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IMPROVED COMPUTATION OF NON-LINEAR ADVECTION IN POROUS MEDIA USING SLIGHTLY MODIFIED BASIC FINITE ELEMENT ALGORITHMS

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SUMMARY

The numerical stability of standard finite element schemes applied to the advection–diffusion equation is evaluated using a space-time eigenvalue analysis. Unlike the usual approaches which only consider temporal aspects of stability, this analysis also describes the spatial stability of the solutions.To this end, the one-dimensional advection–diffusion equation is put into an alternative semi-discrete form which allows the derivation of a very practical stability condition. In multidimensional flow situations the latter is applied along the streamlines by means of a tensorial corrective function that prevents excessive numerical smearing of fronts or phase interfaces. The efficiency of the procedure is illustrated by an example which successfully simulates the coupling of two low miscible fluid phases in a variably saturated porous medium.

INTRODUCTION

In practical groundwater applications of contaminant transport modelling, advective regimes are commonly encountered because of the importance of studied volumes and time periods considered. In such cases, finite element spatial discretizations combined with finite difference time-stepping procedures involve high mesh Peclet (P) and Courant (C) numbers. The difficulties (unstable and oscillatory results, diverging algorithms) encountered by standard approaches when analysing this type of problems are well known (e.g. References 1 and 2) and various higher order improved techniques have been analysed in terms of accuracy and solution stability (e.g. References 3–5).

When standard first order central schemes are applied to the advection–diffusion equation, the solution domain shows a limit beyond which the asymmetric advection operator becomes dominant and destabilizes the matrix equation system. This limit is generally defined by means of the numbers P and C. When both conditions $P \leqslant 2$ and $C \leqslant 1$ are fulfilled, a central scheme is stable and provides non-oscillating numerical solutions. The above two conditions have been established according to theoretical considerations[6,7] and it is frequently recommended in the specialized literature that they should be respected.[8]

For advection dominated transport problems (i.e. $P \rightarrow \infty$) it was recently shown[9] that more sophisticated, highly accurate non-diffusive methods[10,11] still have severe stability difficulties when the transport of sharp fronts has to be simulated. In effect, to perform properly these

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Petrov– or Taylor–Galerkin methods require a careful determination of the numbers P and C (the condition $C < 1$ being imperative) as well as relatively smooth initial conditions (smooth fronts, gaussian hills). However, in most practical groundwater flow problems finite elements are coarse and irregular — making impossible a rigorous check of P and C — and initial solute fronts are often sharp.

To overcome these problems, alternative successful techniques involving Laplace transforms were developed (e.g. References 12 and 13), but their use is strictly restricted to linear equations. Promising adaptive schemes (e.g. Reference 14) have also been put forward to improve stability and accuracy. However, in applications with transient, heterogeneous flow fields and coupled processes, efficient adaptive remeshing becomes a tricky matter, particularly in two or three dimensions.

In this paper a stability criterion for standard first-order time-stepping schemes is derived, based on a revised space-time eigenvalue analysis. This criterion is then put into a very practical multidimensional form which can be easily implemented in standard source codes, allowing numerical treatment of deemed complex problems like advective coupled transfers involving sharp fronts.

BASIC EQUATIONS

In the context of non-reactive, source free solute transport in porous media, the advection–diffusion equation may be written as

$$\frac{\partial \phi}{\partial t} = -\mathbf{v}\cdot\nabla\phi + \nabla\cdot(\mathbf{D}\nabla\phi) \tag{1}$$

subject to Dirichlet or Neuman-type boundary conditions. In this equation ϕ (-) is the solute relative concentration, $\mathbf{v}$ (m/s) the average pore velocity and $\mathbf{D}$ (m^2/s) the macro-dispersion tensor. Classically the latter is defined, in a vector form, by

$$\mathbf{D} = \frac{\alpha_L}{\|\mathbf{v}\|}\mathbf{v}\otimes\mathbf{v} + \frac{\alpha_T}{\|\mathbf{v}\|}(\|\mathbf{v}\|^2\mathbf{I} - \mathbf{v}\otimes\mathbf{v}) + D_m\mathbf{I} \tag{2}$$

where α_L and α_T(m) are longitudinal and transverse dispersivities with respect to flow direction, D_m(m^2/s) is the isotropic molecular diffusion and $\mathbf{I}$ the identity matrix.

