rangle},$$

which is equivalent to

$$\mu \circ \varphi = \alpha^* \circ \mu.$$

The case $\varepsilon(\varphi) = -1$ can be proved similarly. The only notable difference is the fact that if φ is antisymplectic, then the Hamiltonian vector fields transform as follows,

$$X_{H\circ\varphi} = -\varphi_*^{-1}(X_H \circ \varphi).$$

This accounts precisely for the additional minus sign in equation (3.2.8). □

Remark 3.2.6. In the antisymplectic case, we will mostly work with involutions, i.e. diffeomorphisms $R : M \to M$ satisfying $R^*\omega = -\omega$ and $R^2 = \mathrm{id}$. In this case, the maps

β, α and σ are involutions as well.

Definition 3.2.7. *An (anti-)symplectic diffeomorphism φ on a Hamiltonian T^n-space (M, ω, μ) is called* compatible *if one of the equivalent conditions in Proposition 3.2.5 holds.*

Remark 3.2.8. This compatibility condition is a special case of the notion of *real Hamiltonian G-manifold* given in [Sja10], which contains many examples. These ideas go back to Duistermaat's work [Dui83], who considered the case where $\beta(t) = t^{-1}$.

Remark 3.2.9. In later chapters, we will denote the map β corresponding to a compatible map φ by $\varphi_{T^n} : T^n \to T^n$ and thus $\sigma = \varepsilon(\varphi)(\varphi_{T^n}^{-1})^*$ by (3.2.8).

In case the Hamiltonian T^n-space admits symplectic reduction, a given compatible (anti-)symplectic map yields an (anti-)symplectic map on the symplectic quotient. This fact plays a crucial role in the construction of the (anti-)symplectic maps in Section 3.4.

Proposition 3.2.10. *Let φ be a compatible diffeomorphism on a Hamiltonian T^n-space (M, ω, μ) satisfying $\varphi^*\omega = \varepsilon(\varphi)\omega$ for $\varepsilon(\varphi) \in \{-1, 1\}$. Furthermore, suppose that M admits symplectic reduction at the level $0 \in (\mathfrak{t}^n)^*$. Then φ induces a diffeomorphism $\widehat{\varphi} : \widehat{M} \to \widehat{M}$ on the symplectic quotient satisfying $\widehat{\varphi}^*\widehat{\omega} = \varepsilon(\varphi)\widehat{\omega}$ such that the following diagram commutes,*

$$\begin{tikzcd}
\mu^{-1}(0) \arrow[r, hook, "\iota"] \arrow[loop above, "\varphi|_{\mu^{-1}(0)}"] \arrow[d, two heads, "p"', "/T^n"] & M \arrow[loop above, "\varphi"] \\
\widehat{M} \arrow[loop left, "\widehat{\varphi}"] &
\end{tikzcd}$$

Proof.

The diffeomorphism φ preserves the level set $\mu^{-1}(0)$, as can be read off from (3.2.6). Furthermore, the restriction $\varphi|_{\mu^{-1}(0)}$ descends to $\widehat{M}$ by equation (3.2.5) to yield a diffeomorphism $\widehat{\varphi}$. Since $\widehat{\omega}$ is defined by $\iota^*\omega = p^*\widehat{\omega}$, we can compute

$$\begin{aligned}
p^*\widehat{\varphi}^*\widehat{\omega} &= \varphi|_{\mu^{-1}(0)}^* p^*\widehat{\omega} \\
&= \varphi|_{\mu^{-1}(0)}^* \iota^*\omega \\
&= \iota^*\varphi^*\omega \\
&= \iota^*\left(\varepsilon(\varphi)\omega\right) \\
&= p^*\left(\varepsilon(\varphi)\widehat{\omega}\right).
\end{aligned}$$

Since p is a surjective submersion, this implies that $\widehat{\varphi}^*\widehat{\omega} = \varepsilon(\varphi)\widehat{\omega}$. □

3.3 Toric symplectic manifolds

We refer to [Aud04], [CdS03], [Del88], [Gui94] or [McD11] for details on toric symplectic manifolds and the Delzant construction.

3.3.1 Basics

Toric symplectic manifolds are a special case of Hamiltonian T^n-spaces.

Definition 3.3.1. *A Hamiltonian T^n-space (M, ω, μ) is called* toric *if $n = \frac{1}{2} \dim M$ and the action is effective.*

In the case of toric symplectic manifolds the moment map is a quotient map for the torus action, and our choice of normalization in Remark 3.2.2 implies that the barycentre of its moment polytope Δ lies at $0 \in (\mathfrak{t}^n)^*$. Furthermore, by a classical result of Delzant, $\Delta = \mu(M)$ takes a particular form and is, in fact, a sufficient datum to reconstruct (M, ω, μ) along with its T^n-action up to equivariant symplectomorphisms. We will recall Delzant's result and some of the facts surrounding it, since these will be used later on.

Let $\Delta \subset (\mathfrak{t}^n)^*$ be a rational polytope with respect to the standard lattice $(\mathfrak{t}^n_{\mathbb{Z}})^*$ bounded by k hyperplanes. A lattice vector $v \in \mathfrak{t}^n_{\mathbb{Z}}$ is called *primitive* if it cannot be written as a non-trivial integer multiple of another lattice vector. Equivalently, a primitive vector is the first intersection of the line it spans with the standard lattice. We can describe Δ in terms of primitive vectors $v_i \in \mathfrak{t}^n_{\mathbb{Z}}$ and a set of numbers $\kappa_i \in \mathbb{R}$,

$$\Delta = \{\eta \in (\mathfrak{t}^n)^* \mid \langle \eta, v_i \rangle \leq \kappa_i\}. \tag{3.3.9}$$

After identifying $\mathfrak{t}^n$ and $(\mathfrak{t}^n)^*$ with $\mathbb{R}^n$ by the choice of a basis, the vectors v_i correspond to outward pointing primitive normal vectors to the facets. The constants κ_i measure the affine distance of the facets to the origin. Details can be found in [McD11].

3.3.2 The Delzant construction

Definition 3.3.2. *A rational polytope $\Delta \subset (\mathfrak{t}^n)^*$ is called* Delzant *if each of its vertices is formed by the intersection of n hyperplanes whose primitive normal vectors form a $\mathbb{Z}$-basis of $\mathfrak{t}^n_{\mathbb{Z}}$.*

Remark 3.3.3. If we identify $\mathfrak{t}^n_{\mathbb{Z}}$ with $\mathbb{Z}^n$, the Delzant condition on polytopes is equivalent to requiring that the set of primitive normal vectors at any given vertex can be mapped to the standard basis $\{e_1, ..., e_n\} \subset \mathbb{Z}^n$ by an element of $\mathrm{GL}(n, \mathbb{Z})$.

Theorem 3.3.4 ([Del88])**.** *The moment polytope of a toric symplectic manifold is Delzant and there is a bijective correspondence between Delzant polytopes up to $\mathrm{Aut}_{\mathbb{Z}}(\mathfrak{t}^n)^*$-action and toric symplectic manifolds up to T^n-equivariant symplectomorphisms.*

Furthermore, Delzant gave an explicit construction of the toric symplectic manifold (M, ω, μ), starting from a given Delzant polytope $\Delta = \{\eta \in (\mathfrak{t}^n)^* \mid \langle \eta, v_i \rangle \leq \kappa_i\}$ such that $\mu(M) = \Delta$. The desired manifold M is obtained as a symplectic quotient of $(\mathbb{C}^k, \omega_0)$. Since we will heavily rely on the details of this construction, it will be recalled here. Details can be found in the original paper [Del88], or in [CdS03] and [Gui94].

Let $\Delta \subset (\mathfrak{t}^n)^*$ be a Delzant polytope. Up to a translation, we can assume that the normalization convention from Remark 3.2.2 holds. Via the description (3.3.9), the polytope Δ uniquely defines a set of pairs $\{(v_i, \kappa_i)\}_{i \in \{1,...,k\}}$. The *characteristic map* associated to Δ is defined as

$$\pi_* : \mathfrak{t}^k \to \mathfrak{t}^n, \quad \pi_* e_i = v_i, \tag{3.3.10}$$

where e_i denotes the i-th standard basis vector of $\mathfrak{t}^k \cong \mathbb{R}^k$. The characteristic map is thus a linear map of full rank n. Furthermore, it maps $\mathfrak{t}^k_{\mathbb{Z}}$ to $\mathfrak{t}^n_{\mathbb{Z}}$, since the vectors v_i are integral. Hence it descends to the respective tori to yield a map $T^k \to T^n$, which we denote by π. Let $K = \ker \pi \subset T^k$ and denote by $\mathfrak{k}$ and $\mathfrak{k}^*$ its Lie algebra and its dual Lie algebra. We get three short exact sequences,

$$\begin{aligned} 1 \to\ & K \overset{j}{\hookrightarrow} T^k \overset{\pi}{\to} T^n \to 1,\\ 0 \to\ & \mathfrak{k} \overset{j_*}{\to} \mathfrak{t}^k \overset{\pi_*}{\to} \mathfrak{t}^n \to 0,\\ 0 \to\ & (\mathfrak{t}^n)^* \overset{\pi^*}{\to} (\mathfrak{t}^k)^* \overset{j^*}{\to} \mathfrak{k}^* \to 0. \end{aligned} \tag{3.3.11}$$

The desired toric manifold (M, ω) arises as a symplectic quotient of $(\mathbb{C}^k, \omega_0)$ as follows. The moment map

$$\nu_0 : \mathbb{C}^k \to (\mathfrak{t}^k)^* \cong \mathbb{R}^k, \quad (z_1, \dots, z_k) \mapsto \frac{1}{2}\left(|z_1|^2, \dots, |z_k|^2\right) - (\kappa_1, \dots, \kappa_k)$$

generates the standard T^k-action on $\mathbb{C}^k$,

$$t.z = (t_1, \dots, t_k).(z_1, \dots, z_k) = (e^{2\pi i t_1} z_1, \dots, e^{2\pi i t_k} z_k), \quad t \in T^k,\ z \in \mathbb{C}^k. \tag{3.3.12}$$

The inclusion $j : K \hookrightarrow T^k$ induces a K-action on $\mathbb{C}^k$. The moment map corresponding to this K-action is given by

$$\nu : \mathbb{C}^k \to \mathfrak{k}^*, \quad \nu = j^* \circ \nu_0. \tag{3.3.13}$$

One can show that $0 \in \mathfrak{k}^*$ is a regular value of ν and that K acts freely on $\nu^{-1}(0)$. Thus the conditions for symplectic reduction are satisfied. One can show that the symplectic quotient $\nu^{-1}(0)/K$ with its induced symplectic form is the desired toric manifold (M, ω). We will briefly describe how the moment map $\mu : M \to (\mathfrak{t}^n)^*$ defining the toric structure on M is obtained. Combine the symplectic reduction diagram defining M

$$\begin{tikzcd} M & \nu^{-1}(0) \arrow[l, two heads, "p", "/K"'] \arrow[r, hook, "\iota"] & \mathbb{C}^k \end{tikzcd} \tag{3.3.14}$$

with the short exact sequence from (3.3.11) to obtain the commutative diagram

$$\begin{tikzcd} M \arrow[dr, dashed, "\mu"'] & \nu^{-1}(0) \arrow[l, two heads, "p", "/K"'] \arrow[r, hook, "\iota"] \arrow[d, dashed, "\overline{\mu}"] & \mathbb{C}^k \arrow[d, "\nu_0"] \arrow[dr, "\nu"] & & \\ 0 \arrow[r] & (\mathfrak{t}^n)^* \arrow[r, "\pi^*"] & (\mathfrak{t}^k)^* \arrow[r, "j^*"] & \mathfrak{k}^* \arrow[r] & 0. \end{tikzcd} \tag{3.3.15}$$

The map μ is the desired moment map. We will show that both μ and $\overline{\mu}$ are well-defined maps. Since ν is defined as $j^* \circ \nu_0$, the composition $\nu_0 \circ \iota$ maps $\nu^{-1}(0)$ to the kernel of j^* and thus, by exactness of the lower row, to the image of π^*. Since π^* is injective, we obtain a unique map $\overline{\mu} : \nu^{-1}(0) \to (\mathfrak{t}^n)^*$ with

$$\pi^* \circ \overline{\mu} = \nu_0 \circ \iota.$$

Since ι is K-equivariant and ν_0 is T^k-invariant and therefore in particular K-invariant, we obtain that $\overline{\mu}$ is K-invariant. Since $\nu^{-1}(0)$ is a K-principal bundle with base M, this implies that $\overline{\mu}$ factors through M to yield the desired moment map μ defined by the equation

$$\pi^* \circ \mu \circ p = \nu_0 \circ \iota.$$

Example 3.3.5. Let $n = 1$ and take the Delzant polytope $[-1, 1] \subset \mathbb{R} \cong (\mathfrak{t}^1)^*$. Then $k = 2$, the outward pointing normal vectors are given by $v_1 = (1), v_2 = (-1)$, and the corresponding constants are $\kappa_1 = \kappa_2 = 1$. The characteristic map is $\pi = (1, -1)$ and furthermore

$$\nu_0 : \mathbb{C}^2 \to (\mathfrak{t}^2)^* \cong \mathbb{R}^2, \quad (z_1, z_2) \mapsto \left(\frac{1}{2}|z_1|^2 - 1, \frac{1}{2}|z_2|^2 - 1\right).$$

Since $K = \ker \pi = \langle(1, 1)\rangle$, the map j^* is given by projection to the vector $(1, 1)$ and hence

$$\nu(z_1, z_2) = \frac{1}{2}\left(|z_1|^2 + |z_2|^2\right) - 2.$$

Therefore, the level set $\nu^{-1}(0)$ is a 3-sphere on which $K \cong \S^1$ acts diagonally. Hence we obtain the Hopf fibration and the quotient is $M \cong \mathbb{C}P^1 \cong \S^2$ with the K-equivalence classes $[(z_1, z_2)]_K$ corresponding to the homogeneous coordinates $[z_1 : z_2]$ on $\mathbb{C}P^1$.

3.4 Lifting symmetries of the moment polytope

Throughout this section, let (M, ω) be a toric symplectic manifold with moment map μ and moment polytope $\Delta = \mu(M) \subset (\mathfrak{t}^n)^*$ with normalization $\int_M \mu\omega^n = 0 \in (\mathfrak{t}^n)^*$.

3.4.1 Symmetries of the moment polytope

Definition 3.4.1. *Let (M, ω, μ) be a toric symplectic manifold with moment polytope Δ. The group*

$$\mathcal{S}_\Delta = \{\sigma \in \mathrm{Aut}_\mathbb{Z}(\mathfrak{t}^n)^* \mid \sigma(\Delta) = \Delta\} \tag{3.4.16}$$

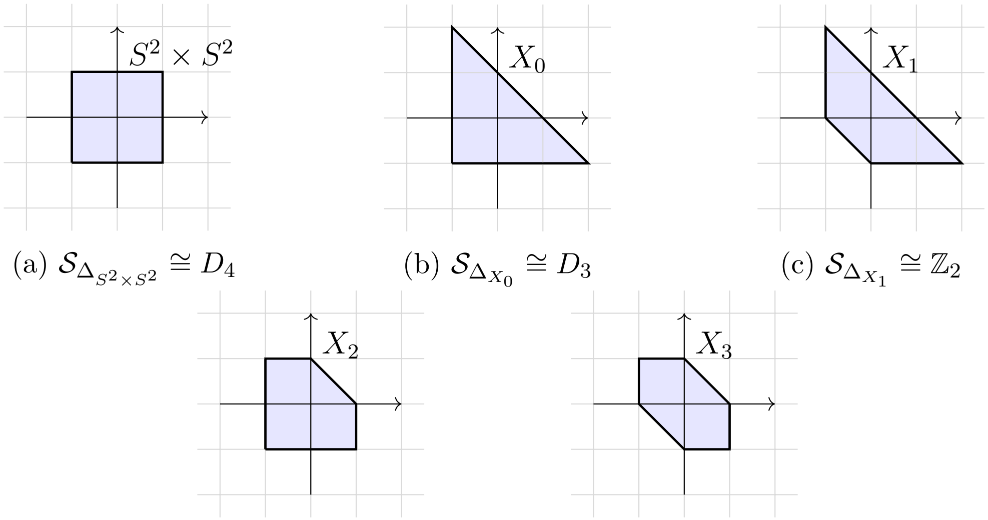
is called the symmetries of Δ.

Example 3.4.2. As discussed in the introduction, there are five toric symplectic del Pezzo surfaces, namely $S^2 \times S^2$ and the blow-ups X_0, X_1, X_2, X_3 of $\mathbb{C}P^2$. Their moment polytopes and the corresponding groups $\mathcal{S}_\Delta$ are given in Figure 3.2, where D_n denotes the dihedral group of order $2n$. These groups are readily found by noting that elements of $\mathrm{GL}(2, \mathbb{Z})$ preserve the affine length of edges. For example after identfying $\mathrm{Aut}_\mathbb{Z}(\mathfrak{t}^2)^*$ with $\mathrm{GL}(2, \mathbb{Z})$, the subgroup $\mathcal{S}_{\Delta_{X_0}} \cong D_3$ is generated by the matrices

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\sigma \in \mathcal{S}_\Delta$. Recall from (3.3.9) that we can associate a unique pair (v_i, κ_i) to each facet of Δ such that

$$\Delta = \{\eta \in (\mathfrak{t}^n)^* \mid \langle \eta, v_i \rangle \leq \kappa_i\}.$$



(a) $\mathcal{S}_{\Delta_{S^2\times S^2}} \cong D_4$ (b) $\mathcal{S}_{\Delta_{X_0}} \cong D_3$ (c) $\mathcal{S}_{\Delta_{X_1}} \cong \mathbb{Z}_2$

(d) $\mathcal{S}_{\Delta_{X_2}} \cong \mathbb{Z}_2$ (e) $\mathcal{S}_{\Delta_{X_3}} \cong D_6$

Figure 3.2: Moment polytope Δ and $\mathcal{S}_\Delta$ for toric symplectic del Pezzo surfaces.

Applying any $\sigma \in \operatorname{Aut}_{\mathbb{Z}}(\mathfrak{t}^n)^*$ to Δ yields

$$\sigma(\Delta) = \{\eta \in (\mathfrak{t}^n)^* \mid \langle \eta, (\sigma^{-1})^* v_i\rangle \leq \kappa_i\}.$$

Hence, applying σ to the moment polytope amounts to applying $(\sigma^{-1})^* \in \operatorname{Aut}_{\mathbb{Z}} \mathfrak{t}^n$ to its associated normal vectors v_i. The hypothesis

$$\sigma(\Delta) = \Delta$$

along with the uniqueness of the set of pairs $\{(v_i, \kappa_i)\}_{1\leq i\leq k}$ thus implies that $(\sigma^{-1})^*$ permutes normal vectors. In conclusion, there is a permutation on k elements $\tau \in S_k$ such that

$$(\sigma^{-1})^* v_i = v_{\tau(i)}, \tag{3.4.17}$$
$$\kappa_i = \kappa_{\tau(i)}. \tag{3.4.18}$$

3.4.2 Lifted symplectomorphisms

Let φ be a compatible symplectomorphism of M. In order to clarify notation, the corresponding homomorphism on T^n will be denoted by φ_{T^n} from now on. It follows from Proposition 3.2.5 that φ descends to a map on Δ. In the following, we will be concerned with proving the opposite direction, namely that symmetries of Δ can be lifted to symplectomorphisms of M.

Lemma 3.4.3. *Let $\sigma \in \mathcal{S}_\Delta$ be a symmetry of the moment polytope of M. Then σ lifts to a compatible symplectomorphism $\varphi^\sigma \in \operatorname{Symp}(M, \omega)$ with*

$$\mu \circ \varphi^\sigma = \sigma \circ \mu. \tag{3.4.19}$$

Our construction below shows that $\varphi^{\sigma\sigma'} = \varphi^\sigma \circ \varphi^{\sigma'}$. Consequently, the symmetries of Δ yield a subgroup of the symplectomorphisms of M, that we identify with $\mathcal{S}_\Delta$.

Corollary 3.4.4. *The group of symmetries* $\mathcal{S}_\Delta$ *is a subgroup of* $\mathrm{Symp}(M, \omega)$.

Remark 3.4.5. The lift φ^σ of σ is not uniquely determined by (3.4.19). Nonetheless, any other $\varphi \in \mathrm{Symp}(M, \omega)$ satisfying $\mu \circ \varphi = \sigma \circ \mu$ is closely related to φ^σ. This will be discussed in Subsection 3.4.4.

Proof of Lemma 3.4.3. The main idea of the proof is to view M as a symplectic quotient of $\mathbb{C}^k$ via Delzant's construction, then to let the permutation $\tau \in S_k$ defined by (3.4.17) act on $\mathbb{C}^k$ by permutation of coordinates, and to check that this map descends to a symplectomorphism on M. Recall that the Delzant construction describes the toric manifold M as a symplectic quotient of $(\mathbb{C}^k, \omega_0)$ by $K = \ker \pi < T^k$. Let $\tau \in S_k$ be the permutation associated to σ via equation (3.4.17). Since we have $\pi_* e_i = v_i$, applying $(\sigma^{-1})^*$ to $\mathfrak{t}^n$ corresponds to permuting coordinates according to τ on $\mathfrak{t}^k$,

$$(\sigma^{-1})^*(\pi_* e_i) = \pi_* e_{\tau(i)}. \tag{3.4.20}$$

Notice that this equation holds on the corresponding tori as well, since all maps involved preserve the corresponding lattices. This leads us to define the following permutations of coordinates

$$\begin{aligned} \Phi : \mathbb{C}^k &\to \mathbb{C}^k, \quad (z_1, \dots, z_k) \mapsto (z_{\tau(1)}, \dots, z_{\tau(k)}), && (3.4.21)\\ \Phi_{T^k} : T^k &\to T^k, \quad (t_1, \dots, t_k) \mapsto (t_{\tau(1)}, \dots, t_{\tau(k)}). && (3.4.22) \end{aligned}$$

The map Φ is symplectic and compatible with the standard T^k-action from (3.3.12),

$$\Phi(t.z) = \Phi_{T^k}(t).\Phi(z), \quad t \in T^k, \ z \in \mathbb{C}^k. \tag{3.4.23}$$

We will prove that Φ is K-compatible as well. This allows us to apply Proposition 3.2.10 to the reduction in the Delzant construction, which yields an induced symplectomorphism $\varphi^\sigma \in \mathrm{Symp}(M, \omega)$ on the quotient M such that the following diagram commutes

$$\begin{tikzcd} M \arrow[loop above, "\varphi^\sigma"] & \nu^{-1}(0) \arrow[l, two heads, "/K"'] \arrow[r, hook] \arrow[loop above, "\Phi|_{\nu^{-1}(0)}"] & (\mathbb{C}^k, \omega_0). \arrow[loop above, "\Phi"] \end{tikzcd} \tag{3.4.24}$$

Since Φ is compatible with the full T^k-action, it suffices to prove that Φ_{T^k} preserves the subgroup K in order to prove that it is K-compatible. The identity (3.4.20) now reads

$$\sigma^* \circ \pi = \pi \circ \Phi_{T^k}^{-1}. \tag{3.4.25}$$

Recall that by definition $K = \ker \pi$ and hence

$$\Phi_{T^k}(K) = \Phi_{T^k}(\ker \pi) = \ker(\pi \circ \Phi_{T^k}^{-1}) = \ker(\sigma^* \circ \pi) = \ker \pi = K, \tag{3.4.26}$$

where we have used elementary properties of $\ker(\cdot)$. This proves that Φ is K-compatible and therefore that $\varphi^\sigma : M \to M$ is a well-defined symplectomorphism.

We now show that the symplectomorphism φ^σ is compatible with the Hamiltonian

T^n-action on M and induces the initially chosen symmetry $\sigma \in \mathcal{S}_\Delta$ on Δ,

$$\mu \circ \varphi^\sigma = \sigma \circ \mu.$$

Since Φ is T^k-compatible, Proposition 3.2.5 yields

$$\nu_0 \circ \Phi = (\Phi_{T^k}^{-1})^* \circ \nu_0. \tag{3.4.27}$$

Adding all of the above maps to (3.3.15) we obtain the following commutative diagram.

$$\begin{tikzcd}
 & M \arrow[loop above, "\varphi^\sigma"] \arrow[dr, "\mu"] & \nu^{-1}(0) \arrow[loop above, "\Phi|_{\nu^{-1}(0)}"] \arrow[l, two heads, "p"', "/K"] \arrow[r, hook, "\iota"] & \mathbb{C}^k \arrow[loop above, "\Phi"] \arrow[d, "\nu_0"] \arrow[dr, "\nu"] & & \\
 & 0 \arrow[r] & (\mathfrak{t}^n)^* \arrow[loop below, "\sigma"] \arrow[r, "\pi^*"] & (\mathfrak{t}^k)^* \arrow[loop below, "(\Phi_{T^k}^{-1})^*"] \arrow[r, "j^*"] & \mathfrak{k}^* \arrow[r] & 0
\end{tikzcd}$$

Since the moment map μ was defined by this diagram, the claim follows. ☐

Example 3.4.6. Let $M = S^2 = \mathbb{C}P^1$ be equipped with its standard toric structure given in Example 3.3.5. The moment polytope is $\Delta = [-1, 1]$ and there is only one non-trivial symmetry $\sigma \in \mathcal{S}_\Delta$ given by $z \mapsto -z$. Since σ exchanges the two normal vectors $v_1 = (1)$ and $v_2 = (-1)$, the corresponding permutation is the non-trivial permuation on two elements and $\Phi(z_1, z_2) = (z_2, z_1)$. This yields the lifted symplectomorphism $\varphi^\sigma([z_1 : z_2]) = [z_2 : z_1]$ on $S^2 \cong \mathbb{C}P^1$. When we view S^2 as embedded in $\mathbb{R}^3$, this symplectomorphism corresponds to the map $(x, y, z) \mapsto (x, -y, -z)$, which obviously induces σ on Δ.

Example 3.4.7. Let $M = \mathbb{C}P^2$ be equipped with its standard toric structure. Then $\mathcal{S}_\Delta$ is isomorphic to the dihedral group D_3. The generators given in Example 3.4.2 correspond to the symplectomorphisms

$$[z_0 : z_1 : z_2] \mapsto [z_2 : z_0 : z_1] \ \text{ and } \ [z_0 : z_1 : z_2] \mapsto [z_0 : z_2 : z_1].$$

3.4.3 Lifted antisymplectic involutions

We restrict our attention to the case where $\sigma \in \mathcal{S}_\Delta$ is an involution and prove Theorem 3.1.1 stated in the introduction.

Theorem 3.4.8. *Let $\sigma \in \mathcal{S}_\Delta$ be an involution on the Delzant polytope of a toric symplectic manifold (M, ω, μ). Then there is an antisymplectic involution R^σ such that*

$$\mu \circ R^\sigma = \sigma \circ \mu. \tag{3.4.28}$$

The idea of proof is as follows. By Lemma 3.4.3, the involution σ lifts to a symplectic involution φ^σ on M. To make this map antisymplectic, we will compose it with a standard antisymplectic involution R^0, coming from the toric structure on M, and apply the following remark

Remark 3.4.9. Let S be a symplectic involution and R an antisymplectic involution on a symplectic manifold M, such that S and R commute. Then $S \circ R$ defines an antisymplectic involution on M.

In order to prove the theorem, we will first construct the antisymplectic involution R^0 on M. This construction is well-known, see for example [DGSD12, Definition 2.6] or [Hau13, Section 2.6]. We will prove it for the convenience of the reader and in order to expose its relation to the Delzant construction.

Proposition 3.4.10. *Let (M, ω, μ) be a toric symplectic manifold. Then there is an antisymplectic involution R^0 which leaves the moment map invariant, i.e.*

$$\mu \circ R^0 = \mu. \tag{3.4.29}$$

Proof. Let

$$\begin{tikzcd} M & \nu^{-1}(0) \arrow[l, two heads, "p"', "/K"] \arrow[r, hook, "\iota"] & \mathbb{C}^k \end{tikzcd}$$

denote the reduction diagram of the Delzant construction. Take the standard antisymplectic involution on $\mathbb{C}^k$ defined by complex conjugation $\rho^0(z_1, ..., z_k) = (\overline{z}_1, ..., \overline{z}_k)$. It descends to an antisymplectic involution on the quotient M satisfying equation 3.4.29. Indeed, we have

$$\rho^0(t.z) = t^{-1}.\rho^0(z), \quad t \in T^k, z \in \mathbb{C}^k.$$

Hence, ρ^0 is compatible in the sense of Definition 3.2.7 and the claim follows from Proposition 3.2.10, since ρ^0 leaves the moment map ν_0 invariant. ☐

Remark 3.4.11. In the context of algebraic geometry, the fixed point set $\operatorname{Fix} R^0$ of the above involution is commonly referred to as *real locus* of the toric variety M. See also [DJ91] for a topological generalization of real toric varieties.

Remark 3.4.12. Alternatively, R^0 can be viewed as follows. The pre-image $\mu^{-1}(\mathring{\Delta}) \subset M$ of the interior of Δ is equivariantly symplectomorphic to the product $T^n \times \mathring{\Delta} \subset T^*T^n$ equipped with the natural symplectic form. In the language of Hamiltonian dynamics, this symplectomorphism is referred to as *global action-angle coordinates*, since it corresponds to Arnold–Liouville coordinates on an open dense subset of M. There is a natural antisymplectic involution on $T^n \times \mathring{\Delta}$ given by group inversion on the T^n-component. This involution extends to all of M and corresponds to R^0.

