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# A REFINEMENT OF THE HOFER–ZEHNDER THEOREM ON THE EXISTENCE OF CLOSED CHARACTERISTICS NEAR A HYPERSURFACE

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## Abstract

The Hofer–Zehnder theorem states that almost every hypersurface in a thickening of a hypersurface $S$ in a symplectic manifold $(M, \omega)$ carries a closed characteristic, provided that $S$ bounds a compact submanifold and $(M, \omega)$ has finite capacity. It is shown that it is enough to assume that the thickening of $S$ has finite capacity.




We consider a smooth symplectic manifold $(M, \omega)$. A *hypersurface* $S$ in $M$ is a smooth compact connected orientable codimension 1 submanifold of $M \setminus \partial M$ without boundary. A closed characteristic on $S$ is an embedded circle in $S$, all of whose tangent lines belong to the distinguished line bundle

$$\mathcal{L}_S = \{(x, \xi) \in TS \mid \omega(\xi, \eta) = 0 \text{ for all } \eta \in T_x S\}.$$

We denote by $\mathcal{P}(S)$ the set of closed characteristics on $S$. Examples show that $\mathcal{P}(S)$ can be empty; see [**2**, **3**]. We therefore follow [**4**] and consider parametrized neighbourhoods of $S$. Since $S$ is orientable, there exist an open neighbourhood $I$ of 0 and a smooth diffeomorphism

$$\psi : S \times I \to U \subset M$$

such that $\psi(x, 0) = x$ for $x \in S$. We call $\psi$ a *thickening of* $S$, and we abbreviate $S_\epsilon = \psi(S \times \{\epsilon\})$.

Given an open subset $U \subset M$, we consider the function space $\mathcal{H}(U)$ of smooth functions $H : U \to [0, \max H]$ such that

(i) $H|_V = 0$ for some non-empty open set $V \subset U$;

(ii) $H|_{U \setminus K} = \max H$ for some compact set $K \subset U$.

We say that $H \in \mathcal{H}(U)$ is HZ-*admissible* if the flow $\varphi_H^t$ has no non-constant $T$-periodic orbit with period $T \leqslant 1$, and we set

$$\mathcal{H}_{\mathrm{HZ}}(U) = \{H \in \mathcal{H}(U) \mid H \text{ is HZ-admissible}\}.$$

The Hofer–Zehnder capacity of $U$ is defined as

$$c_{\mathrm{HZ}}(U) = \sup\{\max H \mid H \in \mathcal{H}_{\mathrm{HZ}}(U)\}.$$

It has been shown in [**4**, Sections 4.1 and 4.2] that for any thickening $\psi : S \times I \to U \subset M$ for which $c_{\mathrm{HZ}}(U) < \infty$, the set $\{\epsilon \in I \mid \mathcal{P}(S_\epsilon) \neq \emptyset\}$ is dense in $I$, and that

$$\mu\{\epsilon \in I \mid \mathcal{P}(S_\epsilon) \neq \emptyset\} = \mu(I)$$

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if $S$ bounds a compact submanifold of $M$ and $c_{\mathrm{HZ}}(M)<\infty$. Here, $\mu$ denotes Lebesgue measure. In this paper we improve both results in the following way.

THEOREM 1. *For any thickening $\psi : S\times I \to U \subset M$, we have*

$$\mu\,\{\epsilon \in I \mid \mathcal{P}\,(S_\epsilon) \neq \emptyset\} = \mu(I),$$

*provided that $c_{\mathrm{HZ}}(U)<\infty$.*

*Proof.* Consider a thickening $\psi : S\times I \to U \subset M$. We can assume that $I=(-1,1)$. Let $G=\{\epsilon\in(-1,1) \mid \mathcal{P}\,(S_\epsilon)\neq\emptyset\}$ be the parameters of the good hypersurfaces.

*Step* 1. The set $G$ is measurable.

