PRICING, IMPLEMENTATION AND CALIBRATION OF CREDIT DERIVATIVES IN INCOMPLETE MARKET

THÈSE

Présentée à la Faculté des Sciences
Institut de Mathématiques
Université de Neuchâtel

Pour l'obtention du grade du docteur ès sciences

par

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Diplomé en Mathématiques Appliquées à la Finance

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Soutenue le 17 Décembre 2010

Institut de Mathématiques, Université de Neuchâtel (Suisse)

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Pricing, implementation and calibration of credit derivatives in incomplete market

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Résumé

Après la crise financière de 2008, on a pu observer que le marché des dérivés de crédit avait fortement perdu son niveau courant de liquidité. L'existence de dérivés de crédit illiquides qui ne peuvent pas être parfaitement couverts signifie que le marché est incomplet. Comme conséquence, dans l'univers risque-neutre les approches classiques de valorisation des dérivés de crédit ne prendront pas en compte les risques non couverts. Dans cette thèse, nous abordons ces questions en modifiant les modèles classiques d'intensité de défaut en les intégrant dans le cadre des problèmes de portefeuille optimal, une méthodologie qui prend en compte l'aversion au risque de l'investisseur.

Grâce à des méthodes de contrôle optimal stochastique, nous avons employé la valorisation par indifférence d'utilité exponentielle pour déterminer les prix des obligations risquées et des primes de Credit Default Swap (CDS). Les équations d'Hamilton-Jacobi-Bellman des fonctions de valeur sont dérivées. Les primes de CDS des acheteurs et des vendeurs sont déterminées grâce à des méthodes numériques, sur la base de l'indifférence des deux problèmes de maximisation d'utilité de l'investisseur. Nous examinons comment ces quantités sont affectées qualitativement par l'aversion au risque, l'intensité de défaut, le coefficient de corrélation et le taux de recouvrement. De plus, le comportement asymptotique des courbes des primes est étudié.

La calibration du modèle de valorisation des CDS dans le cas où l'intensité de defaut suit un processus de CIR est discutée. Cela donne lieu à des problèmes inverses non linéaires, car les primes de CDS dépendent de façon non linéaire des paramètres de l'intensité et du coefficient de l'aversion au risque. Dans une première étape en utilisant les données des primes de CDS de Markit, les estimations des séries temporelles de l'intensité de défaut sont obtenues pour chaque classe de notation donnée. Dans une deuxième étape, en utilisant les séries d'intensité construites dans la première étape, les paramètres du modèle CIR-intensité et le coefficient de l'aversion au risque sont estimés. Les résultats des estimations montrent que pendant la période de crise hypothécaire, plus les investisseurs sont sensibles au risque, plus ces derniers préfèrent acheter la protection contre la faillite des entreprises à faible qualité de crédit. Par ailleurs, les estimations des paramètres d'intensité sont en général significatives et augmentent lorsque la qualité de crédit est en baisse.

Mots clés: Modèles du risque de crédit, Dérivés de crédit, Valorisation par indifférence, Processus de Poisson, EDP, Méthode des différences finies, Régression non linéaire.

Keywords: Credit Risk models, Credit Derivatives, Indifference Pricing, Poisson processes, PDE, Finite difference method, Nonlinear regression.

Abstract

After the financial crisis of 2008, it is observed that the liquidity of many credit derivatives has dried up. The existence of illiquid credit derivatives that can not be perfectly hedged means that the market is incomplete. Therefore, the classical risk-neutral approach to valuing the credit derivatives will not take account the risks that remain after the hedging. In this thesis, we address these issues by modifying the classical intensity-based models for credit derivatives by embedding them into the framework of optimal portfolio problems, a methodology that takes into account the investor's risk aversion.

Through stochastic optimal control methods, we use indifference pricing with exponential utility function to determine the defaultable bond prices and Credit Default Swap (CDS) spreads. The Hamilton-Jacobi-Bellman (HJB) equations for the value functions are derived. The bid and ask spreads are numerically solved based on the indifference between the investor's two utility maximization problems. We examine how these quantities are affected qualitatively by the risk aversion, the default intensity, the correlation coefficient and the recovery rate. Moreover, the asymptotic behavior of the spread curves is studied.

The calibration of the indifference pricing model of CDS spreads in the context of CIR-intensity is investigated. This gives rise to nonlinear inverse problems since the price of the CDS spreads depends in a non-linear way on the intensity's parameters and the risk aversion coefficient. In a first step using the data of CDS spreads from Markit, the estimates of time series for the default intensity for a given rating class are obtained. In a second step, including the default intensities constructed in the first step, the parameters of the CIR intensity model and the absolute risk aversion coefficient are estimated. The estimation results reveal that during the subprime crisis, more risk averse investors prefer to buy the protection against the default of low credit quality firms. In addition, estimates for the parameters of CIR-intensity are in general significant and are increasing when the credit quality is decreasing.

Keywords: Credit Risk models, Credit Derivatives, Indifference Pricing, Poisson processes, PDE, Finite difference method, Nonlinear regression.

Acknowledgements

All thanks first go to my advisor Prof. Olivier Besson, for his encouragement and patience. To him I owe more than I could express. Olivier is a rare breed, his greatness as a mathematician and advisor is matched by his greatness as a person. It was truly an honor to be his student, and working with him has been an immense pleasure and privilige. I hope to repay him by working hard...from now on. I am forever indebted to Olivier for providing his wisdom to me on mathematics, with regards to the material as well as professional development and teaching.

I would like to thank Dr. Akimou Osse for our successful collaboration and for friendly discussions during my thesis.

Furthermore I would like to thank Prof. Michel Dubois from the Institute of Finance and Prof. Michel Benaïm from the Institute of Mathematics of University of Neuchâtel, as well as Prof. Nicole El-Karoui from Ecole Polytechnique Paris, for acting as members on my Ph.D. committee and for providing valuable suggestions for my research.

I am thankful to the members of the Institute of Mathematics who provided a pleasant environment during my thesis.

Finally I would like to thank my parents Rosalie and Moïse for giving the opportunity to undertake my studies and move to Switzerland, as well as for their unbounded support. I keep all my lovely compliments for my wife Marceline and my son Amaël. Their constant patience and support help me a lot during these years, especially during the writing phase. I dedicate to them this dissertation with all my love.

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Chapter 1

Introduction

1.1 Classical Models of Credit Risk

Credit risk models can be partitioned into two groups known as structural models and reduced form models.

Structural models were pioneered by Merton (1974). The basic idea, common to all structural-type models, is that a company defaults on its debt if the value of the assets of the company falls below a certain default point. Such a default can be expected. In these models it has been demonstrated that default can be modelled as an option and, as a result, researchers were able to apply the same principles used for option pricing to the valuation of risky corporate securities. The application of option pricing avoids the use of risk premium and tries to use other marketable securities to price the option. The use of option pricing theory set forth by Merton offers much more accurate prices, but provides information about how to hedge out the default risk. The shortcoming of the structural approach is that it underestimates credit risk in the sense that the corporate bond is cheaper than the default free bond even if the firm value is larger than the default point. Subsequent to the work of Merton (1974), there have been many extensions such as Black and Cox (1976), Longstaff and Schwartz (1995), etc.

The second group of credit models, known as reduced form models, are more recent. These models, including the Artzner and Delbaen (1995), Jarrow and Turnbull (1995), Jarrow et al. (1997) and Duffie and Singleton (1999) models, do not look inside the firm. Instead, they assume that default occurs without warning at an exogenous default rate, or intensity. So, the default event would be unexpected. The dynamics of the intensity are specified under the pricing probability. Instead of asking why the firm defaults, the intensity model is calibrated using the market data such that the credit rating, the stock prices, the bond prices and etc. Several studies have focussed directly or indirectly on estimating the intensity process of reduced form models, see for example Longstaff et al. (2003), Duffee (1999), Driessen (2005). These studies have relied on corporate bond price data and in such contexts, bond prices are modeled as expectations of discounted payoffs under a risk neutral measure. This leads in general to an estimate of the intensity process under some risk-free pricing measure. As the market data come not only from credit risk but also from other factors, thus the reduced form model overrates the credit risk. This is the shortcoming of the reduced approach.

One of the recent trends is to combine both approaches, see for example Cathcart and El-Jahel (2003), Realdon (2006).

In Cathcart and El-Jahel (2003) a Partial Differential Equation (PDE) method is used, providing a semi-analytical pricing formula of defaultable bond combining the two approaches when the short rate follows Cox-Ingersoll-Ross (CIR) model and the default intensity is linearly dependent on the short rate. The case with exogenous default recovery and no correlation between the firm value and the short rate is studied.

Realdon (2006), using PDE method, provided an analytical pricing formula of defaultable bond with both approaches in the case that the default intensity and the short rate are constant and the default recovery is endogenous. He also provided an analytical pricing formula when the short rate is uncorrelated with the firm value and the default recovery is exogenous.

1.2 Credit Risk Modelling in Incomplete Market

A market with stochastic default intensity is incomplete in that the intensity is a source of uncertainty that is not traded. The incompleteness of the market usually gives rise to infinitely many martingales measures, each of which produces a no-arbitrage pricing. So, it is not clear which one to use in the pricing of the defaultable bond. The superreplication price (upper hedging price) is the minimal initial wealth needed for hedging the claim without risk. If the bank decides to charge a superreplication price for selling a bond so that it can trade to eliminate all risks, the price is usually high and unrealistic. There are three major approaches that have been developped in searching for solutions of pricing and hedging in incomplete markets.

One is to choose the risk neutral probability Q as closed as possible to the real world measure P, in the sense that it wants to minimize

$$E\left[f\left(\frac{dQ}{dP}\right)\right]$$

over all equivalent martingale measures. Here f is strictly convex function in $[0;\infty)$. One popular choice is $f(x) = x\ln(x)$, in which case Q is the minimal entropy martingale measure, see Avellaneda (1998) or Hobson (2004). The drawback is this approach sometimes provides results that are not very financially reasonable. Another approach is to identify the martingale measure consistent with the market price. This approach does not provide a hedging strategy at all. Instead of choosing an equivalent martingale measure for valuation, a dynamic utility-based valuation theory has been developed producing the so-called indifference price.

Utility indifference pricing was first introduced by Hodges and Neuberger (1989). It is an alternative where the price is uniquely determined at the cost of depending on the preferences of the pricer. The writer utility indifference price is the value of the initial payment that makes the seller indifferent to whether to sell the contract or not. From practical point of view utility indifference pricing has at least two advantages:

- it does not refer to the market portfolio;
- it generates an optimal hedging strategy in the sense that the resulting utility is equal to that of an optimal pure investment.

The mathematical structure of utility indifference pricing has been well characterized by numerous researchers, cf. Rouge and El Karoui (2000), Delbaen et al. (2002), Becherer (2003), Musiela and Zariphopoulou (2004), Mania and Schweizer (2005), and references therein.

Bielecki et al. (2004), Collin-Dufresne and Hugonnier (2001), Sircar and Zariphopoulou (2007), Shouda (2006) and Leung et al. (2008) apply the utility-based pricing to credit risk.

Bielecki et al. (2004) discussed the utility-indifference price of defaultable claims within the reduced approach by a backward stochastic differential equation. They studied a particular indifference price, based on the quadratic criterion and solved the problem by the duality approach for exponential utilities.

Collin-Dufresne and Hugonnier (2001) study the utility based prices of Event Sensitive Contingent (*ESCC*) claims under two scenarios of resolution of uncertainty for event risk: when the event is continuously monitored or when it is revealed only at the payment date. In both cases, they transformed the incomplete market optimal portfolio choice problem of an agent endowed with an *ESCC* into a complete market problem with a state and possibly path dependent utility function. They also obtain an explicit representation of the utility based prices under both information resolution scenarios in the case of negative exponential utility function.

Shouda (2006) discussed the utility-based pricing of defaultable bonds where their recovery values are unpredictable. He derived a partial integro-differential equations that the utility-based bond price solves. He also extracted credit risk premium from the yield spread of defaultable bonds and classified them to default-timing risk, recovery risk and spread risk.

Sircar and Zariphopoulou (2007) apply the utility indifference valuation in intensity-based models of default risk where the default time is the first jump of a time-changed Poisson process. They derive the Hamilton-Jacobi-Bellman (HJB) equations and analyze resulting yield spreads for single-name defaultable bonds, and a simple representative two-name credit derivative when the default intensity is constant.

Leung et al. (2008) apply the technology of utility-indifference valuation for defaultable bonds in a structural model for Black-Cox-type. They derive the

HJB equations and simplify them to the linear (Feynman-Kac) differential equations. Finally, they find that the utility valuation has a significant impact on the bond prices and yield spreads.

The indifference valuation should be attractive to participants working in the OTC market. It is a direct way for them to quantify the default risks they face in a portfolio of complex instruments, when calibration data is scarce. While one does not have to identify an appropriate risk-neutral measure to apply indifference pricing, it should be mentioned that one has an equally difficult problem, namely determining the appropriate's investor utility function reflecting his personnel risk aversion. Important quantities to classify utility functions are the coefficient of absolute risk aversion of Arrow (1971) and Pratt (1964) defined as $-\frac{u''(x)}{u'(x)}$, and the coefficient of relative risk aversion of Arrow (1971) and Pratt (1964), given by $-x\frac{u''(x)}{u'(x)}$. It is easy to check that exponential utility function $u(x) = -e^{-\gamma x}$, $\gamma > 0$ has constant absolute risk aversion, whereas power utility function $u(x) = \frac{x^p}{p}$, $-\infty < p < 1$ has constant relative risk aversion. According to Cochrane (2001), the latter is more realistic, and consequently, it would be desirable to use power utility instead. This however would be at the cost of analytical tractability.

The absolute risk aversion measures the concavity of the utility function and the idea of how people react to best maximize their utility. Measuring absolute risk aversion is important as it plays an important part in measuring the curvature of the utility function of the investor. Risk aversion is not easily measured, just as utility is not easily measured. Many researchers have focussed on the estimation of the absolute risk aversion, see for example Babcock et al. (1993), Ukhov (2002), Guiso and Paiella (2008), etc.

Babcock et al. (1993) show that assuming appropriate levels of either risk premiums or probability premiums can aid in the selection of reasonable Absolute Risk Aversion (ARA) levels for use in simulation studies using CARA utility functions. They found in the context of risk premiums between 1% and 85% of the gamble size, the appropriate ARA ranges are: 0.000002 to 0.000462 for gamble sizes of $10,000$ dollars, 0.00002 to 0.00462 for gamble sizes of $1,000$ dollars, and 0.0002 to 0.046204 for gamble sizes of 100 dollars.

Ukhov (2002) uses the prices of lottery bonds issued by the Imperial Russian Government in 1864 and 1866 to estimate the time-variation in investor risk aversion. Time variation in investor risk aversion is then compared to the dynamics of the Russian bond market over the period 1889 to 1904, and increases in risk aversion are positively associated with increases in the price of a risk-free asset. The estimated values of constant absolute risk aversion $\gamma \in [-0.00027; 2.5.10^{-6}]$ with a mean value of -0.0001.

Also Guiso and Paiella (2008) confirm that in practice the average value of absolute risk aversion is 0.01981 with a median of 0.000708 using the exponential utility function.

1.3 Credit Derivatives

A derivative is a bilateral agreement that shifts risk from one party to another; its value is derived from the value of an underlying price, rate or index. A credit derivative is an agreement designed explicitly to shift credit risk between the parties; its value is derived from the credit performance of one or more corporation, sovereign entity or security. If its value depends of the credit risk of a single entity, the credit derivative is called single-name; if there are several entities, it is called multi-name.

The most common underlying assets for credit derivatives are loans and their securitized versions, bonds. A bond, loan or mortgage is a contract between two counterparties. At the time of entry into the contract the creditor lends money to the obligor, for which the latter agrees to pay back a predetermined amount (the face value or notional) at maturity. In the case of a zero-coupon bond, these are the only payments agreed upon, while for a coupon-bearing bond, the obligor makes additional periodic predetermined coupon payments.

If the creditor enters this kind of contract, he is exposed to credit risk, namely the risk of losing his investment in the case a credit event occurs. A credit event is defined as the obligor's default, i.e. the failure to meet his obligations. Possible credit events include bankruptcy, failure to make coupon payments, restructuring or downgrade below a certain level.

When a credit event occurs, the assets of the bond issuer are normally liquidated to meet his obligations at least partially. Consequently, bond holders can expect to receive a certain percentage of the notional even in the case of a default. This percentage is called recovery rate and ideally paid at or very shortly after default. In reality however, the settlement process can take quite a long time.

Credit derivative markets have undergone a rapid growth in the last decade. According to the British Banker's Association (BBA), the global outstanding notional volume of credit derivatives was 180 billion USD in 1996. Only ten years later, at the end of 2006, the market size had expanded to a volume of more than 20 trillion USD, roughly 112 times the market size of 1996 (BBA, 2006). Recent informations released by the Bank for International Settlement (BIS) report a huge grow notional amount of outstanding credit derivatives of 57.3 trillion USD per mid-year 2008, reflecting the continuing growth of this market (BIS, 2008). However, such volumes are expected to shrink as a result of the financial crisis. This tremendous market growth was accompanied by the invention of new, innovative products, therewith widening the diversity and the complexity of credit derivative instruments. Market participants are nowadays able to issue and trade in products such as single-name credit default swaps (CDS), credit linked notes (CLN), credit spread options (CSO), collateralized debt obligations (CDO), equity linked products, portfolio products, to name just a few. The principal feature of these instruments is the separation and isolation of credit risk, which facilitates the trading, enables the replication, the transfer and hedging of credit risk.

The most common single-name credit derivative is the Credit Default Swap (CDS). A CDS can be succinctly described as a traded insurance contract which provides protection against a default by a particular company. The buyer of the protection makes periodic payments, analogous to insurance premiums referred to CDS premium. The company whose obligation is insured is known as the reference entity. The face value or notional of the CDS is the face value of the reference obligation whose credit risk is being insured. The protection buyer pays the CDS premium to the protection seller until either the maturity of the contract or a credit event occurs. Upon the credit event, the seller pays the loss incurred by the credit event to the buyer. This payment can be implemented through two settlement mechanisms: cash or physical settlement. The cash settlement consists of a cash payment (from the protection seller to the protection buyer) equal to the difference between the notional and market value of the reference obligation. For example, when Lehman Brothers defaulted, if its debt was worth eight cents on the dollar, the protection seller would have to pay about ninety-two cents for each notional dollar of debt they had guaranteed. In a physical settlement, the protection buyer delivers the protection seller a portfolio of the reference entity's obligations (with face value equal to the CDS notional) and receives in cash, from the protection seller, their face value. The obligations that can be delivered by the protection buyer are called deliverable obligations and may be the reference obligation or one of a broad class of obligations meeting certain specifications, usually in terms of seniority and maturity.

Although credit default swaps can be used as insurance against a default, the buyer of protection is not required to own the reference obligation or to be otherwise exposed to the borrower's default. Both buyers and sellers may use credit default swaps to speculate on a firm's prospects. Some have suggested that investors should not be allowed to purchase CDS protection unless they are hedging exposure to the reference entity. This form of speculation made the CDS markets very liquid and is estimated to be up to 80% of the CDS market according to Federal Reserve Bank of Atlanta (2010).

Numerous researchers have focussed to the pricing of CDS; see Duffie (1999), Hull and White (2000), Skinner and Townend (2002), etc.

Duffie (1999) attempts to derive the basics of CDS pricing and the estimation of the hazard rate.

Hull and White (2000) provides a methodology for valuing credit default swaps when the payoff is contingent on default by a single reference entity and there is no counterparty default risk. The authors also test the sensitivity of credit default swap valuations to assumptions about the expected recovery rate.

Skinner and Townend (2002) argue, by appealing to the put-call parity, that a CDS can be expressed as a put option on the reference obligation. They find that variables affecting option prices such as default-free rates, volatility, underlying asset and time to maturity are also important in determining CDS prices.

1.4 Outline and Contributions of this Thesis

When this thesis was started (2006), credit derivatives market were in rapid expansion, reaching a stunning 28.5 trillion USD at the end of 2006 (BBA, 2006). As it is well known, especially CDSs and CDOs made a significant contribution to the current financial crisis (2008/2009). As it became clear that uncontrolled redistribution of credit risk poses a imminent danger to the worldwide financial system, the outstanding notional of credit derivatives had been reduced to under 20 trillion USD by late 2008, according to Giesecke (2009). Consequently, the liquidity for many credit derivatives has dried up.

The question which certainly raises is whether the credit derivatives market (specifically credit default swaps) still has a future and whether it is still necessary to put effort into their pricing. Since the original purpose of CDSs was to hedge credit risk, and since there will still be need for this in the future, it is safe to say that the answers to both questions are yes. However it is also almost certain that products will be held simple and will be subject to more regulation than in the past to ensure the transparency. Furthermore, the market for CDSs will probably not be as liquid as it was in the previous years. So, the existence of illiquid CDS contracts that can not be perfectly hedged in the market means that the CDS market is incomplete. The common complete-market risk-neutral approach to valuing illiquid CDSs does not take into account the risks that remain after hedging, nor does it take into account the requirement a potential purchaser of the illiquid CDS should make a positive expected profit on the transaction. In this context of illiquid CDS contracts, the classical methodology which consists in practice to work with the mid premium (the average of the bid and ask premium) should be biased.

Motivated by theses issues, the main purpose of this thesis is to price directly the bid and ask CDS premium in the context of exponential utility indifference approach. In addition, we analyze how in practice the model can be implemented and calibrated to market data.

In the first part, we concentrate on the indifference valuation of the zero-coupon defaultable bond and the credit spread since the defaultable bonds are the most common underlyings of the CDS contract. In this context, we implement the indifference pricing model of Sircar and Zariphopoulou (2007) under a Cox et al. (1985) (CIR) intensity model. The default time τ is then defined as the first jump time of a Cox process with a random intensity $\lambda \geq 0$, which is correlated with the firm's stock price S. A finite difference method is used to solve the nonlinear reaction-diffusion equations satisfied by the value functions. The analysis of the bid and ask spread curves clearly shows the nonzero short-term yield-spread reflecting that the default risk comes as a surprise. This effect is not significant in Leung et al. (2008) even if their results enhance short-term yield spreads compared to the standard Black-Cox valuation. In addition, we focus on two main questions:

- It is well-known that when the default intensity increases, the credit spread increases; but how the bid and ask credit spreads curve behaves depending on the values of the default intensity?
- What is the impact of the correlation between the default intensity and the stock price on the credit spread?

To answer both questions, it would be difficult to perform a theoretical analysis with the nonlinear PDE as it stands. Some experiments have been done with the estimates of CIR-intensity parameters of Longstaff et al. (2003) and the estimates of Denault et al. (2009) with arbitrary values of default intensity λ_0.

Concerning the first question, we can answer that although the default intensity increases the bid and ask credit spread, the shape of the spread curve changes with the values of the default intensity. For illustration, it is found with the parameters of Longstaff et al. (2003), an upward sloping bid spread curve for $0 < \lambda_0 \leq 0.06$, a S shape bid curve for $0.06 < \lambda_0 \leq 0.15$ and a downward sloping bid curve for $0.15 < \lambda_0 \leq 1$. The S shape curve is sometimes observed in practice but is not revealed in most studies such as Fons (1994) and Jarrow et al. (1997) which focussed on a split behavior of the credit spread curve according to the credit quality of the firm. For the ask spread curves, an upward sloping curve is observed for $0 < \lambda_0 \leq 0.25$ and a humped sloping curve for $0.25 < \lambda_0 \leq 1$.