Proof of Theorem 3.4.8. Let $\sigma \in \mathcal{S}_\Delta$ be an involution of the moment polytope and φ^σ its symplectic lift to M given by Lemma 3.4.3. We will show that φ^σ and R^0 commute and apply Remark 3.4.9. Recall from the proof of Lemma 3.4.3 that φ^σ is induced by a coordinate permutation $\Phi(z_1, \dots, z_k) = (z_{\tau(1)}, \dots, z_{\tau(k)})$ on $\mathbb{C}^k$. Similarly, R^0 is induced by complex conjugation $\rho^0(z_1, ..., z_k) = (\overline{z}_1, ..., \overline{z}_k)$. Since the maps Φ and ρ^0 commute, the corresponding maps φ^σ and R^0 on M commute as well. Hence we can define the antisymplectic involution

$$R^\sigma = R^0 \circ \varphi^\sigma.$$

The compatibility condition (3.4.28) follows from equations (3.4.19) and (3.4.29). ☐

The following alternative view of R^σ will be used in Section 3.5. Define

$$\rho = \rho^0 \circ \Phi : \mathbb{C}^k \to \mathbb{C}^k, \quad (z_1, \dots, z_k) \mapsto (\overline{z}_{\tau(1)}, \dots, \overline{z}_{\tau(k)}). \tag{3.4.30}$$

This is a compatible antisymplectic involution. The corresponding group involution is given by

$$\rho_{T^k} = \rho^0_{T^k} \circ \Phi_{T^k} : T^k \to T^k, \quad (t_1, \dots, t_k) \mapsto (t^{-1}_{\tau(1)}, \dots, t^{-1}_{\tau(k)}). \tag{3.4.31}$$

Since $\rho^0_{T^k}$ and Φ_{T^k} both preserve K (see equation (3.4.26)), so does ρ_{T^k} and hence ρ is K-compatible. By Proposition 3.2.10, ρ induces an antisymplectic involution R^σ on M. By the compatibility of R^σ, there is an involutive automorphism $R^\sigma_{T^n}$ on the torus such that $R^\sigma(t.p) = R^\sigma_{T^n}(t).R^\sigma(p)$. This automorphism is related to ρ_{T^k} via

$$R^\sigma_{T^n} \circ \pi = \pi \circ \rho_{T^k}. \tag{3.4.32}$$

Note that this is a direct analogue of (3.4.25).

3.4.4 Classification of compatible maps

The lifts constructed in Sections 3.4.2 and 3.4.3 are not unique with respect to their respective compatibility conditions. However, any two maps inducing the same symmetry $\sigma \in \mathcal{S}_\Delta$ on the moment polytope are closely related. We start by a lemma which follows from Proposition 3.2.5.

Lemma 3.4.13. *Let φ be a symplectomorphism on a toric symplectic manifold (M, ω, μ) which leaves the moment map invariant, i.e. $\mu \circ \varphi = \mu$. Then φ is of the form*

$$p \mapsto \vartheta(\mu(p)).p \tag{3.4.33}$$

for a smooth map[1] $\vartheta : \Delta \to T^n$.

Symplectomorphisms of the form 3.4.33 will be denoted by $\psi_\vartheta : M \to M$. These maps rotate a given fibre $\mu^{-1}(b)$ by an angle $\vartheta(b) \in T^n$.

Proof of Lemma 3.4.13. Let φ be a symplectomorphism which leaves the moment map invariant. Hence φ preserves the torus fibres, and since T^n acts transitively on each fibre, there is a smooth map $\tilde{\vartheta} : M \to T^n$ such that φ takes the form

$$\varphi(p) = \tilde{\vartheta}(p).p, \quad p \in M.$$

We will prove that $\tilde{\vartheta}(p) = \tilde{\vartheta}(p')$ whenever p and p' lie in the same fibre, and thus $\tilde{\vartheta}$ factors through Δ to yield a map $\vartheta : \Delta \to T^n$. By (3.2.5), the symplectomorphism φ is T^n-equivariant, i.e.

$$\varphi(t.p) = t.\varphi(p), \quad t \in T^n, p \in M.$$

Now let $p, p' \in M$ be points in the fibre over $b \in \Delta$. Since T^n acts transitively on $\mu^{-1}(b)$,

---

[1] We define $\vartheta : \Delta \to T^n$ to be *smooth* if it comes from a smooth map $\tilde{\vartheta} : M \to T^n$ which satisfies $\vartheta \circ \mu = \tilde{\vartheta}$.

there is $t \in T^n$ such that $p' = t.p$. Using the T^n-equivariance of φ, we compute

$$\tilde{\vartheta}(p').p' = \varphi(p') = \varphi(t.p) = t.\varphi(p) = (t\tilde{\vartheta}(p)).p = (\tilde{\vartheta}(p)t).p = \tilde{\vartheta}(p).p'.$$

Since the action of T^n is free on an open dense subset of M, we deduce that $\tilde{\vartheta}(p) = \tilde{\vartheta}(p')$ and thus $\tilde{\vartheta}$ is constant on fibres. ☐

Lemma 3.4.13 allows us to classify compatible symplectomorphisms as well as compatible antisymplectic involutions. We will use the convention established in 3.4.2 and 3.4.3 and denote the standard lift of $\sigma \in \mathcal{S}_\Delta$ by φ^σ and R^σ, respectively.

Proposition 3.4.14. *Let φ be a compatible symplectomorphism on M such that $\mu \circ \varphi = \sigma \circ \mu$, for $\sigma \in \mathcal{S}_\Delta$. Then there is a smooth map $\vartheta : \Delta \to T^n$ such that*

$$\varphi = \psi_\vartheta \circ \varphi^\sigma.$$

Proof. By the construction of φ^σ, we have $\mu \circ \varphi^\sigma = \sigma \circ \mu$ and hence we can apply Lemma 3.4.13 to the symplectomorphism $\varphi \circ (\varphi^\sigma)^{-1}$ to prove the claim. ☐

Proposition 3.4.15. *Let R be a compatible antisymplectic involution on M such that $\mu \circ R = \sigma \circ \mu$, for $\sigma \in \mathcal{S}_\Delta$. Then there is a smooth map $\vartheta : \Delta \to T^n$ such that*

$$R = \psi_\vartheta \circ R^\sigma. \tag{3.4.34}$$

Furthermore, ϑ satisfies $R^\sigma_{T^n}(\vartheta(\sigma(x))) = \vartheta(x)^{-1}$ for all $x \in \Delta$.

Proof. Again, apply Lemma 3.4.13 to the symplectomorphism $R \circ (R^\sigma)^{-1}$, to prove the first claim. For the condition on ϑ, we use (3.4.34) and compute, for $p \in \mu^{-1}(x)$,

$$p = R(R(p)) = (\psi_\vartheta \circ R^\sigma \circ \psi_\vartheta \circ R^\sigma)(p) = (\vartheta(x)R^\sigma_{T^n}(\vartheta(\sigma(x))).p.$$

The torus T^n acts freely on an open dense set in M, see for example Remark 3.4.12, and thus the claim follows. ☐

Remark 3.4.16. Let $R_1, R_2 \colon M \to M$ be two compatible antisymplectic involutions satisfying $\mu \circ R = \sigma \circ \mu$, for some $\sigma \in \mathcal{S}_\Delta$. Even though they are related by (3.4.34), their respective fixed point sets may have different topology. For example, take $M = \mathbb{C}P^3$ equipped with its standard toric structure and define

$$\begin{aligned} R_1[z_0 : z_1 : z_2 : z_3] &= [\bar{z}_1 : \bar{z}_0 : \bar{z}_3 : \bar{z}_2], \\ R_2[z_0 : z_1 : z_2 : z_3] &= [-\bar{z}_1 : \bar{z}_0 : -\bar{z}_3 : \bar{z}_2]. \end{aligned}$$

Both R_1 and R_2 are compatible with $\sigma(x, y, z) = (-x - y - z, z, y)$, but $\operatorname{Fix} R_1 \cong \mathbb{R}P^3$ and $\operatorname{Fix} R_2 = \varnothing$. See Example 3.5.5 and [Kim21b, Example 2.4] for details.

3.5 Real Delzant construction

In this section, we describe in detail the real Delzant construction stated in Section 3.1. Let (M^{2n}, ω, μ) be a toric symplectic manifold with moment polytope $\Delta = \mu(M)$. By the Delzant construction (Section 3.3.2), we can write

$$M = \nu^{-1}(0)/K,$$

where ν is defined as in (3.3.13) and $\pi \colon T^k \to T^n$ is the characteristic map with kernel $K = \ker \pi$. Let $R = R^\sigma$ be the standard antisymplectic involution on M given by the lift of an involution $\sigma \in \mathcal{S}_\Delta$ from Theorem 3.4.8. By Proposition 3.2.5 there is a group involution R_{T^n} of T^n satisfying (3.2.5). Recall that

$$\rho_{T^k}(t_1, \dots, t_k) = (t_{\tau(1)}^{-1}, \dots, t_{\tau(k)}^{-1})$$

is the group involution of T^k defined in equation (3.4.31). Here $\tau \in S_k$ is the permutation satisfying (3.4.17). By Equation 3.4.32, we see that $\pi\big(\operatorname{Fix}(\rho_{T^k})\big) \subset \operatorname{Fix}(R_{T^n})$ and thus we can define

Definition 3.5.1. *The* characteristic map π_R *associated to* $L = \operatorname{Fix}(R)$ *is the group homomorphism defined as the restriction*

$$\pi_R := \pi|_{\operatorname{Fix}(\rho_{T^k})} \colon \operatorname{Fix}(\rho_{T^k}) \to \operatorname{Fix}(R_{T^n}).$$

We write $K_R := \ker \pi_R = K \cap \operatorname{Fix}(\rho_{T^k})$ and denote its Lie algebra by $\mathfrak{k}_R$. Recall that

$$\rho(z_1, \dots, z_k) = (\bar{z}_{\tau(1)}, \dots, \bar{z}_{\tau(k)})$$

is the antisymplectic involution of $\mathbb{C}^k$ given in (3.4.30).We construct the real analogue of the moment map $\nu \colon \mathbb{C}^k \to \mathfrak{k}^*$ defined in (3.3.13) as follows. We define

$$\nu_R \colon \operatorname{Fix}(\rho) \to (\mathfrak{k}/\mathfrak{k}_R)^*, \quad \nu_R(z)[\xi]_{\mathfrak{k}_R} := \langle \nu(z), \xi \rangle \quad \text{for } [\xi]_{\mathfrak{k}_R} \in \mathfrak{k}/\mathfrak{k}_R.$$

It follows from Lemma 3.5.2 below that ν_R is well-defined. Recall from Section 3.4.3 that ρ_{T^k} preserves K and hence we write

$$\rho_K := \rho_{T^k}|_K \colon K \to K$$

for the group involution on K with $\operatorname{Fix}(\rho_K) = K_R$. Using the fact that the tangent space of the fixed point set of any smooth involution ϕ is the 1-eigenspace of its differential ϕ_*, namely $T_x \operatorname{Fix}(\phi) = \operatorname{Fix}(\phi_*)$ for $x \in \operatorname{Fix}(\phi)$, we deduce that $\operatorname{Fix}((\rho_K)_*) = \mathfrak{k}_R$ and $\operatorname{Fix}(\rho_K^*) = \mathfrak{k}_R^*$.

Lemma 3.5.2. *We have* $\langle \nu(z), \xi \rangle = 0$ *for all* $z \in \operatorname{Fix}(\rho)$ *and* $\xi \in \mathfrak{k}_R$.

Proof. Applying Proposition 3.2.5 to the maps ν_0 and ρ, we have $\nu_0 \circ \rho = -\rho_{T^k}^* \circ \nu_0$.

We verify that for $z \in \mathrm{Fix}(\rho)$,

$$\begin{aligned}\nu(z) &= \nu(\rho(z)) \\ &= (j^* \circ \nu_0 \circ \rho)(z) \\ &= -(j^* \circ \rho_{T^k}^* \circ \nu_0)(z) \\ &= -(\rho_K^* \circ j^* \circ \nu_0)(z) \\ &= -\rho_K^*(\nu(z)).\end{aligned}$$

Now, for any $\xi \in \mathfrak{k}_R$ we see that

$$\langle \nu(z), \xi \rangle = -\langle \rho_K^*\big(\nu(z)\big), \xi \rangle = -\langle \nu(z), (\rho_K)_*(\xi) \rangle = -\langle \nu(z), \xi \rangle,$$

which yields $\langle \nu(z), \xi \rangle = 0$.

We observe that

$$\nu_R^{-1}(0) = \nu^{-1}(0) \cap \mathrm{Fix}(\rho) = \mathrm{Fix}(\rho|_{\nu^{-1}(0)}). \tag{3.5.35}$$

Since $\nu_R^{-1}(0)$ is given by the fixed point set of the involution $\rho|_{\nu^{-1}(0)}$ and $\nu^{-1}(0)$ is compact, $\nu_R^{-1}(0)$ is a closed submanifold of $\nu^{-1}(0)$.

Remark 3.5.3. In the spirit of the Delzant construction, one can also prove that $0 \in (\mathfrak{k}/\mathfrak{k}_R)^*$ is a regular value of $\nu_R\colon \mathrm{Fix}(\rho) \to (\mathfrak{k}/\mathfrak{k}_R)^*$. Hence, $\nu_R^{-1}(0)$ is a closed submanifold of $\nu^{-1}(0)$ of dimension $n + \dim K_R$.

Since the map ν is T^k-invariant and K acts freely on $\nu^{-1}(0)$, the action of K_R on $\nu_R^{-1}(0)$ is well-defined and free. As a result, the quotient $\nu_R^{-1}(0)/K_R$ is a closed manifold. The natural inclusion

$$\nu_R^{-1}(0) \hookrightarrow \nu^{-1}(0)$$

induces the well-defined smooth map on the quotients

$$I\colon \nu_R^{-1}(0)/K_R \to \nu^{-1}(0)/K, \quad I(K_R z) = Kz.$$

Recall that $M = \nu^{-1}(0)/K$.

Lemma 3.5.4. *The map* $I\colon \nu_R^{-1}(0)/K_R \to \nu^{-1}(0)/K$ *is an embedding.*

Proof. Since $\nu_R^{-1}(0)/K_R$ is compact, it suffices to show that I and its differential are injective.

Let $z_1, z_2 \in \nu_R^{-1}(0)$ such that $I(K_R z_1) = I(K_R z_2)$. Then there is an element $t \in K$ such that $t.z_1 = z_2$. Applying the involution ρ to both sides, we get

$$\rho_K(t).z_1 = \rho_K(t).\rho(z_1) = \rho(z_2) = z_2 = t.z_1.$$

Since K acts freely on $\nu^{-1}(0)$, we conclude that $\rho_K(t) = t$ and hence $t \in \mathrm{Fix}(\rho_K) = K_R$. Therefore $K_R z_1 = K_R z_2$.

We are left with showing that the differential of I is injective. Notice that

$$T_z\big(\nu_R^{-1}(0)/K_R\big) \cong T_z \nu_R^{-1}(0)/T_z K_R z$$

and

$$T_z\big(\nu^{-1}(0)/K\big) \cong T_z\nu^{-1}(0)/T_zKz$$

for all $z \in \nu_R^{-1}(0)$. In order to prove that the differential of I is injective, it suffices to prove that T_zK_Rz is a subspace of T_zKz, which follows from

$$T_zK_Rz = T_zKz \cap T_z\nu_R^{-1}(0). \tag{3.5.36}$$

To prove this identity, consider the following representations of tangent spaces

$$\begin{aligned} T_zK_Rz &= \left\{X \in T_z\nu_R^{-1}(0) \mid X = \frac{d}{dt}\exp(t\xi).z,\ \xi \in \mathfrak{k}_R\right\} \\ &= \left\{X \in T_z\nu^{-1}(0) \mid X = \frac{d}{dt}\exp(t\xi).z,\ \xi \in \mathfrak{k}_R \text{ and } \rho_*X = X\right\}, \end{aligned}$$

$$T_zKz \cap T_z\nu_R^{-1}(0) = \left\{X \in T_z\nu^{-1}(0) \mid X = \frac{d}{dt}\exp(t\xi).z,\ \xi \in \mathfrak{k} \text{ and } \rho_*X = X\right\}.$$

Let $\xi \in \mathfrak{k}$ and $X = \frac{d}{dt}\exp(t\xi).z$ such that $\rho_*X = X$. By direct computation, we verify that

$$\frac{d}{dt}\exp(t\xi).z = \frac{d}{dt}\exp((\rho_K)_*t\xi).z.$$

Since the action of K on $\nu^{-1}(0)$ is free, we have that $(\rho_K)_*\xi = \xi$, and so (3.5.36) follows.

We are now in a position to prove the main theorem of the paper. *Proof.* [Proof of Theorem 3.1.2] Since we already know that $I\colon \nu_R^{-1}(0)/K_R \to M$ is an embedding, it suffices to show that $\mathrm{Fix}(R) = I(\nu_R^{-1}(0)/K_R)$. We prove this claim by double inclusion. Recall from Section 3.4.3 that ρ is K-compatible with ρ_{T^k}, whence $R\colon M \to M$ is given by

$$R(Kz) = K\rho(z) \quad \text{for } z \in \nu^{-1}(0).$$

Let $K_Rz \in \nu_R^{-1}(0)/K_R$. Since $\rho(z) = z$, we have $R(Kz) = Kz = I(K_Rz)$. This shows that $I(\nu_R^{-1}(0)/K_R) \subset \mathrm{Fix}(R)$, and hence $\mathrm{Fix}(R)$ is not empty. To prove the other inclusion, let $Kz \in \mathrm{Fix}(R)$. Since $R(Kz) = K\rho(z) = Kz$, there exists $t \in K$ such that $\rho(z) = t.z$. We observe that

$$z = \rho(t.z) = \rho_K(t).\rho(z) = \rho_K(t)t.z.$$

Since the K-action is free, we have $\rho_K(t) = t^{-1}$.

Claim. *There exists $\tilde t \in K$ such that $\tilde t^2 = t$ and $\rho_K(\tilde t) = \tilde t^{-1}$.*

The claim is obvious in case $t = 1$ and thus assume that $t \neq 1$. Denote by $S^1\langle t\rangle$ the subgroup of T^n generated by $t \in T^n$. Since K is a subtorus of T^n, we have $S^1\langle t\rangle \leq K$. Consider the group homomorphism $\phi\colon S^1\langle t\rangle \to K$ defined by $\phi(s) = \rho_K(s)s$. Since $\phi(t) = 1$, we have $\phi(s) = 1$ for all $s \in S^1\langle t\rangle$. Therefore, any choice of $\tilde t \in S^1\langle t\rangle$ with $\tilde t^2 = t$ satisfies $\rho_K(\tilde t) = \tilde t^{-1}$ as claimed.

Finally, we verify that

$$\rho(\tilde t.z) = \rho_K(\tilde t).\rho(z) = \rho_K(\tilde t)t.z = \rho_K(\tilde t)\tilde t^2.z = \tilde t.z,$$

and hence $\tilde t.z \in \nu_R^{-1}(0)$ by (3.5.35). Since $Kz = K(\tilde t.z) = I(K_R(\tilde t.z))$, this implies that

$\mathrm{Fix}(R) \subset I(\nu_R^{-1}(0)/K_R)$, which completes the proof.

Example 3.5.5. Consider complex projective space $\mathbb{C}P^3$ with moment map

$$\mu[z_0 : z_1 : z_2 : z_3] = \frac{4}{\|z\|^2}(|z_1|^2, |z_2|^2, |z_3|^2) - (1,1,1),$$

where $\|z\|^2 = \sum_{j=0}^3 |z_j|^2$. We subtract $(1,1,1)$ in order for the normalization $\int_{\mathbb{C}P^3} \mu\omega^3 = 0$ to hold. The moment polytope is the 3-simplex given as the convex hull of the vectors

$$\{(-1,-1,-1),(-1,-1,3),(-1,3,-1),(3,-1,-1)\}\,.$$

Let $\sigma \in \mathcal{S}_\Delta$ be the involution defined by $\sigma(x,y,z) = (-x-y-z, z, y)$. Its standard lift is given by the antisymplectic involution

$$R^\sigma[z_0 : z_1 : z_2 : z_3] = [\bar{z}_1 : \bar{z}_0 : \bar{z}_3 : \bar{z}_2].$$

Using Theorem 3.1.2, we verify that $\mathrm{Fix}(R^\sigma)$ is diffeomorphic to $\mathbb{R}P^3$. Figure 3.3 describes the fixed point set of σ in the moment polytope Δ of $\mathbb{C}P^3$. We take the primitive outward pointing normal vectors of each facet of Δ,

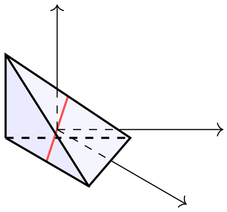


Figure 3.3: $\mathrm{Fix}(\sigma)$ in Δ.

$$v_1 = (0,-1,0), \quad v_2 = (0,0,-1), \quad v_3 = (-1,0,0), \quad v_4 = (1,1,1),$$

with $\kappa_1 = \cdots = \kappa_4 = 1$, and hence

$$\pi = \begin{pmatrix} 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

Since $\sigma^* v_1 = v_2$ and $\sigma^* v_3 = v_4$, we obtain two involutions

$$\begin{aligned} \rho_{T^4}(t_1,t_2,t_3,t_4) &= (t_2^{-1}, t_1^{-1}, t_4^{-1}, t_3^{-1}), \\ \rho(z_1,z_2,z_3,z_4) &= (\bar{z}_2, \bar{z}_1, \bar{z}_4, \bar{z}_3). \end{aligned}$$

One can direct check that

$$\begin{aligned} \mathrm{Fix}(\rho_{T^4}) &= \{(t, t^{-1}, s, s^{-1}) \mid t, s \in S^1\}, \\ \mathrm{Fix}(\rho) &= \{(z, \bar{z}, w, \bar{w}) \mid z, w \in \mathbb{C}\}. \end{aligned}$$

Since

$$\begin{aligned} K &= \ker \pi = \{(t,t,t,t) \mid t \in S^1\} \cong S^1, \\ \nu^{-1}(0) &= \left\{ (z_1,z_2,z_3,z_4) \mid \sum_{j=1}^{4} |z_j|^2 = 2 \right\}, \end{aligned}$$

we obtain

$$\begin{aligned} K_R &= K \cap \mathrm{Fix}(\rho_{T^4}) = \{(1,1,1,1),(-1,-1,-1,-1)\} \cong \mathbb{Z}_2, \\ \nu_R^{-1}(0) &= \nu^{-1}(0) \cap \mathrm{Fix}(\rho) = \{(z,\bar{z},w,\bar{w}) \mid |z|^2+|w|^2=1\} \cong S^3. \end{aligned}$$

Since K_R acts by the antipodal action on $\nu_R^{-1}(0)$, we deduce that $\nu_R^{-1}(0)/K_R \cong \mathbb{R}P^3$.

3.6 Convexity and Tightness

In this section we shall prove Theorem 3.1.3. We follow the same setup as in Section 3.5. Let (M,ω,μ) be a toric symplectic manifold with moment polytope $\Delta = \mu(M)$. By the Delzant construction, we can write $M = \nu^{-1}(0)/K$. Suppose that R is the antisymplectic involution of M which is the lift of an involution $\sigma \in \mathcal{S}_\Delta$, see Theorem 3.4.8.

We know that $\mathrm{Fix}(\sigma) = \Delta \cap \{x \in \mathfrak{t}^* \mid \sigma(x) = x\}$ and that the Delzant polytope is convex. Since the intersection of two convex sets is again convex, so is $\mathrm{Fix}(\sigma)$.

Theorem 3.6.1. *We have* $\mathrm{Fix}(\sigma) = \mu(L)$. *In particular, L is nonempty and $\mu(L)$ is convex.*

Proof. Let $x \in \mu(L)$. Then there is an element $Kz \in M = \nu^{-1}(0)/K$ such that $R(Kz) = Kz$ and $\mu(Kz) = x$. Since R is compatible, we obtain

$$\sigma(x) = \sigma(\mu(Kz)) = \mu(R(Kz)) = \mu(Kz) = x.$$

This imples that $\mu(L) \subset \mathrm{Fix}(\sigma)$.

Let $x \in \mathrm{Fix}(\sigma) \subset \Delta$ and let $Kz \in M$ with $\mu(Kz) = x$. We show that there is $\tilde{t} \in T^n$ such that $\tilde{t}.Kz \in L$. Note that

$$\mu(K\rho(z)) = \mu(R(Kz)) = \sigma(\mu(Kz)) = \sigma(x) = x = \mu(Kz),$$

and hence there is an element $t \in T^n$ such that $K\rho(z) = t.Kz$. This follows from the fact that T^n acts transitively on fibres. Applying the involution R, we obtain

$$Kz = R_{T^n}(t)t.Kz. \tag{3.6.37}$$

Claim. *There exists $\tilde{t} \in T^n$ such that $\tilde{t}^2 = t$ and $Kz = R_{T^n}(\tilde{t})\tilde{t}.Kz$.*

We follow the arguments used in the proof of Theorem 3.1.2, and thus assume that $t \neq 1$. Again, denote by $S^1\langle t\rangle$ the subgroup of T^n generated by $t \in T^n$ and consider the group homomorphism

$$\phi \colon S^1\langle t\rangle \to T^n, \quad \phi(s) = R_{T^n}(s)s.$$

Since the stabilizer $\mathrm{Stab}(Kz)$ of the T^n-action at the point $Kz \in M$ is a subtorus and, by (3.6.37), $\phi(t) = R_{T^n}(t)t \in \mathrm{Stab}(Kz)$, we have $\mathrm{im}\,\phi \leq \mathrm{Stab}(Kz)$. If we choose $\tilde{t} \in S^1\langle t\rangle$ such that $\tilde{t}^2 = t$, then $\phi(\tilde{t}) = R_{T^n}(\tilde{t})\tilde{t} \in \mathrm{Stab}(Kz)$. Hence, the claim follows.

In order to show $\tilde{t}.Kz \in L = \mathrm{Fix}(R)$, we verify

$$R(\tilde{t}.Kz) = R_{T^n}(\tilde{t}).R(Kz) = R_{T^n}(\tilde{t})t.Kz = R_{T^n}(\tilde{t})\tilde{t}^2.Kz = \tilde{t}.Kz.$$

This completes the proof.

We denote the set of critical points of $f \in C^\infty(M)$ by $\mathrm{Crit}(f)$. We recall Duistermaat's tightness theorem [Dui83, Theorem 3.1].

Theorem 3.6.2 (Duistermaat)**.** *Let (M, ω, μ) be a compact connected Hamiltonian T^n-space. Suppose that R is an antisymplectic involution on M such that $\mu \circ R = \mu$ and $L = \mathrm{Fix}(R)$ is nonempty. For any $\xi \in \mathfrak{t}$ we have*

$$\dim H_*(L; \mathbb{Z}_2) = \dim H_*(\mathrm{Crit}(H_\xi|_L); \mathbb{Z}_2).$$

Our tightness result is a corollary of this theorem. *Proof.* [Proof of Theorem 3.1.3] The convexity result follows from Theorem 3.6.1. To prove the tightness, let σ^* denote the transpose of $\sigma \in \mathrm{Aut}_\mathbb{Z}(\mathfrak{t}^n)^*$. Then σ^* is an involution on $\mathfrak{t}^n$ and for any $\xi \in \mathfrak{t}^n$ we can decompose ξ as $\xi = \xi_1 + \xi_2$ with $\sigma^*\xi_1 = -\xi_1$ and $\sigma^*\xi_2 = \xi_2$. Note that $H_\xi = H_{\xi_1} + H_{\xi_2}$. If $x \in \mathrm{Fix}(R)$, we have

$$\langle \mu(x), \xi_1\rangle = \langle \mu(R(x)), \xi_1\rangle = \langle \sigma(\mu(x)), \xi_1\rangle = \langle \mu(x), \sigma^*\xi_1\rangle = \langle \mu(x), -\xi_1\rangle,$$

which implies

$$H_{\xi_1}|_L \equiv 0. \tag{3.6.38}$$

Furthermore, we see that

$$\langle \mu \circ R, \xi_2\rangle = \langle \sigma \circ \mu, \xi_2\rangle = \langle \mu, \sigma^*\xi_2\rangle = \langle \mu, \xi_2\rangle.$$

We take the subtorus $T_0 \overset{j}{\hookrightarrow} T^n$ such that $\mathrm{Lie}(T_0) = \mathfrak{t}_0 = \{\xi \in \mathfrak{t}^n \mid \sigma^*\xi = \xi\}$, i.e., T_0 is the identity component of $\mathrm{Fix}(\sigma^*)$. Then the induced T_0-action on M is Hamiltonian and has moment map $\tilde{\mu} := j^*\mu$ with $\tilde{\mu} \circ R = \tilde{\mu}$. Since $\xi_2 \in \mathfrak{t}_0$ by Theorem 3.6.2 and (3.6.38) we obtain

$$\begin{aligned} \dim H_*(L; \mathbb{Z}_2) &= \dim H_*(\mathrm{Crit}(H_{\xi_2}|_L); \mathbb{Z}_2) \\ &= \dim H_*(\mathrm{Crit}(H_\xi|_L); \mathbb{Z}_2). \end{aligned}$$

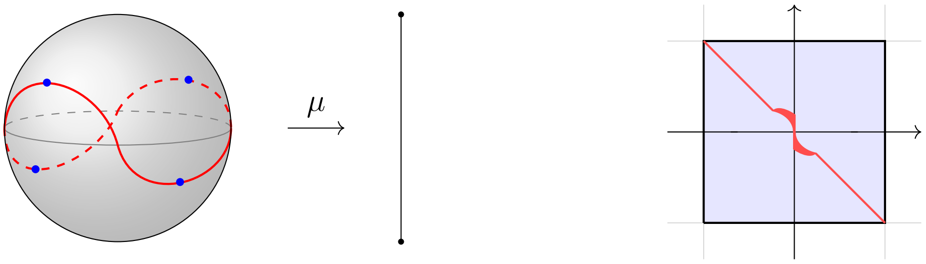
This completes the proof.

The following example illustrates that the tightness and convexity do not hold if we drop the compatible condition on the real Lagrangian $L = \mathrm{Fix}(R)$.

Example 3.6.3. Consider the two-sphere S^2 equipped with the Euclidean area form. Any embedded loop in S^2 dividing S^2 into two discs of equal area is a real Lagrangian. Pick $\xi = 1 \in \mathfrak{t} \cong \mathbb{R}$ so that $\mu = H_\xi$. It is not difficult to find a real Lagrangian L in S^2

such that $\mathrm{Crit}(H_\xi|_L)$ consists of four critical points, see Figure 3.4a. Hence, tightness fails for the real Lagrangian $L \cong S^1$, namely,

$$\dim H_*(L;\mathbb{Z}_2) = 2 \neq 4 = \dim H_*(\mathrm{Crit}(H_\xi|_L);\mathbb{Z}_2).$$



(a) The real Lagrangian L satisfies $\#\,\mathrm{Crit}(H_\xi|_L) = 4$.

(b) The image $\mu(L)$ is not convex

Figure 3.4: Examples in which convexity or tightness fail.

One can easily find a real Lagrangian torus L in $S^2 \times S^2$ such that $\mu(L)$ is not convex. Indeed, let

$$L' = \mathrm{Fix}(R') = \{(x,-x) \mid x \in S^2\}$$

be the real Lagrangian in $S^2 \times S^2$, where $R'(x,y) = (-y,-x)$. Note that $\mu(L') = \{(\xi,-\xi) \mid \xi \in \mathfrak{t}^*\} \cap \square$ with $\square = \mu(S^2 \times S^2)$. Then one can choose a suitable Hamiltonian diffeomorphism ϕ on $S^2 \times S^2$ which is compactly supported in $\mu^{-1}(U_0)$, where $U_0 \subset \square$ is a small open set of the origin, such that the antisymplectic involution $R := \phi \circ R' \circ \phi^{-1}$ has fixed point set $\mathrm{Fix}(R) = \phi(\mathrm{Fix}(R'))$ whose moment image is wiggled near the origin. See Figure 3.4b.