When the space-time domain (x, y, z, t) is discretized, the elementary grid defined by Δx, Δy, Δz and Δt characterizes the simulation by means of the dimensionless P and C numbers. P is a measure of the relative effects of advection and diffusion in (1), and is expressed as the ratio of the length of an elementary representative trajectory Δl (see Figure 1) to the total mixing length $\alpha_{L_{tot}}$ along the flow direction. C is the ratio between the advective distance covered by a water particle during the time step Δt and the distance Δl. For a multidimensional problem, the numbers P and C can be defined as

$$P = \frac{\Delta l}{\alpha_{L_{tot}}} = \frac{\mathbf{v}\cdot\Delta\mathbf{s}}{\|\mathbf{v}\|(\alpha_L + D_m/\|\mathbf{v}\|)} = \frac{\mathbf{v}\cdot\Delta\mathbf{S}}{D} \tag{3}$$

and

$$C = \frac{\|\mathbf{v}\|\Delta t}{\Delta l} = \frac{\|\mathbf{v}\|^2\Delta t}{\mathbf{v}\cdot\Delta\mathbf{s}} \tag{4}$$

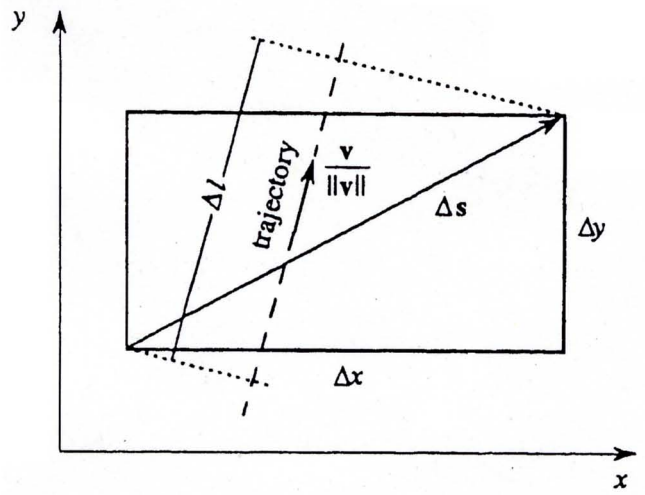

Figure 1. Two-dimensional rectangular finite element with $\mathbf{v}/\|\mathbf{v}\|$, Δs and Δl

with

$$\Delta l = \frac{\mathbf{v} \cdot \Delta \mathbf{s}}{\|\mathbf{v}\|}, \quad \mathbf{v} = (v_x, v_y, v_z)^T, \quad \Delta \mathbf{s} = (\Delta x, \Delta y, \Delta z)^T$$

and where the components of $\mathbf{v}$ and $\Delta \mathbf{s}$ (a representative vector of the spatial discretization) are taken in absolute value to calculate Δl. In equation (3), D is the total longitudinal dispersion. In a linear, one-dimensional problem along an arbitrary direction s, equation (1) becomes

$$\frac{\partial \phi}{\partial t} = -v \frac{\partial \phi}{\partial s} + D \frac{\partial^2 \phi}{\partial s^2} \tag{5}$$

and the total dispersion D is

$$D = \alpha_L v + D_m \tag{6}$$

In this situation, the length of the representative trajectory Δl is equal to the node spacing (Δs) and the two dimensionless numbers are simply

$$P = \frac{v \Delta s}{D} \quad \text{and} \quad C = \frac{v \Delta t}{\Delta s} \tag{7}$$

These two equations show that it is very difficult to meet simultaneously the recommended constraints $P \leqslant 2$ and $C \leqslant 1$ when simulating low diffusive–dispersive transport problems. For high flow velocities and/or low diffusion–dispersion effects, Δs has to be reduced to keep $P \leqslant 2$. With such a high space resolution, reducing the time step to very low values is then the only way to satisfy the constraint $C \leqslant 1$. Such a detailed space-time discretization is probably acceptable when simulating one-dimensional transfers in laboratory experiments or small volumes. This is no longer the case in real practical groundwater problems which have to be analysed on a two or even three-dimensional basis and involve large volumes. However, it is demonstrated further that the practical stability criterion derived in this paper considerably widens the operational solution domain of standard finite element schemes.

EIGENVALUE ANALYSIS FOR FIRST-ORDER ADVECTION–DIFFUSION SCHEMES

In this section the stability domain for standard one-dimensional linear advection–diffusion schemes is defined using the basic concepts relative to Liapounov's stability theory. The eigenvalue

analysis is based on a 'two step' treatment of the differential equation. In the first place, the effect of the finite element discretization on the *temporal* behaviour of the general solutions is evaluated. Secondly, the impact of discretizing time with finite differences on the *spatial* behaviour of the general solutions is assessed.