Concerning the second question, using the estimates of Longstaff et al. (2003) and Denault et al. (2009), it is observed by experimentation that the correlation between the default intensity and the stock price has a little impact on the bid and ask credit spread.

We also investigate our analysis of the credit spread curve when the default intensity is assumed to be independent from the stock price. Although this assumption is not consistent with the reality because the firm's stock performance reflects the firm's well-being, it is observed from the results obtained in the previous analysis that the correlation between the default intensity and the stock has no significatif impact on the credit spread. In conducting this analysis, the limit of the short-term bid and ask spread is found. A closed-form model of the defaultable bond price is developped and the asymptotic behavior for the spread curve is analyzed. Finally, experiments with some values of the default intensity show different shapes of the spread curves according to the level of the default intensity.

In the second part of the thesis, we conduct an examination of the utility indifference valuation for the CDS spreads within the reduced form model. We define the protection buyer utility indifference (bid) spreads as the value of the initial payment that makes the protection buyer indifferent, in the sense of his expected utility between buying the protection or not. In other words, the protection buyer is willing to pay at most this premium for insuring against the default of the bond. Similarly, the protection seller utility indifference (ask) spreads is the value of the initial payment that makes the protection seller indifferent, in the sense of his expected utility between bearing the losses or not. It is the smallest premium the protection seller is willing to accept in order to reimburse the protection buyer the losses derived. The utility indifference pricing for the CDS should be attractive for market-makers in the CDS market; it is a direct way for them to control the trading in comparing the quoted bid-ask CDS spreads to the model spreads. For the economists, the indifference valuation of the CDS can be used to estimate particularly the implicit risk aversion coefficient for the protection buyer and the protection seller using directly the quoted bid and ask spreads. Finally, the indifference CDS model can also be used to estimate the real-world intensity process. The knowledge of the intensity process under the physical measure can be useful in applications such as value-at-risk estimation. In this case, the analyst needs to assess the real-world probabilities of defaulting and their dynamics for computing the estimates of the loss percentage of the portfolio distribution. Estimates of physical intensities are also used in the calculation of bank capital under Basel II.

The pricing of the CDS showed that the bid and ask CDS spreads are the zeros of functions depending of the density function of the default and the survival probability of default at time t. To examine how the model can be implemented in practice, we derived numerically the bid and ask CDS spreads in particular when the intensity is constant or follows a Cox et al. (1985) (CIR) model.

The short-term limits of the bid and ask CDS are found and depend on the current default intensity, the recovery rate and the risk aversion coefficient. In addition, the impacts of the default intensity, the risk aversion parameter and the recovery rate on the CDS spread curve are discussed. It is observed a negative relationship between the protection buyer spread and the risk aversion parameter and a positive relationship between the protection seller spread and the risk aversion whatever the time to maturity. It is shown that both bid and ask spreads decrease when the recovery rate increases. Experiments with arbitrary values of the default intensity show that the shape of the spread curve can be upward sloping or downward sloping according to the value of the default intensity. Finally, the model recovers the classical arbitrage-free pricing for the CDS when the risk aversion goes to zero.

To prove how the indifference CDS model can be calibrated to market data, we focus in the last part of the thesis in the estimation of physical intensity process and the absolute risk aversion parameter during the period of the subprime crisis. The estimation approach uses single-name CDS spreads data from Markit and the treasury zero yield data and proceeds in two steps. In a first step, for each day from September 15, 2008 to October 15, 2008, an estimate of the short-term limit of CDS spread for a whole class of firms grouped by credit ratings is obtained. Two models are considered. In the model I, the quotes CDS spread for the shortest maturity (6 months) are used as proxy for the limit. In the model II, Nelson and Siegel (1987) model is fitted to the observed term stucture of CDS to obtain the short-term limit. These values are then used to estimate the dynamic default intensities. The estimates of the intensity indicate the instantaneous default probabilities during the subprime crisis for typical or average firm withing a rating class. In a second step, relying on the default intensities constructed and using the observations of CDS spreads and risk-free interest rate, we estimate by nonlinear least square the parameters of CIR-intensity model and the investor's risk aversion parameter for each rating class. Generally, most of the estimated parameters are significantly different from zero. The estimates of the risk aversion clearly show during the subprime crisis the protection buyers are risk averse and more they are risk averse more they prefer to protect against the default of low credit quality company. The peaks of the estimated time series for the default intensity are consistent with the fall of the global financial market after the banckruptcy of Lehman Brothers. Finally, it reveals that the approach of Nelson-Siegel interpolation gives more satisfactory results than the approximation by the 6-months spreads.

The rest of the thesis is organized as follows. Chapter 2 presents the credit risk models, in particular the structural models and the reduced form models. Chapter 3 discusses the utility-based pricing of defaultable bond. In chapter 4, the indifference pricing of Credit Default Swap is analyzed and implemented. Chapter 5 focusses on the calibration of the CDS indifference spread.

Chapter 2

Credit Risk models

2.1 General Concepts in Credit Risk

2.1.1 Default Events

Credit risk concerns the possibility of financial losses due to changes in the credit quality of market participants. The most radical change in credit quality is a default event. The default event is then a rare occurrence taking place at a random time and resulting in large financial losses to some sectors of the market.

Regardless of what definition is used for a default event, let us denote the default time by τ that is $[0, \infty]$-valued random variable on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Ω denotes the possible states of the world, $\mathcal{F}$ is the σ-algebra, $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ is the filtration satisfying the usual conditions which are listed in Appendix A, and $\mathcal{F}_T = \mathcal{F}$. P is the probability measure describing the likelihood of certain events. The only mathematical structure assumed for τ is that it should be a *stopping time*, that is, a random variable $\tau : \Omega \rightarrow \mathbb{R}_+ \cup \{\infty\}$ such that $\{\tau \leq t\} \in \mathcal{F}_t$, for $t \geq 0$. Intuitively, one can determine whether or not the default time occurs before a certain deterministic time t by observing the past up to time t, which is encoded in the filtration $(\mathcal{F}_t)$. In other words, τ is a stopping time if its associated counting process

$$N_t(\omega) = 1_{\{\tau \leq t\}} = \begin{cases} 1 \text{ if } \ \tau \leq t \\ 0 \ \text{ else} \end{cases} \tag{2.1}$$

is adapted. For default times, N_t is known as the default indicator process. Let us say that the default time $\tau > 0$ is *predictable stopping time* if there is an

announcing sequence of stopping times $\tau_1 \leq \tau_2 \leq ...$ such that $\lim_{n\to\infty}\tau_n = \tau$, P-a.s. The opposite of a predictable default time is a *totally inaccessible default time*, that is, a stopping time τ such that $P(\tau = \bar{\tau} < \infty) = 0$, for any predictable stopping time $\bar{\tau}$.
Given a default time τ, define the *probability of survival* in t years is as

$$P(\tau > t) = 1 - P(\tau \leq t) = 1 - E(1_{\{\tau \leq t\}})$$

Several other related quantities can be derived from this basic probability. For instance,

$$P(s \leq \tau \leq t) = P(\tau > s) - P(\tau > t)$$

is the unconditional probability of default occuring in the time interval $[s, t]$. Using Bayes's rule for conditional probability, one can deduce that the probability of survival in t years conditioned on survival up to $s \leq t$ years is

$$P(\tau > t|\tau > s) = \frac{P(\{\tau > t\} \cap \{\tau > s\})}{P(\tau > s)} = \frac{P(\tau > t)}{P(\tau > s)}$$

since $\{\tau > t\} \subset \{\tau > s\}$. Assuming that $P(\tau > t)$ is strictly positive and differentiable in t, the *hazard rate process* is defined as

$$h(t) = -\frac{\partial \log P(\tau > t)}{\partial t}$$

It measures the instantaneous rate of default conditioned on survival up to time t. It then follows that

$$P(\tau > t|\tau > s) = e^{-\int_s^t h(u)du}$$

If $h(t)$ is continuous we have

$$h(t)\Delta t \approx P(t \leq \tau \leq t + \Delta t|\tau > t).$$

Basing on $P(\tau > t|\mathcal{F}_s)$, that is, the survival probability in t years conditioned on all the information available at time $s \leq t$, one define the *forward default rate* as

$$f(s, t) = -\frac{\partial \log P(\tau > t|\mathcal{F}_s)}{\partial t}$$

It then follows that

$$P(\tau > t|\mathcal{F}_s) = e^{-\int_s^t f(s,u)du}$$

The indicator process N_t defined in (2.1) is clearly a submartingale. Moreover, it can be shown that it is of *class* $\mathcal{D}$, i.e. $\{N_t$: t is a finite stopping time$\}$ is uniformly integrable; so that it follows from the Doob-Meyer decomposition that there exist an increasing predictable process Λ_t, called the *compensator*, such that $N_t - \Lambda_t$ is a martingale. If the compensator can be written as

$$\Lambda_t = \int_0^t \lambda_s ds$$

for non-negative, progressively measurable process λ_t, then this process is called the *default intensity*. It measures the instantaneous rate of default conditioned on all the information available up to time t. As we will see later, although all default indicators have a compensator, not all of them admit a default intensity. This happens, for example, whenever the stopping time is predictable. Under suitable technical conditions, it follows that

$$\lambda_t = f(t,t)$$

In other words, starting from a sufficiently regular family of survival probabilities, one can obtain the forward default rate and the associated intensity rate.

2.1.2 Implied Survival Probabilities

Let $D(t,T)1_{\{\tau>t\}}$ be the price at time $t \leq T$ of a defaultable zero-coupon bond issued by a certain company with maturity T and face value equal to one unit of currency. Then clearly $D(t,T) > 0$ denotes the price of this bond given that the company has survived up to time t.

$$D(T,T)1_{\{\tau>T\}} = 1_{\{\tau>T\}} \leq 1 = B(T,T),$$

where $B(t,T)$ is the price of a default-free zero-coupon bond with maturity T and face value to one unit of currency. By the Law of one Price

$$D(t,T)1_{\{\tau>t\}} \leq B(t,T)$$

for all earlier times t.
Under a risk-neutral measure Q, the price of the defaultable bond if $\{\tau > t\}$ is

$$D(t,T) = E_t^Q\left(e^{-\int_t^T r_s ds} 1_{\{\tau > T\}}\right) \tag{2.2}$$

where r_t is the intantaneous spot rate on the riskless security at time t. $E_t^Q(.) = E^Q(.|\mathcal{F}_t)$
If one assume that r_t and τ are independent under the risk-neutral measure Q, then equation (2.2) can be rewritten as

$$D(t,T) = E_t^Q\left(e^{-\int_t^T r_s ds}\right) E_t^Q\left(1_{\{\tau > T\}}\right)$$

Therefore, the risk neutral survival probability is given by

$$Q(\tau > T|\mathcal{F}_t) = \frac{D(t,T)}{B(t,T)} \tag{2.3}$$

That is under the independence assumption for r_t and τ, the term structure of risk neutral survival probabilities is completely determined by the term structure of both defaultable and default-free zero-coupon bonds. In the sequel, these will be called *implied survival probability*, emphasizing the fact that they are derived from the market prices and associated to the risk neutral measure Q.
Assuming differentiability with respect to the maturity date, we can define the *implied forward default* rate by

$$f^Q(t,s) = -\frac{\partial \log Q(\tau > s|\mathcal{F}_t)}{\partial s} \tag{2.4}$$

and the *implied default intensity* by

$$\lambda_t^Q = f^Q(t,t)$$

It is reasonable to assume that the prices of defaultable bonds show a sharper decrease as a function of maturity than do prices of default-free bonds. Therefore, the term structure of implied survival probabilities as functions of the maturity date share the properties of the term structure of bond prices, namely, initial value equal to 1, decreasing and approaching zero at infinity.

2.1.3 Credit Spreads

Assume that $\{\tau > t\}$ and let $D(t,s)$ be the price at time t of the defaultable zero-coupon bond with maturity s. The continuously compounding yield $Y(t,s)$ on the bond is the return the investor will receive by holding a bond to maturity. It is defined by

$$Y(t,s) = -\frac{\log\left(D\left(t,s\right)\right)}{s-t}$$

The credit spread is the difference between yields for defaultable bond and the corresponding default-free bond. It gives the excess return demanded by the bond investors to carry the potential default losses. The credit spread $S(t,s)$ of the bond at time t is given by

$$S(t,s) = -\frac{1}{s-t}\log\left(\frac{D(t,s)}{B(t,s)}\right) \tag{2.5}$$

Under the independence assumption for r_t and τ, and using implied survival probabilities in (2.3) and implied forward default rate in (2.4), the credit spread

$$\begin{aligned} S(t,s)(s-t) &= -\log\left(Q\left(\tau > s|\mathcal{F}_t\right)\right) \\ &= \int_t^s f^Q(t,u)du \end{aligned}$$

The term structure of credit spreads also called *spread curve* is the function $S(t,.)$, i.e. the schedule of $S(t,.)$ against s holding t fixed. The spread curves can be essentially monotonic, humped, or, occasionally S shaped. Empirical studies reveal that observed credit spreads for defaultable bonds remain positive even for small time horizons. This follows from the fact that there is always a small probability of immediate default.

2.2 Structural Models of Credit Risk

The structural approach directly refers to economic fundamentals, such as the capital structure of a firm, in order to model credit events. The two major driving concepts in the structural modeling are the total value of the firm's assets and the default triggering barrier. A default event is deemed to occur for a firm when its assets reach a sufficiently low level compared to its liabilities. The models require strong assumptions on the dynamics of the firm's asset, its debt and how its capital is structured. We shall consider three models in this class

- the classical Merton model
- the First passage models
- the excursion models

The main advantage of structural models is that they provide an intuitive picture, as well as an endogenous explanation, for default.

2.2.1 The Merton Model

Merton (1974) uses the Black and Scholes (1973) option pricing model to value corporate liabilities. This is an straightforward application only if we adapt the firm's capital structure and the default assumptions to the requirements of the Black-Scholes model.
Assume that the total value V_t of a firm's assets follows a geometric Brownian motion

$$dV_t = \mu V_t dt + \sigma V_t dW_t,\ V_0 > 0, \tag{2.6}$$

where μ is the mean of return on the assets and σ is the asset volatility.
Let r be the risk-free interest rate, which is assumed to be constant. Consider the capital structure of the firm which is comprised by equity and by a zero-coupon bond with maturity T and face value of K. At Maturity, if the total value of the assets is greater than the debt, the latter is paid in full and the remaining is distributed among shareholders. However, if $V_T < K$ then default is deemed to occur because bondholders exercise a debt covenants giving them the right to liquidate the firm and receive the liquidation value in lieu of the debt. Shareholders receive nothing in this case, but are not required to inject any additional funds to pay for the debt, in which is called limited liability. Therefore shareholders have a cash flow at T equal to

$$(V_T - K)^+ = \max(0, V_T - K)$$

so that equity can be view as an European call option on the firm's assets and its value E_t at earlier times $t < T$ can be calculated using the Black-Scholes formula. Note that in the Merton model, equity value increases with the firm's volatility, so shareholders are generally inclined to press for riskier positions to be taken by their managers. Bond holders, on the other hand receive

$$\min(K, V_T) = K - (K - V_T)^+$$

Therefore the value D_t for the debt at earlier times $t < T$ can be obtained as the value of a riskfree zero-coupon bond minus an European put option. It follows from the put-call parity relation that

$$V_t = E_t + D_t$$

which is the fundamental identity of accounting.
Under this model, the default time τ is a discrete random variable given by

$$\begin{cases} \tau = T \text{ if } \ V_T < K \\ \tau = \infty \ \text{ else} \end{cases}$$

Assuming the market is complete i.e. one can trade the firm value V_t; we note $e^{-rt}V_t$ is a martingale under the unique risk-neutral measure Q with market price of risk $\theta = \frac{\mu - r}{\sigma}$ and Radon-Nikodym derivative

$$\frac{dQ}{dP} = \exp(\theta W_T - \frac{\theta^2}{2}T)$$

The dynamic of the asset's value under the risk neutral measure is

$$dV_t = rV_t dt + \sigma V_t dW_t^Q$$

where

$$W_t^Q = W_t + \theta t$$

Setting $m = r - \frac{\sigma^2}{2}$ and via Itô's lemma, the solution of the above equation can be written as

$$V_t = V_0 e^{mt + \sigma W_t^Q}$$

The risk neutral probability of default is given by

$$\begin{aligned} Q(\tau = T) &= Q(V_T < K) \\ &= Q\left(\sigma W_T^Q < \log L - mT\right) \\ &= \phi(-d_2) \end{aligned}$$

where

$$d_2 = \frac{-\log L + mT}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

$L = \frac{K}{V_0}$ is the initial leverage ratio and ϕ is the standard normal distribution function. The value of the debt at time 0 is then

$$D_0 = V_0 - E_0 = V_0\phi(-d_1) + Ke^{-rT}\phi(d_2)$$

The term structure of the credit spread is given by

$$S(0,T) = -\frac{1}{T}\log\left(\frac{D_0}{Ke^{-rT}}\right)$$

For $V_0 > K$, the spread curve starts at zero for $T = 0$, increases sharply to a maximum, and then start to decrease to a positive plateau. This is in accordance with the diffusive character of the model. For very short maturity times, the asset price diffusion will almost surely never cross the default barrier. The probability of default then increases for longer maturities but start to decrease again as the geometric Brownian motion drifts away from the barrier.

Besides all the nice intuitive definition of default and mathematical attractiveness of Merton's model, there are also several shortcomings. One problem of the model is the restriction of default time to the maturity of the debt, ruling out the possibility of an early default, no matter what happens with the firm's value before the maturity of the debt. Another handicap of the model is that the usual capital structure of a firm is much more complicated than a simple zero-coupon bond. Also, the credit spreads that are produced by the model are 0 for $T = 0$ which is in contradiction with empirical observations.

Despite all these features of the Merton's model, it still serves as a benchmark model for comparison and provides a useful basic framework to develop more complicated models based on it.

2.2.2 The First passage models

First Passage Models were introduced by Black and Cox (1976) extending the Merton model to the case when the firm may default at any time, not only at the maturity date of the debt. More precisely, the time of default is defined as the first instant at which a relevant process, describing for instance the assets value of the firm, falls below a certain level called the default barrier. Such a boundary can be assumed to be either constant as in Longstaff and Schwartz (1995), either a time-dependent barrier as in Black and Cox (1976), or it can be governed by some stochastic process see for example Saà-Requejo and Santa-clara (1999). We concentrate on the model of Black and Cox (1976) and give the main features of it.

Consider again a firm with asset value V given by (2.6) and outstanding debt with face value K at maturity T. Instead of having the possibility of default only at maturity time T, Black and Cox (1976) postulated that default occurs at the first time the firm's asset value drops below a certain time dependent barrier $K(t)$. In this model, the default time is given by

$$\tau_t = \inf\{t > 0 : V_t \leq K(t)\} \tag{2.7}$$

More precisely, Black and Cox (1976) consider a time dependent default barrier given by $K(t) = e^{-k(T-t)}K$, that is, the face value K discounted by a constant rate k.

Suppose V is a traded asset and let $D(V, t, T)$ be the value of the defaultable bond at time t, D satisfies the following partial differential equation

$$\frac{\partial D}{\partial t} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 D}{\partial V^2} + rV\frac{\partial D}{\partial V} - rD = 0 \tag{2.8}$$

with boundary conditions

$$\begin{cases} D(V, T, T) = \min(V, K) \\ D(K, t, T) = K(t) \end{cases}$$

Under the risk neutral measure Q, the default time is given by

$$\tau_t = \inf\left\{t > 0 : (m-k)t + \sigma W_t^Q < \log L - kT\right\}$$

Using the properties of the Brownian motion W^Q, in particular the reflection principle see for example Karatzas and Shreve (1991), one can infer the risk neutral probability of default occurring before time T

$$Q\left(0 \leq \tau \leq T\right) = \phi\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right) + \left(\frac{Ke^{-kT}}{V_0}\right)^{\frac{2(m-k)}{\sigma^2}} \phi\left(\frac{\log L + (m-2k)T}{\sigma\sqrt{T}}\right)$$

The payoff of the equity holders at maturity is $(V_T - K)^+ 1_{\{M_T > K(t)\}}$ where $M_t = \min_{s<t} V_s$ is the running minimum of the diffusion V_t. Its value is then given by the price of a down-and-out call option with a moving barrier $K(t)$, for which closed form expressions are available, see Merton (1974). This is smaller than the equity value obtained in the Merton model, and is not monotone in the volatility. For the bond holders, the payoff is $K - (K - V_T)^+ + (V_T - K)^+ 1_{\{M_T \leq K(t)\}}$. Its value is then given by the price of a zero-coupon bond minus a vanilla put option plus a down-and-in call option.

The principal drawback of the first passage models is the analytical complexity they introduce, which is increased if we consider stochastic interest rates or endogenous default thresholds. This mathematical complexity makes difficult to obtain closed form expressions for the value of the firm's equity and debt, or even for the default probability, forcing us to make use of numerical procedures. The models still suffer from the near zero short-term spreads, simply due to the fact that a continuous process needs some time to reach the level of default boundary.

In figure 2.1, we plot the term structure for implied credit spread for varying the debt-to-asset value ratio L. We fix $r = k = 3\%$ per year, $K = 10$, $V_0 = 20$ and $\sigma = 30\%$ per year. The same behavior is observed in Merton and first passage models, except that in the first passage models, the spreads are more important and exhibit a faster decrease for longer maturities. In addition, for both models the spread curves are higher when the debt-to-asset value ratio L increases.

Although some extensions, including an allowance of jumps in asset process have been introduced for purposes of bond pricing, for example by Hilberink and Rogers (2002) the pricing formulas are not as nice as in this section and there is an issue of calibration about how to interpret the market data for the jumps.

It is important to mention that in the first-passage time modeling of credit risk, there is no distinction between the time at which a firm defaults and the time at which it is liquidated. In the excursion models which will be discussed in the next section, the duration of the default is expressly modeled.

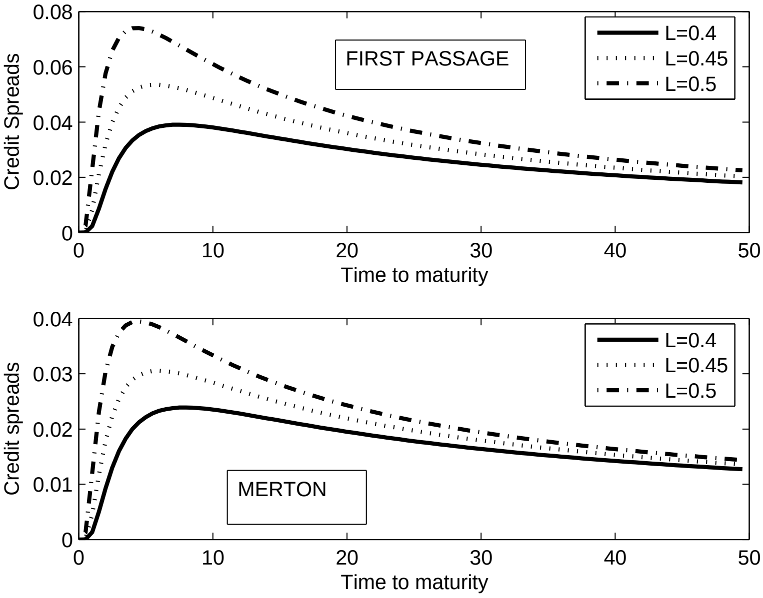


Figure 2.1: Term structure for structural models varying firm leverage L

2.2.3 Excursion models

In the first passage models, a firm defaults when its assets are too low according to some criterion, and is immediately liquidated. This definition of default no longer reflects economic reality; bankruptcy laws often grant an extended time for restructuring. In this section, instead of setting the default time as the firm's asset hitting time for a barrier $K(t)$, one can allow for the asset value to have "excursions" in the region under the barrier and set a default to occur if the time spent in that region is sufficiently long. Haber et al. (1999) and Moraux (2003) suggest to consider the occupation time which is the total time spent by the value process beyond the default threshold, hence it seems more suitable in order to study the financial history and distress periods of the debt issuer.