3.7 Real Lagrangians in toric symplectic del Pezzo surfaces

As an application of our real Delzant construction, we study real Lagrangians in toric symplectic del Pezzo surfaces. Recall that a *symplectic del Pezzo surface* is one the following symplectic 4-manifolds:

1. $Q := S^2 \times S^2$ the product of the 2-sphere (S^2,ω), where ω denotes an area form on S^2,
2. $X_k := \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ the k-fold monotone symplectic blow-up of $\mathbb{C}P^2$ for $0 \leq k \leq 8$.

Every closed monotone symplectic 4-manifold is one of the symplectic del Pezzo surfaces and that the monotone symplectic structures on del Pezzo surfaces are unique, see [Via16, Section 1] for the references.

3.7.1 Topological obstructions on real Lagrangian surfaces

The following is an analogue of [Eva10, Lemma 2.3], which gives a homological obstruction for being Lagrangian in symplectic del Pezzo surfaces.

Lemma 3.7.1. *Symplectic del Pezzo surfaces contain no Lagrangian Σ_g for all $g \geq 2$, where Σ_g denotes the closed oriented surface of genus g. Furthermore, X_1 does not contain any Lagrangian sphere.*

Proof. Let L be an orientable Lagrangian in a symplectic del Pezzo surface X. It suffices to show that $\chi(L) \geq 0$. Since L is Lagrangian and X is monotone, the homology class $[L] \in H_2(M;\mathbb{Z})$ satisfies

$$c_1(X)[L] = 0, \quad [L]\cdot[L] = -\chi(L). \tag{3.7.39}$$

The second property follows from the Weinstein neighborhood theorem, which asserts that the normal bundle of L is isomorphic to T^*L, and from $\dim L = 2$.

Case of Q. The first Chern class $c_1(Q)$ is Poincaré dual to $2\alpha+2\beta$, where $\alpha = [S^2\times\{pt\}]$ and $\beta = [\{pt\}\times S^2]$ are generators of $H_2(Q;\mathbb{Z})$. Let $[L] = a\alpha + b\beta$ for $a, b \in \mathbb{Z}$. Then the identities (3.7.39) become

$$2a + 2b = 0, \quad 2ab = -\chi(L),$$

which shows that $\chi(L) = 2b^2 \geq 0$.

Case of X_k for $1 \leq k \leq 8$. Note that $c_1(X_k)$ is Poincaré dual to the class

$$3H - \sum_{j=1}^{k} E_j \in H_2(X_k;\mathbb{Z}),$$

where $H = [\mathbb{C}P^1]$ and the E_j are the classes of the exceptional spheres. Write $[L] = aH - \sum_{j=1}^{k} b_j E_j$. Equation (3.7.39) becomes

$$3a - \sum_{j=1}^{k} b_j = 0, \quad a^2 - \sum_{j=1}^{k} b_j^2 = -\chi(L),$$

which yields

$$9\sum_{j=1}^{k} b_j^2 - \left(\sum_{j=1}^{k} b_j\right)^2 = 9\cdot\chi(L).$$

This identity can be rewritten as

$$9\chi(L) = (9-k)\sum_{j=1}^{k} b_j^2 + \sum_{i<j}(b_i - b_j)^2.$$

Since $k \leq 8$ we conclude that $\chi(L) \geq 0$ also in this case.

In order to prove the last statement, note that if L were a Lagrangian sphere in X_1 with $[L] = aH - bE_1$, then by the second property in equation (3.7.39) we would have $a^2 - b^2 = -2$.

Remark 3.7.2. Using more sophisticated methods, Welschinger [Wel05, Theorem 2.2] proved a much stronger result, namely that there are no orientable hyperbolic Lagrangian surfaces (i.e., Lagrangian Σ_g with $g \geq 2$) in (not necessarily monotone) rational and ruled symplectic 4-manifolds. The fact that there is no Lagrangian sphere in the blow-up of $\mathbb{C}P^2$ does not require monotonicity.

Smith theory [Bre72, Theorems 4.1 and 4.3, Chapter III] implies that any real Lagrangian L in a symplectic manifold (M, ω) satisfies

$$\begin{aligned} \dim H_*(M;\mathbb{Z}_2) &\geq \dim H_*(L;\mathbb{Z}_2), \\ \chi(M) &= \chi(L) \mod 2. \end{aligned}$$

Together with Lemma 3.7.1 one obtains Table 3.1 for the candidates of diffeomorphism types except for the cases of $\mathbb{R}P^2\#\mathbb{R}P^2$ in $S^2 \times S^2$ and T^2 in X_1. We are only interested in the symplectic del Pezzo surfaces that have a toric structure, namely $S^2 \times S^2$ and X_k for $0 \leq k \leq 3$. By [Kim21b, Lemma 4.4], we can exclude $\mathbb{R}P^2\#\mathbb{R}P^2$ in $S^2 \times S^2$ in Table 3.1.

In order to show that T^2 cannot be a real Lagrangian in X_1, we employ the Arnold lemma which we now explain. We refer to [Arn71] for details. Let τ be an orientation-preserving involution of a closed oriented manifold X^4. Assume that the fixed point set $\mathrm{Fix}(\tau)$ of τ is a closed surface. The involution τ induces the isomorphism $\tau_* \colon H_2(X;\mathbb{Z}_2) \to H_2(X;\mathbb{Z}_2)$. We define the symmetric $\mathbb{Z}_2$-bilinear form Φ_τ (called *the twisted intersection form*) on $H_2(X;\mathbb{Z}_2)$ by

$$\Phi_\tau(\alpha, \beta) = \alpha \cdot \tau_*(\beta) \mod 2,$$

where $\cdot$ denotes the intersection number. Recall that $w \in H_2(X;\mathbb{Z}_2)$ is called *characteristic class (or fundamental class)* of Φ_τ if $\Phi_\tau(w, \alpha) = \Phi_\tau(\alpha, \alpha)$ for all $\alpha \in H_2(X;\mathbb{Z}_2)$. Since Φ_τ is non-degenerate, there exists a unique characteristic class of Φ_τ. Note that the characteristic class of Φ_τ vanishes if and only if $\Phi_\tau(\alpha, \alpha) = 0$ for all $\alpha \in H_2(X;\mathbb{Z}_2)$. The following is the so-called *Arnold lemma*, see [Arn71, Lemma 3] for the proof.

Lemma 3.7.3. *The $\mathbb{Z}_2$-homology class $[\mathrm{Fix}(\tau)]_{\mathbb{Z}_2} \in H_2(X;\mathbb{Z}_2)$ represented by $\mathrm{Fix}(\tau)$ is the characteristic class of Φ_τ.*

We are ready to prove the following lemma.

Lemma 3.7.4. *Assume that L is a real Lagrangian in X_1 that is diffeomorphic to a closed connected surface. Then L must be non-orientable.*

Proof. Let $L = \mathrm{Fix}(R)$ for some antisymplectic involution R of X_1. Assume to the contrary that L is orientable, and hence L represents a $\mathbb{Z}$-homology class $[L]_{\mathbb{Z}} \in H_2(X_1;\mathbb{Z})$. Using the notations in the proof of Lemma 3.7.1, we take the generators H and E_1 on $H_2(X_1;\mathbb{Z})$. Since R is orientation-preserving, R_* preserves the intersection form. Using also that $R_*^2 = \mathrm{id}$ and that $R^*[\omega] = -[\omega]$ on $H^2(X_1;\mathbb{Z})$, one computes that that the induced map R_* on $H_2(X_1;\mathbb{Z})$ is given by $R_* = -\mathrm{id}$. Since $R_*[L]_{\mathbb{Z}} = [L]_{\mathbb{Z}}$, we obtain $[L]_{\mathbb{Z}} = 0$ and hence $[L]_{\mathbb{Z}_2} = 0$ as well. Noting that $R_*^{\mathbb{Z}_2} = \mathrm{id}$ on $H_2(X_1;\mathbb{Z}_2)$, the twisted intersection form Φ_R is the usual mod 2 intersection form of X_1. By Lemma 3.7.3, the characteristic class of Φ_R vanishes and so the intersection form of X_1 must be even, which yields a contradiction.

We conclude that there are no real Lagrangian tori in X_1.

3.7.2 Constructions of explicit real Lagrangians

We now explicitly construct real Lagrangians that realize all diffeomorphism types in Table 3.1.

Remark 3.7.5. On every toric manifold M, we can lift the trivial involution $\sigma_0 = \mathrm{id}$, which yields the natural antisymplectic involution R^0 on M. Its fixed point set corresponds to the *real locus*, see Proposition 3.4.10 and Remark 3.4.11. In particular, the real locus of X_k, that is diffeomorphic to $\#_{k+1}\mathbb{R}P^2$, is the real Lagrangian $\mathrm{Fix}(R^{\mathrm{id}})$ in X_k for each $0 \leq k \leq 3$.

Example 3.7.6. Consider $\mathbb{C}P^2$ equipped with the Fubini-Study form ω_{FS}. Its Delzant polytope Δ is defined by the outward pointing normal vectors

$$v_1 = (-1, 0), \quad v_2 = (0, -1), \quad v_3 = (1, 1).$$

There is one non-trivial involution in $\mathcal{S}_\Delta$, namely the reflection $\sigma(x, y) = (y, x)$ with respect to the diagonal line, see Figure 3.5.

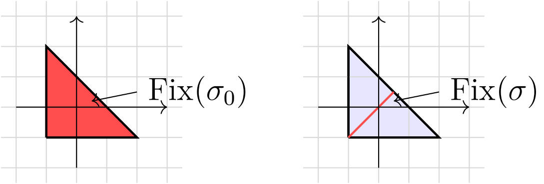


Figure 3.5: Involutions on the 2-simplex Δ

We use the real Delzant construction to prove that $\mathrm{Fix}(R^\sigma)$ is diffeomorphic to $\mathbb{R}P^2$. Since σ exchanges the normal vetors v_1 and v_2, we obtain

$$\begin{aligned} &\rho\colon \mathbb{C}^3 \to \mathbb{C}^3, && \rho(z_1, z_2, z_3) = (\bar{z}_2, \bar{z}_1, \bar{z}_3), \\ &\rho_{T^3}\colon T^3 \to T^3, && \rho_{T^3}(t_1, t_2, t_3) = (t_2^{-1}, t_1^{-1}, t_3^{-1}). \end{aligned}$$

Observe that

$$\begin{aligned} \mathrm{Fix}(\rho) &= \{(z, \bar{z}, x) \mid z \in \mathbb{C},\ x \in \mathbb{R}\}, \\ \mathrm{Fix}(\rho_{T^3}) &= \left\{(t, t^{-1}, s) \mid t \in S^1,\ s = \{1, -1\}\right\} \cong S^1 \oplus \mathbb{Z}_2. \end{aligned}$$

Hence, the kernel of π_R is

$$K_R = K \cap \mathrm{Fix}(\rho_{T^3}) = \{(1,1,1), (-1,-1,-1)\} \cong \mathbb{Z}_2$$

and

$$\nu_R(z, \bar{z}, x) = |z|^2 + \frac{x^2}{2} - 3.$$

Therefore $\nu_R^{-1}(0)$ is a 2-sphere on which K_R acts by the antipodal mapping and hence

$$\mathrm{Fix}(R^\sigma) \cong \nu_R^{-1}(0)/K_R \cong \mathbb{R}P^2.$$

In fact, it follows from Smith theory that any real Lagrangian in $\mathbb{C}P^2$ (not necessarily compatible with the torus action) is diffeomorphic to $\mathbb{R}P^2$.

Example 3.7.7. Consider $(S^2 \times S^2, \omega_0 \oplus \omega_0, \mu)$, where ω_0 is the area form on the sphere and its Delzant polytope is

$$\square := [-1,1]^2 = \mu(S^2 \times S^2)$$

with outward pointing normal vectors

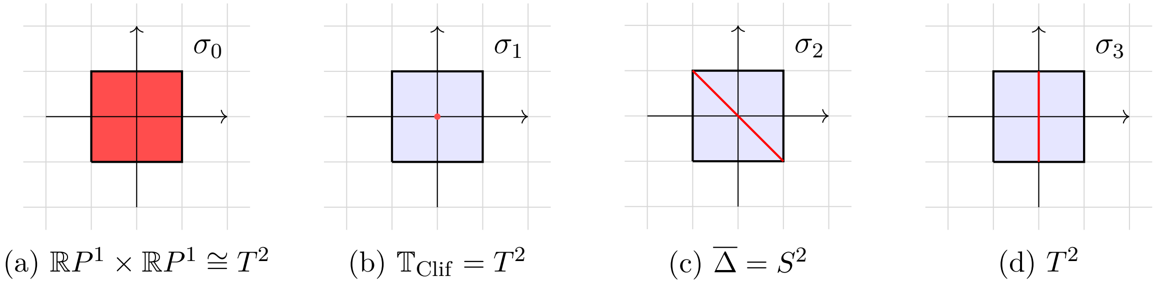
$$v_1 = (1,0), \quad v_2 = (0,1), \quad v_3 = (-1,0), \quad v_4 = (0,-1)$$

and $\kappa_1 = \cdots = \kappa_4 = 1$. Hence,

$$\pi = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad K = \{(e^{2\pi i\alpha}, e^{2\pi i\beta}, e^{2\pi i\alpha}, e^{2\pi i\beta})\}$$

We consider the four involutions in $\mathcal{S}_\square$ given in Figure 3.6, namely

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$



(a) $\mathbb{R}P^1 \times \mathbb{R}P^1 \cong T^2$ (b) $\mathbb{T}_{\text{Clif}} = T^2$ (c) $\overline{\Delta} = S^2$ (d) T^2

Figure 3.6: Involutions of $\square = [-1,1]^2$.

Involution σ_0. Since the real locus of $S^2 \times S^2 = \mathbb{C}P^1 \times \mathbb{C}P^1$ is diffeomorphic to $\mathbb{R}P^1 \times \mathbb{R}P^1 \cong T^2$, so is $\mathrm{Fix}(R^{\sigma_0})$.

Involution σ_1. Since $\mathrm{Fix}(\sigma_1)$ is a singleton, the corresponding real Lagrangian is given by the Lagrangian fibre $\mathrm{Fix}(R^{\sigma_1}) = \mu^{-1}\big((0,0)\big) \cong T^2$, which is called the *Clifford torus* in $S^2 \times S^2$.

Involution σ_2. We observe that

$$\nu^{-1}(0) = \big\{(z_1,z_2,z_3,z_4) \mid |z_1|^2 + |z_3|^2 = 4,\ |z_2|^2 + |z_4|^2 = 4\big\} \cong S^3 \times S^3.$$

Since σ_2 exchanges v_1 with v_4 and v_2 with v_3, we obtain

$$\mathrm{Fix}(\rho) = \{(z, w, \bar{w}, \bar{z}) \mid z, w \in \mathbb{C}\},$$
$$\mathrm{Fix}(\rho_{T^3}) = \{(t, s, s^{-1}, t^{-1}) \mid t, s \in S^1\}.$$

Hence,

$$\nu_R^{-1}(0) = \{(z, w, \bar{w}, \bar{z}) \mid |z|^2 + |w|^2 = 4\} \cong S^3$$

and $K_R = K \cap \mathrm{Fix}(\rho_{T^3}) = \{(t, t^{-1}, t, t^{-1}) \mid t \in S^1\}$. Hence the K_R-action on $\nu_R^{-1}(0)$ can be identified with the Hopf action on S^3, and we obtain

$$\mathrm{Fix}(R^{\sigma_2}) \cong \nu_R^{-1}(0)/K_R \cong S^2,$$

which is called the *antidiagonal sphere* $\mathrm{Fix}(R^{\sigma_2}) = \overline{\Delta} := \{(x, -x) \mid x \in S^2\}$.

Involution σ_3. Similarly, we have

$$\begin{aligned} &\nu_R^{-1}(0) = \{(z, x_1, \bar{z}, x_2) \mid z \in \mathbb{C},\ x_1, x_2 \in \mathbb{R},\ |z|^2 = 2,\ x_1^2 + x_2^2 = 2\} \cong T^2, \\ &K_R = \{(s_1, s_2, s_1, s_2) \mid s_1, s_2 \in \{1, -1\}\} \cong \mathbb{Z}_2^2, \end{aligned}$$

and hence

$$\mathrm{Fix}(R^{\sigma_3}) \cong \mathbb{R}P^1 \times \mathbb{R}P^1 \cong T^2.$$

Recall that any two embedded loops in S^2 are Hamiltonian isotopic if they divide the sphere into two discs with equal area. Using this, one can easily show that the real Lagrangian tori $\mathrm{Fix}(R^{\sigma_0})$, $\mathrm{Fix}(R^{\sigma_1})$, and $\mathrm{Fix}(R^{\sigma_3})$ are (pairwise) Hamiltonian isotopic to each other.

Example 3.7.8. Consider the monotone toric symplectic manifold $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ with moment polytope the isosceles trapezoid Δ depicted in Figure 3.7. Then we have

$$v_1 = (-1, 0), \quad v_2 = (0, -1), \quad v_3 = (1, 1), \quad v_4 = (-1, -1)$$

and $\kappa_1 = \cdots = \kappa_4 = 1$. Note that

$$\pi = \begin{pmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 \end{pmatrix}, \quad K = \{(e^{2\pi i\alpha}, e^{2\pi i\alpha}, e^{2\pi i(\alpha+\beta)}, e^{2\pi i\beta})\},$$

There is only one non-trivial involution on Δ, namely $\sigma(x, y) = (y, x)$. We show that

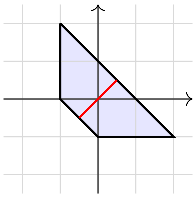


Figure 3.7: Monotone blow-up of $\mathbb{C}P^2$

$\mathrm{Fix}(R^\sigma) \cong \mathbb{R}P^2 \# \mathbb{R}P^2$ using the real Delzant construction. Observe that

$$\nu^{-1}(0) = \left\{ (z_1, z_2, z_3, z_4) \;\middle|\; |z_1|^2 + |z_2|^2 + |z_3|^2 = 6,\ |z_3|^2 + |z_4|^2 = 4 \right\}.$$

The involution σ acts on Δ by exchanging v_1 with v_2 and leaving v_3 and v_4 invariant, whence

$$\begin{aligned}\mathrm{Fix}(\rho) &= \{(z,\bar{z},x,y) \mid z \in \mathbb{C},\ x,y \in \mathbb{R}\},\\ \mathrm{Fix}(\rho_{T^4}) &= \big\{(t,t^{-1},s_1,s_2) \mid t \in S^1,\ s_1,s_2 \in \{1,-1\}\big\} \cong S^1 \oplus \mathbb{Z}_2^2.\end{aligned}$$

We obtain

$$\begin{aligned}\nu_R^{-1}(0) &= \{(z,\bar{z},x,y) \mid 2|z|^2 + x^2 = 6,\ x^2+y^2 = 4\} \cong T^2,\\ K_R &= \{(1,1,1,1),(-1,-1,-1,1),(1,1,-1,-1),(-1,-1,1,-1)\} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.\end{aligned}$$

We claim that the quotient map

$$T^2 \cong \nu_R^{-1}(0) \longrightarrow \nu_R^{-1}(0)/K_R$$

is a 4-fold covering of the Klein bottle $\mathbb{R}P^2\#\mathbb{R}P^2$. This follows from Table 3.1. To see this explicitly, we first identify $\nu_R^{-1}(0)$ with the torus T^2 obtained by the product of two circles, namely

$$S_{xy} := \{x^2+y^2 = 4\} \quad \text{and} \quad S_z := \{2|z|^2 + x^2 = 6\}.$$

Note that S_z varies depending on x. We obtain the identification in Table 3.2. Using this, we see that the quotient map above is a 4-fold covering of $\mathbb{R}P^2\#\mathbb{R}P^2$ as desired, see Figure 3.8.

K_R	S_z	S_{xy}
$(1,1,1,1)$	id	id
$(-1,-1,-1,1)$	antipodal map	y-axis reflection
$(1,1,-1,-1)$	id	antipodal map
$(-1,-1,1,-1)$	antipodal map	x-axis reflection

Table 3.2: Each element in K_R is identified with the composition of the two corresponding maps.

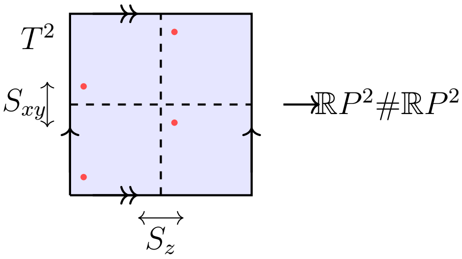


Figure 3.8: The torus T^2 as a 4-fold cover of the Klein bottle $\mathbb{R}P^2\#\mathbb{R}P^2$

Example 3.7.9. Consider the monotone toric symplectic manifold $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ with moment polytope Δ given on the left in Figure 3.9. Since the real locus of $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ is diffeomorphic to $\#_3\mathbb{R}P^2$, the standard antisymplectic involution R^0 yileds the real Lagrangian $\mathrm{Fix}(R^0) \cong \#_3\mathbb{R}P^2$. We claim that the real Lagrangian $\mathrm{Fix}(R^\sigma)$ associated to the involution $\sigma(x,y) = (y,x)$ is diffeomorphic to $\mathbb{R}P^2$. To see this, recall that by Example 3.7.6 the real Lagrangian $\mathrm{Fix}(R^{\sigma_2})$ in $\mathbb{C}P^2$ is diffeomorphic to $\mathbb{R}P^2$. Since the blow-ups of $\mathbb{C}P^2$ were performed away from the real Lagrangian $\mathrm{Fix}(R^{\sigma_2})$, we deduce that $\mathrm{Fix}(R^\sigma) \cong \mathbb{R}P^2$.

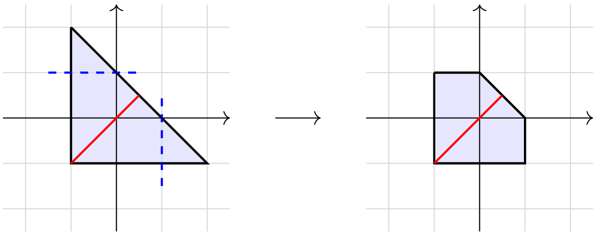


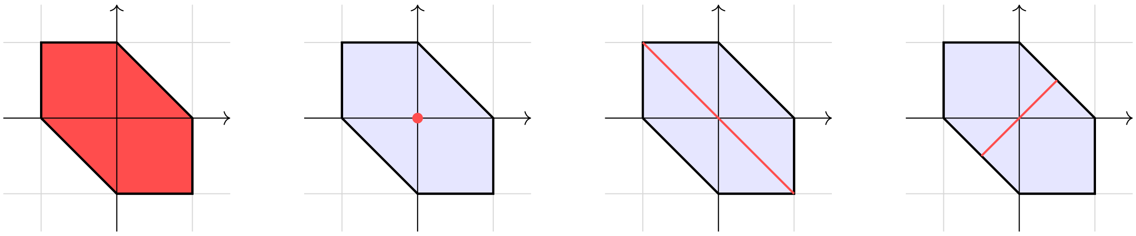
Figure 3.9: Monotone two-fold blow-up of $\mathbb{C}P^2$

Example 3.7.10. Consider the three-fold monotone blow-up of $\mathbb{C}P^2$ as in Figure 3.10. Then we have

$$v_1 = (1,0),\ v_2 = (0,1),\ v_3 = (-1,0),\ v_4 = (0,-1),\ v_5 = (1,1),\ v_6 = (-1,-1),$$
$$\kappa_1 = \cdots = \kappa_6 = 1.$$

We exhibit the real Lagrangians corresponding to the four involutions,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$



(a) $\#_4\mathbb{R}P^2$ (b) T^2 (c) S^2 (d) $\mathbb{R}P^2\#\mathbb{R}P^2$

Figure 3.10: Three-fold monotone blow-up of $\mathbb{C}P^2$

Involution σ_1. Since $\mathrm{Fix}(\sigma_1)$ is a singleton, we have $\mathrm{Fix}(R^{\sigma_1}) = \mu^{-1}\big((0,0)\big) \cong T^2$.

Involution σ_2. Note that the moment polytope of $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$ can be seen as the polytope obtained by two fold blow-up of $S^2 \times S^2$. Since the real Lagrangian in $S^2 \times S^2$ corresponding to the antidiagonal line in the polytope $\square$ is diffeomorphic to S^2 and the blow-ups are performed away from it, we obtain that $\mathrm{Fix}(R^{\sigma_2}) \cong S^2$.

Involution σ_3. In a similar vein, since the real Lagrangian in the one point blow up X_1 of $\mathbb{C}P^2$, corresponding to the diagonal line, is diffeomorphic to $\mathbb{R}P^2 \# \mathbb{R}P^2$, so is $\mathrm{Fix}(R^{\sigma_3}) \cong \mathbb{R}P^2 \# \mathbb{R}P^2$.

Remark 3.7.11. It seems difficult to adapt this topological study of real Lagrangians to higher dimensions. For one, it is hard to understand the topology of real Lagrangians coming from affine involutions on the moment polytope. However in the special case where Δ of a toric symplectic manifold M is *centrally symmetric*, i.e. $\Delta = -\Delta$, the antipodal map of Δ yields a real Lagrangian torus in M. See [Bre20] for further discussion.

Acknowledgement

The authors cordially thank Felix Schlenk for carefully reading the first draft. We also thank the anonymous referee for many useful comments. The paper was carried out when the authors visited the Institut de Mathématiques at Neuchâtel and the Korea Institute for Advanced Study at Seoul. We are grateful for their warm hospitality. JM specially thanks her advisor Suyoung Choi for continued support and encouragement. JB is supported by the grant 200021-181980/1 of the Swiss National Foundation. JK and JM are supported by Samsung Science and Technology Foundation under Project Number SSTF-BA1901-01. JM is supported by NRF-2019R1A2C2010989.

4 Real Lagrangian tori and versal deformations

This chapter consists of the article [Bre20].

Can a given Lagrangian submanifold be realized as the fixed point set of an anti-symplectic involution? If so, it is called *real*. We give an obstruction for a closed Lagrangian submanifold to be real in terms of the displacement energy of nearby Lagrangians. Applying this obstruction to toric fibres, we obtain that the central fibre of many (and probably all) toric monotone symplectic manifolds is real only if the corresponding moment polytope is centrally symmetric. Furthermore, we embed the Chekanov torus in all toric monotone symplectic manifolds and show that it is exotic and not real, extending Kim's result [Kim21a] for $S^2 \times S^2$. Inside products of S^2, we show that all products of Chekanov tori are pairwise distinct and not real either. These results indicate that real tori are rare.

Our methods are elementary in the sense that we do not use J-holomorphic curves. Instead, we rely on symplectic reduction and the displacement energy of product tori in $\mathbb{R}^{2n}$.

4.1 Introduction

A Lagrangian submanifold L in a symplectic manifold (M, ω) is said to be *real* if there is an anti-symplectic involution σ of M such that L is the fixed point set of σ or a connected component thereof. Here, an involution is a map satisfying $\sigma \circ \sigma = \mathrm{id}$, and anti-symplectic means that $\sigma^* \omega = -\omega$. An example is the equator of the 2-sphere with its Euclidean area form, which is the fixed point set of the reflection about the equatorial plane, and taking products of this example we get as real Lagrangian the so-called Clifford torus in $\times_n S^2$. For more examples, see Section 4.2.

Real or *not real* are symplectic invariants in the following sense: If φ is a symplectomorphism of (M, ω) and L is the fixed point set of the anti-symplectic involution σ, then $\varphi(L)$ is the fixed point set of the anti-symplectic involution $\varphi \circ \sigma \circ \varphi^{-1}$. There are many other reasons to study real Lagrangian submanifolds, some of which we give at the end of this introduction.

In this paper we address the question if a given closed Lagrangian submanifold of a symplectic manifold is real. An obstruction to being real has been given by J. Kim in [Kim21a]: If L is real, then the number of J-holomorphic discs $u \colon (D^2, \partial D^2) \to (M, L)$ of Maslov index 2 passing through a generic point in L must be even. In this paper we use a different symplectic invariant as obstruction to being real, namely the displacement energy of nearby Lagrangian submanifolds, a tool invented by Chekanov in [Che96]. While the Lagrangian submanifolds L that we are interested in usually have infinite displacement energy, nearby Lagrangians can be displaced. This leads to the so-called displacement energy germ $S_L \colon (H_1(L, \mathbb{R}), 0) \to \mathbb{R} \cup \{\infty\}$. In our basic result, (M, ω) is any, not necessarily compact, symplectic manifold.

Theorem 4.1.1. *Assume that L is a compact real Lagrangian submanifold of (M, ω). Then the displacement energy germ $S_L\colon (H_1(L, \mathbb{R}), 0) \to \mathbb{R} \cup \{\infty\}$ is even,*

$$S_L(-p) = S_L(p).$$

In general, it is hard to compute the displacement energy germ of a Lagrangian L. However, for the special class of fibers of toric symplectic manifolds we show in Section 4.3 that the displacement energy is intimately related to the moment polytope Δ.

Application I: Toric fibres. Let (M, ω) be a toric symplectic manifold with moment map μ and moment polytope $\Delta = \mu(M)$. For all $x \in \mathring{\Delta}$, the toric fibre $T_x = \mu^{-1}(x)$ is Lagrangian. These Lagrangian tori are especially well-suited to our methods, since they come with a natural versal deformation defined by varying the base point $a \mapsto T_{x+a}$. Hence, we are led to the question of what the displacement energy of toric fibres looks like as a function of the base point. In other words, we want to understand the function

$$e_\Delta\colon \Delta \to \mathbb{R} \cup \{\infty\}, \quad x \mapsto e_M(T_x),$$

where e_M denotes displacement energy. If T_x is real, we get by Theorem 4.1.1 that the function e_Δ is invariant under central symmetry in a neighbourhood of x.

