



# Bifurcating extremal domains for the first eigenvalue of the Laplacian

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## Abstract

We prove the existence of a smooth family of non-compact domains  $\Omega_s \subset \mathbb{R}^{n+1}$ ,  $n \geq 1$ , bifurcating from the straight cylinder  $B^n \times \mathbb{R}$  for which the first eigenfunction of the Laplacian with 0 Dirichlet boundary condition also has constant Neumann data at the boundary: For each  $s \in (-\varepsilon, \varepsilon)$ , the overdetermined system

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega_s, \\ u = 0 & \text{on } \partial\Omega_s, \\ \langle \nabla u, \nu \rangle = \text{const} & \text{on } \partial\Omega_s \end{cases}$$

has a bounded positive solution. The domains  $\Omega_s$  are rotationally symmetric and periodic with respect to the  $\mathbb{R}$ -axis of the cylinder; they are of the form

$$\Omega_s = \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| < 1 + s \cos\left(\frac{2\pi}{T_s} t\right) + O(s^2) \right\}$$

where  $T_s = T_0 + O(s)$  and  $T_0$  is a positive real number depending on  $n$ . For  $n \geq 2$  these domains provide a smooth family of counter-examples to a conjecture of Berestycki, Caffarelli and Nirenberg. We also give rather precise upper and lower bounds for the bifurcation period  $T_0$ . This work improves a recent result of the second author.

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## 1. Introduction and main results

### 1.1. The problem

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$  with smooth boundary, and consider the Dirichlet problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Denote by  $\lambda_1(\Omega)$  the smallest positive constant  $\lambda$  for which this system has a solution (i.e.  $\lambda_1(\Omega)$  is the first eigenvalue of the Laplacian on  $\Omega$  with 0 Dirichlet boundary condition). By the Krein–Rutman theorem, the eigenvalue  $\lambda_1(\Omega)$  is simple, and the corresponding eigenfunction (that is unique up to a multiplicative constant) has constant sign on  $\Omega$ , see [14, Theorem 1.2.5]. One usually takes the eigenfunction  $u$  with  $u > 0$  on  $\Omega$  and  $\int_{\Omega} u^2 = 1$ . The eigenfunctions of higher eigenvalues must change sign on  $\Omega$ , since they are orthogonal to the first eigenfunction. By the Faber–Krahn inequality,

$$\lambda_1(\Omega) \geq \lambda_1(B^n(\Omega)) \tag{2}$$

where  $B^n(\Omega)$  is the round ball in  $\mathbb{R}^n$  with the same volume as  $\Omega$ . Moreover, equality holds in (2) if and only if  $\Omega = B^n(\Omega)$ , see [9] and [17]. In other words, round balls are minimizers for  $\lambda_1$  among domains of the same volume. This result can also be obtained by reasoning as follows. Consider the functional  $\Omega \rightarrow \lambda_1(\Omega)$  for all smooth bounded domains  $\Omega$  in  $\mathbb{R}^n$  of the same volume, say  $\text{Vol}(\Omega) = \alpha$ . A classical result due to Garabedian and Schiffer asserts that  $\Omega$  is a critical point for  $\lambda_1$  (among domains of volume  $\alpha$ ) if and only if the first eigenfunction of the Laplacian in  $\Omega$  with 0 Dirichlet boundary condition has also constant Neumann data at the boundary, see [11]. In this case, we say that  $\Omega$  is an extremal domain for the first eigenvalue of the Laplacian, or simply an *extremal domain*. Extremal domains are then characterized as the domains for which the *overdetermined system*

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \langle \nabla u, \nu \rangle = \text{const} & \text{on } \partial\Omega \end{cases} \tag{3}$$

has a positive solution (here  $\nu$  is the outward unit normal vector field along  $\partial\Omega$ ). By a classical result due to J. Serrin the only domains for which the system (3) has a positive solution are round balls, see [23]. One then checks that round balls are minimizers.

For domains with infinite volume, at first sight one cannot ask for “a domain that minimizes  $\lambda_1$ ”. Indeed, with  $c\Omega = \{cz \mid z \in \Omega\}$  we have

$$\lambda_1(c\Omega) = c^{-2}\lambda_1(\Omega), \quad c > 0.$$

On the other hand, system (3) can be studied also for unbounded domains. Therefore, it is natural to determine all domains  $\Omega$  for which (3) has a positive solution. This is an open problem. We will continue to call such a domain an *extremal domain*. In the non-compact case, this definition does not have a geometric meaning, except for domains which along each coordinate direction of  $\mathbb{R}^n$  are bounded or periodic. In the case of periodic directions, one obtains extremal domains for the first eigenvalue of the Laplacian in flat tori, cf. Remark 1.2(ii) below.

Berestycki, Caffarelli and Nirenberg conjectured in [2] that if  $f$  is a Lipschitz function on a domain  $\Omega$  in  $\mathbb{R}^n$  such that  $\mathbb{R}^n \setminus \overline{\Omega}$  is connected, then the existence of a bounded positive solution to the more general system

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \langle \nabla u, \nu \rangle = \text{const} & \text{on } \partial\Omega \end{cases} \tag{4}$$

implies that  $\Omega$  is a ball, or a half-space, or the complement of a ball, or a generalized cylinder  $B^k \times \mathbb{R}^{n-k}$  where  $B^k$  is a round ball in  $\mathbb{R}^k$ . Serrin’s result mentioned above continues to hold true for the overdetermined system (4): The only bounded domains where one can find a positive solution to (4) are round balls (see [23] for  $f \in C^1$  and [20] for  $f$  Lipschitz). Some extensions of this result to exterior domains have been proven by W. Reichel [22], and A. Aftalion and J. Busca [1]. Assuming that  $\Omega$  is the complement of a bounded region  $W$  with smooth boundary, for some particular classes of functions  $f$  and under some assumptions on the behavior of  $u$  at infinity, they proved that  $W$  must be a ball. In [10], A. Farina and E. Valdinoci look for natural geometric assumptions under which one can conclude that a domain  $\Omega$  admitting a positive solution  $u$  to (4) must be a half-space and  $u$  must be a function of only one variable. In particular, they obtain that if  $\Omega$  is a uniformly Lipschitz epigraph of  $\mathbb{R}^n$ , where  $n = 2$  or  $n = 3$ , then there exists no positive solution  $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$  of (3). In the recent paper [13], F. Hélein, L. Hauswirth and F. Pacard notice that on the 2-dimensional domain

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2 \mid |y| < \frac{\pi}{2} + \cosh(x) \right\}$$

there exists a positive solution to system (4) for  $f \equiv 0$ . This solution is not bounded, nor is  $\mathbb{R}^2 \setminus \overline{\Omega}$  connected, so that this example is not a counter-example to the conjecture of Berestycki, Caffarelli and Nirenberg. In [24], however, the second author constructed a counter-example to this conjecture by showing that the cylinder  $B^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$  (for which it is easy to find a bounded positive solution to (3)) can be perturbed to an unbounded domain whose boundary is a periodic hypersurface of revolution with respect to the  $\mathbb{R}$ -axis and such that (3) has a bounded positive solution. More precisely, *for each  $n \geq 2$  there exist a positive number  $T_* = T_*(n)$ , a sequence of positive numbers  $T_j \rightarrow T_*$ , and a sequence of non-constant  $T_j$ -periodic functions  $v_j \in C^{2,\alpha}(\mathbb{R})$  of mean zero (over the period) that converges to 0 in  $C^{2,\alpha}(\mathbb{R})$  such that the domains*

$$\Omega_j = \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| < 1 + v_j(t) \right\}$$

*have a positive solution  $u_j \in C^{2,\alpha}(\Omega_j)$  to the problem (3). The solution  $u_j$  is  $T_j$ -periodic in  $t$  and hence bounded.*

1.2. Main results

The goal of this paper is to show that these domains  $\Omega_j$  (introduced in [24] by the second author) belong to a smooth bifurcating family of domains, to determine their approximate shape for small bifurcation values, and to determine the bifurcation values  $T_*(n)$ . Our main result is the following.

**Theorem 1.1.** *Let  $\mathcal{C}_{\text{even},0}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$  be the space of even  $2\pi$ -periodic  $\mathcal{C}^{2,\alpha}$  functions of mean zero. For each  $n \geq 1$  there exist a positive number  $T_* = T_*(n)$  and a smooth map*

$$(-\varepsilon, \varepsilon) \rightarrow \mathcal{C}_{\text{even},0}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$$

$$s \mapsto (w_s, T_s)$$

with  $w_0 = 0, T_0 = T_*$  and such that for each  $s \in (-\varepsilon, \varepsilon)$  the system (3) has a positive solution  $u_s \in \mathcal{C}^{2,\alpha}(\Omega_s)$  on the modified cylinder

$$\Omega_s = \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| < 1 + s \cos\left(\frac{2\pi}{T_s}t\right) + s w_s\left(\frac{2\pi}{T_s}t\right) \right\}. \tag{5}$$

The solution  $u_s$  is  $T_s$ -periodic in  $t$  and hence bounded.

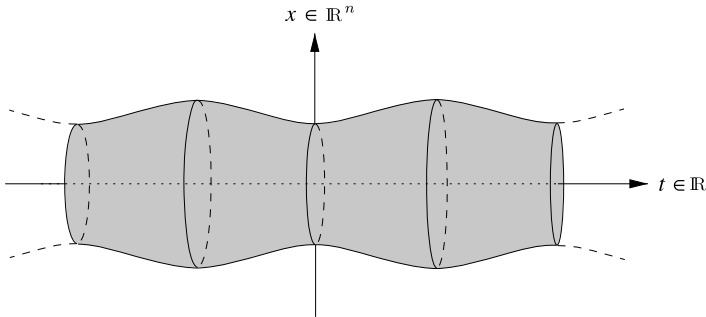


Fig. 1. A domain  $\Omega_s$ .

For  $n = 2$  and for  $|s|$  small enough, the bifurcating domains  $\Omega_s$  look as in Fig. 1. For a figure for  $n = 1$  see Section 8.

Notice that for  $n = 1$ , the domains  $\Omega_s$  do not provide counter-examples to the conjecture of Berestycki, Caffarelli and Nirenberg, because  $\mathbb{R}^2 \setminus \overline{\Omega_s}$  is not connected.

**Remarks 1.2.** (i) From the extremal domains  $\Omega_s \subset \mathbb{R}^{n+1}$  and the solutions  $u_s$  from Theorem 1.1 we obtain other extremal domains by adding an  $\mathbb{R}^k$ -factor: For each  $k \geq 1$  the domains  $\Omega_s^k := \Omega_s \times \mathbb{R}^k$  are extremal domains in  $\mathbb{R}^{n+1+k}$  with solutions  $u_s^k(x, t, y) := u_s(x, t)$  (where  $y \in \mathbb{R}^k$ ). For instance, in  $\mathbb{R}^3$  we then have the “wavy cylinder” in Fig. 1, and the “wavy board” obtained by taking the product of the wavy band in Fig. 2 with  $\mathbb{R}$ . Notice that  $\mathbb{R}^{n+1+k} \setminus \overline{\Omega_s^k}$  is connected if and only if  $n \geq 2$ .

