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Action selectors without Floer-Homology

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par

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Abstract

Hamiltonian systems on symplectic manifolds tend to have many periodic orbits. The "actions" of these orbits form an invariant for the Hamiltonian system. The set of actions can be very large, however. To get useful invariants, one selects for each Hamiltonian function just one action value by some minimax procedure: A so-called action selector associates with every compactly supported Hamiltonian diffeomorphism of a symplectic manifold the action of a 1-periodic orbit, in a continuous and nontrivial way. The mere existence of an action selector has many applications to Hamiltonian dynamics, symplectic geometry and topology: It readily yields a symplectic capacity and thus implies Gromov's nonsqueezing theorem, implies the almost existence of closed characteristics on displaceable hypersurfaces and in particular the Weinstein conjecture for displaceable energy surfaces of contact type, often shows that the diameter of the Hamiltonian diffeomorphism group with Hofer's metric is infinite, etc.

Action selectors were first constructed for the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$ by Viterbo and Hofer–Zehnder. For more general symplectic manifolds (M, ω), action selectors were obtained, up until now, only by means of Floer homology: For symplectically aspherical symplectic manifolds (namely those for which the integral of the symplectic form over spheres vanishes) Schwarz constructed the Floer selector when M is closed, and his construction was adapted to convex symplectic manifolds by Frauenfelder–Schlenk. For some further classes of symplectic manifolds and Hamiltonian functions, the Floer selector was constructed by Lanzat, Oh and Usher.

In this thesis we give a more elementary construction of an action selector for closed or convex symplectically aspherical manifolds. Our construction uses only Gromov compactness and results from Chapter 6 of the text book by Hofer and Zehnder, that also rely on rudimentary Fredholm theory, but none of the more advanced tools in the construction of Floer homology. In this way, the three basic properties of an action selector (spectrality, continuity, and non-triviality) are readily established and their proofs are rather straightforward, since the only tool at our hands is the compactness property of certain spaces of holomorphic cylinders. From these three basic properties of the selector many further properties then follow in an elementary way.

Keywords: Hamiltonian systems, symplectic geometry, action spectrum, action selectors, Floer Homology, Gromov compactness.

Résumé

Les systèmes hamiltoniens sur des variétés symplectiques ont d'habitude beaucoup d'orbites périodiques. Les *actions* des orbites forment un invariant du système hamiltonien. Cependant, l'ensemble des actions peut être très grand. Pour obtenir des invariants utiles, on ne sélectionne pour chaque fonction hamiltonienne qu'une valeur d'action en utilisant une procédure de minimax : on appelle sélecteur d'action la fonction qui associe à chaque difféomorphisme hamiltonien à support compact une seule valeur d'action d'une orbite de période 1, de manière continue et non triviale. L'existence d'un sélecteur d'action a déjà beaucoup d'applications en mécanique hamiltonienne, en topologie symplectique et géométrie. Elle permet de construire une capacité symplectique et implique alors le théorème de non-tassement de Gromov. De plus, l'existence d'un sélecteur d'action implique la quasi-existence de caractéristiques fermées sur des hypersurfaces déplaçables de type contact, implique souvent que le diamètre du groupe des difféomorphismes hamiltoniens muni de la métrique de Hofer est infini, etc.

Des sélecteurs d'action ont été construits d'abord pour l'espace vectoriel symplectique standard $(\mathbb{R}^{2n}, \omega)$ par Viterbo et Hofer–Zehnder. Pour des variétés symplectiques (M, ω) plus générales, la construction d'un sélecteur d'action a jusqu'à présent toujours nécessité l'homologie de Floer : pour des variétés symplectiques asphériques (c.à.d. les variétés symplectiques pour lesquelles l'intégrale de la forme symplectique sur les sphères s'annule), Schwarz a construit le sélecteur de Floer dans le cas où M est compacte et sans bord. Cette construction a été généralisée aux variétés symplectiques convexes par Frauenfelder–Schlenk. Pour quelques autres classes de variétés symplectiques et fonctions hamiltoniennes, le sélecteur de Floer a été construit par Lanzat, Oh et Usher.

Dans cette thèse on donne une construction plus élémentaire d'un sélecteur d'action pour des variétés symplectiques asphériques compactes et sans bord, et pour des variétés symplectiques asphériques convexes. Notre construction utilise seulement la compacité de Gromov et des résultats du chapitre 6 du livre écrit par Hofer et Zehnder, basé sur la théorie de Fredholm rudimentaire. On n'utilise aucun des outils plus avancés qui sont utilisés dans la construction de l'homologie de Floer. Ainsi, les trois propriétés de base d'un sélecteur d'action (spectralité, continuité et non-trivialité) sont démontrables d'une manière plus simple, car le seul outil disponible est la compacité de certains espaces de cylindres holomorphes. En utilisant ces trois propriétés de base, on déduit alors plusieurs autres propriétés de manière élémentaire.

Mots clefs : Systèmes hamiltoniens, géométrie symplectique, spectre d'action, sélecteur d'actions, homologie de Floer, compacité de Gromov

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Contents

Chapter 1

Introduction

A dynamical system describes the evolution over time of one or more objects in a configuration or a phase space. The forces which are acting on these objects can be internal as well as external. Examples for dynamical systems are the motion of a pendulum, the motion of a particle which is subject to electromagnetic forces or the motion of our solar system. In this work we restrict ourselves to Hamiltonian systems. These are special dynamical systems, in which the behavior of the particles is given by a differential equation involving a so-called Hamiltonian function $H(q, p)$. Here q is the position of the system and p is the momentum. The evolution of the system is then given by the so called Hamilton's equations

$$\begin{aligned}\frac{dp}{dt} &= -\frac{\partial H}{\partial q},\\ \frac{dq}{dt} &= +\frac{\partial H}{\partial p}.\end{aligned}$$

Many dynamical systems can be described in such a way. In fact, all the examples mentioned above are Hamiltonian systems. Since Hamiltonian systems describe systems without friction, there orbit structure can be very intricate. In particular it is usually helpless to explicitly solve a Hamiltonian system. This is already the case for the (restricted) 3-body problem. Instead of studying individual orbits of a Hamiltonian system, one therefore looks at the system as a whole, and tries to get some information on the whole system with the help of global invariants. One such invariant associated to Hamiltonian systems is the set of "actions" of periodic orbits of the system. However these systems tend to have many periodic orbits and so the set of actions tends to be very large. To get useful invariants, one selects for each Hamiltonian function just one action value by some minimax procedure: A so-called action selector associates with every compactly supported Hamiltonian function on a symplectic manifold the action of a 1-periodic orbit of its flow, in a continuous and nontrivial way. The mere existence of an action selector has many applications to Hamiltonian dynamics and symplectic geometry and topology: It readily yields a symplectic capacity and thus implies Gromov's nonsqueezing theorem, implies the almost existence of closed characteristics on displaceable hypersurfaces and in particular the Weinstein conjecture for displaceable energy surfaces of contact type, often proves the non-degeneracy of Hofer's metric and its unboundedness, etc., see for instance [3, 4, 6, 12, 18, 21]. Action selectors were first constructed for standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$ by Viterbo [21] and Hofer–Zehnder [6]. For more general symplectic manifolds (M, ω), action selectors were obtained, up until now, only by means of Floer homology: For symplectically aspherical symplectic manifolds (namely those for which $[\omega]|_{\pi_2(M)} = 0$), Schwarz [18] constructed the so-called PSS selector when M is closed, and his construction was adapted to convex symplectic manifolds in [4]. (We refer to Appendix A of [3] for a short description of these selectors.) For some further classes of symplectic manifolds and Hamiltonian functions, the PSS selector was constructed in [9, 11, 20]. In this work we give a more elementary construction of an action selector for closed or convex symplectically aspherical manifolds. Our construction uses only the results from Chapter 6.4 of the text book [6] by Hofer and Zehnder, that rely on Gromov compactness, rudimentary Fredholm theory, and the continuity property of Alexander–Spanier cohomology, but none of the more advanced tools in the construction of Floer homology (such as exponential decay, the spectral flow, unique continuation, gluing, or transversality). In this way, the three basic properties of an action selector (spectrality, continuity, and non-triviality) are readily established, and their proofs are rather straightforward, since the only tool at our hands is the compactness property of certain spaces of holomorphic cylinders.

After explaining some basic notions in Chapter 2, we slightly generalise the famous Gromov compactness theorem in Chapter 3. This allows us to give the construction of our action selector for closed aspherical symplectic manifolds in Chapter 4. In Chapter 5 we generalise this action selector to other cohomology classes and in Chapter 6 we adapt this construction to convex aspherical symplectic manifolds. Examples are cotangent bundles and their fiberwise starshaped subdomains, on which most of classical mechanics takes place, and so these action selectors have many applications to dynamics. In Chapter 7 we show that the three basic properties of the action selector imply many further properties. We treat three applications of our construction in Chapter 8 and in Chapter 9 we compare the minimal selector with the PSS selector.

1.1 Idea of the construction

In the rest of this introduction we outline the construction of our action selector on closed symplectically aspherical manifolds (M, ω). Denote by $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ the circle of length 1. Recall that the Hamiltonian action functional on the space of contractible loops $C^\infty_{\text{contr}}(\mathbb{S}^1, M)$ associated to a Hamiltonian function $H \in C^\infty(\mathbb{S}^1 \times M, \mathbb{R})$ is given by

$$\mathbb{A}_H(x) := \int_{\mathbb{D}} \bar{x}^*(\omega) + \int_{\mathbb{S}^1} H(t, x(t))\, dt,$$

where $\bar{x} \in C^\infty(\mathbb{D}, M)$ is such that $\bar{x}|_{\partial\mathbb{D}} = x$. The critical points of $\mathbb{A}_H$ are the contractible 1-periodic solutions of the Hamiltonian equation $\dot{x}(t) = X_H(x(t))$, where the vector field X_H is defined by $\omega(X_H, \cdot) = dH$, and the set of critical values of $\mathbb{A}_H$ is called the action spectrum of H and denoted by $\operatorname{spec} H$. A first idea to use the action spectrum for defining an action selector is to boldly take the smallest action value of a 1-periodic orbit,

$$\sigma(H) := \min \operatorname{spec} H.$$

Since $\operatorname{spec} H$ is a compact subset of $\mathbb{R}$, this definition makes sense, and yields an invariant with the spectral property. It is nevertheless useless, since σ thus defined dismally fails to be continuous and monotone, two crucial properties for applications. To see why, consider radial functions $H_f(z) := f(\pi|z|^2)$ on $\mathbb{R}^{2n}$, where $f \colon \mathbb{R} \to \mathbb{R}$ is a smooth function with compact support. (For an arbitrary symplectic manifold, such functions can be constructed in a Darboux chart.) The critical points of $\mathbb{A}_H$ are the origin and the (Hopf-)circles on those spheres that have radius $s = \pi|z|^2$ with $f'(s) \in \mathbb{Z}$, and at a critical point,

$$\mathbb{A}_{H_f}(x) = f(s) - s\,f'(s), \tag{1.1}$$

see the left drawing in Figure 1.1. Now take the profile functions f, f_+, f_- as in the right drawing: $f' \in [0,1]$ and $f'(s) = 1$ for a unique s, while f_-, f_+ are C^∞-close to f and satisfy $f_- \leq f \leq f_+$ and $f'_-, f'_+ \in [0,1)$. Then the formula (1.1) shows that $\sigma(H_f)$ is much smaller than $\sigma(H_{f_-}) \approx \sigma(H_{f_+})$, whence σ is neither continuous nor monotone. Or take g with $|g|$ very small and very steep. Then $\sigma(H_g)$ is much smaller than $\sigma(H_f)$, whence monotonicity fails drastically.

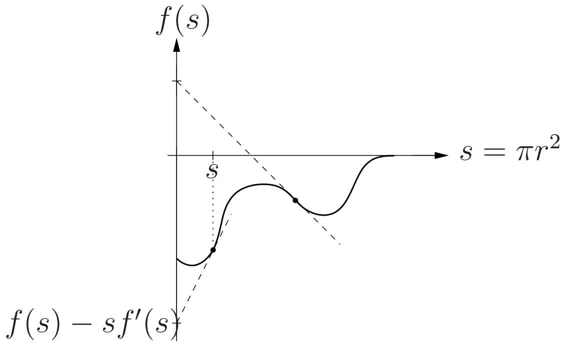


Figure 1.1: Radial functions and their minimal spectral values

The above discussion shows that the continuous, or monotone, selection of an action from $\operatorname{spec} H$ must be done by some kind of minmax procedure for the action functional. This was done for the Hofer–Zehnder selector by minmax over a uniform minimax family, and for the Viterbo selector and the PSS selector by a homological minmax. Our minmax will be over certain spaces of perturbed holomorphic cylinders.

To motivate our construction, we consider the quadratic form $q(x,y) = x^2 - y^2$ on $\mathbb{R}^2$ and its perturbations

$$q_h = q + h$$

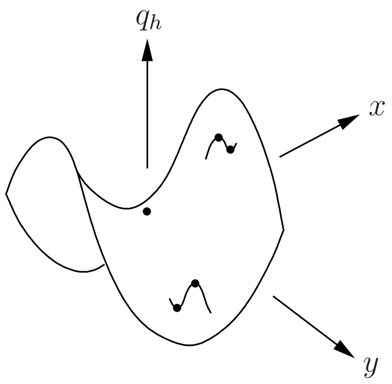


Figure 1.2: A perturbed quadratic form q_h

where h is a compactly supported function on $\mathbb{R}^2$. If $h = 0$, the only critical point of q_h is the origin, with critical value 0. If h consists, for instance, of two little positive bumps, one centered at $(1,0)$ and one at $(0,1)$, then the graph of q_h looks as in Figure 1.2. A continuous selection of critical values should, in our example, choose again 0, by somehow discarding the four new critical values introduced by h. The idea is to take the lowest critical point that "cannot be shaken off": Take a family h^s with $s \in \mathbb{R}$ of compactly supported functions such that $h^s = h$ for s small and $h^s = 0$ for s large, and look at the space $\mathscr{U}(h^s)$ of bounded solutions of the gradient equation

$$\dot{u}(s) \;=\; -\nabla q_{h^s}(u(s)), \qquad s \in \mathbb{R}.$$

The boundedness of u is equivalent to bounded energy

$$E(u) \;:=\; \int_{\mathbb{R}} |\nabla q_{h^s}(u(s))|^2 \, ds \;=\; \lim_{s \to -\infty} q_{h^s}(u(s)) - \lim_{s \to +\infty} q_{h^s}(u(s)) \;<\; \infty,$$

or, since $h^s = h$ in the first limit and $h^s = 0$ in the second limit, to the fact that u starts at a critical level of q_h and ultimately lies on the x-axis and converges to the origin (the only critical point of q). A bounded solution u starts at the critical level

$$q_h^-(u) \;:=\; \lim_{s \to -\infty} q_h(u(s)),$$

and

$$\min_{u \in \mathscr{U}(h^s)} q_h^-(u)$$

is the lowest critical value of q_h from which starts a bounded h^s-flow line. Unbounded solutions that start at a critical level disappear for $s \to +\infty$ to $-\infty$.

In our examples, if we take $h^s = \beta(s)\, h$ with a cut-off function β, then the two low critical points p_1, p_2 near $(0,1)$ are sent to $-\infty$ by the gradient flow, that is, $\mathscr{U}(h^s)$ contains no flow line u emanating from p_1 or p_2. On the other hand, it is easy to construct a family h^s that has a gradient line starting at p_1, then goes up and finally tends to the origin. To be sure that we discard all inessential critical values, we therefore set

$$\sigma(h) \;:=\; \sup_{h^s} \min_{u \in \mathscr{U}(h^s)} q_h^-(u).$$

In the example it is quite clear that for every deformation h^s there exists a flow line in $\mathscr{U}(h^s)$ emanating from the critical point $(0,0)$, that is, $\sigma(h)=0$ as it should be. In general, it is not hard to see that $\sigma(h)$ is a critical value of q_h that depend continuously and in a monotone way on h.

The number $\sigma(h)$ is the lowest critical value c of q_h such that for every deformation h^s of h there exists a bounded flow line $u \in \mathscr{U}(h^s)$ starting at a critical level $\le c$. Or, if we assume that q_h is a Morse function all of whose critical points have different critical values: $\sigma(h)$ is the highest critical value c of q_h such that for every critical point p strictly below c there exists a deformation h^s of h such that all flow lines of q_{h^s} starting at p diverge to $-\infty$; that is: The whole critical set strictly below c can be shaken off.

The relevance of this finite dimensional model for defining an action selector is indicated by the form of the extension of the action functional $\mathbb{A}_H$ to the fractional Sobolev space $H^{1/2} = H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n})$, see [6, §3]: A loop $x \in H^{1/2}$ decomposes as $x_- + x_0 + x_+$ according to its negative, zero, and positive Fourier modes. In terms of the norm on $H^{1/2}$, the action functional has the form

$$\mathbb{A}_H(x) \;=\; \|x_+\|^2 - \|x_-\|^2 + \int_{\mathbb{S}^1} H(x(t))\,dt.$$

This is the sum of a quadratic form $\mathbb{A}_0(x) = \|x_+\|^2 - \|x_-\|^2$ and the term $\int_{\mathbb{S}^1} H(x(t))\,dt$ whose gradient for "reasonable" functions H is a compact operator. Imitating the above construction, and inspired by the proof of the degenerate Arnol'd conjecture in [6, §6.4], we therefore look for any compact symplectically aspherical manifold (M,ω) at perturbations H^s of H, namely at s-dependent Hamiltonians in $C^\infty(\mathbb{R}\times\mathbb{S}^1\times M,\mathbb{R})$ such that $H^s = H$ for s small and $H^s = 0$ for s large, and following Floer's interpretation of the L^2-gradient flow of the action functional, we consider the space $\mathscr{U}(H^s)$ of solutions $u \in C^\infty(\mathbb{R}\times\mathbb{S}^1, M)$ of Floer's equation

$$\partial_s u + J(u)\big(\partial_t u - X_{H^s}(t,u)\big) = 0 \tag{1.2}$$

that have finite energy $E(u) = \int_{\mathbb{R}\times\mathbb{S}^1} |\partial_s u|_J^2 < \infty$. Here, J is a fixed ω-compatible almost complex structure on TM. The space $\mathscr{U}(H^s)$ is C^∞_{loc}-compact by Gromov's compactness theorem. Now define the function

$$a_H^- \colon \mathscr{U}(H^s) \to \mathbb{R}, \qquad a_H^-(u) := \lim_{s\to-\infty} \mathbb{A}_H(u(s))$$

and finally define the action selector of H by

$$\boxed{A_{0,J}(H) \;:=\; \sup_{H^s}\ \min_{u\in\mathscr{U}(H^s)} a_H^-(u)}$$

where the supremum is taken over all deformations H^s of H as above. The number $A_{0,J}(H)$ is the smallest essential action of H in the following sense: It is the lowest critical value c of $\mathbb{A}_H$ (that is, the lowest action of a contractible 1-periodic orbit of H) such that for every deformation H^s of H there exists a finite energy solution of Floer's equation for H^s and J that starts at a critical level $\le c$. For another characterization of $A_{0,J}(H)$ see §4.3.

In our finite dimensional model, we could have allowed for stronger deformations of the gradient flow of q_h, by looking at families h^s that for s large are equal to a fixed function h^+ instead of 0, and by taking the gradient with respect to any family g_s of Riemannian metrics that depend on s on a compact interval. In the symplectic setting, the role of Riemannian metrics is played by ω-compatible almost complex structures. We will thus modify the above definition by looking at families H^s with $H^- = H$ and H^+ an arbitrary fixed function on $\mathbb{S}^1 \times M$, and at families J^s of ω-compatible almost complex structures that depend on s on a compact interval. This has the advantage that $A(H)$ is manifestly independent of the choice of J. It will be clear from the analysis of $A(H)$ that $A_{0,J}(H)$ is also an action selector, cf. Section 4.5.1.

Chapter 2

Basic notions

In this chapter we introduce all concepts which are necessary to understand the following chapters. The reason for some definitions may not be clear in this chapter, but will become clear afterwards.

2.1 Symplectically Aspherical

Definition 2.1.1. *A symplectic manifold* (M, ω) *is symplectically aspherical if it satisfies the following condition:*

$$[\omega]|_{\pi_2(M)} = 0.$$

Remark 2.1.2. *Let us explain the above definition in more detail: Since* ω *is a closed element of* $\Gamma(M, \Lambda^2 T^* M)$*, we can see* $[\omega]$ *as an element in the de Rham cohomology:* $[\omega] \in H^2_{dR}(M)$*. Consider now the homomorphism*

$$[\omega] : H_2(M) \to \mathbb{R}, \ \ [c] \mapsto \int_c \omega.$$

This is a well-defined homomorphism by Stokes theorem. Denote by $h_2 : \pi_2(M) \to H_2(M)$ *the Hurewicz homomorphism. In fact,* (M, ω) *being symplectically aspherical means that*

$$[\omega]|_{h_2(\pi_2(M))} = 0,$$

or in other words, for any sphere u *in the image of* $\pi_2(M)$ *under the Hurewicz homomorphism the following holds true:*

$$0 = \int_u \omega = \int_{\mathbb{S}^2} u^* \omega.$$

A symplectic manifold satisfying this definition contains therefore no 2-spheres with non-zero "symplectic area". The reason why we require (M, ω) *to be symplectically aspherical will become clear soon.*

2.2 Almost Complex Structures

Definition 2.2.1. *Let M be a smooth manifold. A smooth vector bundle isomorphism*

$$J : TM \to TM$$

is called an almost complex structure if it satisfies $J^2 = -1$.

Therefore, an almost complex structure equips every tangent space of M with a linear complex structure. It is a necessary condition that M is even-dimensional for an almost complex structure to exist. Since we only deal with symplectic manifolds, this condition is always satisfied.

Definition 2.2.2. *Let $J : TM \to TM$ be an almost complex structure on a symplectic manifold (M, ω). J is called ω-compatible if*

$$g_J(\xi, \eta) := \omega(J\xi, \eta), \quad \forall \xi, \eta \in T_xM, \forall x \in M,$$

defines a Riemannian metric on M. The associated norm is denoted by $|\cdot|_J$.

The following classical result can be found in [10].

Proposition 2.2.3. *Let (M, ω) be a symplectic manifold. The following holds true:*

1. *There exists an ω-compatible almost complex structure J.*
2. *The space $\mathcal{J}(M, \omega)$ of ω-compatible almost complex structures is contractible.*

2.3 Hamiltonion Vector Field and Action Functional

Let (M, ω) be a closed symplectic manifold.

Definition 2.3.1. *Set $\mathscr{H}(M) := C^\infty(\mathbb{S}^1 \times M, \mathbb{R})$. We then call a function $H \in \mathscr{H}(M)$ a Hamiltonian function, or a Hamiltonian for short.*

A Hamiltonian is not necessarily time-dependent. We consider a smooth map $H \in C^\infty(M, \mathbb{R})$ as a Hamiltonian which is constant in the first factor. To any Hamiltonian function we can assign a smooth vector field X_H by the requirement

$$\omega(X_H, \cdot) = dH(\cdot).$$

Definition 2.3.2. *X_H is called the Hamiltonian vector field associated to the Hamiltonian H. This vector field induces a flow φ_H^t and we will denote the set of all time-1 flows of this form by $\mathcal{D}$,*

$$\mathcal{D} := \{\varphi_H^1 \mid H \in C^\infty(\mathbb{S}^1 \times M, \mathbb{R})\}.$$

The Hamiltonian action functional associated to a Hamiltonian $H \in C^\infty(\mathbb{S}^1 \times M)$ is the following map on the space of contractible loops:

$$\mathbb{A}_H : C^\infty_{\text{contr}}(\mathbb{S}^1, M) \to \mathbb{R},$$

$$\mathbb{A}_H := \int_{\mathbb{D}^2} \overline{x}^*(\omega) + \int_{S^1} H\left(t, x(t)\right)\, dt.$$

Here $\mathbb{D}^2$ denotes the 2-dimensional disk and $\overline{x}$ is a map $\overline{x} \in C^\infty(\mathbb{D}^2, M)$ extending the loop x, meaning $\overline{x}|_{\partial\mathbb{D}^2} = x$. Note that such an $\overline{x}$ exists since x is contractible.

Remark 2.3.3. *1. On first sight this action functional may seem strange to the reader but it is a quite natural construction. It plays the role of the Morse function in Morse theory. We are interested in* 1*-periodic contractible orbits of* X_H*. So the action functional should satisfy* $(d\mathbb{A}_H)_x = 0$ *if and only if* x *is of the desired form. This is exactly the case, since*

$$(d\mathbb{A}_H)_x(Y) = \int_0^1 \omega(\dot{x}(t) - X_H(x(t)), Y)\, dt,$$

where Y *is tangent to* x*, i.e.* $Y(t) \in T_{x(t)}M$ *for* $t \in \mathbb{S}^1$*.*

2. Now it becomes clear why (M, ω) *has to be symplectically aspherical. For* $\mathbb{A}_H$ *to be well-defined we need it to be independent of the choice of the extension* $\overline{x}$ *of* x*. If* $\tilde{x}$ *is another extension, the two extensions form a* 2*-sphere in* M *because they can be glued together along* x*. The symplectic area of this sphere is zero and therefore*

$$\int_{\mathbb{D}^2} \overline{x}^*(\omega) = \int_{\mathbb{D}^2} \tilde{x}^*(\omega).$$

This proves that the Hamiltonian action functional is well-defined.