Considering equation (5) and applying the finite element space discretization to the right-hand side terms, one obtains, for a linear element and after integration, the typical divergence element matrix

$$\mathbf{A}^e = -\begin{bmatrix} \frac{D}{\Delta s} - \frac{v}{2} & -\frac{D}{\Delta s} + \frac{v}{2} \\ -\frac{D}{\Delta s} - \frac{v}{2} & \frac{D}{\Delta s} + \frac{v}{2} \end{bmatrix} \tag{8}$$

Assembling this element matrix for a network with n elements and $N = n + 1$ nodes leads to a global tridiagonal matrix $\mathbf{A}_{N\times N}$. Equation (5) can then be written under the semi-discrete form

$$\frac{\partial(\phi_i)}{\partial t} = -v \begin{bmatrix} \frac{1}{P} - \frac{1}{2} & -\frac{1}{P} + \frac{1}{2} & 0 & \cdots & 0 \\ & & & & \vdots \\ -\frac{1}{P} - \frac{1}{2} & \frac{2}{P} & \ddots & & 0 \\ 0 & \ddots & & \frac{2}{P} & -\frac{1}{P} + \frac{1}{2} \\ \vdots & & & & \\ 0 \cdots & & 0 & -\frac{1}{P} - \frac{1}{2} & \frac{1}{P} + \frac{1}{2} \end{bmatrix} (\phi_i) \tag{9}$$

where ϕ_i represents the concentration at a given point with co-ordinate s_i. The behaviour (i.e. the stability) of the general solutions $\phi(s_i, t)$ depends on the eigen values of the above coefficient matrix. If the number of nodes N is high enough ($N > 15$ according to Reference 7), this tridiagonal coefficient matrix may be assimilated to a Toeplitz matrix (single value in each diagonal) for which the eigen values are

$$\lambda_k = -\frac{2}{P} + 2\sqrt{\frac{1}{P^2} - \frac{1}{4}} \cos\left(\frac{\pi k}{N+1}\right), \quad k = 1, 2, \ldots N \tag{10}$$

Equation (10) shows that the eigen values of (9) become complex when $P > 2$ and that they always present negative real parts (i.e. $\mathrm{Re}(\lambda_k) < 0$). In this case, the general solutions are of a damped oscillatory nature. Small disturbances at points with co-ordinates s_i propagate in time according to periodic damped functions, and it can be thus concluded that the general unknown functions $\phi(s_i, t)$ are unconditionally stable. Similar results are obtained with the von Neumann stability analysis (e.g. Reference 15). Since standard central schemes generate highly unstable and oscillatory results in cases where advection dominates, it seems obvious that the stability of the discretized problem is not unconditional, as stated above, but depends on a criterion which cannot be directly derived from the analysis of system (9).

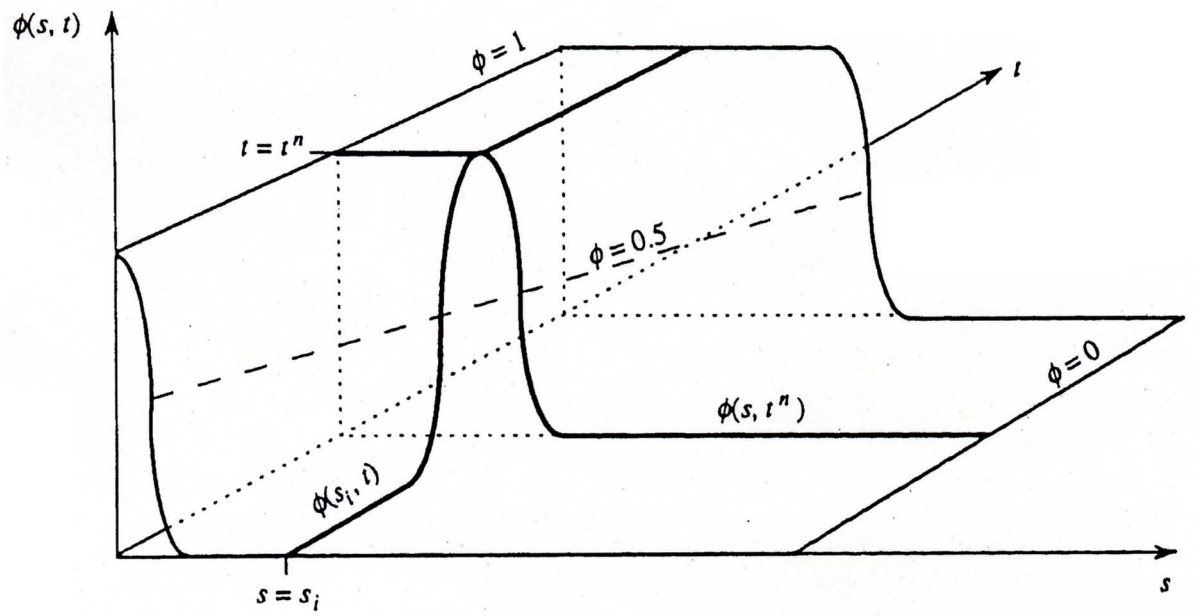

Figure 2. Schematic view of the space-time function $\phi(s, t)$ with 'sections' $\phi(s_i, t)$ and $\phi(s, t^n)$ in the one-dimensional v plug-flow' case

Instead of concentrating only on the analysis of the temporal stability of the function $\phi(s_i, t)$, as most researchers do, it is also important to assess the spatial stability of the functions $\phi(s, t^n)$ (see Figure 2). This procedure appears entirely justified since in most applications both *space* and *time* stability have to be guaranteed. Moreover, in the context of practical problems, engineers are specially interested in obtaining stable concentration profiles.