Consider a fixed barrier K and define the occupation time under the barrier as

$$\mathcal{T}(t) = \int_0^t 1_{\{V_s \leq K\}} ds \tag{2.9}$$

$\mathcal{T}(t)$ is a measure of the amount of time the process V spends below the barrier during the time interval $[0; t]$.
Then, fixing the maximum duration allowed below the barrier $\mathcal{T}^* \geq 0$, the default time can be specified as $\tau = \min(\tau_1, \tau_2)$ where

$$\tau_1 = \inf\{t > 0 : \mathcal{T}(t) \geq \mathcal{T}^*\}$$

and

$$\tau_2 = \begin{cases} T \text{ if } \; V_T < K \\ \infty \text{ if else} \end{cases}$$

The probability of default is defined by

$$P(0 \leq \tau \leq T) = 1 - P(\tau_1 > T, \tau_2 > T)$$

and can be calculated from the joint probability of $(V_t, \mathcal{T}(t))$. In general, one can use Monte Carlo simulation in order to compute an estimate of the average time of default and the default probability.

To summarize, all these structural models relate the default to the process for the firm's asset backing and define the default event in terms of a boundary condition on this process. A major deficiency of these models, as noted in Madan and Unal (1998) is that they treat the value of the asset backing as a primary asset of the economy when in fact it may be a derivative asset in its own right with exposure to other more primitive state variables of the economy. Prices should be reduced to exogenous state variables of the economy and it is unclear that a firm's asset value is such a variable. In addition, it is unrealistic that the firm's asset value is a tradable security.

2.3 Reduced form Models of Credit Risk

Reduced form models, also known as hazard rate models or intensity-based models form an approach to default complementary to the structural models. In structural models, default was directly linked to the value of the firm, and in the simplest versions, default times are predictable in the filtration available to traders. In contrast, reduced form models make the assumption that default is always a surprise, that is, a totally inaccessible stopping time. The firm value is not modelled, but rather attention is focussed on the instantaneous conditional probability of default called the default intensity. In applications, default intensities may be allowed to depend on observable variables that are linked with the likelihood of default, such as debt-to-equity ratios, volatility measures, other accounting measures of indebtedness, market equity prices, bond yield spreads, industry performance measures, and macroeconomic variables related to the business cycle, as Duffie and Wang (2003).

We fix a complete probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ of sub-σ-algebras of $\mathcal{F}$ satisfying the usual conditions, which are listed in appendix A.

A *counting process N has an intensity* λ if λ is a predictable non-negative process satisfying $\int_0^t \lambda_s ds < \infty$ almost surely for all t, such that the compensated counting process M given by

$$M_t = N_t - \int_0^t \lambda_s ds \tag{2.10}$$

is a local martingale. The accompanying intuition is that, at any time t, the $\mathcal{F}_t$-conditional probability of an event between t and $t + \Delta t$ is approximately $\lambda_t \Delta t$ for small Δt. This intuition is justified in the sense of derivatives if λ is bounded and continuous, and under weaker conditions.

We will say that a *stopping-time τ has an intensity* λ if τ is the first jump time of a nonexplosive counting process whose intensity process is λ.

If a stopping time τ is predictable and if the filtration $\mathbb{F}$ is the standard filtration of some Brownian Motion B, then τ could not have an intensity. We know this from the fact that if $\mathbb{F}$ is the standard filtration of B then the associated compensated counting process M of (2.10) could be represented as a stochastic integral with respect to B, and therefore cannot jump, but M must jump at τ. In order to have an intensity, a stopping time must be totally inaccessible, a property whose definition suggests arrival as a sudden surprise, but there are no such surprises on a Brownian filtration.

As an example, Duffie and Lando (2001) model the firms equityholders or managers are equipped with some Brownian filtration for purposes of determining their optimal default time τ but that bondholders have imperfect monitoring, and may view τ having an intensity with respect to the bondholders own filtration $\mathbb{F}$, which contains less information than the Brownian filtration.

2.3.1 Homogeneous Poisson Event

Let $N = (N_t)_{t \geq 0}$ a counting process i.e. for each $t > 0$ they count the number of default events that happen between time 0 and time t. We say that N is a *homogeneous Poisson process* with intensity λ if the increments $N_t - N_s$ are independent and have a Poisson distribution with parameter $\lambda(t-s)$ for $s < t$, i.e

$$P(N(t) - N(s) = k) = \frac{e^{-\lambda(t-s)}(\lambda(t-s))^k}{k!} \tag{2.11}$$

The fundamental assumption of the intensity based approach consists of setting the default time equal to the first jump time of a Poisson process N with given intensity λ. Thus, τ is exponentially distributed with parameter λ and the default probability is given by

$$F(T) = P(\tau \leq T) = 1 - e^{-\lambda T} \tag{2.12}$$

The intensity, or hazard rate, λ is the conditional default arrival rate given no default:

$$\lambda = -\frac{\partial \log P(\tau > t)}{\partial t}$$

which may also be expressed as

$$P(t \leq \tau \leq t + dt | \tau > t) \approx \lambda dt$$

Let f denotes the density of F, from (2.12) we obtain

$$\lambda = \frac{f(t)}{1 - F(t)}$$

The valuation of defaultable zero bonds is straightforward in the intensity based approach. Let us assume that in the event of a default, bond investors recover some constant fraction $R \in [0,1]$ of the unit face value of the bond with maturity T. This convention is called *recovery of face value* or simply *constant recovery*. Assume that interest rates $r > 0$ are constant; hence non-defaultable zero bond prices are given by $B(0,T) = e^{-rT}$. Supposing that the recovery is paid at T, the defaultable bond price at time zero is given by

$$\begin{aligned} D(0,T) &= E^Q\left(e^{-rT}\left(1_{\{\tau > T\}} + R1_{\{\tau \leq T\}}\right)\right) \\ &= e^{-rT}\left(e^{-\tilde{\lambda}T} + R\left(1 - e^{-\tilde{\lambda}T}\right)\right) \\ &= B(0,T)\left[R + (1-R)Q(\tau > T)\right] \end{aligned}$$

When $R = 0$, we have that

$$D(0,T) = B(0,T)Q(\tau > T) = e^{-(r+\tilde{\lambda})T} \tag{2.13}$$

i.e. we can value a defaultable bond as if it were default free by simply adjusting the discounting rate. Instead of discounting with the risk-free interest rate r, we discount with the default-adjusted rate $r + \tilde{\lambda}$, where $\tilde{\lambda}$ is the risk neutral intensity. This is a central and important feature of the intensity based approach.
Given the bond price, one can now look at the resulting credit spreads, given by

$$S(0,T) = -\frac{1}{T}\log\left[R + (1-R)Q(\tau > T)\right] \tag{2.14}$$

When $R = 0$, we have

$$S(0,T) = -\frac{1}{T}\log e^{-\tilde{\lambda}T} = \tilde{\lambda} \tag{2.15}$$

so that the credit spread is in fact given by the risk-neutral intensity $\tilde{\lambda}$. Obviously, even for maturities going to zero the spread is bounded away from zero.
With a constant intensity, the term structure of credit spreads is of course flat. For richer term structures, we need more sophisticated intensity models. An extension towards time variation and stochastic variation in intensities is considered in the following sections.

2.3.2 Inhomogeneous Poisson Event

$N = (N_t)_{t\geq 0}$ is called an *inhomogeneous Poisson process* with deterministic intensity function λ_t, if the increments $N_t - N_s$ are independent and for $s < t$

$$P(N(t) - N(s) = k) = \frac{e^{-\int_s^t \lambda_u du}(\int_s^t \lambda_u du)^k}{k!} \tag{2.16}$$

The default probability is then given by

$$P(\tau \leq T) = 1 - P(N_T = 0) = 1 - e^{-\int_0^T \lambda_u du} \tag{2.17}$$

Assuming constant interest rate r and constant recovery rate R, the price of the defaultable bond at time zero is

$$D(0,T) = B(0,T)\left[R + (1-R)Q(\tau > T)\right] \tag{2.18}$$

where

$$Q(\tau > T) = e^{-\int_0^T \tilde{\lambda}_u du} \tag{2.19}$$

As in (2.14), we can now look at the resulting credit spreads. With a zero recovery convention,

$$S(0,T) = -\frac{1}{T}\text{log}e^{-\int_0^T \tilde{\lambda}_u du} = \frac{1}{T}\int_0^T \tilde{\lambda}_u du \tag{2.20}$$

where $\tilde{\lambda}$ is the risk neutral intensity. Analogously to the constant intensity case, we find for short spreads

$$\text{lim}_{t\to 0} S(0,t) = \tilde{\lambda}_0 > 0 \tag{2.21}$$

For derivatives pricing purposes in practice, one often assumes that risk neutral intensities are piecewise constant:

$$\tilde{\lambda_t} = \alpha_i, \;\; t \in \left[T_{i-1} - T_i\right], \;\; i = 1, 2, ..., n$$

for some $n \geq 1$ and constants T_i which represent the maturities of n traded default-contingent instruments such as bonds or default swaps of the same issuer. Given some estimate of riskless interest rates r and some recovery assumption, the prices of these instruments can then be used to calibrate the risk neutral intensities $\tilde{\lambda_t}$ from observed market prices. This is a bootstrapping approach.
Figure 2.2 simulates the homogeneous Poisson process for $\lambda \in \{0.3, 0.6, 0.9\}$ and the inhomogeneous Poisson process when $\lambda(t) = t + 2$. As we would expect the number of jumps increases when the default intensity rises.

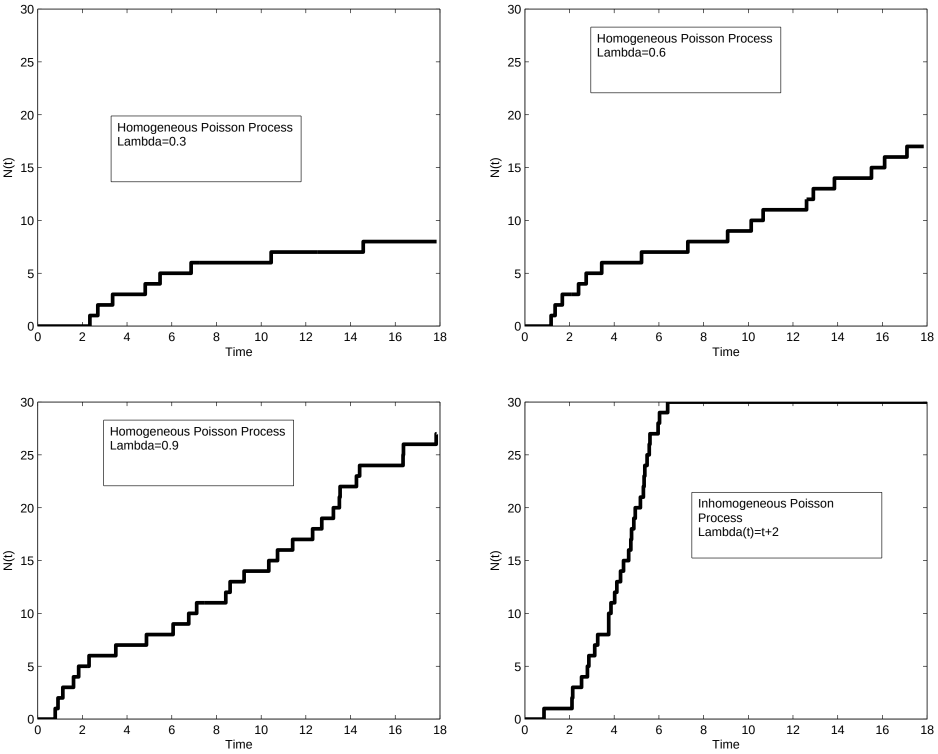


Figure 2.2: Simulation of Poisson Processes

2.3.3 Doubly-Stochastic Poisson Event

Let us suppose that all the background information available in the economy, except for the default times, is expressed through the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$. For example, $\mathcal{G}_t$ might be generated by a d-dimensional driving process X_t. We assume that all the default-free economic factors, including the risk-free interest rates, are adapted to $\mathbb{G}$. Assume further that there exists a non-negative process λ_t which is also adapted to $\mathbb{G}$ that plays the role of a stochastic intensity, generally correlated with the different components of the driving process X_t. Next, assume that $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ is the filtration generated by the default indicator process $1_{\{\tau \leq t\}}$. The full filtration for the model is obtained as

$$\mathbb{F} = \mathbb{G} \vee \mathbb{H}$$

We say that a point process $N = (N_t)_{t \geq 0}$ is a *doubly-stochastic Poisson process* or a *Cox process* if, conditioned on the background information $\mathcal{G}_t$ available at time t, N_t is an inhomogeneous Poisson process with a time-varying intensity λ_t. In other words, each realization of the process λ_t determines the local jump probabilities for the process N_t. The intuition of the doubly-stochastic assumption is that $\mathcal{G}_t$ contains enough information to reveal the intensity λ_t, but not enough information to reveal the event times of the counting process N. Therefore, the default time τ is a $\mathbb{F}$-stopping time but not necessarily a $\mathbb{G}$-stopping time. Letting A be the event $\{N_s - N_t\}$ of no arrivals, the law of iterated expectations implies that, for $\tau > t$

$$\begin{aligned} P\left(\tau > s|\mathcal{F}_t\right) &= E\left(1_A|\mathcal{F}_t\right) \\ &= E\left[E\left(1_A|\mathcal{G}_s \vee \mathcal{H}_t\right)|\mathcal{F}_t\right] \\ &= E\left[P\left(N_s - N_t = 0|\mathcal{G}_s \vee \mathcal{H}_t\right)|\mathcal{F}_t\right] \end{aligned}$$

$$P\left(\tau > s|\mathcal{F}_t\right) = E\left(e^{-\int_t^s \lambda_u du}|\mathcal{F}_t\right) \tag{2.22}$$

Equation (2.22) is convenient for calculations, because evaluating the expectation in (2.22) is computationally equivalent to the standard financial calculation of default-free zero-coupon bond price, treating λ as a short-term interest process. Indeed, this analogy is also quite helpful for intuition when extending (2.22) to pricing applications.
Let $\lambda_t = \Phi(X_t)$ where the driving process X_t solves a stochastic differential equation of the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t \tag{2.23}$$

for some $\mathcal{F}_t$-standard Brownian motion B in $\mathbb{R}^d$. Here, $\mu(.)$ and $\sigma(.)$ are functions on the state space of X that satisfy enough regularity for (2.23) to have a unique solution. With this, the survival probability calculation (2.22) is of the form

$$\begin{aligned} P(\tau > s|\mathcal{F}_t) &= E\left(e^{-\int_t^s \Phi(X_u)du}|X_t\right) \qquad (2.24) \\ &= f(X_t, t) \qquad (2.25) \end{aligned}$$

where, under the usual regularity for the Feynman-Kac approach, $f(.)$ solves the partial differential equation (PDE)

$$\frac{\partial f(x,t)}{\partial t} + \mathcal{C}f(x,t) - \Phi(x)f(x,t) = 0, \tag{2.26}$$

for the generator $\mathcal{C}$ of X given by

$$\mathcal{C}f(x,t) = \sum_i \frac{\partial f(x,t)}{\partial x_i}\mu_i(x) + \frac{1}{2}\sum_{i,j}\frac{\partial^2 f(x,t)}{\partial x_i \partial x_j}\gamma_{ij}(x)$$

and where $\gamma(x) = \sigma(x)\sigma(x)'$, with the boundary condition $f(x,s) = 1$. Parametric assumptions are often used to get an explicit solution to this PDE. Although very natural, the doubly-stochastic assumption excludes several plausible situations. For example, the process N_t cannot be adapted with respect to the background information $\mathcal{G}_t$, nor can it be directly triggered by any of the driving processes as is the case with structural models. It also excludes the possibility of N_t directly influencing the background processes. We shall now introduce the following two key properties of the reduced model which are known to be equivalent to one another.

- *$\mathcal{H}$-condition*: Under the physical measure P, $\mathcal{H}_t$ and $\mathcal{G}_\infty$ are independent conditioned on $\mathcal{G}_t$, that is, for any $\mathcal{H}_t$-measurable random variable X and $\mathcal{G}_\infty$-measurable random variable Y,

  $$E\left(XY|\mathcal{G}_t\right) = E\left(X|\mathcal{G}_t\right)E\left(Y|\mathcal{G}_t\right)$$

- The *martingale invariance property*: any $\mathcal{G}_t$-martingale is also a $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$-martingale.

Taken together, the $\mathcal{H}$-condition and martingale invariance property have several modelling implications: First, the default event cannot cause any observable effect on the background filtration; second, in multifirm versions, the $\mathcal{H}$-condition forbids "'contagion"', namely the effect that the default of one firm influences the default intensity of another firm. Therefore reduced form models exclude several effects that are often viewed as important in credit risk: this is one of the primary weaknesses pointed out by detractors. Consider a defaultable security (X, T) paying off a $\mathbb{G}_T$-measurable random variable X at time T if no default occurs and zero otherwise (for $X = 1$ this is a defaultable zero bond with zero recovery). The price at time t of the defaultable security under a risk neutral measure Q is

$$\begin{aligned}
& E^Q\left(e^{-\int_t^T r_s ds} X_T 1_{\{\tau>T\}} | \mathcal{F}_t\right) \\
= \ & E^Q\left[E^Q\left(e^{-\int_t^T r_s ds} X_T 1_{\{\tau>T\}} | \mathcal{G}_T \vee \mathcal{H}_t\right) | \mathcal{F}_t\right] \\
= \ & E^Q\left[e^{-\int_t^T r_s ds} X_T E^Q\left(1_{\{\tau>T\}} | \mathcal{G}_T \vee \mathcal{H}_t\right) | \mathcal{F}_t\right] \\
= \ & 1_{\{\tau>t\}} E^Q\left[e^{-\int_t^T \left(r_s+\tilde{\lambda}_s\right) ds} X_T | \mathcal{F}_t\right]
\end{aligned}$$

where r is the risk-free short rate process and $\tilde{\lambda}$ is the risk-neutral intensity process for default. Also in the most general case, the defaultable claim (X, T) can be valued as if it were default-free by simply adjusting the rate used for discounting by the risk-neutral intensity. In the intensity based framework, defaultable term structure modeling exactly parallels ordinary non-defaultable term structure modeling.
In the general setting under technical conditions on $\tilde{\lambda}$, short spreads satisfy

$$\lim_{T \to t} S(t, T) = \tilde{\lambda}_t$$

almost surely, as expected. In the general intensity based approach, credit spreads are bounded away from zero, and short spreads are given by the risk neutral intensity. That is, for very short maturities bond investors still demand a premium for default risk. This empirically confirmed property is due to modeling default as a Poisson event, which implies that the default is totally unpredictable. That means default is a complete surprise event; there is no way to anticipate it as was the case in the structural approach. The unpredictability of defaults leads to markets being incomplete in the intensity based framework. As long as there is no asset having default contingent payoffs available for trading, defaultable bonds cannot be perfectly hedged. An intuitive explanation for this is that unpredictable jumps in bond prices cannot be duplicated with predictable trading strategies. As a consequence we have many prices for the same defaultable claim since there is many martingale measures. Instead of choosing an equivalent martingale measure for valuation, an alternative approach based on utility valuation theory has been developed to produce a unique price. This is called the utility indifference valuation.

Chapter 3

Indifference Pricing for Defaultable bonds

3.1 Utility maximization: The General Set-Up

Utility maximization or the maximization of expected utility is a very basic topic in economics. A rigorous treatment of this subject within the framework of mathematical finance was initiated by the two seminal papers by Merton (1969) and Samuelson (1969). Since then the problem of utility maximization has found its way into numerous textbooks : from the more economic point of view, e.g., Huang and Litzenbenger (1988), as well as from the more mathematical side, e.g., Karatzas and Shreve (1998) or Föllmer and Shied (2002).

We start with a locally bounded vector semi-martingale $S = (S_t)_{0 \leq t \leq T}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The market we consider is furthermore assumed to be *frictionless*. This means that the agents who trade in the market, i.e., buy and sell assets, do not face any transaction costs. There are $d+1$ assets: one savings account and d stocks, so we write $S = (S^0, ..., S^d)$, with each S^i a locally bounded semi-martingale, $S^i = (S^i_t)_{0 \leq t \leq T}$. The zeroth asset is riskless, with $S^0_t = 1$ for all $t \in [0, T]$, so for instance, we are assuming the interest rate is zero.

A probability measure Q is called an *equivalent martingale measure* if it is equivalent to P and if S is a local martingale under Q. We denote by $\mathcal{M}$ the family of all such measures and assume that

$$\mathcal{M} \neq \phi.$$

This condition is essentially equivalent to the absence of arbitrage opportunities in the market; see Delbaen and Schachermayer (1994) for precise statements and more details.

A *self-financing portfolio* is defined by a pair (x, H) where $x \in \mathbb{R}$ defines the initial wealth and $H = (H^i)_{i=1}^d$ is a predictable and S-integrable process specifying the number of shares of each of the stocks held in the portfolio. Hence, for each $i = 1, ..., d$, $H^i = (H_t^i)_{0 \leq t \leq T}$. The value process of a self-financing portfolio evolves in time as the stochastic integral of the process H with respect to the stock price:

$$X_t = x + (H.S)_t = x + \int_0^t H_s dS_s, \;\; t \in [0, T]$$

The market is in general incomplete, so not every contingent claim C can be replicated by a self-financing portfolio. Then there is no unique martingale measure, and the possible no-arbitrage prices span an interval given by

$$\left(\inf_{Q \in \mathcal{M}} E^Q C, \sup_{Q \in \mathcal{M}} E^Q C\right). \tag{3.1}$$

This was shown by Kramkov (1996).

When the *market is complete*, there exists a unique self-financing portfolio satisfying $X_T = C$. We say the *strategy H replicates C*, and H is the unique hedging strategy for the claim. In this case, there is a unique martingale measure Q, the interval in (3.1) reduces to a single point, and the no-arbitrage price of the claim at time 0 is $P_0^A = E^Q C$.

In an *incomplete market*, one is faced with choosing one of the martingale measures $Q \in \mathcal{M}$ as a pricing measure. At first sight, this choice appears to have little to do with optimal investment. But the incompleteness means that selling a claim entails opening oneself up to non-zero terminal risk, as represented by the difference $X_T - C$, where X_T is the terminal wealth of any self-financing portfolio. The question arises as to how one should deal with the residual risk $X_T - C$. This can only be answered by specifying the risk preferences of the financial agent selling the claim.