2.4 The Action Spectrum

Definition 2.4.1. *Let* (M, ω) *be a closed symplectically aspherical manifold,* J *an* ω*-compatible almost complex structure and* H *a Hamiltonian. The set of critical points of the action functional* $\mathbb{A}_H$ *is denoted by* $\mathcal{P}(H)$*.*

Proposition 2.4.2. $\mathcal{P}(H)$ *is a compact set.*

Proof. As already mentioned above, the elements of $\mathcal{P}(H)$ are contractible maps $x : \mathbb{S}^1 \to M$ satisfying $\dot{x}(t) = X_H(x(t))$. Denote by φ^t_H the flow of the Hamiltonian vector field X_H. Since the diagonal $\triangle \subset M \times M$ is compact, also its preimage $A := f^{-1}(\triangle)$ under the map

$$f := (id, \varphi^1_H) : M \to M \times M$$

is closed and therefore compact in M. Since $\mathcal{P}(H)$ is the image of A under the continuous map

$$\begin{aligned} &M \to C^\infty(I, M), \\ &x \mapsto (t \mapsto \varphi_H^t(x)) \end{aligned}$$

the claim follows. □

Definition 2.4.3. *Denote the set of critical values of* $\mathbb{A}_H$ *by* spec (H)*:*

$$\operatorname{spec}(H) = \{\mathbb{A}_H(x) \mid x \in \mathcal{P}(H)\}.$$

The idea behind an action selector is to choose for every Hamiltonian H an element from spec (H). However, it is a priori not clear how to do this in a "good" way (whatever "good" may mean), since the spectrum does not change continuously if we vary H (see Section 1.1).

2.5 Connecting Orbits

The space $C^\infty(\mathbb{R}\times\mathbb{S}^1, M)$ is endowed with the C^∞_{loc}-topology (i.e. $u_n \to u$ if u_n and all its derivatives converge uniformly on all compact sets $[-T,T]\times\mathbb{S}^1, T\in\mathbb{R}$), which is metrizable and complete. We shall identify $C^\infty(\mathbb{R}\times\mathbb{S}^1, M)$ with $C^\infty(\mathbb{R}, C^\infty(\mathbb{S}^1, M))$, and we use the notation

$$u(s) = u(s,\cdot) \in C^\infty(\mathbb{S}^1, M), \qquad \forall s \in \mathbb{R}.$$

The additive group $\mathbb{R}$ acts on $\mathbb{C}^\infty(\mathbb{R}\times\mathbb{S}^1, M)$ by translations

$$(\sigma, u) \mapsto \tau_\sigma u, \qquad \text{where } (\tau_\sigma u)(s) := u(\sigma + s).$$

As in Morse theory, where one is interested in flow lines connecting two critical points of the Morse function, we are interested in Floer "sausages": Flow lines $u : \mathbb{S}^1 \times \mathbb{R} \to M$ to the following L^2-negative gradient equation for the action functional $\mathbb{A}_H$.

$$\partial_s u + J(u)\big(\partial_t u - X_H(t,u)\big) = 0. \tag{2.1}$$

We will call this equation the Floer equation. If u is a solution of (2.1), then the function $s \mapsto \mathbb{A}_H(u(s,\cdot))$ is non-increasing (see Lemma 3.1.2 in the next chapter) and

$$\lim_{s\to-\infty} \mathbb{A}_H(u(s,\cdot)) - \lim_{s\to+\infty} \mathbb{A}_H(u(s,\cdot)) = E(u) := \int_{\mathbb{R}\times\mathbb{S}^1} \left|\partial_s u\right|_J^2 ds\, dt$$

is called the energy of u. The Gromov compactness theorem states that the space of all solutions of (2.1) with finite energy is a compact and non-empty subset of $C^\infty(\mathbb{R}\times\mathbb{S}^1, M)$. However we are going to change the Floer equation slightly and therefore generalize the proof in the next chapter.

2.6 Hofer norm and displacement energy

Definition 2.6.1. *Let $H \in C^\infty(\mathbb{S}^1 \times M, \mathbb{R})$ be a Hamiltonian function. We then abbreviate*

$$E^+(H) := \int_{S^1} \max_{x \in M} H(t,x)\, dt \quad \textit{and} \quad E^-(H) := \int_{S^1} \min_{x \in M} H(t,x)\, dt.$$

Proposition 2.6.2. *The map*

$$\| \cdot \| : C^\infty(\mathbb{S}^1 \times M, \mathbb{R}) \to \mathbb{R}, \quad H \mapsto \|H\| := E^+(H) - E^-(H)$$

defines a norm on $C^\infty(\mathbb{S}^1 \times M, \mathbb{R})$. This norm is called the Hofer norm.

Definition 2.6.3. *Let $\varphi \in \mathcal{D}$ be a symplectic diffeomorphism. We then denote by*

$$E(\varphi) := \inf\{\|H\| \ \mid \varphi_H^1 = \varphi\}$$

its energy. Furthermore for a compact subset K of M we define its displacement energy by

$$e(K,M) := \inf\{E(\varphi) \,|\, \varphi \in \mathcal{D},\, \varphi(K) \cap K = \emptyset\}.$$

The concept of Hofer energy $E(\varphi)$ is remarkable since $E(\varphi) > 0$ for $\varphi \neq \mathrm{id}$ and so $d_H(\varphi, \psi) := E(\varphi \circ \psi^{-1})$ defines a metric on the group $\mathcal{D}$ of Hamiltonian diffeomorphisms.

2.7 Symplectic Contraction

We introduce now a technique which was used by Polterovich in [15]. To do so we need first two definitions.

Definition 2.7.1 (Liouville domain and Liouville vecor field)**.** *A Liouville domain $U \subset M$ is an open connected subset with smooth boundary such that the closure $\overline{U}$ admits a vector field Y (called Liouville vector field) that is transverse to the boundary ∂U and is symplectically conformal: $\mathscr{L}_Y \omega = \omega$.*

This definition is equivalent to the fact that $\omega = d\lambda$ is exact on $\overline{U}$, and $\lambda|_{\partial U}$ is a contact form (Y and λ are obtained from each other by $\lambda(\cdot) = \omega(Y, \cdot)$). A Liouville domain is thus a convex domain for which Y exists on all of U, not just near ∂U. Examples of Liouville domains are symplectic images in (M, ω) of starshaped domains in $\mathbb{R}^{2n}$ or of fiberwise starshaped neighborhoods of the zero section of a cotangent bundle T^*Q.

Definition 2.7.2. *A set $U \subset M$ is called incompressible if the map*

$$i_* : \pi_1(U) \to \pi_1(M)$$

induced by the inclusion $i : \mathcal{U} \to M$ is injective.

Let $U \subset M$ be an incompressible Liouville domain. Let Y be a Liouville vector field associated to $U \subset M$ and Ψ_s its flow. Note that Y points outwards along ∂U and therefore Ψ_s exists for all times $s \leq 0$. Furthermore let $H : M \times \mathbb{S}^1 \to \mathbb{R}$ be a Hamiltonian with support in $U \times \mathbb{S}^1$. Now we compute:

$$\frac{d}{ds} (\Psi_s)^* \omega = (\Psi_s)^* \mathscr{L}_X \omega = (\Psi_s)^* \omega$$

and $\Psi_0 = \mathrm{id}$,

$$(\Psi_s)^* \omega = e^s \omega.$$

The idea is thus to deform the Hamiltonian H by using Ψ_s. To formalize this idea we define for $s \leq 0$,

$$H_s(t,x) := \begin{cases} e^s H(t, \Psi_s^{-1}(x)) & \text{if } x \in \Psi_s(U), \\ 0 & \text{if } x \notin \Psi_s(U). \end{cases}$$

Hence we obtain the following flow for the Hamiltonian H_s:

$$\Phi^t_{H_s}(x) := \begin{cases} \Psi_s \varphi^t_H \Psi_s^{-1}(x) & \text{if } x \in \Psi_s(U), \\ x & \text{if } x \notin \Psi_s(U). \end{cases}$$

Proposition 2.7.3. *There is a bijection between the* 1*-periodic contractible orbits of* H *and* H_s*.*

Proof. Let x be a 1-periodic orbit of H, then $x_s(t) := \Psi_s(x(t))$ is a 1-periodic orbit of H_s. □

Here is now the reason why we need U to be incompressible: There exists a disc $\mathbb{D}^2$ in U (not just in M) such that $\partial \mathbb{D}^2 = x$ and therefore $\Psi_s(\mathbb{D}^2)$ is a disk with boundary $\Psi_s(x)$. We can thus deduce the following:

Proposition 2.7.4. $\mathbb{A}_{H_s}(x_s) = e^s \mathbb{A}_H(x)$.

For the proof, we compute

$$\begin{aligned} \mathbb{A}_{H_s}(x_s) &= \int_0^1 e^s H(t, \Psi_s^{-1}(x_s(t)))dt - \int_{\Psi_s(\mathbb{D}^2)} \omega \\ &= \int_0^1 e^s H(t, x(t))dt - \int_{\mathbb{D}^2} e^s \omega \\ &= e^s \mathbb{A}_H(x). \end{aligned}$$

This technique is called symplectic contraction and can be very usefull to actually calculate the value of an action selector of a given Hamiltonian with support in a incompressible Liouville domain. We will make use of it in Chapter 8.

Chapter 3

Gromov compactness

In this chapter we modify some of the well known objects used in Floer homology which we introduced in Chapter 2 in order to use them in the next chapter. The first important difference to the usual theory is that we will consider perturbations of a given Hamiltonian H. Such a perturbation will not only depend on $t \in \mathbb{S}^1$ but also on $s \in \mathbb{R}$. In consistency with the variational approach to construct an action selector, we will also consider s-dependent families $\{J^s\}$ of ω-compatible almost complex structures. After giving the definitions, the main point of this chapter will be to prove that all of the most important facts such as Gromov compactness still hold in this more general setting.

3.1 Closedness of $\mathcal{U}(K, J)$

Definition 3.1.1.

1. *We will denote by $\mathscr{K}(M)$ the set of functions $K \in C^\infty(\mathbb{R} \times \mathbb{S}^1 \times M)$ such that $\partial_s K$ has compact support. Furthermore, for $s \in \mathbb{R}$ large enough, we denote $K(\pm s, ., .)$ by $K^\pm$ respectively.*

2. *By $\mathscr{J}_\omega(M)$ we denote the set of smooth families $J = \{J^s\}$ of ω-compatible almost complex structures on M such that $\partial_s J$ has compact support.*

3. *Let $\mathscr{K}(H)$ be the subset of those $K \in \mathscr{K}(M)$ for which $K^- = H$, and abbreviate $\mathscr{D}(H) = \mathscr{K}(H) \times \mathscr{J}_\omega(M)$.*

4. *Finally, for $(K, J) \in \mathscr{D}(H)$ let $\mathscr{U}(K, J)$ be the space of finite energy solutions of Floer's equation*

$$\partial_s u + J^s(u)\big(\partial_t u - X_H(s, t, u)\big) = 0 \tag{3.1}$$

defined by K and J. Recall Section 2.5 *for the definition of energy.*

Of course this now changes all notions previously defined in Chapter 2. Also equation (3.1) is no longer Floer's "classical" equation since the almost complex structure and the Hamiltonian vector field depend on $s \in \mathbb{R}$.

We recall that a subset $\mathscr{U}$ of $C^\infty(\mathbb{R} \times \mathbb{S}^1, M)$ is said to be bounded if for every multi-index $\alpha \in \mathbb{N}^2$, $|\alpha| \geq 1$, it holds that

$$\sup_{u \in \mathscr{U}} |\partial_s^{\alpha_1} \partial_t^{\alpha_2} u|_J^s < \infty.$$

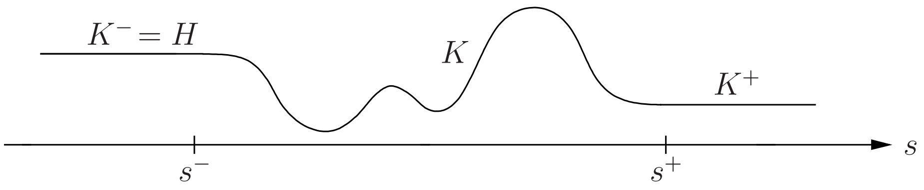


Figure 3.1: A function $K \in \mathscr{K}(H)$ deforming H, for (t, x) fixed

Bounded subsets are relatively compact in the C^∞_{loc}-topology. Therefore, to ultimatively prove that $\mathcal{U}(K, J)$ is a compact subset of $C^\infty(\mathbb{R} \times \mathbb{S}^1, M)$ in the C^∞_{loc}-topology, we show in this section that $\mathcal{U}(K, J)$ is bounded. First we can assume by the Whitney embedding theorem that M is embedded into $\mathbb{R}^N$ for some $N \in \mathbb{N}$ and therefore we can view $C^\infty(\mathbb{R} \times \mathbb{S}^1, M)$ as a subspace of $C^\infty(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^N)$. Since $\partial_s H$ has compact support, there exists $s_0 \in \mathbb{R}^+$ with $\partial_s H(s, ., .) = 0$ for $s \in (-\infty, -s_0] \cup [s_0, \infty)$ and we define

$$H^-(t, x) := H(-s, t, x) \text{ and } H^+(t, x) := H(s, t, x) \text{ for } s \geq s_0.$$

In addition, we take again $J = \{J^s\}$ to be an s-dependent family of ω-compatible almost complex structures such that $\partial_s J$ has compact support, and we set

$$J^-(x) := J_{-s}(x) \quad \text{and} \quad J^+(x) := J^s(x) \quad \text{for } s \text{ large.}$$

Lemma 3.1.2. *For $u \in \mathcal{U}(H, J)$ the following two identities hold:*

$$\begin{aligned}
\frac{d}{ds}\mathbb{A}_{H(s,.,.)}(u(s)) &= \int_{\mathbb{S}^1} -g_s(\partial_t u - X_{H(s)}, \partial_t u - X_{H(s)})\, dt + \int_0^1 \partial_s H(s, t, u(s, t))\, dt \\
&= \int_{\mathbb{S}^1} -|\partial_s u|^2_{J^s}\, dt + \int_0^1 \partial_s H(s, t, u(s, t))\, dt
\end{aligned}$$

Proof. The first identity follows from the computation

$$\begin{aligned}
\frac{d}{ds}\mathbb{A}_{H(s,.,.)}(u(s)) &= \frac{d}{ds}\Big(\int_{\mathbb{D}^2} \overline{u(s)}^*\omega + \int_0^1 H(s, t, u(s, t))dt\Big) \\
&= \int_{\mathbb{S}^1} \omega(\partial_s u, \partial_t u)\, dt + \int_0^1 dH_s(\partial_s u)\, dt + \int_0^1 \partial_s H(s, t, u(s, t))\, dt \\
&= \int_{\mathbb{S}^1} \omega(\partial_s u, \partial_t u)\, dt + \int_0^1 \omega(X_{H(s)}, \partial_s u)\, dt + \int_0^1 \partial_s H(s, t, u(s, t))\, dt \\
&= \int_{\mathbb{S}^1} \omega(\partial_s u, \partial_t u - X_{H(s)})\, dt + \int_0^1 \partial_s H(s, t, u(s, t))\, dt \\
&\overset{*}{=} \int_{\mathbb{S}^1} \omega(-J^s \partial_t u + J^s X_{H(s)}, \partial_t u - X_{H(s)})\, dt + \int_0^1 \partial_s H(s, t, u(s, t))\, dt \\
&= \int_{\mathbb{S}^1} -g_s(\partial_t u - X_{H(s)}, \partial_t u - X_{H(s)})\, dt + \int_0^1 \partial_s H(s, t, u(s, t))\, dt,
\end{aligned}$$

where at $*$ we used that u satisfies the Floer equation (3.1). The second identity follows easily from the first in view of Floer's equation. ☐

Proposition 3.1.3. *There exists a constant* $C > 0$ *such that for every* $u \in \mathcal{U}(H,J)$ *the following holds:*

$$\mathcal{A}_{H(s,.,.)}(u(s,.)) \in [-C, C] \text{ for all } s \in \mathbb{R}.$$

Proof. Let $u \in \mathcal{U}(H,J)$ be arbitrary. Recall that $\partial_s H$ and $\partial_s J$ are compactly supported. Using this and the fact that the energy of u is bounded, we obtain from Lemma 3.1.2 that for s big enough there exists $c > 0$ such that

$$\int_s^\infty \left| \frac{d}{ds} \mathbb{A}_{H^+}(u(s)) \right| ds < c_1.$$

This implies that there exists a sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \to \infty$ and a real number a such that

$$\mathbb{A}_{H^+}(u(s_n)) \to a \ \text{ and } \ \frac{d}{ds}\mathbb{A}_{H^+}(u(s_n)) \to 0.$$

By fixing the s-variable of the Floer-cylinder u we define a sequence of circles

$$x_n(\cdot) := u(s_n, \cdot) \in C^\infty(\mathbb{S}^1, M).$$

Since $\dot{x}_n(t) = \partial_t u(s_n, t)$ by definition, the first identity of Lemma 3.1.2 implies

$$|\dot{x}_n(t) - X_{H^+}(x_n(t))|_{J^+, L^2} \to 0. \tag{3.2}$$

Here we did not take into consideration the s-dependency of H and J, since for s_n large enough we leave the compact support of $\partial_s H$ and $\partial_s J$. For convenience we will now consider M to be embedded in $\mathbb{R}^N$ for some $N \in \mathbb{N}$. Since M is compact, $\|X_{H^+}\|_{L^2(\mathbb{S}^1,\mathbb{R}^N)}$ is bounded and consequently there exists $c_2 > 0$ such that $\|\dot{x}_n\|_{L^2(\mathbb{S}^1,\mathbb{R}^N)} \le c_2$ for n large enough. Here we used the standard L^2-inner product. For $s, t \in [0,1]$ we can therefore estimate, with Cauchy–Schwarz,

$$|x_n(t) - x_n(s)| = |\int_s^t \dot{x}_n(\sigma) d\sigma| \le \sqrt{|t-s|} \int_s^t |\dot{x}_n(\sigma)|^2 d\sigma \le \sqrt{|t-s|} c_2^2,$$

uniformly in n. The family $\{x_n\}$ is therefore equi-continuous. Hence the Arzelà–Ascoli theorem implies the existence of a subsequence of (s_n) and of a function $x \in C^0(\mathbb{S}^1, M)$ satisfying

$$x_n \to x \ \text{ uniformly in } \ C^0(\mathbb{S}^1, M).$$

This implies $\mathbb{A}_H(x_n) \to \mathbb{A}_{H^+}(x)$ and hence $\mathbb{A}_{H^+}(x) = a$.

Lemma 3.1.4. *The limit x satisfies $x(t) - x(0) = \int_0^t X_{H^+}(x(\tau))d\tau$.*

Proof. Since we consider M to be embedded into $\mathbb{R}^N$, which is a vector space, the lemma follows from a calculation:

$$\begin{aligned} x(t) - x(0) - \int_0^t X_H(x(\tau))d\tau &= \lim_{n\to\infty}\left(\int_0^t \dot{x}_n(\tau)d\tau - \int_0^t X_H(x(\tau))d\tau\right) \\ &= \lim_{n\to\infty}\int_0^t \left(\dot{x}_n(\tau) - X_H(x_n(\tau))\right)d\tau \\ &+ \lim_{n\to\infty}\int_0^t \left(X_H(x_n(\tau)) - X_H(x(\tau))\right)d\tau, \end{aligned}$$

and both terms tend to zero (use (3.2) and Cauchy–Schwarz for the first term). □

By a bootstrapping argument, the lemma implies $x \in C^\infty(S^1, M)$ and also $\dot{x}(t) = X_{H^+}(x(t))$. Hence $\mathbb{A}_{H^+}(x) = a \in \operatorname{spec}(H^+)$. Recall from Section 2.4 that $\operatorname{spec}(H^+)$ is a compact set. Of course we can argue the same way for sequences $x_n = u(s_n)$, where $s_n \to -\infty$ and obtain the same result. Therefore we find a constant C_1 such that

$$C_1 \geq \mathbb{A}_{H^-}(u(-\infty)) \text{ and } -C_1 \leq \mathbb{A}_{H^+}(u(\infty)),$$

for all $u \in \mathcal{U}(H)$. From the first equation of Lemma 3.1.2 it is clear that $\mathcal{A}_{H(s,.)}(u(s,.))$ is not necessarily decreasing on $[-s_0, s_0]$. But it can at most increase by

$$\begin{aligned} &\int_{[-s_0,s_0]\times\mathbb{S}^1} \partial_s H(s,t,u(s,t))\,dt\,ds \\ &\leq \left(\max_{(s,t,p)\in[-s_0,s_0]\times\mathbb{S}^1\times M} \partial_s H(s,t,p)\right)\int_{[-s_0,s_0]\times\mathbb{S}^1} 1\,dt\,ds =: C_2. \end{aligned}$$

Since C_2 does not depend on u, we define $C := C_1 + C_2$ and it follows that

$$\mathbb{A}_{H(s,.,.)}(u(s)) \in [-C, C].$$

□

Lemma 3.1.5. *Let $u \in C^\infty(\mathbb{R}\times\mathbb{S}^1, M)$ be a solution of* (3.1) *and take $s_0 < s_1 \in \mathbb{R}$. Then*

$$\begin{aligned} \mathbb{A}_{H(s_0,\cdot,\cdot)}(u(s_0)) - \mathbb{A}_{H(s_1,\cdot,\cdot)}(u(s_1)) &= \int_{[s_0,s_1]\times\mathbb{S}^1} |\partial_s u|^2_{J^s}\,ds\,dt \\ &- \int_{[s_0,s_1]\times\mathbb{S}^1} \partial_s H(s,t,u(s,t))\,ds\,dt \end{aligned}$$

holds. Especially for $u \in \mathcal{U}(H)$:

$$\lim_{s_0\to-\infty}\mathbb{A}_{H(s_0,\cdot,\cdot)}(u(s_0)) - \lim_{s_1\to\infty}\mathbb{A}_{H(s_1,\cdot,\cdot)}(u(s_1)) = E(u) - \int_{\mathbb{R}\times\mathbb{S}^1} \partial_s H(s,t,u(s,t))\,ds\,dt.$$

Proof. This is a direct consequence of Lemma 3.1.2. □

Lemma 3.1.5 will be useful in the following proposition. But it gives also insight into the behavior of connecting orbits $u \in \mathcal{U}(H, J)$: In the s-independent case the energy of such orbits can only decrease. Now it is also possible that the energy of u increases! However, since the s-dependence of the Hamiltonian is compactly supported, it cannot diverge.

Proposition 3.1.6. *$\mathcal{U}(H, J)$ is closed in $C^\infty(\mathbb{R} \times \mathbb{S}^1, M)$.*

Proof. Take any sequence $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}(H, J)$. Assume $u_n \to u$ in the C^∞_{loc}-topology. We need to verify that u has finite energy and satisfies (3.1). Since (3.1) holds for every u_n this is also true for the limit u. We know by Proposition 3.1.3 for every $r \in \mathbb{R}$ that

$$\mathbb{A}_{H(r,.,.)}(u_n(r)) \in [-C, C].$$

By taking the limit $n \to \infty$ we find $\mathbb{A}_{H(r,.,.)}(u(r)) \in [-C, C]$ for all $r \in \mathbb{R}$. Therefore, by using Lemma 3.1.5 and taking the limit, we see now that $E(u)$ is finite. □

3.2 Compactness of $\mathcal{U}(K, J)$

We will use that for the metric space $C^\infty(\mathbb{S}^1 \times \mathbb{R}, M)$, the properties compact and sequentially compact are equivalent. The following lemma corresponds to Lemma 6, page 238 in [6]. The differences are that X_H depends now not only on $t \in \mathbb{S}^1$, but also on $s \in [-s_0, s_0]$ and that the almost complex structured J is not fixed but also s-dependent. The proof of this lemma will also be of importance in the next section, which is why we will carry it out completely, even though there are only minor adjustments to the proof in [6] needed.