In order to apply the Liapounov principles to a space stability analysis, it is necessary to write a system of equations which is similar to that in (9), but this time with first space derivatives $\partial\phi^n/\partial s$ in the left-hand side and only time terms in the right-hand side. Since the form of (5) is not readily suitable for such an operation, the following transformation is performed.

Differentiating the reference equation (5) with respect to time and reversing the derivative indexes gives

$$\phi_{tt} = -v\partial_t\phi_s + D\partial_t\phi_{ss} = -v\partial_s\phi_t + D\partial_{ss}\phi_t \tag{11}$$

Substituting in this equation the first time-derivative ϕ_t by its expression in (5) yields

$$\phi_{tt} = v^2\phi_{ss} - 2vD\phi_{sss} + D^2\phi_{ssss} \tag{12}$$

which can be written as

$$\phi_{ss} = \frac{1}{v^2}\phi_{tt} + \frac{2D}{v}\phi_{sss} - \frac{D^2}{v^2}\phi_{ssss} \tag{13}$$

If (13) is now substituted into (5), one obtains a alternative form for the transport equation which reads after rearranging terms

$$\phi_s = -\frac{1}{v}\phi_t + \frac{D}{v^3}\phi_{tt} + \frac{2D^2}{v^2}\phi_{sss} - \frac{D^3}{v^3}\phi_{ssss} \tag{14}$$

In groundwater hydrology, terms involving derivatives such as ϕ_{sss} and ϕ_{ssss} are generally neglected. In effect, for typical values of v and D, and for relatively large space-increments Δs

(resulting in high Peclet numbers), it can easily be shown that these terms are several orders of magnitude smaller than the terms involving ϕ_t and ϕ_{tt}. Consequently equation (14) is approximated by

$$\frac{\partial \phi}{\partial s} = -\frac{1}{v}\frac{\partial \phi}{\partial t} + \frac{D}{v^3}\frac{\partial^2 \phi}{\partial t^2} \tag{15}$$

which is the form required to perform the *space* stability analysis of the solution $\phi(s, t^n)$. Applying a first-order finite difference treatment to the right-hand side terms of (15), one obtains for time level t^n the typical difference equation

$$\frac{\partial \phi^n}{\partial s} = -\frac{1}{v}\left((1-\varepsilon)\left(\frac{\phi^{n+1}-\phi^n}{\Delta t}\right) + \varepsilon\left(\frac{\phi^n - \phi^{n-1}}{\Delta t}\right)\right) + \frac{D}{v^3}\left(\frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{\Delta t^2}\right) \tag{16}$$

where ε is a time weighting factor ($0 \leqslant \varepsilon \leqslant 1$). Hence, for a sufficient number M of temporal levels, equation (15) may be expressed in the semi-discrete form

$$\frac{\partial(\phi^n)}{\partial s} = -\frac{1}{\Delta x}$$

$$\begin{bmatrix} \frac{2}{PC^2} - \frac{(1-2\varepsilon)}{C} & -\frac{1}{PC^2} + \frac{(1-\varepsilon)}{C} & 0 & \cdots & 0 \\ & & & & \vdots \\ -\frac{1}{PC^2} - \frac{\varepsilon}{C} & \frac{2}{PC^2} - \frac{(1-2\varepsilon)}{C} & \ddots & & 0 \\ 0 & \ddots & \frac{2}{PC^2} - \frac{(1-2\varepsilon)}{C} & & -\frac{1}{PC^2} + \frac{(1-\varepsilon)}{C} \\ \vdots & & & & \\ 0 & \cdots & 0 & -\frac{1}{PC^2} - \frac{\varepsilon}{C} & \frac{2}{PC^2} - \frac{(1-2\varepsilon)}{C} \end{bmatrix} (\phi^n) \tag{17}$$

in which the dimensionless tridiagonal Toeplitz matrix has eigen values given by

$$\lambda_k = -\frac{2}{PC^2} + \frac{(1-2\varepsilon)}{C} + \frac{2}{C}\sqrt{\frac{1}{P^2C^2} + \frac{(2\varepsilon-1)}{PC} + \varepsilon(\varepsilon-1)}\cos\left(\frac{\pi k}{M-1}\right), \quad k = 1, 2, \ldots, M-2 \tag{18}$$

In this case, if the solutions $\phi(s, t^n)$ have to be stable and non-oscillatory space functions, the λ_k have to be real and negative numbers, and therefore the condition is

$$PC \leqslant \frac{1}{1-\varepsilon} \tag{19}$$

Within the validity limits of the simplifications inherent in equation (15), a limiting constraint that preserves the stability of first-order schemes has been established from basic considerations.