Assume furthermore that (M, ω) is monotone. In the toric case, this means that we can assume that each facet of the moment polytope lies at affine distance one from the origin, in particular the origin is the only lattice point in the interior. We call the corresponding fibre T_0 the *central fibre*. The moment polytope of a toric monotone symplectic manifold is called *monotone*, see [McD11] for details. In this case, the function e_Δ can often be explicitely computed on an open dense subset of Δ and there is equal to the affine distance to the boundary. In particular the level sets of e_Δ are simply given by rescalings of $\partial\Delta$, see Figure 4.1. As noticed in [BCS22], this geometric property is implied by the following combinatorial property of the moment polytope: Let $\mathcal{S}(\Delta) = \Delta \cap (-\Delta) \cap \mathbb{Z}^n \setminus \{0\}$ be the set of non-zero symmetric lattice points in Δ. We say that Δ has property FS if every facet of Δ contains a point of $\mathcal{S}(\Delta)$. This property, which is closely related to the Ewald conjecture, is known to hold for monotone polytopes in dimensions $n \leq 9$ and is conjectured to hold in all dimensions, in which case requiring property FS becomes obsolete in all following statements. See Subsection 4.3.4 for a discussion.

Monotonicty has another useful consequence. Since real Lagrangians in monotone symplectic manifolds are automatically monotone as Lagrangian submanifolds, see for example [Oh93a], the only candidate to be real among all T_x is the central fibre T_0. For this torus we obtain the following.

Theorem 4.1.2. *Let (M, ω) be a toric monotone symplectic manifold whose moment polytope Δ has property FS. If the central fibre T_0 is real, then Δ is centrally symmetric, $\Delta = -\Delta$.*

Together with J. Kim and J. Moon, we show in [BKM19] that central symmetry of the moment polytope is a sufficient condition for the central fibre T_0 to be real. Under property FS, Theorem 4.1.2 is therefore an equivalence. For example, the central fibre in $S^2 \times S^2$ is real, whereas the central fibre in $\mathbb{C}P^2$ is not, see Figure 4.1.

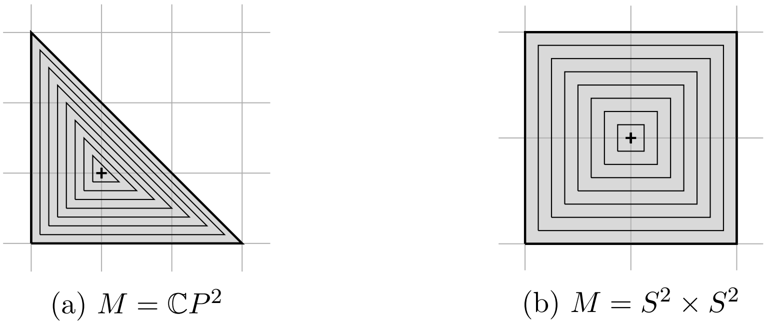

(a) $M = \mathbb{C}P^2$ (b) $M = S^2 \times S^2$

Figure 4.1: Level sets of the function $e_\Delta(x) = e_M(T_x)$.

Remark 4.1.3. We outline an alternative approach to Theorem 4.1.2 in appendix 4.6 based on the count of Maslov 2 J-holomorphic disks with boundary on T_0 which disposes of property FS. This approach was suggested to us by Grigory Mikhalkin and an anonymous referee.

Centrally symmetric polytopes. The set of centrally symmetric monotone Delzant polytopes is known. For any natural number n we define the *del Pezzo polytope* $\mathrm{DP}(n) \subset \mathbb{R}^n$ as the monotone polytope defined by the $2n+2$ inequalities

$$\pm x_1 \leqslant 1,\ \pm x_2 \leqslant 1, \dots, \pm x_n \leqslant 1,\ \pm(x_1 + \ldots + x_n) \leqslant 1.$$

For example $\mathrm{DP}(1) = [-1, 1]$ and $\mathrm{DP}(2)$ is the moment polytope of the monotone three-fold blow-up of $\mathbb{C}P^2$, see Figure 4.3 in Section 4.3. In general, these correspond to two-fold blow-ups of $\times_n S^2$. For n even or $n = 1$, the del Pezzo polytopes are centrally symmetric monotone Delzant polytopes. It is thus clear that products of such polytopes $\mathrm{DP}(n)$ are again centrally symmetric monotone Delzant polytopes. It was proved in [VK84] that the converse is true: The centrally symmetric monotone Delzant polytopes of dimension n are exactly the products of del Pezzo polytopes $\mathrm{DP}(n_j)$ with $n_j \in \{1, 2, 4, 6, \dots\}$ and $n = \sum_j n_j$. In order to determine the number $\nu_c(n)$ of centrally symmetric monotone Delzant polytopes in a given dimension n, we thus only need to count the number of ways in which n can be written as the sum of ones and even numbers. Let $p(n)$ be the partition function, i.e. the function counting the number of ways in which n can be written as the sum of natural numbers. Then for even $n = 2k$,

$$\nu_c(2k) = \sum_{j=0}^{k} p(j).$$

This can be seen as follows. Suppose a decomposition of $2k$ contains $2m$ ones. Omitting the ones induces a decomposition of $2(k-m)$ into strictly positive even numbers. This is equivalent to a decomposition of $k-m$ into strictly positive integers, whence there are $p(k-m)$ possibilities if the decomposition of $2k$ contains $2m$ ones. Summing over the possible number of ones yields the result. Furthermore $\nu_c(2k+1) = \nu_c(2k)$, since the odd del Pezzo polytopes $\mathrm{DP}(n)$ for $n > 1$ are not Delzant.

The number $\nu(n)$ of all monotone Delzant polytopes of dimension n is much larger than $\nu_c(n)$: For small values of n we have

n	1	2	3	4	5	6	7	8	9
$\nu_c(n)$	1	2	2	4	4	7	7	12	12
$\nu(n)$	1	5	18	124	866	7 622	72 256	749 892	8 229 721

(4.1.1)

The next few values for $\nu_c(2k)$ are $19, 30, 45, 67, 97, 139$. The growth of ν_c is subexponential. Indeed, since the partition function $p(n)$ grows like $e^{\sqrt{n}}$, $\nu_c(n)$ grows like $e^{\sqrt{n}}$ as well. On the other hand, for $\nu(2) = 5$ see Figure 4.3. The value $\nu(3) = 18$ was found in [Bat81, WW82], see also [Oda88, pp. 90], and $\nu(4) = 124$ was found in [Bat99, Sat00]. The values $\nu(n)$ for $5 \leq n \leq 8$ were computed by Øbro [Obr07], and $\nu(9)$ by Paffenholz [Paf]. The asymptotic behaviour of $\nu(n)$ is unfortunately not known, but based on discussions with Benjamin Nill and Andreas Paffenholz we expect that $\nu(n)$ grows at least exponentially. It follows that the property of a toric monotone symplectic manifold to have real central fiber is very restrictive.

Application II: Chekanov tori. The Chekanov torus was defined in [Che96] as the first example of monotone Lagrangian tori in $\mathbb{R}^{2n}$ which is not symplectomorphic to a product torus. We show that it can be embedded into any toric monotone symplectic manifold M and compute its displacement energy germ, by closely following the ideas used in [CS10]. In particular, its germ shows that the Chekanov torus is exotic in M. Furthermore, the polytope which is obtained as level set of the displacement energy germ is never centrally symmetric, and hence the Chekanov torus in M is not real, see for example Figure 4.8 in Section 4.5.

Theorem 4.1.4. *Let M be a toric monotone symplectic manifold satisfying property FS. Then the Chekanov torus can be embedded into M to yield an exotic Lagrangian torus which is not real.*

In the case of $M = \times_n S^2$, we prove that arbitrary products of Chekanov tori are pairwise not symplectomorphic and hence we get a collection of non real exotic Lagrangian tori in $\times_n S^2$ whose cardinality grows like the partition function and hence like $e^{\sqrt{n}}$ with n. The fact that these tori are exotic can be seen as an extension of earlier work by Chekanov–Schlenk [CS10] for $S^2 \times S^2$. In case the moment polytope of M is centrally symmetric, we furthermore prove that the Chekanov torus and products thereof can be realized as the fixed point set of a smooth involution. Hence in that case, Theorem 4.1.4 exhibits a symplectic phenomenon. We recall that Kim showed in [Kim21a] that the Chekanov torus in $S^2 \times S^2$ is not real by using that the count of Maslov-index two holomorphic disks on this torus is five, while this count on real Lagrangians must be even. For Chekanov tori in other toric monotone symplectic manifolds, this condition for realness seems to be less useful than our condition, since the count of Maslov-index two disks is difficult, see Remark 4.6.1.

Remark and Questions. In this paper we look at monotone Lagrangian tori that appear as the fibre of a torus fibration. In del Pezzo surfaces there exist many more monotone Lagrangian tori, that are the fibre of an *almost* toric fibration. An infinity of such tori were constructed by Vianna [Via17], and for del Pezzo surfaces different from

$\mathbb{C}P^2$ many more are found in [BCS22]. One can show that the level sets of displacement energy are still given by the shape of the almost toric base polygons and hence only those tori with centrally symmetric base polygons may be real, by Theorem 4.1.1. Details will be given in [BCS22], see also the fourthcoming [BMS22]. Since none of the almost toric base polygons of the exotic tori is centrally symmetric, our main result still holds, showing that none of the new tori is real. In view of this and our present work we ask:

Question 4.1.5. Let (M, ω) be a toric monotone symplectic manifold which contains a real Lagrangian torus.

(1) Is the moment polytope of M necessarily centrally symmetric?

(2) Suppose the moment polytope of M is centrally symmetric. Is the central fibre the unique real Lagrangian torus, up to Hamiltonian isotopy?

In dimension four, the only closed toric monotone symplectic manifolds are the toric del Pezzo surfaces $S^2 \times S^2$ and the k-fold blow-up X_k of $\mathbb{C}P^2$ for $k = 0, 1, 2, 3$, and the only closed monotone symplectic manifolds are the del Pezzo surfaces $S^2 \times S^2$ and X_k with $k \leq 8$, with unique symplectic structure up to scaling, see [Via17] for references.

(3) Is it true that the only real tori up to Hamiltonian diffeomorphism in a closed monotone symplectic 4-manifold are the Clifford torus in $S^2 \times S^2$ and in X_3?

We note that the uniqueness in $S^2 \times S^2$ was proved by Kim [Kim20].

A few motivations for the study of real Lagrangian submanifolds. We conclude this introduction by mentioning some of the strands that lead to the study of real Lagrangian submanifolds.

1. A related theme is the study of real algebraic varieties, namely the fixed point set of an anti-holomorphic involution of a complex algebraic variety. The study of their topologocial properties has a rich history with an impressive body of results, see [DK00]. It is interesting to see which of these results have analogues in the symplectic setting.

2. Let ι be a *smooth* involution of a manifold X. Classical Smith theory

$$\chi(\mathrm{Fix}(\iota)) = \chi(X) \mod 2, \tag{4.1.2}$$

$$\dim H(\mathrm{Fix}(\iota); \mathbb{Z}_2) \leq \dim H(X; \mathbb{Z}_2) \tag{4.1.3}$$

relates the homology of the fixed point set of a smooth involution to the homology of the ambient manifold. We refer to [Bor60] for details. It is interesting to find invariants of real Lagrangian manifolds that go beyond the Smith inequalities, and thus describe a genuine symplectic pheonomenon. As was noted by Kim [Kim21a], the Chekanov torus in $S^2 \times S^2$ can be realized as the fixed point set of a smooth involution, but not of an anti-symplectic one. In the general context of toric monotone symplectic manifolds (see table (4.1.1)), Theorem 4.1.2 seems to yield a significantly stronger obstruction than Smith theory, which only excludes two of the five manifolds in dimension 4 and five of the eighteen manifolds in dimension 6.

In symplectic geometry, real Lagrangians have appeared quite a while ago under different forms:

3. Several time-honoured systems in classical mechanics, like the planar circular restricted 3-body problem, are invariant under several anti-symplectic involutions. Their fixed point sets can be used to find special orbits, see [FvK18].

4. The Arnold–Givental conjecture generalizes the classical Arnold conjecture on the number of Lagrangian intersections in terms of real Lagrangian manifolds, see e.g. [MS17, §11.3].

5. The presence of an anti-symplectic involution simplifies J-holomorphic disk counts in some situations, see for example Oh [Oh93b], or the more recent work by Fukaya–Oh–Ohta–Ono [FOOO17].

6. Welschinger [Wel05] has applied techniques from Gromov-Witten theory to real Lagrangians to gain new insights in real enumerative geometry, which has generated a lot of interest and subsequent work.

As of now, very little is known about which topological types of Lagrangians can be realized as fixed point sets of anti-symplectic involutions and if real Lagrangians have some uniqueness properties. Some progress has been made by Kim [Kim21b], who proved that real Lagrangians in a given compact symplectic manifold are unique up to cobordism, and that the only real Lagrangian in $\mathbb{C}P^2$ is $\mathbb{R}P^2$ up to Hamiltonian isotopy. More recent are [Kim21a, Kim20] that we discussed earlier. In collaboration with J. Kim and J. Moon [BKM19], we construct many real Lagrangians of different topological types in toric symplectic manifolds by lifting symmetries of the moment polytope.

Organization of the paper. In Section 4.2 we discuss real Lagrangians and versal deformations. We prove Theorem 4.1.1 on the displacement energy germ of real Lagrangians. In Section 4.3, we discuss the displacement energy of toric fibres with a focus on the case in which the moment polytope has property FS. This discussion is instrumental for both our applications. In Section 4.4, we discuss whether fibres of toric symplectic manifolds are real and establish a criterion in terms of the geometry of the corresponding moment polytope. In particular, we prove Theorem 4.1.2. In Section 4.5, we deal with Chekanov tori in toric monotone symplectic manifolds and show that none of them are real, see Theorem 4.1.4. The appendix in Section 4.6 outlines an alternate approach to our results using J-holomorphic curves.

Acknowledgements. I thank Joontae Kim for introducing me to real symplectic geometry and for his invitation to KIAS in May 2019, where the main idea of this paper arose from numerous stimulating discussions. I also wish to thank Yuri Chekanov for useful remarks and for agreeing that I use some of our joint results from [BCS22] in this paper. Many thanks to an anonymous referee for careful reading of the manuscript; to Andreas Paffenholz for having written a script checking that all of the Delzant polytopes in dimension 9 have the required property FS and to Grigory Mikhalkin and an anonymous referee for suggesting the approach to the problem outlined in the appendix. I am grateful to Felix Schlenk for his generous support and countless remarks from which this paper has greatly benefited.

4.2 Versal Deformations of real Lagrangians

In this section, we will discuss real Lagrangians, displacement energy and versal deformations. In particular, we will prove Theorem 4.1.1, the proof of which relies on two key observations. Firstly, the displacement energy is invariant under anti-symplectic involutions, see Proposition 4.2.8. Secondly, if we combine this invariance with a $\mathbb{Z}_2$-equivariant Weinstein neighbourhood Theorem, we obtain the desired result.

4.2.1 Real Lagrangians

Let (M, ω) be a symplectic manifold and let σ be an anti-symplectic involution on M, i.e. a smooth map satisfying $\sigma \circ \sigma = \mathrm{id}$ and $\sigma^* \omega = -\omega$. Its fixed point set $\operatorname{Fix} \sigma$ is a (possibly not connected) Lagrangian submanifold whenever it is not empty.

Definition 4.2.1. *A Lagrangian submanifold L in (M, ω) is called* real *if there is an anti-symplectic involution of M having L as a connected component of its fixed point set.*

Example 4.2.2. The equator in the standard symplectic 2-sphere (S^2, ω) is real. The corresponding involution is given by reversing the height $z \mapsto -z$. By taking the product of this example, we can describe the product of equators (also known as the Clifford torus) as the fixed point set of an anti-symplectic involution on $\times_n S^2$.

Example 4.2.3. Let $(\mathbb{C}P^n, \omega_{\mathrm{FS}})$ be the complex projective space equipped with the Fubini–Study form. Then $\mathbb{R}P^n \subset \mathbb{C}P^n$ is real since it is the fixed point set of the anti-symplectic involution

$$\sigma \colon \mathbb{C}P^n \to \mathbb{C}P^n, \quad [z_0 : \ldots : z_n] \mapsto [\overline{z}_0 : \ldots : \overline{z}_n].$$

It is well-known that this example can be generalized to all toric manifolds, an observation which gives rise to so-called real toric geometry. See for example [AM13], [Hau13] or [DK00].

Example 4.2.4. Any symplectic manifold (M, ω) can be seen as a real Lagrangian submanifold in $(M \times M, \omega \oplus -\omega)$. The embedding is given by the diagonal map $p \mapsto (p, p)$ and the corresponding anti-symplectic involution is given by exchanging the two coordinates in $M \times M$.

Example 4.2.5. Let $(T^*Q, \omega_0 = -d\lambda)$ be the cotangent bundle of a smooth manifold Q equipped with its canonical symplectic form. The map which reverses momenta,

$$\sigma_0 \colon T^*Q \to T^*Q, \quad (q, p) \mapsto (q, -p),$$

satisfies $\sigma_0^* \lambda = -\lambda$ and is therefore an anti-symplectic involution. Its fixed point set is the zero section

$$\operatorname{Fix} \sigma_0 = Q \subset T^*Q.$$

By Weinstein's Lagrangian neighbourhood theorem and Example 4.2.5, any Lagrangian submanifold admits a *locally defined* anti-symplectic involution of which it is the fixed point set. Of course, locally defined involutions might not extend globally. On the other hand, Meyer [Mey81] proved that any anti-symplectic involution σ with non-empty fixed point set is locally of the form described in Example 4.2.5. This can be viewed as a $\mathbb{Z}_2$-equivariant version of Weinstein's theorem.

Theorem 4.2.6. *(Meyer [Mey81]) Let σ be an anti-symplectic involution of a symplectic manifold (M, ω) containing a Lagrangian $L \subseteq \operatorname{Fix} \sigma \neq \varnothing$. Furthermore let T^*L be equipped with its canonical symplectic form and the anti-symplectic involution σ_0 which reverses momenta. Then there is a σ-invariant neighbourhood V of L, a σ_0-invariant neighbourhood U of the zero-section in T^*L and a symplectomorphism*

$$g \colon (U, \omega_0|_U) \to (V, \omega|_V),$$

which maps the zero section to L and which intertwines the anti-symplectic involutions σ and σ_0,

$$g \circ \sigma_0 = \sigma \circ g. \tag{4.2.4}$$

4.2.2 Displacement energy

Recall that the displacement energy of a compact subset A of a symplectic manifold (M, ω) is defined as

$$e_M(A) = \inf \left\{ \|H\| \;\middle|\; H \in C_c^\infty([0,1] \times M),\, \varphi_H^1(A) \cap A = \varnothing \right\},$$

where

$$\|H\| = \int_0^1 \left(\max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) \right) dt$$

is the Hofer norm on $C_c^\infty([0,1] \times M)$. By convention, we put $e_M(A) = \infty$ whenever the set of displacements is empty.

Example 4.2.7. Let $T(a) \subset (\mathbb{R}^2, \omega_0)$ be the circle enclosing area $a > 0$ in the plane. Its displacement energy is

$$e_{\mathbb{R}^2}(T(a)) = a.$$

By taking products, we obtain Lagrangian *product tori* $T(a_1, \dots, a_n) = T(a_1) \times \cdots \times T(a_n) \subset (\mathbb{R}^{2n}, \omega_0)$. Their displacement energy is (see Remark 4.3.3)

$$e_{\mathbb{R}^{2n}}(T(a_1, \dots, a_n)) = \min\{a_1, \dots, a_n\}.$$

Given a symplectomorphism ψ of (M, ω) we have $\varphi^t_{H \circ \psi^{-1}} = \psi \circ \varphi^t_H \circ \psi^{-1}$. The set A is thus displaced by the time-one map of H if and only if $\psi(A)$ is displaced by the time-one map of $H \circ \psi^{-1}$. Since the Hofer norm satisfies $\|H \circ \psi^{-1}\| = \|H\|$, it follows that displacement energy is invariant under symplectomorphisms,

$$e_M(\psi(A)) = e_M(A).$$

The same is true for anti-symplectic involutions.

Proposition 4.2.8. *Let σ be an anti-symplectic involution on a symplectic manifold (M, ω). Then the displacement energy is invariant under σ in the sense that*

$$e_M(\sigma(A)) = e_M(A)$$

for any compact subset $A \subset M$.

Proof. Let $H \in C_c^\infty([0,1] \times M)$ be a Hamiltonian, X_H^t and φ_H^t its associated vector field and flow. Since σ is an anti-symplectic involution, we have

$$X_{H\circ\sigma}^t = -\sigma_*(X_H^t \circ \sigma).$$

Define $H_t' = -H_t \circ \sigma$. Its Hamiltonian vector field is

$$X_{H'}^t = \sigma_*(X_H^t \circ \sigma)$$

and thus we get for the respective flows

$$\varphi_{H'}^t = \sigma \circ \varphi_H^t \circ \sigma.$$

This proves that a set A is displaced by φ_H^1 if and only if $\sigma(A)$ is displaced by $\varphi_{H'}^1$. Since $\|H'\| = \|H\|$, the claim follows. □

4.2.3 Versal Deformations

Versal deformations were introduced in symplectic geometry by Chekanov [Che96] and subsequently used in [CS10] and [CS16] as a tool to distinguish Lagrangian submanifolds. The idea is to look at the behaviour of known symplectic invariants on neighbouring Lagrangians of the submanifolds in question. Let us outline the construction. Since we will only use the displacement energy as an invariant, we will restrict ourselves to this case. We refer to [CS16] for details.

In every cotangent bundle T^*L of a closed Lagrangian submanifold, Lagrangians which are C^1-close to the zero section can be identified with the graphs of closed one-forms. Using Weinstein's theorem, one can translate this identification to the case of any Lagrangian $L \subset (M,\omega)$ as follows. For a given Weinstein chart $g : T^*L \supset U \to V \subset M$ there is a C^1-neighbourhood $\widehat{\mathcal{U}} \subset \Omega^1_{cl}(L)$ of the zero section in the space of closed one-forms, a C^1-neighbourhood $\widehat{\mathcal{V}}$ of L in the space of Lagrangian submanifolds in M, and a bijection

$$\widehat{w}_L^g \colon \widehat{\mathcal{U}} \to \widehat{\mathcal{V}}, \quad \alpha \mapsto g(\Gamma_\alpha),$$

where we denote the graph of $\alpha \in \Omega^1(L)$ by Γ_α. Furthermore, C^1-small Hamiltonian perturbations of the zero section in T^*L are in one-to-one correspondence with C^1-small exact one-forms, and hence the above map descends to

$$w_L^g \colon \mathcal{U} \to \mathcal{V},$$

where we divide out exact one-forms on the left-hand side and Hamiltonian isotopies on the right-hand side. In particular we can view $\mathcal{U}$ as a neighbourhood of zero in $H^1(L,\mathbb{R})$. Up to Hamiltonian isotopy, neighbouring Lagrangians of L are thus parametrized by a neighbourhood of zero in the vector space $H^1(L,\mathbb{R})$.

As displacement energy is invariant under Hamiltonian isotopies we can compose it with the above map w_L^g to obtain a function on $\mathcal{U}$

$$H^1(L,\mathbb{R}) \supset \mathcal{U} \to \mathbb{R} \cup \{\infty\}, \quad [\alpha] \mapsto e_M(g(\Gamma_\alpha)).$$

The germ at 0 associated to this function corresponds to the displacement energy of neighbouring Lagrangians of L and will be denoted by

$$S_L^g \colon (H^1(L,\mathbb{R}),0) \to \mathbb{R} \cup \{\infty\}.$$

The following remark is crucial for what will follow.

Remark 4.2.9. The germ of the bijection w_L^g is independent of the choice of Weinstein chart g and thus so is the germ S_L^g. Hence we will write $S_L = S_L^g$. See [CS16] for details.

We are now in a position to prove Theorem 4.1.1, which we recall for the reader's convenience.

Theorem 4.2.10. *Assume that $L \subseteq \operatorname{Fix} \sigma$ is a compact real Lagrangian submanifold of (M,ω). Then the displacement energy germ S_L is even,*

$$S_L(-p) = S_L(p).$$

Proof. By Theorem 4.2.6 we can pick a Weinstein neighbourhood g such that $g \circ \sigma_0 = \sigma \circ g$. Let $\alpha \in \Omega^1_{cl}(L)$ be a one-form representing p, then $\sigma_0(\Gamma_\alpha) = \Gamma_{-\alpha}$. Hence, using the invariance of the displacement energy under anti-symplectic involutions, we find

$$\begin{aligned} S_L^g(-\alpha) &= e_M(g(\Gamma_{-\alpha})) \\ &= e_M(g(\sigma_0(\Gamma_\alpha))) \\ &= e_M(\sigma(g(\Gamma_\alpha))) \\ &= e_M(g(\Gamma_\alpha)) \\ &= S_L^g(\alpha). \end{aligned}$$

Since $S_L^g = S_L$ is independent of the choice of g, the claim follows. □

4.3 Displacement energy of toric fibres

In this section we compute the displacement energy of toric fibres. We begin by proving that displacement energy can only increase under symplectic reduction. This observation was already made in [AM13] and will be used here to prove the existence of a lower bound as well as an upper bound on the displacement energy of toric fibres. For the lower bound, we will use the fact that any toric symplectic manifold can be seen as a symplectic quotient of some $\mathbb{C}^k$ via Delzant's construction. For the upper bound, we will give a slightly modified version of McDuff's method by probes, see [McD11]. In the last part of this section we will apply these results to compute the displacement energy of toric fibres in toric monotone symplectic manifolds. This is a crucial ingredient for Sections 4.4 and 4.5.

4.3.1 Displacement energy and symplectic reduction.

Let $(\widehat{M}, \widehat{\omega}, \nu)$ be a Hamiltonian G-space. Recall that this a symplectic manifold $(\widehat{M}, \widehat{\omega})$ equipped with a Hamiltonian action of a compact Lie group G generated by a moment map $\nu \colon \widehat{M} \to \mathfrak{g}^*$. We furthermore assume that $(\widehat{M}, \widehat{\omega}, \nu)$ admits symplectic reduction at the level $0 \in \mathfrak{g}^*$, i.e. 0 is a regular value and G acts freely on $Z = \nu^{-1}(0)$. This means that we have the following reduction diagram

$$\begin{tikzcd}
Z = \nu^{-1}(0) \arrow[r, hook] \arrow[d, "p"] & (\widehat{M}, \widehat{\omega}) \\
(M, \omega) &
\end{tikzcd}$$

with $\widehat{\omega}|_{TZ} = p^*\omega$. Furthermore, assume that the symplectic quotient (M, ω) is compact.

Lemma 4.3.1. *Under the above hypotheses, we have*

$$e_{\widehat{M}}(p^{-1}(A)) \leqslant e_M(A)$$

for any set $A \subset M$.

In other words, symplectic reduction can only increase displacement energy. The proof of Lemma 4.3.1 runs as follows. For any Hamiltonian $H \in C^\infty(M \times [0,1])$ which displaces A we will construct a compactly supported Hamiltonian $\widehat{H} \in C^\infty_c(\widehat{M} \times [0,1])$ which displaces $p^{-1}(A)$ and which has the same Hofer norm as H. The Hamiltonian $\widehat{H}$ is obtained as an extension of the lift $p^*H \in C^\infty(Z \times [0,1])$ which is zero outside of a tubular neighbourhood of $Z \subset \widehat{M}$. Although this was already outlined in [AM13], we give a full proof for the reader's convenience.

Proof of Lemma 4.3.1. If A is not displaceable, there is nothing to show. Therefore let H be a Hamiltonian on M which displaces A. We can assume that $\min_{p \in M} H_t(p) = 0$ for all $t \in [0,1]$. Now fix a time $t \in [0,1]$ and pick a tubular neighbourhood of Z, i.e. a diffeomorphism

$$\chi : NZ \supset U \to V \subset \widehat{M}$$

from a neighbourhood U of the zero section inside the normal bundle $\pi : NZ \to Z$ to a neighbourhood V of $Z \subset \widehat{M}$ mapping the zero section to Z. Let $\rho \in C^\infty(U)$ be a function such that

1. $\rho = 1$ on the zero section and $\rho \leqslant 1$ elsewhere,
2. ρ is compactly supported.

We can now define $\widehat{H}_t$ on U by putting $\widehat{H}_t(v) = \rho(v) p^* H_t(\pi(v))$. By using χ, we transport this function to a function $\widehat{H}_t$ on V, which can be smoothly extended to all of $\widehat{M}$ by zero since ρ has compact support. Notice that the Hofer norm of $\widehat{H}$ is equal to the Hofer norm of H. For $\widehat{H}_t \in C^\infty(\widehat{M})$ we have

$$\widehat{H}_t|_Z = p^* H_t. \tag{4.3.5}$$

In particular, $\widehat{H}_t|_Z$ is invariant under the G-action on Z. We will show that the restriction of the Hamiltonian vector field $X^t_{\widehat{H}}$ to Z

1. is tangent to Z,
$$(X^t_{\widehat{H}})_z \in T_z Z \quad \forall z \in Z; \tag{4.3.6}$$
2. projects to the Hamiltonian vector field of H on M,
$$p_*(X^t_{\widehat{H}}|_Z) = X^t_H. \tag{4.3.7}$$

In order to prove (4.3.6), we use the invariance of $\widehat{H}_t|_Z$ under the action of G, which implies that the following equivalent conditions hold

$$\begin{array}{rl} & d\widehat{H}_t(z)(X_\zeta)_z = 0 \quad \forall \zeta \in \mathfrak{g}, \\ \Leftrightarrow & \langle d\nu(z)(X^t_{\widehat{H}})_z, \zeta\rangle = 0 \quad \forall \zeta \in \mathfrak{g}, \\ \Leftrightarrow & (X^t_{\widehat{H}})_z \in T_z Z. \end{array}$$

Here X_ζ denotes the fundamental vector field of the G-action associated to $\zeta \in \mathfrak{g}$. The last line follows from the fact that $T_z Z = T_z \nu^{-1}(0) = \ker d\nu(z)$. Let $Y \in TM$ and pick $\widehat{Y} \in TZ$ so that $p_*\widehat{Y} = Y$. Using (4.3.5), we find $d(\widehat{H}_t|_Z)(\widehat{Y}) = dH_t(Y)$, which we use to compute

$$\begin{array}{rcl} \omega(p_* X^t_{\widehat{H}}, Y) & = & \omega(p_* X^t_{\widehat{H}}, p_*\widehat{Y}) \\ & = & (p^*\omega)(X^t_{\widehat{H}}, \widehat{Y}) \\ & = & \widehat{\omega}(X^t_{\widehat{H}}, \widehat{Y}) \\ & = & d(\widehat{H}_t|_Z)(\widehat{Y}) \\ & = & dH_t(Y) \\ & = & \omega(X^t_H, Y). \end{array}$$

This proves (4.3.7). Now let φ^t_H and $\varphi^t_{\widehat{H}}$ denote the corresponding Hamiltonian flows. Since equation (4.3.7) holds for all $t \in [0,1]$, we have

$$p \circ \varphi^t_{\widehat{H}}|_Z = \varphi^t_H \circ p. \tag{4.3.8}$$

Since $\varphi^1_H(A) \cap A = \varnothing$, take the pre-image under p of both sides to get $p^{-1}(\varphi^1_H(A)) \cap p^{-1}(A) = \varnothing$. Together with equation (4.3.8),

$$\varphi^1_{\widehat{H}}(p^{-1}(A)) \cap p^{-1}(A) \subseteq p^{-1}(\varphi^1_H(A)) \cap p^{-1}(A) = \varnothing$$

and hence $\varphi^1_{\widehat{H}}$ displaces $p^{-1}(A)$. □

4.3.2 Lower bound for toric fibres

Let (M^{2n}, ω) be a compact toric symplectic manifold. By this we mean that T^n acts effectively on M by Hamiltonian diffeomorphisms which are generated by a moment map $\mu\colon M \to \mathfrak{t}^*$. We identify the dual $\mathfrak{t}^*$ of the Lie algebra $\mathfrak{t}$ of T^n with $\mathbb{R}^n$ by choice of a basis. As is the case for all Hamiltonian torus actions, the image of μ is a convex polytope $\Delta = \mu(M) \subset \mathbb{R}^n$, called *moment polytope*. Since M is toric, the corresponding moment polytope has the Delzant property, see [Del88] or [Aud04] for details. Furthermore, Delzant showed that M can be reconstructed from such Δ by taking a suitable symplectic quotient of $\mathbb{C}^k$ by the action of a linear subtorus of T^k acting by the standard action on $\mathbb{C}^k$. Let ν be the moment map of this action. The situation is summarized by the following reduction diagram

$$\begin{tikzcd}
Z = \nu^{-1}(0) \arrow[r, hook] \arrow[d, two heads, "p"] & (\mathbb{C}^k, \omega_0) \\
(M, \omega) \arrow[r, "\mu"] & \Delta.
\end{tikzcd}$$

We describe the moment polytope $\Delta \subset \mathbb{R}^n$ of M by a set of inequalities

$$\langle x, v_i \rangle \leqslant \kappa_i, \quad i \in \{1, \dots, k\},$$

where the v_i are the unique outward-pointing normal vectors to the facets of Δ which are primitive in the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Define the functionals on $\mathbb{R}^n$

$$\ell_i(x) = \kappa_i - \langle x, v_i \rangle$$

for all $i \in \{1, \dots, k\}$. Every ℓ_i defines a half-space $\{\ell_i \geq 0\}$ and the moment polytope Δ is given by the intersection of these half-spaces. Using Lemma 4.3.1, we will give a lower bound for the displacement energy of any toric fibre $T_x = \mu^{-1}(x)$.