(ii) The characterization of extremal domains described in Section 1.1 more generally holds for domains in Riemannian manifolds: Given a Riemannian manifold  $(M, g)$ , a domain  $\Omega \subset M$

of given finite volume is a critical point of  $\Omega \rightarrow \lambda_1(\Omega)$ , where  $\lambda_1(\Omega)$  is the first eigenvalue of the Laplace–Beltrami operator  $-\Delta_g$ , if and only if the overdetermined system

$$\begin{cases} \Delta_g u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ g(\nabla u, \nu) = \text{const} & \text{on } \partial\Omega \end{cases} \quad (6)$$

has a positive solution (here  $\nu$  is the outward unit normal vector to  $\partial\Omega$  with respect to  $g$ ), see [8] and [19]. Theorem 1.1 thus implies that the full tori

$$\tilde{\Omega}_s = \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R}/T_s\mathbb{Z} \mid \|x\| < 1 + s \cos\left(\frac{2\pi}{T_s}t\right) + s\nu_s\left(\frac{2\pi}{T_s}t\right) \right\}$$

are extremal domains in the manifold  $\mathbb{R}^n \times \mathbb{R}/T_s\mathbb{Z}$  with the metric induced by the Euclidean metric.  $\square$

**Open problem 1.** Are the extremal domains  $\tilde{\Omega}_s$  in  $\mathbb{R}^n \times \mathbb{R}/T_s\mathbb{Z}$  (local) minima for the functional  $\Omega \rightarrow \lambda_1(\Omega)$ ?

It follows from our proof of Theorem 1.1 and from the Implicit Function Theorem that the family  $\Omega_s$  is unique among those smooth families of extremal domains bifurcating from the straight cylinder that are rotationally symmetric with respect to  $\mathbb{R}^n$  and periodic with respect to  $\mathbb{R}$ . A much stronger uniqueness property should hold. Indeed, the existence problem of extremal domains near the solid cylinder, say in  $\mathbb{R}^3$ , is tightly related to the existence problem of positive constant mean curvature surfaces near the cylinder, see Sections 2 and 3. Any positive constant mean curvature surface with two ends (that is properly embedded and complete) must be a Delaunay surface, by a result of Korevaar, Kusner, and Solomon [16]. We thus ask:

**Open problem 2.** Assume that  $\Omega$  is an unbounded extremal domain in  $\mathbb{R}^{n+1}$  that is contained in a solid cylinder. Is it then true that  $\Omega$  belongs to the family  $\Omega_s$ ?

We also determine the bifurcation values  $T_* = T_*(n)$ . It has been proved in [24] that  $T_*(n) < \frac{2\pi}{\sqrt{n-1}}$ . In particular,  $T_*(n) \rightarrow 0$  as  $n \rightarrow \infty$ . We shall show in Section 8 that  $T_*(1) = 4$ . Fix now  $n \geq 2$  and define  $\nu = \frac{n-2}{2}$ . Write  $T_\nu$  for  $T_*(n)$ .

**Theorem 1.3.** Let  $J_\nu : (0, +\infty) \rightarrow \mathbb{R}$  be the Bessel function of the first kind. Let  $j_\nu$  be its smallest positive zero. Then the function  $sJ_{\nu-1}(s) + J_\nu(s)$  has a unique zero on the interval  $(0, j_\nu)$ , say  $\rho_\nu$ , and

$$T_\nu = \frac{2\pi}{\sqrt{j_\nu^2 - \rho_\nu^2}}.$$

In particular,

$$T_\nu = \sqrt{2\pi} \nu^{-1/2} + O(\nu^{-7/6}).$$

Furthermore, the sequence  $T_\nu$  is strictly decreasing to 0.

The numbers  $T_\nu$  for  $\nu \leq 10$  are given in Section 9. In particular, for  $n = 2, 3$  and 4 (corresponding to the bifurcation of the straight cylinder in  $\mathbb{R}^3, \mathbb{R}^4$  and  $\mathbb{R}^5$ ) the values of  $T_\nu$  are

$$T_0 \approx 3.06362, \quad T_{\frac{1}{2}} \approx 2.61931, \quad T_1 \approx 2.34104.$$

**Open problem 3.** Is the bifurcation at  $T_*(n)$  sub-critical, critical, or super-critical? In other words,  $\partial_s(T_s)|_{s=0} < 0$ ,  $\partial_s(T_s)|_{s=0} = 0$ , or  $\partial_s(T_s)|_{s=0} > 0$ ?

The paper is organized as follows. In Section 2 we show how the existence of Delaunay surfaces (i.e., constant mean curvature surfaces of revolution in  $\mathbb{R}^3$  that are different from the cylinder) can be proved by means of a bifurcation theorem due to Crandall and Rabinowitz. We will follow the same line of arguments to prove Theorem 1.1 in Sections 3–8. In Section 9 we prove Theorem 1.3 on the bifurcation values  $T_*(n)$ .

## 2. The Delaunay surface via the Crandall–Rabinowitz theorem

Our proof of Theorem 1.1 is motivated by the following argument that proves the existence of Delaunay surfaces by means of the Crandall–Rabinowitz bifurcation theorem. The material of this section was explained by Frank Pacard to the second author when he was his PhD student.

We start with some generalities. Let  $\Sigma$  be an embedded hypersurface in  $\mathbb{R}^{n+1}$  of codimension 1. We denote by  $II$  its second fundamental form defined by

$$II(X, Y) = -\langle \nabla_X N, Y \rangle$$

for all vector fields  $X, Y$  in the tangent bundle  $T\Sigma$ . Here  $N$  is the unit normal vector field on  $\Sigma$ , and  $\langle \cdot, \cdot \rangle$  denotes the standard scalar product of  $\mathbb{R}^{n+1}$ . The mean curvature  $H$  of  $\Sigma$  is defined to be the average of the principal curvatures, i.e. of the eigenvalues  $k_1, \dots, k_n$  of the shape operator  $A : T\Sigma \rightarrow T\Sigma$  given by the endomorphism

$$\langle AX, Y \rangle = -II(X, Y).$$

Hence

$$H(\Sigma) = \frac{1}{n} \sum_{i=1}^n k_i.$$

Given a sufficiently smooth function  $w$  defined on  $\Sigma$  we can define the normal graph  $\Sigma_w$  of  $w$  over  $\Sigma$ ,

$$\Sigma_w = \{p + w(p)N(p) \in \mathbb{R}^{n+1} \mid p \in \Sigma\},$$

and consider the operator  $w \mapsto H(\Sigma_w)$  that associates to  $w$  the mean curvature of  $\Sigma_w$ . The linearization of this operator at  $w = 0$  is given by the Jacobi operator:

$$D_w H(\Sigma_w)|_{w=0} = \frac{1}{n} \left( \Delta_g + \sum_{i=1}^n k_i^2 \right),$$

where  $g$  is the metric induced on  $\Sigma$  by the Euclidean metric and  $-\Delta_g$  is the Laplace–Beltrami operator on  $\Sigma$ . All these facts are well known, and we refer to [3] for further details.

In 1841, C. Delaunay discovered a beautiful one-parameter family of complete, embedded, non-compact surfaces  $D_\sigma$  in  $\mathbb{R}^3$ ,  $\sigma > 0$ , whose mean curvature is constant, see [5]. These surfaces are invariant under rotation about an axis and periodic in the direction of this axis. The Delaunay surface  $D_\sigma$  can be parametrized by

$$X_\sigma(\theta, t) = (y(t) \cos \theta, y(t) \sin \theta, z(t)) \tag{7}$$

for  $(\theta, t) \in S^1 \times \mathbb{R}$ , where the function  $y$  is the smooth solution of

$$(y'(t))^2 = y^2(t) - \left(\frac{y^2(t) + \sigma}{2}\right)^2 \tag{8}$$

and  $z$  is the solution (up to a constant) of

$$z'(t) = \left(\frac{y^2(t) + \sigma}{2}\right). \tag{9}$$

When  $\sigma = 1$ , the Delaunay surface is nothing but the cylinder  $D_1 = S^1 \times \mathbb{R}$ . It is easy to compute the mean curvature of the family  $D_\sigma$  and to check that it is equal to 1 for all  $\sigma$ . One can obtain each Delaunay surface  $D_\sigma$  by taking the surface of revolution generated by the roulette of an ellipse, i.e. the trace of a focus of an ellipse  $\ell$  as  $\ell$  rolls along a straight line in the plane. In particular, these surfaces are periodic in the direction of the axis of revolution. When the ellipse  $\ell$  degenerates to a circle, the roulette of  $\ell$  becomes a straight line and generates the straight cylinder, and when  $\sigma \rightarrow 0$ ,  $D_\sigma$  tends to the singular surface which is the union of infinitely many spheres of radius  $1/2$  centered at the points  $(0, 0, n)$ ,  $n \in \mathbb{Z}$ . For further details about this geometric description of Delaunay surfaces we refer to [6].

We now prove the existence of Delaunay surfaces by a bifurcation argument, using a bifurcation theorem due to M. Crandall and P. Rabinowitz. Their theorem applies to Delaunay surfaces in a simple way. We shall use the same method to prove Theorem 1.1. The phenomenon underlying our existence proof of Delaunay surfaces is the Plateau–Rayleigh instability of the cylinder [21].