Lemma 3.2.1. *Let $(s_k, t_k) \subset \mathbb{R}^2$ be a sequence and $D_k := \{(s, t) \in \mathbb{R}^2 \mid |(s, t)| \leq A_k\} \subset \mathbb{R}^2$, where $A_k \to \infty$. Furthermore let $(v_k) \subset C^\infty(\mathbb{R}^2, M)$ have the following properties:*

$$\begin{aligned} &|\nabla v_k(s,t)|_{J^s} \leq 2 \text{ for } (s,t) \in D_k, \\ &|\nabla v_k(0,0)|_{J^0} = 1. \end{aligned}$$

Moreover we suppose that v_k satisfies the following partial differential equation

$$\frac{\partial}{\partial s} v_k(s,t) + J^s(v_k(s,t)) \frac{\partial}{\partial t} v_k(s,t) = \frac{1}{R_k} J^s(v_k(s,t)) X_H(s_k + \frac{s}{R_k}, t_k + \frac{t}{R_k}, v_k),$$

where R_k is a sequence satisfying $R_k \to \infty$. Then there exists $v \in C^\infty(\mathbb{R}^2, M)$ and a subsequence of (v_k) such that

$$v_k \to v \text{ in } C^\infty_{loc}(\mathbb{R}^2, M).$$

Proof. Let $\tilde{x} = (s, t, p) \in \mathbb{R}^2 \times M$ and $(a, b, \xi) \in T_{\tilde{x}}(\mathbb{R}^2 \times M)$. Then we define

$$\begin{aligned}&J_k : T(\mathbb{R}^2 \times M) \to T(\mathbb{R}^2 \times M)\\ &J_k(\tilde{x})(a, b, \xi) := (-b, a, J^s(p)\xi + aX_{H,k}(s, t, p) - bJ^s(p)X_{H,k}(s, t, p))\end{aligned}$$

where $X_{H,k}(s, t, p) := \frac{1}{R_k} X_H(s_k + \frac{s}{R_k}, t_k + \frac{t}{R_k}, p)$. An easy computation shows that $J_k^2 = -\mathbb{1}$. Hence we have a sequence J_k of almost complex structures. Since $\mathbb{S}^1 \times M$ is compact and $\partial_s H = 0$ outside $[-s_0, s_0]$, the sequence of vector fields $(X_H(s_k + \frac{s}{R_k}, t_k + \frac{t}{R_k}, p))$ is uniformly bounded. Since $R_k \to \infty$ for $k \to \infty$, we find that $X_{H,k}(s, t, p)$ converges uniformly to 0 and therefore $J_k \to i \oplus J^s$ for $k \to \infty$. Now we define the sequence $\tilde{v}_k \in C^\infty(\mathbb{R}^2, \mathbb{R}^2 \times M)$ by

$$\tilde{v}_k(s, t) := (s, t, v_k(s, t)).$$

Then on $D_k \subset \mathbb{R}^2$,

$$\begin{aligned}\frac{\partial \tilde{v}_k}{\partial s} + J_k(\tilde{v}_k)\frac{\partial \tilde{v}_k}{\partial t} &= 0\\ |\nabla \tilde{v}_k(s, t)| &\leq 4 \text{ for} (s, t) \in D_k\\ |\nabla \tilde{v}_k(0, 0)| &\geq 1\end{aligned}$$

By this trick we got rid of the Hamiltonian term, but unlike in the original proof in [6] J_k depends now on s not only because of the $X_{H,k}(s, t, p)$ used in its definition but also because of the s-dependence of J. Our aim is to use the Arzela–Ascoli Theorem to find a subsequence such that $\tilde{v}_k \to \tilde{v}$ in $C^\infty(\mathbb{R}^2, \mathbb{R}^2 \times M)$ for a map $\tilde{v}$ in $C^\infty(\mathbb{R}^2, \mathbb{R}^2 \times M)$. For this it is sufficient to establish C^∞_{loc}-bounds for $\tilde{v}_k$. Fix now $p > 2$. By Sobolev's embedding theorem we only need for every compact subset $K \subset \mathbb{R}^2$ and every $l \in \mathbb{N}$ uniform $W^{l,p}(K)$-bounds instead of $C^l(K)$-bounds. We will give a proof by induction. The initial step, the $W^{1,p}(K)$-bounds, follows directly from $|\nabla \tilde{v}_k(s, t)| \leq 4$. Let us now assume that there exists $l \geq 1$ such that there are $W^{l,p}_{\text{loc}}$-bounds but no $W^{l+1,p}_{\text{loc}}$-bounds. Then we find sequences $(x_k) \subset \mathbb{R}^2$ and $\varepsilon_k \to 0$ satisfying

$$x_k \to x_0 \text{ and } |\tilde{v}_k|_{W^{l+1,p}(B_{\varepsilon_k}(x_k))} \to \infty,$$

where we view $M \subset \mathbb{R}^N$ for some $N \in \mathbb{N}$. Using again the assumption $|\nabla \tilde{v}_k(s, t)| \leq 4$ and applying the Arzela–Ascoli Theorem, we obtain for a subsequence of $\tilde{v}_k$:

$$\tilde{v}_k(x_0) \to (x_0, m) \text{ and } \tilde{v}_k \to \tilde{v} \text{ in } C^0_{\text{loc}}.$$

Now we are able to localize this problem around (x_0, m). We take a coordinate chart $U \subset \mathbb{R}^2 \times M$ and a coordinate map $\alpha : U \to \mathbb{R}^{2+2n}$ such that $\alpha(x_0, m) = 0$ and $\tilde{v}_k(\overline{B_{\varepsilon_0}(x_k)}) \subset U$, for some ε_0 and all k large enough. Then we obtain a new sequence of functions $u_k := \alpha \circ \tilde{v}_k : \overline{B_{\varepsilon_0}(x_0)} \to \mathbb{R}^{2+2n}$ satisfying on $B_{\varepsilon_0}(x_0)$ the following:

$$\begin{aligned} \frac{\partial u_k}{\partial s} + \widehat{J}_k(u_k)\frac{\partial u_k}{\partial t} &= 0 \\ u_k(x_0) &\to 0 \\ |\nabla u_k(x)| &\leq C \\ |u_k|_{W^{l+1,p}(B_{\varepsilon_k(x_k)})} &\to \infty, \end{aligned}$$

with the new almost complex structure

$$\widehat{J}_k(y) := D\alpha(\alpha^{-1}(y)) \circ J_k^{\tilde{y}}(\alpha^{-1}(y)) \circ D\alpha^{-1}(y), \tag{3.3}$$

$y \in \alpha(U) \subset \mathbb{R}^{2+2n}$ and $\tilde{y} := \mathrm{pr}_1(\alpha^{-1}(y))$, the first component of $\alpha^{-1}(y) \in \mathbb{R}^2$. Here there is again a slight difference to the s-independent case. Because of the trick to get rid of the Hamiltonian term we were obliged to add two dimensions to M and consider $\mathbb{R}^2 \times M$, hence the almost complex structure became dependent on s. But now we "added" even more s-dependence as can be seen in (3.3). Next we will use the following fact for the Cauchy–Riemann operator. For every $1 < p < \infty$ and $n, m > 0$ there exists a positive constant C such that

$$C|u|_{m+1,p} \leq \left| \frac{\partial u}{\partial s} + J_0 \frac{\partial u}{\partial t} \right|_{m,p}, \tag{3.4}$$

for every smooth function $u : \mathbb{R}^2 \to \mathbb{R}^{2n+2}$ having compact support in the unit ball. Here J_0 is a constant almost complex structure. We will use this estimate with $\widehat{J}_0 := \lim_{k\to\infty} \widehat{J}_k(0,0)$. Since our functions u_k are not compactly supported in the unit ball, we have to adjust them by using a cutoff function. Let $\beta : \mathbb{R} \to [0,1]$ be a smooth function such that $\beta(s) = 1$ for $|s| \leq \frac{1}{2}$ and $\beta(s) = 0$ for $|s| \geq 1$. For $\lambda > 0$ we define

$$\alpha_\lambda(x) := \beta\left(\frac{x - x_0}{\lambda}\right).$$

Hence $\alpha_\lambda(x) = 0$ if $|x - x_0| \geq \lambda$ and therefore $\alpha_\lambda \cdot u_k$ is supported in a λ-ball around x_0. To simplify the following calculation we abbreviate

$$|u|_l := |u|_{W^{l,p}(B_{\varepsilon_0}(x_0))},$$

where $p > 2$ is fixed. First by (3.4) we get

$$C|\alpha_\lambda u_k|_{l+1} \leq |(\alpha_\lambda u_k)_s + \widehat{J}_0(\alpha_\lambda u_k)_t|_l.$$

Using the Cauchy–Riemann equation, $(u_k)_s = -\widehat{J}_k(u_k)(u_k)_t$ we obtain

$$\begin{aligned} C|\alpha_\lambda u_k|_{l+1} &\leq |(\alpha_\lambda u_k)_s + \widehat{J}_0(\alpha_\lambda u_k)_t|_l \\ &= |(\alpha_\lambda)_s u_k + \alpha_\lambda (u_k)_s + \widehat{J}_0(\alpha_\lambda u_k)_t|_l \\ &= |(\alpha_\lambda)_s u_k - \alpha_\lambda \widehat{J}_k(u_k)(u_k)_t + \widehat{J}_0(\alpha_\lambda u_k)_t|_l \\ &= |(\alpha_\lambda)_s u_k - \widehat{J}_k(u_k)(\alpha_\lambda u_k)_t + \widehat{J}_k(\alpha_\lambda)_t u_k + \widehat{J}_0(\alpha_\lambda u_k)_t|_l \\ &= |(\alpha_\lambda)_s u_k + \widehat{J}_k(\alpha_\lambda)_t u_k + \left(\widehat{J}_0 - \widehat{J}_k(u_k)\right)(\alpha_\lambda u_k)_t|_l \\ &\leq |(\alpha_\lambda)_s u_k + \widehat{J}_k(\alpha_\lambda)_t u_k|_l + \left|\left(\widehat{J}_0 - \widehat{J}_k(u_k)\right)(\alpha_\lambda u_k)_t\right|_l. \end{aligned}$$

Using now the inductive assumption $|u_k|_l \leq C_l$, we find

$$\begin{aligned} C|\alpha_\lambda u_k|_{l+1} &\leq C(\lambda) + \left|\left(\widehat{J}_0 - \widehat{J}_k(u_k)\right)(\alpha_\lambda u_k)_t\right|_l \\ &\leq C(\lambda) + \left|\left(\widehat{J}_0 - \widehat{J}_k(u_k)\right)(\alpha_\lambda u_k)\right|_{l+1}, \end{aligned}$$

where $C(\lambda)$ is a constant depending only on λ. We can simplify this to

$$C|\alpha_\lambda u_k|_{l+1} \leq C(\lambda) + |\widehat{J}_0 - \widehat{J}_k(u_k)|_{L^\infty(B_\lambda(x_0))} \cdot |\alpha_\lambda u_k|_{l+1}.$$

Here we wrote $L^\infty(B_\lambda(x_0))$ instead of $L^\infty(B_{\varepsilon_0}(x_0))$, using that the support of $(\alpha_\lambda u_k)$ is a subset of $B_\lambda(x_0)$. Recall that we want to show that $|u_k|_{W^{l+1,p}(B_{\varepsilon_k}(x_k))}$ is bounded. Since $u_k \to \alpha \circ \tilde{v}$ and therefore $u_k(x_0) \to 0$ for k large enough, the factor $|\widehat{J}_0 - \widehat{J}_k(u_k)|_{L^\infty(B_\lambda(x_0))}$ is smaller than $\frac{C}{2}$ for λ small enough. This is because, by definition, $\widehat{J}_0 := \lim_{k\to\infty} \widehat{J}_k(0,0)$ and for λ small, (s,t) becomes arbitrarily close to $(0,0)$, therefore the additional s-dependency of $\widehat{J}_k$ is not a problem. Hence we find C' such that

$$C'|\alpha_\lambda u_k|_{l+1} \leq C(\lambda).$$

Since $B_{\varepsilon_k}(x_k) \subset B_\lambda(x_0)$ for k large, we find that $\alpha_\lambda \equiv 1$ on $B_{\varepsilon_k}(x_k)$ and therefore

$$C'|u_k|_{l+1} \leq C(\lambda).$$

This contradicts the inductive assumption. □

The following proposition is the crucial step to obtain the compactness result for $\mathcal{U}(K,J) \subset C^\infty(\mathbb{R} \times \mathbb{S}^1, M)$. It corresponds to Theorem 8 on page 236 in [6].

Proposition 3.2.2. *There exists a constant* $A \geq 0$ *such that*

$$|\nabla u(s,t)|_{J^s} \leq A$$

for all $u \in \mathcal{U}(K,J)$ *and* $(s,t) \in \mathbb{R} \times \mathbb{S}^1$.

Proof. Let us suppose that we find sequences $(s_k,t_k) \in \mathbb{R} \times \mathbb{S}^1$ and $(u_k) \in \mathcal{U}(K,J)$ such that

$$|\nabla u_k(s_k,t_k)|_{J^{s_k}} \to \infty.$$

Then there exists a sequence (ε_k) satisfying

$$\varepsilon_k > 0,\ \varepsilon_k \to 0,\ \varepsilon_k|\nabla u_k(s_k,t_k)|_{J^{s_k}} \to \infty.$$

We wish this sequence to meet additional conditions and for this use a little trick due to Ekeland (cf. Lemma 6.6.3.f in [1]). By this trick, we can replace the sequences (s_k,t_k) and (ε_k) by slightly modified sequences. We may then assume in addition that

$$|\nabla u_k(s,t)|_{J^s} \leq 2|\nabla u_k(s_k,t_k)|_{J^{s_k}},\ \text{if } |s|^2 + |t-t_k|^2 \leq \varepsilon_k^2,\ 0 \leq t_k \leq 1.$$

Here we view the maps u_k as maps defined on $\mathbb{R} \times \mathbb{R}$ by a 1-periodic extension in the t-variable. Rescaling we define a new sequence $v_k \in C^\infty(\mathbb{R}^2, M)$ by

$$v_k(s,t) := u_k((s_k, t_k) + \frac{1}{R_k}(s,t)),$$

where $R_k := |\nabla u_k(s_k, t_k)|_{J^{s_k}}$. Denote by D_k the disk $D_k := \{(s,t) \in \mathbb{R}^2 \mid |(s,t)| \leq \varepsilon_k R_k\}$. Then the sequence v_k has the following properties:

$$\begin{aligned} v_k &\in C^\infty(D_k, M), \\ |\nabla v_k(s,t)|_{J^s} &\leq 2 \text{ for } x \in D_k, \\ |\nabla v_k(0)|_{J^0} &= 1, \\ \varepsilon_k R_k &\to \infty. \end{aligned}$$

Moreover, v_k satisfies the partial differential equation

$$\frac{\partial}{\partial s} v_k(s,t) + J^s(v_k(s,t)) \frac{\partial}{\partial t} v_k(s,t) = \frac{1}{R_k} J^s(v_k(s,t)) X_H \left(s_k + \frac{s}{R_k}, t_k + \frac{t}{R_k}, v_k(s,t) \right)$$

on D_k. Now we use Lemma 3.2.1 and find a subsequence of (v_k) and a map $v \in C^\infty(\mathbb{R}^2, M)$ such that $v_k \to v$ in $C^\infty(\mathbb{R}^2, M)$. We are now interested in

$$\Phi(R) := \left| \int_{D_R} v^* \omega \right|.$$

Abbreviating $B_k := B_{\varepsilon_k}(s_k, t_k)$, we estimate

$$\begin{aligned}
&\int_{D_k} |\nabla v_k|^2_{J^s} \, ds \, dt = \int_{B_k} |\nabla u_k|^2_{J^s} \, ds \, dt \\
&= \int_{B_k} \left(\left| \frac{\partial u_k}{\partial s} \right|^2_{J^s} + \left| \frac{\partial u_k}{\partial t} - X_H(s,t,u_k(s,t)) + X_H(s,t,u_k(s,t)) \right|^2_{J^s} \right) ds \, dt \\
&\leq \int_{B_k} \left(\left| \frac{\partial u_k}{\partial s} \right|^2_{J^s} + \left| \frac{\partial u_k}{\partial t} - X_H(s,t,u_k(s,t)) \right|^2_{J^s} \right) ds \, dt \\
&+ \int_{B_k} \left(|X_H(s,t,u_k(s,t))|^2_{J^s} + 2 \left| \frac{\partial u_k}{\partial t} - X_H(s,t,u_k(s,t)) \right|_{J^s} |X_H(s,t,u_k(s,t))|_{J^s} \right) ds \, dt \\
&\leq \int_{B_k} \left(\left| \frac{\partial u_k}{\partial s} \right|^2_{J^s} + 2 \left| \frac{\partial u_k}{\partial t} - X_H(s,t,u_k(s,t)) \right|^2_{J^s} + 2 \, |X_H(s,t,u_k(s,t))|^2_{J^s} \right) ds \, dt \\
&\leq 3E(u_k) + 2 \int_{B_k} |X_H(s,t,u_k(s,t))|^2_{J^s} \, ds \, dt.
\end{aligned}$$

We know that $E(u_k)$ is uniformly bounded, but we still have to take care of the second term. Consider the map

$$X_{H,k} : \mathbb{R}^2 \to \mathbb{R}, \quad (s,t) \mapsto |X_H(s,t,u_k(s,t))|^2_{J^s}.$$

This has compact support in the first variable and is periodic in the second. Hence

$$\max_{(s,t)\in\mathbb{R}^2} |X_{H,k}(s,t)|_{J^s} \leq \max_{(s,t,p)\in[-s_0,s_0]\times\mathbb{S}^1\times M} |X_H(s,t,p)|_{J^s} =: M < \infty,$$

independent of k. Therefore,

$$\int_{B_k} |X_H(s,t,u_k(s,t))|^2_{J^s}\, ds\, dt \leq M^2 \int_{B_k} ds\, dt,$$

and the right-hand side is bounded since the radius ε_k of B_k tends to 0. So we have found

$$\int_{D_k} |\nabla v_k|^2_{J^s}\, ds\, dt \leq C$$

for a suitable $C > 0$ independent of k, and this estimate holds also for the limit v of v_k. In particular we have found that the symplectic area of v is finite,

$$0 \leq \left|\int_{\mathbb{R}^2} v^*\omega\right| = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 \leq C < \infty.$$

In summary, we have shown that v has the following properties:

$$\begin{aligned} |\nabla v(0)| &= 1, \\ |\nabla v(x)| &\leq 2 \text{ for all } x \in \mathbb{R}^2, \\ v_s + J(v)v_t &= 0, \\ 0 \leq \left|\int_{\mathbb{R}^2} v^*\omega\right| &= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 < \infty. \end{aligned}$$

From this point we can copy the proof in [6]. □

Theorem 3.2.3. *$\mathcal{U}(K,J)$ is compact.*

Proof. As already mentioned, we will show that $\mathcal{U}(K,J)$ is sequentially compact. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}(K,J)$ be any sequence. By Proposition 3.2.2 we can apply the Arzelà–Ascoli theorem and obtain a convergent subsequence of $(u_n)_{n\in\mathbb{N}}$ with limit $u \in C^0(\mathbb{S}^1 \times \mathbb{R}, M)$. What is left is to verify that in fact $u \in C^\infty(\mathbb{S}^1 \times \mathbb{R}, M)$ and $u_n \to u$ in the C^∞_{loc}-topology. This works entirely analogous to the proof of Lemma3.2.1 and is the reason why we explained this lemma in detail.

□

3.3 Non-emptyness of $\mathcal{U}(K,J)$

The compactness of $\mathcal{U}(K,J)$ will be crucial for the construction of our action selector. However, we also need to verify that it is non-empty.

Definition 3.3.1. *We denote by*

$$\mathrm{ev} : C^\infty(\mathbb{R}\times\mathbb{S}^1, M) \to M, \qquad \mathrm{ev}(u) := u(0,1)$$

the evaluation map at the point $(0,1)\in\mathbb{R}\times\mathbb{S}^1$.

Theorem 3.3.2. *Let* H *be a Hamiltonian function,* $K\in\mathcal{K}(H)$ *and* $J=\{J^s\}$ *a family of* ω*-compatible almost complex structures such that* $\partial_s J$ *has compact support. Then the compact set* $\mathcal{U}(K,J)$ *is non-empty.*

For the proof of this theorem we first need some definitions.

Definition 3.3.3. *Let* $T>0$ *be such that* $\mathrm{supp}\,(\partial_s J)\subset[-T,T]$ *and* $\mathrm{supp}\,(\partial_s H)\subset[-T,T]$. *We then define*

$$\begin{aligned}\mathcal{U}_T(K,J) := \{(a,u,b)\mid & \, u \text{ is a solution of (3.1) on } [-T,T],\\ & a,b : D\to M\\ & \partial_s a - J^-\partial_t a = 0 \text{ and } \partial_s b + J^+\partial_t b = 0,\\ & u(-T,t)=a(e^{2\pi i t}) \text{ resp. } u(T,t)=b(e^{2\pi i t}) \text{ for } t\in\mathbb{S}^1\},\end{aligned}$$

where $D\subset\mathbb{R}^2$ *is the disc of radius* 1.

The set $\mathcal{U}_T(K,J)$ can be seen as the cutoff of the infinite cylinders $\mathcal{U}(K,J)$ by a J^--anti-holomorphic disc on the left and a J^+-holomorphic disc on the right. In the following we describe the analytical setup to understand the space $\mathcal{U}_T(K,J)$ better. The first step is to equip the space

$$S_T := D_1 \bigcup Z_T \bigcup D_2,$$

where $Z_T := [-T,T]\times\mathbb{S}^1$ is the cylinder of length $2T$, with the structure of a complex manifold. We obtain a complex structure j on S_T such that (S_T,j) is bi-holomorphic to $(\mathbb{S}^2,i)$. The notations of the discs by D_1 and D_2 are just for convenience. For the details see for example [6]. Next we consider the vector bundle $\pi : A_M\to S_T\times M$ whose fiber over $(z,m)\in S_T\times M$ consists of all complex anti-linear maps $\gamma : T_zS_T\to T_mM$, i.e. satisfying $\gamma j = -J\gamma$. For a map $v\in W^{1,p}(S_T,M)$ we denote by v^*A_M the pullback-bundle of A_M by the graph $(z,v(z))$ and by $L^p(v^*A_M)$ the space of L^p-sections of the bundle $v^*A_M\to S_T$. Finally we define the bundle

$$L^p(W^{1,p}(S_T,M)^*A_M) := \bigcup_{v\in W^{1,p}(S_T,M)} \{v\}\times L^p(v^*A_M)\to W^{1,p}(S_T,M),$$

which we will abbreviate by $\mathcal{E}\to\mathcal{B}$. We will not go into details on how to equip the space $\mathcal{B}$ with a differentiable structure nor on how to equip $\mathcal{E}$ with the structure of a Banach space bundle. Let now $v\in W^{1,p}(S_T,M)$ be arbitrary and $T>0$ such that $\mathrm{supp}\,(\partial_s J)\in[-T,T]$. Then we define the function $Lv\in L^p(v^*A_M)$ by

$$Lv := Dv + J\circ(Dv)\circ j + \chi[dK(v)ds - JdK(v)dt],$$

where $\chi : S_T \to \mathbb{R}$ is the characteristic function of $Z_T = [-T, T] \times \mathbb{S}^1$. Because of the s-dependency of J and K we distinguish cases:

$$Lv(z) = \begin{cases} Dv_z + J^- \circ (Dv_z) \circ j, & \text{for } z \in D_1 \\ Dv_z + J^s \circ (Dv_z) \circ j + [dK(s,t,v)ds - J^s dK(s,t,v)dt], & \text{for } z = (s,t) \in [-T,T] \times \mathbb{S}^1 \\ Dv_z + J^+ \circ (Dv_z) \circ j, & \text{for } z \in D_2 \end{cases}$$

Note that $(Lv)(z)$ actually satisfies

$$-J \circ (Lv) = (Lv) \circ j.$$

Therefore we can see Lv as a section of $v^* A_M \to S_T$, and if v moreover satisfies $Lv = 0$, then Lv is the zero-section of $v^* A_M \to S_T$. In addition, still for the case $Lv = 0$, considering the vector field $\frac{\partial}{\partial s}$ on $(-T, T) \times \mathbb{S}^1$ we obtain

$$0 = (Lv) \left(\frac{\partial}{\partial s} \right) = \partial_s v + J^s \partial_t v + dK(s,t,v).$$

With this set-up we can give another description of the space $\mathcal{U}_T(K, J)$:

$$\mathcal{U}_T(K,J) = \{v \in W^{1,p}(S_T, M) | \; Lv = 0\}.$$

In the following we are going to see that we can approximate the space $\mathcal{U}(K, J)$ by the space $\mathcal{U}_T(K, J)$ for large T.

Definition 3.3.4. *Let $\beta : \mathbb{R} \to [0,1]$ be a smooth cutoff function satisfying $\beta(s) = 1$ for $s \leq 1$, $\beta'(s) < 0$ for $1 < s < 2$ and $\beta(s) = 0$ for $s \geq 2$. For every $T > 5$ we then define the family of cutoff functions $\beta_T : \mathbb{R} \to [0,1]$ by*

$$\begin{array}{lll} \beta_T(s) = 1 & \textit{for} & |s| \leq T - 3 \\ \beta_T(s) = \beta(s - T + 3) & \textit{for} & s \geq T - 3 \\ \beta_T(s) = \beta_T(-s) & \textit{for} & s \geq 0. \end{array}$$

Using this we can now define the map

$$\sigma_T : \mathcal{U}_T(K,J) \to C^\infty(\mathbb{R} \times \mathbb{S}^1, M)$$
$$\sigma_T(u)(s,t) = u\left(s\beta_T(s), t\right).$$

The proof of Theorem 3.3.2 relies on the following two propositions, which can be found in [6] on pages 245 − 249 for the case that H and J do not depend on s.

Proposition 3.3.5. *For every open subset $\mathcal{U}$ of $\mathbb{C}^\infty(\mathbb{R} \times \mathbb{S}^1, M)$ satisfying $\mathcal{U}(K, J) \subset \mathcal{U}$, there exists $T' > 0$ such that for all $T > T'$*

$$\sigma_T \left(\mathcal{U}_T(K, J) \right) \subset \mathcal{U}.$$

Proof. We argue indirectly and suppose that we find a sequence $T_n \to \infty$ and a sequence $(u_n) \subset \mathcal{U}_{T_n}(K,J)$ such that $v_n := \sigma_{T_n}(u_n) \notin \mathcal{U}$. By copying the proof of Lemma 3.2.1 we know that

$$\max\{|D^\alpha v_n(s,t)| \,|\, (s,t) \in [-T_n+2, T_n-2] \times \mathbb{S}^1\} \leq C_\alpha$$

and hence we find a convergent subsequence of (v_n) by the Arzelà–Ascoli theorem. We denote this subsequence again by (v_n). Its limit $v : \mathbb{R} \times \mathbb{S}^1 \to M$ satisfies $v \notin \mathcal{U}$. Let us suppose for now that there exists a constant $C > 0$ such that for every $R > 0$ the following holds:

$$\int_{-R}^{R} \int_0^1 |\partial_s v|_{J^s}^2 \, dt \, dt \leq 2C.$$

Hence, for $R \to \infty$, we find

$$E(v) = \int_{-\infty}^{+\infty} \int_0^1 |\partial_s v|_{J^s}^2 \, dt \, dt \leq 2C.$$

In addition, v clearly satisfies the Floer-equation 3.1:

$$\partial_s v + J^s(v)\big(\partial_t v - X_K(s,t,v)\big) = 0.$$

Therefore $v \in \mathcal{U}(K,J)$, in contradiction to the fact $v \notin \mathcal{U}$. □

To finish to proof of Proposition 3.3.5 we have to prove the following lemma.