One notes from equation (19) the well known unconditional stability of the fully implicit scheme ($\varepsilon = 1$) due to its diffusive second-order time truncation error. On the other hand, the anti-diffusive explicit scheme ($\varepsilon = 0$) requires the respect of a stability condition on the time-step size. Both of these schemes, however, suffer second-order inaccuracies and these are drastically reduced when a central procedure is adopted (i.e. $\varepsilon = 0.5$), in which case the stability requirement becomes

$$PC \leqslant 2 \tag{20}$$

This condition, for which the conventional constraints ($P \leqslant 2$ and $C \leqslant 1$) are only a particular case, indicates that advection dominated transport problems (high P numbers) can be analyzed with reasonable accuracy with standard central methods provided C is consequently reduced.

For obvious mathematical reasons the above theoretical one-dimensional analysis cannot be performed for multidimensional practical applications. However, numerical experiments strongly suggest that the enforcement of condition (20) along the flow lines is sufficient to ensure transient numerical stability in two- or three-dimensional situations. This means that the one-dimensional theoretical stability criterion (20) can be empirically but safely 'streamlined' for general cases. It can be noted at this point that an extension of this kind was also enforced in the classical work by Reference 16 on anisotropic 'upwind schemes' applied to the steady-state transport equation.

Here, addressing the transient case, this multidimensional extension is simply achieved by recalling equations (3) and (4) and substituting into condition (20). In doing so the latter becomes

$$\frac{\|\mathbf{v}\|^2 \Delta t}{\|\mathbf{v}\|(\alpha_L + D_m/\|\mathbf{v}\|)} \leqslant 2 \tag{21}$$

and it is obvious that this condition is satisfied, without requiring the determination of P and C, if

$$\alpha_L \geqslant \frac{\|\mathbf{v}\| \Delta t}{2} - D_m/\|\mathbf{v}\| \tag{22}$$

For a given time-step Δt, the above condition indicates that a stable solution is obtained if the total mixing length is at least equal to half of the advective distance $\|\mathbf{v}\| \Delta t$. In regions of the model domain where the physical longitudinal dispersivity α_L is not enough, an additional longitudinal dispersivity α_L^* defined as

$$\alpha_L^* = \frac{\|\mathbf{v}\| \Delta t}{2} - \alpha_L - D_m/\|\mathbf{v}\|, \quad \alpha_L^* \geqslant 0 \tag{23}$$

can be added up to the minimum required following the 'smart upwind' philosophy. However, questions related to the accurate definition of locations requiring upwinding, as well as to the quantity of upwind needed and to operational upwind implementation techniques are readily answered with the proposed approach.

Hence, for a purely advective case ($\alpha_L = \alpha_T = 0$, $D_m = 0$, $P = \infty$) the differential equation will include a corrective diffusive term with the dispersion tensor $\mathbf{D}^*$ such as

$$\mathbf{D}^* = \frac{\alpha_L^*}{\|\mathbf{v}\|} \mathbf{v} \otimes \mathbf{v} = \frac{\Delta t}{2} \mathbf{v} \otimes \mathbf{v} \tag{24}$$

Moreover, when small residual oscillations can be tolerated in the solutions, the factor 2 in equations (20)–(24) can be replaced by a performance index γ ($2 \leqslant \gamma \leqslant 10$), the value $\gamma = 10$ corresponding to the experimental stability limit $P \leqslant 10$ given in Reference 8 for C numbers approaching unity.

APPLICATION TO COUPLED TRANSFERS

To demonstrate the applicability of the above elements in a general non-linear context, a two-dimensional density driven flow problem is solved involving two very low miscible fluid phases and a rather sharp interface. The flow domain is a ditch-drain system with variably saturated areas in which the unsaturated hydraulic conductivity is defined according to Reference 17, with saturated and residual moisture contents of 0.42 and 0.12 and a saturated conductivity of 10^{-5} m/s. Bi-quadratic isoparametric elements were used in the simulation.

Steady state flow of a reference fresh water (with density $\rho_o = 10^3$ Kg/m^3 and viscosity $\mu_o = 10^{-3}$ Kg/m/s) is initially specified between the ditch and the drain (drain at atmospheric pressure, other boundaries assumed impervious) and transport simulation starts when water in the ditch is replaced by a pollutant phase (with density ρ_p and viscosity μ_p) of unit relative concentration. These conditions as well as the geometry of the problem are illustrated in Figure 3.