Consider the straight cylinder of radius 1, in cylindrical coordinates:

$$C_1 = \{(\rho, \theta, t) \in (0, +\infty) \times S^1 \times \mathbb{R} \mid \rho = 1\}.$$

Let  $w$  be a  $C^2$ -function on  $S^1 \times \mathbb{R}/2\pi\mathbb{Z}$ . In Fourier series,

$$w(\theta, t) = \sum_{j,k \geq 0} (\alpha_j \cos(j\theta) + \beta_j \sin(j\theta))(a_k \cos(kt) + b_k \sin(kt)).$$

If  $w(\theta, t) > -1$  for all  $\theta, t$ , we consider, for each  $T > 0$ , the normal graph  $C_{1+w}^T$  over the cylinder  $C_1$  of  $w$  rescaled to period  $T$ ,

$$C_{1+w}^T := \left\{ (\rho, \theta, t) \in (0, +\infty) \times S^1 \times \mathbb{R} \mid \rho = 1 + w\left(\theta, \frac{2\pi}{T}t\right) \right\}.$$

Define the operator

$$\tilde{F}(w, T) = 1 - H(C_{1+w}^T)$$

where  $H$  is the mean curvature. Then  $\tilde{F}(w, T)$  is a function on  $S^1 \times \mathbb{R}$  of period  $T$  in the second variable. Therefore,

$$F(w, T)(\theta, t) := \tilde{F}(w, T)\left(\theta, \frac{T}{2\pi}t\right) \tag{10}$$

is a function on  $S^1 \times \mathbb{R}/2\pi\mathbb{Z}$ . Note that  $F(0, T) = 0$  for all  $T > 0$ , because for  $w = 0$  the surface  $C_{1+w}^T$  is the cylinder  $C_1$  whose mean curvature is 1. If we found a non-trivial solution  $(w, T)$  of the equation  $F(w, T) = 0$ , we would obtain a constant mean curvature surface different from  $C_1$ . In order to solve this equation, we consider the linearization of the operator  $F$  with respect to  $w$  and computed at  $(w, T) = (0, T)$ . As mentioned above, the linearization of the mean curvature operator for normal graphs over a given surface with respect to  $w$  computed at  $w = 0$  is the Jacobi operator. Since the Laplace–Beltrami operator on  $C_1$  (with the metric induced by the Euclidean metric) is  $-\partial_\theta^2 - \partial_t^2$ , and since the principal curvatures  $k_i$  of  $C_1$  are equal to 0 and 1, we find that

$$D_w F(0, T) = -\frac{1}{2}\left(\partial_\theta^2 + \left(\frac{2\pi}{T}\right)^2 \partial_t^2 + 1\right).$$

For each  $j, k \in \mathbb{N} \cup \{0\}$  and each  $T > 0$ , the four 1-dimensional spaces generated by the functions

$$\cos(j\theta)\cos(kt), \quad \cos(j\theta)\sin(kt), \quad \sin(j\theta)\cos(kt), \quad \sin(j\theta)\sin(kt)$$

are eigenspaces of  $D_w F(0, T)$  with eigenvalue

$$\sigma_{j,k}(T) = \frac{1}{2}\left(j^2 - 1 + \left(\frac{2\pi k}{T}\right)^2\right).$$

Clearly,

- $\sigma_{j,k}(T) \neq 0$  for all  $T > 0$  if  $j \geq 2$ , or if  $j = 1$  and  $k \geq 1$ ;
- $\sigma_{1,0}(T) = 0$  for all  $T > 0$ ;
- $\sigma_{0,k}(T) = 0$  only for  $T = 2\pi k$  and  $k \geq 1$ ; moreover  $\sigma_{0,k}(T)$  changes sign at these points.

It follows that  $\text{Ker } D_w F(0, T)$  is 2-dimensional (spanned by  $\cos \theta, \sin \theta$ ) if  $T > 0$  and  $T \notin 2\pi\mathbb{N}$ , and that  $\text{Ker } D_w F(0, T)$  is 4-dimensional (spanned by  $\cos \theta, \sin \theta, \cos(kt), \sin(kt)$ ) if  $T \in 2\pi\mathbb{N}$ .

We will now bring into play an abstract bifurcation theorem, which is due to Crandall and Rabinowitz. For the proof and for many other applications we refer to [15,25] and to the original exposition [4].

**Theorem 2.1** (*Crandall–Rabinowitz bifurcation theorem*). *Let  $X$  and  $Y$  be Banach spaces, and let  $U \subset X$  and  $\Lambda \subset \mathbb{R}$  be open subsets, where we assume  $0 \in U$ . Denote the elements of  $U$  by  $w$  and the elements of  $\Lambda$  by  $T$ . Let  $F : U \times \Lambda \rightarrow Y$  be a  $C^\infty$ -smooth function such that*

- i)  $F(0, T) = 0$  for all  $T \in \Lambda$ ;
- ii)  $\text{Ker } D_w F(0, T_0) = \mathbb{R}w_0$  for some  $T_0 \in \Lambda$  and some  $w_0 \in X \setminus \{0\}$ ;
- iii)  $\text{codim Im } D_w F(0, T_0) = 1$ ;
- iv)  $D_T D_w F(0, T_0)(w_0) \notin \text{Im } D_w F(0, T_0)$ .

Choose a linear subspace  $\dot{X} \subset X$  such that  $\mathbb{R}w_0 \oplus \dot{X} = X$ . Then there exists a  $C^\infty$ -smooth curve

$$(-\varepsilon, \varepsilon) \rightarrow \dot{X} \times \mathbb{R}, \quad s \mapsto (w(s), T(s))$$

such that

- 1)  $w(0) = 0$  and  $T(0) = T_0$ ;
- 2)  $s(w_0 + w(s)) \in U$  and  $T(s) \in \mathbb{R}$ ;
- 3)  $F(s(w_0 + w(s)), T(s)) = 0$ .

Moreover, there is a neighborhood  $\mathcal{N}$  of  $(0, T_0) \in X \times \mathbb{R}$  such that  $\{s(w_0 + w(s)), T(s)\}$  is the only branch in  $\mathcal{N}$  that bifurcates from  $\{(0, T) \mid T \in \Lambda\}$ .

The theorem is useful for finding non-trivial solutions of an equation  $F(x, \lambda) = 0$ , where  $x$  belongs to a Banach space and  $\lambda$  is a real number. It says that under the given hypothesis, there is a smooth bifurcation into the direction of the kernel of  $D_w F$  for the solution of  $F(x, \lambda) = 0$ , and that there is no other nearby bifurcation.

In order to apply Theorem 2.1, we now restrict the operator  $F$  defined in (10) to functions that are independent of  $\theta$  (so as to get rid of the functions  $\cos \theta, \sin \theta$  in the kernel of  $D_w F(0, T)$ ) and that are even (so as to have a 1-dimensional kernel for  $T \in 2\pi\mathbb{N}$ ). We can also assume that the functions  $w$  have zero mean. In other words, we look for new constant mean curvature surfaces among deformations of  $C_1$  that are surfaces of revolution, even in the  $t$ -direction. We hence consider the Banach space

$$X = C_{\text{even},0}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$$

of even  $2\pi$ -periodic functions of zero mean whose second derivative is Hölder continuous. Moreover, define the open subset  $U = \{w \in X \mid w(t) > -1 \text{ for all } t\}$  of  $X$ , and the Banach space

$$Y = C_{\text{even},0}^{0,\alpha}(\mathbb{R}/2\pi\mathbb{Z}).$$

Furthermore, chose  $\Lambda = (0, +\infty) \subset \mathbb{R}$ . Then the operator  $F$  defined as above restricts to the operator

$$F : U \times \Lambda \rightarrow Y.$$

With

$$\sigma_k(T) := \sigma_{0,k}(T) = \frac{1}{2} \left( -1 + \left( \frac{2\pi k}{T} \right)^2 \right),$$

its linearization with respect to  $w$  at  $T_0 := 2\pi$  is

$$D_w F(0, T_0) \left( \sum_{k \geq 1} a_k \cos(kt) \right) = \sum_{k \geq 1} \sigma_k(T_0) a_k \cos(kt) = \sum_{k \geq 1} \frac{1}{2} (k^2 - 1) a_k \cos(kt).$$

Hence,

$$\text{Ker } D_w F(0, T_0) = \mathbb{R} \cos t.$$

Moreover, the image  $\text{Im } D_w F(0, T_0)$  is the closure of  $\bigoplus_{k \geq 2} \mathbb{R} \cos(kt)$  in  $Y$ ; its complement in  $Y$  is the 1-dimensional space spanned by  $\cos t$ . Finally,

$$D_T D_w F(0, T_0)(\cos t) = \frac{\partial \sigma_1(T)}{\partial T} \Big|_{T=T_0} \cos t = -\frac{1}{2\pi} \cos t \notin \text{Im } D_w F(0, T_0).$$

With  $w_0 = \cos t$  and  $\dot{X}$  the closure of  $\bigoplus_{k \geq 2} \mathbb{R} \cos(kt)$  in  $X$ , the Crandall–Rabinowitz bifurcation theorem applies and yields the existence of a  $C^\infty$ -smooth curve

$$(-\varepsilon, \varepsilon) \rightarrow \dot{X} \times \mathbb{R}, \quad s \mapsto (w(s), T(s)) \tag{11}$$

such that

- 1)  $w(0) = 0$  and  $T(0) = T_0$ ;
- 2)  $F(s(w_0 + w(s)), T(s)) = 0$ ,

i.e. (by the definition of the operator  $F$ ) the existence of a  $C^\infty$ -smooth family of surfaces of revolution that have mean curvature constant and equal to 1, bifurcating from the cylinder  $C_1$ . That these surfaces are Delaunay surfaces follows from Sturm’s variational characterization of constant mean curvature surfaces of revolution [5,6].

**Remarks 2.2.** (i) The boundaries of the new domains  $\Omega_s \subset \mathbb{R}^3$  described in Theorem 1.1 are not Delaunay surfaces (at least not for  $|s|$  small). Indeed, Delaunay surfaces bifurcate from the cylinder at  $T_0 = 2\pi$ , while the domains  $\Omega_s$  bifurcate from the cylinder at  $T_*(2) \approx 3.06362$ .

(ii) It follows from the representation (7), (8), (9) of the Delaunay surfaces  $D_\sigma$ , or also from their geometric description, that the family  $\sigma \mapsto D_\sigma$  is real analytic. This does not follow from their construction using the Crandall–Rabinowitz theorem, which gives only the  $C^\infty$ -smooth curve (11). Accordingly, we do not know whether the  $C^\infty$ -smooth curve  $(-\varepsilon, \varepsilon) \rightarrow C_{\text{even},0}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}$  of Theorem 1.1, that describes the new extremal domains  $\Omega_s$ , is real analytic.  $\square$

### 3. Rephrasing the problem for extremal domains

We want to follow the proof of the existence of Delaunay surfaces given in the previous section in order to prove the existence of a smooth family of normal graphs over the straight cylinder such that the first eigenfunction of the Dirichlet Laplacian has constant Neumann data. In this section we recall the set-up from [24], where the second author studied the Dirichlet-to-Neumann operator that associates to a periodic function  $v$  the normal derivative of the first eigenfunction of the domain defined by the normal graph of  $v$  over the straight cylinder, and

computed the linearization of this operator. The novelty of this paper is the analysis of the kernel of the linearized operator; it will be carried out in Sections 4–7.