Lemma 3.3.6. *There exists a positive constant C, independent of $T > 0$ such that every solution $(a,u,b) \in \mathcal{U}_T(K,J)$ satisfies*

$$\int_{-T}^{T} \int_0^1 |\partial_s u|_{J^s}^2 \, dt \, dt \leq C.$$

Proof. Let $C_1 > 0$ be a constant bounding $|\max_{(s,t,p)} K(s,t,p)|$, $|\min_{(s,t,p)} K(s,t,p)|$ and

$$\left| \left(\max_{(s,t,p) \in [-s_0,s_0] \times S^1 \times M} \partial_s K(s,t,p) \right) \int_{[-s_0,s_0] \times S^1} 1 \, dt \, ds \right|,$$

where $\operatorname{supp}(\partial_s K) \subset [-s_0, s_0]$. Then we find

$$\mathbb{A}_{K(-T,\cdot,\cdot)}(u(-T)) = \int_D a^*\omega + \int_0^1 K(-T,t,u(-T,t)) \, dt \leq \max_{(s,t,p)} K(s,t,p) \leq C_1,$$

where we have used $\int_D a^*\omega = -\frac{1}{2}\int_D |\nabla a|_{J^s}^2 ds\, dt \leq 0$, since a is anti-holomorphic. Similarily,

$$\mathbb{A}_{K(T,\cdot,\cdot)}(u(T)) = \int_D b^*\omega + \int_0^1 K(T,t,u(T,t)) \, dt \geq \min_{(s,t,p)} K(s,t,p) \geq -C_1,$$

because $\int_D b^*\omega = \frac{1}{2}\int_D |\nabla b|_{J^s}^2 ds\, dt \geq 0$, since b is holomorphic. Putting this together we obtain

$$\begin{aligned} 2C_1 &\geq \mathbb{A}_{K(-T,\cdot,\cdot)}(u(-T)) - \mathbb{A}_{K(T,\cdot,\cdot)}(u(T)) \\ &= \int_{[-T,T] \times \mathbb{S}^1} |\partial_s u|_{J^s}^2 \, ds \, dt - \int_{[-T,T] \times \mathbb{S}^1} \partial K(s,t,u(s,t)) \, ds \, dt. \end{aligned}$$

With $C := 3C_1$ we therefore find

$$C \geq \int_{[-T,T]\times\mathbb{S}^1} |\partial_s u|^2_{J^s}\, ds\, dt.$$

□

Proposition 3.3.7. *The homomorphism*

$$(\mathrm{ev}_T|_{\mathcal{U}_T(K,J)})^* : H^*(M) \cong \check{H}^*(M) \to \check{H}^*(\mathcal{U}_T(K,J))$$

which is induced by the map

$$\mathrm{ev}_T|_{\mathcal{U}_T(K,J)} : \mathcal{U}_T(K,J) \to M, \qquad \mathrm{ev}_T(u) := u(0,1)$$

is injective. Here $\check{H}^*$ denotes the Alexander–Spanier cohomology with coefficients in $\mathbb{Z}_2$.

Proof. The idea of the proof is to homotop the section $L : \mathcal{B} \to \mathcal{E}$ to the Cauchy–Riemann type section $\partial_s + \tilde{J}\partial_t$, where $\tilde{J}$ is a fixed almost-complex structure. We do this by two homotopies ρ_1 and ρ_2. For $0 \leq \rho_1 \leq 1$ we consider the homotopy

$$L_{\rho_1} v := Dv + J \circ (Dv) \circ j + \rho_1 \chi_{[-T,T]\times\mathbb{S}^1}[dK(v)ds - JdK(v)dt],$$

and therefore L_0 is of Cauchy–Riemann type. But there is still the problem that J is s-dependent. To deal with this problem, we consider now the homotopy

$$\rho_2 \mapsto L_{\rho_2} v := Dv + J^{\rho_2} \circ (Dv) \circ j,$$

where $J^{\rho_2}(s) := J^{s+\rho_2(T+s)}$ and $-1 \leq \rho_2 \leq 0$. By composing the two homotopies we obtain a homotopy $\rho \mapsto L_\rho$, $-1 \leq \rho \leq 1$ such that $L_1 = L$, the original section, and L_{-1} is of Cauchy–Riemann type with fixed almost complex structure J^-. Therefore, the proof of injectivity now works exactly as in [6] with the exception of the following lemma. □

Lemma 3.3.8. *Assume $2 < p < \infty$. Then the set*

$$\tilde{\mathcal{U}}_T(K,J) := \{(\rho, v) \in [-1,1] \times W^{1,p}(S_T, M) \mid L_\rho(v) = 0\}$$

is compact in $[-1,1] \times W^{1,p}(S_T, M)$.

Proof. For each $v \in \tilde{\mathcal{U}}_T(K,J)$ we define the number

$$\varepsilon(v) := \inf\{\varepsilon > 0 \mid \exists z \in S_T \,\text{such that}\, |\nabla v|_{p,B_\varepsilon(z)} = \varepsilon^{\frac{2-p}{p}}\}.$$

Note that since $\frac{2-p}{p}$ is negative, the function $\varepsilon \mapsto \varepsilon^{\frac{2-p}{p}}$ is decreasing. On the other hand the function $\varepsilon \mapsto |\nabla v|_{p,B_\varepsilon(z)}$ is increasing for every $z \in S_T$. Hence, for every z in S_T, these two functions will intersect for one ε. The number $\varepsilon(v)$ can be understood as the infimum over all of them. Assume that

$$\varepsilon := \inf_{v \in \tilde{\mathcal{U}}_T(K,J)} \varepsilon(v) > 0. \tag{3.5}$$

The proof of this assumption does not alter for s-dependent H and J and therefore we will not go into the details but show how this implies compactness of $\tilde{\mathcal{U}}_T(K, J)$. Assumption (3.5) readily implies

$$|\nabla v|_{p,B_\varepsilon(z)} \leq \varepsilon^{\frac{2-p}{p}},$$

for all $v \in \tilde{\mathcal{U}}_T(K, J)$ and all $z \in S_T$ and hence a $W^{1,p}$-bound for $\tilde{\mathcal{U}}_T(K, J)$. In the following we will consider M as a subset of $\mathbb{R}^N$ for some N and therefore $W^{1,p}(S_T, M) \subset W^{1,p}(S_T, \mathbb{R}^N)$. Let now (ρ_k, v_k) be a sequence in $\tilde{\mathcal{U}}_T(K, J)$. By taking a subsequence we can assume that $\rho_k \to \rho$ and, because of the compact embedding of $W^{1,p}(S_T, \mathbb{R}^N)$ into $C^0(S_T, \mathbb{R}^N)$, also $v_k \to v$ in $C^0(S_T, \mathbb{R}^N)$. In addition, due to the reflexivity of $W^{1,p}(S_T, \mathbb{R}^N)$, we can suppose that $v_k \rightharpoonup v$ weakly in $W^{1,p}(S_T, \mathbb{R}^N)$. The idea is to show that v_k is a Cauchy sequence in $W^{1,p}(S_T, \mathbb{R}^N)$. This is a local problem because of the convergence in $C^0(S_T, \mathbb{R}^N)$ and we can pick charts for S_T and of M. The maps $u \in \tilde{\mathcal{U}}_T(K, J)$ are smooth away from the boundaries of the cylinder $\partial Z_T \subset S_T$ and hence it is possible to obtain C^∞-bounds using the same arguments as in Lemma 3.2.1. So the only problems that may arise are on the boundaries of the cylinder $\partial Z_T \subset S_T$. The important point is now that we choose T large enough and therefore both $\partial_s H = 0$ and $\partial_s J^s = 0$ near these boundaries. Consequently we can copy from this point on the proof in [6]. □

Proposition 3.3.9. *The map* $\mathrm{ev}_T : \mathcal{U}_T(K, J) \to M$ *is surjective.*

Proof. Since M is a compact $2n$-dimensional manifold we know that

$$\check{H}^{2n}(M) \cong \mathbb{Z}_2.$$

Furthermore, by using the continuity property of the Alexander–Spanier cohomology and the fact that we can approximate the set $\mathcal{U}_T(K, J)$ by compact $2n-$dimensional manifolds, we conclude that

$$\check{H}^{2n}(\mathcal{U}_T(K, J)) \cong \mathbb{Z}_2.$$

In view of the injectivity of the map

$$(\mathrm{ev}_T|_{\mathcal{U}_T(K,J)})^{2n} : \check{H}^{2n}(M) \to \check{H}^{2n}(\mathcal{U}_T(K, J))$$

we know from Proposition 3.3.7 that the generator of $\check{H}^{2n}(M)$ is mapped to the generator of $\check{H}^{2n}(\mathcal{U}_T(K, J))$. Thus the mapping degree of ev_T is 1. Note that this is only the $\mathbb{Z}_2$-mapping degree, but it is enough for our purpose: The number of preimages of an arbitrary regular value $p \in M$ is congruent to the mapping degree modulo 2. Therefore every regular value has a preimage. Suppose now that $q \in M$ is a critical value without preimage. By definition, q is then a regular value and we have a contradiction. □

Corollary 3.3.10. *The map*

$$\mathrm{ev}|_{\mathcal{U}(K)} : \mathcal{U}(K, J) \to M$$

is surjective. In particular $\mathcal{U}(K, J)$ *is non-empty.*

Proof. Endow M with a metric induced by a Riemannian metric. Suppose that the point $p \in M$ has no preimage. Because ev is continuous and $\mathcal{U}(K, J)$ is compact, we conclude that $\mathrm{ev}(\mathcal{U}(K, J))$ is compact and therefore there exists $\varepsilon > 0$ such that the open ball $B_\varepsilon(p)$ centered at p, does not lie in the image of ev. It follows that

$$V := \mathrm{ev}^{-1}(M \setminus \overline{B_\varepsilon(p)}) \subset C^\infty(\mathbb{R} \times \mathbb{S}^1, M)$$

is an open set which contains $\mathcal{U}(K, J)$. In view of Proposition 3.3.5 we know that there exists $T > 0$, $\mathcal{U}_T(K, J)$ and $\sigma_T : \mathcal{U}_T(K, J) \to C^\infty(\mathbb{R} \times \mathbb{S}^1, M)$ such that

$$\sigma_T(\mathcal{U}_T(K, J)) \subset V.$$

Consider now the following commutative diagram:

$$\begin{tikzcd}
\mathcal{U}_T(K, J) \arrow[r, "\mathrm{ev}_T"] \arrow[d, "\sigma_T"'] & M \\
V \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}$$

Since we already know that ev_T is surjective, we get $\mathrm{ev} \circ \sigma_T(\mathcal{U}_T(K, J)) = M$. This is a contradiction to $\mathrm{ev}(V) \subset M \setminus \overline{B_\varepsilon}(p) \subsetneq M$. □

3.4 Non-emptyness of $\alpha - \lim(u)$

Now that we have established the Gromov compactness theorem in our more general setting, we are almost ready to construct an action selector. The following theorem is the last missing piece and relies also on Gromov compactness.

Theorem 3.4.1. *For every $u \in \mathcal{U}(K, J)$ the sets*

$$\alpha - \lim(u) := \{ \lim_{n\to\infty} \tau_{s_n} u \mid s_n \to -\infty \textit{ is such that } \tau_{s_n} u \textit{ converges } \}$$

and

$$\beta - \lim(u) := \{ \lim_{n\to\infty} \tau_{s_n} u \mid s_n \to +\infty \textit{ is such that } \tau_{s_n} u \textit{ converges } \}$$

are non-empty subsets of $\mathcal{U}(K^\mp, J^\mp)$, which consist of trivial cylinders of the form $v(s, t) = x(t) \in \mathcal{P}(K^\mp)$.

Proof. The proof can be obtained by adapting Propositions 8 and 9 in [6, §6.3] and by using Lemma 2 in [6, §6.4]. We write out a proof, since we wish to make clear that the unique continuation Lemma 2 in [6, §6.4], that relies on the Carleman similarity principle, can be avoided. Let $u \in \mathscr{U}(K, J)$. Assume that $v = \lim_{n\to\infty} \tau_{s_n} u$ with $s_n \to -\infty$. Since $v_n := \tau_{s_n} u$ solves the equation

$$\partial_s v_n + (\tau_{s_n} J)(v_n)\big(\partial_t v_n - X_{\tau_{s_n} K}(s, t, v_n)\big) = 0,$$

and since $\tau_{s_n} K$ converges to K^- and $\tau_{s_n} J$ converges to J^-, the limit v is a solution of the s-independent Floer equation defined by K^- and J^-. Moreover, since

$$\int_{[-T,T]\times\mathbb{S}^1} |\partial_s v|^2_{J^-}\, ds\, dt = \lim_{n\to\infty} \int_{[-T,T]\times\mathbb{S}^1} |\partial_s v_n|^2_{\tau_{s_n} J}\, ds\, dt \le \liminf_{n\to\infty} E_{\tau_{s_n} J}(v_n)$$

for every $T > 0$ and since $E_{\tau_{s_n}J}(v_n) = E_J(u)$ for all n, we have

$$E_{J^-}(v) \;\le\; \liminf_{n\to\infty} E_{\tau_{s_n}J}(v_n) \;=\; E_J(u).$$

Hence $v \in \mathscr{U}(K^-, J^-)$. We next show that for $u \in \mathscr{U}(K, J)$ the set α-lim(u) is non-empty. For $T > 0$ let $\mathscr{U}_T(K^-, J^-)$ be the space of solutions of Floer's equation

$$\partial_s w + J^-(w)\big(\partial_t w - X_{K^-}(t, w)\big) = 0, \qquad s \in [-T, T] \tag{3.6}$$

such that $E_{J^-}(w) \le E_J(u)$. The proof showing that $\mathscr{U}(K^-, J^-)$ is C^∞_{loc}-compact in $C^\infty(\mathbb{R}\times\mathbb{S}^1, M)$ in particular shows that $\mathscr{U}_T(K^-, J^-)$ is C^∞-compact in $C^\infty([-T, T]\times\mathbb{S}^1, M)$. To simplify notation we assume that $K = K^-$ and $J = J^-$ on $\{s \le 0\}$. Now take any monotone decreasing sequence $s_n \to -\infty$ of negative numbers. Then $\tau_{s_n}u$ solves (3.6) on $[-s_n, s_n]$ and $E_{J^-}\big(\tau_{s_n}u|_{[-s_n,s_n]}\big) \le E_J(u)$, whence $\tau_{s_n}u \in \mathscr{U}_{s_n}(K^-, J^-)$. Since $\mathscr{U}_{s_1}(K^-, J^-)$ is compact, we find a subsequence (s_n^1) of (s_n) such that the restriction of $\tau_{s_n^1}u$ to $[-s_1, s_1]$ converges in $\mathscr{U}_{s_1}(K^-, J^-)$. Since $\mathscr{U}_{s_2}(K^-, J^-)$ is compact, we find a subsequence (s_n^2) of (s_n^1) with $s_1^2 \le s_2$ such that the restriction of $\tau_{s_n^2}u$ to $[-s_2, s_2]$ converges in $\mathscr{U}_{s_2}(K^-, J^-)$, etc. The diagonal sequence $\tau_{s_n^n}u$ converges in C^∞_{loc} to an element of $C^\infty(\mathbb{R}\times\mathbb{S}^1, M)$, which therefore belongs to the set α-lim(u). For the proof of the last assertion, define the function

$$a_{K^-} \colon \mathscr{U}(K^-, J^-) \to \mathbb{R}, \qquad a_{K^-}(w) = \mathbb{A}_K(w(0,\cdot)).$$

Since K^- and J^- do not depend on s, the shifts τ_σ act on $\mathscr{U}(K^-, J^-)$, and a_{K^-} is non-increasing under this action, since by (3.1.5) for all $\sigma < \sigma'$,

$$a_{K^-}(\tau_\sigma w) - a_{K^-}(\tau_{\sigma'} w) \;=\; \mathbb{A}_{K^-}(w(\sigma)) - \mathbb{A}_{K^-}(w(\sigma')) \;=\; \int_{[\sigma,\sigma']\times\mathbb{S}^1} |\partial_s w|^2_{J^-}\, ds\, dt \;\ge\; 0. \tag{3.7}$$

Since a_{K^-} is bounded on $\mathscr{U}(K^-, J^-)$ (see Proposition 3.1.3) and the function $\sigma \mapsto a_{K^-}(\tau_\sigma u)$ is non-increasing,

$$\lim_{\sigma\to-\infty} a_{K^-}(\tau_\sigma u) \;=:\; d$$

exists in $\mathbb{R}$. It follows that $a_K^-(v) = d$ for every $v \in \alpha$-lim(u). Now note that the set α-lim(u) is invariant under the shifts τ_σ, since for $v = \lim_{n\to\infty} \tau_{s_n}u$ we have $\tau_\sigma v = \lim_{n\to\infty} \tau_{s_n+\sigma}u$. With (3.7) we conclude that for $v \in \alpha$-lim(u) and $\sigma < \sigma'$,

$$0 \;=\; a_{K^-}(\tau_\sigma v) - a_{K^-}(\tau_{\sigma'} v) \;=\; \int_{[\sigma,\sigma']\times\mathbb{S}^1} |\partial_s v|^2_{J^-}\, ds\, dt.$$

Therefore, $\partial_s v$ vanishes on $\mathbb{R}\times\mathbb{S}^1$, whence v is of the form $x(t)$. Since we already know that $v \in \mathscr{U}(K^-, J^-)$, we can read off from (3.1) that $\partial_t v = X_{K^-}(t, v(t))$, that is, $v \in \mathscr{P}(K^-)$. □

Chapter 4

The minimal selector

Let $H \in C^\infty(\mathbb{S}^1 \times M)$ be a Hamiltonian. We would like to define an action selector for H.

4.1 The definition

Let $(K, J) \in \mathscr{D}(H)$ and assume that $\partial_s K$ and $\partial_s J$ are supported in $[s^-, s^+] \times \mathbb{S}^1 \times M$. If $u \in \mathscr{U}(K, J)$, then on $(-\infty, s^-]$ the function $s \mapsto \mathbb{A}_H(u(s))$ is non-increasing (see Lemma 3.1.2) and bounded (see Proposition 3.1.3). Therefore, the function

$$a_H^- \colon \mathscr{U}(K, J) \to \mathbb{R}, \qquad a_H^-(u) := \lim_{s \to -\infty} \mathbb{A}_H(u(s)) = \sup_{s \leq s^-} \mathbb{A}_H(u(s)),$$

is well-defined. Being the supremum of a family of continuous functions, the function a_H^- is lower semi-continuous. As such, it has a minimum on the compact space $\mathscr{U}(K, J)$.

Definition 4.1.1. *Let $H \in C^\infty(\mathbb{S}^1 \times M)$ and $(K, J) \in \mathscr{D}(H)$. We set*

$$A^-(K, J) := \min_{u \in \mathscr{U}(K,J)} a_H^-(u), \qquad A(H) := \sup_{(K,J) \in \mathscr{D}(H)} A^-(K, J).$$

We refer to the function

$$A \colon C^\infty(\mathbb{S}^1 \times M) \to \mathbb{R}$$

as to the minimal action selector.

4.2 First properties

Proposition 4.2.1 (Spectrality). $A(H) \in \operatorname{spec}(H)$

Proof. The number $A^-(K, J)$ is a critical value of $\mathbb{A}_H$. Indeed, take $u \in \mathscr{U}(K, J)$ such that $a_H^-(u) = A^-(K, J)$. By Theorem 3.4.1, we find $v \in \alpha\text{-lim}(u)$, and v is of the form $v(s, t) = x(t)$ with $x \in \mathscr{P}(H)$. Hence $a_H^-(u) = \mathbb{A}_H(x)$. Hence $A^-(K, J)$ is a critical value of $\mathbb{A}_H$. Since the set of critical values of $\mathbb{A}_H$ is closed, also $A(H)$ is a critical value of $\mathbb{A}_H$, and so $A(H) \in \operatorname{spec}(H)$. □

Two very simple properties of the action selector A are:

$$A(H) = 0 \qquad \text{if } H \equiv 0, \tag{4.1}$$

$$A(H+r) = A(H) + \textstyle\int_{\mathbb{S}^1} r(t)\,dt \qquad \forall\, r \in C^\infty(\mathbb{S}^1),\ H \in C^\infty(\mathbb{S}^1 \times M). \tag{4.2}$$

Indeed, the first property follows from the fact that for the Hamiltonian $H \equiv 0$, the set $\mathscr{P}(H)$ consists of all the constant loops, which have action zero. The second property follows from the identities $\mathscr{K}(H+r) = \mathscr{K}(H) + r$ and $a^-_{H+r} = a^-_H + \int_{\mathbb{S}^1} r(t)\,dt$. Less trivial is the following:

Proposition 4.2.2 (Monotonicity). *If $H_0, H_1 \in C^\infty(\mathbb{S}^1 \times M)$ are such that*

$$\int_{\mathbb{S}^1} \max_{x \in M} \big(H_1(t,x) - H_0(t,x)\big)\,dt \ \le\ 0,$$

then $A(H_0) \ge A(H_1)$.

Proof. Fix $\varepsilon > 0$. We shall prove that

$$\sup_{(K_0,J_0)\in\mathscr{D}(H_0)} \ \min_{\mathscr{U}(K_0,J_0)} a^-_{H_0} \ \ge \sup_{(K_1,J_1)\in\mathscr{D}(H_1)} \ \min_{\mathscr{U}(K_1,J_1)} a^-_{H_1} - \varepsilon, \tag{4.3}$$

and the claim will follow from the arbitrariness of ε. Proving (4.3) is equivalent to showing that for every (K_1, J_1) in $\mathscr{D}(H_1)$ there exists (K_0, J_0) in $\mathscr{D}(H_0)$ such that

$$\min_{\mathscr{U}(K_0,J_0)} a^-_{H_0} \ \ge \ \min_{\mathscr{U}(K_1,J_1)} a^-_{H_1} - \varepsilon. \tag{4.4}$$

Up to a translation, we may assume that

$$K_1(s,t,x) \ =\ H_1(t,x) \quad \text{and} \quad J_1(s,t,x) \ =\ J_1^-(t,x), \qquad \forall s \le 0. \tag{4.5}$$

Let $\varphi \in C^\infty(\mathbb{R})$ be a real function such that $\varphi' \ge 0$, $\varphi(s) = 0$ for $s \le 0$ and $\varphi(s) = 1$ for $s \ge 1$. For $\lambda \in \mathbb{R}$ we define $K_0^\lambda \in \mathscr{K}(H_0)$ by

$$K_0^\lambda(s,t,x) \ :=\ \varphi(s-\lambda)K_1(s,t,x) + \big(1 - \varphi(s-\lambda)\big)H_0(t,x). \tag{4.6}$$

We claim that there exists $\lambda \le -1$ such that (4.4) holds with $(K_0, J_0) = (K_0^\lambda, J_1)$. Arguing by contradiction, we assume that for every $\lambda \le -1$ there is a u_λ in $\mathscr{U}(K_0^\lambda, J_1)$ such that

$$a^-_{H_0}(u_\lambda) \ <\ \min_{\mathscr{U}(K_1,J_1)} a^-_{H_1} - \varepsilon. \tag{4.7}$$

Let $(\lambda_n) \subset (-\infty, -1]$ be such that $\lambda_n \to -\infty$. By Theorem 3.2.3, $\mathscr{U}(K_1, J_1)$ is compact. Arguing by a diagonal sequence argument, we see that after replacing (λ_n) by a subsequence, (u_{λ_n}) converges to some u in $\mathscr{U}(K_1, J_1)$.

We fix a number $s \le 0$. If $\lambda_n \le s - 1$, then by (4.5) and the action-energy identity (3.1.5),

$$\begin{aligned}
&a^-_{H_0}(u_{\lambda_n}) \ \ge\ \mathbb{A}_{H_0}\big(u_{\lambda_n}(\lambda_n)\big)\\
&\ =\ \mathbb{A}_{H_1}\big(u_{\lambda_n}(s)\big) + \int_{[\lambda_n,s]\times\mathbb{S}^1} \big|\partial_\sigma u_{\lambda_n}\big|^2_{J_1^-}\,d\sigma\,dt - \int_{[\lambda_n,s]\times\mathbb{S}^1} \varphi'(\sigma-\lambda_n)(H_1 - H_0)(t,u_{\lambda_n})\,d\sigma\,dt\\
&\ \ge\ \mathbb{A}_{H_1}\big(u_{\lambda_n}(s)\big),
\end{aligned}$$

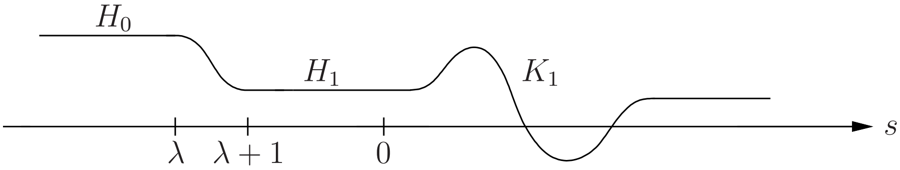


Figure 4.1: The function K_0^λ, for (t,x) fixed

where at the end we have used the hypothesis of the proposition. By taking the limit for $n \to \infty$, we deduce that

$$\liminf_{n\to\infty} a^-_{H_0}(u_{\lambda_n}) \geq \mathbb{A}_{H_1}\big(u(s)\big),$$

and by taking the supremum over all $s \leq 0$,

$$\liminf_{n\to\infty} a^-_{H_0}(u_{\lambda_n}) \geq a^-_{H_1}(u).$$

Together with (4.7), this implies the chain of inequalities

$$a^-_{H_1}(u) \leq \liminf_{n\to\infty} a^-_{H_0}(u_{\lambda_n}) \leq \min_{\mathscr{U}(K_1,J_1)} a^-_{H_1} - \varepsilon,$$

which is the desired contradiction because $u \in \mathscr{U}(K_1, J_1)$. □

Monotonicity and property (4.2) imply the following form of continuity.