We define the smooth function $K : U \to (-1,1)$ by

$$K(z)=\epsilon \qquad \text{if } z\in S_\epsilon.$$

The set $\mathcal{P}_K(\epsilon)$ of periodic orbits of the Hamiltonian flow $\varphi_K^t$ of $K$ on $S_\epsilon$ corresponds to $\mathcal{P}\,(S_\epsilon)$. For each periodic orbit $x$ of $\varphi_K^t$ we denote by $T(x)$ its period, defined as

$$T(x) = \min\left\{t>0 \mid \varphi_K^t(p)=p\right\},$$

where $p$ is any point on $x$. For each $n\in\mathbb{N}$, we set

$$G_n = \{\epsilon\in(-1,1) \mid \text{there exists } x\in\mathcal{P}_K(\epsilon) \text{ with } T(x)\leqslant n\}\,.$$

It follows from the Arzelà–Ascoli theorem that $G_n$ is a closed subset of $(-1,1)$; see [**4**, p. 109, Proposition 1]. The set $G=\bigcup_{n\geqslant 1} G_n$ is thus measurable.

*Step* 2. The set $G$ has full measure.

The arguments in the proof of [**4**, Theorem 3 in Section 4.2] imply the following lemma.

LEMMA 1. *For each $\epsilon\in(0,1)$, set $U_\epsilon=\psi\,(S\times(-\epsilon,\epsilon))$. If the function $t\mapsto c_{\mathrm{HZ}}\,(U_t)$ is Lipschitz continuous at $\epsilon$, then $\mathcal{P}\,(S_{-\epsilon})\neq\emptyset$ or $\mathcal{P}\,(S_\epsilon)\neq\emptyset$.*

*Proof.* Let $L>0$ be such that

$$c_{\mathrm{HZ}}\,(U_{\epsilon^*}) \leqslant c_{\mathrm{HZ}}\,(U_\epsilon) + L(\epsilon^*-\epsilon) \tag{1}$$

for all $\epsilon^*>\epsilon$ close enough to $\epsilon$. Fix such an $\epsilon^*\in(\epsilon,1)$, and choose an admissible Hamiltonian function $H_1\in\mathcal{H}_{\mathrm{HZ}}\,(U_\epsilon)$ satisfying

$$\max(H_1) > c_{\mathrm{HZ}}\,(U_\epsilon) - (\epsilon^*-\epsilon). \tag{2}$$

Using (1) and (2), we find a smooth function $f:(0,1)\to\mathbb{R}$ satisfying

$$\begin{aligned} f(t) &= \max H_1 && \text{if } t\leqslant\epsilon,\\ f(t) &= c_{\mathrm{HZ}}\,(U_{\epsilon^*}) + (\epsilon^*-\epsilon) && \text{if } t\geqslant\epsilon^*,\\ f'(t) &\in [0, L+1] && \text{if } t\in(\epsilon,\epsilon^*). \end{aligned}$$

We now define $H\in\mathcal{H}\,(U_{\epsilon^*})$ by setting $H=H_1$ on $U_\epsilon$,

$$H(x) = f\,(|K(x)|) \qquad \text{if } x\in S_{-t}\cup S_t,\ \epsilon\leqslant t\leqslant\epsilon^*,$$

and $H = c_{\mathrm{HZ}}\,(U_{\epsilon^*}) + (\epsilon^*-\epsilon)$ off $U_{\epsilon^*}$; see Figure 1.

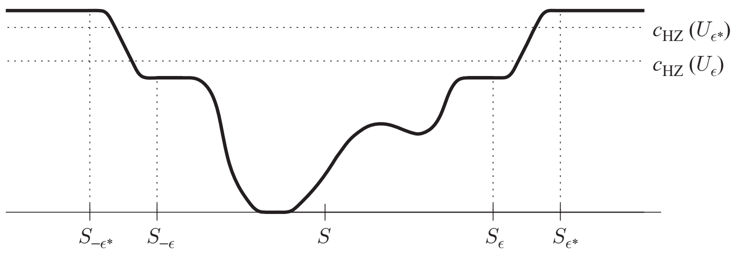


FIGURE 1.