The manifold  $\mathbb{R}/2\pi\mathbb{Z}$  will always be considered with the metric induced by the Euclidean metric. Motivated by the previous section, we consider the Banach space  $C_{\text{even},0}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$  of even functions on  $\mathbb{R}/2\pi\mathbb{Z}$  of mean 0. For each function  $v \in C_{\text{even},0}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$  with  $v(t) > -1$  for all  $t$ , the domain

$$C_{1+v}^T := \left\{ (x, t) \in \mathbb{R}^n \times \mathbb{R}/T\mathbb{Z} \mid 0 \leq \|x\| < 1 + v\left(\frac{2\pi}{T}t\right) \right\}$$

is well defined for all  $T > 0$ . The domain  $C_{1+v}^T$  is relatively compact. According to standard results on the Dirichlet eigenvalue problem (see [12]), there exist, for each  $T > 0$ , a unique positive function

$$\phi = \phi_{v,T} \in C^{2,\alpha}(C_{1+v}^T)$$

and a constant  $\lambda = \lambda_{v,T} \in \mathbb{R}$  such that  $\phi$  is a solution to the problem

$$\begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in } C_{1+v}^T, \\ \phi = 0 & \text{on } \partial C_{1+v}^T \end{cases} \tag{12}$$

which is normalized by

$$\int_{C_{1+v}^{2\pi}} \left( \phi\left(x, \frac{T}{2\pi}t\right) \right)^2 \text{dvol} = 1. \tag{13}$$

Furthermore,  $\phi$  and  $\lambda$  depend smoothly on  $v$ . We denote  $\phi_1 := \phi_{0,T}$  and  $\lambda_1 := \lambda_{0,T}$ . Notice that  $\phi_1$  does not depend on the  $t$  variable and is radial in the  $x$  variable. (Indeed,  $\phi_1$  is nothing but the first eigenfunction of the Dirichlet Laplacian over the unit ball  $\mathbb{B}^n$  in  $\mathbb{R}^n$  normalized to have  $L^2$ -norm  $\frac{1}{2\pi}$ .) We can thus consider  $\phi_1$  as a function of  $r := \|x\|$ , and we write

$$\varphi_1(r) = \phi_1(x). \tag{14}$$

We define the Dirichlet-to-Neumann operator

$$\tilde{F}(v, T) = \langle \nabla\phi, \nu \rangle|_{\partial C_{1+v}^T} - \frac{1}{\text{Vol}(\partial C_{1+v}^T)} \int_{\partial C_{1+v}^T} \langle \nabla\phi, \nu \rangle \text{dvol},$$

where  $\nu$  denotes the unit normal vector field on  $\partial C_{1+v}^T$  and where  $\phi = \phi_{v,T}$  is the solution of (12). The function

$$\tilde{F}(v, T) : \partial C_{1+v}^T \cong \partial(\mathbb{B}^n) \times \mathbb{R}/T\mathbb{Z} \rightarrow \mathbb{R}$$

depends only on the variable  $t \in \mathbb{R}/T\mathbb{Z}$ , since  $v$  has this property. It is an even function; indeed,  $v$  is even, and hence  $\phi_{v,T}$  is even, since the first eigenvalue  $\lambda_{v,T}$  is simple. Moreover,  $\tilde{F}(v, T)$  has mean 0. We rescale  $\tilde{F}$  and define

$$F(v, T)(t) = \tilde{F}(v, T)\left(\frac{T}{2\pi}t\right).$$

Schauder’s estimates imply that  $F$  takes values in  $C_{\text{even},0}^{1,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ . With

$$U := \{v \in C_{\text{even},0}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z}) \mid v(t) > -1 \text{ for all } t\}$$

we thus have

$$F : U \times (0, +\infty) \rightarrow C_{\text{even},0}^{1,\alpha}(\mathbb{R}/2\pi\mathbb{Z}).$$

Also notice that  $F(0, T) = 0$  for all  $T > 0$ , and that  $F$  is smooth.

The following result is proved in [24].

**Proposition 3.1.** *The linearized operator*

$$H_T := D_w F(0, T) : C_{\text{even},0}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z}) \rightarrow C_{\text{even},0}^{1,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$$

is a formally self adjoint, first order elliptic operator. It preserves the eigenspaces

$$V_k = \mathbb{R} \cos(kt)$$

for all  $k$  and all  $T > 0$ , and we have

$$H_T(w)(t) = \left( \partial_r \psi + \partial_r^2 \phi_1 \cdot w\left(\frac{2\pi t}{T}\right) \right) \Big|_{\partial C_1^T} \tag{15}$$

where  $\psi$  is the unique solution of

$$\begin{cases} \Delta \psi + \lambda_1 \psi = 0 & \text{in } C_1^T, \\ \psi = -\partial_r \phi_1 \cdot w(2\pi t/T) & \text{on } \partial C_1^T \end{cases} \tag{16}$$

which is  $L^2(C_1^T)$ -orthogonal to  $\phi_1$ , and where  $r = \|x\|$ .

Write

$$w(t) = \sum_{k \geq 1} a_k \cos(kt).$$

Since  $H_T$  preserves the eigenspaces,

$$H_T(w)(t) = \sum_{k \geq 1} \sigma_k(T) a_k \cos(kt). \tag{17}$$

We use (15) and (16) to describe  $\sigma_k(T)$  as the solution of an ordinary differential equation: The solution  $\psi$  of (16) is differentiable, and even with respect to  $x$  for fixed  $t$ . Therefore, for each  $t$ , the derivative of  $\psi$  with respect to  $r$  vanishes at 0:  $\partial_r \psi|_{r=0} = 0$ . Hence,

$$\sigma_k(T) = c'_k(1) + \varphi''_1(1) \tag{18}$$

where for  $n \geq 2$ ,  $c_k$  is the continuous solution on  $[0, 1]$  of the ordinary differential equation

$$\left( \partial_r^2 + \frac{n-1}{r} \partial_r + \lambda_1 - \left( \frac{2\pi k}{T} \right)^2 \right) c_k = 0$$

such that  $c_k(1) = -\varphi'_1(1)$ , while for  $n = 1$ ,  $c_k$  is the solution on  $[0, 1]$  of the ordinary differential equation

$$\left( \partial_r^2 + \lambda_1 - \left( \frac{2\pi k}{T} \right)^2 \right) c_k = 0$$

such that  $c_k(1) = -\varphi'_1(1)$  and  $c'_k(0) = 0$ . Notice that for all  $k \geq 1$  and all  $n \geq 1$

$$\sigma_k(T) = \sigma_1\left(\frac{T}{k}\right).$$

Our next aim is to find an explicit expression for the function  $\sigma_1$  in order to describe the spectrum of the linearized operator, to read off its kernel, and to find the codimension of its image. We first consider the case  $n \geq 2$ , for which we need Bessel functions. The case  $n = 1$  is discussed in Section 8.

#### 4. Recollection on Bessel functions

In what follows we shall use several basic properties of Bessel functions. For the readers convenience, we recall the definition of the Bessel functions  $J_\tau$  and  $I_\tau$ , and state their principal properties. For proofs we refer to [26, Chapter III].

##### 4.1. The functions $J_\tau$

For  $\tau \geq 0$  the Bessel function of the first kind  $J_\tau : \mathbb{R} \rightarrow \mathbb{R}$  is the solution of the differential equation

$$s^2 y''(s) + s y'(s) + (s^2 - \tau^2) y(s) = 0 \tag{19}$$

whose power series expansion is

$$J_\tau(s) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}s\right)^{\tau+2m}}{m! \Gamma(\tau + m + 1)}. \tag{20}$$

We read off that

$$J_0(0) = 1, \quad J_\tau(0) = 0 \quad \text{for all } \tau > 0. \tag{21}$$

The power series (20) defines a solution  $J_\tau : (0, \infty) \rightarrow \mathbb{R}$  of (19) also for  $\tau < 0$ . If  $\tau = n$  is an integer, then

$$J_{-n}(s) = (-1)^n J_n(s)$$

and  $J_n$  is bounded near 0. If  $\tau$  is not an integer, then the function  $J_\tau(s)$  is bounded near 0 if  $\tau > 0$  but diverges as  $s \rightarrow 0$  if  $\tau < 0$ . The functions  $J_\tau(s)$  and  $J_{-\tau}(s)$  are therefore linearly independent, and hence are the two solutions of the differential equation (19) on  $(0, \infty)$ .

For all  $\tau \in \mathbb{R}$  and all  $s > 0$  we have the recurrence relations

$$J_{\tau-1}(s) + J_{\tau+1}(s) = \frac{2\tau}{s} J_\tau(s), \tag{22}$$

$$J_{\tau-1}(s) - J_{\tau+1}(s) = 2J'_\tau(s), \tag{23}$$

$$sJ'_\tau(s) + \tau J_\tau(s) = sJ_{\tau-1}(s), \tag{24}$$

$$sJ'_\tau(s) - \tau J_\tau(s) = -sJ_{\tau+1}(s). \tag{25}$$

Another important property that we will use often is that the first eigenvalue  $\lambda_1$  of the Dirichlet Laplacian on the unit ball of  $\mathbb{R}^n$ ,  $n \geq 2$ , is equal to the square of the first positive zero of  $J_\nu$  for  $\nu = \frac{n-2}{2}$ . Notice that  $\lambda_1$  depends on  $n$ . Moreover, the function  $J_\nu$  is positive on the interval  $(0, \sqrt{\lambda_1})$ , and  $J'_\nu(\sqrt{\lambda_1}) < 0$ .

#### 4.2. The functions $I_\tau$

For  $\tau \in \mathbb{R}$  the modified Bessel function of the first kind  $I_\tau : \mathbb{R} \rightarrow \mathbb{R}$  is the solution of the differential equation

$$s^2 y''(s) + s y'(s) - (s^2 + \tau^2) y(s) = 0$$

whose power series expansion is

$$I_\tau(s) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}s)^{\tau+2m}}{m! \Gamma(\tau + m + 1)}. \tag{26}$$

We read off that  $I_\tau(s) > 0$  for all  $\tau \in \mathbb{R}$  and  $s > 0$ , and that

$$I_0(0) = 1, \quad I_\tau(0) = 0 \quad \text{for all } \tau > 0. \tag{27}$$

Comparing coefficients readily shows that for all  $\tau \in \mathbb{R}$  and all  $s > 0$  we have the recurrence relations

$$I_{\tau-1}(s) - I_{\tau+1}(s) = \frac{2\tau}{s} I_{\tau}(s), \tag{28}$$

$$I_{\tau-1}(s) + I_{\tau+1}(s) = 2I'_{\tau}(s), \tag{29}$$

$$sI'_{\tau}(s) + \tau I_{\tau}(s) = sI_{\tau-1}(s), \tag{30}$$

$$sI'_{\tau}(s) - \tau I_{\tau}(s) = sI_{\tau+1}(s). \tag{31}$$

We shall also make use of the asymptotics

$$\lim_{s \rightarrow \infty} \frac{I_{\tau}(s)}{\frac{1}{\sqrt{2\pi s}} e^s} = 1. \tag{32}$$

**5. A formula for  $\sigma_1(T)$  when  $n \geq 2$**

In this and the next section we analyze the first eigenvalue  $\sigma_1(T)$  of the linearized operator  $H_T$  given by (17). We assume that  $n \geq 2$  throughout. Our goal is to show that the function  $\sigma_1(T)$  has negative derivate and vanishes once, say at  $T_v$ . From this and from  $\sigma_k(T) = \sigma_1(T/k)$ ,  $k \geq 2$ , we shall readily conclude in Section 7 that the operator  $H_{T_v} = D_w F(0, T_v)$  satisfies the assumptions of the Crandall–Rabinowitz theorem, implying Theorem 1.1 for  $n \geq 2$ .