Proposition 4.2.3 (Lipschitz continuity). *For all $H_0, H_1 \in C^\infty(\mathbb{S}^1 \times M)$,*

$$\int_{\mathbb{S}^1} \min_{x\in M} \big(H_1(t,x) - H_0(t,x)\big)\,dt \;\leq\; A(H_1) - A(H_0) \;\leq\; \int_{\mathbb{S}^1} \max_{x\in M} \big(H_1(t,x) - H_0(t,x)\big)\,dt.$$

In particular, the action selector A is 1-Lipschitz with respect to the sup-norm on $C^\infty(\mathbb{S}^1 \times M)$:

$$\big|A(H_1) - A(H_0)\big| \;\leq\; \|H_1 - H_0\|_\infty.$$

Proof. Set

$$c_-(t) \;=\; \min_{x\in M}\big(H_1(t,x) - H_0(t,x)\big), \quad c_+(t) \;=\; \max_{x\in M}\big(H_1(t,x) - H_0(t,x)\big).$$

Then

$$H_0(t,x) + c_-(t) \;\leq\; H_1(t,x) \;\leq\; H_0(t,x) + c_+(t), \qquad \forall\, t \in \mathbb{S}^1,\ x \in M.$$

Applying Proposition 4.2.2 and (4.2) we obtain

$$A(H_0) + \int_{\mathbb{S}^1} c_-(t)\,dt \;\leq\; A(H_1) \;\leq\; A(H_0) + \int_{\mathbb{S}^1} c_+(t)\,dt$$

as we wished to prove. □

4.3 An equivalent definition

By now we know that our action selector A is spectral, monotone, and continuous. These properties already imply many further properties, see Proposition 7.1.3. For most applications of an action selector, such as the non-squeezing theorem or (almost) existence of periodic orbits, one also needs that the selector is negative on functions that are non-positive and do not vanish identically. To prove this property for our selector A we shall describe A by a minmax in which the space $\mathscr{U}(K,J)$ is replaced by a certain space of solutions of Floer's equation for H. Recall that $(\tau_\sigma u)(s) := u(\sigma+s)$. Given $(K,J) \in \mathscr{D}(H)$, consider the set

$$\begin{aligned}\mathscr{U}_{\text{ess}}(K,J) \;:=\; \{u \in C^\infty(\mathbb{R}\times\mathbb{S}^1, M) \mid \\ u = \lim_{n\to\infty} \tau_{s_n} u_n \text{ where } s_n \to -\infty \text{ and } (u_n) \subset \mathscr{U}(K,J)\}.\end{aligned}$$

Example 4.3.1. *Assume that neither* H *nor* J *depend on* s. *Then* $\mathscr{U}_{\text{ess}}(H,J) = \mathscr{U}(H,J)$.

Proof. The inclusion $\mathscr{U}_{\text{ess}}(H,J) \subset \mathscr{U}(H,J)$ holds since with u_n also $\tau_{s_n}u_n \in \mathscr{U}(H,J)$ and since $\mathscr{U}(H,J)$ is closed. Further, $\mathscr{U}(H,J) \subset \mathscr{U}_{\text{ess}}(H,J)$ since for $u \in \mathscr{U}(H,J)$ we have $u_n := \tau_n u \in \mathscr{U}(H,J)$ and $\lim_{n\to\infty} \tau_{-n}(u_n) = u$. □

As we shall see in Proposition 4.3.2, $\mathscr{U}_{\text{ess}}(K,J)$ is a compact τ-invariant subspace of $\mathscr{U}(H,J^-)$. The space $\mathscr{U}_{\text{ess}}(K,J)$ is therefore the space of those cylinders in $\mathscr{U}(H,J^-)$ which are essential with respect to K, in the sense that they survive through the homotopy K. We shall prove that the minimal action selector

$$A(H) \;=\; \sup_{(K,J)\in\mathscr{D}(H)} \; \min_{\mathscr{U}(K,J)} a_H^-$$

can be expressed as

$$A(H) \;=\; \sup_{(K,J)\in\mathscr{D}(H)} \; \min_{\mathscr{U}_{\text{ess}}(K,J)} a_H \tag{4.8}$$

where $a_H(u) = \mathbb{A}_H(u(0,\cdot))$. We begin with the following result.

Proposition 4.3.2. *The set* $\mathscr{U}_{\text{ess}}(K,J)$ *is a non-empty compact* τ*-invariant subspace of* $\mathscr{U}(H,J^-)$.

Proof. The set $\mathscr{U}_{\text{ess}}(K,J)$ contains all the sets α-lim(u) of $u \in \mathscr{U}(K,J)$ and therefore is non-empty by Theorem 3.4.1. The inclusion $\mathscr{U}_{\text{ess}}(K,J) \subset \mathscr{U}(H,J^-)$ is shown in the same way as the inclusion α-lim$(u) \subset \mathscr{U}(H,J^-)$ in Theorem 3.4.1: Let $u = \lim \tau_{s_n}u_n$ be an element of $\mathscr{U}_{\text{ess}}(K,J)$. Since $v = \tau_{s_n}u_n$ solves the equation

$$\partial_s v + (\tau_{s_n}J)(v)\big(\partial_t v - X_{\tau_{s_n}K}(s,t,v)\big) = 0,$$

and since $\tau_{s_n}K$ converges to $K^- = H$ and $\tau_{s_n}J$ converges to J^-, the map u is a solution of the s-independent Floer equation defined by H and J^-. Moreover,

$$E_{J^-}(u) \;\le\; \liminf_{n\to\infty} E_{\tau_{s_n}J}(\tau_{s_n}u_n) \;=\; \liminf_{n\to\infty} E_J(u_n) \;\le\; \sup_{v\in\mathscr{U}(K,J)} E_J(v) \;<\; +\infty.$$

Therefore, $\mathscr{U}_{\mathrm{ess}}(K,J) \subset \mathscr{U}(H,J^-)$. If $\sigma \in \mathbb{R}$, then

$$\tau_\sigma u \;=\; \lim_{n\to\infty} \tau_{s_n+\sigma} u_n$$

is in $\mathscr{U}_{\mathrm{ess}}(K,J)$, which is therefore τ-invariant. If

$$v^h = \lim_{n\to\infty} \tau_{s_n^h} u_n^h, \qquad \text{where} \quad \lim_{n\to\infty} s_n^h = -\infty, \quad \forall h \in \mathbb{N},$$

and (v^h) converges to $v \in C^\infty(\mathbb{R}\times\mathbb{S}^1, M)$, a standard diagonal argument implies the existence of a diverging sequence $(n_h) \subset \mathbb{N}$ such that

$$\lim_{h\to\infty} \operatorname{dist}\left(\tau_{s^h_{n_h}} u^h_{n_h}, v^h\right) = 0, \qquad \lim_{h\to\infty} s^h_{n_h} = -\infty,$$

where dist is a distance on $C^\infty(\mathbb{R}\times\mathbb{S}^1, M)$. Therefore, $\tau_{s^h_{n_h}} u^h_{n_h}$ converges to v, which hence belongs to $\mathscr{U}_{\mathrm{ess}}(K,J)$. This shows that $\mathscr{U}_{\mathrm{ess}}(K,J)$ is a closed subspace of $\mathscr{U}(H,J^-)$. Since $\mathscr{U}(H,J^-)$ is compact, so is $\mathscr{U}_{\mathrm{ess}}(K,J)$. □

Formula (4.8) is an immediate consequence of the following:

Proposition 4.3.3. $A^-(K,J) := \min\limits_{\mathscr{U}(K,J)} a_H^- = \min\limits_{\mathscr{U}_{\mathrm{ess}}(K,J)} a_H.$

Proof. Let $u \in \mathscr{U}(K,J)$ be a minimizer of a_H^-. By Theorem 3.4.1 there exists $v \in \alpha\text{-lim}(u)$, and $v(s,t) = x(t)$, $x \in \mathscr{P}(H)$. Then

$$a_H(v) \;=\; \mathbb{A}_H(x) \;=\; a_H^-(u).$$

Since $v \in \alpha\text{-lim}(u) \subset \mathscr{U}_{\mathrm{ess}}(K,J)$, we conclude

$$\min_{\mathscr{U}_{\mathrm{ess}}(K,J)} a_H \le a_H(v) \;=\; a_H^-(u) \;=\; \min_{\mathscr{U}(K,J)} a_H^-.$$

Conversely, let $v \in \mathscr{U}_{\mathrm{ess}}(K,J)$ be a minimizer of a_H. Then

$$v = \lim_{n\to\infty} \tau_{s_n} u_n, \qquad \text{where } s_n \to -\infty \text{ and } (u_n) \subset \mathscr{U}(K,J).$$

Up to a subsequence, we may assume that (u_n) converges to some $u \in \mathscr{U}(K,J)$. For every fixed $s \le s^-(K,J)$,

$$\mathbb{A}_H(u(s)) \;=\; \lim_{n\to\infty} \mathbb{A}_H(u_n(s)) \;\le\; \lim_{n\to\infty} \mathbb{A}_H(u_n(s_n)) \;=\; \lim_{n\to\infty} a_H(\tau_{s_n} u_n) \;=\; a_H(v).$$

By taking the limit for $s \to -\infty$, we find

$$a_H^-(u) \;\le\; a_H(v),$$

which implies that

$$\min_{\mathscr{U}(K,J)} a_H^- \;\le\; a_H^-(u) \;\le\; a_H(v) \;=\; \min_{\mathscr{U}_{\mathrm{ess}}(K,J)} a_H.$$

□

The space $\mathscr{U}_{\mathrm{ess}}(K,J)$ satisfies the analogue of Proposition 3.3.7 and Corollary 3.3.10:

Proposition 4.3.4. *For every* $z \in \mathbb{R} \times \mathbb{S}^1$ *the homomorphism*

$$\big(\mathrm{ev}_z|_{\mathscr{U}_{\mathrm{ess}}(K,J)}\big)^*\colon H^*(M) \cong \check{H}^*(M) \to \check{H}^*\big(\mathscr{U}_{\mathrm{ess}}(K,J)\big)$$

which is induced by the evaluation map

$$\mathrm{ev}_z|_{\mathscr{U}_{\mathrm{ess}}(K,J)}\colon \mathscr{U}_{\mathrm{ess}}(K,J) \to M \tag{4.9}$$

is injective. In particular, the map (4.9) *is surjective.*

Proof. Let $\mathscr{W} \subset C^\infty(\mathbb{R} \times \mathbb{S}^1, M)$ be a neighborhood of $\mathscr{U}_{\mathrm{ess}}(K,J)$. We claim that if $s_0 \in \mathbb{R}$ is small enough, then $\tau_{s_0}\mathscr{U}(K,J) \subset \mathscr{W}$. If not, we could find sequences $s_n \to -\infty$ and $(u_n) \subset \mathscr{U}(K,J)$ such that $\tau_{s_n}u_n \notin \mathscr{W}$. But since $\tau_{s_n}K \to H$ and by compactness, $(\tau_{s_n}u_n)$ has a convergent subsequence, whose limit is by definition an element of $\mathscr{U}_{\mathrm{ess}}(K,J)$. Therefore, this subsequence must eventually belong to $\mathscr{W}$, which is a contradiction.

If s_0 is as above, we denote by

$$i\colon \mathscr{U}_{\mathrm{ess}}(K,J) \hookrightarrow \mathscr{W} \quad \text{and} \quad j\colon \tau_{s_0}\mathscr{U}(K,J) \hookrightarrow \mathscr{W}$$

the inclusion mappings. Then, if $z = (s,t) \in \mathbb{R} \times \mathbb{S}^1$ and $z' = (s + s_0, t)$, we get the commutative diagram

$$\begin{tikzcd}
\check{H}^*(\mathscr{W}) \arrow[r, "j^*"] \arrow[d, "i^*"'] & \check{H}^*\big(\tau_{s_0}\mathscr{U}(K,J)\big) \arrow[r, "\tau_{s_0}^*"] & \check{H}^*\big(\mathscr{U}(K,J)\big) \\
\check{H}^*\big(\mathscr{U}_{\mathrm{ess}}(K,J)\big) & \check{H}^*(M) \arrow[l, "\mathrm{ev}_z^*"'] \arrow[ul, "\mathrm{ev}_z^*"] \arrow[u, "\mathrm{ev}_z^*"'] \arrow[ur, "\mathrm{ev}_{z'}^*"'] &
\end{tikzcd}$$

Proposition 3.3.7 applied to $\mathrm{ev}_{z'}^*$ and the fact that $\tau_{s_0}^*$ is an isomorphism imply that the vertical map $(\mathrm{ev}_z|_{\tau_{s_0}\mathscr{U}(K,J)})^*$ is injective. Then so is $(\mathrm{ev}_z|_{\mathscr{W}})^*$. Since this is true for every neighborhood $\mathscr{W}$ of $\mathscr{U}_{\mathrm{ess}}(K,J)$, the continuity of Alexander–Spanier cohomology implies that the homomorphism $(\mathrm{ev}_z|_{\mathscr{U}_{\mathrm{ess}}(K,J)})^*$ is also injective. ☐

4.4 Autonomous Hamiltonians

Let $H \in C^\infty(M)$ be an autonomous Hamiltonian. In this case, the critical points of H are the constant orbits of X_H, and in particular they are elements of $\mathscr{P}(H)$. In general, the vector field X_H can have other non-constant contractible orbits, but if this does not happen we can often calculate the value of the minimal action selector. We first prove

Proposition 4.4.1. *Let* $H \in C^\infty(M)$ *be an autonomous Hamiltonian with exactly two critical values. Assume also that* $\mathscr{P}(H)$ *consists only of constant orbits. Then*

$$A(H) = \min_M H.$$

Proof. In this case $\mathbb{A}_H$ has exactly two critical values, $\min_M H$ and $\max_M H$. Hence $A(H)$ is one of these two numbers. For every $(K, J) \in \mathscr{D}(H)$ we have

$$\min_{\mathscr{U}_{\mathrm{ess}}(K,J)} a_H \leq \max_{\mathscr{U}_{\mathrm{ess}}(K,J)} a_H \leq \max_{\mathscr{U}(H,J^-)} a_H = \max_{\mathscr{P}(H)} \mathbb{A}_H = \max_M H. \tag{4.10}$$

Therefore, by Proposition 4.3.3, $\min_{\mathscr{U}_{\mathrm{ess}}(K,J)} a_H = A^-(K, J)$ belongs to $\operatorname{spec} H = \{\min_M H, \max_M H\}$. Therefore, if $A(H) = \sup_{(K,J)\in\mathscr{D}H)} A^-(K, J) = \max_M H$, we find $(K, J) \in \mathscr{D}(H)$ such that all inequalities in (4.10) are equalities. Hence $\mathscr{U}_{\mathrm{ess}}(K, J)$ consists only of constant cylinders defined by the maximum points of H. This violates the surjectivity of the evaluation map $\mathrm{ev}_z |_{\mathscr{U}_{\mathrm{ess}}(K,J)}$ from Proposition 4.3.4. □

As in the introduction take a compactly supported negative bump $H_f(x) = f(\pi\|x\|^2)$ on $\mathbb{R}^{2n}$ with exactly two critical values $H(0) = f(0)$ and 0. If also $|f'| < 1$, then the only 1-periodic orbits of X_{H_f} are constant. Transporting such functions H_f into M by a Darboux chart we obtain functions satisfying the hypothesis of Proposition 4.4.1. In particular, the action selector A is non-trivial.

The proof of Proposition 4.4.1 relies on Proposition 4.3.4, that uses Proposition 3.3.7, and hence only relies on compactness and Fredholm theory. We now prove a variant of Proposition 4.4.1 whose proof appeals, in addition, to transversality and gluing analysis from Floer theory. However, the Proposition only holds true for a fixed s-independent $J \in \mathscr{J}_\omega(M)$. Therefore we define

$$A(H, J) := \sup_{K\in\mathscr{K}(H)} \min_{\mathscr{U}(K,J)} a_H^-.$$

This is also an action selector, for details see Section 4.5.

Proposition 4.4.2. *Let $H \in C^\infty(M)$ be an autonomous Hamiltonian such that X_H has no non-trivial contractible closed orbits of period $T \in (0, 1]$. Then for every s-independent $J \in \mathscr{J}_\omega(M)$,*

$$A(H, J) = \min_M H.$$

Proof. Fix an s-independent $J \in \mathscr{J}_\omega(M)$. The solutions of the Floer equation (2.1) which do not depend on t, that is $u(s, t) = u(s)$, are the solutions of the ODE

$$u'(s) + \nabla H(u(s)) = 0,$$

so they are the negative gradient flow lines of H. In general, the Floer equation could have other, t-dependent, bounded energy solutions. If this is not the case, we say that H is J-trivial.

Step 1. Assume in addition that H is J-trivial: Every solution of (2.1) with bounded energy does not depend on t. Since $\min_M H$ is the minimal critical value of $\mathbb{A}_H$, we must show that for every $K \in \mathscr{K}(H)$ there holds $A^-(K, J) \leq \min_M H$. Assume by contradiction that there is a $K \in \mathscr{K}(H)$ with

$$A^-(K, J) > \min_M H.$$

By our assumption that $\mathscr{U}(H,J)$ is the set of gradient flow lines of H, and by the characterization of $A^-(K,J)$ in Proposition 4.3.3, we deduce that $\mathscr{U}_{\mathrm{ess}}(K,J)$ is a set of gradient flow lines of H which is contained in

$$\big\{x \in M \mid H(x) \geq A^-(K,J)\big\} \subsetneq M.$$

But this violates the surjectivity of the map $\mathrm{ev}_z\,|_{\mathscr{U}_{\mathrm{ess}}(K,J)}$ of Proposition 4.3.4.

Step 2. Since X_H has no contractible closed orbits of period $T \in (0,1]$, the same is true for X_{sH}, $s \in (0,1]$. In particular, $\mathscr{P}(sH) = \mathscr{P}(H) = \mathrm{crit} H$ for all $s \in (0,1]$ and hence $\mathrm{spec}\,(sH) = s\,\mathrm{spec}\,(H)$ for $s \in (0,1]$. Since $\mathrm{spec}\,(H)$ is nowhere dense and $A(\cdot,J)$ is continuous, it follows that $A(sH,J) = sA(H,J)$ for these s. It thus suffices to prove the proposition for sH with s as small as we like.

Step 3. Let $G\colon M \to \mathbb{R}$ be a Morse function such that $-\nabla_{g_J} G$ is a Morse–Smale vector field. Then there exists $k_0(G)$ such that $\frac{1}{k}G$ is J-trivial for all $k \geq k_0(G)$. This is proved in [16, §7] for Floer trajectories between critical points of index difference ≤ 2, see also [1, pp. 378–381]. Their argument extends to Floer trajectories between any pair of critical points x, y, since the compactification $\overline{\mathcal{M}}(x,y)$ of the space of these trajectories by broken flow lines is a manifold with corners, see e.g. the appendix of [2].

Step 4. Now let H be as in the proposition, and fix $\varepsilon > 0$. Also fix a Riemannian metric g on M and denote by $\|X\|_1$ the C^1-norm with respect to g of a smooth vector field X on M. There exists $\delta > 0$ such that any vector field X with $\|X\|_1 < \delta$ has no non-constant closed orbits of period $T \in (0,1]$, see [1, p. 154]. Take $s_0 > 0$ so small that $\|X_{s_0 H}\|_1 = s_0 \|X_H\|_1 < \frac{\delta}{2}$. Choose a Morse function $G\colon M \to \mathbb{R}$ such that $-\nabla_{g_J} G$ is a Morse–Smale vector field and such that $\|G - H\| < \varepsilon$ and $\|X_G - X_H\|_1 < \frac{\delta}{2s_0}$. Then $\|X_{sG}\|_1 \leq \|X_{sH}\|_1 + \|X_{sG} - X_{sH}\|_1 < \delta$ for all $s \leq s_0$. By Step 3 we find $k \in \mathbb{N}$ such that $\frac{s_0}{k}G$ is J-trivial. Since $X_{\frac{s_0}{k}G}$ has no non-trivial closed orbits of period $T \in (0,1]$, Step 1 shows that $A(s_0 G, J) = \min s_0 G$. Hence

$$\begin{aligned}\big|A(s_0H,J) - \min_M s_0 H\big| \;&\leq\; |A(s_0H,J) - A(s_0G,J)| + \big|\min_M s_0G - \min_M s_0H\big| \\ &\leq\; 2\,\|s_0H - s_0G\| \;<\; 2\varepsilon.\end{aligned}$$

Sine $\varepsilon > 0$ was arbitrary, $A(s_0H,J) = \min_M s_0 H$, and so $A(H,J) = \min H$ by Step 2. □

Remark 4.4.3. *AS we shall see in Section* 4.5.2*,*

$$\hat{A}(H) := \sup_J A(H,J),$$

where the supremum is taken over all s-independent $J \in \mathscr{J}_\omega(M)$, is also an action selector. Proposition 4.4.2 *computes $\hat{A}(H)$ for autonomous Hamiltonians such that X_H has no non-trivial contractible closed orbits of period $T \in (0,1]$.*

4.5 Variations of the minimal selector

4.5.1 Smaller classes of deformations

The set $\mathscr{K}(H)$ is a large class of deformations of H, and it is useful to look at smaller classes. Let $\mathscr{K}$ be the union of all sets $\mathscr{K}(H)$, where $H \in C^\infty(\mathbb{S}^1 \times M, \mathbb{R})$.

Definition 4.5.1. A subset $\mathscr{K}' = \bigcup_H \mathscr{K}'(H)$ of $\mathscr{K}$ is *admissible* if the following holds: For any pair $H_0 \geq H_1$ and for any $K_1 \in \mathscr{K}'(H_1)$ with $\operatorname{supp} \frac{\partial K_1}{\partial s} \subset [s_-, s_+]$, every $K_0 \in \mathscr{K}(H_0)$ with $\frac{\partial K_0}{\partial s} \leq 0$ for $s \leq s_-$ and $K_0 = K_1$ for $s \geq s_-$ belongs to $\mathscr{K}'(H_0)$.

Going through this section, we see that for every admissible set $\mathscr{K}' \subset \mathscr{K}$,

$$A'(H,J) = \sup_{K \in \mathscr{K}'(H)} \min_{\mathscr{U}(K,J)} a_H^-$$

defines a minimal action selector: $A'(H,J) \in \operatorname{spec}(H)$, A' is monotone (Proposition 4.2.2) and hence Lipschitz continuous (Proposition 4.2.3), and $A'(H) = \min_M H$ for small negative bump functions (Proposition 4.4.1). Examples of admissible sets are given by the monotone decreasing deformations ($\frac{\partial K}{\partial s} \leq 0$), and for every real number c by the set $\mathscr{K}_c = \{K \in \mathscr{K} \mid K_+ = c\}$, i.e., $K \in \mathscr{K}_c$ if there exists $s_0 = s_0(K)$ such that $K(s,t,x) = 0$ for $s \geq s_0$ and all t, x. Of course, $A' \leq A$ for every admissible subset $\mathscr{K}'$ of $\mathscr{K}$. For the classes $\mathscr{K}_c$, equality holds:

Proposition 4.5.2. *For every $c \in \mathbb{R}$ and every $J \in \mathscr{J}_\omega(M)$ we have $A_c(H,J) = A(H,J)$ for all $H \in C^\infty(\mathbb{S}^1 \times M, \mathbb{R})$.*

Proof. Fix $H \in C^\infty(\mathbb{S}^1 \times M, \mathbb{R})$. We need to show that given $\varepsilon > 0$ and $K \in \mathscr{K}(H)$ there exists $K_c \in \mathscr{K}_c(H)$ such that

$$\min_{\mathscr{U}(K_c,J)} a_H^- \geq \min_{\mathscr{U}(K,J)} a_H^- - \varepsilon. \tag{4.11}$$

For every $n \in \mathbb{N}$ let $\beta_n \colon \mathbb{R} \to [0,1]$ be a smooth function such that

$$\beta_n(s) = 1 \text{ for } s \leq n, \quad \beta_n(s) = 0 \text{ for } s \geq n+1,$$

and let $\gamma_n \colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\gamma_n(s) = 0 \text{ for } s \in [-\infty, n+1], \quad \gamma_n(s) = c \text{ for } s \geq n+2.$$

Then $K_n := \beta_n K + \gamma_n \in \mathscr{K}_c(H)$ for every $n \in \mathbb{N}$. We claim that (4.11) holds for at least one of the functions K_n. If not, for every $n \in \mathbb{N}$ there is a $u_n \in \mathscr{U}(K_n, J)$ such that

$$a_H^-(u_n) \leq \min_{\mathscr{U}(K,J)} a_H^- - \varepsilon. \tag{4.12}$$

Hence, by a diagonal sequence argument we find a subsequence of (u_n) that converges to some $u \in \mathscr{U}(K,J)$ in the C^∞_{loc}-topology. Since the function a_H^- is lower semicontinuous, we deduce that

$$a_H^-(u) \leq \liminf_{n\to\infty} a_H^-(u_n) \leq \min_{\mathscr{U}(K,J)} a_H^- - \varepsilon,$$

which is the desired contradiction because $u \in \mathscr{U}(K,J)$. □

Remark 4.5.3. For functions $K \in \mathscr{K}_0(H)$, removal of singularities shows that the elements of $\mathscr{U}(K,J)$ are actually open discs (or equivalently, punctured spheres), which are J-holomorphic near the origin and satisfy the Floer equation on a collar of the boundary equipped with cylindrical coordinates. If J does not depend on s, these are exactly the objects which are used in the PSS isomorphism from [13], see Chapter 9. In this approach, capping long cylinders at the positive end becomes unnecessary.