According to the early work described in References 18 and 19, such a two-phase flow system can be equivalently described by association of an advection–diffusion equation and a saturated–unsaturated flow equation (i.e. Richard's equation) in terms of pressure head. These two equations (which are given in the papers mentioned above) are coupled by means of two constitutive relations expressing density and viscocity as functions of the relative concentration in one of the two phases. In order to test the robustness of the method, capillary effects between phases are deliberately assumed low or negligible, and therefore the process may be simulated by keeping only the advective term in the transport equation (i.e. coupled 'plug-flow' $P = \infty$).

Hence, this deemed complex problem is simulated by sequential solutions of both Richard's equation and the transport equation (1). In the latter, the advective velocity field $\mathbf{v}$ is derived at each time step from the solution of Richard's equation and the dispersion tensor $\mathbf{D}$ is replaced by the stabilizing tensor function $\mathbf{D}^*$ given in equation (24). In the presence of strong density/viscosity effects (coupled transfers), several iterations (typically three to six) are required to update both the flow field and the concentration distribution simultaneously.

In order to show the 'net' effects of the stabilizing dispersivity α_L^* defined in (23), the injection of a pollutant with $\rho_p = \rho_o$ and $\mu_p = \mu_o$ is simulated first in Figure 4 (decoupled, linear advection). Figure 4(a) shows the oscillatory results obtained by a standard central scheme ($\varepsilon = 0.5$) with no stabilizing dispersivity at $t = 600{,}000$ s, and Figure 4(b) illustrates the beneficial effects achieved by α_L^*. The smooth development and migration of the front through the saturated–unsaturated domain, in spite of the rather coarse discretization, point to the robustness of the suggested

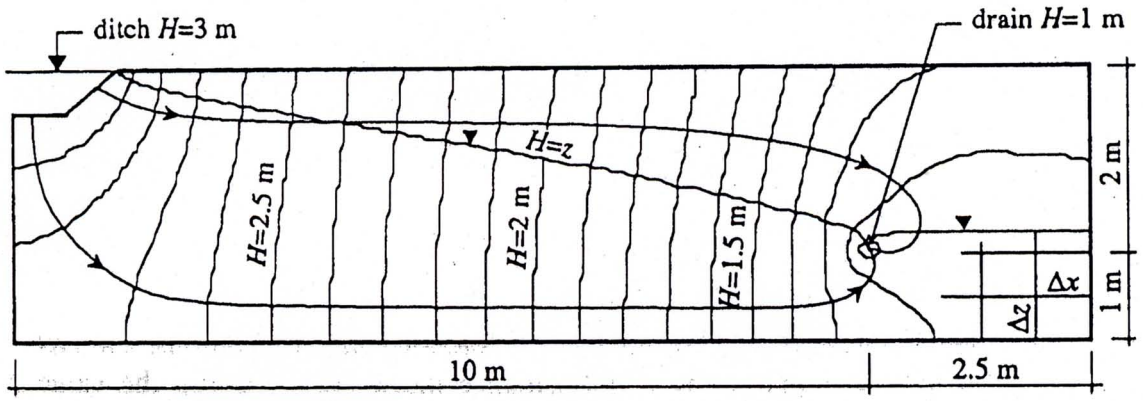

Figure 3. Geometry of the variably saturated ditch-drain system with initial distribution of hydraulic heads (H), trajectories and watertable level $H = z$