To simplify the notation, we denote the previously defined function  $c_1$  by  $c$ . Recall that for  $n \geq 2$ ,

$$\sigma_1(T) = c'(1) + \varphi_1''(1) \tag{33}$$

where  $c$  is the continuous solution on  $[0, 1]$  of the ordinary differential equation

$$\left( \partial_r^2 + \frac{n-1}{r} \partial_r + \lambda_1 - \left( \frac{2\pi}{T} \right)^2 \right) c = 0 \tag{34}$$

such that  $c(1) = -\varphi_1'(1)$ . We shall distinguish three cases, according to whether the term

$$\lambda_1 - \left( \frac{2\pi}{T} \right)^2$$

is negative, zero or positive. Recall that  $\lambda_1$  depends on  $n$ . In order to simplify notation, we put  $v = \frac{n-2}{2}$  and write  $\lambda_v$  for  $\lambda_1 = \lambda_1(n)$ . As mentioned in the previous section,  $\sqrt{\lambda_v}$  is the first zero of  $J_v$ . Denote

$$j_v = \sqrt{\lambda_v}$$

and  $\mu = \frac{2\pi}{j_v}$ . Our aim is to find a formula for  $\sigma_1$ , and then to study its derivative. Unfortunately, we are not able to find a single formula for this function. However, we find a formula for  $\sigma_1$  holding for  $T < \mu$ , another formula for  $\sigma_1$  holding for  $T > \mu$ , and the value of  $\sigma_1$  at  $\mu$ . To simplify the reading, we define the two functions

$$\begin{aligned} \sigma_{\text{Left}}(T) &: (0, \mu) \rightarrow \mathbb{R}, & T &\rightarrow \sigma_1(T), \\ \sigma_{\text{Right}}(T) &: (\mu, +\infty) \rightarrow \mathbb{R}, & T &\rightarrow \sigma_1(T). \end{aligned} \tag{35}$$

In this section we shall find explicit formulae for the functions  $\sigma_{\text{Left}}$  and  $\sigma_{\text{Right}}$ . In view of (33), (34) and the definition of  $\varphi_1$ , it is not surprising that these formulae will be given in terms of Bessel functions. These formulae will readily imply that  $\sigma_{\text{Left}}(T) > 0$  and  $\sigma_1(\mu) > 0$ . In Section 6 we shall use the formula for  $\sigma_{\text{Right}}$  to show that  $\sigma_{\text{Right}}$  has negative derivative and vanishes once, say at  $T_\nu$ .

5.1. A formula for  $\sigma_{\text{Left}}$

Assume that  $T < \mu$ . This allows us to define

$$\xi = \sqrt{\left(\frac{2\pi}{T}\right)^2 - \lambda_\nu}. \tag{36}$$

We rescale the function  $c$  by defining

$$\tilde{c}(s) = c\left(\frac{s}{\xi}\right).$$

In view of (34),  $\tilde{c}$  is the continuous solution on  $[0, \xi]$  of

$$\left(\partial_s^2 + \frac{n-1}{s}\partial_s - 1\right)\tilde{c} = 0$$

with  $\tilde{c}(\xi) = -\varphi'_1(1)$ . This equation is very similar to a modified Bessel equation. In order to obtain exactly a modified Bessel equation, we define the function  $\hat{c}$  by

$$\tilde{c}(s) = s^{-\nu}\hat{c}(s).$$

Note that  $-\nu \leq 0$  because  $n \geq 2$ . Hence  $\hat{c}$  is the continuous solution on  $[0, \xi]$  of

$$\left[\partial_s^2 + \frac{1}{s}\partial_s - \left(1 + \frac{\nu^2}{s^2}\right)\right]\hat{c} = 0$$

with  $\hat{c}(\xi) = -\xi^\nu\varphi'_1(1)$ . The solution of this ordinary differential equation is given by  $\alpha I_\nu(s)$ , where the constant  $\alpha$  (depending on  $\nu$  and  $T$ ) is chosen such that

$$\alpha I_\nu(\xi) = -\xi^\nu\varphi'_1(1).$$

Returning to the function  $c$ , we get

$$c(r) = -\frac{\varphi'_1(1)}{I_\nu(\xi)}r^{-\nu}I_\nu(\xi r)$$

and from (18) and (35), using the identities (29), (30) and (31), we obtain

$$\begin{aligned} \sigma_{\text{Left}}(T) &= -\varphi_1'(1) \frac{1}{I_\nu(\xi)} \frac{1}{2} (\xi I_{\nu-1}(\xi) - 2\nu I_\nu(\xi) + \xi I_{\nu+1}(\xi)) + \varphi_1''(1) \\ &= \varphi_1''(1) - \varphi_1'(1) \frac{\xi I_{\nu+1}(\xi)}{I_\nu(\xi)}. \end{aligned} \tag{37}$$

To better understand  $\sigma_{\text{Left}}(T)$  we shall need the values  $\varphi'(1)$  and  $\varphi''(1)$ . From (14) and the definition of  $\phi_1$  we have that  $\phi_1$  is the continuous solution on  $[0, 1]$  of

$$\left( \partial_r^2 + \frac{n-1}{r} \partial_r + \lambda_\nu \right) \phi_1 = 0$$

such that  $\phi_1(1) = 0$ , with normalization

$$\int_0^1 \phi_1^2(r) \, dr = \frac{1}{2\pi \text{Vol}(S^{n-1})}.$$

We rescale the function  $\phi_1$  and define

$$\tilde{\varphi}_1(s) = \phi_1\left(\frac{s}{j_\nu}\right).$$

Hence,  $\tilde{\varphi}_1$  is the continuous solution on  $[0, j_\nu]$  of

$$\left( \partial_s^2 + \frac{n-1}{s} \partial_s + 1 \right) \tilde{\varphi}_1 = 0 \tag{38}$$

with  $\tilde{\varphi}_1(j_\nu) = 0$  and normalization

$$\int_0^{j_\nu} \tilde{\varphi}_1^2(s) \, ds = \frac{j_\nu}{2\pi \text{Vol}(S^{n-1})}.$$

Equation (38) is very similar to a Bessel equation. In order to obtain exactly a Bessel equation, we define the function  $\hat{\varphi}_1$  by

$$\tilde{\varphi}_1(s) = s^{-\nu} \hat{\varphi}_1(s).$$

Since  $-\nu \leq 0$  because  $n \geq 2$ , we get that  $\hat{\varphi}_1$  is the continuous solution on  $[0, j_\nu]$  of

$$\left[ \partial_s^2 + \frac{1}{s} \partial_s + \left( 1 - \frac{\nu^2}{s^2} \right) \right] \hat{\varphi}_1 = 0$$

with  $\hat{\varphi}_1(j_\nu) = 0$  and normalization

$$\int_0^{j_\nu} s^{2-n} \hat{\varphi}_1^2(s) \, ds = \frac{j_\nu}{2\pi \text{Vol}(S^{n-1})}.$$

The solution of this ordinary differential equation is  $\kappa_n J_\nu(s)$ , where the constant  $\kappa_n$  is chosen such that

$$\int_0^{j_\nu} \kappa_n^2 s^{2-n} J_\nu^2(s) \, ds = \frac{j_\nu}{2\pi \text{Vol}(S^{n-1})}.$$

Returning to the function  $\varphi_1$ , we get

$$\varphi_1(r) = \kappa_n j_\nu^{-\nu} r^{-\nu} J_\nu(j_\nu r).$$

It follows that

$$\varphi_1'(r) = \kappa_n j_\nu^{-\nu} \left( (-\nu) r^{-\nu-1} J_\nu(j_\nu r) + r^{-\nu} j_\nu J_\nu'(j_\nu r) \right).$$

Since  $J_\nu(j_\nu) = 0$  we obtain

$$\varphi_1'(1) = \kappa_n j_\nu^{-\nu+1} J_\nu'(j_\nu). \tag{39}$$

Furthermore,

$$\varphi_1''(r) = \kappa_n j_\nu^{-\nu} \left( (-\nu)(-\nu-1) r^{-\nu-2} J_\nu(j_\nu r) + 2(-\nu) r^{-\nu-1} j_\nu J_\nu'(j_\nu r) + r^{-\nu} j_\nu^2 J_\nu''(j_\nu r) \right)$$

and hence

$$\varphi_1''(1) = \kappa_n j_\nu^{-\nu+1} \left( -2\nu J_\nu'(j_\nu) + j_\nu J_\nu''(j_\nu) \right).$$

To rewrite this further note that, by (23),

$$2J_\nu''(s) = J'_{\nu-1}(s) - J'_{\nu+1}(s).$$

Together with (25) and (24) we find

$$\begin{aligned} 2s J_\nu''(s) &= s J'_{\nu-1}(s) - s J'_{\nu+1}(s) \\ &= \left( (\nu-1) J_{\nu-1}(s) - s J_\nu(s) \right) - \left( -(\nu+1) J_{\nu+1}(s) + s J_\nu(s) \right). \end{aligned}$$

At  $s = j_\nu$  we obtain, together with (22) and (23),

$$2j_\nu J_\nu''(j_\nu) = J_{\nu+1}(j_\nu) - J_{\nu-1}(j_\nu) = -2J_\nu'(j_\nu).$$

Altogether,

$$\varphi_1''(1) = -\kappa_n j_\nu^{-\nu+1} (2\nu+1) J_\nu'(j_\nu). \tag{40}$$

In view of (37), (39) and (40) the function  $\sigma_{\text{Left}}(T)$  is equal to

$$\sigma_{\text{Left}}(T) = -\kappa_n j_\nu^{-\nu+1} J'_\nu(j_\nu) \left( (2\nu + 1) + \frac{\xi I_{\nu+1}(\xi)}{I_\nu(\xi)} \right). \tag{41}$$

Using also (30) and (31) we can rewrite this as

$$\sigma_{\text{Left}}(T) = -\kappa_n j_\nu^{-\nu+1} J'_\nu(j_\nu) \left( 1 + \frac{\xi I_{\nu-1}(\xi)}{I_\nu(\xi)} \right). \tag{42}$$

Since  $\kappa_\nu, j_\nu$  are positive,  $J'_\nu(j_\nu)$  is negative, and the functions  $I_\nu$  are positive at all  $\xi > 0$ , formula (42) implies

**Lemma 5.1.** *In the interval of definition  $(0, \mu)$  of the function  $\sigma_{\text{Left}}$ , we have*

$$\sigma_{\text{Left}}(T) > 0.$$

Moreover, by (26) we have

$$\lim_{\xi \rightarrow 0} \frac{\xi I_{\nu+1}(\xi)}{I_\nu(\xi)} = 2 \frac{\Gamma(\nu + 2)}{\Gamma(\nu + 1)}.$$