Proposition 4.3.2. *Let (M, ω, μ) be a toric symplectic manifold with moment polytope Δ. Then for every $x \in \Delta$ the displacement energy of the corresponding toric fibre is bounded from below by*

$$e_\Delta(x) = e_M(T_x) \geqslant \min\{\ell_1(x), \dots, \ell_k(x)\},$$

where $\ell_i(x)$ is the affine distance of x to the i-th facet of Δ.

Proof. As is clear from the Delzant construction, $p^{-1}(T_x) = p^{-1}(\mu^{-1}(x)) \subset \mathbb{C}^k$ is the product torus

$$T(a_1, ..., a_k) = \{(z_1, ..., z_k) \in \mathbb{C}^k \mid \pi |z_i|^2 = a_i\},$$

with $a_i = \ell_i(x)$. Since $e_{\mathbb{C}^k}(T(a_1, ..., a_k)) = \min\{a_1, \dots, a_k\}$ by Remark 4.3.3, the claim follows from Lemma 4.3.1. $\square$

Remark 4.3.3. In order to compute the displacement energy of a product torus in $\mathbb{C}^n$, we use the inequalities

$$\min\{a_1, \dots, a_k\} \leq c_1(T(a_1, ..., a_k)) \leq e_{\mathbb{C}^k}(T(a_1, ..., a_k)) \leq \min\{a_1, \dots, a_k\},$$

where c_1 denotes the first Ekeland-Hofer capacity. The first inequality follows from Theorem (b) on page 43 of [Sik90] and the second from [Hof90, Theorem 1.6], which are both obtained by applying the calculus of variations to the action functional of classical mechanics. The third inequality follows from Proposition 4.3.4. This is the only *hard* symplectic result we use and hence our methods do not rely on J-holomorphic curves, with the obvious exception of the complementary appendix.

4.3.3 Upper bound for toric fibres

In order to prove displaceability in toric symplectic manifolds, McDuff introduced probes in [McD11], a technique independently found in [CS10]. We will show that probes can be interpreted in the framework of Lemma 4.3.1 by performing symplectic reduction on the pre-image of the probe. Let (M, ω) be a toric symplectic manifold with moment map μ and moment polytope $\Delta = \{\ell_i \geqslant 0,\ \forall i\}$. A probe $P_{i,u}(w)$ is determined by a facet $F_i = \{\ell_i = 0\} \cap \Delta$ of Δ, a point $w \in F_i$ and a vector $u \in \mathbb{Z}^n$ which is integrally transverse to F_i. By this we mean that u can be completed to a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ by vectors parallel to F_i. The set $P_{i,u}(w) \subset \mathbb{R}^n$ is the half open line segment obtained as the union of $\{w\}$ with the open segment defined by the intersection of $\mathring{\Delta}$ with the line emanating from w in direction u, see Figure 4.2. Displaceability of toric fibres lying on a suitable probe was proved in [McD11].

Proposition 4.3.4. *Let $x \in \Delta$ be a point in a probe $P_{i,u}(w)$ lying in the same half of $P_{i,u}(w)$ as w and not on the midpoint of the probe. Then*

$$e_\Delta(x) \leqslant \ell_i(x).$$

Proof. Since u is integrally transverse to F_i we can assume, up to applying a transformation in $\mathrm{SL}(n, \mathbb{Z})$, that $u = e_1$ and that F_i lies in the hyperplane spanned by $e_2, \ldots, e_n$. Indeed, the integral transversality condition states that there is a $\mathbb{Z}$-basis $u, f_2, \ldots, f_n$ of the lattice $\mathbb{Z}^n$, where the f_j are parallel to F_i. This means that there is an integral change of basis mapping the ordered set of vectors $u, f_2, \ldots, f_n$ to the ordered set of vectors $e_1, \pm e_2, e_3, \ldots, e_n$, where we choose the sign of the second vector such that the determinant of the transformation is $+1$. Hence we obtain $w = (0, w')$ for some $w' \in \mathbb{R}^{n-1}$ and $x = (\ell_i(x), w')$. Let $U = \mu^{-1}(\mathring{\Delta} \cup \mathring{F}_i) \subset M$. The subtorus $T^{n-1} = \{1\} \times S^1 \times \cdots \times S^1 \subset T^n$ acts freely[1] on U. The moment map of this action $\mu' : U \to \mathbb{R}^{n-1}$ is obtained by restricting μ to U and by dropping the first coordinate

$$\mu'(y_1, \ldots, y_{n-1}) = (\mu_2|_U(y_1), \ldots, \mu_n|_U(y_{n-1})).$$

We get $(\mu')^{-1}(w') = \mu^{-1}(P_{i,u}(w))$ and since T^{n-1} acts freely on this set, we can consider the following symplectic reduction

$$\begin{tikzcd}
(\mu')^{-1}(w') \arrow[r, hook] \arrow[d, two heads, "p"] & (U, \omega|_U) \\
(D^2(a), \omega_0).
\end{tikzcd}$$

Here, the reduced space is an open disk of area a equal to the affine length of the probe. The fibre we are interested in is

$$T_x = \mu^{-1}(x) = p^{-1}(T(\ell_i(x))),$$

---

[1] This can be seen as follows. For toric manifolds (M, ω, μ) the stabilizer of any point $p \in M$ can be read off from the moment polytope (viewed as $\Delta \subset \mathfrak{t}^*$) by taking the subtorus inside T^n which is generated by the annihilator of the smallest face of Δ which contains $\mu(p)$. In our situation, the annihilator is generated by e_1 and hence the stabilizer is the first coordinate circle in T^n.

where $T(\ell_i(x)) \subset D^2(a)$ is the circle bounding area $\ell_i(x)$. By our assumption on x, we have $\ell_i(x) < \frac{a}{2}$ and therefore $T(\ell_i(x)) \subset D^2(a)$ has displacement energy $\ell_i(x)$. Hence by Lemma 4.3.1 and the fact that $U \subset M$, we find

$$e_\Delta(x) = e_M(T_x) \leqslant e_U(T_x) \leqslant e_{D^2(a)}(T(\ell_i(x))) = \ell_i(x).$$

□

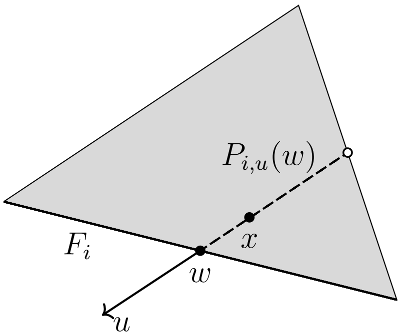


Figure 4.2: The probe $P_{i,u}(w)$.

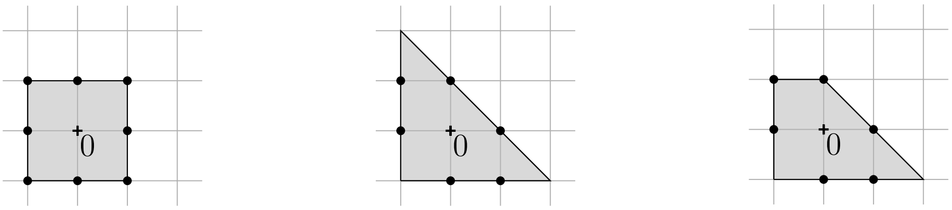
4.3.4 Probes in monotone polytopes.

Let $\Delta = \{\ell_i \geqslant 0, \forall i\}$ be the moment polytope of a toric monotone symplectic manifold (M, ω). We can assume that the barycentre lies in $0 \in \mathbb{R}^n$ and that $\ell_i(0) = \kappa_i = 1$. McDuff [McD11] discovered that displaceability by probes is related to the Ewald conjecture, which we briefly state here. Let $\mathcal{S}(\Delta) = \Delta \cap (-\Delta) \cap \mathbb{Z}^n \setminus \{0\}$ be the set of non-zero symmetric integral points of Δ. The Ewald conjecture states that the set $\mathcal{S}(\Delta)$ contains an integral basis. Mcduff proved that every point except the barycentre is displaceable by probes if and only if Δ satisfies a slightly stronger property than the one conjectured by Ewald. For more dtails, we refer to the paper [McD11].

In our case, we only need to know the function $e_\Delta \colon x \mapsto e_M(T_x)$ on an open and dense subset of Δ and thus we can work directly with a variation of the Ewald conjecture which has been checked by Øbro [Obr07] for dimensions $\leqslant 8$ and by Paffenholz [Paf] for dimension 9. This approach is also used in [BCS22].

Definition 4.3.5. *The polytope Δ has property FS if every facet $F \subset \Delta$ contains a point of the set $\mathcal{S}(\Delta)$.*

Øbro and Paffenholz checked that all monotone polytopes in dimensions $\leqslant 9$ satisfy property FS. We therefore expect property FS to hold for all monotone Delzant polytopes. The two-dimensional case is obvious by the classification of four-dimensional toric monotone symplectic manifolds, see Figure 4.3. Let Δ_0 be the set of points $x \in \Delta$ such that $\min\{\ell_1(x), \dots, \ell_n(x)\}$ is attained by exactly one $\ell_i(x)$. This is an open, dense subset of Δ which is subdivided into chambers Δ_i by the hyperplanes $\ell_i = \ell_j$, see Figure 4.5 in Section 4.4.



(a) $S^2 \times S^2$ (b) $\mathbb{C}P^2$ (c) $X_1 = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

(d) $X_2 = \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ (e) $X_3 = \mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$

Figure 4.3: The set $\mathcal{S}(\Delta)$ for the moment polytopes of the five toric del Pezzo surfaces.

Lemma 4.3.6. *Let* (M, ω) *be a toric symplectic manifold whose moment polytope* Δ *satisfies property* FS*. Then*

$$e_\Delta(x) = \min\{\ell_1(x), \dots, \ell_k(x)\}$$

for all $x \in \Delta_0$*.*

Proof. The lower bound on the displacement energy follows from Proposition 4.3.2. For the upper bound, let $x \in \Delta_i$, which means that $\min\{\ell_1(x), \dots, \ell_k(x)\} = \ell_i(x)$. The set Δ_i is the cone $\{ty \,|\, t \in (0, 1],\ y \in \mathring{F}_i\}$ over the interior $\mathring{F}_i$ of the i-th facet F_i of Δ. We are going to construct a probe with respect to F_i and apply Proposition 4.3.4 for the upper bound. By the property FS, we can pick $u \in F_i \cap \mathcal{S}(\Delta)$. Since u is integrally transverse to F_i and $-u \in \Delta$, this yields a probe with the barycentre $0 \in \Delta$ as its midpoint. Take the unique probe $P_{i,u}(w)$ parallel to u which contains x, see Figure 4.4. We claim that the point x lies in the same half of $P_{i,u}(w)$ as w, which finishes the proof. If x lies on the segment with endpoints u and $-u$, then this is obvious. Otherwise, the three points u, $-u$ and x define a unique plane V. Now let $v \in V \cap \Delta$ be the endpoint of the segment $V \cap F_i$ which lies on the same side of the segment $[u, -u]$ as x. By convexity of Δ, the intersection $V \cap \Delta$ is convex and hence the segment $[-u, v]$ lies in $V \cap \Delta$. Therefore x lies in the same half of $P_{i,u}(w)$ as w. Figure 4.4 illustrates this construction in $V \cap \Delta$. □

4.4 Application I: Toric fibres

An important class of examples for Lagrangian tori are moment fibres in toric symplectic manifolds. In this Section we use Theorem 4.1.1 to give a criterion to exclude toric fibres from being real in terms of the function e_Δ. We assume that e_Δ is given by the affine distance to the boundary of the moment polytope, see Assumption 4.4.2. In Section 4.3, we proved that this assumption is reasonable in case the ambient manifold is monotone. In the present section, we do not assume monotonicity except for the proof of Theorem 4.1.2.

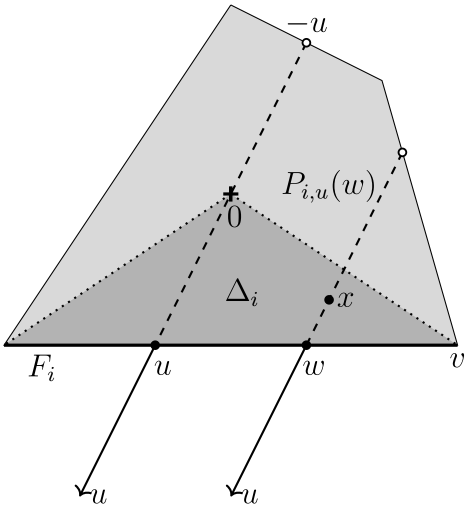


Figure 4.4: Construction of the probe $P_{i,u}(w)$.

Let (M^{2n}, ω) be a compact toric symplectic manifold with moment map μ and moment polytope Δ. For every point x in the interior $\mathring{\Delta}$ of the moment polytope, the set $T_x = \mu^{-1}(x)$ is a Lagrangian torus in M called toric fibre. Furthermore, the map

$$\begin{aligned} (H^1(T^n, \mathbb{R}^n), 0) \cong (\mathbb{R}^n, 0) &\rightarrow \{\text{Lagrangian tori in } M\} \\ a &\mapsto T_{x+a} = \mu^{-1}(x+a), \end{aligned}$$

that is defined for all a such that $x + a \in \mathring{\Delta}$, yields a versal deformation of T_x. Indeed, the components of μ give action coordinates on $\mu^{-1}(\mathring{\Delta})$ and thus T_{x+a} and T_{x+b} are related by a C^1-small Hamiltonian isotopy if and only if $a = b$. Varying x in $\mathbb{R}^n$ as above therefore yields an n-dimensional family of Hamiltonian isotopy classes of Lagrangian tori and hence a versal deformation of T_x.

As a warm-up example and as an illustration to Theorem 4.2.10, we consider the Clifford torus in products of S^2.

Example 4.4.1. Let (S^2, ω) be the unit 2-sphere in $\mathbb{R}^3$ equipped with the rescaled Euclidean area form $\omega = \frac{1}{2\pi}\text{area}$, for which $\int_{S^2} \omega = 2$. Let $H : S^2 \to \mathbb{R}$ be the projection to the z-axis $H(p) = z$. Since the Hamiltonian flow of H is 1-periodic, it defines a toric structure on S^2 with moment polytope $[-1, 1] \subset \mathbb{R}$. The level sets of $T_c = H^{-1}(c)$ are circles of fixed height and have displacement energy

$$e_{S^2}(T_c) = \begin{cases} 1 - |c| & \text{if } c \in [-1,1] \setminus \{0\}, \\ \infty & \text{if } c = 0. \end{cases}$$

Recall from Example 4.2.2 that the equator T_0 is real. In accordance with Theorem 4.2.10, the displacement energy germ $S_{T_0}(c) = e_{S^2}(T_c)$ is invariant under $c \mapsto -c$. Consider the n-fold product of this example. The corresponding moment map μ is given as the n-fold product of the above Hamiltonian H. The moment polytope is the unit square $\Delta = [-1, 1] \times \cdots \times [-1, 1] \subset \mathbb{R}^n$. The level sets of μ are products of circles of fixed height.

Their displacement energy is

$$e_{\times_n S^2}(T_{(c_1,\dots,c_n)}) = \begin{cases} \min\limits_{1\leqslant i\leqslant n}\{1-|c_i|\} & \text{if } (c_1,\dots,c_n)\in\Delta\setminus\{0\};\\ \infty & \text{if } (c_1,\dots,c_n)=0. \end{cases}$$

The *Clifford torus* T_0 is real, and its displacement energy germ

$$S_{T_0}(c_1,\dots,c_n) = e_{\times_n S^2}(T_{(c_1,\dots,c_n)})$$

is invariant under $(c_1,\dots,c_n)\mapsto(-c_1,\dots,-c_n)$.

We will now turn to the class of toric symplectic manifolds for which the level sets of the function

$$e_\Delta\colon\Delta\to\mathbb{R}\cup\{\infty\},\quad x\mapsto e_M(T_x)$$

look as in Figure 4.1 in Section 4.1, namely like scalings of $\partial\Delta$. Let

$$\ell_i(x)=\kappa_i-\langle x,v_i\rangle$$

be the functionals on $\mathbb{R}^n$ which define $\Delta=\{\ell_i\geqslant 0,\,\forall i\}$, where the v_i are the primitive outward pointing normal vectors to the facets, see Subsection 4.3.2. The facets F_i of Δ are given by the intersection of the moment polytope and the affine hyperplanes bounding the half-spaces, $F_i=\Delta\cap\{\ell_i=0\}$. For every $x\in\mathbb{R}^n$, the value $\ell_i(x)$ is equal to the affine distance of x to the corresponding facet F_i. See [McD11] for details.

Assumption 4.4.2. *For all x in an open dense subset of Δ, the displacement energy of the toric fibre over $x\in\Delta$ is given by the affine distance of x to the boundary $\partial\Delta$, i.e.*

$$e_\Delta(x)=\min\{\ell_1(x),\dots,\ell_k(x)\}.$$

If a variation of the Ewald conjecture holds, then this assumption is true for all monotone symplectic toric manifolds. See Section 4.3.4 for details.

For any $x\in\mathring{\Delta}$ define the set I_x of indices i for which the minimal affine distance to $\partial\Delta$ is attained by the corresponding ℓ_i, i.e. $i\in\{1,\dots,k\}$ belongs to I_x if and only if $\ell_i(x)=\min\{\ell_1(x),\dots\ell_k(x)\}$. Notice that if I_x is not a singleton, then x lies in a finite union of hyperplanes, see Figure 4.5.

Proposition 4.4.3. *Let $x\in\Delta$. Under Assumption 4.4.2, the displacement energy germ of the corresponding toric fibre is given by*

$$S_{T_x}(a)=\min_{i\in I_x}\{\ell_i(x+a)\},$$

for $a\in\mathbb{R}^n$ in an open dense subset around 0.

Proof. By Assumption 4.4.2, we have

$$S_{T_x}(a)=e(T_{x+a})=\min_{1\leq i\leq k}\{\ell_i(x+a)\}=\min_{i\in I_x}\{\ell_i(x+a)\}.$$

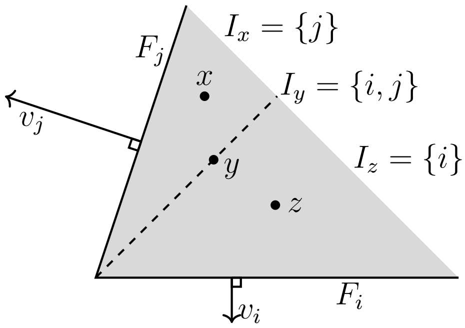


Figure 4.5: The set I_x for three different points.

The last equality holds since $I_{x+a} \subseteq I_x$ for small enough a. □

Proposition 4.4.4. *Let $x \in \Delta$ be such that $T_x \subset M$ is a real Lagrangian. Under Assumption 4.4.2 the moment polytope has to satisfy the following symmetry condition. For each $i \in I_x$ there is $j \in I_x$ such that $v_i = -v_j$. In particular, I_x contains an even number of elements.*

Proof. By Theorem 4.2.10, if T_x is real, then $S_{T_x}(a) = S_{T_x}(-a)$. By Proposition 4.4.3, this translates to

$$\min_{r \in I_x}\{\ell_r(x+a)\} = \min_{s \in I_x}\{\ell_s(x-a)\}$$

for a in an open neighbourhood of $0 \in \mathbb{R}^n$. For every $i \in I_x$, there is an open set U_i (which may not contain 0) such that $\ell_i(x+a) = \min_{r \in I_x}\{\ell_r(x+a)\}$ for all $a \in U_i$. Hence, there is $j \in I_x$ such that, possibly after shrinking the subset U_i, we have

$$\ell_i(x+a) = \ell_j(x-a), \quad \forall a \in U_i.$$

Using $\ell_i(x+a) = \ell_i(x) - \langle v_i, a\rangle$, we deduce that $\langle v_i + v_j, a\rangle = 0$ for all $a \in U_i$ and hence $v_i = -v_j$. □

We are now in a position to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. Assume that M is monotone and that its moment polytope satisfies property FS. Lemma 4.3.6 implies that Assumption 4.4.2 holds. Furthermore, since M is monotone, one can choose the moment polytope s.th. $\ell_i(0) = 1$ for all $i \in \{1, \dots, k\}$ and thus $I_0 = \{1, \dots, k\}$. Hence the theorem follows from Proposition 4.4.4. □

4.5 Application II: Chekanov tori

Chekanov tori were defined in [Che96] as the first examples of monotone Lagrangian tori in $\mathbb{C}^n$ which are not symplectomorphic to a product torus. In this section, we recall an alternative construction given in [CS10], see also [EP97], and show that the Chekanov torus can be embedded into any toric monotone symplectic manifold. Under the property FS, we compute its displacement energy germs and show that it is exotic and not real.

4.5.1 Embedding Chekanov tori

Let T^n act on $\mathbb{C}^n$ by the standard Hamiltonian torus action generated by the moment map

$$\nu\colon \mathbb{C}^n \to \mathbb{R}^n, \quad (z_1, \dots, z_n) \mapsto \pi(|z_1|^2, \dots, |z_n|^2) + (-1, \dots, -1).$$

The image of ν is the positive quadrant in $\mathbb{R}^n$ translated by the vector $(-1, \dots, -1)$. By $\widehat{T}^{n-1}$ we will denote the linear subtorus

$$\widehat{T}^{n-1} = \big\{ (e^{i\alpha_1}, \dots, e^{i\alpha_n}) \,|\, \alpha_1 + \ldots + \alpha_n = 0 \big\} \subset T^n, \tag{4.5.9}$$

which has a natural Hamiltonian action on $\mathbb{C}^n$. Now take a smooth embedded curve $\gamma(t) = r(t)e^{2\pi i \vartheta(t)}$ in $\mathbb{C}$ which encloses area 1 and for which

$$0 < \vartheta(t) < \frac{1}{n} \quad \text{and} \quad 0 < r(t) < \sqrt{\frac{n}{\pi}} + \delta, \tag{4.5.10}$$

for a small $\delta > 0$. From γ construct the curve $\Gamma(t) = \frac{1}{\sqrt{n}}(\gamma(t), \dots, \gamma(t))$ lying in the diagonal plane in $\mathbb{C}^n$.

Definition 4.5.1. *The* Chekanov torus Θ^n *in* $\mathbb{C}^n$ *is the torus swept out by* Γ *under the action of* $\widehat{T}^{n-1}$,

$$\Theta^n = \left\{ \frac{1}{\sqrt{n}} \left(e^{i\alpha_1}\gamma(t), \dots, e^{i\alpha_n}\gamma(t) \right) \in \mathbb{C}^n \,\middle|\, \alpha_1 + \ldots + \alpha_n = 0 \right\}.$$

The Chekanov torus is embedded, Lagrangian and monotone. Notice that $\nu(\Theta^n)$ is contained in the diagonal line, and by the choice of γ in (4.5.10) every component satisfies

$$\varepsilon - 1 < \nu_i(\Theta^n) < \varepsilon \tag{4.5.11}$$

for a small $\varepsilon > 0$, see Figure 4.6.

Remark 4.5.2. The Chekanov torus $\Theta^n \subset \mathbb{C}^n$ is not real. In fact, by the Smith inequality (4.1.3), tori in $\mathbb{C}^n$ cannot be realized as the fixed point set of a *smooth* involution.

Let M^{2n} be a toric monotone symplectic manifold with moment map μ. We show that Θ^n can be embedded into M. Pick a vertex v of its moment polytope $\Delta = \mu(M) = \{\ell_i \geqslant 0\}$. Since Δ is a Delzant polytope, we can assume (up to applying a transformation in $\mathrm{SL}(n, \mathbb{Z})$) that the facets meeting at v are parallel to the coordinate hyperplanes. By monotonicity, one can choose these hyperplanes to lie at affine distance 1 to the origin and hence $v = (-1, \dots, -1)$. In other words, we assume that the n first functionals defining Δ satisfy

$$\ell_1(x) = 1 + x_1, \; \dots \;, \ell_n(x) = 1 + x_n. \tag{4.5.12}$$

Equivalently, the moment polytope Δ near v has the same structure as $\nu(\mathbb{C}^n)$ near $\nu(0)$. This can be used to construct an embedding of Θ^n into M. By convexity of Δ, the line segment between the origin and v is contained in Δ. By (4.5.11) we can thus choose a neighbourhood $U \subset \nu(\mathbb{C}^n)$ of the segment $\nu(\Theta^n)$ which fits into Δ, see again Figure 4.6. This yields a T^n-equivariant symplectic embedding of the neighbourhood $\nu^{-1}(U)$ of Θ^n

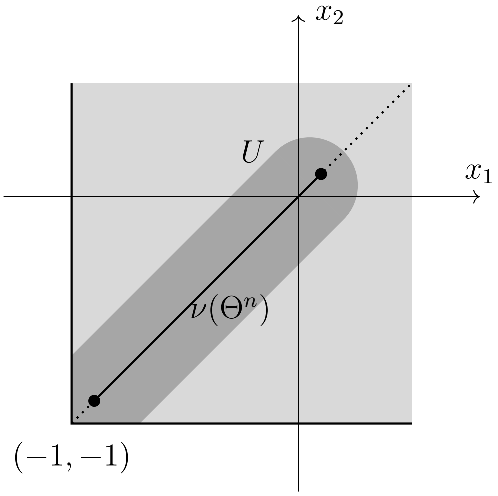


Figure 4.6: The image of $\Theta^n \subset \mathbb{C}^n$ under ν.

into M. Denote the so obtained Chekanov torus by Θ^n_M. By equivariance of the embedding, it is invariant under the $\widehat{T}^{n-1}$-action on M induced by μ.

Proposition 4.5.3. *Let M be a toric monotone symplectic manifold. Then the Chekanov torus embeds into M to yield a monotone Lagrangian torus $\Theta^n_M \subset M$.*

Proof. We prove that Θ^n_M is monotone. This means that the Maslov index and the area class are proportional on disks with boundary on Θ^n_M, i.e. that there is a $C > 0$ such that

$$\mathrm{Maslov}(D) = C \int_D \omega, \quad \forall D \in \pi_2(M, \Theta^n_M).$$

The homotopy long exact sequence yields

$$0 \to \pi_2(M) \to \pi_2(M, \Theta^n_M) \to \pi_1(\Theta^n_M) \to 0.$$

As a basis for $\pi_1(\Theta^n_M)$ we choose $[\Gamma]$ and the orbits of the $\widehat{T}^{n-1}$-action. The Maslov index and the area class vanish on the latter and $\mathrm{Maslov}([\Gamma]) = 2\int_{[\Gamma]} \omega = 2$. On spheres, the Maslov index is equal to twice the first Chern class of the ambient manifold. Recall that M is itself monotone with $c_1 = [\omega]$ (because of our choice of normalization $\kappa_i = 1$) and thus we obtain $\mathrm{Maslov}(D) = 2c_1(D) = 2\int_D \omega$ for all $D \in \pi_2(M)$. This proves that Θ^n_M is monotone with $C = 2$. □

Remark 4.5.4. In general, the tori Θ_M may depend on the choice of the vertex v. However, in the cases of $\times_n S^2$ and $\mathbb{C}P^n$ all vertices of the corresponding moment polytopes are interchangeable by an element of $\mathrm{SL}(n,\mathbb{Z})$ and hence we obtain a unique torus Θ_M up to symplectomorphism.

4.5.2 Versal deformations

Assume that M has property FS. For readability, we will write $\overset{\otimes}{=}$ for equalities that hold on an open dense subset of a neighbourhood of the origin of a vector space. By monotonicity, we can assume $\ell_i(0) = 1$ for all i and hence, by Proposition 4.4.3,

$$S_{T_0}(a) = e_\Delta(a) \overset{\otimes}{=} \min\{\ell_1(a), \dots, \ell_n(a)\}.$$

In particular, the displacement energy germ of the central fibre T_0 is determined by the moment polytope. The displacement energy germ of the corresponding Chekanov torus Θ_M is closely related to the one of T_0.

Lemma 4.5.5. *Let M be a toric monotone symplectic manifold satisfying property FS. Then the displacement energy germ of the Chekanov torus Θ^n_M is given by*

$$S_{\Theta^n_M} \overset{\otimes}{=} S_{T_0} \circ \phi. \tag{4.5.13}$$

Here $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is the piece-wise linear homeomorphism defined by (4.5.18) and (4.5.19), which does not depend on M.