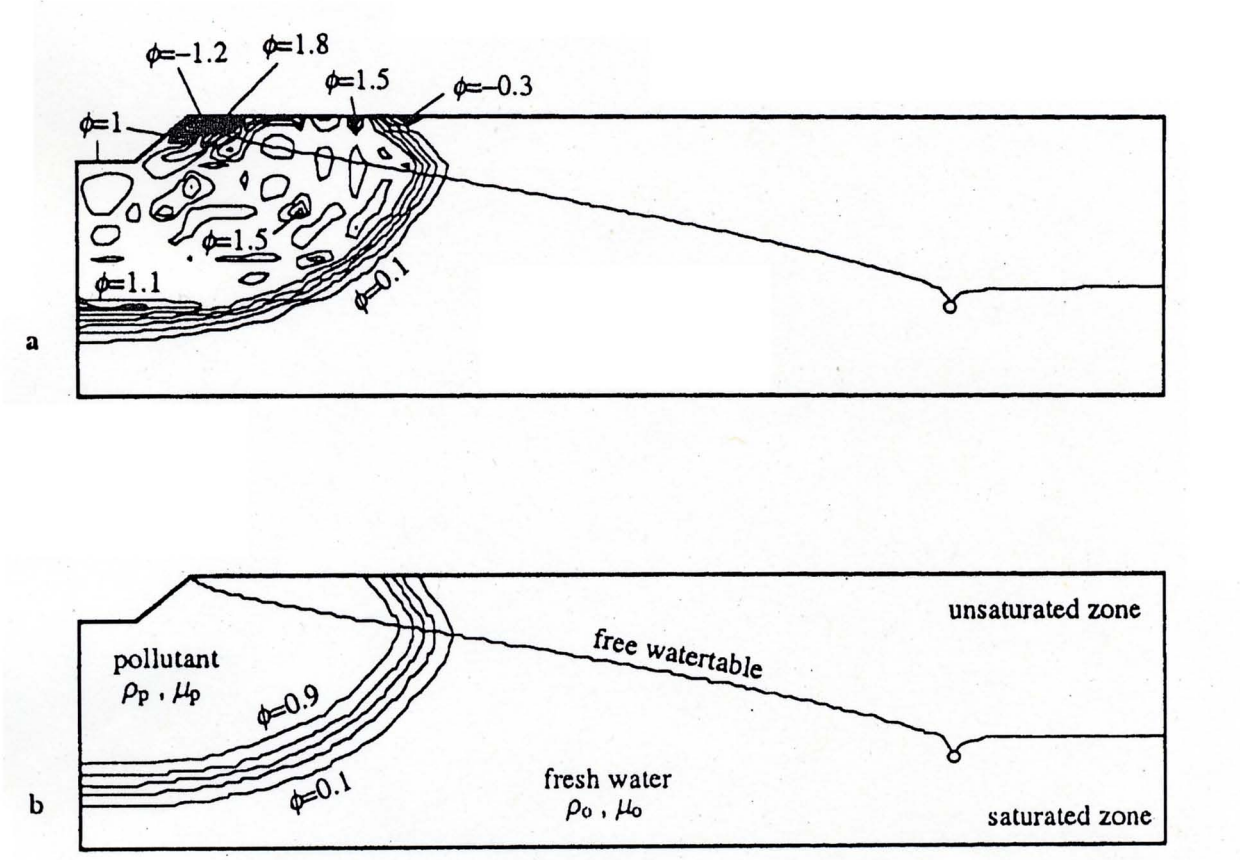

Figure 4. Central first order solutions ($\varepsilon = 0.5$) of linear advection at $t = 600{,}000$ s: (a) standard unstable solution; (b) solution after implementation of the suggested stabilizing tensor function using $\gamma = 10$

criterion. Using a performance index $\gamma = 10$ the quantity $\alpha_L^* = \|\mathbf{v}\| \Delta t/\gamma$, which acts differentially at each integration point, allows the simulation of an interface which is only slightly over-diffused. In fact, only some of the streamlined additional dispersivity acts as artificial diffusion—the cost to pay for the simplicity of the approach—because of a partial cancellation of the truncation errors (second order and above) that remain in central first-order schemes.

Seeking comparative results, this linear example was also run using a Laplace Transform Galerkin scheme (valid exclusively for the linear case) which confirmed the pertinence of the approach. Other more sophisticated time-stepping schemes of the Petrov- and Taylor–Galerkin type were also tested but failed to produce reasonable results. In fact, severe difficulties were encountered due to $C > 1$ conditions prevailing in certain areas of the flow domain and to the sharp interface specified.

Concerning coupled transfers it is evident that, due to the spurious oscillations appearing in Figure 4(a), standard algorithms which do not include implementation of the additional longitudinal dispersivity α_L^* will never reach convergence. On the other hand, when the additional dispersivity is introduced, non-linear advection can be safely simulated. Figure 5(a) and 5(b) illustrate, respectively, the propagation of a 'light' and a 'heavy' pollutant (for instance gasoline and carbon tetrachloride) into the system using the same simulation parameters as in Figure 4(b). The only difference here being the density/viscosity contrast (i.e. $\rho_p = 0.79\rho_o$, $\mu_p = 0.33\mu_o$ and $\rho_p = 1.60\rho_o$, $\mu_p = 0.97\mu_o$).

In the case of Figure 4(b), given the homogeneity of the two fluids (linear transport), the pollutant propagates without affecting the steady-state regime initially specified. When the injected fluid is lighter than the reference water (Figure 5(a)), it tends to float on the watertable