Since  $\xi \rightarrow 0$  as  $T \nearrow \mu$  by (36), we find together with (41) that for all  $\nu \geq 0$ ,

$$\sigma_1(\mu) = \lim_{T \nearrow \mu} \sigma_{\text{Left}}(T) = -\kappa_n j_\nu^{-\nu+1} J'_\nu(j_\nu) \left( 2\nu + 1 + 2 \frac{\Gamma(\nu + 2)}{\Gamma(\nu + 1)} \right).$$

In particular,

**Lemma 5.2.**  $\sigma_1(\mu) > 0$ .

5.2. A formula for  $\sigma_{\text{Right}}$

We follow the reasoning that we used to find a formula for the function  $\sigma_{\text{Left}}(T)$ . We skip the technical details. Assume that  $T > \mu$ . This allows us to define

$$\rho = \sqrt{\lambda_\nu - \left( \frac{2\pi}{T} \right)^2}. \tag{43}$$

The function  $\hat{c}(s) := s^\nu c(\frac{s}{\rho})$  is the continuous solution on  $[0, \rho]$  of

$$\left[ \partial_s^2 + \frac{1}{s} \partial_s - \left( 1 + \frac{\nu^2}{s^2} \right) \right] \hat{c} = 0$$

with  $\hat{c}(\rho) = -\rho^\nu \varphi'_1(1)$ . The solution of this ordinary differential equation is given by  $\beta J_\nu(s)$ , where the constant  $\beta$  (depending on  $\nu$  and  $T$ ) is chosen such that

$$\beta J_\nu(\rho) = -\rho^\nu \varphi'_1(1).$$

Returning to the function  $c$ , we get

$$c(r) = -\frac{\varphi_1'(1)}{J_\nu(\rho)} r^{-\nu} J_\nu(\rho r)$$

and from (18) and (35), using the identities (23), (24) and (25), we obtain

$$\begin{aligned} \sigma_{\text{Right}}(T) &= -\varphi_1'(1) \frac{1}{J_\nu(\rho)} \frac{1}{2} (\rho J_{\nu-1}(\rho) - 2\nu J_\nu(\rho) - \rho J_{\nu+1}(\rho)) + \varphi_1''(1) \\ &= \varphi_1''(1) + \varphi_1'(1) \frac{\rho J_{\nu+1}(\rho)}{J_\nu(\rho)}. \end{aligned} \tag{44}$$

In view of (39) and (40) this becomes

$$\begin{aligned} \sigma_{\text{Right}}(T) &= -\kappa_n j_\nu^{-\nu+1} J_\nu'(j_\nu) \left( (2\nu + 1) - \frac{\rho J_{\nu+1}(\rho)}{J_\nu(\rho)} \right) \\ &= -\kappa_n j_\nu^{-\nu+1} J_\nu'(j_\nu) \left( 1 + \frac{\rho J_{\nu-1}(\rho)}{J_\nu(\rho)} \right), \end{aligned} \tag{45}$$

where we used the identities (24) and (25) to get the second equality.

### 6. Study of the derivative of $\sigma_1(T)$

Throughout this section we assume again that  $n \geq 2$ . We start with

**Lemma 6.1.** *The function  $\sigma_1 : (0, \infty) \rightarrow \mathbb{R}$  has the asymptotics*

$$\lim_{T \rightarrow 0} \sigma_1(T) = +\infty \quad \text{and} \quad \lim_{T \rightarrow \infty} \sigma_1(T) = -\infty.$$

**Proof.** The first asymptotics is already proven in [24]. We give an easier proof: By (36) we have  $\xi \rightarrow \infty$  as  $T \rightarrow 0$ . Using (42) and (32) we therefore find

$$\lim_{T \rightarrow 0} \sigma_1(T) = \lim_{\xi \rightarrow \infty} \frac{\xi I_{\nu+1}(\xi)}{I_\nu(\xi)} = \lim_{\xi \rightarrow \infty} \xi = \infty.$$

To prove the second asymptotics, we read off from (43) that  $\rho \nearrow \sqrt{\lambda_\nu} = j_\nu$  as  $T \rightarrow \infty$ . As is well known,  $j_\nu < j_{\nu+1}$  (see e.g. [26, §15.22]). Therefore  $J_{\nu+1}(j_\nu) > 0$ . Together with (45) we thus find

$$\lim_{T \rightarrow \infty} \sigma_1(T) = -\lim_{\rho \nearrow j_\nu} \frac{\rho J_{\nu+1}(\rho)}{J_\nu(\rho)} = -\infty,$$

as claimed.  $\square$

It is shown in [24, p. 336] that the function  $\sigma_1$  is analytic and hence differentiable. For our purposes, it would be enough to know that  $\sigma_1$  has exactly one zero  $T_\nu$  and that  $\sigma_1'(T_\nu) \neq 0$ . This follows from Lemma 5.1, Lemma 5.2, Lemma 6.1 and Lemma 6.5 below, that states that

$\sigma'_1(T) < 0$  for all  $T \in (\mu, \infty)$ . We shall prove a somewhat stronger statement, namely that  $\sigma'_1(T) < 0$  for all  $T \in (0, \infty)$ .

**Proposition 6.2.** *Let  $n \geq 2$ . The function  $\sigma_1 : (0, \infty) \rightarrow \mathbb{R}$  has negative derivative. Moreover,  $\sigma_1$  has exactly one zero, say  $T_\nu$ .*

**Proof.** We show that  $\sigma_{\text{Left}}$  has negative derivative (Lemma 6.3), that  $\sigma_{\text{Right}}$  has negative derivative (Lemma 6.5), and that  $\sigma'_1(\mu) < 0$  (Lemma 6.7). The fact that  $\sigma_1$  has exactly one zero then follows together with Lemma 6.1.

**Lemma 6.3.**  *$\sigma'_{\text{Left}}(T) < 0$  for all  $T \in (0, \mu)$ .*

**Proof.** Recall from (39) that

$$-\varphi'_1(1) > 0.$$

Set  $f(s) = \frac{sI_{\nu+1}(s)}{I_\nu(s)}$ . In view of (37) we need to show that  $\frac{d}{dT} f(\xi(T)) < 0$  for all  $T \in (0, \mu)$ . Since  $\frac{d}{dT} f(\xi(T)) = f'(\xi(T))\xi'(T)$  and  $\xi'(T) < 0$  for all  $T \in (0, \mu)$ , this is equivalent to

$$f'(s) > 0 \quad \text{for all } s \in (0, \infty). \tag{46}$$

By (24) we have  $sI'_{\nu+1} = -(\nu + 1)I_{\nu+1} + sI_\nu$  and  $sI'_\nu = -\nu I_\nu + sI_{\nu-1}$ . Therefore,

$$f'(s) = \frac{s(I_\nu^2 - I_{\nu-1}I_{\nu+1})}{I_\nu^2}.$$

The lemma now follows from the following claim.

**Claim 6.4.**  *$I_\nu^2(s) > I_{\nu-1}(s)I_{\nu+1}(s)$  for all  $\nu \in \mathbb{R}$  and all  $s > 0$ .*

**Proof.** In view of (27) we have  $I_\nu^2(0) \geq I_{\nu-1}(0)I_{\nu+1}(0)$  for all  $\nu \geq 0$ . It therefore suffices to show that for all  $s > 0$ ,

$$\frac{d}{ds} I_\nu^2 > \frac{d}{ds} (I_{\nu-1}I_{\nu+1}).$$

Multiplying by  $s$ , we see that this is true if and only if

$$2I_\nu s I'_\nu > sI'_{\nu-1} I_{\nu+1} + I_{\nu-1} s I'_{\nu+1}. \tag{47}$$

In view of (29), (31), (30) we have

$$\begin{aligned} 2sI'_\nu &= sI_{\nu-1} + sI_{\nu+1}, \\ sI'_{\nu-1} &= (\nu - 1)I_{\nu-1} + sI_\nu, \\ sI'_{\nu+1} &= -(\nu + 1)I_{\nu+1} + sI_\nu. \end{aligned}$$

Therefore, (47) holds if and only if

$$sI_{v-1}I_v + sI_vI_{v+1} > (v - 1)I_{v-1}I_{v+1} + sI_vI_{v+1} - (n + 1)I_{v-1}I_{v+1} + sI_{v-1}I_v$$

i.e.,

$$0 > -2I_{v-1}I_{v+1}$$

which is true because  $I_v(s) > 0$  for all  $v \in \mathbb{R}$  and  $s > 0$ .  $\square$

**Lemma 6.5.**  $\sigma'_{\text{Right}}(T) < 0$  for all  $T \in (\mu, \infty)$ .

**Proof.** Recall that  $-\phi'_1(1) > 0$ . Note that the function

$$\rho : (\mu, \infty) \rightarrow (0, j_v), \quad \rho(T) = \sqrt{\lambda_v - \left(\frac{2\pi}{T}\right)^2}$$

is strictly increasing. Set  $h(s) = \frac{sJ_{v+1}(s)}{J_v(s)}$ . In view of (44) we need to show that

$$h'(s) > 0 \quad \text{for all } s \in (0, j_v). \tag{48}$$

Since  $j_v$  is the first positive zero of  $J_v$ , we see as in the proof of Lemma 6.3 that (48) is equivalent to

**Claim 6.6.**  $J_v^2(s) > J_{v-1}(s)J_{v+1}(s)$  for all  $s \in (0, j_v)$ .

**Proof.** Let again  $j_{v-1}, j_v, j_{v+1}$  be the first positive zero of  $J_{v-1}, J_v, J_{v+1}$ , respectively. Moreover, denote by  $j_{v-1}^{(2)}$  the second positive zero of  $J_{v-1}$ . Then

$$j_{v-1} < j_v < j_{v+1}, \quad j_v < j_{v-1}^{(2)}, \tag{49}$$

see e.g. [26, §15.22]. It follows from the power series expansion (20) that

$$J_v(s) > 0 \quad \text{for } s \in (0, j_v). \tag{50}$$

Assume first that  $s \in [j_{v-1}, j_v)$ . Then (49) and (50) show that  $J_v(s) > 0, J_{v-1}(s) \leq 0, J_{v+1}(s) > 0$ , whence the claim follows. Assume now that  $s \in (0, j_{v-1})$ . In view of (21) we have  $J_v^2(0) \geq J_{v-1}(0)J_{v+1}(0)$  for all  $v \geq 0$ . It therefore suffices to show that

$$\frac{d}{ds} J_v^2 > \frac{d}{ds} (J_{v-1}J_{v+1}) \quad \text{on } (0, j_{v-1}). \tag{51}$$

Using (23), (25) and (24) we see as in the proof of Claim 6.4 that (51) is equivalent to

$$0 > -2J_{v-1}(s)J_{v+1}(s)$$

which is true because  $J_{v-1}$  and  $J_{v+1}$  are positive on  $(0, j_{v-1})$ .  $\square$