Proof. We will closely follow the ideas used in [CS10] to compute $S_{\Theta_{S^2\times S^2}}$. Since there is no risk of confusion here, we denote the Chekanov torus by $\Theta = \Theta^n_M$. Let $\mu\colon M \to \mathbb{R}^n$ be the moment map for which Δ has the form (4.5.12). Notice that the subtorus $\widehat{T}^{n-1}$ defined by equation (4.5.9) has a natural Hamiltonian action on M via the inclusion $\widehat{T}^{n-1} \subset T^n$ and that Θ is invariant under this torus action. The moment map $\widehat{\mu}\colon M \to \mathbb{R}^{n-1}$ corresponding to the $\widehat{T}^{n-1}$-action is given by

$$\widehat{\mu} = (\mu_1 - \mu_n, \dots, \mu_{n-1} - \mu_n). \tag{4.5.14}$$

As a basis of $H_1(\Theta, \mathbb{Z})$, we choose the class $[\Gamma]$ of the curve lying in the diagonal and the classes $[\tau_1], \dots, [\tau_{n-1}]$ of the orbits of the $\widehat{T}^{n-1}$-action. The latter can also be seen as the closed orbits of the Hamiltonians $\mu_i - \mu_n$. By the equivariant Weinstein neighbourhood theorem, we can choose a versal deformation of Θ which preserves the $\widehat{T}^{n-1}$-orbit structure. Let $t_1, \dots, t_{n-1}$ and s be the deformation parameters corresponding to the classes $[\tau_1], \dots, [\tau_{n-1}]$ and $[\Gamma]$. This means that the deformation parameters $t_1, \dots, t_{n-1}$ measure the change in symplectic area of the disks with boundary $[\tau_1], \dots, [\tau_{n-1}]$ and s the change in area of the disk bounding $[\Gamma]$. For small deformation parameters, the resulting deformation yields an embedded torus. For convenience we denote $\mathbf{t} = (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$. Since $\widehat{T}^{n-1}$-orbits are preserved, we find that the Lagrangian neighbour $\Theta_{\mathbf{t},s}$ of Θ maps to a line segment $\mu(\Theta_{\mathbf{t},s})$ parallel to $\mu(\Theta)$. Furthermore, by equation (4.5.14), the line segment $\mu(\Theta_{\mathbf{t},s})$ is contained in the line

$$L_{\mathbf{t}} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 - x_n = t_1, \dots, x_{n-1} - x_n = t_{n-1}\}. \tag{4.5.15}$$

See Figure 4.7. We prove that whenever $t_i \neq 0$ for all $1 \leqslant i \leqslant n-1$, the versal deformation $\Theta_{\mathbf{t},s}$ of Θ is Hamiltonian isotopic to a toric fibre $T_x = \mu^{-1}(x)$ for a suitable $x = (x_1, \dots, x_n)$. Since the displacement energy is preserved under Hamiltonian isotopies, property FS and Lemma 4.3.6 yield the displacement energy germ of Θ. Notice that if $t_i \neq 0$ for all i, then $\widehat{T}^{n-1}$ acts freely on the set

$$Z_{\mathbf{t}} = \mu^{-1}(L_{\mathbf{t}} \cap \Delta \setminus \{y_2\}) = \widehat{\mu}^{-1}(\mathbf{t}) \setminus \mu^{-1}(y_2).$$

The intersection $L_{\mathbf{t}} \cap \Delta$ consists of two points. We call y_1 the point in this intersection

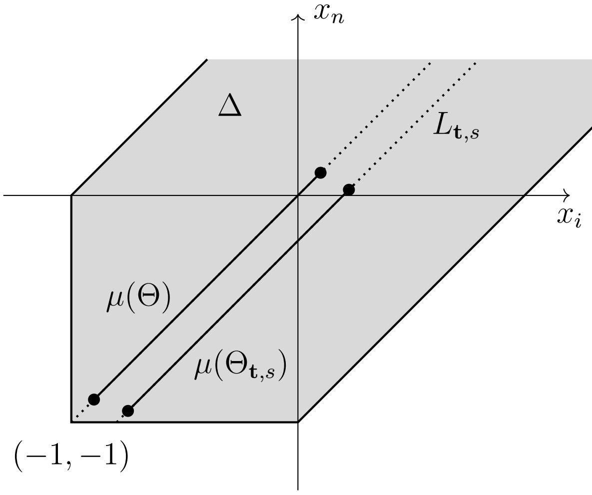


Figure 4.7: Versal deformation of $\Theta \subset M$.

closest to the vertex $(-1,\dots,-1)$ and y_2 the point furthest away. Since the intersection $L_{\mathbf{t}} \cap \Delta$ in y_1 is integral transversal as in the proof of Proposition 4.3.4, the action is indeed free. Note that this is in general not the case in y_2 and this is why we remove it. Hence, we can perform symplectic reduction by $\widehat{T}^{n-1}$ on $Z_{\mathbf{t}}$

$$\begin{tikzcd}
\Theta_{\mathbf{t},s} \subset Z_{\mathbf{t}} \arrow[r, hook, "i"] \arrow[d, two heads, "p"] & (M,\omega) \\
c_{\mathbf{t},s} \subset (M_{\mathbf{t}},\omega_{\mathbf{t}}). &
\end{tikzcd}$$

The symplectic quotient $(M_{\mathbf{t}},\omega_{\mathbf{t}})$ is symplectomorphic to a disk of radius equal to the affine length of $L_{\mathbf{t}} \cap \Delta$. Indeed, since there is a Hamiltonian T^n-action on $Z_{\mathbf{t}}$ the reduced space has an induced Hamiltonian S^1-action with moment polytope $L_{\mathbf{t}} \cap \Delta \setminus \{y\}$. Since $\Theta_{\mathbf{t},s}$ is $\widehat{T}^{n-1}$-invariant, it projects to a circle $c_{\mathbf{t},s} = p(\Theta_{\mathbf{t},s})$. We claim that this circle encloses symplectic area $1+s$. Since the $\widehat{T}^{n-1}$-orbits $\tau_1,\dots,\tau_{n-1}$ are divided out by the above symplectic reduction, the circle $c_{\mathbf{t},s}$ corresponds to the class $[\Gamma]$ in $\Theta_{\mathbf{t},s}$. The latter class bounds a disk of area $1+s$ in M since s is the deformation parameter of $[\Gamma]$. By symplectic reduction we have $p^*\omega_{\mathbf{t}} = i^*\omega$ and hence $c_{\mathbf{t},s}$ encloses area $1+s$. It is thus Hamiltonian isotopic to the circle $S^1(1+s)$ centered in the origin of the disk $M_{\mathbf{t}}$ which bounds the same area. The pre-image $p^{-1}(S^1(1+s))$ is a toric fibre T_x and thus the Hamiltonian isotopy in the quotient can be lifted to M to yield a Hamiltonian isotopy between $\Theta_{\mathbf{t},s}$ and T_x.

Now, let $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ be the map that takes $(\mathbf{t},s)$ to x such that $\Theta_{\mathbf{t},s}$ and T_x are Hamiltonian isotopic. Note that this defines ϕ only on an open dense subset of a neighbourhood of $0 \in \mathbb{R}^n$ on which we have

$$S_\Theta(\mathbf{t},s) = e_M(\Theta_{\mathbf{t},s}) \overset{\otimes}{=} e_M(T_{\phi(\mathbf{t},s)}) = e_\Delta(\phi(\mathbf{t},s)) \overset{\otimes}{=} S_{T_0}(\phi(\mathbf{t},s)).$$

We now determine the map ϕ. Let $(\mathbf{t},s) \in \mathbb{R}^n$ be such that ϕ is defined. The point $x = \phi(\mathbf{t},s)$ lies on $L_{\mathbf{t}}$ and hence

$$t_1 = x_1 - x_n, \dots, t_{n-1} = x_{n-1} - x_n.$$

Recall that $y_1 \in \partial\Delta$ is the point close to $(-1, \ldots, -1)$ in which $L_{\mathbf{t}}$ intersects the boundary $\partial\Delta$. The area enclosed by $S^1(1+s) \subset M_{\mathbf{t}}$ is equal to the affine length of the line segment $[z, x]$, which in turn is equal to $1 + \min\{x_1, \ldots, x_n\}$ and hence

$$s = \min\{x_1, \ldots, x_n\}. \tag{4.5.16}$$

The map ϕ we are looking for is thus given as the inverse of

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_n \\ \vdots \\ x_{n-1} - x_n \\ \min\{x_1, \ldots, x_n\} \end{pmatrix}. \tag{4.5.17}$$

There is a unique extension to a piece-wise linear homeomorphism on all of $\mathbb{R}^n$. By distinguishing cases we obtain

$$\phi(\mathbf{t}, s) = \begin{pmatrix} s + t_1 \\ \vdots \\ s + t_{n-1} \\ s \end{pmatrix}, \tag{4.5.18}$$

whenever all $t_i \geqslant 0$ and

$$\phi(\mathbf{t}, s) = \begin{pmatrix} s + t_1 - t_i \\ \vdots \\ s + t_{n-1} - t_i \\ s - t_i \end{pmatrix}, \tag{4.5.19}$$

if $t_i < 0$ and t_i is minimal among all t_j. □

Instead of working directly with the displacement energy germ S_L of a Lagrangian L, it is often useful to look at its level sets $S_L^{-1}(c)$ for some $c > 0$. In particular, if L is real, then these level sets are centrally symmetric, by Theorem 4.2.10. In the case of T_0, the level sets are rescalings of Δ,

$$S_{T_0}^{-1}(c) \overset{\otimes}{=} \lambda\Delta, \quad \lambda > 0. \tag{4.5.20}$$

Here we mean that both sets agree when intersected with a set which is open and dense in the neighbourhood of the origin. Since $S_{\Theta_M^n} \overset{\otimes}{=} S_{T_0} \circ \phi$, we obtain

$$S_{\Theta_M^n}^{-1}(c) \overset{\otimes}{=} \phi^{-1}(S_{T_0}^{-1}(c)) \overset{\otimes}{=} \lambda\phi^{-1}(\Delta). \tag{4.5.21}$$

This allows us to understand the versal deformation of Θ_M^n by applying ϕ^{-1} to the moment polytope Δ. The inverse of ϕ is given by equation (4.5.17) and its image $\phi^{-1}(\Delta)$ is a gain a convex polytope.

We will now prove that one can pick a suitable vertex v for which the embedding of the Chekanov torus constructed in Subsection 4.5.1 yields an exotic Lagrangian torus in M, i.e. a torus which is not symplectomorphic to a toric fibre. For this, let F_0 be a facet of the moment polytope Δ which contains the maximal number of integral points among all facets of Δ, let v to be any vertex contained in F_0 and let Θ^n_M be the Chekanov torus embedded with respect to v. A priori, Θ^n_M can only be symplectomorphic to the central toric fibre, since all other fibres are not monotone. By (4.5.20) and (4.5.21), it suffices to show that the polytopes Δ and $\phi^{-1}(\Delta)$ are not $\mathrm{GL}(n,\mathbb{Z})$-equivalent in order to show that T_0 and Θ^n_M are not symplectomorphic. This follows from the invariance of versal deformations under symplectomorphisms, see for example [CS16, §2.2]. Note that the maximal number of lattice points in a facet is a $\mathrm{GL}(n,\mathbb{Z})$-invariant of polytopes and thus it suffices to show that this invariant strictly increases when we apply ϕ^{-1} with respect to v. Assume that Δ is given in the normal form (4.5.12) with respect to v and hence the minimum $\min\{x_1,\dots,x_n\}$ is constant and equal to -1 on all facets containing $v=(-1,\dots,-1)$. Therefore ϕ^{-1} maps all facets containing v (in particular F_0) to the same facet of $\phi^{-1}(\Delta)$. Since ϕ^{-1} is a bijection on the lattice, the facet maximal number of integral points in a facet strictly increases when we pass from Δ to $\phi^{-1}(\Delta)$. We have shown

Proposition 4.5.6. *Let M be a toric monotone symplectic manifold satisfying property FS. Then M contains an exotic copy of the Chekanov torus.*

Remark 4.5.7. The following example shows that the right choice of the vertex v is crucial for the obtained Chekanov torus to be distinguishable from the central fibre by versal deformations. The polytope in $\mathbb{R}^2$ defined by the functionals

$$1+x_1,\ 1\pm x_2,\ 1-x_1+x_2$$

is the moment polytope of the one-fold blow-up X_1 of $\mathbb{C}P^2$. The level sets of $S_{\Theta^2_{X_1}}$ when Θ^2 is embedded with respect to the vertex $(-1,-1)$ are rescalings of the polytope defined by

$$1-t,\ 1\pm s,\ 1+t-s.$$

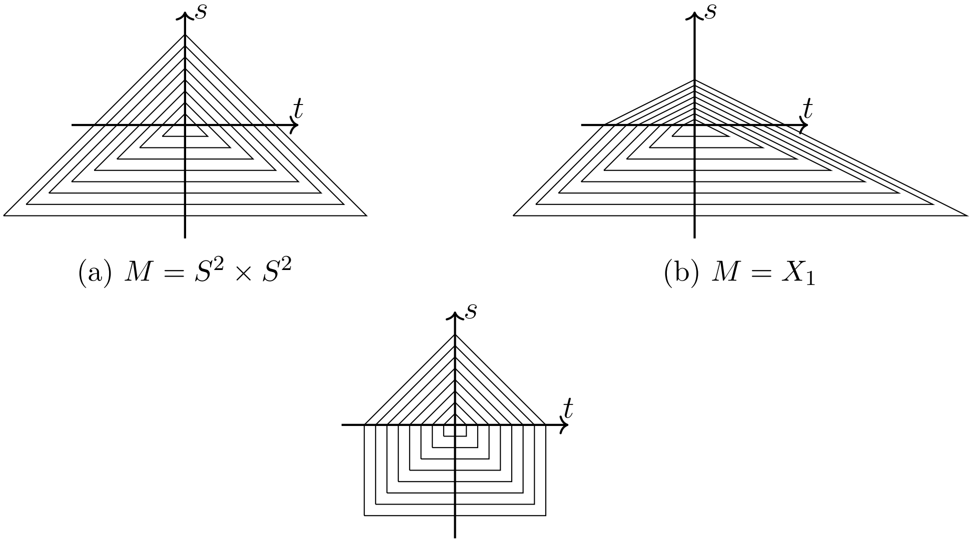
Since these two polytopes are related by an element in $\mathrm{GL}(2,\mathbb{Z})$, versal deformations cannot distinguish between $T^2_0\subset X_1$ and $\Theta^2_{X_1}$.

4.5.3 Chekanov tori are not real

As a warm-up, let $M\in\{S^2\times S^2,\mathbb{C}P^2,X_1,X_2,X_3\}$ be one of the five toric monotone symplectic manifolds in dimension 4. See Figure 4.3 in Section 4.3 for their moment polytopes. Then Θ^2_M is not real. The existence of real Lagrangian tori in $M=\mathbb{C}P^2$ and $M=X_2$ is excluded by the Smith inequality, see (4.1.3) in Section 4.1. Applying ϕ^{-1} to the moment polytopes of the remaining three cases shows that the corresponding Chekanov tori are not real either, since the level sets of their displacement energy germs are not centrally symmetric, see Figure 4.8. This can be generalized to all Θ^n_M.

Theorem 4.5.8. *Let M be a toric monotone symplectic manifold satisfying property FS. Then the Chekanov torus Θ^n_M is not real.*

Proof. Again, we suppose that Δ is in the form (4.5.12) with distinguished vertex $v=(-1,\dots,-1)$. In order to understand the versal deformation of Θ^n_M, we apply ϕ^{-1} to the moment polytope as in (4.5.21). The vertex v is mapped to $-e_n$ and all facets surrounding



(a) $M = S^2 \times S^2$ (b) $M = X_1$

(c) $M = X_3$

Figure 4.8: Level sets of the function $S_{\Theta^n_M}$.

it to the hypersurface $\{s = -1\}$. Hence, if $U \subset \mathbb{R}^n$ is a neighbourhood of v, then there is a neighbourhood $V \subset \mathbb{R}^n$ of $-e_n$ such that

$$\phi^{-1}(U \cap \partial\Delta) = V \cap \{s = -1\} \subset \partial\phi^{-1}(\Delta).$$

Now suppose that Θ^n_M is real and hence, by (4.5.21) that $\phi^{-1}(\Delta)$ is centrally symmetric. This implies that

$$(-V) \cap \{s = 1\} \subset \partial\phi^{-1}(\Delta)$$

Since $-V$ is a neighbourhood of e_n, points of the form $e_n + re_i$ belong to $(-V) \cap \{s = 1\}$ and hence to $\phi^{-1}(\Delta)$ for small $r > 0$ and $i \neq n$. This implies that $\phi(e_n + re_i) \in \Delta$. Observe that $\phi(e_n + re_i) = (1, \dots, 1) + re_i$ by equation (4.5.18). Since $(1, \dots, 1)$ is integral, it does not belong to the interior of Δ and hence $\phi(e_n + re_i) \in \Delta$ contradicts the convexity of the moment polytope. See Figure 4.9, the grey areas belong to the respective polytopes in case Θ^n_M is real. □

One may wonder whether Theorem 4.5.8 reflects a symplectic phenomenon or a smooth one. This is not obvious in general, but we discuss the case in which the moment polytope of M is centrally symmetric. See Section 4.1 for a discussion and the classification of manifolds having this property. Although Θ^n_M is not real (M has property FS whenever Δ is centrally symmetric), we prove that it can be realized as the fixed point set of a smooth involution.

Proposition 4.5.9. *Let M be a toric monotone symplectic manifold which has a centrally symmetric moment polytope $\Delta = -\Delta$. Then the Chekanov torus Θ^n_M is the fixed point set of a smooth involution.*

Proof. It is proved in [BKM19] that the central fibre T_0 is real whenever $\Delta = -\Delta$. Hence we can take an anti-symplectic involution σ of M such that $\operatorname{Fix}\sigma = T_0$. We claim that

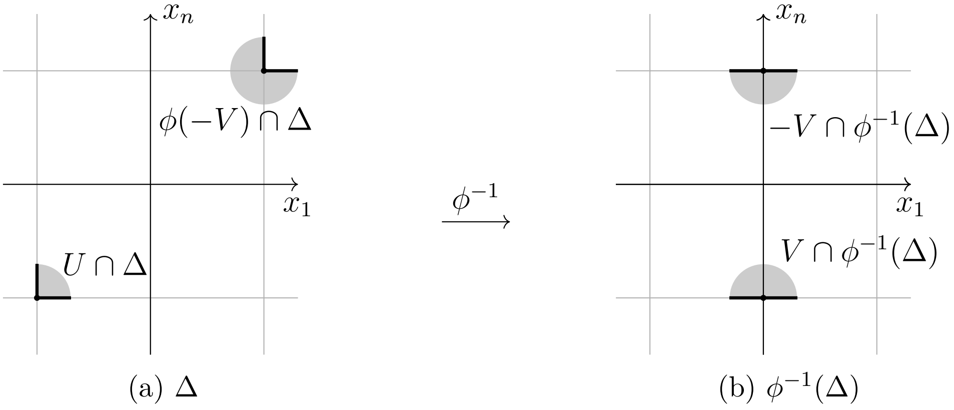


Figure 4.9: Idea of the proof of Theorem 4.5.8, the set $\phi(-V) \cap \Delta$ is not convex.

there is a $\psi \in \mathrm{Diff}(M)$ such that $\psi(\Theta^n_M) = T_0$. Then

$$\Theta^n_M = \mathrm{Fix}(\psi^{-1} \circ \sigma \circ \psi)$$

is the fixed point set of a smooth involution. The existence of ψ follows from the proof of Lemma 4.5.5. Indeed, Θ^n_M is smoothly isotopic to all versal deformations $\Theta_{\mathbf{t},s}$ and whenever $\mathbf{t} \neq 0$, we have proved that $\Theta_{\mathbf{t},s}$ is isotopic to a toric fibre T_x. Since all toric fibres are isotopic, so are Θ^n_M and T_0. ☐

4.5.4 More examples in $\times_n S^2$

In order to obtain more than only one example of non-real exotic Lagrangian torus in a given toric manifold, one may try to embed higher twist tori, see [CS10] or products of Chekanov tori. We will discuss the second case here. For $\mathbf{k} = (k_1, \ldots, k_s)$ with $k_i \geqslant 2$ and $s \geqslant 1$, define the product

$$\Theta^{\mathbf{k},m} = \Theta^{k_1} \times \ldots \times \Theta^{k_s} \times T^m_0 \subset \mathbb{C}^n, \quad \sum_{i=1}^{s} k_i + m = n,$$

where T^m_0 denotes the Clifford torus in $\mathbb{C}^m$. The image of such products under the standard moment map ν in $\mathbb{C}^n$ is given by a hypercube formed by the product of diagonal segments

$$\nu(\Theta^{\mathbf{k},m}) = \{(\underbrace{r_1, \ldots, r_1}_{k_1}, \ldots, \underbrace{r_s, \ldots, r_s}_{k_s}, \underbrace{0, \ldots, 0}_{m}) \in \mathbb{R}^n | \, \varepsilon - 1 < r_i < \varepsilon\}.$$

In order to embed $\Theta^{\mathbf{k},m}$ in a toric monotone symplectic manifold M with moment polytope Δ, one may try to apply the same strategy as for Θ^n, namely put Δ in the normal form (4.5.12) and see if $\nu(\Theta^{\mathbf{k},m})$ lies inside Δ. If it is so, the resulting torus is not real.

Proposition 4.5.10. *Let M be a toric monotone symplectic manifold satisfying property FS. Assume furthermore that $\Theta^{\mathbf{k},m}$ can be embedded as described above. Then the image $\Theta^{\mathbf{k},m}_M \subset M$ is a monotone Lagrangian torus which is not real.*

Proof. Monotonicity follows from the same arguments as in the proof of Proposition 4.5.3.

In order to prove that $\Theta_M^{\mathbf{k},m}$ is not real, we compute its displacement energy germ

$$S_{\Theta_M^{\mathbf{k},m}} \overset{\otimes}{=} S_{T_0^n} \circ \phi_{\mathbf{k},m}. \tag{4.5.22}$$

Here $\phi_{\mathbf{k},m} : \mathbb{R}^n \to \mathbb{R}^n$ is the piece-wise linear homeomorphism given as a product of the map ϕ defined as in the proof of Lemma 4.5.5,

$$\phi_{\mathbf{k},m} = \phi_{k_1} \times \ldots \times \phi_{k_s} \times \mathrm{id}_m.$$

Indeed, note that the normal form (4.5.12) of v splits in $\mathbb{R}^n = \mathbb{R}^{k_1} \times \ldots \times \mathbb{R}^{k_s} \times \mathbb{R}^m$ as the product of vertices in normal form. Hence the argument given in Lemma 4.5.5 can be carried out on the factors. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{k_1}$ be the projection to the first k_1 coordinates. Assume that the polytope $\phi_{\mathbf{k},m}^{-1}(\Delta)$ is centrally symmetric. Then so is its projection $\pi(\phi_{\mathbf{k},m}^{-1}(\Delta))$. By the product structure of $\phi_{\mathbf{k},m}$, we have $\pi(\phi_{\mathbf{k},m}^{-1}(\Delta)) = \phi_{k_1}^{-1}(\pi(\Delta))$. By convexity of Δ, the projection $\pi(\Delta)$ is in normal form at the vertex $\pi(v)$ and hence we can apply the same argument as in the proof of Theorem 4.5.8 to get a contradiction to the convexity of $\pi(\Delta)$. □

To enumerate the non-real product tori $\Theta_M^{\mathbf{k},m}$ that one obtains by this method up to symplectomorphism, one should now solve the following two problems: First, for which vertices v does $\Theta^{\mathbf{k},m}$ fit into the normal form of Δ at v? Second, which of the so-obtained tori $\Theta_M^{\mathbf{k},m}$ (also depending on the vertex v) are exotic and which are pairwise non-symplectomorphic? For a general M, both problems seem to involve complicated combinatorics outside of the scope of the present paper, whence we will only carry out the details for $M = \times_n S^2$. In that case, all tori $\Theta^{\mathbf{k},m}$ embed, the embedding does not depend on the vertex v, and all tori $\Theta_M^{\mathbf{k},m}$ turn out to be pairwise distinct.

Let $M = \times_n S^n$. As we have seen in (4.5.21), we can understand the versal deformation of Θ_M^n by applying ϕ^{-1} to the moment polytope $\Delta = [-1,1]^n$ of M. We call the resulting polytopes **Chekanov polytopes** and denote them by

$$\mathrm{CP}_n = \phi^{-1}(\Delta).$$

We have a closer look at the geometry and the combinatorics of CP_n. Notice that $s = \min\{x_1, \ldots, x_n\}$ is equal to -1 on all facets that contain the vertex $(-1, \ldots, -1)$. In other words, all of these facets are mapped to the hyperplane $\{s = -1\}$ by ϕ^{-1}. The one remaining vertex $(1, \ldots, 1)$ is mapped to e_n. Hence CP_n has the structure of a convex cone over the $(n-1)$-dimensional polytope $P_{-1} = \{s = -1\} \cap \mathrm{CP}_n$. In order to understand CP_n, we thus need to understand P_{-1} in the hyperplane $\{s = -1\} \cong \mathbb{R}^{n-1}$. We claim that P_{-1} is equal to the polytope obtained by sweeping out the standard $(n-1)$-hypercube along $r(1, \ldots, 1)$ for all $r \in [-1,1]$. This follows from equation (4.5.17), which yields

$$P_{-1} = \{(x_1 - x_n, \ldots, x_{n-1} - x_n) \,|\, x_i \in [-1,1] \text{ and } \min\{x_i\} = -1\}.$$

The polytope CP_n has $2^n - 1$ vertices, since ϕ^{-1} maps vertices to vertices except for $(-1, \ldots, -1)$ which is mapped to the interior of P_{-1}. The valencies of the vertices are given by

$$V(\mathrm{CP}_n) = ((2^n-2)^{\times 1}, (n+1)^{\times(2^n-2n-2)}, n^{\times 2n}), \tag{4.5.23}$$

for $n \geqslant 3$ where $l^{\times k}$ means that there are k vertices of valency l. We also view $V(P)$ as a vector with as many entries (in decreasing order) as the polytope P has vertices and call $V(P)$ the *valency vector* of P. For $n=2$, we have $V(\mathrm{CP}_2) = (2^{\times 3})$, as illustrated by Figure 4.8. The general case can be seen as follows. The valency of the vertex at the apex of the cone is equal to the number of vertices of P_{-1} and hence equal to 2^n-2. As we have seen, any other vertex w that is not the apex lies in $\{s=-1\}$ and hence it is a vertex of $P_{-1} \subset \mathbb{R}^{n-1}$. Furthermore, since CP_n is a cone over P_{-1}, we can count the valency of $w \in \mathrm{CP}_n$ by adding 1 to the valency of the same vertex considered in P_{-1}. Thus we have reduced the problem to counting valencies in P_{-1}.

Recall that P_{-1} is obtained by sweeping out the standard $(n-1)$-hypercube $C_0 = [-1,1]^{n-1}$ along the main diagonal,

$$P_{-1} = \bigcup_{r\in[-1,1]} C_r = \bigcup_{r\in[-1,1]} [-1+r, 1+r]^{n-1}.$$

In the two extremal cases $r=1$ and $r=-1$, we obtain the two shifted hypercubes $C_1 = [0,2]^{n-1}$ and $C_{-1} = [-2,0]^{n-1}$ and P_{-1} is the convex hull of the union $C_1 \cup C_{-1}$. We deduce that every vertex w of P_{-1} is a vertex of C_1 or of C_{-1}. By symmetry of the polytope P_{-1}, we can restrict our attention to understanding the vertices of P_{-1} that are also vertices of C_1. Note also that we create at most one more outgoing edge at each such vertex when passing from C_1 to P_{-1} by sweeping, namely along the sweeping direction $d = (-1, ..., -1)$. Since each vertex in C_1 has valency $n-1$, the valency of each of the vertices of P_{-1} is either $n-1$ or n. There are essentially four cases:

1. The vertex $(2,\dots,2) \in C_1$ is a vertex with the same edges in P_{-1}. Hence it appears with valency $n-1$ in the valency count of P_{-1}.

2. The vertex $(0,\dots,0) \in C_1$ lies in the interior of P_{-1}, thus it does not appear in the valency count of P_{-1}.

Now let $\varepsilon = (\varepsilon_1,\dots,\varepsilon_{n-1}) \in C_1$ be any other vertex of C_1, meaning that $\varepsilon_i \in \{0,2\}$ with at least one $\varepsilon_i = 0$ and one $\varepsilon_j = 2$. These vertices all form vertices of P_{-1} and one new outgoing edge is created by the sweeping, since the vector $d = (-1,\dots,-1)$ (along which we sweep) points outwards of C_1 at ε. However, in some cases, the sweeping also deletes one edge when passing from C_1 to P_{-1}, meaning that this edge of C_1 points into the interior of P_{-1}.

3. Suppose that the vertex $\varepsilon \in P_{-1}$ has all components $=0$, except for one $\varepsilon_i = 2$. Then the outgoing edges of C_1 in ε have directional vectors $e_1,\dots,e_{i-1},-e_i,e_{i+1},\dots,e_{n-1}$, where e_j denotes the j-th standard basis vector. In P_{-1}, there is an additional edge vector $d = -e_1 - e_2 - \ldots - e_{n-1}$. Note however that we can write

$$-e_i = d + e_1 + \ldots + e_{i_1} + e_{i+1} + \ldots + e_{n-1}, \tag{4.5.24}$$

meaning that $-e_i$ is a positive linear combination of the other edges. Hence the edge $-e_i$ at ε in C_1 is not an edge anymore in P_{-1}, since it points into the interior of P_{-1}. Therefore, the valency of the $n-1$ vertices of this type is $n-1$.

4. In all other cases, the directional vectors of the edges are $\pm e_1,\dots,\pm e_n$ with at

least two minus signs. This means that none of these vectors can be written as a positive linear combination of the remaining ones and d as was the case in (4.5.24). Hence the valency of these remaining $2^{n-1}-n-1$ vertices is n.

By doubling all of these valency counts due to the symmetry of P_{-1}, we obtain

$$V(P_{-1}) = (n^{\times 2^n-2n-2}, (n-1)^{2n})$$

Since CP_n is a cone over P_{-1}, we obtain the valency vector (4.5.23).