4.5.2 Dependence on J

There are various possibilities to deal with the dependency of the almost complex structure J. In Definition 4.1.1 we did not choose the easiest solution: We could have chosen a fixed ω-compatible almost complex structure J to define an action selector $A(H,J)$ in the following way:

$$A(H,J) := \sup_{K\in\mathscr{K}(H)} \min_{\mathscr{U}(K,J)} a_H^-.$$

We chose not to do that for two reasons. First, Definition 4.1.1 corresponds better to the variational approach and second, we unfortunately do not know wether $A(H) = A(H,J)$ depends on the choice of J. However this potential defect of the construction can be fixed by looking at

$$\widehat{A}(H) := \sup_{J\in\mathscr{J}(\omega)} A(H,J)$$

where $\mathscr{J}(\omega)$ denotes the set of ω-compatible almost complex structures on M. This is again a minimal action selector, since the properties of being spectral, monotone and equal to the minimum of H for little negative bumps is preserved under taking supremum (or infimum). Yet another way to deal with the issue of J-dependence is to look at families $\{J^s\}$ just as we did in Definition 4.1.1 and set

$$A(H,\{J^s\}) = \sup_{K\in\mathscr{K}(H)} \min_{\mathscr{U}(K,\{J^s\})} a_H^-.$$

Given $J_0, J_1 \in \mathscr{J}(\omega)$ and deformations $\{J_0^s\}, \{J_1^s\}$ thereof, the proof of Proposition 4.2.2 with $H_0 = H_1$ readily adapts to showing that $A(H,\{J_0^s\}) \geq A(H,\{J_1^s\})$. (One uses that the space $\mathscr{J}(\omega)$ is path-connected, and the action-energy identity which we adapted in (3.1.5) for s-dependent ω-compatible almost complex structures.) We conclude that $A(H,\{J_s\})$ does not depend on the family $\{J_s\}$ and hence is equal to $A(H)$.

Open Problem 4.5.4. Is it true that the inequalities in the diagram below are equalities, for all H?

$$\begin{tikzcd}
A(H) = \sup\limits_{\{J_s\}\in\mathscr{J}_\omega(M)} A(H,\{J_s\}) \arrow[r, phantom, "\geq"] \arrow[d, phantom, "\parallel"] & \sup\limits_{J\in\mathscr{J}(\omega)} A(H,J) \arrow[d, phantom, "\rotatebox{-90}{$\geq$}"] \\
A(H,\{J_s\}) \arrow[r, phantom, "\geq"] & A(H,J)
\end{tikzcd}$$

Chapter 5

Generalisation to other cohomology classes

Let $H \in C^\infty(\mathbb{S}^1 \times M)$ and $(K, J) \in \mathscr{D}(H)$. Let ξ be a non-zero cohomology class in $H^*(M) \cong \check{H}^*(M)$, where as before we take $\mathbb{Z}_2$ as coefficients. By Proposition 3.3.10, the cohomology class

$$\xi_{K,J} \;:=\; \big(\mathrm{ev}_z|_{\mathscr{U}(K,J)}\big)^*\xi \;\in\; \check{H}^*\big(\mathscr{U}(K,J)\big)$$

is non-zero. Note that the class $\xi_{K,J}$ does not depend on the choice of $z \in \mathbb{R} \times \mathbb{S}^1$, since a continuous path $z(t)\colon [0,1] \to \mathbb{R} \times \mathbb{S}^1$ yields a continuous homotopy of maps $\mathrm{ev}_{z(t)}|_{\mathscr{U}(K,J)}\colon \mathscr{U}(K,J) \to M$. We shall thus omit the subscript z in the sequel. For $a \in \mathbb{R}$ abbreviate

$$\mathscr{U}^a(K,J) \;=\; \{u \in \mathscr{U}(K,J) \mid a_H^-(u) \le a\}.$$

Since a_H^- is lower semicontinuous, $\mathscr{U}^a(K,J)$ is closed, and since $\mathscr{U}(K,J)$ is compact, $\mathscr{U}^a(K,J)$ is compact. Let $\iota_a^*\colon \check{H}^*(\mathscr{U}(K,J)) \to \check{H}^*(\mathscr{U}^a(K,J))$ be the homomorphism induced by the inclusion $\iota_a\colon \mathscr{U}^a(K,J) \to \mathscr{U}(K,J)$.

Definition 5.1.1. *Set* $A^-(\xi, K, J) \;:=\; \inf\big\{a \in \mathbb{R} \mid \iota_a^*(\xi_{K,J}) \in \check{H}^*(\mathscr{U}^a(K,J)) \textit{ is non-zero}\big\}$, *and*

$$A(\xi, H) \;:=\; \sup_{(K,J)\in\mathscr{D}(H)} A^-(\xi, K, J).$$

Recall that $\mathrm{spec}\,(H) = \mathrm{crit}\mathbb{A}_H \subset \mathbb{R}$ is compact. Since $a_H^-(\mathscr{U}(K,J)) \subset \mathrm{crit}\mathbb{A}_H$, the sets $\mathscr{U}^a(K,J)$ are empty for $a < \min \mathrm{crit}\mathbb{A}_H$, and so $A^-(\xi,K,J) \in \mathbb{R}$. Further, for $a < b$ we have $\iota_a^* = \iota_{a,b}^* \circ \iota_b^*$, where $\iota_{a,b}\colon \mathscr{U}^a(K,J) \to \mathscr{U}^b(K,J)$ is the inclusion. The set

$$\big\{a \in \mathbb{R} \mid \iota_a^*(\xi_{K,J}) \in \check{H}^*(\mathscr{U}^a(K,J)) \text{ is non-zero}\big\}$$

in the definition of $A^-(\xi, K, J)$ is therefore an interval which is unbounded from above and bounded from below.

Lemma 5.1.2. *$A(\xi, H)$ is a critical value of $\mathbb{A}_H$.*

Proof. Since $\mathrm{crit}\mathbb{A}_H$ is closed, it suffices to check each $A^-(\xi,K,J)$ is a critical value of $\mathbb{A}_H$. Since $a_H^-(\mathscr{U}(K,J)) \subset \mathrm{crit}\mathbb{A}_H$ and $\mathbb{A}_H$ is closed, it is enough to prove that

$$A^-(\xi,K,J) \in \overline{a_H^-(\mathscr{U}(K,J))}.$$

Arguing by contradiction, we can find $\varepsilon > 0$ such that

$$\big[A^-(\xi, K, J) - \varepsilon, A^-(\xi, K, J) + \varepsilon\big] \cap a_H^-(\mathscr{U}(K, J)) = \emptyset.$$

Therefore,

$$\mathscr{U}^{A^-(\xi,K,J)+\varepsilon}(K, J) \;=\; \mathscr{U}^{A^-(\xi,K,J)-\varepsilon}(K, J),$$

in contradiction with the definition of $A^-(\xi, K, J)$, which requires that the image of $\xi_{K,J}$ by the homomorphism

$$\check{H}^*\big(\mathscr{U}(K, J)\big) \to \check{H}^*\big(\mathscr{U}^{A^-(\xi,K,J)+\varepsilon}(K, J)\big)$$

is non-zero while its image by the homomorphism

$$\check{H}^*\big(\mathscr{U}(K, J)\big) \to \check{H}^*\big(\mathscr{U}^{A^-(\xi,K,J)-\varepsilon}(K, J)\big)$$

is zero. □

In the particular case $\xi = 1 \in H^0(M)$, we have

$$A^-(1, K, J) \;=\; \min_{u \in \mathscr{U}(K,J)} a_H^-(u),$$

and we find the minimal action selector of the previous section: $A(1, H) = A(H)$. Generalizing Properties (4.1) and (4.2) we have that for all non-zero $\xi \in H^*(M)$,

$$A(\xi, H) = 0 \qquad \text{if } H \equiv 0, \tag{5.1}$$

$$A(\xi, H + r) = A(\xi, H) + \int_{\mathbb{S}^1} r(t)\, dt \qquad \forall\, r \in C^\infty(\mathbb{S}^1),\ H \in C^\infty(\mathbb{S}^1 \times M) \tag{5.2}$$

The following proposition generalizes Propositions 4.2.2 and 4.2.3.

Proposition 5.1.3. *Let $H_0, H_1 \in C^\infty(\mathbb{S}^1 \times M)$ and $0 \neq \xi \in H^*(M)$.*

(i) **(Monotonicity)** *If $\displaystyle\int_{\mathbb{S}^1} \max_{x \in M}\big(H_1(t, x) - H_0(t, x)\big)\, dt \;\le\; 0$, then $A(\xi, H_0) \ge A(\xi, H_1)$.*

(ii) **(Lipschitz continuity)**

$$\int_{\mathbb{S}^1} \min_{x \in M} \big(H_1(t, x) - H_0(t, x)\big)\, dt \;\le\; A(\xi, H_1) - A(\xi, H_0)$$
$$\le\; \int_{\mathbb{S}^1} \max_{x \in M} \big(H_1(t, x) - H_0(t, x)\big)\, dt.$$

In particular, the action selector $A(\xi, \cdot)$ is 1-Lipschitz with respect to the sup-norm on $C^\infty(\mathbb{S}^1 \times M)$:

$$\big|A(\xi, H_1) - A(\xi, H_0)\big| \;\le\; \|H_1 - H_0\|_\infty.$$

Proof. The continuity statement (ii) follows from the monotonicity (i) and (5.2) in the same way as Proposition 4.2.3 followed from Proposition 4.2.2 and (4.2). For the proof of (i) we fix $(K_1, J_1) \in \mathscr{D}(H_1)$. As in the proof of Proposition 4.2.2 we can assume that $K_1(s, t, x) = H_1(t, x)$ for $s \le 0$, and for $\lambda \in \mathbb{R}$ we define $K_0^\lambda \in \mathscr{K}(H_0)$ by (4.6) and tke $J_0 = J_1$.

Lemma 5.1.4. *Let $a \in \mathbb{R}$, and let $\mathscr{W}$ be an open neighbourhood of $\mathscr{U}^a(K_1, J_1)$ in $C^\infty(\mathbb{R} \times \mathbb{S}^1, M)$. Then $\mathscr{U}^a(K_0^\lambda, J_1) \subset \mathscr{W}$ for all small enough λ.*

Proof. If not, there is a sequence $(\lambda_n) \subset (-\infty, -1]$ with $\lambda_n \to -\infty$, and a sequence (u_{λ_n}) with $u_{\lambda_n} \in \mathscr{U}^a(K_0^{\lambda_n}, J_1)$ but $u_{\lambda_n} \notin \mathscr{W}$. Since $\lim_{n\to\infty} K_0^{\lambda_n} = K_1$ in C^∞_{loc} and by compactness, after passing to a subsequence of (λ_n) we can assume that $u_{\lambda_n} \to u$ in C^∞_{loc} for some $u \in \mathscr{U}(K_1, J_1)$. As in the proof of Proposition 4.2.2 we find that

$$a^-_{H_1}(u) \;\le\; \liminf_{n\to\infty} a^-_{H_0}(u_{\lambda_n}) \;\le\; a.$$

Hence $u \in \mathscr{U}^a(K_1, J_1)$. But then $u_{\lambda_n} \in \mathscr{W}$ for n large enough, a contradiction. □

For the proof of (i) it suffices to show that given $\varepsilon > 0$ there exists λ such that $A^-(\xi, K_0^\lambda, J_1) \ge A^-(\xi, K_1, J_1) - \varepsilon$. Assume, instead, that

$$A^-(\xi, K_0^\lambda, J_1) \;<\; A^-(\xi, K_1, J_1) - \varepsilon \;=:\; a \qquad \text{for all } \lambda \in \mathbb{R}. \tag{5.3}$$

Given an open neighbourhood $\mathscr{W}$ of $\mathscr{U}^a(K_1, J_1)$ we choose $\lambda = \lambda(\mathscr{W})$ as in Lemma 5.1.4. We then have the commutative diagram

$$\begin{tikzcd}
& M & \\
\mathscr{U}^a(K_1, J_1) \arrow[ur, "\mathrm{ev}_1"] \arrow[r, "\iota_1"] & \mathscr{W} \arrow[u, "\mathrm{ev}_{\mathscr{W}}"'] & \mathscr{U}^a(K_0^\lambda, J_1) \arrow[ul, "\mathrm{ev}_0"'] \arrow[l, "\iota_0"']
\end{tikzcd}$$

and hence the commutative diagram

$$\begin{tikzcd}
& H^*(M) \arrow[dl, "\mathrm{ev}_1^*"'] \arrow[d, "\mathrm{ev}_{\mathscr{W}}^*"] \arrow[dr, "\mathrm{ev}_0^*"] & \\
\check{H}^*\big(\mathscr{U}^a(K_1, J_1)\big) & \check{H}^*(\mathscr{W}) \arrow[l, "\iota_1^*"'] \arrow[r, "\iota_0^*"] & \check{H}^*\big(\mathscr{U}^a(K_0^\lambda, J_1)\big)
\end{tikzcd}$$

Define the classes

$$\begin{aligned}
\xi_1 \;&:=\; \mathrm{ev}_1^*(\xi) \;=\; \iota_a^*\left(\mathrm{ev}\,|_{\mathscr{U}(K_1,J_1)}\right)^*(\xi) \;\in\; \check{H}^*\big(\mathscr{U}^a(K_1, J_1)\big),\\
\xi_0 \;&:=\; \mathrm{ev}_0^*(\xi) \;=\; \iota_a^*\left(\mathrm{ev}\,|_{\mathscr{U}(K_0^\lambda,J_1)}\right)^*(\xi) \;\in\; \check{H}^*\big(\mathscr{U}^a(K_0^\lambda, J_1)\big).
\end{aligned}$$

By assumption (5.3), $\xi_1 = 0$, while $\xi_0 \neq 0$. Since $\mathscr{U}^a(K_1, J_1)$ is compact, $\check{H}^*$ is continuous at $\mathscr{U}^a(K_1, J_1)$. In particular, we can choose the neighbourhood $\mathscr{W}$ of $\mathscr{U}^a(K_1, J_1)$ so small that ξ_1 vanishes already in $\check{H}^*(\mathscr{W})$, i.e.,

$$\xi_{\mathscr{W}} \;:=\; \mathrm{ev}_{\mathscr{W}}^*(\xi) = 0 \in \check{H}^*(\mathscr{W}),$$

see Definition 1 (ii) in [6, Appendix 8]. Then the commutativity of the right part of the above diagram yields

$$\xi_0 \;=\; \mathrm{ev}_0^*\,\xi \;=\; \iota_0^*\xi_{\mathscr{W}} \;=\; 0,$$

a contradiction. The proof of monotonicity is complete. □

Lemma 5.1.5. *For $\xi, \eta \in H^*(M) \setminus \{0\}$ with $\xi \cup \eta \neq 0$,*

$$A(\xi \cup \eta, H) \geq \max\{A(\xi, H), A(\eta, H)\}$$

Proof. For any $(K, J) \in \mathscr{D}(H)$,

$$\begin{aligned} A^-(\xi \cup \eta, K, J) &= \inf\left\{a \in \mathbb{R} \mid \iota_a^* \circ (\mathrm{ev}\,|_{\mathscr{U}(K,J)})^*(\xi \cup \eta) \neq 0\right\} \\ &= \inf\{a \in \mathbb{R} \mid (\iota_a^* \xi_{K,J}) \cup (\iota_a^* \eta_{K,J}) \neq 0\} \\ &\geq \inf\{a \in \mathbb{R} \mid \iota_a^* \xi_{K,J} \neq 0\} \\ &= A^-(\xi, K, J). \end{aligned}$$

Taking the supremum over $(K, J) \in \mathscr{D}(H)$ we find $A(\xi \cup \eta, H) \geq A(\xi, H)$, and in the same way $A(\xi \cup \eta, H) \geq A(\eta, H)$. □

As for the minimal action selector, it will be useful to describe, or at least estimate, the selectors $A(\xi, H)$ in terms of the essential spaces $\mathscr{U}_{\mathrm{ess}}(K, J) \subset \mathscr{U}(H)$. For $(K, J) \in \mathscr{D}(H)$ and $a \in \mathbb{R}$ we abbreviate

$$\mathscr{U}^a_{\mathrm{ess}}(K, J) = \{u \in \mathscr{U}_{\mathrm{ess}}(K, J) \mid a_H(u) = \mathbb{A}_H(u(0)) \leq a\}.$$

Note that here, the sublevels are with respect to a_H, while in the definition of $\mathscr{U}^a(K, J)$ they are with respect to a_H^-. Abusing notation we denote the inclusion $\mathscr{U}^a_{\mathrm{ess}}(K, J) \to \mathscr{U}_{\mathrm{ess}}(K, J)$ also by ι_a, and define

$$A^-_{\mathrm{ess}}(\xi, K, J) := \inf\left\{a \in \mathbb{R} \mid \iota_a^*\big(\mathrm{ev}_z\,|_{\mathscr{U}_{\mathrm{ess}}(K,J)}\big)^* \xi \in \check{H}^*(\mathscr{U}^a_{\mathrm{ess}}(K, J)) \text{ is non-zero}\right\},$$

and

$$A_{\mathrm{ess}}(\xi, H) := \sup_{(K,J) \in \mathscr{D}(H)} A^-_{\mathrm{ess}}(\xi, K, J).$$

By Proposition 4.3.4, $A^-_{\mathrm{ess}}(\xi, K, J) \in \mathbb{R}$. Generalizing half of Proposition 4.3.3, we have

Proposition 5.1.6. *$A^-(\xi, K, J) \geq A^-_{\mathrm{ess}}(\xi, K, J)$.*

Proof. Let $a \in \mathbb{R}$ and let $\mathscr{W}$ be an open neighbourhood of $\mathscr{U}^a_{\mathrm{ess}}(K, J)$. Then $\tau_{s_0}\big(\mathscr{U}^a(K, J)\big) \subset \mathscr{W}$ for s_0 small enough. Otherwise, we find as in the proof of Proposition 4.3.4 a sequence $s_n \to -\infty$ and a sequence $(u_n) \subset \mathscr{U}^a(K, J)$ such that $\tau_{s_n} u_n \notin \mathscr{W}$ and $\tau_{s_n} u_n \to u \in \mathscr{U}_{\mathrm{ess}}(K, J)$. Then

$$a_H(u) = \lim_{n \to \infty} a_H(\tau_{s_n} u_n) = \lim_{n \to \infty} a_H(u_n(s_n)) \leq \liminf_{n \to \infty} a_H^-(u_n) \leq a,$$

whence $u \in \mathscr{U}^a_{\mathrm{ess}}(K, J)$. But then $\tau_{s_n} u_n \in \mathscr{W}$ for n large.

Now consider the commutative diagram

$$\begin{tikzcd}
\check{H}^*(\mathscr{W}) \arrow[r, "j^*"] \arrow[d, "i^*"'] & \check{H}^*\big(\tau_{s_0}\mathscr{U}^a(K, J)\big) \arrow[r, "\tau_{s_0}^*"] & \check{H}^*\big(\mathscr{U}^a(K, J)\big) \\
\check{H}^*\big(\mathscr{U}^a_{\mathrm{ess}}(K, J)\big) & \check{H}^*(M) \arrow[l, "\mathrm{ev}_z^*"'] \arrow[ul, "\mathrm{ev}_z^*"] \arrow[u, "\mathrm{ev}_z^*"'] \arrow[ur, "\mathrm{ev}_{z'}^*"']
\end{tikzcd}$$

where again $z' = (s + s_0, t)$. Take $a > A^-(\xi, K, J)$. Then $\mathrm{ev}_{z'}^* \xi \neq 0$. Hence $\big(\mathrm{ev}_z|_{\mathscr{W}}\big)^* \xi \neq 0$ in $\check{H}^*(\mathscr{W})$. Since this holds true for every open neighbourhood $\mathscr{W}$ of $\mathscr{U}_{\mathrm{ess}}^a(K, J)$, the continuity of $\check{H}^*$ implies that $\big(\mathrm{ev}_z|_{\mathscr{U}_{\mathrm{ess}}^a(K,J)}\big)^* \xi \neq 0$. Therefore $A_{\mathrm{ess}}^-(\xi, K, J) \leq a$. Since $a > A^-(\xi, K, J)$ was arbitrary, it follows that

$$A_{\mathrm{ess}}^-(\xi, K, J) \leq A^-(\xi, K, J).$$

□

Given a subset $A \subset M$ and $\xi \in H^*(M)$ we from now on abbreviate $\xi|_A = \iota_A^*(\xi)$, where $\iota\colon A \to M$ is the inclusion. Further, a smooth map $H\colon M \to \mathbb{R}$ understood, we write M^a for the sublevel set $\{x \in M \mid H(x) \leq a\}$. The following lemma is almost tautological.

Lemma 5.1.7. *Let X be a topological space and $f\colon X \to M$ a continuous map. Let $\xi \in H^*(M)$ and $H\colon M \to \mathbb{R}$ a smooth function. If $f(X) \subset M^a$ and $\xi|_{M^a} = 0$, then $f^*\xi = 0$ in $\check{H}^*(X)$.*

Proof. Recall that $H^*(M) = \check{H}^*(M)$. Since M^a is a (deformation retract of a) manifold with boundary, also $H^*(M^a) = \check{H}^*(M^a)$. Let $\varphi \in \check{C}^*(M)$ be an Alexander–Spanier cocycle on M with $[\varphi] = \xi$. Then $\varphi|_{M^a}$ is an Alexander–Spanier cocycle for $\xi|_{M^a}$. Since $\xi|_{M^a} = 0$, there exists a coboundary $\psi \in \check{C}^{*-1}(M^a)$ with $d\psi = \varphi|_{M^a}$. Since $f(X) \subset M^a$, $f^*\psi$ is a coboundary for $f^*\varphi \in \check{C}^*(X)$:

$$d(f^*\psi) \;=\; f^*(d\psi) \;=\; f^*(\varphi|_{M^a}) \;=\; f^*\varphi.$$

Hence $f^*\xi = [f^*\varphi] = 0$ in $\check{H}^*(X)$. □

For $\xi \in H^*(M) \setminus \{0\}$ define

$$c(\xi, H) \;:=\; \inf\left\{a \in \mathbb{R} \mid \xi_{M^a} \neq 0 \text{ in } H^*(M^a)\right\}.$$

As is well-known, $c(\xi, H)$ is a critical value of H. Since we work with $\mathbb{Z}_2$ coefficients, $H^*(M^a) = \mathrm{Hom}(H_*(M^a); \mathbb{Z}_2)$ for all a. The value $c(\xi, H)$ is therefore the smallest number a such that there exists a homology class $C \in H_*(M)$ that can be represented by a cycle in M^a and such that $\xi(C) \neq 0$. We denote the generator of $H^{2n}(M)$ by $[M]$.

Proposition 5.1.8. *Let $H \in C^\infty(M)$ be an autonomous Hamiltonian. Assume that $\mathscr{P}(H)$ consists only of constant orbits and that for some s-independent J every solution of the Floer equation* (2.1) *with bounded energy does not depend on t. Then*

$$A(\xi, H) \;\geq\; c(\xi, H).$$

In particular, $A([M], H) = \max_M H$.

Proof. We need to show that $A(\xi, H, J) \geq c(\xi, H)$. Arguing by contradiction we assume that for all $K \in \mathscr{K}(H)$,

$$A^-(\xi, K, J) < c(\xi, H).$$

We fix $K \in \mathscr{K}(H)$ and, using Proposition 5.1.6, find $a \in \mathbb{R}$ with

$$A^-_{\text{ess}}(\xi, K, J) \;<\; a \;<\; c(\xi, H).$$

Then $\big(\mathrm{ev}\,|_{\mathscr{U}^a_{\text{ess}}(K,J)}\big)^*\xi \neq 0$. On the other hand, by Proposition 4.3.2 and by the assumption,

$$\mathrm{ev}\,(\mathscr{U}^a_{\text{ess}}(K,J)) \;\subset\; \mathrm{ev}\,(\mathscr{U}^a(H,J)) \;\subset\; M^a,$$

and $\xi|_{M^a} = 0$. Hence $\big(\mathrm{ev}\,|_{\mathscr{U}^a_{\text{ess}}(K,J)}\big)^*\xi = 0$ by Lemma 5.1.7, a contradiction. □

Proposition 5.1.9. *Let $H \in C^\infty(M)$ be an autonomous Hamiltonian with exactly two critical values. Assume also that X_H has no non-constant contractible 1-periodic orbits. Then*

$$A(\xi, H) \;\geq\; c(\xi, H).$$

More explicitely: $A(\xi, H) = \max_M H$ if $\xi|_{M^a} = 0$ for one and hence any $a \in (\min H, \max H)$. In particular, $A([M], H) = \max_M H$.