We assume that his preferences have an expected utility representation, following the theory introduced by von Neumann and Morgenstern (1953). This

means that the agent tries to maximize his expected utility over all admissible trading strategies H. This gives rise to the agent's primal value function. The von Neumann and Morgenstern (1953) utility function $U : \mathbb{R} \to \mathbb{R}$, is assumed to be strictly increasing, strictly concave, continuously differentiable, and is assumed to satisfy the Inada conditions:
$U'(-\infty) = \lim_{x \to -\infty} U'(x) = \infty$, $U'(\infty) = \lim_{x \to \infty} U'(x) = 0$.
$U'(x) > 0$ corresponds that investors prefer more wealth to less. The concavity reflects the issue of risk aversion: people are assumed to dislike risk and thus prefer for example to get 100 currency unit for sure compared to having a 50-50 chance of getting 200 currency unit or nothing. The concavity implies :

$$\lambda U(x) + (1-\lambda)U(y) < U(\lambda x + (1-\lambda)y), \ \forall \lambda \in (0,1).$$

In our example $\frac{1}{2}U(200) + \frac{1}{2}U(0) < U(100)$.
Let us introduce the coefficient of absolute risk aversion which was introduced by Arrow (1971) and Pratt (1964). It is defined by

$$R_a(x) = -\frac{U''(x)}{U'(x)}.$$

Since $U'(x) > 0$ and $U''(x) < 0$, the risk aversion $R_a(x)$ is greater than 0. As indicated above it is very reasonable to use the curvature i.e. the degree of concavity $U''(x)$ as a measure for the risk aversion. The reason why $U''(x)$ is divided by $U'(x)$, which corresponds to a normalization, is the fact that von Neumann and Morgenstern (1953) utility functions are only unique up to positive linear transformations. That is $U(x)$ and $\bar{U} = aU(x) + b$ for $a, b \in \mathbb{R}$, $a > 0$ correspond to the same preference relation of the agent. Dividing $U''(x)$ by the derivative $U'(x)$ respectively $\bar{U}'(x)$ ensures the same risk aversion coefficient.
The utility function that we shall employ in this thesis is the *exponential utility function* $U(x) = -e^{-\gamma x}$ which has a constant absolute risk aversion (CARA) of $\gamma > 0$. A Bigger γ corresponds to a higher degree of risk aversion. In particular, $\gamma = \infty$ indicates absolute risk aversion while $\gamma = 0$ corresponds to risk neutrality.
To be specific about the permissible trading strategies H for an agent with the exponential utility function, we first introduce $\mathcal{M}f$, the set of measures in $\mathcal{M}$ with finite relative entropy with respect to P, where the *relative entropy*

of a measure, $H(Q|P)$ is defined as

$$\mathcal{H}(Q|P) = \begin{cases} \mathbb{E}\left\{\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)\right\}, Q << P, \\ \infty, \qquad \text{otherwise} \end{cases}$$

where $Q << P$ denotes Q is absolutely continuous with respect P.
As we will see later, this quantity will play a role in the dual to a primal utility maximization problem under exponential utility.
For an agent with exponential utility function, we follow Becherer (2004) and define the set $\mathcal{A}$ for *admissible strategies* H by

$$\mathcal{A} = \{H : (H.S) \text{ is a Q-martingale for all Q} \in \mathcal{M}f\}$$

The primal value function $V(x)$ of the agent who sells the claim C with initial wealth x, is given by

$$V(x) = \sup_{H\in\mathcal{A}} E(U(X_T - C)) = \sup_{H\in\mathcal{A}} E(U(x + \int_0^T H_s dS_s - C)) \quad (3.2)$$

Similarly, the primal value function $W(x)$ of the agent who buys the claim C with initial wealth x, is given by

$$W(x) = \sup_{H\in\mathcal{A}} E(U(X_T + C)) = \sup_{H\in\mathcal{A}} E(U(x + \int_0^T H_s dS_s + C)) \quad (3.3)$$

For an initial wealth x, we also define the Merton value function $M(x)$ of the agent who sells or buys no claim

$$M(x) = \sup_{H\in\mathcal{A}} E(U(X_T)) = \sup_{H\in\mathcal{A}} E(U(x + \int_0^T H_s dS_s)) \quad (3.4)$$

The seller indifference price of the claim, v, is defined as:

$$V(x+v) = M(x)$$

Using the exponential utility function,

$$\sup_{H\in\mathcal{A}} E(-e^{-\gamma(X_T+v-C)}) = \sup_{H\in\mathcal{A}} E(-e^{-\gamma X_T})$$

which means that the seller is indifferent in the sense of the maximum expected utility between selling the contingent claim for the premium v and selling no claim. Due to the desirable separability of the exponential function, we have the following explicit expression for the indifference price

$$v(\gamma, C) = \frac{1}{\gamma}\log \inf_{H\in\mathcal{A}} E\left(e^{-\gamma(X_T - C)}\right) - \frac{1}{\gamma}\log \inf_{H\in\mathcal{A}} E\left(e^{-\gamma X_T}\right) \qquad (3.5)$$

Similarly, the buyer indifference price of the claim, w, is defined as:

$$W(x - w) \quad = \quad M(x)$$

which means that the buyer is indifferent in the sense of the maximum expected utility between buying the contingent claim for the premium w and buying no claim.

Under the exponential utility function, the following explicit expression for the buyer indifference price is

$$w(\gamma, C) = -\frac{1}{\gamma}\log \inf_{H\in\mathcal{A}} E\left(e^{-\gamma(X_T + C)}\right) + \frac{1}{\gamma}\log \inf_{H\in\mathcal{A}} E\left(e^{-\gamma X_T}\right) \qquad (3.6)$$

In general, for a given underlying price process of S_t, the two terms in (3.5) and in (3.6) can at least be evaluated by numerical methods.

Under certain technical conditions, maximizing the expected utility of exponential function has an interesting dual problem, (Frittelli (2000) and Delbaen et al. (2002)). This gives rise to the study of the so-called duality approach for utility indifference pricing, and leads to a linkage between the minimal entropy martingale measure price and $v(0;.) = w(0;.)$ (i.e., the limit of the utility indifference price as the risk aversion parameter tends to 0). We refer interested readers to those references listed in the bibliography.

The utility indifference price has the following desirable properties (cf. Rouge and El Karoui (2000), Henderson and Hobson (2004)):

1. If the market is complete or the claim is perfectly replicable, the utility indifference price is equivalent to the complete market price.

2. Monotonicity: $v(\gamma_2, C) \geq v(\gamma_1, C)$ for $\gamma_2 \geq \gamma_1 > 0$; $w(\gamma_2, C) \leq w(\gamma_1, C)$ for $\gamma_2 \geq \gamma_1 > 0$.

3. Convexity for sell price: for $\mu \in (0, 1)$,

$$v(\gamma, \mu C_1 + (1-\mu)C_2) \leq \mu v(\gamma, C_1) + (1-\mu) v(\gamma, C_2)$$

4. Concavity for buy price: for $\mu \in (0,1)$,

$$w(\gamma, \mu C_1 + (1-\mu)C_2) \geq \mu w(\gamma, C_1) + (1-\mu) w(\gamma, C_2)$$

5. Non-linear pricing: in contrast to the arbitrage-free prices, utility indifference prices are non-linear in the number of claims. This means that the seller requires more than twice the price for taking on twice the risk. Alternatively, the buyer is not willing to pay twice as much for twice as many claims, but requires a reduction in this price to take an additional risk.

In the next section, we will apply the utility-based approach to the pricing of defaultable bonds in the case of stochastic default intensity.

3.2 Indifference Valuation for Defaultable bonds: PDE Approach

3.2.1 Maximal Expected Utility Problem

Let us start with several basic assumptions

Assumption 1. *(The firm)*

> Consider a zero-coupon defaultable bond of a firm with expiration date $T < \infty$. Unlike in a traditional structural approach, the default time τ is the first jump time of a counting process with a random intensity $\lambda \geq 0$, which is correlated with the firm's stock price S. The price of the asset S is described by a geometric Brownian motion. The intensity process is $\lambda(Y_t)$ where $\lambda(.)$ is a non-negative, locally Lipschiptz, smooth and bounded function. The dynamics of S and Y are
>
> $$\begin{cases} dS_t &= \mu S_t dt + \sigma S_t dW_t^1, \ S_0 = S > 0, \\ dY_t &= b(Y_t)dt + a(Y_t)\left(\rho dW_t^1 + \rho' dW_t^2\right), \ Y_0 = \lambda, \end{cases} \tag{3.7}$$
>
> with ρ is the correlation coefficient and $\rho' = \sqrt{1-\rho^2}$.

The processes W^1 and W^2 are two independent Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathcal{F} = \mathcal{F}_T$ is the σ-algebra at time T. Probability space is assumed to be large enough to support both a stochastic process $Z = (Z_t = (S_t, Y_t), 0 \leq t \leq T)$ which is right continuous with left limits and a Poisson process $N = (N_t)_{t\geq 0}$ independent of Z.

Z is the background driving process that generates the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0} = (\sigma(Z_s = (S_s, Y_s) : 0 \leq s \leq t))_{t\geq 0}$ representing the flow of all background information except default itself and $\mathcal{G} = \mathcal{G}_T$ is the sub σ-algebra at time T.

The Poisson process N_t has a non negative and right-continuous stochastic intensity $\lambda(Y_t)$ which is independent of N_t and is assumed to be adapted to $\mathbb{G}$. The assumption of time dependent intensity implies the existence of an inhomogeneous Poisson process.

In this subfiltration setting it is natural to consider a $\mathcal{G}$-conditional Poisson process in such a way that a Cox process results in association with the state variables process Z. So, let $N_t = \sum_{i=1}^{\infty} 1_{\{\tau_i \leq t\}}$ be a Cox process with intensity $\lambda(Y_t)$. The default time τ is a stopping time, $\tau : \Omega \rightarrow \mathbb{R}^+$, defined as the first jump time of N,

$$\tau = \inf\left\{t \in \mathbb{R}^+ | N_t > 0\right\}$$

The right continuous default indicator process $1_{\{\tau\leq t\}}$ generates the subfiltration $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0} = (\sigma\left(1_{\{\tau\leq s\}} : 0 \leq s \leq t\right))_{t\geq 0}$, that is assumed to be a component of the filtration $\mathbb{F}$. Since obviously $\mathcal{H}_t \subset \mathcal{F}_t$, $\forall t \in \mathbb{R}^+$, τ is a stopping with respect to $\mathbb{F}$, but it is not necessarily a stopping time with respect $\mathbb{G}$.

We define the full filtration as $\mathbb{F} = \mathbb{G} \vee \mathbb{H}$, that is $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$, $\forall t \in \mathbb{R}^+$ where $\mathcal{G}_t = \sigma(Z_s = (S_s, Y_s) : 0 \leq s \leq t)$ and $\mathcal{H}_t = \sigma\left(1_{\{\tau\leq s\}} : 0 \leq s \leq t\right)$. Following Lando (1998), a way to define the default time τ of N is to let E_1 be an exponential random variable and define

$$\tau = \inf\left\{s \geq 0 : \int_0^s \lambda(Y_u) du \geq E_1\right\}.$$

Assumption 2. *(Regularity of the driving process Y)*

> The coefficients $a(.)$ and $b(.)$ are such that (3.7) has a unique strong solution Y_t which lies in $\mathbb{R}$ for all $t \in [0, T]$, P-a.s.

Observe that Assumption 2 is weaker than the usual global Lipschitz continuity and growth conditions that are sufficient for the existence of a strong solution Y_t. This avoids ruling out from the stochastic intensity model such that the CIR model, for which Y_t is square root diffusion that does not satisfy the global Lipschipz condition.

Assumption 3. *(The investor)*

> Assume that a bank account is also traded in the market and the risk free interest rate is r. The investor with initial wealth x will choose a self-financing trading strategy in order to maximize his utility with respect to terminal wealth. Besides trading the stocks, he has two choices, either holding corporate bonds until maturity T or not holding them. The investor's control process is π_t, the amount held in the stock at time t, until $\tau \wedge T$. Following Musiela and Zariphopoulou (2004), the set of admissible policies $\mathcal{A}$ in this model is the set of trading strategies π that are $\mathcal{F}_t$-measurable and satisfy the integrability contraint $E\{\int_0^T \pi_s^2 ds\} < \infty$. The set of all admissible strategies over $[t, \tau \wedge T]$ is denoted by $\mathcal{A}_{t,T}$. For $t < \tau \wedge T$, the investor's wealth process X_t follows
>
> $$\begin{aligned} dX_t &= \pi_t \frac{dS_t}{S_t} + r(X_t - \pi_t)dt && (3.8) \\ &= (rX_t + \pi_t(\mu - r))dt + \sigma \pi_t dW_t^1. && (3.9) \end{aligned}$$

Assumption 4. *(Default event reducing the trading opportunities of the investor)*

> If the default event occurs before T, the investor has to liquidate holdings in the stock and deposit money in the bank account. For simplicity, we assume he receives full pre-default market value on his stock holdings on liquidation.

One might extend to consider some jumps or loss in the stock price at the default; however this would be more complex to solve because one cannot reduce the dimension of the optimization problem at hand. Also, including some jumps would violate the independence assumption of doubly-stochastic Poisson process. Therefore, given that $\tau < T$, for $\tau \leq t \leq T$, we have $X_t = X_\tau e^{r(t-\tau)}$.

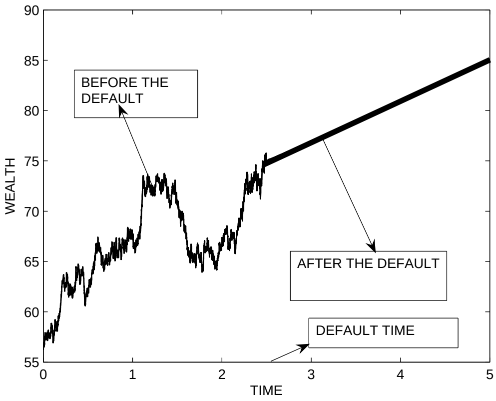


Figure 3.1: Evolution of Wealth before and after the default event

Now consider first the Merton problem i.e. the optimal investment problem up to time T of the investor who does not hold the defaultable bond. At time zero, the maximum expected utility of discount payoff takes the form:

$$\sup_{\pi\in\mathcal{A}_{0,T}} E\left(U(e^{-rT}X_T)1_{\{\tau>T\}} + U(e^{-r\tau}X_\tau)1_{\{\tau\le T\}}\right) \tag{3.10}$$

To simplify the discounted variable, let us apply the Itô Lemma to $e^{-rt}X_t$

$$d(e^{-rt}X_t) = e^{-rt}(\pi_t(\mu-r)dt + \sigma\pi_t dW_t^1).$$

Transforming the discount variable $X_t \to e^{-rt}X_t$, the excess return $\mu \to \mu - r$ and $\pi_t \to \pi_t e^{-rt}$, we get

$$dX_t = \pi_t\left(\mu dt + \sigma dW_t^1\right). \tag{3.11}$$

Next, consider the stochastic control problem initiated at time $t \le T$, and define the default time as

$$\tau_t = \inf\left\{s \ge t : \int_t^s \lambda_u du \ge E_1\right\}.$$

Using the exponential utility of discounted wealth, the Merton value function becomes

$$M(t,x,y) = \sup_{\pi\in\mathcal{A}_{t,T}} E_t\left(-e^{-\gamma X_T}1_{\{\tau_t>T\}} + (-e^{-\gamma X_{\tau_t}})1_{\{\tau_t\le T\}}\right), \tag{3.12}$$

where $E_t(\cdot) = E(\cdot|X_t = x, Y_t = y)$.

We are now interested to the same problem of the investor who owns the defaultable bond of the firm, which pays 1 currency unit on date T if the firm has survived till then. The bond's holder value function is:

$$H^b(t,x,y) = \sup_{\pi\in\mathcal{A}_{t,T}} E_t\left(-e^{-\gamma(X_T+C)}1_{\{\tau_t>T\}} + (-e^{-\gamma X_{\tau_t}})1_{\{\tau_t\leq T\}}\right), \quad (3.13)$$

where $C = e^{-rT}$. In the same way, the bond's seller value function is :

$$H^s(t,x,y) = \sup_{\pi\in\mathcal{A}_{t,T}} E_t\left(-e^{-\gamma(X_T-C)}1_{\{\tau_t>T\}} + (-e^{-\gamma X_{\tau_t}})1_{\{\tau_t\leq T\}}\right). \quad (3.14)$$

Remark 1.

The value functions M, H^b, H^s are concave, increasing in x and uniformly bounded in y. See Sircar and Zariphopoulou (2007).

In the next section, we derive the Hamilton-Jacobi-Bellman equations satisfied by these value functions.

3.2.2 The Hamilton-Jacobi-Bellman Equations

Proposition 1. *Consider the Merton value function*

$$M(t,x,y) = sup_{\pi\in\mathcal{A}_{t,T}} E_t\left(-e^{-\gamma X_T}1_{\{\tau_t>T\}} + (-e^{-\gamma X_{\tau_t}})1_{\{\tau_t\leq T\}}\right),$$

with

$$M(T,x,y) = -e^{-\gamma x}$$

subject to the wealth and state variable contraints

$$dX_t = \pi_t\left(\mu dt + \sigma dW_t^1\right)$$

and

$$dY_t = b(Y_t)dt + a(Y_t)\left(\rho dW_t^1 + \rho' dW_t^2\right)$$

with the coefficients $a(.)$ and $b(.)$ verifying the assumption 2.
Then the Merton value function M is the unique viscosity solution in the class of functions that are concave and increasing in x, and uniformly bounded in y of the Hamilton-Jacobi-Bellman (HJB) equation

$$sup_\pi\left\{\frac{E_t\left[dM(t,x,y)\right]}{dt}\right\} = 0 \quad (3.15)$$

Proof. Assuming that an optimal amount exists, we derive from

$$\sup_{\pi\in\mathcal{A}_{t,T}} E_t\left(-e^{-\gamma X_T}1_{\{\tau_t>T\}}+(-e^{-\gamma X_{\tau_t}})1_{\{\tau_t\leq T\}}\right)-M(t,x,y)=0$$

that for some small $h>0$,

$$\sup_{\pi\in\mathcal{A}_{t,T}} E_t\left[E_{t+h}\left(-e^{-\gamma X_T}1_{\{\tau_t>T\}}+(-e^{-\gamma X_{\tau_t}})1_{\{\tau_t\leq T\}}\right)\right]-M(t,x,y)=0.$$

The expression $E_{t+h}\left(-e^{-\gamma X_T}1_{\{\tau_t>T\}}+(-e^{-\gamma X_{\tau_t}})1_{\{\tau_t\leq T\}}\right)$ is nothing else than the expected utility for an investor starting with wealth X_{t+h} and state variable Y_{t+h} at time $t+h$. Therefore, for any control π_s with $s\geq t+h$,

$$E_{t+h}\left(-e^{-\gamma X_T}1_{\{\tau_t>T\}}+(-e^{-\gamma X_{\tau_t}})1_{\{\tau_t\leq T\}}\right)\leq M(t+h,X_{t+h},Y_{t+h}),$$

and equality holds for the optimal π^*. Hence,

$$\sup_{\pi\in\mathcal{A}_{t,t+h}} E_t\left[M\left(t+h,X_{t+h},Y_{t+h}\right)\right]-M(t,x,y)=0.$$

Assuming an optimal behavior from $t+h$ on T, the optimal π^* has only to be determined until $t+h$, and not on the whole time horizon. Dividing by h and applying the limit $h\to 0$, the above equation becomes

$$\sup_{\pi}\left\{\lim_{h\to 0} E_t\frac{1}{h}\left[M\left(t+h,X_{t+h},Y_{t+h}\right)-M(t,x,y)\right]\right\}=0$$

Using the theorem of bounded convergence, this equation becomes

$$\sup_{\pi}\left\{E_t\lim_{h\to 0}\frac{1}{h}\left[M\left(t+h,X_{t+h},Y_{t+h}\right)-M(t,x,y)\right]\right\}=0,$$

but $\lim_{h\to 0}\frac{1}{h}\left[M\left(t+h,X_{t+h},Y_{t+h}\right)-M(t,x,y)\right]=\frac{d}{dt}M(t,x,y)$ and the value function M verifies

$$\sup_{\pi\in\mathcal{A}}\left\{\frac{E_t\left[dM(t,x,y)\right]}{dt}\right\}=0$$

(see Sennewald (2007) or Malliaris and Brock (1982) for more details).
The value function M is bounded, see appendix B; so M is a viscosity solution of the HJB equation. Using the comparison principle see (Evans (1998)), we deduce that the value function M is the unique viscosity solution of the HJB. □

Let us now derive the explicit form of the HJB equations satisfied by the Merton function M. We have

$$E_t\left(dM(t,x,y)\right) = \\ \left(M_t + \pi\mu M_x + b(y)M_y + \frac{1}{2}\sigma^2\pi^2 M_{xx} + \frac{1}{2}a(y)^2 M_{yy} + \rho\sigma\pi a(y)M_{xy}\right)dt+ \\ \lambda(y)\left(-e^{-\gamma x} - M\right)dt.$$

For more details about Itô's formula see appendix A. Dividing by dt and applying the supremum, we obtain

$$M_t + \mathcal{L}_y M + \sup_\pi\left(\frac{1}{2}\sigma^2\pi^2 M_{xx} + \pi\left(\rho\sigma a(y)M_{xy} + \mu M_x\right)\right) \\ +\lambda(y)\left(-e^{-\gamma x} - M\right) = 0 \qquad (3.16)$$

with $M(T,x,y) = -e^{-\gamma x}$ and $\mathcal{L}_y = \frac{1}{2}a(y)^2\frac{\partial^2}{\partial y^2} + b(y)\frac{\partial}{\partial y}$

Similarly, the value function H^b of the investor, who owns the defaultable bond, satisfies the HJB equation:

$$H^b_t + \mathcal{L}_y H^b + \sup_\pi\left(\frac{1}{2}\sigma^2\pi^2 H^b_{xx} + \pi\left(\rho\sigma a(y)H^b_{xy} + \mu H^b_x\right)\right) \\ +\lambda(y)\left(-e^{-\gamma x} - H^b\right) = 0 \qquad (3.17)$$

with $H^b(T,x,y) = -e^{-\gamma(x+C)}$.

The value function H^s of the seller of the bond also satisfies :

$$H^s_t + \mathcal{L}_y H^s + \sup_\pi\left(\frac{1}{2}\sigma^2\pi^2 H^s_{xx} + \pi\left(\rho\sigma a(y)H^s_{xy} + \mu H^s_x\right)\right) \\ +\lambda(y)\left(-e^{-\gamma x} - H^s\right) = 0, \qquad (3.18)$$

with $H^s(T,x,y) = -e^{-\gamma(x-C)}$.

The buyer's indifference price $p^b_0(T)$ and the seller's indifference price $p^s_0(T)$ (at time 0) of the defaultable bond with maturity T are defined respectively by

$$M(0,x,y) = H^b(0,x-p^b_0,y) \qquad (3.19)$$

and

$$M(0,x,y) = H^s(0,x+p^s_0,y). \qquad (3.20)$$

Remark 2.

It is well-known see Delbaen et al. (2002), that the indifference price under exponential utility does not depend on the investor's initial wealth x, this is an attractive feature of using this utility function.

In the next section we factorize the value functions M, H^b, H^s, that leads to some reaction-diffusion equations.