Proposition 4.5.11. *If two Lagrangian tori $\Theta_M^{\mathbf{k},m}$ and $\Theta_M^{\mathbf{k}',m'}$ in $M = \times_n S^2$ are symplectomorphic, then $\mathbf{k}=\mathbf{k}'$ and $m=m'$.*

Together with Proposition 4.5.10 we conclude that all the tori $\Theta_M^{\mathbf{k},m}$ in $\times_n S^2$ are not real and mutually not symplectomorphic. The number of such tori is $p(n)-1$, where $p(n)$ is the number of partitions of n.

Proof of Proposition 4.5.11. By the product structure of M and (4.5.22), the level sets of the displacement energy germ of $\Theta_M^{\mathbf{k},m}$ are given by the product of Chekanov polytopes and intervals $I=[-1,1]$,

$$S^{-1}_{\Theta_M^{\mathbf{k},m}}(c) \overset{\otimes}{=} \mathrm{CP}_{k_1} \times \ldots \times \mathrm{CP}_{k_s} \times I^m.$$

Hence, it suffices to show that $\mathbf{k}=(k_1,\ldots,k_s)$ and m are determined by the $\mathrm{GL}(n,\mathbb{Z})$-equivalence class of $\times_i \mathrm{CP}_{k_i} \times I^m$. This again follows from the $\mathrm{GL}(n,\mathbb{Z})$-invariance of versal deformations of tori, see [CS16, §2.2]. In order to prove this, we associate to the latter polytopes the vector counting emanating edges at its vertices in decreasing order as in (4.5.23). This datum is a $\mathrm{GL}(n,\mathbb{Z})$-invariant of polytopes. Note that if P and P' are polytopes, we have for the respective valency vectors

$$V(P\times P') = V(P)\oplus V(P'), \tag{4.5.25}$$

where the operation $\oplus$ on vectors $a=(a_1,\ldots,a_{k_1})$ and $b=(b_1,\ldots,b_{k_2})$ with $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_{k_1}$ and $b_1 \geqslant b_2 \geqslant \ldots \geqslant b_{k_2}$ is defined as the vector of all possible sums in decreasing order

$$a\oplus b = (a_1+b_1,\ldots,a_{k_1}+b_{k_2}),$$

This operation is commutative and associative, and hence we obtain

$$V(\mathrm{CP}_{k_1}\times\ldots\times\mathrm{CP}_{k_s}\times I^m) = V(\mathrm{CP}_{k_1})\oplus\cdots\oplus V(\mathrm{CP}_{k_s})\oplus V(I^m).$$

Furthermore, this operation is invertible in the following sense. Let $c=(c_1,\ldots,c_{k_1k_2})$ denote $a\oplus b$. Then a is determined by c and b; in other words, there is an operation $\ominus$ with $c\ominus b = a$. We will prove this by induction on the length k_1 of a. The case $k_1=1$ is obvious. In case $k_1=l+1$, note that a_1, b_1 and $c_1=a_1+b_1$ are by convention the maximal components of the corresponding vectors and hence a_1 is given by c_1-b_1. The situation can be reduced to the case $k_1=l$ by removing the value a_1 from a and the values $a_1+b_1,\ldots,a_1+b_{k_2}$ from c.

We will now successively split off factors from the product polytope using the operation $\ominus$. First, notice that the multiplicity of the maximal entry of $V(\times_i \mathrm{CP}_{k_i}\times I^m)$ determines

m and p, where p is the number of times we have $k_i = 2$. Indeed, we have $V(I) = (1^{\times 2})$ and $V(\mathrm{CP}_2) = (2^{\times 3})$ and by equation (4.5.25) the multiplicity of the maximal entry is given by $2^m 3^p$. Indeed, by equation (4.5.23), the maximal entry in each $V(\mathrm{CP}_{k_i})$ is unique for $k_i \geqslant 3$. Since the maximal entry in $V(\times_i \mathrm{CP}_{k_i} \times I^m)$ is obtained as a sum of maximal entries of the valency vectors of all factors, its multiplicity in $V(\times_i \mathrm{CP}_{k_i} \times I^m)$ can only be larger than one if there are factors of the type I or CP_2 and in that case it will be $2^m 3^p$. Hence the prime decomposition of this multiplicity yields m and p. After splitting off the corresponding factors, we can assume that $m = 0$ and $k_i \geqslant 3$. Let M_1 and M_2 be the largest and the second largest component of the valency vector. Then we have $M_1 = \sum_{i=1}^{s} 2^{k_i} - 2s$ and $M_1 - M_2 = 2^{k_{\min}} - k_{\min} - 3$, where $k_{\min}$ is minimal among all k_i. Therefore $M_1 - M_2$ determines $k_{\min}$ and we can split off $V(\mathrm{CP}_{k_{\min}})$ from the valency vector by using formula (4.5.23). □

4.6 Appendix: Alternate approach using J-holomorphic disks

In this appendix, we outline an alternate approach to Theorem 4.1.2 based on the count of J-holomorphic Maslov 2 disks with boundary on the Lagrangian, which was introduced in [EP97] and [Che97] and was used in [Kim21a] to determine whether a given Lagrangian is real. This approach is less elementary than the above, but has the advantage of avoiding property FS.

Let T_0 be the central fibre in a toric monotone symplectic manifold (M, ω) with moment polytope $\Delta = \{\langle x, v_i \rangle \leqslant 1\}$. Assuming that T_0 is the fixed point set of an anti-symplectic involution σ, we will show that Δ is centrally symmetric. Fix an ω-compatible almost-complex structure J on M and a homology class $\xi \in H_1(T_0, \mathbb{Z})$ and define the moduli space

$$\begin{aligned} \mathcal{M}(T_0, J, \xi) = \{u\colon (D, \partial D) \to (M, T_0) \quad | \quad & u \ J\text{-holomorphic}, \\ & \mathrm{Maslov}(u) = 2, \\ & [\partial u] = \xi\}/\sim, \end{aligned}$$

where $\sim$ denotes the equivalence relation induced by reparametrizing the domain D by biholomorphisms fixing the point $1 \in \partial D$. We can count (mod 2) the elements of $\mathcal{M}(T_0, J, \xi)$ whose boundary passes through a given point on T_0 by taking the degree $n(T_0, J, \xi) \in \mathbb{Z}$ of the evaluation map

$$\mathrm{ev}\colon \mathcal{M}(T_0, J, \xi) \to T_0, \quad [u] \mapsto u(1).$$

See for example [Aur15] for details. As in [Kim21a], we now assume in addition that $\sigma^* J = -J$ and associate to every element $[u] \in \mathcal{M}(T_0, J, \xi)$ its image under the anti-symplectic involution

$$\mathcal{R}\colon [u] \mapsto [\sigma \circ u \circ \rho],$$

where ρ denotes complex conjugation on the disk. Note that the involution $\mathcal{R}$ maps the moduli space $\mathcal{M}(T_0, J, \xi)$ to $\mathcal{M}(T_0, J, -\xi)$ since T_0 is the fixed point set of σ. By Cho and Oh [CO06], there exists a J_0-holomorphic disk in $\mathcal{M}(T_0, J_0, \xi)$ if and only if ξ coincides with one of the primitive vectors v_i normal to the facets of the moment polytope Δ. Here, J_0 denotes the Kähler complex structure. The regularity of J_0 was shown in [CO06] and that of J by Kim [Kim21a]. Hence the two counts $n(T_0, J, \xi)$ and $n(T_0, J_0, \xi)$ are well-defined and agree. Since the involution $\mathcal{R}$ maps $\mathcal{M}(T_0, J, \xi)$ to $\mathcal{M}(T_0, J, -\xi)$, we find that a given vector v appears as orthogonal vector to one of the facets if and only if $-v$ does as well. Hence Δ is invariant under central symmetry.

Remark 4.6.1. One can make a similar argument to show that the Chekanov tori are not real by reformulating the information given by J-holomorphic disks in terms of the Landau-Ginzburg potential (see [Aur09] or [PT20]). The so-called wall-crossing formulae describe how this potential behaves when passing from the Clifford to the Chekanov torus. We note that our technique using versal deformations is more elementary than the use of wall-crossing, which has only been shown in [PT20].

5 Squeezing via degenerations of the complex projective plane

This is an appendix to the paper On certain quantifications of Gromov's non-squeezing theorem *by Kevin Sackel, Antoine Song, Umut Varolgunes and Jonathan J. Zhu [SSVZ21].*

5.1 Introduction and main theorem

Our goal is to show the following.

Theorem 5.1.1. *For every $\alpha < 3$, there is a set $\Sigma \subset B^4(\alpha)$ of Minkowski dimension* 2 *such that $B^4(\alpha) \setminus \Sigma$ symplectically embedds into $Z^4(1) = \mathbb{R}^2 \times D^2(1) \subset \mathbb{R}^4$.*

Our notation corresponds to that of the main body of the text by setting $\alpha = \pi R^2$. The idea of the proof is to view $B^4(\alpha)$ as $\mathbb{C}P^2(\alpha) \setminus \mathbb{C}P^1$ and to use almost toric fibrations of $\mathbb{C}P^2$. As observed in [Via16] for every *Markov triple* (a, b, c), there is a triangle $\Delta_{a,b,c}(\alpha) \subset \mathbb{R}^2$ and an almost toric fibration on $\mathbb{C}P^2(\alpha)$ with a base diagram whose underlying polytope is $\Delta_{a,b,c}(\alpha)$. Now note that the toric moment map image of $Z^4(1)$ is the half-strip $\mathcal{S} = \mathbb{R}_{\geqslant 0} \times [0, 1)$. We shall show that if the triangle $\Delta_{a,b,c}(\alpha)$ fits into $\mathcal{S}$ (after applying an integral affine transformation), then there is a symplectic embedding of $\mathbb{C}P^2(\alpha)$ into $Z^4(1)$ at the cost of removing a certain subset Σ' from $\mathbb{C}P^2(\alpha)$. The point here is that one can get a good understanding of the subset one needs to remove. Indeed, we show that Σ' is a union of three Lagrangian pinwheels (defined as in [ES18]) and a symplectic torus. In particular, this set has Minkowski dimension 2. A combinatorial argument shows that for every $\alpha < 3$, there is a Markov triple (a, b, c) and an inclusion $\Delta_{a,b,c}(\alpha) \subset \mathcal{S}$, see Lemma 5.2.2.

Remark 5.1.2. As was pointed out to us by Leonid Polterovich, our results can be combined with Gromov's non-squeezing to show that any symplectic ball $B^4(1 + \varepsilon) \subset \mathbb{C}P^2(\alpha)$ intersects the set $\Sigma' \subset \mathbb{C}P^2$ discussed above. See Corollary 5.3.4 for more details.

Remark 5.1.3. The same strategy may work to produce symplectic embeddings $D^2(\alpha) \times D^2(\alpha) \setminus \Sigma \hookrightarrow Z^4(1)$ of the polydisk of capacity $\alpha < 2$ minus a union of some two-dimensional manifolds into the cylinder. Indeed, one can view the polydisk as the affine part of $S^2 \times S^2$ and use almost toric fibrations of the latter space to carry out the same argument.

The relationship between Markov triples and the complex and symplectic geometry of $\mathbb{C}P^2$ has generated a lot of interest in recent years. It first appeared in the work of Galkin–Usnich [GA10], where the authors conjectured that for every Markov triple there is an exotic Lagrangian torus in $\mathbb{C}P^2$. This conjecture was proved and generalized by Vianna [Via16], [Via17] by the use of almost toric fibrations, see also Symington [Sym03]. On the algebro-geometric side, Hacking–Prokhorov [HP10] showed that a complex surface X with quotient singularities admits a $\mathbb{Q}$-Gorenstein smoothing to $\mathbb{C}P^2$ if and only if X is a weighted projective space $\mathbb{C}P(a^2,b^2,c^2)$ and (a,b,c) forms a Markov triple. In [ES18], Evans–Smith studied embeddings of *Lagrangian pinwheels* into $\mathbb{C}P^2$. This is directly related to [HP10], since Lagrangian pinwheels appear naturally as vanishing cycles of the smoothings of $\mathbb{C}P(a^2,b^2,c^2)$ to $\mathbb{C}P^2$. See also the recent work by Casals–Vianna [CV20] and the forthcoming paper joint with Mikhalkin and Schlenk [BMS22] for other applications of almost toric fibrations to symplectic embedding problems.

Acknowledgements. We heartily thank Jonny Evans, Grisha Mikhalkin, Leonid Polterovich, Kevin Sackel and Umut Varolgunes for valuable discussions and careful comments on a previous draft. We are grateful to the attentive referee for pointing out a gap in the proof. We thank Felix Schlenk for constant encouragement and generous help.

5.2 Some geometry of Markov triangles

Let us recall some facts about Markov numbers and their associated triangles.

Definition 5.2.1. *A triple of natural numbers $a,b,c \in \mathbb{N}_{>0}$ is called a* Markov triple *if it solves the* Markov equation

$$a^2+b^2+c^2 = 3abc. \tag{5.2.1}$$

If (a,b,c) is a Markov triple, then so is $(a,b,3ab-c)$. Starting from the solution $(1,1,1)$, we obtain the so-called *Markov tree* by mutations $(a,b,c) \to (a,b,3ab-c)$. The first few Markov triples are

$$(1,1,1) \quad (1,1,2) \quad (1,2,5) \quad (1,5,13) \quad (2,5,29) \quad (1,13,34) \quad \cdots \tag{5.2.2}$$

Given $\alpha > 0$, let $\mathbb{C}P^2$ be equipped with the Fubini–Study symplectic form ω normalized such that $\int_{\mathbb{C}P^1}\omega = \alpha$. For every Markov triple (a,b,c), there is an almost toric fibration of $\mathbb{C}P^2$ with almost toric base diagram a rational triangle $\Delta_{a,b,c}(\alpha) \subset \mathbb{R}^2$, which we call the *Markov triangle* associated to the Markov triple (a,b,c). The first Markov triangle is the (honest) toric moment map image of $\mathbb{C}P^2$ in our normalization,

$$\Delta_{1,1,1}(\alpha) = \{(x,y) \in \mathbb{R}^2_{\geqslant 0} \,|\, x+y \leqslant \alpha\}. \tag{5.2.3}$$

In fact, we will slightly abuse notation and sometimes think of $\Delta_{a,b,c}(\alpha)$ as a triangle in $\mathbb{R}^2$ and sometimes as the equivalence class of triangles $\mathbb{R}^2$ under the group of toric symmetries given by *integral affine transformations*, i.e. elements in $\mathrm{Aff}(2;\mathbb{Z}) = \mathbb{R}^2 \rtimes \mathrm{GL}(2;\mathbb{Z})$. For every mutation of Markov triples $(a,b,c) \to (a,b,3ab-c)$, there is a corresponding mutation of triangles $\Delta_{a,b,c}(\alpha) \to \Delta_{a,b,3ab-c}(\alpha)$, defined by cutting the triangle in two halves and applying a shear map to one of the halves and glueing it back to the other half. This is called a *branch move* and we refer to Symington [Sym03, Sections 5.3 and 6] and Vianna [Via16, Section 2] for details. For a concrete description of the triangles $\Delta_{a,b,c}(\alpha)$, see (5.4.23) and the surrounding discussion. The area of the Markov triangles is well-defined, since it is invariant under $\mathrm{Aff}(2;\mathbb{Z})$. Furthermore, the area is invariant under the mutation of triangles and hence we obtain

$$\operatorname{area}(\Delta_{a,b,c}(\alpha)) = \operatorname{area}(\Delta_{1,1,1}(\alpha)) = \frac{\alpha^2}{2}. \tag{5.2.4}$$

Recall that we are interested in *embedding Markov triangles into the half-strip* $\mathcal{S} =$

$\mathbb{R}_{\geqslant 0} \times [0,1)$. This means that, given $\alpha > 0$, we look for a Markov triple (a,b,c) such that $\Delta_{a,b,c}(\alpha) \subset \mathcal{S}$ up to applying an element in $\mathrm{Aff}(2;\mathbb{Z})$. We will prove the following.

Lemma 5.2.2. *For every* $0 < \alpha < 3$*, there is a Markov triple* (a,b,c) *such that the Markov triangle* $\Delta_{a,b,c}(\alpha) \subset \mathbb{R}^2$ *is in* $\mathcal{S} = \mathbb{R}_{\geqslant 0} \times [0,1)$ *up to applying an element in* $\mathrm{Aff}(2;\mathbb{Z})$*. This result is sharp in the sense that for* $\alpha \geqslant 3$*, there is no such Markov triple.*

Let us introduce some definitions from integral affine geometry. A vector $v \in \mathbb{Z}^2$ is called *primitive* if $\beta v \notin \mathbb{Z}^2$ for all $0 < \beta < 1$. Note that to every vector $w \in \mathbb{R}^2$ with rational slope, we can associate a unique primitive vector v such that $w = \gamma v$ for $\gamma > 0$. We call γ the *affine length* of w and denote it by $\ell_{aff}(w)$. Let $l \subset \mathbb{R}^2$ be a rational affine line (or line segment) with primitive directional vector $v \in \mathbb{Z}^2$ and $p \in \mathbb{R}^2$ be a point. Then the *affine distance* is defined as $d_{aff}(p,l) = |\det(v,u)|$, where u is any vector such that $p + u \in l$. This does not depend on any of the choices we have made and these quantities are $\mathrm{Aff}(2;\mathbb{Z})$-invariant. See McDuff [McD11] for more details.

For a given rational (not necessarily Markov) triangle Δ, we denote by E_1, E_2, E_3 its edges and by v_1, v_2, v_3 its vertices such that v_i lies opposite to the edge E_i. We call $\ell_{aff}(E_1)+\ell_{aff}(E_2)+\ell_{aff}(E_3)$ the *affine perimeter* of Δ. Note that the affine perimeter of a Markov triangle $\Delta_{a,b,c}(\alpha)$ is equal to 3α. We have the following formula for the area of Δ,

$$\mathrm{area}(\Delta) = \frac{1}{2}\ell_{\mathrm{aff}}(E_i)\, d_{\mathrm{aff}}(v_i, E_i). \tag{5.2.5}$$

This follows from the definition of affine distance and affine length.

Proof of Lemma 5.2.2. For the first part of the proof we only need one branch of the Markov tree, namely the one where the maximal entry grows the fastest. More precisely, let Δ_n be the Markov triangle associated to the triple (m_{n+2}, m_{n+1}, m_n), where m_k is recursively defined as

$$m_0 = m_1 = m_2 = 1, \quad m_{k+2} = 3m_{k+1}m_k - m_{k-1}. \tag{5.2.6}$$

The first few terms of this sequence are given by $\{m_k\}_{k\in\mathbb{N}} = \{1,1,1,2,5,29,433,\dots\}$. We clearly have $m_k \to \infty$. Recall from [Via16] that the affine side lengths of a Markov triangle $\Delta_{a,b,c}(\alpha)$ are given by $\lambda a^2, \lambda b^2, \lambda c^2$ for a proportionality constant $\lambda > 0$. Since the affine perimeter of the Markov triangle is 3α, we obtain $\lambda(a^2+b^2+c^2) = 3\alpha$. Together with the Markov equation (5.2.1) this yields that the longest edge E_n in $\Delta_n(\alpha)$ has affine length

$$\ell_{aff}(E_n) = \frac{\alpha m_{n+2}}{m_{n+1}m_n} \overset{(5.2.6)}{=} \alpha\left(3 - \frac{m_{n-1}}{m_{n+1}m_n}\right). \tag{5.2.7}$$

Since the second summand goes to 0 for large n, we obtain $\ell_{aff}(E_n) \to 3\alpha$. Let v_n be the vertex opposite to E_n. By (5.2.4) and the area formula (5.2.5) we obtain

$$\ell_{aff}(E_n)\, d_{aff}(v_n, E_n) = \alpha^2. \tag{5.2.8}$$

This implies that $d_{aff}(v_n, E_n) \to \alpha/3$. Now note that we can assume, up to $\mathrm{Aff}(2;\mathbb{Z})$, that the maximal edge E_n lies in $\mathbb{R}_{\geqslant 0} \times \{0\}$ and that $\Delta_n(\alpha)$ lies in the upper half-plane. Then the (Euclidean) height of $\Delta_n(\alpha)$ is equal to $d_{aff}(v_n, E_n)$. This implies that for all $\alpha < 3$, we have $\Delta_n(\alpha) \subset \mathcal{S}$ for large enough $n \in \mathbb{N}$.

Now let $\alpha \geqslant 3$ and suppose that there is a Markov triangle $\Delta_{a,b,c}(\alpha) \subset \mathcal{S}$. Let E be the longest edge of $\Delta_{a,b,c}(\alpha)$ and v the opposite vertex. We first prove that E is parallel to $e_1 = (1,0)$. Indeed, suppose it is not. Then we can write $E = \ell_{aff}(E)v$ for a primitive vector $v = (v_1, v_2) \in \mathbb{Z}^2$ with $v_2 \geqslant 1$. But since the affine perimeter of $\Delta_{a,b,c}(\alpha)$ is 3α and E is the longest edge, we obtain $\ell_{aff}(E) \geqslant \alpha$ and thus $\Delta_{a,b,c}(\alpha)$ is not contained in $\mathcal{S}$. Now if E is parallel to e_1, then the (Euclidean) height of $\Delta_{a,b,c}(\alpha)$ is $d_{aff}(v, E)$. The affine perimeter 3α is strictly larger than $\ell_{aff}(E)$ from which we deduce $d_{aff}(v, E) > \alpha/3 \geqslant 1$ by the area formula (5.2.5). □

5.3 Proof of the main theorem

Definition 5.3.1. *Let p be a positive integer. The topological space obtained from the unit disk D by quotienting out the action of the group of the p-th roots of unity on ∂D is called p*-pinwheel.

For example the 2-pinwheel is $\mathbb{R}P^2$. The image of ∂D in the quotient is called the *core circle*. For all $p > 2$ the p-pinwheels are not smooth at points of the core circle. A *Lagrangian pinwheel* in a symplectic manifold M is a Lagrangian *embedding* of a p-pinwheel into M, see [ES18, Definition 2.3] for the meaning of *embedding* in this context. As it turns out, for every Lagrangian p-pinwheel, there is an additional extrinsic parameter $q \in \{1, \dots, p-1\}$ measuring the twisting of the pinwheel around its core circle. We call such an object (p, q)-pinwheel and denote it by $L_{p,q}$ when this causes no confusion. For us, the following result is key.

Proposition 5.3.2. *Let (a, b, c) be a Markov triple and let $s_a, s_b, s_c \in \mathbb{C}P(a^2, b^2, c^2)$ be the orbifold singular points of the corresponding weighted projective space. Then there is a surjective map $\tilde{\phi}\colon \mathbb{C}P^2 \to \mathbb{C}P(a^2, b^2, c^2)$ and there are (mutually disjoint) Lagrangian pinwheels $L_{a,q_a}, L_{b,q_b}, L_{c,q_c} \subset \mathbb{C}P^2$ with*

$$\tilde{\phi}(L_{a,q_a}) = s_a, \quad \tilde{\phi}(L_{b,q_b}) = s_b, \quad \tilde{\phi}(L_{c,q_c}) = s_c, \tag{5.3.9}$$

such that $\tilde{\phi}$ restricts to a symplectomorphism

$$\phi\colon \mathbb{C}P^2 \setminus (L_{a,q_a} \sqcup L_{b,q_b} \sqcup L_{c,q_c}) \to \mathbb{C}P(a^2, b^2, c^2) \setminus \{s_a, s_b, s_c\}. \tag{5.3.10}$$

Furthermore, the preimage of the set $\mathscr{D} \subset \mathbb{C}P(a^2, b^2, c^2)$, see (5.3.11), consists of the union of three Lagrangian pinwheels and a symplectic two-torus intersecting the pinwheels in their respective core circles.

Remark 5.3.3. For the construction of this symplectomorphism, it seems plausible that one can use the existence of a $\mathbb{Q}$-Gorenstein smoothing of $\mathbb{C}P(a^2, b^2, c^2)$ to $\mathbb{C}P^2$ with vanishing locus consisting of a union of three pinwheels. Such smoothings were constructed in [HP10], by showing that there are no obstructions to piecing together local smoothings of the cyclic quotient singularities holomorphically. Our proof follows the same strategy in the symplectic set-up, see also [ES18, Examples 2.5/2.6] for a discussion of this, [LM14, Section 3] and [Eva19, Section 1] for the local smoothings.

We now turn to the proof of Theorem 5.1.1 using Proposition 5.3.2, the proof of which we postpone to §5.4.

Proof of Theorem 5.1.1. Step 1: Let $\alpha < 3$ and choose α' such that $\alpha < \alpha' < 3$. Pick a Markov triple (a, b, c) and an associated Markov triangle $\Delta_{a,b,c} = \Delta_{a,b,c}(\alpha')$ which lies in $\mathcal{S} = \mathbb{R}_{\geqslant 0} \times [0, 1)$. This is possible by Lemma 5.2.2. Note that $\Delta_{a,b,c}$ is the image of the toric orbifold moment map $\mu \colon \mathbb{C}P(a^2, b^2, c^2) \to \Delta_{a,b,c}$, provided we normalize the orbifold symplectic form appropriately. See the discussion surrounding (5.4.26) for details on the toric structure on weighted projective space. Let

$$\mathscr{D} = \mu^{-1}(\partial \Delta_{a,b,c}) \subset \mathbb{C}P(a^2, b^2, c^2) \tag{5.3.11}$$

be the preimage of the boundary. The set $\mathscr{D}$ is a union of complex suborbifolds $\mathbb{C}P(a^2, b^2) \cup \mathbb{C}P(b^2, c^2) \cup \mathbb{C}P(a^2, c^2)$ such that each of these suborbifolds projects to one edge of the triangle $\Delta_{a,b,c}$. The complement of $\mathscr{D}$ admits a symplectic embedding,

$$\psi \colon \mathbb{C}P(a^2, b^2, c^2) \setminus \mathscr{D} \hookrightarrow Z^4(1). \tag{5.3.12}$$

Indeed, this follows from the inclusion $\operatorname{int}(\Delta_{a,b,c}) \subset \operatorname{int}(\mathcal{S})$ and the fact that inclusions of toric moment map images which respect the boundary stratifications yield (equivariant) symplectic embeddings, see for example [Sym03].

Step 2: By Proposition 5.3.2 there is a symplectomorphism ϕ from the complement of Lagrangian pinwheels $L_{a,q_a}, L_{b,q_b}, L_{c,q_c}$ to the complement of the orbifold points of $\mathbb{C}P(a^2, b^2, c^2)$. By restricting ϕ, we obtain the symplectomorphism,

$$\phi' \colon \mathbb{C}P^2 \setminus \Sigma' \to \mathbb{C}P(a^2, b^2, c^2) \setminus \mathscr{D}. \tag{5.3.13}$$

Again by Proposition 5.3.2, the set Σ' consists of the union of three pinwheels and a symplectic two-torus.

Step 3: The standard embedding $B^4(\alpha) \subset \mathbb{C}P^2(\alpha')$ together with ϕ' and ψ yields an embedding $B^4(\alpha) \setminus \Sigma \hookrightarrow Z^4(1)$. Here Σ denotes $\Sigma = \Sigma' \cap B^4(\alpha)$. The set Σ has Minkowski dimension two. Indeed, the embedding $B^4(\alpha) \subset \mathbb{C}P^2(\alpha')$ is bilipschitz (its image being contained in a closed ball) and volume preserving and the set Σ' consists of the union of three pinwheels and a symplectic two-torus. □

As was pointed out to us by Leonid Polterovich, one can combine Theorem 5.1.1 with Gromov's non-squeezing theorem to get certain rigidity results, reminiscent of [Bir01, Theorem 1.B].

Corollary 5.3.4. *Let $\Sigma' \subset \mathbb{C}P^2(\alpha)$ be one of the above sets such that $\mathbb{C}P^2(\alpha) \setminus \Sigma'$ embeds into $Z^4(1)$ for some $1 < \alpha < 3$. Then every symplectic ball $B^4(1 + \varepsilon) \subset \mathbb{C}P^2$ for $\varepsilon > 0$ intersects Σ'.*

Proof. Assume $B^4(1+\varepsilon) \subset \mathbb{C}P^2(\alpha)$ does not intersect Σ'. The embedding $\mathbb{C}P^2(\alpha) \setminus \Sigma' \hookrightarrow Z^4(1)$ yields a symplectic embedding $B^4(1+\varepsilon) \hookrightarrow Z^4(1)$, contradicting non-squeezing. □

Note that for a fixed $1 < \alpha < 3$ we get infinitely many sets $\Sigma' \subset \mathbb{C}P^2$ to which Corollary 5.3.4 applies and all of these consist of a union of a symplectic torus and Lagrangian pinwheels.

5.4 Proof of Proposition 5.3.2

Following the exposition in [ES18, Example 2.5] we consider smoothings of certain orbifold quotients of $\mathbb{C}^2$. This yields the local version from Lemma 5.4.1 of the symplectomorphism in Proposition 5.3.2.