Proof. By assumption, $A(\xi, H) \in \{\min_M H, \max_M H\}$. Since this is also the case for $c(\xi, H)$, we can assume that $c(\xi, H) = \max H$. We wish to show that also $A(\xi, H) = \max H$. If this is not true, then $A(\xi, H) = \min H$. So $A^-(\xi, K, J) = \min H$ for all $(K, J) \in \mathscr{D}(H)$. Taking $K = H \in \mathscr{K}(H)$ and J s-independent, we in particular have

$$A^-(\xi, H, J) \;=\; \min H. \tag{5.4}$$

Fix $a \in (\min H, \max H)$. Let $u \in \mathscr{U}^a(H, J)$. Then $a^-_H(u)$ and $a^+_H(u)$ are critical values of $\mathbb{A}_H$, hence belong to $\{\min H, \max H\}$. Since H does not depend on s, we have $\min H = a^-_H(u) \geq a^+_H(u)$, and so

$$\min H = a^-_H(u) = a^+_H(u).$$

Therefore, by Lemma 3.1.5,

$$0 \;=\; a^-_H(u) - a^+_H(u) \;=\; E(u) \;=\; \int_{\mathbb{R}\times\mathbb{S}^1} \big|\partial_s u\big|^2_J \, ds\, dt,$$

and so $\partial_s u \equiv 0$. It follow that $u(t)$ is a negative gradient flow line of H whose α- and ω-limit sets belong to min-set$(H) = \{x \in M \mid H(x) = \min H\}$. Hence u is constant. Summarizing, We have shown that $\mathscr{U}^a(H, J) = \text{min-set}(H)$. It follows that $\mathrm{ev}\big(\mathscr{U}^a(H, J)\big) = \text{min-set}(H) \subset M^a$. Further, we assumed that $c(\xi, H) = \max H$, and so $\xi|_{M^a} = 0$. Lemma 5.1.7 now implies that $\big(\mathrm{ev}\,|_{\mathscr{U}^a(H,J)}\big)^*\xi = 0$, in contradiction with (5.4). □

Chapter 6

Action selectors on convex manifolds

A compact symplectic manifold is called *convex* if it has non-empty boundary and near the boundary one can find a Liouville vector field (see Section 2.7 for the definition). Also assume that $[\omega]|_{\pi_2(M)} = 0$. The Liouville vector field Y points outward along the boundary. Since the boundary is compact, we find $\varepsilon > 0$ such that the flow $\phi_Y^{\log r}(x)$ exists for $r \in (1-\varepsilon, 1]$ and $x \in \partial M$. The embedding

$$j\colon (1-\varepsilon, 1] \times \partial M \to M, \quad (r, x) \mapsto \phi_Y^{\log r}(x),$$

is such that $j^*\lambda = r\alpha$ and $j^*Y = r\frac{\partial}{\partial r}$, where $\lambda = \iota_Y \omega$ and $\alpha = \lambda|_{\partial M}$. The *completion* of M is the manifold

$$\widehat{M} \;=\; M \cup_{\partial M} ([1, \infty) \times \partial M).$$

The Liouville form λ and hence the symplectic form ω naturally extend to $\widehat{M}$ by setting $\lambda|_{[1,\infty)\times\partial M} = r\alpha$.

Let $\mathscr{F}$ be the set of smooth functions $f\colon (1, \infty) \to \mathbb{R}$ such that $f'(r) > 0$ and $f''(r) \geq 0$, and such that $f'(r)$ is so small that the Hamiltonian flow of the function $F(x, r) = f(r)$ has no 1-periodic orbits on $(1, \infty) \times \partial M$. Let $\mathscr{K}(\widehat{M})$ be the set of smooth functions $K\colon \mathbb{R} \times \mathbb{S}^1 \times \widehat{M} \to \mathbb{R}$ such that

- on M: there exists $s(K) > 0$ such that $K(s, t, x)$ does not depend on s for $|s| \geq s(K)$.
- on $\widehat{M} \setminus M$: K neither depends on s nor t and is given by a function $f \in \mathscr{F}$.

The families $J = \{J^s\} \in \mathscr{J}_\omega(\widehat{M})$ of ω-compatible almost complex structures that we consider on $\widehat{M}$ depend on s only on a compact interval and only on M, and fulfill $dr \circ J = \lambda$ on $\widehat{M} \setminus M$. For $K \in \mathscr{K}(\widehat{M})$ and $J \in \mathscr{J}_\omega(\widehat{M})$ let $\mathscr{U}(K, J)$ be the set of finite energy solutions of the Floer equation (3.1) on $\widehat{M}$.

Lemma 6.1.1. *If $u \in \mathscr{U}(K, J)$, then the image of u is contained in M.*

Proof. Assume that u is not contained in M. Since by Proposition 3.2.3 (ii) the α-limit set and the ω-limit set of u are non-empty, and since $K = F$ has no 1-periodic orbits on $\widehat{M} \setminus M$, there must be a point q on the cylinder $\mathbb{R} \times \mathbb{S}^1$ such that

$u(q) \notin M$ and $F \circ u$ attains a local maximum at q. We then find an open connected set $D \subset \mathbb{R} \times \mathbb{S}^1$ containing q with $u(D) \subset \widehat{M} \setminus M$ such that $F \circ u$ is not constant. But our choice of J implies that $F \circ u$ is subharmonic on D, whence it cannot have a non-constant local maximum. □

Lemma 6.1.1 implies that $\mathscr{U}(K, J)$ is a compact subset of $C^\infty(\mathbb{R} \times \mathbb{S}^1, \widehat{M})$, and that $\mathscr{U}(K, J)$ does not depend on the precise form of the function $f \in \mathscr{F}$. Now consider the set $\mathscr{H}(M)$ of smooth functions $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$ such that X_H has compact support in $\mathbb{S}^1 \times (M \setminus \partial M)$. In other words, for every component C_i of the boundary ∂M there exists an open neighbourhood U_i of C_i in M and a constant $c_i \in \mathbb{R}$ such that $H(t, x) = c_i$ for all $(t, x) \in \mathbb{S}^1 \times U_i$. For $H \in \mathscr{H}(M)$ let

$$\mathscr{K}(H) = \big\{K \in \mathscr{K}(\widehat{M}) \mid K|_M = H \text{ for } s \le -s(K)\big\}.$$

For $(K, J) \in \mathscr{K}(H) \times \mathscr{J}_\omega(\widehat{M}) =: \mathscr{D}(H)$ define the real number $A^-(K, J)$ as in §4.1, and set

$$A(H) = \sup_{(K,J) \in \mathscr{D}(H)} A^-(K, J).$$

As in §4.2 we find that $A(H)$ is spectral, monotone and Lipschitz continuous. To see that A is non-trivial, we wish to prove the analogue of Proposition 3.3.10:

Proposition 6.1.2. *For every $z \in \mathbb{R} \times \mathbb{S}^1$ and every $(K, J) \in \mathscr{D}(H)$ the homomorphism*

$$\big(\mathrm{ev}_z|_{\mathscr{U}(K,J)}\big)^*\colon H^*(M) \cong \check{H}^*(M) \to \check{H}^*\big(\mathscr{U}(K, J)\big)$$

induced by the evaluation map

$$\mathrm{ev}_z|_{\mathscr{U}(K,J)}\colon \mathscr{U}(K, J) \to M \tag{6.1}$$

is injective. In particular, the map (6.1) *is surjective.*

Given this result, we can proceed exactly as in §4.3 and in particular prove the analogue of Proposition 4.4.1, that implies that A is non-trivial. Proposition 6.1.2 can be proved along the same lines as Proposition 3.3.10: Again, by the continuity of Alexander–Spanier cohomology it suffices to prove Proposition 6.1.2 for the spaces $\mathscr{U}_T(K, J)$, $T > s(K)$, of Floer cylinders $u\colon [-T, T] \times \mathbb{S}^1 \to \widehat{M}$ capped off by $J^\pm$-holomorphic discs. As in the proof of Lemma 6.1.1 one sees that the elements of $\mathscr{U}_T(K, J)$ are contained in M, and so $\mathscr{U}_T(K, J)$ is compact. This time, Fredholm theory allows to think of $\mathscr{U}_T(K, J)$ as a compact $2n$-dimensional manifold with boundary ∂M. In the last step of the argument in [5] (see also page 213 of [6]) one should use Lefschetz duality (instead of Poincaré duality) to show that every map between equi-dimensional compact manifolds with boundary of degree 1 induces an injection in cohomology.

Exhaustions. A symplectic manifold (M, ω) without boundary is called convex if it can be exhausted by compact convex symplectic manifolds: $M = \bigcup_{i=1}^\infty M_i$ where $M_1 \subset M_2 \subset \ldots$ are compact convex submanifolds of M. Also assume that $[\omega]|_{\pi_2(M)} = 0$. Examples are $(\mathbb{R}^{2n}, \omega_0)$, cotangent bundles with their usual symplectic form, and, more generally, Weinstein manifolds. Given a function $H\colon \mathbb{S}^1 \times M \to \mathbb{R}$

with X_H of compact support, choose i so large that the support of X_H is contained in the interior of M_i. Then $A(H;M_j)$ is well-defined for $j \geq i$, and $A(H;M_j) \leq A(H;M_k)$ if $i \leq j \leq k$. We can thus define

$$A(H) \,:=\, \lim_{j\to\infty} A(H;M_j) \,\in\, \mathbb{R}.$$

Note that $A(H)$ is indeed finite, since with $A(H;M_j)$ also $A(H)$ belong to spec$\,(H)$. Also note that $A(H)$ does not depend on the exhaustion $M_1 \subset M_2 \subset \ldots$ of M. One readily checks that $A(H)$ is an action selector on the space of functions $H \colon \mathbb{S}^1 \times M \to \mathbb{R}$ with X_H of compact support.

Chapter 7

Axiomatisation and formal consequences

It is useful to define an action selector by a few properties ("axioms"), and to formally derive other properties from these axioms. In this way, it becomes clearer which properties of an action selector are fundamental, and which other properties are just a formal consequence of these fundamental properties. The axiomatic approach also makes clear that properties that hold for some action selectors but do not follow from the axioms, rely on the specific construction of the selectors for which they hold. For example, the "triangle inequality" $\sigma(H_1+H_2) \geq \sigma(H_1)+\sigma(H_2)$, and the minimum formula $\sigma(H_1+H_2) = \min\{\sigma(H_1), \sigma(H_2)\}$ for functions supported in disjoint incompressible Liouville domains, both hold for the Viterbo selector and the Floer selector, but are unknown for general minimal selectors.
An attempt to axiomatize action selectors was made in [3], and a very nice and slender set of four axioms was given in [7]. We here give an even smaller list of axioms, that retains the first two axioms in [7], but alters their non-triviality axiom and discards the minimum formula axiom.
Assume that $[\omega]|_{\pi_2(M)} = 0$. Remember that for M closed we set $\mathscr{H}(M) = C^\infty(\mathbb{S}^1 \times M, \mathbb{R})$. In addition, if M is convex, let $\mathscr{H}(M)$ as in Chapter 6 be the set of functions in $C^\infty(\mathbb{S}^1 \times M, \mathbb{R})$ such that X_H has compact support in the interior of $\mathbb{S}^1 \times M$. We define the support of $H \in \mathscr{H}(M)$ as the closure of $\{x \in M \mid H(t,x) \neq 0 \text{ for some } t \in \mathbb{S}^1\}$.

Definition 7.1.1. An action selector for a symplectic manifold is a map $\sigma\colon \mathscr{H}(M) \to \mathbb{R}$ that satisfies the following two axioms.

A1 (Spectrality) $\sigma(H) \in \operatorname{spec}(H)$ for all $H \in \mathscr{H}(M)$.

A2 (C^∞-continuity) σ is continuous with respect to the C^∞-topology on $\mathscr{H}(M)$.

An action selector is called minimal if, in addition,

A3 (Local non-triviality) There exists a function $H \in \mathscr{H}(M)$ with $H \leq 0$ and support in a symplectically embedded ball in M such that $\sigma(H) < 0$.

Remark 7.1.2. Assume that $\sigma\colon \mathscr{H}(M) \to \mathbb{R}$ satisfies the spectrality axiom A1. Then C^∞-continuity of σ is equivalent to C^0-continuity of σ, and continuity of σ implies its monotonicity, see assertions 5 and 4 of Proposition 7.1.3 below. On the other hand, it is not clear that monotonicity of σ, together with spectrality, implies its continuity, but this is so if σ in addition has the shift property $\sigma(H+c) = \sigma(H)+c$ for all H and $c \in \mathbb{R}$, cf. the proof of Proposition 4.2.3.

Our minimal selector $A(H) = A(1, H)$ is indeed a minimal action selector, since it is spectral, C^∞-continuous since even Lipschitz continuous with respect to the C^0-norm by Proposition 4.2.3, and non-trivial by Proposition 4.4.1. We note that the proof of monotonicity of A can be readily altered near the end to show directly that A is C^∞-continuous. Proposition 7.1.3 implies many other properties of A, some of which we have already verified.
Recall that the function $(H_1\#H_2)(t,x) = H_1(t,x) + H_2(t,(\phi_{H_1})^{-1}(x))$ generates the isotopy $\phi^t_{H_1} \circ \phi^t_{H_2}$. We write $H_1 \leq H_2$ if $H_1(t,x) \leq H_2(t,x)$ for all $(t,x) \in \mathbb{S}^1 \times M$. Following [21] and [7] we have

Proposition 7.1.3. *Assume that* (M,ω) *is a symplectically aspherical manifold that is closed or convex (with or without boundary). Then every action selector* σ *on* $\mathscr{H}(M)$ *has the following properties.*

1. **Zero:** $\sigma(H) = 0$ *if* $H \equiv 0$.

2. **Shift:** $\sigma(H + r) = \sigma(H) + \int_{\mathbb{S}^1} r(t)\,dt$ *if* $r \colon \mathbb{S}^1 \to \mathbb{R}$ *is a function of time.*

3. **Coordinate change:** *If* ψ *is a symplectomorphism of* (M,ω) *that is isotopic to the identity, then* $\sigma(H) = \sigma(H \circ \psi)$.

4. **Monotonicity:** $\sigma(H_1) \leq \sigma(H_2)$ *if* $H_1 \leq H_2$.

5. **Lipschitz continuity:** $E^-(H_1 - H_2) \leq \sigma(H_1) - \sigma(H_2) \leq E^+(H_1 - H_2)$. *In particular,* $E^-(H) \leq \sigma(H) \leq E^+(H)$.

6. **Energy-Capacity inequality:** $|\sigma(H_1)| \leq \|H_2\|$ *if* ϕ_{H_2} *displaces the support of* H_1.

7. **Composition:** $\sigma(H_1) + E^-(H_2) \leq \sigma\left(H_1\#H_2\right) \leq \sigma(H_1) + E^+(H_2)$.

If, in addition, σ *is a minimal action selector, then:*

8. **Non-degeneracy:** *If* $H \leq 0$ *and* $H \neq 0$*, then* $\sigma(H) < 0$.

9. **Non-positivity:** *If* H *has support in an incompressible Liouville domain, then* $\sigma(H) \leq 0$. *In particular,* $\sigma(H) = 0$ *if* $H \geq 0$.

Outline of the proof. Most properties are proven in [7, §3.1]. We focus on the new parts. The first seven properties follow from the spectrality and the continuity axiom, together with the fact that the spectrum is nowhere dense. This is immediate for Properties 1, 2, 3, and is shown for Properties 4, 5, 6 in [7]. For Property 7 we compute, using Lipschitz continuity,

$$\sigma(H_1\#H_2) - \sigma(H_1) \;\leq\; E^+(H_1\#H_2 - H_1) \;=\; E^+(H_2 \circ (\phi^t_{H_1})^{-1}) \;=\; E^+(H_2)$$

and similarly $\sigma(H_1\#H_2) - \sigma(H_1) \geq E^-(H_1\#H_2 - H_1) = E^-(H_2)$.
For the proof of Properties 8 and 9 we need to work a bit more. Let $U \subset M$ be a Liouville domain. (The equality $U = M$ is not excluded.) Fix a Liouville vector

field Y and denote by $\delta_\tau\colon U \to U$ its flow. By the symplectic contraction principle (see Section 2.7) we obtain for the Hamiltonian

$$H_\tau(t,x) := \begin{cases} e^\tau H(t, \delta_\tau^{-1}(x)) & \text{if } x \in U_\tau, \\ 0 & \text{if } x \notin U_\tau, \end{cases}$$

$$\sigma(H_\tau) = e^\tau \sigma(H) \tag{7.1}$$

for all $\tau \leq 0$.

Lemma 7.1.4. *If G is autonomous with $G \leq 0$ and $G \neq 0$, then $\sigma(G) < 0$.*

Proof. Choose an open set $U \subset M$ such that $G|_{\overline{U}} < 0$. Let H and $B \subset M$ be a function and a symplectically embedded ball as in Axiom A3, and let $0 \in B$ be the center of B. Take $x \in U$, and choose a Hamiltonian isotopy ψ of M with $\psi(0) = x$. Then we find $\tau < 0$ such that $\psi(B_\tau) \subset U$. Choosing τ smaller if necessary, we have $G \leq H_\tau \circ \psi$. Using Properties 3 and 4 and the identity (7.1) we obtain $\sigma(G) \leq \sigma(H_\tau \circ \psi) = \sigma(H_\tau) = e^\tau \sigma(H) < 0$. □

Property 8 now readily follows: Given $H \leq 0$ with $H \neq 0$ we find $t_0 \in (0,1)$ and $x_0 \in M$ with $H(t_0, x_0) < 0$. We can thus construct a function of the form $\alpha(t)G(x)$ with α a non-negative bump function around t_0 and G as in Lemma 7.1.4 such that $H \leq \alpha G$. The time-1 path generated by αG is isotopic to the time-1 path of cG, where $c = \int_0^1 \alpha(t)\,dt > 0$. Hence $\sigma(H) \leq \sigma(\alpha G) = \sigma(cG) < 0$.
To prove Property 9 we choose $\varepsilon > 0$ so small that we find a smooth function $f\colon M \to \mathbb{R}$ such that

$$f(x) = \varepsilon \text{ on } U_{-2}, \quad f(x) = 0 \text{ on } M \setminus U_{-1}, \quad f(x) = f(r) \text{ on } U_{-1} \setminus U_{-2},$$

such that ε and 0 are the only critical values of f, and such that ϕ_f^t has no nontrivial 1-periodic orbits. By spectrality, $\sigma(f) \in \{0, \varepsilon\}$. Take G as in Lemma 7.1.4 with support in $M \setminus U_{-1}$ and such that $G \geq -\varepsilon$. Then $f - \varepsilon \leq G$ and hence $\sigma(f - \varepsilon) \leq \sigma(G) < 0$. Together with the shift property, $\sigma(f) = \sigma(f - \varepsilon) + \varepsilon < \varepsilon$, whence $\sigma(f) = 0$. Given $H \in \mathscr{H}(U)$ we find $\tau < 0$ so small that $H_\tau \leq f$. Then $\sigma(H_\tau) \leq \sigma(f) = 0$ by monotonicity, and so $\sigma(H) = e^\tau \sigma(H_\tau) \leq 0$ by using symplectic contraction (see section 2.7). □

Path independence

Let $\sigma\colon \mathscr{H}(M) \to \mathbb{R}$ be an action selector. Does σ induce a map $\operatorname{Ham}(M, \omega) \to \mathbb{R}$? By $\operatorname{Ham}(M, \omega)$ we denote the group of those Hamiltonian diffeomorphisms that are time-1 maps ϕ_H^1 generated by functions $H \in \mathscr{H}(M)$.
If two functions in $\mathscr{H}(M)$ differ by a constant, they have the same time-1 map. In this paragraph we therefore restrict σ to the set $\mathscr{H}_0(M)$ of normalized functions, by which we mean that $\int_M H(t, \cdot)\, \omega^n = 0$ if M is closed and that each partial function $H(t, \cdot)$, $t \in \mathbb{S}^1$, has compact support in the interior of M if M is open. Write $H_0 \sim H_1$ if H_0, H_1 are the endpoints of a smooth path H_s of functions in $\mathscr{H}_0(M)$ that all generate the same Hamiltonian diffeomorphism.

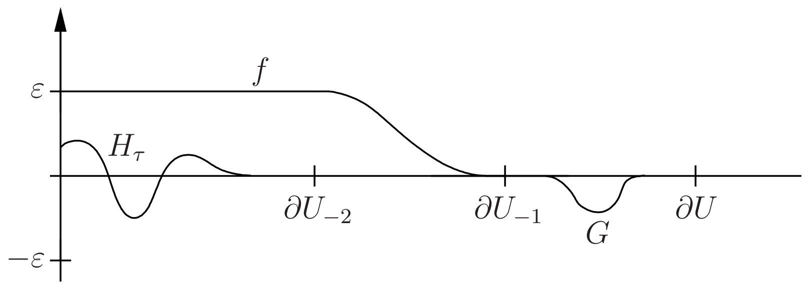


Figure 7.1: The functions f, G, and H_τ

Lemma 7.1.5. *If $H_0 \sim H_1$, then $\sigma(H_0) = \sigma(H_1)$.*

Proof. This follows from the continuity of σ if one knows that the sets spec (H_s) are independent of s, which in turn easily follows if the flow of H_0 has a contractible 1-periodic orbit, see [18, §3.1] and [4, Cor. 6.2]. The existence of such an orbit is obvious if M is open (take a point off the support of H_0) but relies on Floer homology if M is closed. □

The lemma implies that σ descends from $\mathscr{H}_0(M)$ to the universal cover $\widetilde{\mathrm{Ham}}(M, \omega)$. Does σ further descend to $\mathrm{Ham}(M, \omega)$? This is so if one knows that $\phi_G = \phi_H$ implies that spec (G) = spec (H), and that σ satisfies the triangle inequality, see [4, proof of Prop. 7.1]. The first requirement is easy to verify for M open [4, Cor. 6.2], and it holds true by a difficult argument also for M closed if one further assumes that also the first Chern class of (M, ω) vanishes on $\pi_2(M)$, see [18, Theorem 1.1]. We do not know the triangle inequality for our A. At least for Liouville domains one can go around the triangle inequality:

Proposition 7.1.6. *Assume that (M, ω) is a Liouville domain (or an exhaustion thereof). Then for any action selector σ on $\mathscr{H}(M)$ it holds that $\sigma(G) = \sigma(H)$ whenever G and H are normalized Hamiltonians with $\phi_G = \phi_H$.*

Proof. The claim is shown for $(\mathbb{R}^{2n}, \omega_0)$ in [6, Proposition 11 in §5.4]. Their proof can be generalized to any Liouville domain. We give a somewhat streamlined argument. Let $L \in \mathscr{H}_0(M)$ be such that $\phi_L = \mathrm{id}$, that is, ϕ_L^t, $t \in [0,1]$, is a loop in $\mathrm{Ham}(M, \omega)$. Then for every $\tau \leq 0$ the function $L_\tau(x) := e^\tau L(t, \delta_\tau^{-1}(x))$ generates the loop

$$\phi^t_{L_\tau}(x) \;=\; \begin{cases} \delta_\tau \circ \phi_L^t \circ \delta_\tau^{-1}(x) & \text{if } \; x \in M_\tau, \\ x & \text{if } \; x \notin M_\tau. \end{cases}$$

Assume that $\phi_G = \phi_H$. Let $G^-(t,x) = -G(t, \phi_G^t(x))$ be the function generating ϕ_G^{-t}. Then $L = G^- \# H$ generates the loop $\phi_G^{-t} \circ \phi_H^t$. Now note that

$$H \;\sim\; G \# G^- \# H \;=\; G \# L \;\sim\; G \# L_\tau$$

and hence $\sigma(H) = \sigma(G \# L_\tau) \leq \sigma(G) + E^+(L_\tau) = \sigma(G) + e^\tau E^+(L)$ for every $\tau \leq 0$, where we have used Property 7. Hence $\sigma(H) \leq \sigma(G)$. In the same way, $\sigma(G) \leq \sigma(H)$. □

Corollary 7.1.7. *The conclusion of Proposition 7.1.6 also holds for all 2-dimensional symplectic manifolds (M, ω) with M not diffeomorphic to the 2-sphere.*

Proof. If (M, ω) is closed and different from the sphere, then $\pi_1(\mathrm{Ham}(M, \omega))$ is trivial [14, §7.2.B], and the claim follows. If (M, ω) is not closed, then we find a compact surface $M' \subset M$ that contains the support of G and H. The surface (M', ω) is a Liouville domain, see [10, Exercise 3.5.30], and so the claim follows from Lemma 7.1.6. □

Example 7.1.8. We conclude this section with an instructive example. Let (M, ω) be the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ with its usual area form. Choose a symplectic embedding $\varphi\colon D(9\varepsilon) \to (M, \omega)$ of the open 2-disc of area 9ε. Let $H = H_1 + H_2\colon M \to \mathbb{R}$ be a function such that H_1 is a bump function supported in $\varphi(D(2\varepsilon))$ with $\operatorname{spec}(H_1) = \{0, -\varepsilon\}$, and $H_2 \equiv 0$ on $\varphi(D(3\varepsilon))$ and $H_2 \equiv 1$ on $\varphi(M \setminus D(4\varepsilon))$, see Figure 7.2.