3.2.3 Reaction-Diffusion Equations

In order to simplify the above HJB equations let us introduce the distortion scaling:

$$M(t,x,y) = -e^{-\gamma x}u(t,y)^{\frac{1}{1-\rho^2}},$$

$$H^b(t,x,y) = -e^{-\gamma(x+C)}w^b(t,y)^{\frac{1}{1-\rho^2}},$$

and

$$H^s(T,x,y) = -e^{-\gamma(x-C)}w^s(t,y)^{\frac{1}{1-\rho^2}},$$

with $u, w^b, w^s : [0,T] \times \mathbb{R} \to \mathbb{R}^+$, see Sircar and Zariphopoulou (2005). Plugging these expressions in the HJB equations, we get the following reaction-diffusion equations:

$$u_t + \mathcal{K}_y u - \left(1-\rho^2\right)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right)u + \left(1-\rho^2\right)\lambda(y)u^{-\theta} = 0, \quad (3.21)$$

with the final condition $u\,(T,y) = 1$.

$$w^b_t + \mathcal{K}_y w^b - \left(1-\rho^2\right)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right)w^b + \left(1-\rho^2\right)e^{\gamma C}\lambda(y)(w^b)^{-\theta} = 0, \quad (3.22)$$

with the final condition $w^b\,(T,y) = 1$.

$$w^s_t + \mathcal{K}_y w^s - \left(1-\rho^2\right)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right)w^s + \left(1-\rho^2\right)e^{-\gamma C}\lambda(y)(w^s)^{-\theta} = 0, \quad (3.23)$$

with the final condition $w^s(T,y) = 1$
where

$$\theta = \frac{\rho^2}{1-\rho^2},$$

and

$$\mathcal{K}_y = \mathcal{L}_y - \frac{\rho\mu}{\sigma} a(y) \frac{\partial}{\partial y}.$$

With the simplified form of the value functions, the buyer indifference's price is given by,

$$\begin{aligned} M(0,x,y) &= H^b(0, x - p_0^b, y) \\ e^{-\gamma x} u(0,y)^{\frac{1}{1-\rho^2}} &= e^{-\gamma(x-p_0^b+C)} w^b(0,y)^{\frac{1}{1-\rho^2}}, \end{aligned}$$

this yields to

$$p_0^b(T) = e^{-rT} - \frac{1}{\gamma(1-\rho^2)} \log\left(\frac{w^b(0,y)}{u(0,y)}\right). \tag{3.24}$$

It is shown in Sircar and Zariphopoulou (2007) that $w^s(t,y) < u(t,y) \leq w^b(t,y)$ for $(t,y) \in [0,T] \times \mathbb{R}$, so $p_0^b(T) \leq e^{-rT}$. The yield spread of the bond is defined by

$$S_0^b(T) = -\frac{1}{T} \log(p_0^b(T)) - r, \tag{3.25}$$

which is non-negative for all T.
Similarly, the seller indifference's price is

$$p_0^s(T) = e^{-rT} - \frac{1}{\gamma(1-\rho^2)} \log\left(\frac{u(0,y)}{w^s(0,y)}\right), \tag{3.26}$$

and the seller's yield spread is non-negative for all T.
The buyer and seller indifference prices are characterized as the solution of the nonlinear reaction-diffusion equations related to the functions u, w^b and w^s. In the next section, we present the corresponding dual problem which is an alternative approach to evaluate the indifference price. The following result of the dual problem would be relevant for the limit of our model when γ goes to zero.

3.2.4 Dual Problem: Relative entropy Minimization

We recall the primal optimal investment problem for the buyer at time zero,

$$\mathcal{P} = \sup_{\pi \in A} E\left(-e^{-\gamma(X_T+C)}1_{\{\tau>T\}} + (-e^{-\gamma X_\tau})1_{\{\tau\leq T\}}\right).$$

The dual of the maximization of expected exponential utility of discounted wealth over trading strategies is the problem of minimizing the bond's payoff over a space of measures penalized by the relative entropy of the measure. Under the framework that the underlying processes are locally bounded semimartingales and the utility function is exponential, Delbaen et al. (2002) proved the duality relation

$$\mathcal{P} = -e^{-\gamma x - \gamma \mathcal{D}}$$

where x is the initial wealth and $\mathcal{D}$ is the value of the dual optimization problem

$$\mathcal{D} = \inf_{Q \in \mathcal{P}_f}\left(\mathbb{E}^Q\left\{C\right\} + \frac{1}{\gamma}\mathcal{H}(Q|P)\right) \tag{3.27}$$

where $\mathcal{H}(Q|P)$ is the relative entropy between Q and P, namely,

$$\mathcal{H}(Q|P) = \begin{cases} \mathbb{E}\left\{\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)\right\}, Q << P, \\ \infty, \qquad \text{otherwise} \end{cases} \tag{3.28}$$

In (3.27), $\mathcal{P}_f$ denotes the set of absolutely continuous local martingale measures with finite relative entropy with respect to P.

Assumption 5.

> There is an equivalent local martingale measure with finite relative entropy. Thus,
>
> $$\mathcal{P}_f \cap \mathcal{M} \neq \phi \tag{3.29}$$
>
> where $\mathcal{M}$ is the set of equivalent local martingale measures.

Using the exponential utility for discounted wealth, it is easy to show that

$$p_0^b(T) = \frac{1}{\gamma}\log\left(\frac{M(0,x,y)}{H^b(0,x,y)}\right) \tag{3.30}$$

and substituting the specific expressions of the duals for the buyer and the Merton problems, we can write

$$p_0^b(T) = \mathcal{D} - \frac{1}{\gamma}\inf_{Q\in\mathcal{P}_f}\mathcal{H}(Q|P) \tag{3.31}$$

The entropy term in (3.31) can be combined into one entropy term with a different prior measure, the minimal entropy martingale measure, which is the measure minimizing the relative entropy in $\mathcal{P}_f$. Let

$$Q^0 = \arg\min_{Q\in\mathcal{P}_f}\mathcal{H}(Q|P) \tag{3.32}$$

Theorem 1. *(Frittelli (2000) and Delbaen et al. (2002)) Under assumption (5), Q^0 exists, is unique, is in $\mathcal{P}_f \cap \mathcal{M}$ and its density has the form*

$$\frac{dQ^0}{dP} = \alpha_0 e^{-\gamma X_T^0} \tag{3.33}$$

where X_T^0 is the optimal terminal wealth associated with the solution of the Merton optimization problem (3.12) and $\log \alpha_0 = \mathcal{H}(Q^0|P) < \infty$

Under the assumption that $\frac{dQ^0}{dP} \in L^2(P)$, we can apply Proposition 2.2 of Ilhan et al. (2004) and the bid price is

$$p_0^b(T) = \inf_{Q\in\mathbb{P}_f(Q^0)}\left(\mathbb{E}^Q\left\{C\right\} + \frac{1}{\gamma}\mathcal{H}(Q|Q^0)\right). \tag{3.34}$$

Next, we define the Radon-Nikodym derivative between Q^0 and P for our model. This approach is taken in Ilhan et al. (2004) and Bielecki et al. (2004). The so-called minimal martingale measure, P^0, is defined by the following Girsanov transformation.

$$\frac{dP^0}{dP} = \exp(-\frac{\mu}{\sigma}W_T^1 - \frac{\mu^2}{2\sigma^2}T)$$

The dynamic of (S, Y) under P^0 are:

$$\left\{\begin{array}{lcl} dS_t &=& rS_t dt + \sigma S_t dW_t^{P_0(1)} \\ dY_t &=& \left(b(Y_t) - \frac{\rho\mu}{\sigma}a(Y_t)\right)dt + a(Y_t)\left(\rho dW_t^{P_0(1)} + \rho' dW_t^{P_0(2)}\right), \end{array}\right. \tag{3.35}$$

where

$W_t^{P_0(1)} = W_t^1 + \frac{\mu}{\sigma} t$ and $W_t^{P_0(2)} = W_t^2$

P^0 has finite relative entropy and is equivalent to P. Therefore, $Q^0 \in \mathcal{P}_f \cap \mathcal{M}$ and is unique. The parameters μ and σ are independent of Y and by Ilhan et al. (2004), the minimal entropy martingale measure coincides with the minimal martingale measure.

3.2.5 Comparison with the classical reduced model

In this section, we compare the indifference price for both buyer and seller with the classical reduced price. In fact, the latter is just the limit case of our model. Let us recall the classical results of the intensity model. See Bielecki and Rutkowski (2002).

Let Q be a risk neutral probability and $\lambda(Y_t^Q)$ the corresponding risk-neutral intensity. The price at time $t = 0$ of the defaultable bond denoted $p_0^{ar}(T)$ is thus equal to the expectation under Q of the discount pay-off,

$$p_0^{ar}(T) = E^Q(e^{-rT} 1_{\{\tau > T\}}) = e^{-rT} E^Q(e^{-\int_0^T \lambda(Y_s^Q) ds})$$

Proposition 2. *Under the assumption that the market price of risk for the non-traded asset Y under Q is equal to zero,*

$$\lim_{\gamma \to 0} p_0^b(T) = \lim_{\gamma \to 0} p_0^s(T) = p_0^{ar}(T) \ \ \textit{if } \mu = 0 \tag{3.36}$$

Proof. When the risk aversion parameter γ tends to zero, Becherer (2001) proved the indifference price goes to the arbitrage free pricing under the minimal entropy martingale measure:

$$\begin{aligned} \lim_{\gamma \to 0} p_0^b(T) = \lim_{\gamma \to 0} p_0^s(T) &= E^{Q^0}(e^{-rT} 1_{\{\tau > T\}}) \\ &= E^{P^0}(e^{-rT} 1_{\{\tau > T\}}) \end{aligned}$$

If the excess return $\mu = 0$, the minimal martingale measure P^0 coincides with the the investor's subjective measure P and the result follows.

□

3.2.6 Solving the PDE

With the change of time $t \to T - t$, the equation (3.21) reduces to:

$$\begin{cases} -u_t + F_1(y)u_{yy} + F_2(y)u_y + F_3(y)u + F_4(y)u^{-\theta} = 0 \\ u(0,y) = 1 \end{cases} \tag{3.37}$$

with :

$$F_1(y) = \frac{1}{2}a(y)^2$$

$$F_2(y) = b(y) - \frac{\rho\mu}{\sigma}a(y)$$

$$F_3(y) = -\left(1-\rho^2\right)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y)\right)$$

$$F_4(y) = \left(1-\rho^2\right)\lambda(y)$$

Finite Differencing of PDE

Our aim is to solve (3.21)-(3.23) by implicit finite difference schemes. To this end, we divide the interval $[y_{min}, y_{max}]$ into the sub intervals

$$y_{min} = y_0 < y_1 ... < y_{N+1} = y_{max}$$

and we assume for convenience for the mesh-points $\{y_j\}_{j=0}^{N+1}$ are equidistant that is,

$$y_j = y_{j-1} + \Delta y,\ j = 1, ..., N+1\ (\Delta y = \frac{y_{max} - y_{min}}{N+1}).$$

Furthermore, we divide the interval $[0, T]$ into $M + 1$ equal sub-intervals

$$0 = t_0 < t_1 ... < t_{M+1} = T$$

where

$$t_i = t_{i-1} + \Delta t,\ i = 1, ..., M+1\ (\Delta t = \frac{T}{M+1})$$

Let $u(t_{i+1}, y_j) = u_j^{i+1}$, the essence of the finite difference approach lies in replacing the derivatives in (3.21) by divided differences at the mesh-points

(t_i, y_j). The following equations are used to approximate the first and second order derivatives:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{u_j^{i+1} - u_j^i}{\Delta t} + O(\Delta t) \\
\frac{\partial u}{\partial y} &= \frac{u_{j+1}^{i+1} - u_{j-1}^{i+1}}{2\Delta y} + O(\Delta y^2) \\
\frac{\partial^2 u}{\partial y^2} &= \frac{u_{j+1}^{i+1} - 2u_j^{i+1} + u_{j-1}^{i+1}}{(\Delta y)^2} + O(\Delta y^2)
\end{aligned}$$

To avoid the difficulty of the nonlinear term, the following first order approximation is used

$$\begin{aligned}
& (u_j^{i+1})^{-\theta} \\
= \; & (u_j^i)^{-\theta} + \Delta t \left[-\theta (u_j^i)^{-\theta-1} \left(\frac{u_j^{i+1} - u_j^i}{\Delta t} + O(\Delta t) \right) \right] + O(\Delta t^2) \\
= \; & (u_j^i)^{-\theta} - \Delta t \theta (u_j^i)^{-\theta-1} \left(\frac{u_j^{i+1} - u_j^i}{\Delta t} + O(\Delta t) \right) + O(\Delta t^2) \\
= \; & (u_j^i)^{-\theta} - \theta (u_j^i)^{-\theta-1} \left(u_j^{i+1} - u_j^i \right) + O(\Delta t^2) \\
= \; & (1+\theta)(u_j^i)^{-\theta} - \theta (u_j^{i+1})(u_j^i)^{-\theta-1} + O(\Delta t^2)
\end{aligned}$$

Applying the above approximations to (3.37), we get

$$\alpha_j u_{j-1}^{i+1} + \beta_j u_j^{i+1} + \nu_j u_{j+1}^{i+1} = -u_j^i - \psi_j (u_j^i)^{-\theta},$$

with

$$\begin{aligned}
\alpha_j &= \frac{\Delta t F_{1,j}}{\Delta y^2} - \frac{\Delta t F_{2,j}}{2\Delta y}, \\
\beta_j &= -1 - \frac{2\Delta t F_{1,j}}{\Delta y^2} + \Delta t F_{3,j} - \theta \Delta t F_{4,j} (u_j^i)^{-\theta-1}, \\
\nu_j &= \frac{\Delta t F_{1,j}}{\Delta y^2} + \frac{\Delta t F_{2,j}}{2\Delta y}, \\
\psi_j &= (1+\theta)\Delta t F_{4,j}.
\end{aligned}$$

Here $F_{k,j} = F_k(y_j)$ is described as follows:

$$\begin{aligned}
F_{1,j} &= \frac{1}{2}a(y_j)^2 \\
F_{2,j} &= b(y_j) - \frac{\rho\mu}{\sigma}a(y_j) \\
F_{3,j} &= -\left(1-\rho^2\right)\left(\frac{\mu^2}{2\sigma^2} + \lambda(y_j)\right) \\
F_{4,j} &= \left(1-\rho^2\right)\lambda(y_j)
\end{aligned}$$

Thus we obtain the linear system $Ru^{i+1} = d^i$, $0 \le i \le M$ where $R = [r_{j,k}]_{(N+2)\times(N+2)}$ is a tridiagonal matrix structured as follows

$$r_{j,k} = \begin{cases} \beta_j \text{ if } & j = k \\ \alpha_j \text{ if } & j = k+1 \\ \nu_j \text{ if } & j = k-1 \\ 0 \text{ else} & \end{cases}$$

$d^i = (d^i_j)_{1\le j\le N+2}$ is a vector with $d^i_j = -u^i_j - \psi_j(u^i_j)^{-\theta}$. The calculations are similar for w^b and w^c, the only changes are $F_4(y) = e^{\gamma c}\left(1-\rho^2\right)\lambda(y)$ for w^b and $F_4(y) = e^{-\gamma c}\left(1-\rho^2\right)\lambda(y)$ for w^c.

Choice of parameters

The choice of the appropriate level of risk aversion γ (> 0) is important in the indifference pricing within an exponential utility function. If a high value is chosen, the investor will be too risk averse and therefore the results of the pricing will converge to the super-replication's prices. On the other hand if the value of risk aversion is too small, the decisions of the investor will not differ appreciably from the decisions of a risk neutral investor. We use in our study different values of risk aversion coefficients $\gamma \in \{0.0001, 0.01, 0.2, 0.7\}$. We choose as Sircar and Zariphopoulou (2007) $\mu = 0.09$, $\sigma = 0.15$, $r = 0.03$. We work with negative correlation coefficients $\rho \in \{-0.90, -0.5, -0.20, -0.05\}$ in order to specify that the intensity tends to rise when the stock price falls. We define $\lambda(Y_t) = \lambda_t$ when λ follows Cox-Ingersoll-Ross dynamics defined below.

Cox-Ingersoll-Ross intensities and Boundaries In this model, the default intensity follows the stochastic differential equation

$$d\lambda_t = \alpha(\bar{\lambda} - \lambda_t)dt + \phi\sqrt{\lambda_t}dW_t^3 \tag{3.38}$$

where α, $\bar{\lambda}$ and ϕ are positive constants and $dW_t^3 = \rho dW_t^1 + \rho' dW_t^2$. The Cox-Ingersoll-Ross (CIR) model is one of the most popular and commonly used stochastic intensity model in both academic researches and practical applications. When we impose the condition $2\alpha\bar{\lambda} > \phi^2$ then the intensity λ is always positive, otherwise we can only guarantee that it is non-negative (with a positive probability to terminate to zero). In fact, if the default intensity approaches zero then the volatility $\phi\sqrt{\lambda_t}$ approaches zero cancelling the effect of the randomness, so the intensity rate remains always non-negative.

From (3.38), we fix the smallest value $\lambda_{min} = 0$. The question which arises is how to determine the maximal value λ_{max}. As shown by Feller (1954), under the physical probability measure, the distribution of $\lambda_t = \lambda(t)$, conditioned to the observed value λ_0 is the non-central χ^2 type, i.e.

$$P(\lambda_t|\lambda_0) \text{ follows } \chi^2(2c\lambda_t; 2q+2, 2u)$$

with

$$c = \frac{2\alpha}{\phi^2(1 - e^{-\alpha t})}$$
$$u = c\lambda_0 e^{-\alpha t}$$
$$q = \frac{2\alpha\bar{\lambda}}{\phi^2} - 1$$

For a given time to maturity t, the maximal value λ_{max} of λ_t is such that $P(\lambda_t \leq \lambda_{max}) = 1 - \kappa$ with $\kappa \approx 0$. Then, $\lambda_{max} = \dfrac{F^{-1}(1-\kappa)}{2c}$ with F is the cumulative distribution function of $\chi^2(2c\lambda_t; 2q+2, 2u)$.
The maximal value λ_{max} is sensitive to the changes of the observed value λ_0 and the time to maturity t. The figures 3.2 and 3.3 show this sensitivity with $\lambda_0 \in [0, 2]$ and the time to maturity $t \in [0, 50]$. The maximal value λ_{max} increases with the observed value λ_0 and with the time to maturity t. It means that when the observed value λ_0 increases, λ_{max} is sufficiently large

to truncate the range of the default intensity λ_t to $[\lambda_{min}, \lambda_{max}]$. For $t \neq 0$ even if $\lambda_0 = 0$, $\lambda_{max} \neq 0$ and increases with the maturities t as it is shown in the figures 3.2 and 3.3. But when $t \to 0$ and $\lambda_0 = 0$, $\lambda_{max} \to 0$ as it is shown in the figure 3.3.

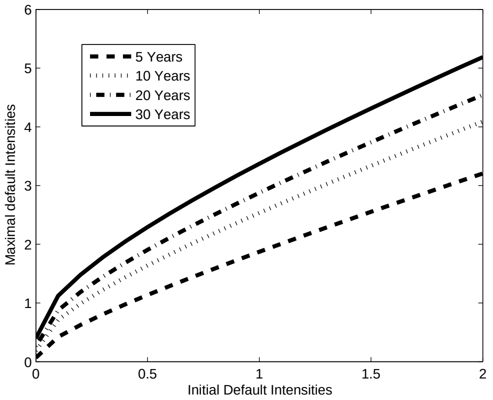


Figure 3.2: Sensitivity of maximal value λ_{max} to initial value λ_0

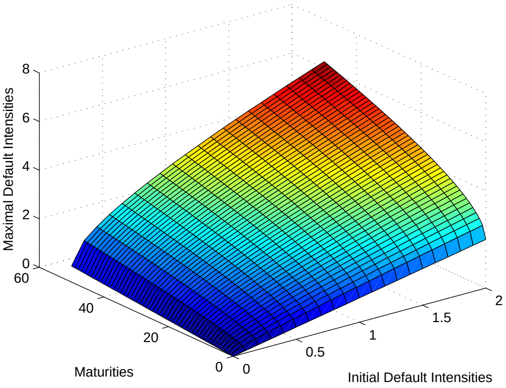


Figure 3.3: Sensitivity of λ_{max} to initial value λ_0 and to time to maturity t

When $\lambda_t = \lambda_{max}$, then default is highly likely, so the investor is likely to follow the sub-optimal policy in investing exclusively in the default-free bank account. Indeed, as $\lambda_t = \lambda_{max}$, we have $\tau_t \to t$ and $\pi_t = 0$ almost sure. For simplicity, we impose that $\pi_t = 0$ and define the following Dirichlet boundary condition,

$$\begin{aligned} M(t,x,\lambda_t = \lambda_{max}) &= E\left(-e^{-\gamma x} 1_{\{\tau_t > T\}} + (-e^{-\gamma x}) 1_{\{\tau_t \leq T\}} | X_t = x, \lambda_t = \lambda_{max}\right) \\ &= -e^{-\gamma x} \end{aligned}$$

and therefore $u(t, \lambda_{max}) = 1$

$$\begin{aligned} H^b(t,x,\lambda_t = \lambda_{max}) &= E\left(-e^{-\gamma(x+c)} 1_{\{\tau_t > T\}} + (-e^{-\gamma x}) 1_{\{\tau_t \leq T\}} | X_t = x, \lambda_t = \lambda_{max}\right) \\ &= -e^{-\gamma x} + (e^{-\gamma x} - e^{-\gamma(x+c)}) P(\tau_t > T | \lambda_t = \lambda_{max}) \\ P(\tau_t > T | \lambda_t = \lambda_{max}) &= 1_{\{\tau > t\}} E(e^{-\int_t^T \lambda_s ds} | \lambda_t = \lambda_{max}) \\ &= 1_{\{\tau > t\}} F(t, \lambda_t = \lambda_{max}, T) \end{aligned}$$

The term $1_{\{\tau > t\}} F(t, \lambda_t, T)$ is the probability of default not occuring in T years conditioned on all the information available at time $t < T$.
Appendix C shows that

$$F(0, \lambda, T) = A(T) e^{-B(T)\lambda} \tag{3.39}$$

with

$$A(T) = \left\{ \frac{2\xi e^{(\alpha+\xi)(\frac{T}{2})}}{2\xi + (\alpha+\xi)(e^{\xi T} - 1)} \right\}^{\frac{2\alpha\bar{\lambda}}{\phi^2}} \tag{3.40}$$

$$B(T) = \frac{2(e^{\xi T} - 1)}{2\xi + (\alpha+\xi)(e^{\xi T} - 1)} \tag{3.41}$$

and

$$\xi = \sqrt{\alpha^2 + 2\phi^2} \tag{3.42}$$

Therefore

$$P(\tau_t > T | \lambda_t = \lambda_{max}) = 1_{\{\tau > t\}} A(T-t) e^{-B(T-t)\lambda_{max}}$$

Then for the buyer,

$$w^b(t,\lambda_{max}) = \left(e^{\gamma c} - (e^{\gamma c}-1)1_{\{\tau>t\}}A(T-t)e^{-B(T-t)\lambda_{max}}\right)^{1-\rho^2}$$

Similarly for the seller,

$$w^s(t,\lambda_{max}) = \left(e^{-\gamma c} - (e^{-\gamma c}-1)1_{\{\tau>t\}}A(T-t)e^{-B(T-t)\lambda_{max}}\right)^{1-\rho^2}$$

When $\lambda_t = 0$, (3.21) gives the following equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \alpha\bar{\lambda}\dfrac{\partial u}{\partial \lambda} - (1-\rho^2)\dfrac{\mu^2}{2\sigma^2}u &= 0 \\ u(T,0) &= 1. \end{cases} \tag{3.43}$$

The equation at hand is a linear Hamilton-Jacobi equation with the hamiltonian is a function of $\frac{\partial u}{\partial \lambda}$ and u. The solution of (3.43) is

$$u(t,0) = e^{-(1-\rho^2)\frac{\mu^2}{2\sigma^2}(T-t)}$$

Similarly, for the buyer and seller

$$w^b(t,0) = w^s(t,0) = e^{-(1-\rho^2)\frac{\mu^2}{2\sigma^2}(T-t)}$$

For the implementation of the model, we use the values for CIR-intensity estimated by Longstaff et al. (2003): $\alpha = 0.2060$, $\bar{\lambda} = 0.0646$, $\phi = 0.0303$ and by Denault et al. (2009): $\alpha = 0.034$, $\bar{\lambda} = 0.00043$, $\phi = 0.014$.