Then $H$ has a non-constant periodic orbit with period $T \leqslant 1$, which is necessarily contained in some hypersurface $S_{-t}$ or $S_t$ with $\epsilon < t < \epsilon^*$. This means that $S_{-t}$ or $S_t$ carries a non-constant periodic orbit of $K$ with period at most $L+1$. By letting $\epsilon^*$ go to $\epsilon$, we obtain, by the Arzelà–Ascoli theorem, a closed characteristic on $S_{-\epsilon}$ or $S_\epsilon$. □

Since the function $t \mapsto c_{\mathrm{HZ}}(U_t)$ is monotone increasing, it is differentiable almost everywhere, and thus Lipschitz continuous almost everywhere. Lemma 1 thus yields

$$\mu\left\{\epsilon \in [0,1) \mid \mathcal{P}(S_{-\epsilon}) \neq \emptyset \text{ or } \mathcal{P}(S_\epsilon) \neq \emptyset\right\} = 1. \tag{3}$$

Let $B = (-1,1) \setminus G$ be the parameters of the bad hypersurfaces. Arguing by contradiction, we assume that $\mu(B) > 0$. In view of the definition of the outer Lebesgue measure $\overline{\mu}$ and $\overline{\mu}(B) = \mu(B) > 0$, we find an interval $(a,b) \subset (-1,1)$ such that

$$\mu\left((a,b) \cap B\right) \geqslant \tfrac{2}{3}(b-a). \tag{4}$$

Applying (3) to $(a,b)$ instead of $(-1,1)$, we see that

$$\mu\left((a,b) \cap G\right) \geqslant \tfrac{1}{2}(b-a). \tag{5}$$

Combining (4) with (5), we find that

$$\begin{aligned} b-a &= \mu((a,b)) \\ &= \mu((a,b)\cap B) + \mu((a,b)\cap G) \\ &\geqslant \tfrac{2}{3}(b-a) + \tfrac{1}{2}(b-a) \\ &> b-a. \end{aligned}$$

This contradiction shows that $\mu(B) = 0$, and so $\mu(G) = \mu\left((-1,1)\right)$. □

It is often of interest to find closed characteristics in a restricted set of homotopy classes. Following [**5**, **6**, **9**], we thus fix a subset $\Gamma$ of the fundamental group $\pi_1(S)$, and given a thickening $\psi : S \times I \to U \subset M$, we denote by $\mathcal{P}^\Gamma(S_\epsilon)$ the set of closed characteristics on $S_\epsilon$ representing an element of $\Gamma \subset \pi_1(S_\epsilon) = \pi_1(S)$. As before, we can assume that $I = (-1,1)$, and for each $\epsilon \in (0,1)$ we set $U_\epsilon = \psi(S \times (-\epsilon, \epsilon))$. We say that $H \in \mathcal{H}(U_\epsilon)$ is $\mathrm{HZ}^\Gamma$-*admissible* if the flow $\varphi_H^t$ has no non-constant $T$-periodic orbit with period $T \leqslant 1$ which represents an element of $\Gamma \subset \pi_1(U_\epsilon) = \pi_1(S)$, and we set

$$\mathcal{H}^\Gamma_{\mathrm{HZ}}(U_\epsilon) = \left\{H \in \mathcal{H}(U_\epsilon) \mid H \text{ is } \mathrm{HZ}^\Gamma\text{-admissible}\right\}.$$

The $\Gamma$-sensitive Hofer–Zehnder capacity of $U_\epsilon$ is defined as

$$c^\Gamma_{\mathrm{HZ}}(U_\epsilon) = \sup\left\{\max H \mid H \in \mathcal{H}^\Gamma_{\mathrm{HZ}}(U_\epsilon)\right\}.$$

As the inclusion $U_\epsilon \subset U_{\epsilon^*}$ induces an isomorphism of fundamental groups, the map $\epsilon \mapsto c^{\Gamma}_{\mathrm{HZ}}(U_\epsilon)$ is monotone increasing. Repeating the above proof with $c_{\mathrm{HZ}}$ replaced by $c^{\Gamma}_{\mathrm{HZ}}$, we find the following refinement of Theorem 1.

THEOREM 2. *For any thickening* $\psi : S \times I \to U \subset M$ *of a hypersurface* $S$ *in* $(M, \omega)$, *we have*

$$\mu\left\{\epsilon \in I \mid \mathcal{P}^{\Gamma}(S_\epsilon) \neq \emptyset\right\} = \mu(I),$$

*provided that* $c^{\Gamma}_{\mathrm{HZ}}(U) < \infty$.

While a version of Theorem 1 is used in [**1**] and [**8**] to prove the almost existence of closed characteristics on (stably) displaceable hypersurfaces, Theorem 2 is used in [**7**] to show that for almost all sufficiently small $c > 0$, the flow describing the dynamics of a unit charge of speed $c$ on a closed Riemannian manifold subject to a symplectic magnetic field has a contractible periodic orbit.

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