To complete the proof of Proposition 6.2 we also show

**Lemma 6.7.**  $\sigma'_1(\mu) < 0$ .

**Proof.** Since the function  $\sigma_1$  is smooth,

$$\sigma'_1(\mu) = \lim_{T \searrow \mu} \sigma'_{\text{Right}}(T).$$

For  $T > \mu$  we have  $\sigma'_1(T) = h'(\rho(T))\rho'(T)$ . We compute

$$\rho'(T) = \frac{2\pi}{\rho(T)T^3}$$

and

$$h'(s) = \frac{s(J_v^2 - J_{v-1}J_{v+1})}{J_v^2}. \tag{52}$$

Since  $\lim_{T \searrow \mu} \rho(T) = 0$  we obtain

$$\sigma'_1(\mu) = \lim_{T \searrow \mu} \sigma'_{\text{Right}}(T) = \frac{2\pi}{\mu^3} \phi'_1(1) \left( 1 - \lim_{s \rightarrow 0} \frac{J_{v-1}J_{v+1}}{J_v^2} \right).$$

In view of the power series expansion (20),

$$J_v(s) = \frac{(\frac{1}{2}s)^v}{\Gamma(v+1)} + O(s^{s+\nu}).$$

Therefore,

$$\lim_{s \rightarrow 0} \frac{J_{v-1}J_{v+1}}{J_v^2} = \frac{\Gamma(v+1)}{\Gamma(v)\Gamma(v+2)} < 1 \quad \text{for all } v \geq 0$$

and thus  $\sigma'_1(\mu) < 0$ .  $\square$

**7. Extremal domains via the Crandall–Rabinowitz theorem**

We are now in position to prove our main result when  $n \geq 2$ : The hypotheses of the Crandall–Rabinowitz bifurcation theorem are satisfied by the operator  $F$  defined in Section 3. For  $n \geq 2$ , Theorem 1.1 follows at once from the following proposition and the Crandall–Rabinowitz theorem. As before,  $\nu = \frac{n-2}{2}$ .

**Proposition 7.1.** *For  $n \geq 2$ , there exists a real number  $T_*(n) = T_\nu$  such that the kernel of the linearized operator  $D_\nu F(0, T_\nu)$  is 1-dimensional and is spanned by the function  $\cos t$ ,*

$$\text{Ker } D_\nu F(0, T_\nu) = \mathbb{R} \cos t.$$

The cokernel of  $D_v F(0, T_v)$  is also 1-dimensional, and

$$D_T D_v F(0, T_v)(\cos t) \notin \text{Im } D_v F(0, T_v).$$

**Proof.** Let  $v \in \mathcal{C}_{\text{even},0}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ ,

$$v = \sum_{k \geq 1} a_k \cos(kt).$$

We know that

$$D_v F(0, T) = \sum_{k \geq 1} \sigma_k(T) a_k \cos(kt). \tag{53}$$

Let  $V_k$  be the space spanned by the function  $\cos(kt)$ . By Proposition 6.2, the function  $\sigma_1(T)$  has exactly one zero  $T_v$ . By (53), the line  $V_1$  belongs to the kernel of  $D_v F(0, T_v)$ . Moreover,  $V_1$  is the whole kernel, because for  $k \geq 2$  we have

$$\sigma_k(T_v) = \sigma_1\left(\frac{T_v}{k}\right) \neq 0$$

(because  $T_v$  is the only zero of  $\sigma_1$ ). Recall from Proposition 3.1 that  $D_v F(0, T_v)$  is a first order elliptic operator. Elliptic estimates yield that there is a constant  $c > 0$  such that

$$\|D_v F(0, T_v)(w)\|_{\mathcal{C}_{\text{even},0}^{1,\alpha}(\mathbb{R}/2\pi\mathbb{Z})} \geq c \|w\|_{\mathcal{C}_{\text{even},0}^{2,\alpha}(\mathbb{R}/2\pi\mathbb{Z})}$$

for all  $w$  that are  $L^2(\mathbb{R}/2\pi\mathbb{Z})$ -orthogonal to  $V_1$ . It follows that  $D_v F(0, T_v)$  has closed range. Therefore, and in view of (53), the image of  $D_v F(0, T_v)$  is the closure of

$$\bigoplus_{k \geq 2} V_k$$

in  $\mathcal{C}_{\text{even},0}^{1,\alpha}(\mathbb{R}/2\pi\mathbb{Z})$ , and its codimension is equal to 1. More precisely,

$$\mathcal{C}_{\text{even},0}^{1,\alpha}(\mathbb{R}/2\pi\mathbb{Z}) = \text{Im } D_v F(0, T_v) \oplus V_1.$$

Again by (53),

$$D_T D_v F(0, T)(v) = \sum_{k \geq 1} \sigma'_k(T) a_k \cos(kt)$$

and in particular

$$D_T D_v F(0, T_v)(\cos t) = \sigma'_1(T_v) \cos t \notin \text{Im } D_v F(0, T_v)$$

because  $\sigma'_1(T_v) < 0$  by Proposition 6.2. This completes the proof of the proposition.  $\square$

### 8. The problem in $\mathbb{R}^2$

Assume that  $n = 1$ , i.e., the ambient space of the cylinder  $C_1^T$  is  $\mathbb{R}^2$ . Recall from Section 3 that in this case,

$$\sigma_1(T) = c'(1) + \varphi_1''(1)$$

where  $c$  is the solution of

$$\left(\partial_r^2 + \lambda_1 - \left(\frac{2\pi}{T}\right)^2\right)c = 0, \tag{54}$$

with  $c(1) = -\varphi_1'(1)$  and  $c'(0) = 0$ , where  $\varphi_1$  is the first eigenfunction of the Dirichlet problem on  $[-1, 1]$  normalized to have  $L^2$ -norm  $\frac{1}{2\pi}$ . (Here and in the sequel,  $c$  denotes again the function  $c_1$ .) For  $\varphi_1$  and  $\lambda_1$  we thus have

$$\lambda_1 = \frac{\pi^2}{4} \quad \text{and} \quad \varphi_1(r) = \frac{1}{\sqrt{2\pi}} \cos\left(\frac{\pi}{2}r\right).$$

Hence

$$-\varphi_1'(1) = \sqrt{\frac{\pi}{8}} \quad \text{and} \quad \varphi_1''(1) = 0.$$

**Lemma 8.1.** *The only zero of the function  $\sigma_1(T)$  is at  $T = 4$ . Moreover  $\sigma_1'(4) < 0$ .*

**Proof.** We abbreviate  $\alpha(T) := \lambda_1 - \left(\frac{2\pi}{T}\right)^2 = \left(\frac{\pi}{2}\right)^2 - \left(\frac{2\pi}{T}\right)^2$ . The solution to (54) is

$$c(r) = \begin{cases} \sqrt{\frac{\pi}{8}} \frac{\cosh \sqrt{-\alpha(T)}r}{\cosh \sqrt{-\alpha(T)}} & \text{if } T \in (0, 4), \\ \sqrt{\frac{\pi}{8}} & \text{if } T = 4, \\ \sqrt{\frac{\pi}{8}} \frac{\cos \sqrt{\alpha(T)}r}{\cos \sqrt{\alpha(T)}} & \text{if } T \in (4, \infty). \end{cases}$$

Hence,

$$\sigma_1(T) = c'(1) = \begin{cases} -\sqrt{\frac{\pi}{8}} \sqrt{-\alpha(T)} \tanh \sqrt{-\alpha(T)} & \text{if } T \in (0, 4), \\ 0 & \text{if } T = 4, \\ -\sqrt{\frac{\pi}{8}} \sqrt{\alpha(T)} \tan \sqrt{\alpha(T)} & \text{if } T \in (4, \infty). \end{cases}$$

In particular,  $\sigma_1(T) > 0$  on  $(0, 4)$  and  $\sigma_1(T) < 0$  on  $(4, \infty)$ . It remains to show that  $\sigma_1'(4) < 0$ . For  $T > 4$  define  $h(T) := \sqrt{\alpha(T)}$ . Then

$$\sigma_1'(T) = -\sqrt{\frac{\pi}{8}} \frac{d}{dT} (h(T) \tan h(T)) = -\sqrt{\frac{\pi}{8}} h'(T) (\tan h(T) + h(T) (1 + \tan^2(h(T)))).$$

Since  $\sigma_1(T)$  is smooth on  $(0, \infty)$  and since  $\lim_{T \rightarrow 4^+} h(T) = 0$  and  $h'(T) = \frac{\alpha'(T)}{2h(T)}$ , we find

$$\begin{aligned} \sigma_1'(4) &= -\sqrt{\frac{\pi}{8}} \lim_{T \rightarrow 4^+} h'(T)(\tan h(T) + h(T)) \\ &= -\sqrt{\frac{\pi}{8}} \lim_{T \rightarrow 4^+} h'(T)2h(T) \\ &= -\sqrt{\frac{\pi}{8}} \lim_{T \rightarrow 4^+} \alpha'(T) = -\sqrt{\frac{\pi}{8}} \frac{\pi^2}{8} < 0. \quad \square \end{aligned}$$

**Remark 8.2.** A computation shows that  $\sigma_1'(T) < 0$  for all  $T \in (0, \infty)$ .  $\square$

Using the previous lemma, the proof of Proposition 7.1 applies also for  $n = 1$ , and we obtain

**Proposition 8.3.** *Proposition 7.1 is true also for  $n = 1$  and  $T_*(1) = 4$ .*

Together with the Crandall–Rabinowitz theorem we now obtain our main Theorem 1.1 also for  $n = 1$ . Fig. 2 shows the shape of the new extremal domains in  $\mathbb{R}^2$ .

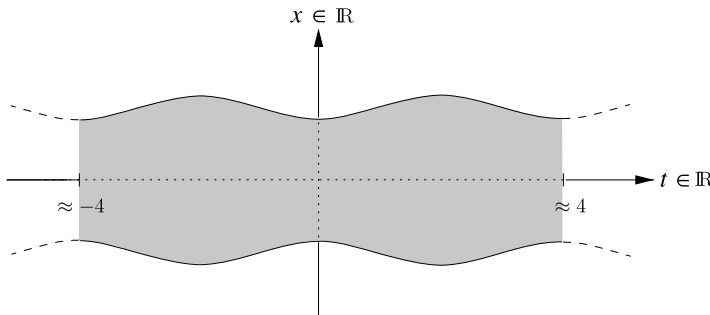


Fig. 2. A domain  $\Omega_S \subset \mathbb{R}^2$ .

### 9. Estimates on the bifurcation period

Recall from Section 8 that  $T_*(1) = 4$ . In this section we study the bifurcation values  $T_\nu = T_*(n)$  for  $n \geq 2$ , and in particular prove Theorem 1.3.