Numerical Results and Comments In this paragraph, we present the results of the indifference pricing of the defaultable bond solved by the finite difference method. The yield-spread curve which reflects the relation between yield spreads and the time to maturity is plotted and analyzed. The yield-spread is the difference between the yield on the defaultable bond and the yield on the risk free bond (in our situation r). The yield spread is the indication of the risk premium required for investing in the risky bond. When it is high, this means that the investors require a higher risk premium, so the bond is more risky. Conversely when it is small, this means that the bond is riskless.

In the figures 3.4 and 3.5 using the estimates of Longstaff et al. (2003), we present the spread curves for the buyer and the seller respectively for various risk aversion coefficients. When $\lambda_0 \neq 0$ the short term limit of the yield spread for the buyer and seller is nonzero, reflecting the presence of non-predictable defaults. Even if the default intensity and the risk aversion parameter are small, the investor requires at short term a risk premium. We can remark for the maturity $T = 0$, $\lambda_0 = 0.02$ and $\gamma = 0.01$, the bid and ask spreads are equal to 0.045. When the default intensity increases, the bid and ask spread curves are in fact high in order to compensate the additional risk; but the shape of the curves changes depending of the values of the default intensity. An experiment on values of default intensity between 0 and 1 shows the following shapes for the buyer's spread curve in the figure 3.4,

- $0 < \lambda_0 \leq 0.06$, the spread curves are upward sloping reflecting a low default risk at short term, whereas the forecast of the credit quality of the firm over longer maturities is less certain. This shape is illustrated for $\lambda_0 = 0.02$.

- $0.06 < \lambda_0 \leq 0.15$, the spread curves are S shaped that is the buyer expects that the credit quality of the firm will increase until a certain maturity and after that it will be low for longer maturity. This particular shape is shown for $\lambda = 0.12$.

- $0.15 < \lambda_0 \leq 1$, the spread curves are downward sloping reflecting a high default probability of the firm at short term; once the firm has survived a certain period of time without a default, it faces a lower default probability in the long term. The shape is shown for $\lambda_0 = 0.5$.

For the seller's spread curves, we observe two kinds of shape in the figure 3.5,

- $0 < \lambda_0 \leq 0.25$, the curve is upward sloping and is shown for $\lambda_0 = 0.02$.

- $0.25 < \lambda_0 \leq 1$, the curves are humped meaning that the seller expects to offer a high risk premium until a certain maturity. Once the firm doesn't default until this reference maturity, the ask yield-spread decreases. We show the humped curves for $\lambda_0 = 0.50$ and $\lambda_0 = 0.90$.

For $\lambda_0 = 0$, both buyer and seller's spreads are closed to zero whatever the maturity since there is no default probability. The impact of the risk aversion γ on the yield-spread is also analyzed. For the buyer, we found a positive relationship between the spread and the risk aversion coefficient in contrast to the negative relationship for the seller. It means that if the buyer is more risk averse, he will require a high risk premium for investing in the defaultable bond. In contrast, a more risk averse seller will offer a low risk premium for selling the defaultable bond. This result is also a consequence of the monotonicity of the indifference price as a function of risk aversion parameter. For more details see Ilhan et al. (2004).

The impact of the correlation between the default intensity and the stock price on the spread curves is analyzed. As in Roncalli (2009), we test by experimentation the impact of $\rho \in \{0, -0.20, -0.5, -0.9\}$ on the spread curves. The figures 3.6 and 3.7 show the analysis obtained with the parameters of Longstaff et al. (2003) and the figures 3.8 and 3.9 show the analysis obtained with the parameters of Denault et al. (2009). In figures 3.6 and 3.7, when the maturity T goes to zero, the correlation between the default intensity and the stock price has no impact on the yield-spread for both buyer and seller. But for longer maturities, ρ (in absolute value) increases slightly the yield-spreads for the investors. In figures 3.8 and 3.9, there is also no impact of the correlation coefficient on the bid and ask spread for shorter maturities, but for longer maturities, ρ (in absolute value) decreases slightly the yield-spreads for the investors. Then, we can conclude that the correlation coefficient between the default intensity and the stock price has a little impact on the credit spread. It is important to notify that we present the results of the analysis with a high maturity (50 years) to show that the impact of the correlation is not significant even if the maturity is high. When $|\rho| = 1$, the intensity and the stock price are perfectly linearly dependent and the unique source of uncertainty is the stock price which is a traded asset. This leads to the complete market. In the figures 3.10 and 3.11, we show the convergence of the indifference spread to the classical spread for $\mu = 0$ and $\gamma = 0.0001$.

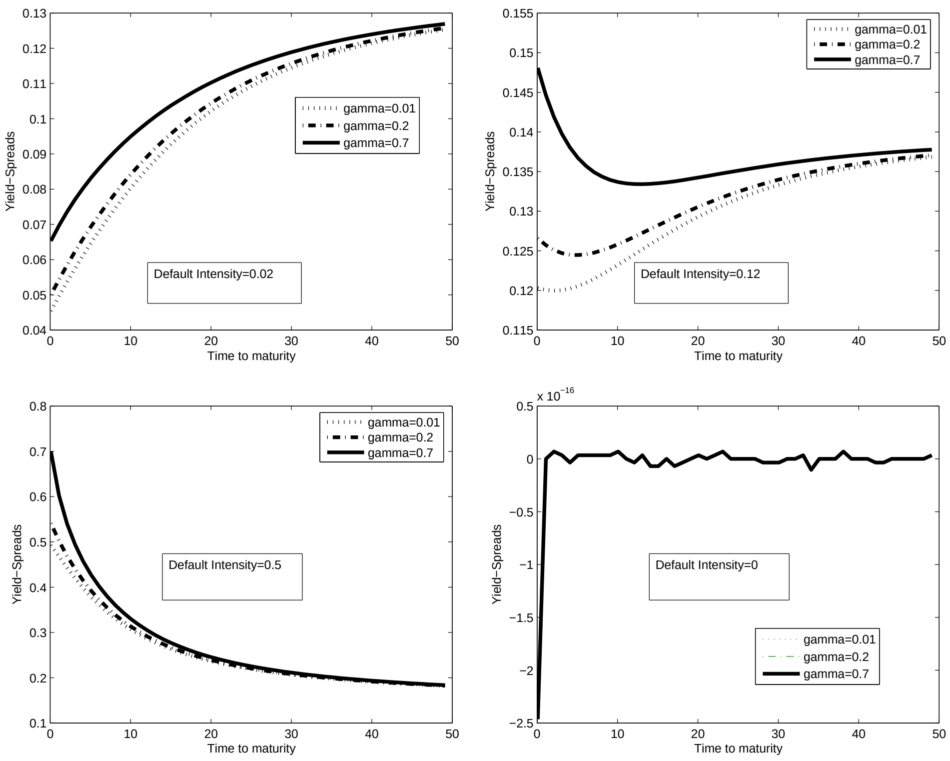


Figure 3.4: Buyer's yield-Spreads for CIR intensities, $\rho = -0.10$, $\alpha = 0.2060$, $\bar{\lambda} = 0.0646$, $\phi = 0.0303$.

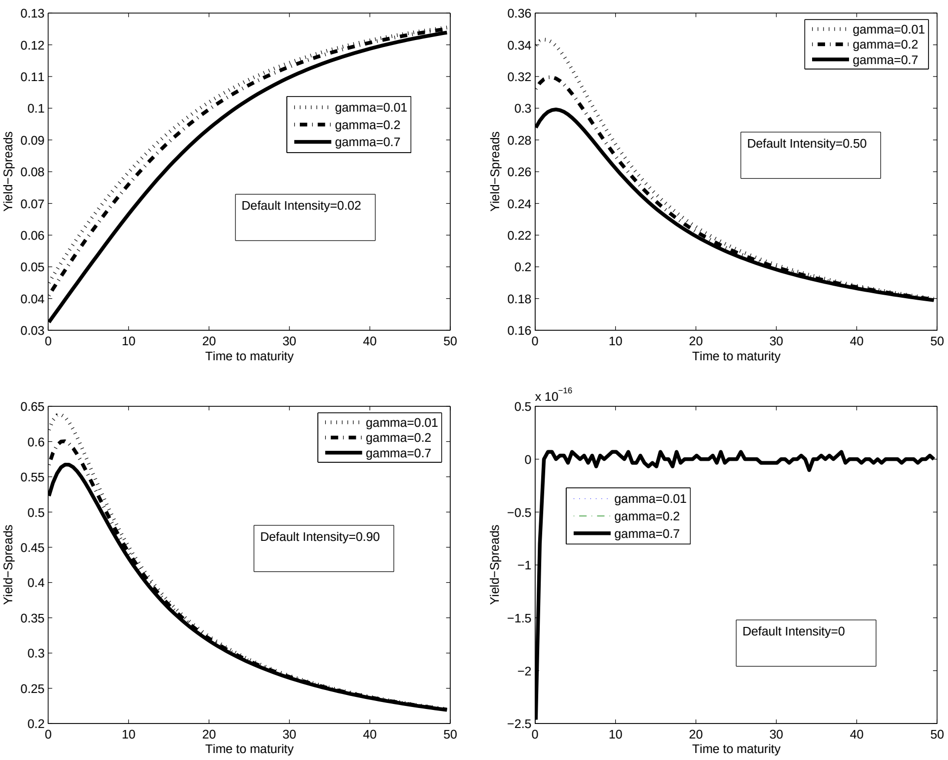


Figure 3.5: Seller's yield-Spreads for CIR intensities, $\rho = -0.10$, $\alpha = 0.2060$, $\bar{\lambda} = 0.0646$, $\phi = 0.0303$.

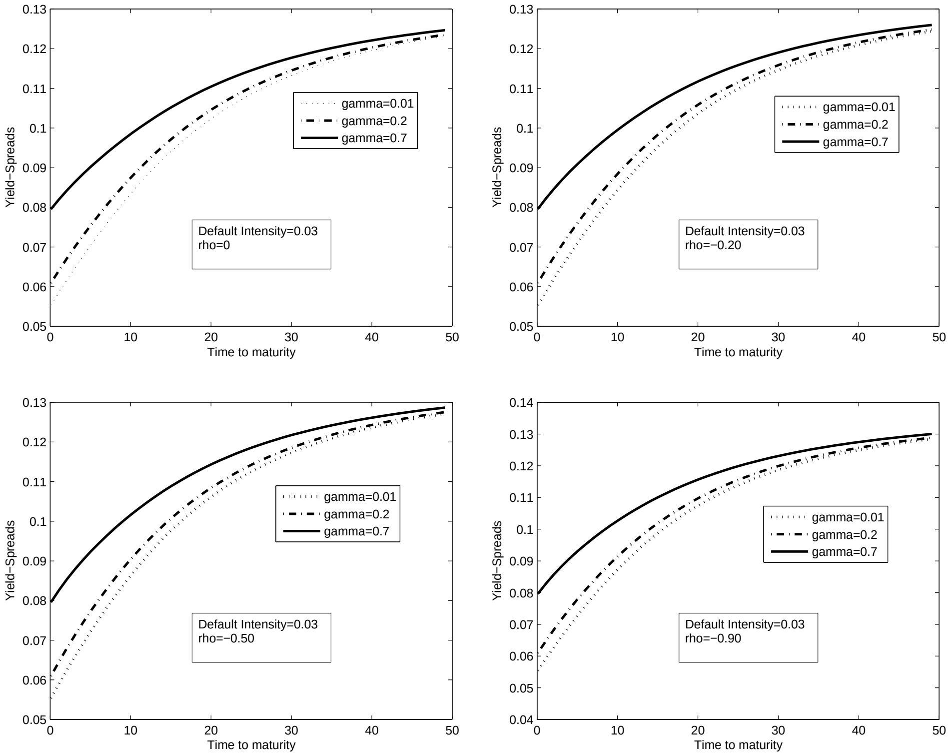


Figure 3.6: Impact of ρ on the bid spread-curves: $\alpha = 0.2060$, $\bar{\lambda} = 0.0646$, $\phi = 0.0303$.

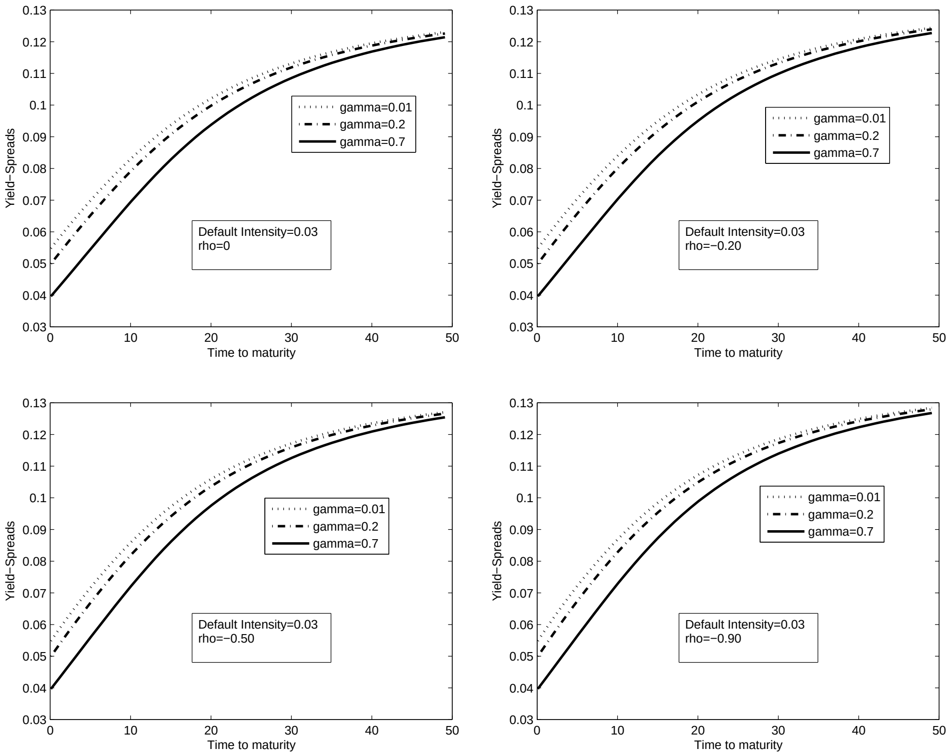


Figure 3.7: Impact of ρ on the ask spread-curves: $\alpha = 0.2060$, $\bar{\lambda} = 0.0646$, $\phi = 0.0303$.

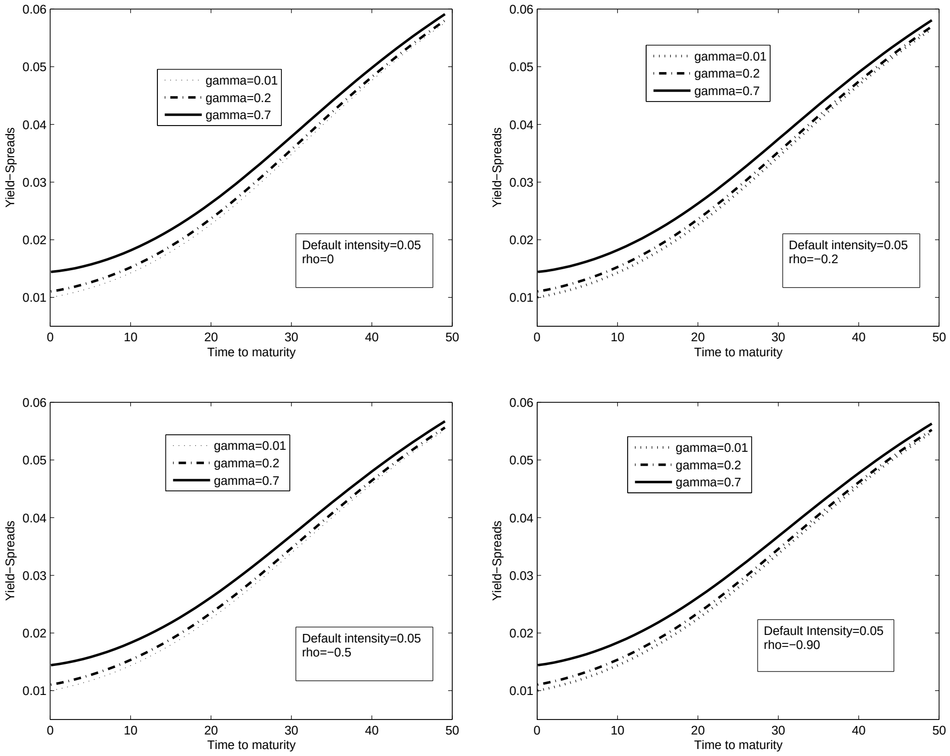


Figure 3.8: Impact of ρ on the bid spread-curves: $\alpha = 0.034$, $\bar{\lambda} = 0.00043$, $\phi = 0.014$.

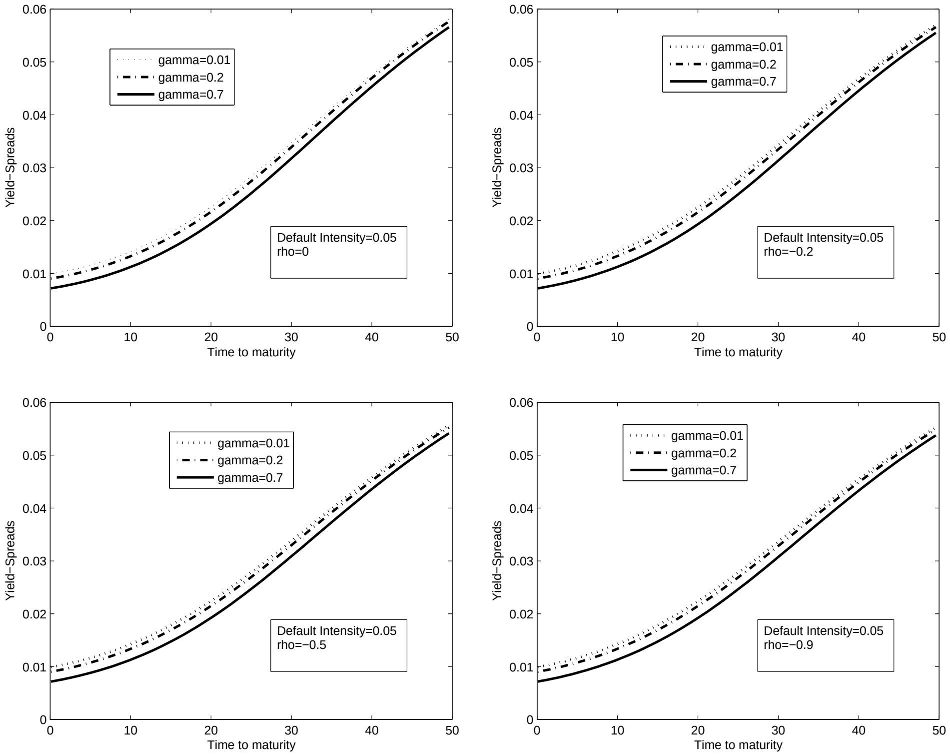


Figure 3.9: Impact of ρ on the ask spread-curves: $\alpha = 0.034$, $\bar{\lambda} = 0.00043$, $\phi = 0.014$.

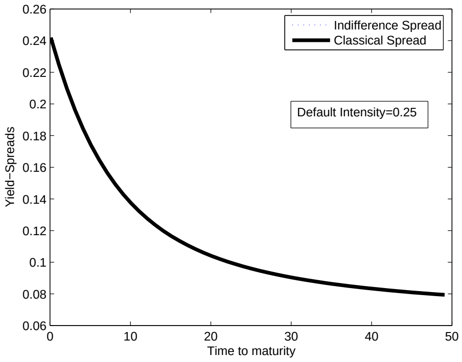


Figure 3.10: Indifference Spreads and Classical Spreads for buyer, $\mu = 0$, $\gamma = 0.0001$, $\alpha = 0.2060$, $\bar{\lambda} = 0.0646$, $\phi = 0.0303$.

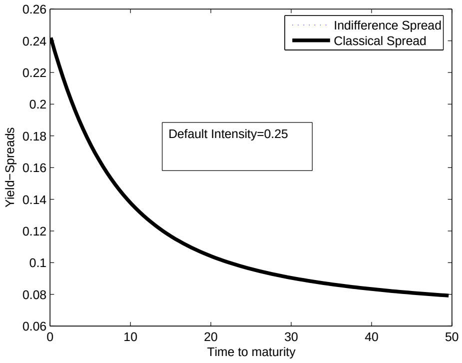


Figure 3.11: Indifference Spreads and Classical Spreads for seller, $\mu = 0$, $\gamma = 0.0001$, $\alpha = 0.2060$, $\bar{\lambda} = 0.0646$, $\phi = 0.0303$.

3.3 Indifference Valuation for Defaultable bonds: Closed form solutions

Here, we keep the assumptions 1 to 3 of the previous section. We add the following one

Assumption 6. *(Independence between the default intensity and the firm's stock price)*

> From the results obtained in the previous section, it is observed that the correlation between the default intensity and the stock price has a little impact on the yield-spreads. As the main purpose of this thesis is to model the yield-spread, we neglect the correlation effect and suppose the independence between the stock price and the default intensity.

Although this assumption is unrealistic because when the firm's stock decreases the default intensity increases, it greatly simplifies the calculations and gives rise to closed form solutions for indifference prices. The closed formula would help to extend more analysis such as the asymptotic behaviour of the spread curve which is difficult to investigate with the PDE approach. In addition, one supposes a full recovery of pre-default market value of the stock in the event of default. Under this assumption, the correlation coefficient ρ is removed from (3.7) and we have

$$\left\{\begin{array}{lll} dS_t & = & \mu S_t dt + \sigma S_t dW_t^1, \ S_0 = S > 0, \\ dY_t & = & b(Y_t)dt + a(Y_t)dW_t^2, \ Y_0 = \lambda, \end{array}\right. \tag{3.44}$$

Where W^1 and W^2 are two independent Brownian motions.