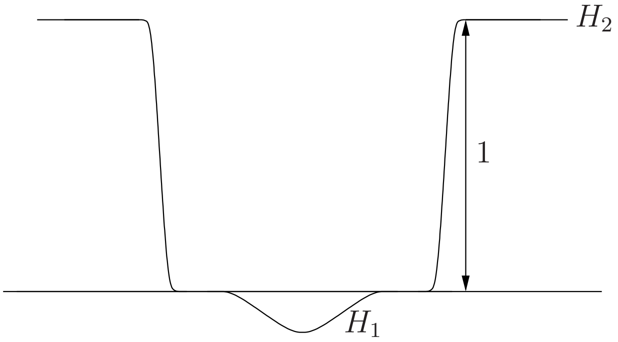


Figure 7.2: The function $H = H_1 + H_2$

Let σ be a minimal action selector on $\mathscr{H}(M)$.

Claim. $\sigma(H_1) = -\varepsilon,\ \ \sigma(H_2) \geq 1 - 5\varepsilon,\ \ \sigma(H_1 + H_2) \geq 1 - 6\varepsilon$

Proof. $\sigma(H_1) = -\varepsilon$ follows from Property 8. The function $H_2 - 1$ is supported in $\varphi(D(4\varepsilon))$, which is displaceable in $\varphi(D(9\varepsilon))$ by a Hamiltonian G with $\|G\| \leq 5\varepsilon$, see e.g. [6, p. 171]. Hence $|\sigma(H_2 - 1)| \leq 5\varepsilon$ by Property 6, and so, by the shift property, $\sigma(H_2) = \sigma(H_2 - 1) + 1 \geq 1 - 5\varepsilon$. Finally, $\sigma(H_1 + H_2) \geq \sigma(-\varepsilon + H_2) = \sigma(H_2) - \varepsilon \geq 1 - 6\varepsilon$. □

Since $\varepsilon < \frac{1}{9} < \frac{1}{6}$, the claim shows that

$$-\varepsilon \;=\; \min\{\sigma(H_1), \sigma(H_2)\} \;<\; \sigma(H_1 + H_2).$$

On the other hand, the minimum formula for the PSS-selector from [7] says that

$$\min\{\sigma_{\mathrm{PSS}}(H_1), \sigma_{\mathrm{PSS}}(H_2)\} \;=\; \sigma_{\mathrm{PSS}}(H_1 + H_2)$$

for any two functions H_1, H_2 supported in disjoint incompressible Liouville domains. Recall from the proof of Corollary 7.1.7 that the domains $U_1 = \varphi(D(2\varepsilon))$ and

$U_2 = M \setminus \varphi(D(3\varepsilon))$, that contain the supports of H_1 and H_2, are Liouville domains. However, U_2 is not incompressible in M, since $\pi_1(U_2)$ is the free group of two generators, while $\pi_1(M)$ is the free Abelian group of two generators. We conclude that the incompressibility assumption is essential for the minimum formula to hold true.

Chapter 8

Three applications

In this section we illustrate by three examples how minimal action selectors give short proofs of theorems in Hamiltonian mechanics. Our three examples are Gromov's nonsqueezing theorem, the Weinstein conjecture for displaceable hypersurfaces, and the unboundedness of Hofer's metric.

8.1 The non-squeezing theorem

Definition 8.1.1. *Throughout this section we denote by* $B^{2n}(r) \subset \mathbb{R}^{2n}$ *the open ball with radius* r *and by* $Z^{2n}(R) := B^2(R) \times \mathbb{C}^{n-1}$ *the infinite cylinder of radius* R.

Now we are able to state Gromov's famous non-squeezing theorem.

Theorem 8.1.2. *Suppose there is a symplectic embedding*

$$s:\ B^{2n}(r) \hookrightarrow Z^{2n}(R),$$

then $r \leq R$.

Definition 8.1.3. *Let* σ *be a minimal action selector and*

$$\mathcal{H}_c(M) := \{H \in C^\infty(M \times \mathbb{S}^1, \mathbb{R}) \mid \operatorname{supp} X_H \subset \operatorname{Int}(M \times \mathbb{S}^1)\}.$$

Then we assign to any subset $U \subset (M, \omega)$ *the number*

$$c(U) := \sup_{H \in \mathcal{H}_c(U)} |\sigma(H)|.$$

Proposition 8.1.4. *Let* $V \subset \mathbb{R}^{2n}$ *be such that there exists a symplectic embedding* $\psi : B^{2n}(r) \hookrightarrow V$. *Then* $c(B^{2n}(\rho)) \leq c(V)$ *for* $\rho < r$.

The proof of this proposition follows directly from the invariance of σ under coordinate change (see 7.1.3) and the following lemma, whose elementary proof can be found in Appendix A of [17].

Lemma 8.1.5. *For every* $\rho < r$ *there exists a symplectic isotopy* ψ^t *of* $\mathbb{R}^{2n}$ *between* id *and* ψ^1 *such that* ψ^1 *restricts to* ψ *on* $B^{2n}(\rho)$.

Proof of Theorem 8.1.2. The first step is to prove $\pi r^2 \leq c(B^{2n}(r))$. To see this we define a special radial Hamiltonian in the following way. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with support in $[0, r^2]$ that is constant and minimal near 0 and satisfies $f'(x) \in [0, 1)$. Then we define the radial Hamiltonian H by

$$H_f : \mathbb{R}^{2n} \to \mathbb{R} \;\; H_f(x) := \pi f(\|x\|^2).$$

X_{H_f} has a closed orbit if and only if $f'(\|x\|^2) \in \mathbb{Z}$. Therefore the action spectrum of H_f contains only 0 and $\min(H_f)$. Since we can choose f such that $\min(H_f)$ is arbitrarily close to $-\pi r^2$, we find

$$\pi r^2 \leq c(B^{2n}(r)).$$

In the second step we will show $c(Z^{2n}(R)) \leq \pi R^2$. This follows directly from the Energy-Capacity inequality (see Chapter 7), since it is possible to displace the cylinder $Z^{2n}(R)$ with energy $\pi R^2 + \varepsilon$, where ε is arbitrarily small. Using Proposition 8.1.4 we can finish the proof: For every $\rho < r$,

$$\pi \rho^2 \leq c(B^{2n}(\rho)) \leq c(Z^{2n}(R)) \leq \pi R^2.$$

Hence $r \leq R$.

□

8.2 The Weinstein conjecture

In this section, (M, ω) is a closed or convex symplectically aspherical manifold. The following theorem will readily imply the Weinstein conjecture.

Theorem 8.2.1 (Almost existence of closed orbits)**.** *Let $H \in C^\infty(M, \mathbb{R})$ be a Hamiltonian and $c \in \mathbb{R}$ a regular value such that $S := H^{-1}(c)$ is displaceable. Then the Hamiltonian vector field X_H admits a periodic orbit on a hypersurface S', arbitrarily close to S.*

Proof. Let $K \in C^\infty(M, \mathbb{R})$ that displaces S:

$$\varphi_K(S) \cap S = \emptyset.$$

Since S is compact and c is a regular value, we find $\varepsilon > 0$ such that φ_K displaces the whole neighbourhood $U = H^{-1}((c - \varepsilon, c + \varepsilon))$. Define a non-positive function $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ with support in the intervall $(c-\varepsilon, c+\varepsilon)$ and only minimum $f_\varepsilon(c) = -\|K\| - 1$ such that its only critical values are 0 and $-\|K\| - 1$. Now we choose a minimal action selector σ. Since σ is locally non-trivial, we find a function $G \in C^\infty(M, \mathbb{R})$ such that $F_\varepsilon := f_\varepsilon \circ H \leq G$ and $\sigma(G) < 0$. Then

$$\sigma(F_\varepsilon) \leq \sigma(G) < 0.$$

In addition, by the Energy-Capacity inequality (see Chapter 7), we find

$$|\sigma(F_\varepsilon)| \leq \|K\|.$$

Hence $-\|K\|-1<\sigma(F_\varepsilon)<0$. Since $\sigma(F_\varepsilon)$ is in the spectrum, it follows that there exists a non-trivial orbit of F_ε and therefore of H. By choosing ε arbitrarily small, we obtain a non-trivial orbit of X_H arbitrarily close to S. □

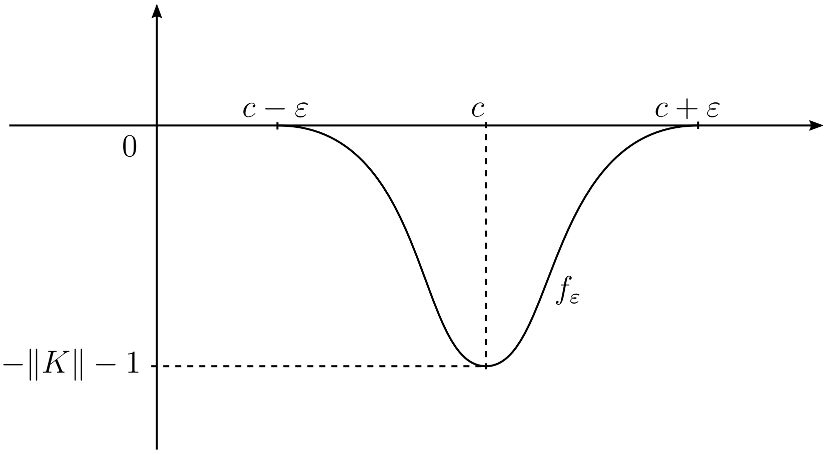


Figure 8.1: The function f_ε

Definition 8.2.2. *1. Let $S\subset(M,\omega_0)$ be a closed hypersurface. Then we denote by*

$$\mathcal{L}_S := \{(p,\xi)\in T_pS \mid \omega_0(\xi,\eta)=0 \;\; \forall\eta\in T_pS\}$$

the characteristic line bundle of S.

2. *We call a smooth map $x:\mathbb{S}^1\to S$ a closed characteristic if $(x(t),\dot{x}(t))\in\mathcal{L}_S$ for all t in $\mathbb{S}^1$.*

3. *Finally we denote the set of all closed characteristics by $\mathcal{P}(S)$.*

Theorem 8.2.3 (Weinstein). *Every compact displaceable hypersurface of contact type $S\subset(M,\omega)$ carries a closed characteristic.*

Proof. The fact that S is of contact type implies the existence of a special vector field Y on U which is transverse to S and satisfies $\mathscr{L}_Y\omega=\omega$. For a proof we refer to Proposition 4 in [6]. This implies the existence of a whole family (S_ε) of hypersurfaces of contact type close to S: We denote the flow of Y by φ_Y^t and obtain therefore, for ε small enough, a diffeomorphism

$$\psi: S\times(-\varepsilon,\varepsilon)\to U, \quad \psi(p,t):=\varphi_Y^t(p),$$

and, in addition

$$\left(\varphi_Y^t\right)^*\omega_0=e^t\omega_0.$$

For details, see Section 2.7. Let $\xi\in\mathcal{L}_S(p)$ and $\eta\in T_pS$ arbitrary, then

$$0=\omega_0(\xi,\eta)=e^t\omega_0(\xi,\eta)=\left(\varphi_X^t\right)^*\omega_0(\xi,\eta)=\omega_0(d\varphi_Y^t\xi,d\varphi_Y^t\eta).$$

This computation shows that $d\varphi_X^t\xi\in\mathcal{L}_{S_t}(\varphi_X^t(p))$, so we have a bijection between $\mathcal{P}(S)$ and $\mathcal{P}(S_t)$. By Theorem 8.2.1 we know that $\mathcal{P}(S_t)\neq\emptyset$ for a t arbitrarily close to 0 and therefore also $\mathcal{P}(S)\neq 0$. □

8.3 Unboundedness of Hofer's metric

Hofer's metric on $\mathrm{Ham}(M,\omega)$ is the biinvariant metric defined by

$$d(\phi, \mathrm{id}) \;=\; \inf_H \|H\|$$

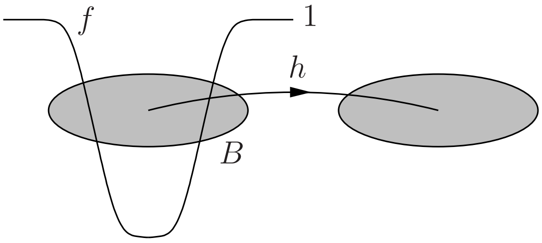
where H varies over those $H \in \mathscr{H}(M)$ with $\phi_H^1 = \phi$ and where $\|H\|$ is the Hofer norm defined in Section 2.6. The following result is due to Ostrover [12].

Theorem 8.3.1. *Let (M,ω) be a closed symplectically aspherical manifold such that $[\omega]$ and c_1 vanish on (M,ω). Then the Hofer metric on $\mathrm{Ham}(M,\omega)$ is unbounded.* [1]

Proof. Let $B \subset M$ be a symplectically embedded ball in M, so small that there exists a Hamiltonian diffeomorphism h of M with $h(B) \cap B = \emptyset$. Let $f\colon M \to \mathbb{R}$ be a function on B such that $f = 1$ on $M \setminus B$ and $\int_M f\,\omega^n = 0$. For $s \in \mathbb{R}$ consider the Hamiltonian diffeomorphism

$$\phi_s \;=\; h \circ \phi_{sf} \;=\; h \circ \phi_f^s.$$

We can assume that h is the time-1 map of an autonomous Hamiltonian H with $\int_M H\,\omega^n = 0$.



Let G_s be any normalized Hamiltonian generating ϕ_s. We shall prove that

$$\mathrm{spec}\,(G_s) \;=\; \mathrm{spec}\,(H) + s. \tag{8.1}$$

Now let σ be any action selector on $\mathscr{H}(M)$. Since $\mathrm{spec}\,H$ is nowhere dense and σ is continuous, (8.1) implies that $\sigma(G_s) = s_0 + s$ for some $s_0 \in \mathbb{R}$ and every $s \in \mathbb{R}$. Further, since G_s is normalized, $E^-(G_s) \le 0 \le E^+(G_s)$ for every s. Property 5 of Proposition 7.1.3 thus implies that

$$\|G_s\| \;=\; E^+(G_s) - E^-(G_s) \;\ge\; E^+(G_s) \;\ge\; \sigma(G_s) = s_0 + s.$$

This holds for all normalized Hamiltonians generating ϕ_s, and so $d(\phi_s, \mathrm{id}) \ge s_0 + s \to +\infty$ as $s \to +\infty$. In order to prove (8.1) we use the deep fact from [18] already mentioned in Chapter 7 that under our assumptions on (M,ω), the set $\mathrm{spec}\,(G_s)$ does not depend on the specific choice of the normalized Hamiltonian G_s generating ϕ_s.

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[1] If $(M, d\lambda)$ is exact and of finite volume, the unboundedness of Hofer's metric follows at once from the Calabi homomorphism.

It therefore suffices to prove (8.1) for the "natural" Hamiltonian generating $h \circ \phi_{sf}$ given by

$$\widehat{G}_s(t,x) \;=\; \begin{cases} \alpha(t+\frac{1}{2})\, s\, f(x) & \text{if } t \in \left[0, \frac{1}{2}\right], \\ \alpha(t)\, H(x) & \text{if } t \in \left[\frac{1}{2}, 1\right], \end{cases}$$

that first generates the map ϕ_{sf} in time $\frac{1}{2}$ and then generates the map h in time $\frac{1}{2}$, yielding $h \circ \phi_{sf}$ in time 1. Here, $\alpha\colon \mathbb{R} \to \mathbb{R}$ is a smooth non-negative function with support in $\left[\frac{1}{2}, 1\right]$ and $\int_{\mathbb{R}} \alpha(t)\, dt = 1$. At first reading one should take $\alpha \equiv 2$, but this would result in a Hamiltonian G_s not smooth at $t = \frac{1}{2}$. Since $h(B) \cap B = \emptyset$, the contracible 1-periodic orbits of $\phi^t_{\widehat{G}_s}$ are exactly the contracible 1-periodic orbits of $\phi^t_{\alpha H}$: Such an orbit γ must start outside the ball B and does not move for $t \in \left[0, \frac{1}{2}\right]$, hence $f \equiv 1$ along γ. The autonomous Hamiltonian H is also constant along γ. After reparametrization, γ corresponds to a 1-periodic γ_H of ϕ^t_H. Given such an orbit γ, and a disc $\overline{\gamma}$ that restricts to γ along its boundary, we compute the actions

$$\begin{aligned} \mathbb{A}_{\widehat{G}_s}(\gamma) &= \int_{\overline{\gamma}} \omega + s \int_0^{\frac{1}{2}} \alpha(t+\tfrac{1}{2})\, f(\gamma(t))\, dt + \int_{\frac{1}{2}}^1 \alpha(t)\, H(\gamma(t))\, dt \\ &= \int_{\overline{\gamma}} \omega + s + \int_0^1 H(\gamma_H(t))\, dt \\ &= \mathbb{A}_H(\gamma_H) + s. \end{aligned}$$

Claim (8.1) follows. □

Chapter 9

The minimal selector versus the PSS selector

In [18] an action selector A_{PSS} for compact aspherical symplectic manifolds was constructed using Floer homology and the PSS isomorphism, and this selector can also be constructed for convex symplectic manifolds [4]. There are two important properties of the PSS selector that we were not able to prove (directly) for our selector: The triangle inequality $A_{\mathrm{PSS}}(G\#H) \leq A_{\mathrm{PSS}}(G) + A_{\mathrm{PSS}}(H)$, and the max formula. The triangle inequality leads to a biinvariant metric on the group $\mathrm{Ham}(M,\omega)$, and the max formula leads to an algorithm for computing A_{PSS} on autonomous Hamiltonians on surfaces [7]. These two and all further properties of the PSS selector would of course hold for the minimal selector if we could show that they agree. For now we have

Proposition 9.1.1. *For every s-independent $J \in \mathscr{J}_\omega(M)$ it holds that*

$$A(H,J) \leq A_{\mathrm{PSS}}(H)$$

for all $H \in C^\infty(\mathbb{S}^1 \times M, \mathbb{R})$.

Proof. We briefly need to recall the definition of the PSS selector. Since A and A_{PSS} are C^0-continuous, we can assume that all 1-periodic orbits of H are nondegenerate, in the sense that for every such orbit x, 1 is not in the spectrum of the linearized return map $d\phi_H^1(x(0))$. There are then finitely many 1-periodic orbits of ϕ_H^t. The Floer chain group is the graded $\mathbb{Z}_2$ vector space $\mathrm{CF}_*(H)$ generated by these orbits, where the grading is defined by the Conley–Zehnder index μ_{CZ}. Fix a generic compatible almost complex structure J on (M,ω), and let $\partial_k\colon \mathrm{CF}_k(H) \to \mathrm{CF}_{k-1}(H)$ be the Floer boundary operator defined by counting finite energy solutions of (2.1). Its homology $\mathrm{HF}_*(H,J)$ is called the Floer homology of (H,J). Now choose a Morse function $f\colon M \to \mathbb{R}$ and a Riemannian metric g on M such that the flow of $-\nabla_g f$ is Morse–Smale, and let $\mathrm{CM}_*(f,g)$ be the Morse chain complex over $\mathbb{Z}_2$ generated by the critical points of f and graded by the Morse index $\operatorname{ind} p$. Its homology $\mathrm{HM}_*(f,g)$, the Morse homology of f and g, is independent of (f,g), and in fact is isomorphic to the homology $\mathrm{H}_*(M)$. Fix $K \in \mathscr{K}_0(H)$, that is, a smooth function $K\colon \mathbb{R}\times\mathbb{S}^1\times M \to \mathbb{R}$ such that for some $s_-, s_+ \in \mathbb{R}$,

$$K(s,t,x) = H(t,x) \text{ for } s \leq s_-, \quad K(s,t,x) = 0 \text{ for } s \geq s_+.$$

Given a finite energy solution u of the s-dependent Floer equation (1.2) for (K, J), the limit $\mathrm{ev}(u) := \lim_{s\to+\infty} u(s,t) \in M$ exists. For critical points x of $\mathbb{A}_H$ and p of f denote by $\mathcal{M}_{x,p}(K,J,f,g)$ the space of solutions (u,v), where u is as above, and v is a flow line of $-\nabla_g f$ such that $v(0) = \mathrm{ev}(u)$. This space is a manifold of dimension $\mu_{\mathrm{CZ}}(x) - \mathrm{ind}\,(p)$. Define the map $\psi_K\colon \mathrm{CF}_*(H,J) \to \mathrm{CM}_*(f,g)$ on generators by counting mod 2 the elements of $\mathcal{M}_{x,p}(K,J,f,g)$ between critical points x, p of equal index $*$. The map ψ_K commutes with the boundary operators on $\mathrm{CF}_*(H,J)$ and $\mathrm{CM}_*(f,g)$. Let $\Psi_K\colon \mathrm{HF}_*(H,J) \to \mathrm{HM}_*(f,g)$ be the induced map.

Lemma 9.1.2. *The map Ψ_K is independent of the choice of (K,J) and (f,g). It is called the PSS map.*

Proof. For $i = 0,1$ let (K_i, f_i, g_i) be two triples as in the above construction. Choose a generic path (K_s, f_s, g_s), $s \in [0,1]$, between these two triples, where we assume that the support of $\partial_s K_s$ is in a uniform compact interval. Fix critical points x of $\mathbb{A}_H$ and p of f of equal index $*$. Consider now the universal moduli space

$$\bigcup_{0\le s\le 1} \{s\} \times \mathcal{M}_{x,p}(K_s, J, f_s, g_s).$$

Since our choice of (K_s, J, f_s, g_s), $s \in [0,1]$, was generic, this is a 1-dimensional compact manifold, for every pair x, p of index $*$, see Figure 9.1. It follows that the parity of $\#\mathcal{M}_{x,p}(K_i, J, f_i, g_i)$ is the same for $i = 0$ and $i = 1$. Therefore, given an element $x = [\sum_j x_j] \in \mathrm{HF}_*(H,J)$,

$$\Psi_{K_i}(x) = \left[\psi_{K_i}\Big(\sum x_j\Big)\right] = \left[\sum \psi_{K_i}(x_j)\right]$$

does not depend on i. Finally, a usual continuation argument shows that Ψ_K neither depends on J. □

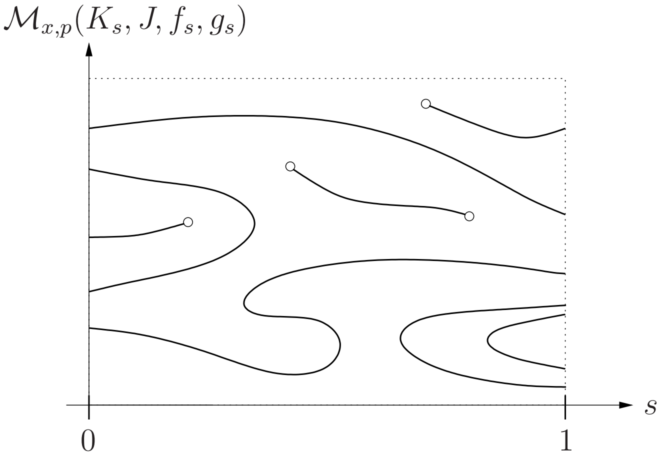

Figure 9.1: The union of moduli spaces $\bigcup_{0\le s\le 1}\{s\} \times \mathcal{M}_{x,p}(K_s, J, f_s, g_s)$.

For $a \in \mathbb{R}$ let $\mathrm{HF}^a(H,J)$ be the subvector space of $\mathrm{HF}(H,J)$ generated by those elements of $\mathrm{HF}(H,J)$ that can be represented by a linear combination of 1-periodic orbits x_j of action $\mathbb{A}_H(x_j) \le a$. Let $[m]$ be the generator of $\mathrm{H}_0(M) \cong \mathbb{Z}_2$. Define

$$A_{K,J}(H) := \inf \{a \mid [m] \in \Psi_K (\mathrm{HF}^a(H,J))\}.$$

By the previous lemma we have

Lemma 9.1.3. *The value $A_{K,J}(H)$ is independent of the choice of (K,J) and (f,g). It is called the PSS selector $A_{\mathrm{PSS}}(H)$.*

The PSS map Ψ is constructed in [18] by using only a special deformation $K(s,t,x) = \beta(s)\, H(t,x)$, where $\beta\colon \mathbb{R} \to [0,1]$ is a smooth function with $\beta(s) = 1$ for $s \le -1$, $\beta(s) = 0$ for $s \ge 0$. We used more general deformations in order to prove Proposition 9.1.1: Fix $K \in \mathscr{K}_0(H)$. We first note that we can rewrite $A_{K,J}(H)$ as

$$A_{K,J}(H) = \inf_{\mathscr{U}_0(K,J)} a_H^-$$

where $\mathscr{U}_0(K,J)$ is a subset of $\mathscr{U}(K,J)$, namely the set of those $u \in \mathscr{U}(K,J)$ that start at $x \in \mathrm{crit}\mathbb{A}_H$ of index zero and evaluate a point through which runs a gradient flow line v of $-\nabla_g f$ to a local minimum of f, and in addition is such that the number of such pairs (u,v) starting at x and ending in a local minimum of f is odd. Therefore,

$$A_{K,J}(H) = \inf_{\mathscr{U}_0(K,J)} a_H^- \ge \inf_{\mathscr{U}(K,J)} a_H^-.$$

Further, by Lemma 9.1.3, $A_{K,J}(H)$ does not depend on the choice of $K \in \mathscr{K}_0(H)$, nor on the choice of the s-independent J. Hence

$$A_{\mathrm{PSS}}(H) = \sup_{K \in \mathscr{K}_0(H)} \inf_{\mathscr{U}_0(K,J)} a_H^- \ge \sup_{K \in \mathscr{K}_0(H)} \inf_{\mathscr{U}(K,J)} a_H^- = \sup_{K \in \mathscr{K}(H)} \inf_{\mathscr{U}(K,J)} a_H^- =: A(H,J)$$

where for the second equality we have invoked Proposition 4.5.2. The proof of Proposition 9.1.1 is complete. □

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