We recall that  $J'_\nu(j_\nu) \neq 0$ , and from (48) that the function  $h(s) = \frac{s J_{\nu+1}(s)}{J_\nu(s)}$  is strictly increasing on  $(0, j_\nu)$  from 0 to  $\infty$ . By (45) the unique zero  $T_\nu$  of  $\sigma_{\text{Right}}$  is therefore determined by

$$\rho_\nu := \rho(T_\nu) = \sqrt{\lambda_\nu - \left(\frac{2\pi}{T_\nu}\right)^2} \tag{55}$$

and

$$\frac{\rho_\nu J_{\nu+1}(\rho_\nu)}{J_\nu(\rho_\nu)} = 2\nu + 1.$$

In other words, the bifurcation value is

$$T_\nu = \frac{2\pi}{\sqrt{\lambda_\nu - \rho_\nu^2}} \tag{56}$$

where  $\rho_\nu$  is the unique zero on  $(0, j_\nu)$  of  $sJ_{\nu+1} - (2\nu + 1)J_\nu$  or, by (22), of  $sJ_{\nu-1} + J_\nu$ .

For fixed  $\nu$ , the value  $\rho_\nu$  and hence  $T_\nu$  can be computed by the computer (using, for instance, Mathematica). The first few and some larger values of  $T_\nu$  (rounded to five decimal places) are

$2\nu$	0	1	2	3	4	5	6	7
$T_\nu$	3.06362	2.61931	2.34104	2.14351	1.99308	1.87315	1.77429	1.69088
$2\nu$	8	9	10	11	12	13	14	15
$T_\nu$	1.61924	1.55650	1.50123	1.45180	1.40735	1.36697	1.33003	1.2963
$2\nu$	16	17	18	19	20	40	200	2000
$T_\nu$	1.2650	1.23616	1.20927	1.18411	1.16058	0.87348	0.4229	0.13888

(57)

To study  $T_\nu$  for  $\nu \geq 10$  define

$$\rho_\nu^- = j_{\nu-1} + \frac{1}{j_{\nu-1} + 2}, \quad \rho_\nu^+ = j_{\nu-1} + \frac{1}{j_{\nu-1}}.$$

**Proposition 9.1.** *The sequence  $T_\nu$  is strictly decreasing to 0. For  $\nu \geq 10$  we have*

$$\frac{2\pi}{\sqrt{\lambda_\nu - (\rho_\nu^-)^2}} < T_\nu < \frac{2\pi}{\sqrt{\lambda_\nu - (\rho_\nu^+)^2}}. \tag{58}$$

**Remark 9.2.** The zeros  $j_\nu$  (and hence the eigenvalues  $\lambda_\nu = j_\nu^2$ ) are rather well known [18], namely

$$\nu - \frac{a_1}{\sqrt[3]{2}}\nu^{1/3} + \frac{3}{20}a_1^2\sqrt[3]{2}\nu^{-1/3} - 0.061\nu^{-1} < j_\nu < \nu - \frac{a_1}{\sqrt[3]{2}}\nu^{1/3} + \frac{3}{20}a_1^2\sqrt[3]{2}\nu^{-1/3}$$

for all  $\nu \in \frac{1}{2}\mathbb{N}$  with  $\nu \geq 10$ . Here,  $a_1 \approx -2.33811$  is the first negative zero of the Airy function  $\text{Ai}(x)$ . Therefore,

$$\nu + a\nu^{1/3} + b\nu^{-1/3} - c\nu^{-1} < j_\nu < \nu + a\nu^{1/3} + b\nu^{-1/3} \tag{59}$$

with positive constants  $a \approx 1.8557$ ,  $b \approx 1.0331$ ,  $c < \frac{1}{16}$ . For  $\lambda_\nu$  we obtain the estimate

$$\begin{aligned} \nu^2 + 2a\nu^{4/3} + (2b + a^2)\nu^{2/3} + 2ab + b^2\nu^{-2/3} - C(\nu) &< \lambda_\nu < \\ \nu^2 + 2a\nu^{4/3} + (2b + a^2)\nu^{2/3} + 2ab + b^2\nu^{-2/3} \end{aligned} \tag{60}$$

where  $C(\nu) = c(2 + 2a\nu^{-2/3} + 2b\nu^{-4/3} + c\nu^{-2})$  is strictly decreasing, and  $C(9) < 1/5$ .  $\square$

We start with proving the estimate (58), which by (56) is equivalent to

$$\rho_v^- < \rho_v < \rho_v^+. \tag{61}$$

Recall that

$$h(s) = \frac{sJ_{v+1}}{J_v} = 2v - \frac{sJ_{v-1}}{J_v}. \tag{62}$$

Since  $J_{v-1}(j_{v-1}) = 0$  we have  $h(j_{v-1}) = 2v$ . This and  $h'(s) > 0$  on  $(0, j_v)$  show that

$$j_{v-1} < \rho_v < j_v.$$

In order to improve these bounds on  $\rho_v$  we need to better understand  $h$  on the interval  $I_v := [j_{v-1}, j_v]$ . The identities (52) and (62) show that

$$h'(s) = s + \frac{h}{s}(h - 2v). \tag{63}$$

In particular,

$$h'(j_{v-1}) = j_{v-1}, \quad h'(\rho_v) = \rho_v + \frac{2v + 1}{\rho_v}. \tag{64}$$

Moreover, using (63)

$$\begin{aligned} h''(s) &= 1 + \frac{h}{s}h' + \frac{h's - h}{s^2}(h - 2v) \\ &= 1 + h + (h - 2v) \left( \frac{h^2 - h}{s^2} + 1 + \frac{h}{s^2}(h - 2v) \right). \end{aligned}$$

It follows that  $h'' > 0$  on  $I_v$ . Therefore, the straight line of slope  $h'(j_{v-1})$  passing through  $(j_{v-1}, 2v)$  reaches the height  $2v + 1$  on the left of the graph of  $h$ , while the straight line of slope  $h'(\rho_v)$  passing through  $(j_{v-1}, 2v)$  reaches the height  $2v + 1$  on the right of the graph of  $h$ , see Fig. 3.

Together with (64) we conclude that

$$j_{v-1} + \frac{1}{\rho_v + \frac{2v+1}{\rho_v}} < \rho_v < j_{v-1} + \frac{1}{j_{v-1}} =: \rho_v^+.$$

Using now that  $j_{v-1} < \rho_v < j_{v-1} + \frac{1}{j_{v-1}}$  and that  $j_{v-1} > v + \frac{1}{2}$  for all  $v \geq 10$  by (59), we find

$$\rho_v + \frac{2v + 1}{\rho_v} < j_{v-1} + \frac{2v + 2}{j_{v-1}} < j_{v-1} + 2,$$

and hence (61) follows.

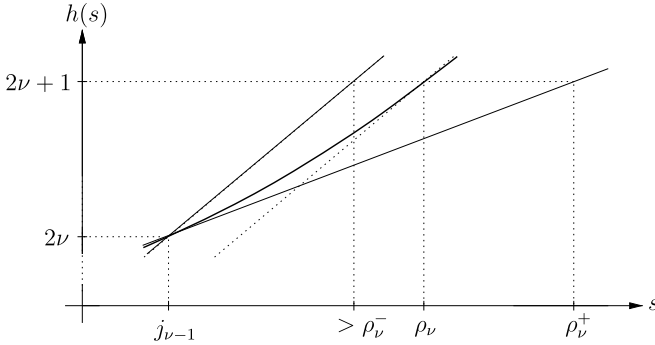


Fig. 3. The two straight lines and the graph of  $h$ .

It has been shown in [24] that  $T_\nu < \frac{\sqrt{2\pi}}{\sqrt{\nu}}$ , whence  $T_\nu$  converges to 0. This also follows from (58) and

$$\lambda_\nu - (\rho_\nu^+)^2 = j_\nu^2 - j_{\nu-1}^2 - 2 - \frac{1}{j_{\nu-1}^2} = 2\nu + O(\nu^{1/3}) \tag{65}$$

where for the last identity we used (60). Note that (65) and  $\lambda_\nu - (\rho_\nu^-)^2 = 2\nu + O(\nu^{1/3})$  imply that

$$\frac{2\pi}{T_\nu} = \sqrt{2}\nu^{1/2} + O(\nu^{-1/6}) \quad \text{or} \quad T_\nu = \sqrt{2\pi}\nu^{-1/2} + O(\nu^{-7/6}).$$

We finally show that the sequence  $T_\nu$  is strictly decreasing. In view of the table (57) we can assume that  $\nu \geq 10$ . By (58) we need to show that for each such  $\nu$ ,

$$\lambda_\nu - (\rho_\nu^-)^2 < \lambda_{\nu+\frac{1}{2}} - (\rho_{\nu+\frac{1}{2}}^+)^2,$$

i.e.,

$$\lambda_{\nu+\frac{1}{2}} - \lambda_\nu > \lambda_{\nu-\frac{1}{2}} - \lambda_{\nu-1} + \left(2 - 2\frac{j_{\nu-1}}{j_{\nu-1} + 2}\right) + \left(\frac{1}{j_{\nu-\frac{1}{2}}^2} - \frac{1}{(j_{\nu-1} + 2)^2}\right). \tag{66}$$

The first bracket on the RHS is equal to

$$\frac{4}{j_{\nu-1} + 2} \stackrel{(59)}{<} \frac{4}{\nu + 2} \leq \frac{1}{3},$$

and the second bracket is less than  $\frac{1}{100}$ . It therefore suffices to show that

$$\lambda_{\nu+\frac{1}{2}} - \lambda_\nu > \lambda_{\nu-\frac{1}{2}} - \lambda_{\nu-1} + \frac{1}{3} + \frac{1}{100}. \tag{67}$$

The function  $v \mapsto v^\alpha$  is convex for  $\alpha = \frac{4}{3}$  and  $\alpha = -\frac{2}{3}$ , but concave for  $\alpha = \frac{2}{3}$ . At  $v = 10$  we have

$$(a^2 + 2b) \left( \left( v + \frac{1}{2} \right)^{2/3} - v^{2/3} \right) > (a^2 + 2b) \left( \left( v - \frac{1}{2} \right)^{2/3} - (v-1)^{2/3} \right) - \frac{1}{30}.$$

Furthermore,

$$\left( v + \frac{1}{2} \right)^2 - v^2 = \left( v - \frac{1}{2} \right)^2 - (v-1)^2 + 1,$$

and  $C(v-1) \leq C(9) < \frac{1}{5}$  for  $v \geq 10$ . Since  $\frac{1}{3} + \frac{1}{100} + \frac{1}{30} + 2\frac{1}{5} < 1$ , the estimate (60) now implies that (67) holds true.  $\square$

**Remark 9.3.** It is known that the function  $v \mapsto \lambda_v$  is strictly convex on  $(0, \infty)$ , see [7]. In particular,

$$\lambda_{v+\frac{1}{2}} - \lambda_v > \lambda_{v-\frac{1}{2}} - \lambda_{v-1}.$$

This is not quite enough to prove inequality (66).  $\square$

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