3.3.1 Maximal Expected Utility Problem

Using the exponential utility of discounted wealth, we are first interested to the classical Merton portfolio optimization problem. Its value function is:

$$M(t,x) = \sup_{\pi \in \mathcal{A}_{t,T}} E(-e^{-\gamma X_T} | X_t = x) \tag{3.45}$$

where $\mathcal{A}_{t,T}$ is the set of admissible strategies over the period $[t,T]$.
$M(t,x)$ is the unique viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation:

$$\left\{\begin{array}{l} M_t + \sup_\pi \left\{ \frac{1}{2}\pi_t^2 \sigma^2 M_{xx} + \pi_t \mu M_x \right\} = 0, \\ M(T,x) = -e^{-\gamma x} \end{array}\right. \tag{3.46}$$

For the proof, see Proposition 1.
Following Merton (1969), the HJB equation can be simplified by the classical

distortion scaling $M(t,x) = -e^{-\gamma x} m(t)$. Substituting $M(t,x)$ in the (HJB) equation, we get $m(t) = e^{-\frac{\mu^2}{2\sigma^2}(T-t)}$ and

$$M(t,x) = -e^{-(\gamma x + \frac{\mu^2}{2\sigma^2}(T-t))} \tag{3.47}$$

We are now interested in the optimal investment problem of the buyer of the defaultable bond who receives 1 currency unit on date T if the firm has survived till then.

Theorem 2. *The bid price of the defaultable bond at time* 0 *is*

$$p^b = -\frac{log\big(F(0,Y_0,T)(e^{-\gamma C}-1)+1\big)}{\gamma} \tag{3.48}$$

where $F(0,Y_0,T) = E\left(e^{-\int_0^T \lambda(Y_s)ds}\middle| Y_0\right)$

Proof. The value function of the buyer is

$$H(t,x,y) = \sup_{\pi\in\mathcal{A}_{t,T}} E_t\left(U\left(X_T + C1_{\{\tau_t>T\}}\right)\right) \tag{3.49}$$

with $E_t(\cdot) = E(\cdot|X_t = x, Y_t = y)$ and $C = e^{-r(T-t)}$.

$$\begin{aligned} H(t,x,y) &= \sup_{\pi\in\mathcal{A}_{t,T}} E_t\left(-e^{-\gamma\left(X_T + C1_{\{\tau_t>T\}}\right)}\right) \\ &= \sup_{\pi\in\mathcal{A}_{t,T}} \left(E\left(-e^{-\gamma X_T}|X_t = x\right) \times E\left(e^{-\gamma C1_{\{\tau_t>T\}}}\right)|Y_t = y\right) \\ &= E\left(e^{-\gamma C1_{\{\tau_t>T\}}}|Y_t = y\right) \times \sup_{\pi\in A} E\left(-e^{-\gamma X_T}|X_t = x\right) \\ &= E\left(e^{-\gamma C1_{\{\tau_t>T\}}}|Y_t = y\right) \times M(t,x) \end{aligned}$$

$$\begin{aligned} E\left(e^{-\gamma C1_{\{\tau_t>T\}}}|Y_t = y\right) &= e^{-\gamma C} P\left(\tau_t > T|Y_t = y\right) + P\left(\tau_t \leq T|Y_t = y\right) \\ &= e^{-\gamma C} 1_{\{\tau_t>t\}} E\left(e^{-\int_t^T \lambda(Y_s)ds}|Y_t = y\right) + 1 \\ &\quad -1_{\{\tau_t>t\}} E\left(e^{-\int_t^T \lambda(Y_s)ds}|Y_t = y\right) \\ &= 1_{\{\tau_t>t\}} F(t,y,T)\left(e^{-\gamma C} - 1\right) + 1 \end{aligned}$$

where $F(t,y,T) = E\left(e^{-\int_t^T \lambda(Y_s)ds}\middle| Y_t = y\right)$. The term $1_{\{\tau>t\}}F(t,y,T)$ is the probability of default not occuring in T years conditioned on all the information available at time $t < T$.

Then

$$H(0,X_0,Y_0) = \big(F(0,Y_0,T)(e^{-\gamma C}-1)+1\big)\, M(0,X_0). \tag{3.50}$$

The buyer indifference price p^b of the defaultable bond is the price at which the investor is indifferent, in terms of maximum expected utility, between paying nothing and not having the claim C and paying p^b now to receive the claim C of the bond at time T. The buyer indifference price p^b at time $t = 0$ is such that $M(0, X_0) = H(0, X_0 - p^b, Y_0)$; using the fact that $M(t, X_0)$ is proportional to $U(x)$ in (3.47), we have

$$p^b = -\frac{\log\left(F(0, Y_0, T)(e^{-\gamma C} - 1) + 1\right)}{\gamma}$$

□

The yield-spread is defined by

$$S^b = -\frac{1}{T}\log(p^b) - r \tag{3.51}$$

From (3.48), p^b does not depend of the investor's intial wealth and is a function of the current driving process Y_0 of the default intensity, the risk aversion coefficient γ, the interest rate r and the maturity T. In addition, p^b is a decreasing function of T and $p^b \leq C = e^{-rT}$; therefore S^b is non negative for all $T > 0$.

We are also interested in the optimization investment problem for the seller of the defaultable bond.

Theorem 3. *The ask price of the defaultable bond at time* 0 *is*

$$p^s = \frac{log\left(F(0, Y_0, T)(e^{\gamma C} - 1) + 1\right)}{\gamma} \tag{3.52}$$

Proof. The value function is:

$$\tilde{H}(t, x, y) = \sup_{\pi \in A} E_t\left(U\left(X_T - C1_{\{\tau_t > T\}}\right)\right) \tag{3.53}$$

Similarly to the buyer, the seller's value function is

$$\tilde{H}(0, X_0, Y_0) = \left(F(0, Y_0, T)\left(e^{\gamma C} - 1\right) + 1\right) M(0, X_0). \tag{3.54}$$

The seller indifference price p^s at time $t = 0$ is such that $M(0, X_0) = H(0, X_0 + p^s, Y_0)$; then

$$p^s = \frac{\log\left(F(0, Y_0, T)(e^{\gamma C} - 1) + 1\right)}{\gamma}$$

□

The yield-spread is defined by

$$S^s = -\frac{1}{T}\log(p^s) - r \tag{3.55}$$

The yield spread S^s is also non-negative for all $T > 0$.
It is straightforward to show that

$$\lim_{\gamma\to 0} p^b = \lim_{\gamma\to 0} p^s = E^P\left(e^{-rT}1_{\{\tau>T\}}|Y_0\right) \tag{3.56}$$

The quantity in (3.56) is nothing else than the arbitrage free valuation of the defaultable bond under the physical measure. Therefore, when γ goes to zero, the reduced model constructed under the utility indifference valuation can recover the classical reduced model of the defaultable zero-coupon bond. The short term limit of the yield-spread is nonzero for both buyer and seller. For the buyer,

$$\lim_{T\to 0}\ S^b = \frac{e^\gamma - 1}{\gamma}\lambda(Y_0)$$

which is greater than $\lambda(Y_0)$ since $\gamma > 0$. For the seller,

$$\lim_{T\to 0}\ S^s = \frac{1 - e^{-\gamma}}{\gamma}\lambda(Y_0)$$

which is less than $\lambda(Y_0)$. In other words, large default intensity would allow the buyer to rise the short term risk premium; in contrast at short-term the seller would offer a lower premium for small intensities. This effect is amplified as γ becomes larger.

When the risk aversion coefficient goes to infinity, the bid spread goes to infinity whereas the offer spread goes to zero; for the buyer,

$$\lim_{\gamma\to\infty} p^b = 0 \text{ and } \lim_{\gamma\to\infty} S^b = \infty$$

for the seller

$$\lim_{\gamma\to\infty} p^s = e^{-rT} \text{ and } \lim_{\gamma\to\infty} S^s = 0$$

The results also confirm the convergence of the utility indifference prices to the superreplication price. For more details, see Dufresne and Hugonnier (2007).

3.3.2 Analysis of the yield-spreads for the constant intensity

For a constant intensity λ, the default time follows an exponential distribution with parameter λ. Substituting λ in (3.48) and (3.52) gives

$$p^b = -\frac{\log\left(e^{-\lambda T}(e^{-\gamma C} - 1) + 1\right)}{\gamma} \tag{3.57}$$

and

$$p^s = \frac{\log\left(e^{-\lambda T}(e^{\gamma C} - 1) + 1\right)}{\gamma} \tag{3.58}$$

We plot in figure 3.12 the buyer and the seller's spread curves, for various risk aversion coefficients and for the intensity $\lambda = 0.04$. While the buyer's spread curve is downward sloping, the seller's spread curve is upward sloping when the risk aversion is large enough. So the curves are not flat as in the case of arbitrage free pricing when the intensity is constant. For both curves, the yield spreads go to λ when maturity goes to infinity. So, for longer maturity the level of risk aversion has no impact on both buyer and seller's yield-spreads:

$$\lim_{T\to\infty} S^b = \lim_{T\to\infty} S^s = \lambda$$

When the risk aversion coefficient goes to zero, i.e. when the investors are risk neutral, both buyer and seller's spread curves are flat and equal to λ.

$$\lim_{\gamma\to 0} p^b = \lim_{\gamma\to 0} p^s = e^{-(r+\lambda)T}$$

and

$$\lim_{\gamma\to 0} S^b = \lim_{\gamma\to 0} S^s = \lambda$$

In the next section, the yield-spread is analyzed in the case of mean reverting square root intensity process.

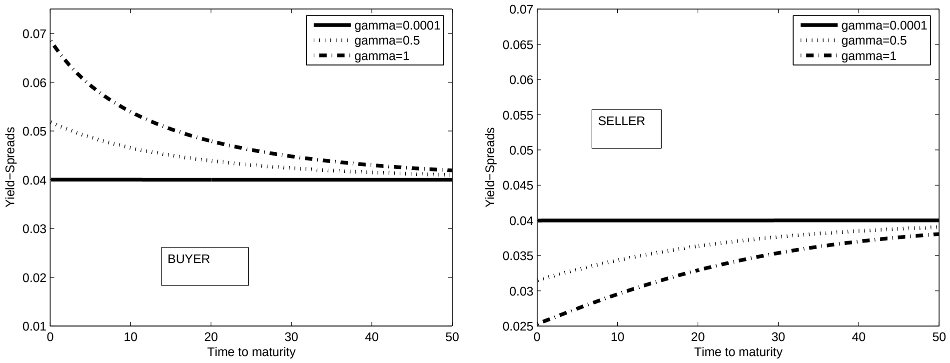


Figure 3.12: Buyer and seller's spread curves for various risk aversion coefficients, $r = 0.03$, $\lambda = 0.04$

3.3.3 Analysis of the yield-spreads for the CIR-intensity

The default intensity $\lambda(Y) = \lambda$ follows a mean-reverting square root process,

$$d\lambda_t = \alpha(\bar{\lambda} - \lambda_t)dt + \phi\sqrt{\lambda_t}dW_t$$

α, $\bar{\lambda}$ and ϕ are constant; α represents the rates of mean reverting, $\bar{\lambda}$ the long run average and ϕ the volatility. Given the dynamic of the default intensity,

$$F(0, \lambda_0, t) = A(t)e^{-B(t)\lambda_0} \tag{3.59}$$

The expressions of (3.48) and (3.52) become,

$$p^b = -\frac{\log\left[A(T)e^{-B(T)\lambda_0}(e^{-\gamma C} - 1) + 1\right]}{\gamma} \tag{3.60}$$

and

$$p^s = \frac{\log\left[A(T)e^{-B(T)\lambda_0}(e^{\gamma C} - 1) + 1\right]}{\gamma} \tag{3.61}$$

The prices p^b and p^s are decreasing functions of maturity; and

$$\lim_{T\to 0}\; p^b = \lim_{T\to 0}\; p^s = 1$$

$$\lim_{T\to \infty}\; p^b = \lim_{T\to \infty}\; p^s = 0$$

To implement the bid and offer yield-spreads of the defaultable bond, the estimated intensity's parameters of Longstaff et al. (2003) are used.

$$d\lambda_t = 0.2060(0.0646 - \lambda_t)dt + 0.0303\sqrt{\lambda_t}dW_t \tag{3.62}$$

The bid and ask spread curves are plotted in figures 3.13 and 3.14 for various risk aversion coefficients $\gamma \in \{0.01,\, 0.2,\, 0.7\}$ and $r = 0.03$. Both curves are upward sloping when the default intensity λ_0 is approximately less than 0.093. The figure 3.13 shows the shape of the spread curves for the intensity $\lambda_0 = 0.02$. When λ_0 is greater than 0.093, the spread curves become downward sloping as is shown in the figure 3.14 for $\lambda_0 = 0.20$.

As we consider longer and longer maturities, both buyer's and seller's yield spreads approach a limit which is independent of the current value of intensity.

$$\lim_{T\to \infty} S^b = \lim_{T\to \infty} S^s = \frac{2\alpha\bar{\lambda}}{\xi + \alpha} \leq \bar{\lambda}$$

This means that for smaller mean intensity level $\bar{\lambda}$, the investors claim small long-term yield-spreads.

When γ goes to zero, then both buyer and seller's bond prices coincide to the expectation of the discount payoff under the physical measure P.

$$\lim_{\gamma\to 0} p^b = \lim_{\gamma\to 0} p^s = A(T)e^{-(rT+B(T)\lambda_0)}$$

and

$$\lim_{\gamma\to 0} S^b = \lim_{\gamma\to 0} S^s = \frac{-\log(A(T)) + B(T)\lambda_0}{T}$$

Figure 3.15 shows the convergence of the bid and ask spreads for $\gamma = 0.0001$.

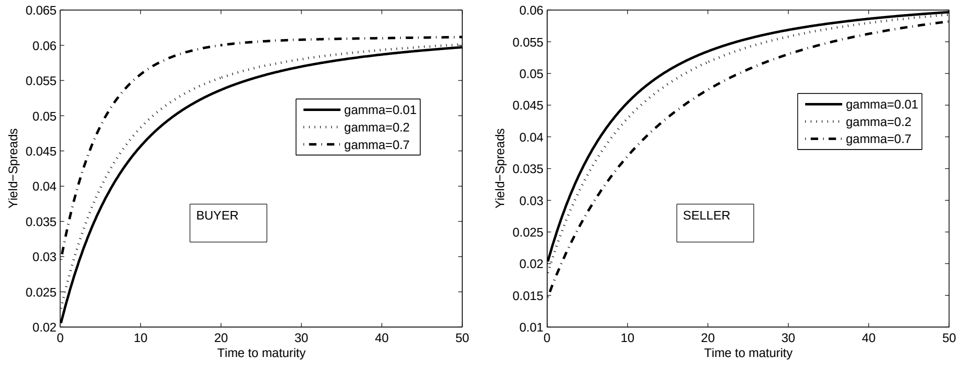


Figure 3.13: Buyer and seller's Yield spreads, $\lambda_0 = 0.02$,$\alpha = 0.2060$, $\bar{\lambda} = 0.0646$, $\phi = 0.0303$

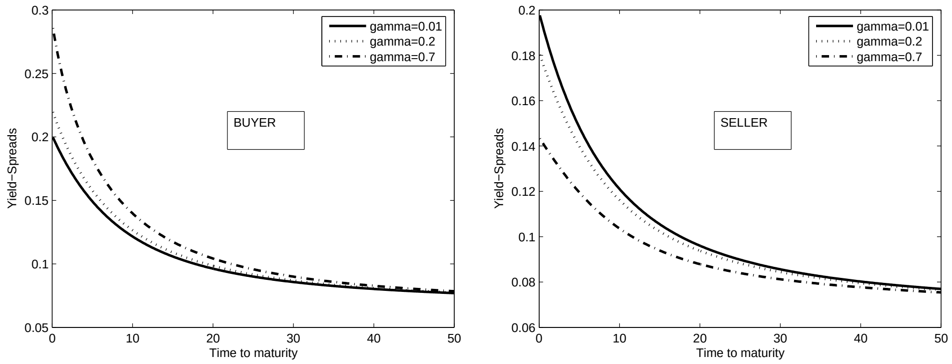


Figure 3.14: Buyer and seller's Yield spreads, $\lambda_0 = 0.2$,$\alpha = 0.2060$, $\bar{\lambda} = 0.0646$, $\phi = 0.0303$

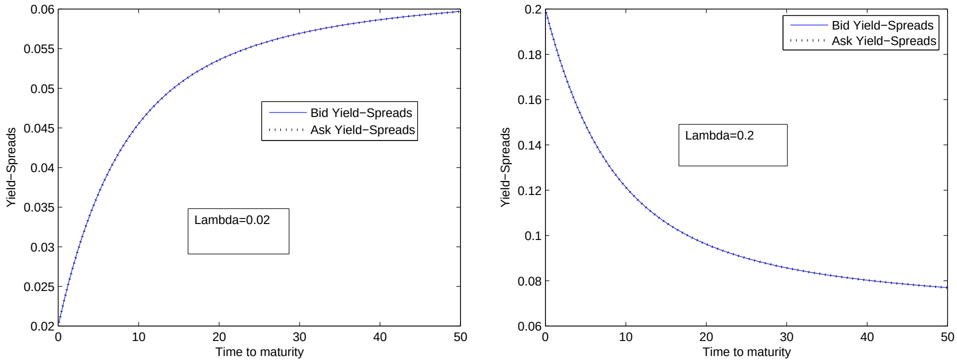


Figure 3.15: Convergence of buyer and seller's Yield spreads, $\gamma = 0.0001$,$\alpha = 0.2060$, $\bar{\lambda} = 0.0646$, $\phi = 0.0303$

Chapter 4

Indifference Valuation of Credit Default Swap (CDS)

In the chapter 3, we discussed the utility indifference valuation for zero-coupon defaultable bonds. In this chapter, we examine the indifference pricing for the CDS where the underlying is the zero-coupon bond. In a first step, we review briefly the martingale approach for the valuation of CDS. In a second step we examine the exponential utility indifference pricing of CDS and some numerical results are presented.

4.1 Martingale approach for the valuation of Credit Default Swap

We briefly review the arbitrage-free pricing model for valuing the credit-default swaps within the well-known reduced-form framework of Duffie (1998), Lando (1998) and others, along the lines of Longstaff et al. (2003).
Let r_t the instantaneous riskless rate and λ_t the stochastic default intensity of the Cox process governing default, with λ_t is generally correlated with r_t. Let s denotes the fixed premium paid by the protection buyer until either the reference entity defaults or the maturity date T of the bond; and $1 - R$ the contingent payment that is made by the protection seller if the reference entity defaults. More precisely, R is the recovery rate defined as the value of the bond just after default; it is assumed to be constant and exogenous. The fixed premium payment s is only paid when the reference entity has not defaulted. To account for this risk, the probability that the reference entity

has not defaulted by the payment date must be considered. Assuming the premium is paid continuously, the value at time 0 of the premium leg of a credit-default swap $P_1(s,T)$ is expressed as

$$\begin{aligned} P_1(s,T) &= E^Q\left(s\int_0^T e^{-\int_0^t r_s ds} 1_{\{\tau>t\}} dt\right) \\ &= E^Q\left(s\int_0^T e^{-\int_0^t r_s ds} E^Q\left(1_{\{\tau>t\}}|\mathcal{G}_T\right) dt\right) \\ &= E^Q\left(s\int_0^T e^{-\int_0^t (r_s+\lambda_s) ds} dt\right) \end{aligned}$$

where Q is a risk neutral probability. Similarly, the payment of the contingent is only made if there is a default and, consequently, has to be adjusted by the probability that default occurs. The value of the protection leg of a credit-default swap $P_2(R,T)$ can be expressed as

$$\begin{aligned} P_2(R,T) &= E^Q\left((1-R)e^{-\int_0^\tau r_s ds} 1_{\{\tau\le T\}}\right) \\ &= E^Q\left(E^Q\left((1-R)e^{-\int_0^\tau r_s ds} 1_{\{\tau\le T\}}|\mathcal{G}_T\right)\right) \\ &= E^Q\left((1-R)\int_0^T \lambda_t e^{-\int_0^t (r_s+\lambda_s) ds} dt\right) \end{aligned}$$

The principle of the CDS arbitrage-free valuation relies on the fact that at inception, the value of the CDS is equal to zero. Equalizing the values of the premium and protection legs yields the following formula for the CDS spread.

$$s = \frac{E^Q\left((1-R)\int_0^T \lambda_t e^{-\int_0^t (r_s+\lambda_s) ds} dt\right)}{E^Q\left(\int_0^T e^{-\int_0^t (r_s+\lambda_s) ds} dt\right)} \tag{4.1}$$

Assuming that λ_t and r_t are independent,

$$s = \frac{(1-R)\int_0^T B(0,t)G(0,\lambda_0,t)dt}{\int_0^T B(0,t)F(0,\lambda_0,t)dt} \tag{4.2}$$

where $B(0,t) = E^Q\left(e^{-\int_0^t r_s ds}\right)$ is the price at time 0 of a riskless zero-coupon bond with maturity t, $G(0,\lambda_0,t) = E\left(\lambda_t e^{-\int_0^t \lambda_s ds}\right)$ is the density of the default occuring at time t and $F(0, \lambda_0, t) = E\left(e^{-\int_0^t \lambda_s ds}\right)$ is the survival probability at time t. Closed form solutions for B, G and F are derived for given processes of λ_t and r_t. If λ is constant, the premium reduces to $\lambda(1 - R)$. Intuitively, Duffie (1999) shows that the CDS premium is equal to the fixed spread over the riskless rate that a corporate floating rate note would need to pay to be able to sell at par.

4.2 Indifference Pricing of Credit Default Swap

4.2.1 The model

The assumptions 1, 2, 3 and 6 are kept. We also need other basic assumptions useful for this model.

Assumption 7. *(Specification of the CDS)*

> We consider a Credit-Default Swap contract where the first party of the contract, the protection buyer, wishes to insure against the possibility of default by the reference entity. The second party of the contract, the protection seller, is willing to bear the risk associated with the default by the entity. The protection seller agrees to pay a fraction $1 - R$ of the par value to the protection buyer in the event of default on the reference bond. In return, the protection buyer will pay a fix premium s to the protection seller if there is no default until the maturity of the bond. In fact, s is the arbitrage free valuation that makes the value of the CDS equal to zero. We define s_b the bid spread i.e. the maximum amount the protection buyer is able to pay to insure against the default of the reference entity; and s_s the ask spread i.e. the smallest amount the protection seller is willing to accept in taking the risk. s_b and s_s equalize the maximum expected utility of discount wealth for both protection buyer and protection seller. Our goal is to find s_b and s_s and to etablish their relation with the arbitrage free spread s. In addition, We assume for simplicity that all payments are done continuously.

Assumption 8. *(Default about the investor)*

> Neither the protection seller nor the protection buyer default during the life of the CDS

While the protection seller default risk would reduce the value of the CDS for the protection buyer, reducing the premium, the protection buyer default risk would have the opposite effect. The net effect would depend on the default risk level of both agents and the credit risk correlations between them and the reference entity.

The next task is to introduce and analyze the three fundamental optimal investment problems via which the bid and ask CDS spreads will be constructed. Throughout the analysis, it is assumed that the individual preferences are modelled via an exponential utility function and that they remain the same, independently of whether the bond is bought, written or not traded at all. Using the exponential utility of discounted wealth, we are first interested to the classical Merton portfolio optimization problem. Its value function is:

$$M(t,x) = \sup_{\pi \in \mathcal{A}_{t,T}} E(-e^{-\gamma X_T} | X_t = x) \tag{4.3}$$

and from Merton (1969),

$$M(t,x) = -e^{-(\gamma x + \frac{\mu^2}{2\sigma^2}(T-t))} \tag{4.4}$$

We are now interested to the optimal investment problem up to time T of the protection buyer who is able to pay the CDS spread s_b until $\tau \wedge T$ and in return receives a fraction $1-R$ of the face value of the bond when $\tau \leq T$.