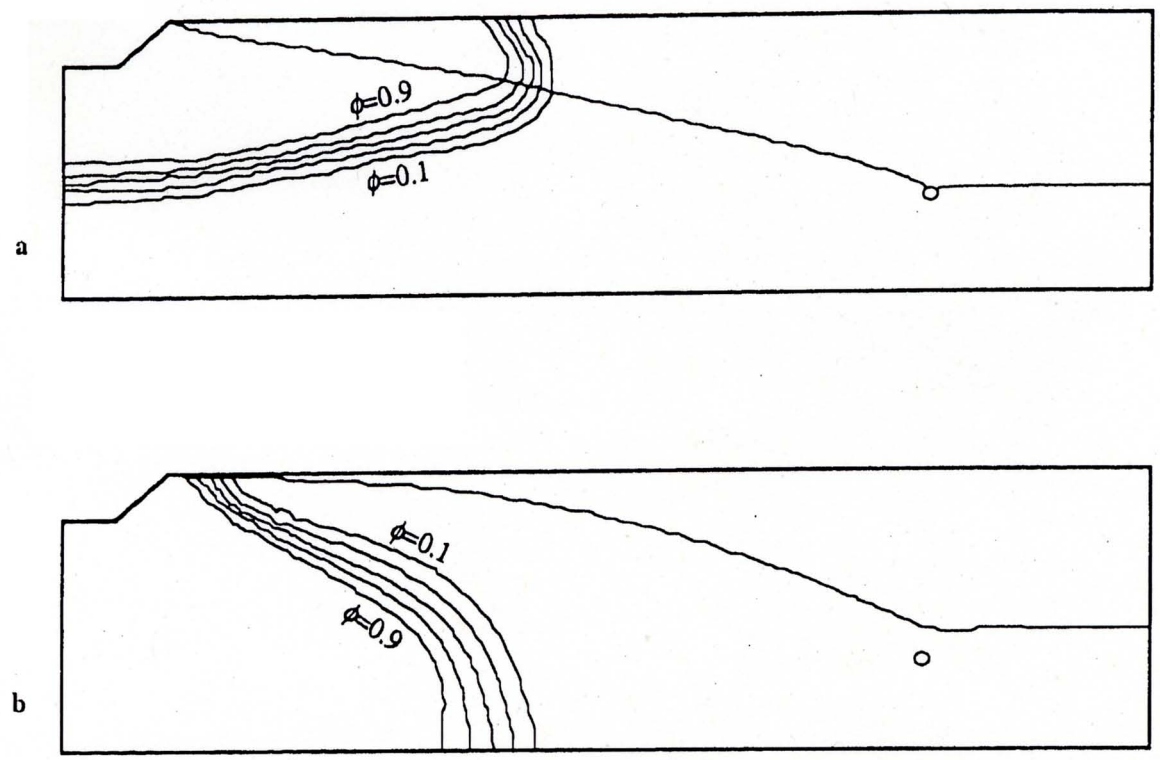

Figure 5. Coupled advection at $t = 600{,}000$ s using the suggested stabilizing tensor function ($\gamma = 10$): (a) gasoline–fresh water ($\rho_p = 0.79\rho_o$, $\mu_p = 0.33\mu$); (b) carbon tetrachloride–fresh water ($\rho_p = 1.60\rho_o$, $\mu_o = 0.97\mu_o$)

and extends laterally. This lateral extension being due to the lower viscosity (higher velocity) of the injected fluid, and also to the fact that fresh water is drained more slowly because of a lower upstream pressure (lower density). In the case of Figure 5(b), the heavier pollutant rapidly sinks towards the bottom of the system, pushing the fresh water upwards. Figure 5(b) also indicates that the saturated volume is significantly increased (watertable level rises, drain under a positive head) because the volume taken by the pollutant cannot be evacuated swiftly enough at the drain.

CONCLUSION

The stability domain of the standard first-order schemes applied to the one-dimensional advection–diffusion equation has been reassessed using an alternative eigenvalue analysis, the outcome of which has been empirically and successfully extended to two-dimensional and three-dimensional situations. The resulting stability condition has a very practical form which makes it easy to implement in standard source codes. Allowing the simulation of advection dominated transport problems using central time-stepping, the method has significant advantages with respect to traditional diffusive approaches—such as the fully implicit or early upwind finite element techniques[20]—particularly in multidimensional modelling.

Unlike the implicit scheme in which every point in the flow domain is subjected to the implicit treatment, the association of constraint (22) with a central finite element scheme precisely limits to a strict minimum the quantity of artificial diffusion. In other words, the latter only acts at points and along directions for which a corrective action is needed. For this reason, it can be stated that the proposed criterion brings the performances of standard central schemes to an optimum level.

With respect to upwind finite element techniques, the proposed method requires neither the definition of asymmetric weighting functions nor the calculation of P and C numbers which in most practical cases is not very easy. Moreover, since the asymmetric weighting functions can only generate a limited quantity of stabilizing diffusion, approaches of this type are no longer effective when local values of the Courant number exceed one.

The simple technique presented here can make standard codes operational for a wide range of practical problems with variable space-time discretization, such that no control over P and C number is needed. Again one has to keep in mind that this is achieved at the expense of some optimal streamlined smearing (absence of cross-wind artificial dissipation), though at levels much less than the excessive anisotropic or isotropic smearing that generally mars the applications of the classic upwind methods.

Apart from the two-dimensional coupled flow problem described herein, other large scale linear and non-linear applications have also proven the efficiency of the method.[21] The ease of generating robust first-order central algorithms using this technique makes it advantageous for many practical problems, such as regional fluid flow simulations where a reasonable approximation is generally acceptable.

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