Let a, q be coprime integers with $1 \leqslant q < a$ and take the quotient of $\mathbb{C}^2$ by the following action of a^2-th roots of unity,

$$\zeta.(z_1, z_2) = (\zeta z_1, \zeta^{aq-1} z_2), \quad \zeta^{a^2} = 1. \tag{5.4.14}$$

We denote this quotient by $\mathbb{C}^2/\Gamma_{a,q}$. It can be embedded as $\{w_1 w_2 = w_3^a\}$ into the quotient $\mathbb{C}^3/\mathbb{Z}_a$ by the action

$$\eta.(w_1, w_2, w_3) = (\eta w_1, \eta^{-1} w_2, \eta^q w_3), \quad \eta^a = 1. \tag{5.4.15}$$

The smoothing is given by

$$\mathcal{X} = \{w_1 w_2 = w_3^a + t\} \subset \mathbb{C}^3/\mathbb{Z}_a \times \mathbb{C}_t, \tag{5.4.16}$$

which we view as a degeneration by projecting to the t-component, $\pi\colon \mathcal{X} \to \mathbb{C}_t$. We denote the fibres by $X_t = \pi^{-1}(t)$. The smooth fibre X_1 is a rational homology ball and the vanishing cycle of the degeneration is a Lagrangian pinwheel $L_{a,q}$. This follows from the description of $\mathcal{X}$ as $\mathbb{Z}_a$-quotient of an A_{a-1}-Milnor fibre. Let $s \in X_0$ be the unique isolated singularity of X_0, and $\mathcal{X}^{\mathrm{reg}} = \mathcal{X} \setminus \{s\}$ its complement. The restriction of the standard symplectic form ω_0 on $\mathbb{C}^4 = \mathbb{C}^3 \times \mathbb{C}_t$ yields a symplectic manifold $(\mathcal{X}^{reg}, \Omega)$. Note that the smooth loci of the fibres $X_{t \neq 0}$ and $X_0 \setminus \{s\}$ are symplectic submanifolds. Let us now construct a symplectomorphism

$$\psi\colon X_1 \setminus L_{a,q} \to X_0 \setminus \{s\} = \mathbb{C}^2/\Gamma_{a,q} \setminus \{0\}. \tag{5.4.17}$$

For this, we take the connection on $\mathcal{X}^{reg}$ defined as the symplectic complement to the vertical distribution,

$$\xi_x = (\ker(\pi_*)_x)^{\Omega} = \left\{ v \in T_x \mathcal{X}^{\mathrm{reg}} \mid \Omega(v, w) = 0 \text{ for all } w \in T_x \pi^{-1}(\pi(x)) \right\}. \tag{5.4.18}$$

This connection is symplectic in the sense that its parallel transport maps are symplectomorphisms whenever they are defined. In particular, we get a symplectomorphism between any two regular fibres X_{t_1} and X_{t_2} by picking a curve in $\mathbb{C}^\times$ with endpoints t_1 and t_2. Since we are interested in the singular fibre for the construction of (5.4.17), take the curve $\gamma(r) = 1 - r \in \mathbb{C}$. For every $r < 1$, this yields a symplectomorphism $\psi^r\colon X_1 \to X_{1-r}$. As it turns out, setting $\psi(x) = \lim_{r \to 1} \psi^r(x)$ for $x \in X_1$ yields a well-defined surjective map $\tilde{\psi}\colon X_1 \to X_0$ with vanishing cycle a Lagrangian pinwheel, $L_{a,q} = \tilde{\psi}^{-1}(s)$, and which restricts to the desired symplectomorphism (5.4.17). For the fact that $\tilde{\psi}$ is well-defined, see the unpublished notes by Evans [Eva19, Lemma 1.2]. We refer to [LM14, Section 3.1] for more details on the specific degeneration we consider above and to [Eva19, Lemma 1.20] for the fact that the vanishing cycle is a Lagrangian pinwheel.

Lemma 5.4.1. (Evans [Eva19]) *Let $\mathbb{C}^2/\Gamma_{a,q}$ the quotient by the action (5.4.14) and $S_{a,q}$ its smoothing as above. Then there is a surjective map $\tilde{\psi}_a\colon S_{a,q} \to \mathbb{C}^2/\Gamma_{a,q}$ with $\tilde{\psi}_a(L_{a,q}) =$*

$\{0\}$ *and which restricts to a symplectomorphism*

$$\psi_a \colon S_{a,q} \setminus L_{a,q} \to (\mathbb{C}^2/\Gamma_{a,q}) \setminus \{0\}. \tag{5.4.19}$$

Furthermore, the preimage of the set $\mathscr{D}_{a,q_a} = \{z_1 z_2 = 0\}/\Gamma_{a,q}$ *under* $\tilde{\psi}_a$ *is the union of* $L_{a,q}$ *and a symplectic cylinder intersecting* $L_{a,q}$ *in its core circle.*

Proof. As explained above, the main statement of the lemma follows from [Eva19]. We only need to identify the preimage of $\mathscr{D}_{a,q_a} = \{z_1 z_2 = 0\}/\Gamma_{a,q}$, which can be done by keeping track of the parallel transport in the explicit model (5.4.16). Under the identification of $\mathbb{C}^2/\Gamma_{a,q}$ with X_0 (which we will tacitly use throughout), the set $\{z_1 z_2 = 0\}/\Gamma_{a,q}$ corresponds to $\{w_3 = 0\}/\mathbb{Z}_a \cap X_0$. We claim that the set $\{w_3 = 0\}/\mathbb{Z}_a \cap X_t \subset \mathcal{X}$ is invariant under the symplectic parallel transport induced by (5.4.18). This proves that $\tilde{\psi}_a^{-1}(\{w_3 = 0\}/\mathbb{Z}_a \cap X_0) = (\{w_3 = 0\}/\mathbb{Z}_a \cap X_1) \cup L_{a,q}$. Indeed, the preimage of $0 \in \mathbb{C}^2/\Gamma_{a,q}$ is $L_{a,q}$ and on the complement of $L_{a,q}$, the map $\tilde{\psi}_a$ is a diffeomorphism. To see that $\{w_3 = 0\}/\mathbb{Z}_a \cap X_1$ is a symplectic cylinder intersecting $L_{a,q}$ in its core circle, one can consider the singular fibration structure of the map $X_s \to \mathbb{C}, (w_1, w_2, w_3) \mapsto w_3^p$ see [Eva19, §1.2.3] for more details. Thus it remains to show the invariance of $\{w_3 = 0\}/\mathbb{Z}_a \cap X_t$.

Let $x \in \{w_3 = 0\}/\mathbb{Z}_a \cap X_t$, meaning that x is a $\mathbb{Z}_a$-class of a point $(x_1, x_2, 0) \in \mathbb{C}^3$ with $x_1 x_2 = t$. This gives a natural inclusion $T_{(x,t)}\mathcal{X} = T_x X_t \oplus T_t\mathbb{C} \subset \mathbb{C}^3 \oplus \mathbb{C}$. Since $\Omega = \omega_{\mathbb{C}^3} \oplus \omega_{\mathbb{C}}$, the horizontal lift by the symplectic connection (5.4.18) of a vector $v \in T_t\mathbb{C} = \mathbb{C}$ is given by $u + v \in \xi_{(x,t)}$, where $u \in (T_x X_t)^{\omega_{\mathbb{C}^3}}$. Note that the subspace $\{u_1 = u_2 = 0\} \subset T_x\mathbb{C}^3 = \mathbb{C}^3$ is contained in $T_x X_t$ and hence its symplectic complement $\{u_3 = 0\} \subset \mathbb{C}^3$ contains the symplectic complement $(T_x X_t)^{\omega_{\mathbb{C}^3}}$. Since u is contained in $(T_x X_t)^{\omega_{\mathbb{C}^3}}$, this proves that u is tangent to the subset $\{w_3 = 0\}$ and hence this subset is preserved under parallel transport. □

We turn to the proof of Proposition 5.3.2. The main idea is to use the fact that $\Delta_{a,b,c}(\alpha)$ is both the almost toric base polytope of $\mathbb{C}P^2(\alpha)$ and the toric base of $\mathbb{C}P(a^2, b^2, c^2)$ with a suitably normalized symplectic form. This shows that there is a symplectomorphism which intertwines the (almost) toric structures on the preimages of a complement of neighbourhoods of the vertices. For example one can choose $W \subset \Delta_{a,b,c}$ as in Figure 5.1. We use Lemma 5.4.1 to extend this symplectomorphism.

Let us make a few preparations. In particular, we discuss how to use the quotient $\mathbb{C}^2/\Gamma_{a,q}$ as a local toric model for $\mathbb{C}P(a^2, b^2, c^2)$. This is the orbifold version of the toric ball embedding into $\mathbb{C}P^2$ one obtains by the inclusion of the simplex with one edge removed into the standard simplex $\Delta_{1,1,1}$. The toric structure on $\mathbb{C}^2/\Gamma_{a,q}$ is induced by the standard toric structure $(z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2)$ on $\mathbb{C}^2$. Indeed, note that the $\Gamma_{a,q}$-action is obtained by restricting the standard $T^2 = (\mathbb{R}/\mathbb{Z})^2$-action to a discrete subgroup $\mathbb{Z}_{a^2}$. This implies that we obtain an induced action by $T^2/\Gamma_{a,q}$ on $\mathbb{C}^2/\Gamma_{a,q}$. This action is Hamiltonian and its moment map image, under a suitable identification $T^2 \cong T^2/\Gamma_{a,q}$, is given by

$$\angle_{a,q} = \{x_1 w_1 + x_2 w_2 \,|\, x_1, x_2 \geqslant 0\}, \quad w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} aq - 1 \\ a^2 \end{pmatrix}. \tag{5.4.20}$$

See for example [ES18, Remark 2.7] or [Sym03, Section 9]. Note that a ball $B^4(d) =$

$\{\pi(|z_1|^2 + |z_2|^2) < d\} \subset \mathbb{C}^2$ quotients to an orbifold ball $B^4(d)/\Gamma_{a,q} \subset \mathbb{C}^2/\Gamma_{a,q}$ which is fibred by the induced toric structure on the quotient. Furthermore, the boundary sphere $S^3(d) \subset \mathbb{C}^2$ quotients to a lens space $\Sigma_{a,q}(d) = S^3(d)/\Gamma_{a,q}$ of type $(aq-1, a^2)$ equipped with its canonical contact structure and which fibers over a segment in $\angle_{a,q}$. We will use this fact in the proof of Proposition 5.3.2.

Let us now show that, for a suitable choice of q, the toric system $\mathbb{C}^2/\Gamma_{a,q} \to \angle_{a,q}$ can be used as a local model around one of the orbifold points of the toric system on $\mathbb{C}P(a^2, b^2, c^2)$. In order to get a concrete description of this toric system on the weighted projective space, recall that the symplectic orbifold $\mathbb{C}P(a^2, b^2, c^2)$ can be defined as a symplectic quotient of $\mathbb{C}^3$,

$$\mathbb{C}P(a^2,b^2,c^2) = H^{-1}(a^2b^2c^2)/S^1, \quad H = a^2|z_1|^2 + b^2|z_2|^2 + c^2|z_3|^2. \tag{5.4.21}$$

This description (5.4.21) has the advantage that it is naturally equipped with a Hamiltonian T^2-action inherited from the standard T^3-action on $\mathbb{C}^3$. This induced action is toric and its moment map image is given by the intersection of the plane defined by H and the positive orthant,

$$\tilde{\Delta}_{a,b,c} = \{a^2y_1 + b^2y_2 + c^2y_3 = a^2b^2c^2\} \cap \mathbb{R}^3_{\geqslant 0}. \tag{5.4.22}$$

Note that this is a polytope in $\mathbb{R}^3$ and not $\mathbb{R}^2$. We get a Markov triangle $\Delta_{a,b,c}(abc) \subset \mathbb{R}^2$ as in §5.2 by setting

$$\Delta_{a,b,c}(abc) = \Phi^{-1}(\tilde{\Delta}_{a,b,c}), \tag{5.4.23}$$

for Φ an integral affine embedding (see Definition 5.4.2) containing $\tilde{\Delta}_{a,b,c}$ in its image. Recall that this produces the same triangles (up to integral affine equivalence) as those obtained from almost toric fibrations of $\mathbb{C}P^2$ as discussed in §5.2, see [Via16, Section 2]. Hence it makes sense to denote them by $\Delta_{a,b,c}(abc)$. The normalization $\alpha = abc$ of the triangle comes from the choice of level at which we have reduced in (5.4.21).

Definition 5.4.2. *An affine map $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^3, x \mapsto Ax + b$ for $A \in \mathbb{Z}^{3\times 2}$ and $b \in \mathbb{R}^3$ is called* integral affine embedding *if it is injective and if $A(\mathbb{Z}^2) = A(\mathbb{R}^2) \cap \mathbb{Z}^3$.*

Note that by the definition of integral affine embedding, the definition (5.4.23) makes sense and the polytope it defines is independent of the choice of Φ up to applying an integral affine transformation. We now show that there is a natural number q and an integral affine embedding $\Phi_{a,q}$ such that the triangle $\Phi^{-1}_{a,q}(\tilde{\Delta}_{a,b,c}) \subset \mathbb{R}^2$ is obtained by intersecting $\angle_{a,q}$ with a half-plane. Indeed, set

$$\Phi_{a,q}\colon \mathbb{R}^2 \to \mathbb{R}^3, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} -b^2 & 1 + \frac{b}{a}(bq - 3c) \\ a^2 & 1 - aq \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b^2c^2 \\ 0 \\ 0 \end{pmatrix}, \tag{5.4.24}$$

where q satisfies $bq = 3c \mod a$, see also [ES18, Example 2.6]. The map $\Phi_{a,q}$ has image $\{a^2y_1 + b^2y_2 + c^2y_3 = a^2b^2c^2\}$ and it is an integral affine embedding, as can be checked by a computation. Furthermore Φ maps 0 to the vertex $(b^2c^2, 0, 0)$ (corresponding to a^2) of $\tilde{\Delta}_{a,b,c}$ and v_1 and v_2 to the outgoing edges at $(b^2c^2, 0, 0)$. This means that there is an integral vector (ξ_1, ξ_2) defining a half-plane $K = \{\xi_1x_1 + \xi_2x_2 \leqslant k\}$ such that

$$\Phi^{-1}_{a,q}(\tilde{\Delta}_{a,b,c}) = K \cap \angle_{a,q}. \tag{5.4.25}$$

From this we deduce the desired toric model. Let $\tilde{E}_a \subset \tilde{\Delta}_{a,b,c}$ be the edge opposite the vertex $(b^2c^2, 0, 0)$.

Proposition 5.4.3. *The subset in* $\mathbb{C}P(a^2, b^2, c^2)$ *fibering over* $\tilde{\Delta}_{a,b,c} \setminus \tilde{E}_a$ *is fibred orbifold-symplectomorphic to the subset in* $\mathbb{C}^2/\Gamma_{a,q}$ *fibering over* $\operatorname{int} K \cap \angle_{a,q}$*.*

Proof. We have shown above that $\tilde{\Delta}_{a,b,c} \setminus \tilde{E}_a$ and $\operatorname{int} K \cap \angle_{a,q}$ are integral affine equivalent. This implies the claim by the classification of compact toric orbifolds by their moment map images, see [LT97]. Compactness is not a problem here, since we can compactify the subset fibering over $K \cap \angle_{a,q}$ by performing a symplectic cut at $\{\xi_1 x_1 + \xi_2 x_2 = k\}$. □

Let us now fix a moment map

$$\mu \colon \mathbb{C}P(a^2, b^2, c^2) \to \Delta_{a,b,c} = \Delta_{a,b,c}(abc) \subset \mathbb{R}^2, \tag{5.4.26}$$

by composing the moment map $\mathbb{C}P(a^2, b^2, c^2) \to \tilde{\Delta}_{a,b,c}$ with the inverse of a suitable integral affine embedding as described above. Until the end of the proof of Proposition 5.3.2 we simplify notation by writing $\Delta_{a,b,c} = \Delta_{a,b,c}(abc)$.

Proof of Proposition 5.3.2. The main part of the proof will be concerned with proving the existence of the symplectomorphism (5.3.10) and for readability, we postpone the proof of the existence of the global map $\tilde{\phi}$ and the computation of $\tilde{\phi}^{-1}(\mathscr{D})$ to *Step 5.*

Step 1. We start by setting up some notation on the side of the weighted projective space. The orbifold points $s_a, s_b, s_c \in \mathbb{C}P(a^2, b^2, c^2)$ are mapped to the vertices $v_a, v_b, v_c \in \Delta_{a,b,c}$ under the moment map $\mu \colon \mathbb{C}P(a^2, b^2, c^2) \mapsto \Delta_{a,b,c}$. Let us first focus on the orbifold point s_a. Denote edge opposite to v_a by E_a. By Proposition 5.4.3, there is an orbifold symplectomorphism

$$\rho_a \colon \mu^{-1}(\Delta_{a,b,c} \setminus E_a) \to \mu^{-1}_{\mathbb{C}^2/\Gamma_{a,q}}(\operatorname{int} K \cap \angle_{a,q}) \tag{5.4.27}$$

which intertwines the toric structures. Now let $\overline{B^4}(d)/\Gamma_{a,q} \subset \mathbb{C}^2/\Gamma_{a,q}$ be a closed orbifold ball for $\overline{B^4}(d) = \{\pi(|z_1|^2 + |z_2|^2) \leqslant d\}$ and $d > 0$. Its boundary $S^3(d)/\Gamma_{a,q} \subset \mathbb{C}^2/\Gamma_{a,q}$ is a lens space equipped with the standard contact structure. Note that both the orbifold ball and its boundary are fibred by the moment map $\mu_{\mathbb{C}^2/\Gamma_{a,q}}$. Since ρ_a intertwines the toric structures, the image sets

$$B_a^{\mathrm{orb}} = \rho_a^{-1}(\overline{B^4}(d)/\Gamma_{a,q}), \quad \Sigma_a = \rho_a^{-1}(S^3(d)/\Gamma_{a,q}), \tag{5.4.28}$$

are fibred by μ. Then the image of the pair (B_a, Σ_a) under μ is a pair (V_a, ℓ_a) consisting of a segment contained in a triangle around the vertex v_a. Note also that the lens space Σ_a is naturally equipped with its standard contact structure. We do the same procedure around the remaining vertices v_b, v_c and denote the corresponding sets by $B_b^{\mathrm{orb}}, B_c^{\mathrm{orb}}, \Sigma_b, \Sigma_c \subset \mathbb{C}P(a^2, b^2, c^2)$ and by $V_b, V_c, \ell_b, \ell_c \subset \Delta_{a,b,c}$. We choose the sizes such that $B_a^{\mathrm{orb}}, B_b^{\mathrm{orb}}, B_c^{\mathrm{orb}}$ are mutually disjoint. Furthermore, we choose a set $W \subset \Delta_{a,b,c}$ such that $\Delta_{a,b,c} = W \cup V_a \cup V_b \cup V_c$ and such that the overlap $W \cap V_j$ is a strip around ℓ_j for all $j \in \{a, b, c\}$. Again, see Figure 5.1.

Step 2: Now consider the almost-toric fibration of $\mathbb{C}P^2$ associated to the triangle $\Delta_{a,b,c}$. In the conventions of [Sym03] and [Via16] the triangle $\Delta_{a,b,c}$ is decorated with three dashed line segments of prescribed slope between the vertices and the nodal points.

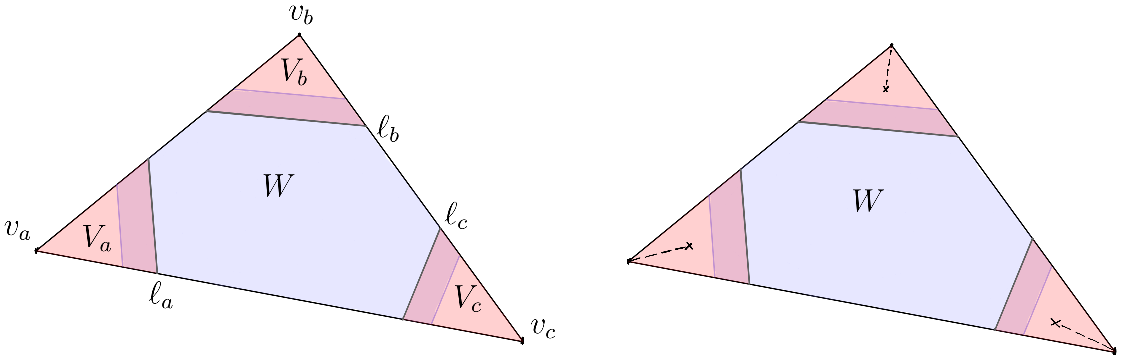


Figure 5.1: The triangle $\Delta_{a,b,c}$ as union $W \cup V_a \cup V_b \cup V_c$, on the left as the toric moment polytope of $\mathbb{C}P(a^2, b^2, c^2)$ and on the right as almost-toric base diagram of $\mathbb{C}P^2$. In both cases the fibration is toric over W and lens spaces fibre over the segments ℓ_a, ℓ_b, ℓ_c.

The latter are usually marked by a cross. There is a map $\pi\colon \mathbb{C}P^2 \to \Delta_{a,b,c}$ which is a standard toric fibration away from the dashed lines, but which is only continuous on the pre-images of the dashed lines (which encode monodromy of the integral affine structure). By applying *nodal slides* if necessary, we may assume that the dashed lines lie outside of the subset $W \subset \Delta_{a,b,c}$. Since the projection π is standard toric away from the dashed lines, this implies that there is a symplectomorphism

$$\phi_0\colon \mathbb{C}P^2 \supset \pi^{-1}(W) \to \mu^{-1}(W) \subset \mathbb{C}P(a^2, b^2, c^2), \tag{5.4.29}$$

which intertwines π and μ. Define the preimages

$$B'_a = \pi^{-1}(V_a), \quad \Sigma'_a = \pi^{-1}(\ell_a). \tag{5.4.30}$$

By [Sym03, Section 9], the set B'_a is a closed rational homology ball and Σ'_a is a lens space of type $(aq-1, a^2)$ equipped with its standard contact structure. In fact, ϕ_0 maps the copy Σ'_a of the lens space to the copy Σ_a. However, contrary to B^{orb}_a, the rational homology ball B'_a is smooth. The same discussion holds for b and c.

Step 3: The key part of the proof is finding extensions

$$\phi_j\colon B'_j \setminus L_{j,q_j} \to B^{\mathrm{orb}}_j \setminus \{s_j\} \tag{5.4.31}$$

of the map $\phi_0|_{\Sigma'_j}$, where L_{j,q_j} are Lagrangian pinwheels for $j \in \{a, b, c\}$. For this we use Lemma 5.4.1. Again, restricting our attention to a, let ψ_a be the symplectomorphism from Lemma 5.4.1. Note that we have already established the correspondence between B^{orb}_a and $\mathbb{C}^2/\Gamma_{a,q}$ by ρ_a and that this correspondence is compatible with the toric picture. We now establish a correspondence between the rational homology sphere B'_a and the space $S_{a,q}$ coming from the smoothing in Lemma 5.4.1. Define yet another copy Σ''_a of the lens space by setting $\Sigma''_a = \psi_a^{-1}(\rho_a(\Sigma_{a,q}))$. This lens space is also equipped with the standard contact structure and it bounds a rational homology ball B''_a by [ES18, Example 2.5]. We now have two pairs (B'_a, Σ'_a) and (B''_a, Σ''_a) consisting of a rational homology ball bounded by a lens space carrying its standard contact structure. By [GS21, Proposition A.2] which relies on [Lis08], this implies that (B'_a, Σ'_a) and (B''_a, Σ''_a) are equivalent up to symplectic deformation. Let $\chi_a \colon (B'_a, \Sigma'_a) \to (B''_a, \Sigma''_a)$ be the diffeomorphism we obtain from this. Note that we cannot directly use the symplectic deformation to conclude, since the symplectomorphism obtained from a Moser-type argument may not restrict to the desired map on the boundary. More precisely, we obtain a diagram of diffeomorphisms of lens spaces,

$$\begin{tikzcd}
B'_a \supset \Sigma'_a \arrow[r, "\phi_0|_{\Sigma'_a}"] \arrow[d, "\chi_a|_{\Sigma'_a}"] & \Sigma_a \subset B_a^{orb} \arrow[d, "\rho_a|_{\Sigma_a}"] \\
B''_a \supset \Sigma''_a \arrow[r, "\psi_a|_{\Sigma''_a}"] & S^3(d)/\Gamma_{a,q} \subset B^4(d)/\Gamma_{a,q},
\end{tikzcd} \tag{5.4.32}$$

and this diagram *does not commute*. We may however correct the diffeomorphism χ_a so that (5.4.32) commutes. Recall that $\Sigma_{a,q}$ is a lens space of type $(aq-1, a^2)$. Since $(aq-1)^2 \neq \pm 1 \mod a^2$, it follows from [Bon83, Théorème 3 (a)] that the space of diffeomorphisms of Σ''_a has two components, namely the one of the identity and the one of the involution τ induced by the involution $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$ of S^3. The diffeomorphism τ extends to a diffeomorphism $\tilde{\tau}$ of $B''_{p,q}$. Up to post-composing χ_a with $\tilde{\tau}$, we may thus assume that $(\psi_a^{-1}\rho_a\phi_0\chi_a^{-1})|_{\Sigma''_a}$ is isotopic to the identity by an isotopy φ_t. Using this isotopy, we can correct the diffeomorphism χ_a such that the diagram (5.4.32) commutes. Indeed, the set $\pi^{-1}(V_a \cap W)$ is a collar neighbourhood $\Sigma'_a \times [0,2)$ and thus we can use the collar coordinate together with the isotopy φ_t to define a corrected diffeomorphism $\tilde{\chi}_a$ which coincides with the original diffeomorphism χ_a on $\mu^{-1}(V_a \setminus W)$ and with $(\psi_a^{-1}\rho_a\phi_0)|_{\Sigma'_a}$ on Σ'_a. Recall that two collar neighbourhoods which agree on the boundary coincide up to applying a smooth isotopy, see Munkres [Mun16, Lemma 6.1]. This means that, after applying an isotopy in (B''_a, Σ''_a), we can assume that the corrected version of χ_a and $\psi_a^{-1}\rho_a\phi_0$ agree on a smaller collar $\Sigma_a \times [0,1)$. Denoting the diffeomorphism we obtain in this way by $\tilde{\chi}_a$, this allows us to define a diffeomorphism

$$\phi_a = \rho_a^{-1} \circ \psi_a \circ \tilde{\chi}_a|_{B'_a \setminus L_{a,q_a}} \colon B'_a \setminus L_{a,q_a} \to B_a^{orb} \setminus \{s_a\}, \tag{5.4.33}$$

which extends ϕ_0 in the sense that it agrees with ϕ_0 on a collar of Σ'_a. Since ψ_a is defined outside of a Lagrangian pinwheel $L_{a,q_a} \subset B''_a$, the diffeomorphism ϕ_a is defined outside of a pinwheel (which we again denote by L_{a,q_a}) in B'_a. We repeat this procedure for b and c to obtain diffeomorphisms ϕ_b and ϕ_c.

Step 4: By construction, the diffeomorphisms ϕ_a, ϕ_b, ϕ_c extend the initial symplectomorphism $\phi_0|_{\pi^{-1}(W \setminus (V_a \cup V_b \cup V_c))}$ and hence we obtain a diffeomorphism

$$\widehat{\phi} \colon \mathbb{C}P^2 \setminus (L_{a,q_a} \sqcup L_{b,q_b} \sqcup L_{c,q_c}) \to \mathbb{C}P(a^2, b^2, c^2) \setminus \{s_a, s_b, s_c\}. \tag{5.4.34}$$

We now turn to the symplectic forms. On $\mathbb{C}P^2$, we define a symplectic form $\widehat{\omega}$ which turns $\widehat{\phi}$ into a symplectomorphism as follows. On $\pi^{-1}(W \setminus (V_a \cup V_b \cup V_c))$ we define $\widehat{\omega}$ to be the usual Fubini-Study form ω. On V_j we define $\widehat{\omega}$ as the pullback form $\tilde{\chi}_j{}^*\omega_{B''_j}$, where $\tilde{\chi}_j$ is the corrected diffeomorphism constructed at the end of *Step 3.* This yields a well-defined symplectic form which turns $\widehat{\phi}$ into a symplectomorphism. Indeed, this follows from the fact that the maps ϕ_0, ψ_j and ρ_j are symplectomorphisms and ϕ_j is defined as their composition (5.4.33). This also implies that the symplectic form $\widehat{\omega}$ has the same total volume as the Fubini–Study form. By the Gromov–Taubes theorem [MS12, Remark 9.4.3 (ii)], the form $\tilde{\omega}$ is symplectomorphic to the the Fubini–Study form and hence post-composing $\widehat{\phi}$ from (5.4.34) with this symplectomorphism yields the desired symplectomorphism (5.3.10).

Step 5: The definition of the global map $\tilde{\phi}\colon \mathbb{C}P^2 \to \mathbb{C}P(a^2, b^2, c^2)$ is obtained by replacing ϕ_a from (5.4.33) by $\tilde{\phi}_a = \rho_a^{-1} \circ \tilde{\psi}_a \circ \tilde{\chi}_a$ and carrying out the rest of the construction as above. The map $\tilde{\psi}_a$ is given by Lemma 5.4.1. Let us now identify $\tilde{\phi}^{-1}(\mathscr{D})$, where $\mathscr{D} = \mu^{-1}(\partial \Delta_{a,b,c})$. Let $\tilde{W} \subset W$ be the subset of W where $\tilde{\phi}$ coincides with ϕ_0. Since ϕ_0 intertwines the toric structures on $\pi^{-1}(W) \subset \mathbb{C}P^2$ and $\mu^{-1}(W) \subset \mathbb{C}P(a^2, b^2, c^2)$, the set $\tilde{\phi}^{-1}(\mathscr{D}) \cap \pi^{-1}(\tilde{W})$ fibers over the three pieces of the boundary given by $\tilde{W} \cap \partial\Delta_{a,b,c}$ and hence consists of three disjoint symplectic cylinders.

We use Lemma 5.4.1 to prove that the missing pieces $\tilde{\phi}^{-1}(\mathscr{D}) \cap B_j'$ for $j \in \{a, b, c\}$ are also given by symplectic cylinders and that the union of the six cylinders is given by a torus. We again discuss the case of $j = a$ since the other two are completely analogous. Recall that $\rho_a\colon B_a^{orb} \to B^4(d)/\Gamma_{a,q}$ is compatible with the toric structure and thus $\rho_a(\mathscr{D} \cap B_a^{orb}) = \mathscr{D}_{a,q_a} \cap \rho_a(B_a^{orb})$, where $\mathscr{D}_{a,q_a} = \{z_1 z_2 = 0\}/\Gamma_{a,q_a}$ as in Lemma 5.4.1. Indeed, this follows from the fact that $\mathscr{D}_{a,q_a}$ fibers over the boundary of the moment map image $\angle_{a,q_a}$. By Lemma 5.4.1, we deduce that $\tilde{\chi}_a^{-1}(\tilde{\phi}_a^{-1}(\rho_a(\mathscr{D} \cap B_a^{orb})))$ is given by the union of a pinwheel with a piece of a cylinder. Furthermore, recall that near the lens spaces at the respective boundaries, the map $\tilde{\rho}_a^{-1} \tilde{\psi}_a \chi_a$ coincides with ϕ_0. This implies that the cylinder contained in B_a' has two boundary components at $\partial B_a' = \Sigma_a'$ which are smoothly identified in a collar neighbourhood with boundary components of the set $\tilde{\phi}^{-1}(\mathscr{D}) \cap \pi^{-1}(\tilde{W})$ discussed in the previous paragraph. This proves the claim. □

Remark 5.4.4. We suspect that there are shorter and more natural proofs of Proposition 5.3.2. In particular, one should be able to avoid Gromov–Taubes. One possibility we have hinted at above is working with a global degeneration and trying to analyze its vanishing cycle. This would completely avoid the use almost-toric fibrations. Another possibility, in the spirit of [Rua01] and [HK15] is to equip the explicit local degeneration from [LM14] with a family of integrable systems avoiding the pinwheel and extending the given toric structure on the boundary. The symplectomorphism from Proposition 5.3.2 then follows from the usual toric arguments and it is automatically equivariant. Although this construction is elementary, it is somewhat outside the scope of this appendix and we hope to carry out the details elsewhere. This is also reminiscent of [GV21, Section 7] and it is plausible one can apply results from this paper to prove the same result.

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