Theorem 4. *The bid CDS spread* s_b *at time* 0 *is the zero of the function*

$$\begin{aligned} H(z) = \int_0^T e^{\gamma\left(\frac{z}{r} - \frac{ze^{-ru}}{r} - (1-R)e^{-ru}\right)} G(0, Y_0, u) du \\ + e^{\frac{\gamma z}{r}\left(1-e^{-rT}\right)} F(0, Y_0, T) - 1 \end{aligned} \tag{4.5}$$

where $G(0, Y_0, u)$ *is the density of the default occuring at time* u *conditioned on* Y_0 *at time* 0*, and* $F(0, Y_0, u)$ *is the probability of default not occuring in* T *years conditioned on* Y_0 *at time* 0.

Proof. See the Appendix D. ☐

Hence, s_b verifies

$$\begin{aligned} &-\int_0^T e^{\gamma\left(\frac{s_b}{r}-\frac{s_b e^{-ru}}{r}-(1-R)e^{-ru}\right)} G(0,Y_0,u)du \\ &\qquad\qquad - e^{\frac{\gamma s_b}{r}\left(1-e^{-rT}\right)} F(0,Y_0,T) = -1 \end{aligned} \tag{4.6}$$

The upper part of the above equation expresses the expected utility of the promised payments in the event of default, and the lower part is the expected utility of the promised payments if there is survival in T. The sum of the two terms is equal to -1 which the utility of the arbitrage free value of the CDS. This implies a relation between the indifference CDS spread and the arbitrage free spread of the CDS.

We are also interested to the optimal investment problem up to time T of the protection seller who requires the CDS premium s_s until $\tau \wedge T$ and in return pays a fraction $1-R$ of the par value of the bond when $\tau \leq T$.

Theorem 5. *The ask CDS spread s_s at time* 0 *is then the zero of the function,*

$$\begin{aligned} \tilde{H}(z) = &\int_0^T e^{-\gamma\left(\frac{z}{r}-\frac{z e^{-ru}}{r}-(1-R)e^{-ru}\right)} G(0,Y_0,u)du \\ &\qquad\qquad + e^{\frac{-\gamma z}{r}\left(1-e^{-rT}\right)} F(0,Y_0,T) - 1 \end{aligned} \tag{4.7}$$

Proof. See the Appendix D ☐

It is also straightforward to show that

$$\lim_{\gamma\to 0} s_b = \lim_{\gamma\to 0} s_s = \frac{(1-R)\int_0^T e^{-rt} G(0,Y_0,t)dt}{\int_0^T e^{-rt} F(0,Y_0,t)dt} \tag{4.8}$$

The quantity in (4.8) is nothing else than the CDS arbitrage free spread described in section 4.1 under the physical measure. Therefore, when γ goes to zero, the reduced model constructed under the utility indifference valuation can recover the classical reduced model of the CDS.
The short-term limit of the bid and ask CDS spread are nonzero as we would expect in the presence of non-predictable defaults. For the bid CDS spread,

$$\lim_{T\to 0} s_b = \frac{1-e^{-(1-R)\gamma}}{\gamma}\lambda(Y_0) \tag{4.9}$$

which is less than $(1-R)\lambda(Y_0)$ since $(1-R)\gamma > 0$.
For the ask CDS spread,

$$\lim_{T\to 0} s_s = \frac{e^{(1-R)\gamma}-1}{\gamma}\lambda(Y_0) \tag{4.10}$$

which is greater than $(1-R)\lambda(Y_0)$. In other words, large recovery rate R will allow the protection buyer to reduce the short-term CDS spread; in contrast the protection seller will rise the short-term CDS spread when the recovery rate is small. The effects are amplified when $(1-R)\gamma$ becomes larger. As we will see later, the limits of the short-term CDS spreads would help for the estimation of the time series of the default intensity.
Numerical algorithms are used to approximate the integral and to find the zeros of the functions H and $\tilde{H}$. These methods are presented in the next section.

4.3 Numerical algorithms

4.3.1 Trapezoidal rule

The trapezoidal rule is used to approximate the integral. It involves dividing the area into a number of strips of equal width. Then, approximating the area of each strip by the area of the trapezium formed when the upper end is replaced by a chord. The sum of these approximations gives the final numerical result of the area under the curve. The trapezoidal rule can be presented as follows:
Let the definite integral $\int_a^b f(x)dx$. The points of subdivision of the domain of integration $[a,b]$ are labelled $x_0, x_1, .., x_n$ where $x_0 = a$, $x_n = b$, $x_j = x_0 + j\frac{b-a}{n}$.
Applying the trapezoidal rule on each of the strip gives

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\left[\frac{f(a)+f(b)}{2}\sum_{j=1}^{n-1} f\left(a+j\frac{b-a}{n}\right)\right]$$

This can be alternatively be written as

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left(f(x_0)+2f(x_1)+2f(x_2)+...+2f(x_{n-1})+f(x_n)\right)$$

The error of the trapezoid rule is the difference between the value of the integral and the numerical result:

$$\text{error} = \int_a^b f(x)dx - \frac{b-a}{n}\left[\frac{f(a)+f(b)}{2}\sum_{j=1}^{n-1} f\left(a + j\frac{b-a}{n}\right)\right]$$

It can be written as

$$\text{error} = -\frac{(b-a)^3}{12n^2} f''(c)$$

with c is a number between a and b. For more details see Bradie (2005) or Sauer (2006).

4.3.2 Newton's method

The Newton's method is used for approximating the zeros of the functions H and $\tilde{H}$. Suppose we wish to find a solution of the equation $f(x) = 0$ for a given function f. The Newton's method starts with an initial guess x_0 which is usually found by graphing the curve $y = f(x)$ and letting x_0 be a point close to where the curve crosses the x-axis. Given the initial guess x_0, let T_0 be the best affine approximation of f at x_0. That is,

$$T_0(x) = f'(x_0)(x - x_0) + f(x_0)$$

The idea behind Newton's method is to obtain an improved estimate of a solution to $f(x) = 0$ by replacing the equation $f(x) = 0$ with the simpler equation $T_0(x) = 0$. Let x_1 denotes the solution for the latter equation, then $T_0(x_1) = 0$, from which it follows

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

To improve upon this approximation, we solve the equation $T_1(x) = 0$, where T_1 is the best affine approximation of f at x_1. Let x_2 the solution of this equation, then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

This generates a sequence of approximations $x_0, x_1, x_2, x_3, ...$ until the desired degree of accuracy is reached. In the general case given the approximation x_n, x_{n+1} is found by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The major preoccupation is to know when to stop generating successive approximations using the last equation. Since the exact solution of the equation is not known, in practice people generate terms until the difference between successive terms is less than a predetermined tolerance level ξ, i.e. $|x_{n+1} - x_n| < \xi$. The Newton's method is a powerful technique, in general the error is quadratic i.e. the error is squared at each step. However, there are three situations where Newton's method may not converge quickly. First, the initial value x_0 is too far from the true zero; second, the derivative of the function is not continuous; third $f'(x_n)$ is closed to zero. For more details see Deuflhard (2004).
In the next section, we will implement the model of the CDS in the case of constant and stochastic intensities.

4.4 Implementation in the Constant Intensity case

For constant intensity λ, the default time τ follows an exponential distribution with parameter λ. Substituting λ in (4.5) and (4.7), the bid CDS spread is the zero of

$$H(z) = \lambda \int_0^T e^{\gamma\left(\frac{z}{r} - \frac{ze^{-ru}}{r} - (1-R)e^{-ru}\right) - \lambda u} du + e^{\frac{\gamma z}{r}\left(1-e^{-rT}\right) - \lambda T} - 1 \qquad (4.11)$$

and the ask CDS spread is the zero of

$$\tilde{H}(z) = \lambda \int_0^T e^{-\gamma\left(\frac{z}{r} - \frac{ze^{-ru}}{r} - (1-R)e^{-ru}\right) - \lambda u} du + e^{\frac{-\gamma z}{r}\left(1-e^{-rT}\right) - \lambda T} - 1 \qquad (4.12)$$

When the risk aversion coefficient goes to zero, the bid and ask CDS spreads converge to $s = \lambda(1 - R)$ which is the arbitrage free valuation of the CDS spread of section 4.1 when the default intensity is constant.
Figures 4.1 and 4.2 show the bid and ask CDS spreads for various risk aversion coefficients $\gamma \in \{0.0001, 0.5, 1\}$. While the buyer's spread curve is upward sloping, the seller's spread curve is downward sloping whatever the default intensity. In addition, the CDS spreads increase with the default intensity; this is observed by comparing the spread curves for $\lambda = 0.2$ in figure 4.2 to the spread curves for $\lambda = 0.0356$ in the figure 4.1. Although the bid spread is decreasing as risk aversion is increasing, the ask spread is increasing as risk

aversion is increasing. In particular, there is a negative correlation between the risk aversion and the CDS rate of the protection buyer. For small risk aversion coefficient ($\gamma = 0.0001$), we observe the convergence of the bid and ask swap spread to $s = (1-R)\lambda$ which is 0.0249 in figure 4.1 and 0.14 in figure 4.2.
The impact of the recovery rate R on the CDS spreads is analyzed. Four different recovery rate are used: 10, 30, 50, 70%. As we would expect in the classical reduced model, there is a negative correlation between both bid and ask CDS spreads and the recovery rate. The figure 4.3 shows the bid spread sensitivity to the recovery rate.

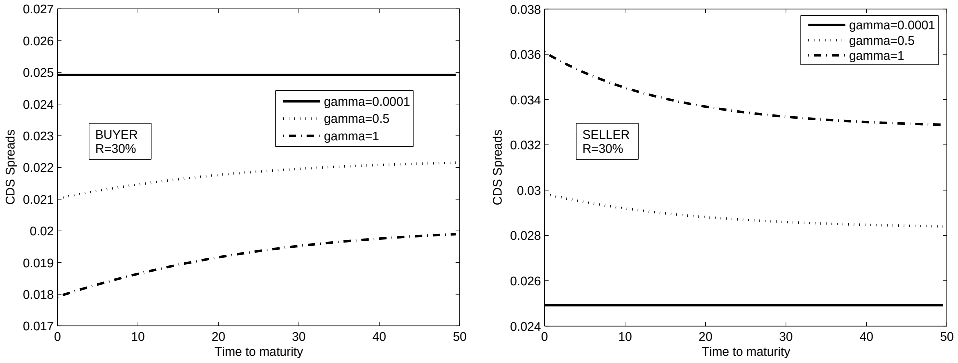


Figure 4.1: Buyer and seller's CDS spread curve, $r = 0.03$, $\lambda = 0.0356$

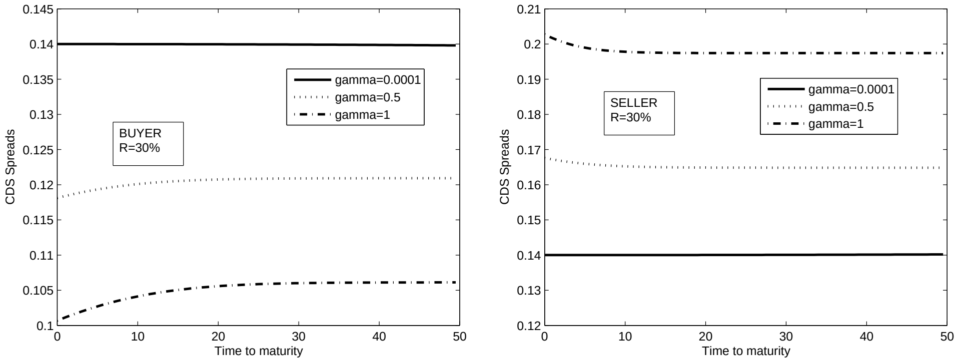


Figure 4.2: Buyer and seller's CDS spread curve, $r = 0.03$, $\lambda = 0.2$

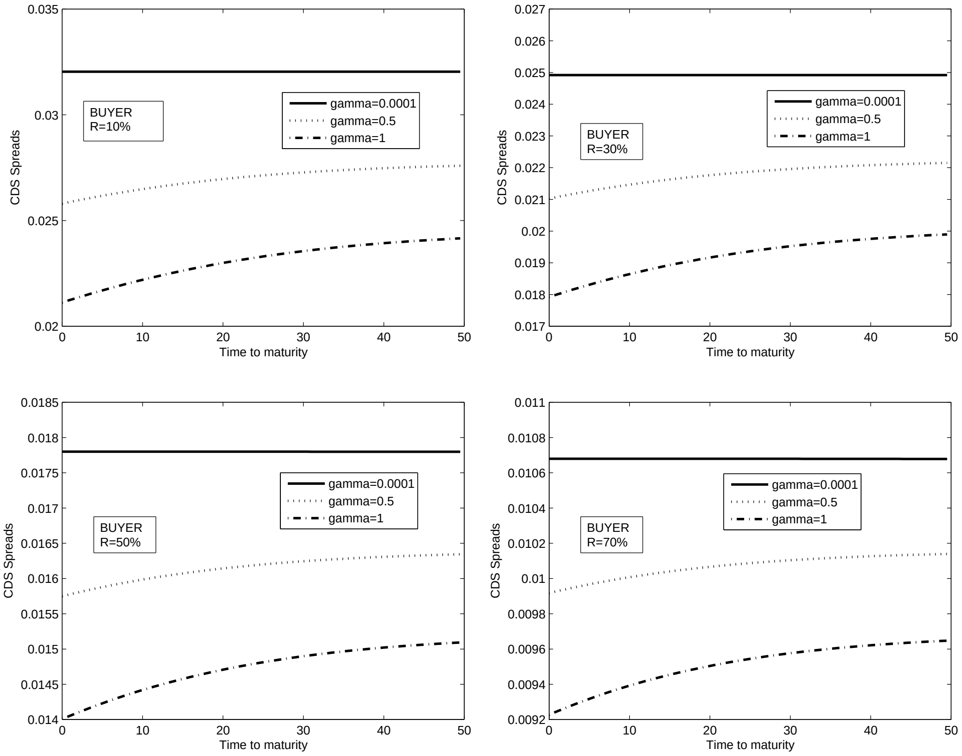


Figure 4.3: Impact of the recovery rate on the bid spread curves, $r = 0.03$, $\lambda = 0.0356$

4.5 Implementation in the CIR-Intensity case

In this section, the default intensity $\lambda(Y) = \lambda$ follows a mean-reverting square root process,

$$d\lambda_t = \alpha(\bar{\lambda} - \lambda_t)dt + \phi\sqrt{\lambda_t}dW_t$$

Under this process of the default intensity,
Appendix E shows a closed form solutions for $G(0, \lambda_0, t)$;

$$G(0, \lambda_0, t) = (C(t) + H(t)\lambda_0)e^{-B(t)\lambda_0} \tag{4.13}$$

with

$$C(t) = \frac{\alpha\bar{\lambda}}{\xi}\left(e^{\xi t}-1\right)e^{\left(\alpha\bar{\lambda}(\alpha+\xi)\frac{t}{\phi^2}\right)}\left(\frac{1-\kappa}{1-\kappa e^{\xi t}}\right)^{\frac{2\alpha\bar{\lambda}}{\phi^2}+1} \tag{4.14}$$

$$H(t) = e^{\left((\alpha\bar{\lambda}(\alpha+\xi)+\xi\phi^2)\frac{t}{\phi^2}\right)}\left(\frac{1-\kappa}{1-\kappa e^{\xi t}}\right)^{\frac{2\alpha\bar{\lambda}}{\phi^2}+2} \tag{4.15}$$

$$\kappa = \frac{\alpha+\xi}{\alpha-\xi} \tag{4.16}$$

with $B(t)$ and ξ defined in (3.41) and (3.42). So, the bid CDS spread s_b is the zero of

$$\begin{aligned} H(z) = \int_0^T e^{\gamma\left(\frac{z}{r}-\frac{ze^{-ru}}{r}-(1-R)e^{-ru}\right)}(C(u)+H(u)\lambda_0)e^{-B(u)\lambda_0}du \\ + e^{\frac{\gamma z}{r}\left(1-e^{-rT}\right)}A(T)e^{-B(T)\lambda_0}-1 \end{aligned} \tag{4.17}$$

The ask CDS spread s_s is the zero of

$$\begin{aligned} \tilde{H}(z) = \int_0^T e^{-\gamma\left(\frac{z}{r}-\frac{ze^{-ru}}{r}-(1-R)e^{-ru}\right)}(C(u)+H(u)\lambda_0)e^{-B(u)\lambda_0}du \\ + e^{\frac{-\gamma z}{r}\left(1-e^{-rT}\right)}A(T)e^{-B(T)\lambda_0}-1 \end{aligned} \tag{4.18}$$

Thus,

$$\lim\nolimits_{\gamma\to 0} s_b = \lim\nolimits_{\gamma\to 0} s_s = \frac{(1-R)\int_0^T e^{-rt-B(t)\lambda}\left(C(t)+H(t)\lambda_0\right)dt}{\int_0^T e^{-rt-B(t)\lambda_0}A(t)dt}$$

Figures 4.4 to 4.11 show the bid and ask CDS spreads for various risk aversion coefficients $\gamma \in \{0.0001, 0.5, 1\}$. The parameters of the intensity have been taken in Longstaff et al. (2003) and the recovery rates $R \in \{10\%, 30\%, 50\%, 70\%\}$. As in the constant case, there is a negative (positive) relationship between the risk aversion coefficient and the CDS spreads for the protection buyer (protection seller). In other words, a more risk averse protection buyer would pay less premium to ensure against the default of the bond. Even if the default intensity increases the bid and offer CDS spreads because it increases the probability of the reference entity defaulting, many experiments show that the shape of the spread curves changes with the values of the default intensity. It is observed in general when the default intensity is small and less than a threshold, the bid and ask spread curves are upward sloping. In fact, the credit risk for a high quality reference entity will not deteriorate in the near term. This explains why bid and ask CDS spreads are low for relatively short maturities. As time passes, credit quality deterioration is more likely than credit quality improvements and this is reflected in higher CDS spreads as maturity increases. Obviously, the value of the threshold depends of the parameters of the intensity process. For example, it is observed with the parameters of Longstaff et al. (2003) that the value of the threshold is approximately 0.065. That is when the initial default intensity λ_0 is less than approximately 0.065, the spread curve is upward sloping. The figures 4.4 et 4.5 show this effect respectively for $\lambda_0 = 0.0356$ et $\lambda_0 = 0.05$. With the parameters of Denault et al. (2009), the value of the threshold is approximately 0.001. The figures 4.6 et 4.7 show the upward sloping curve respectively for $\lambda_0 = 0.0001$ et $\lambda_0 = 0.0002$.

When the default intensity becomes high and greater than the threshold, the bid and ask spread curve becomes downward sloping. In fact, for a lower credit quality issuer, the inverted spread curve caracterized by a short-term CDS spreads higher than long-term spreads was explained by Merton (1974); default risk for this class of issuers is very high in the near term, but it is believed than once the current difficulties are overcome, chances are that the bond issuer would be able to meet its obligation. Hence, default risk in the medium and long term is lower than the near term, and is reflected in the downward slope of the CDS curve. For example, with the estimates of Longstaff et al. (2003), when the default intensity is greater than 0.065, the bid and ask spread curves are downward sloping. It is shown in the figures 4.8 to 4.9 respectively for $\lambda_0 = 0.08$ and $\lambda_0 = 0.2$. Similarly, with the parameters of Denault et al. (2009), it is observed a downward sloping curve when the default intensity is greater than 0.001. The figures 4.10 et 4.11 show this effect respectively for $\lambda_0 = 0.002$ et $\lambda_0 = 0.008$.

The figure 4.12 illustrates the points discussed above, through the CDS spread curves on October 1, 2008 of Allstate Corp rated AA, Highwoods Realty LP rated A, General Electric Capital Corp rated BB and iStar Financial Inc rated CCC. The spread curves for the classes AA and A companies exhibited an upward sloping shape since the companies are likely enough to meet payment obligations that banks are allowed to invest in them. In contrast, the spread curve for the BB and CCC companies are downward sloping because the companies have a higher default probability and have less chance to meet their obligation.

The sensitivity of the CDS spreads to the recovery rate is analyzed and it reveals the bid and ask CDS spreads are decreasing function of recovery rates. Intuitively, a higher recovery rate reduces the bid and ask premium because it reduces the payment of the protection seller in case of default. The figure 4.13 shows the negative relationship between the ask CDS spreads and the recovery rate. The CDS premium sensitivity to the recovery rates is in line with Elizalde (2005) who empirically found the same result.

When γ goes to 0.0001, we observe from figures 4.4 to 4.11 the convergence of the bid and ask CDS spread curve to the classical reduced spread s.

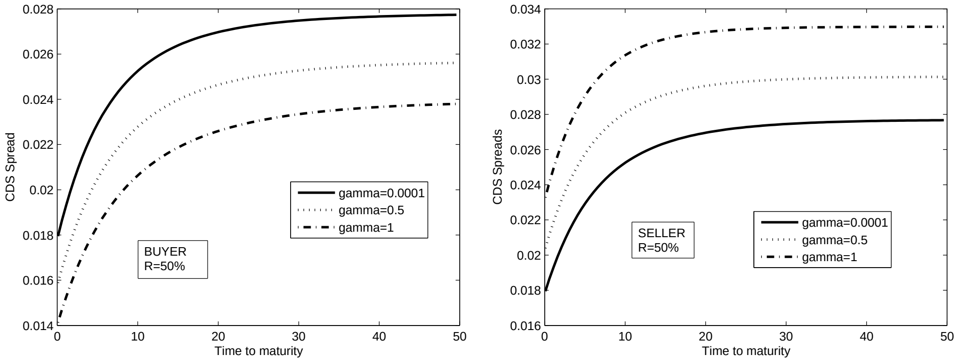


Figure 4.4: Buyer and seller's CDS spread curve, $r = 0.03$, $\alpha = 0.2060$, $\phi = 0.0303$, $\bar{\lambda} = 0.0646$, $\lambda_0 = 0.0356$

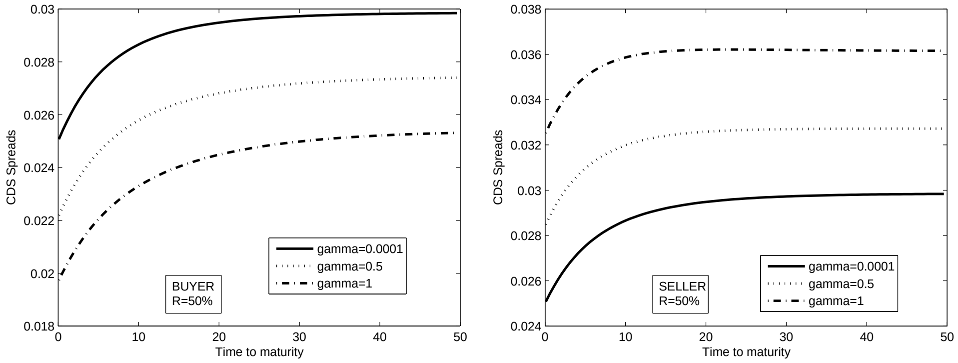


Figure 4.5: Buyer and seller's CDS spread curve, $r = 0.03$, $\alpha = 0.2060$, $\phi = 0.0303$, $\bar{\lambda} = 0.0646$, $\lambda_0 = 0.05$

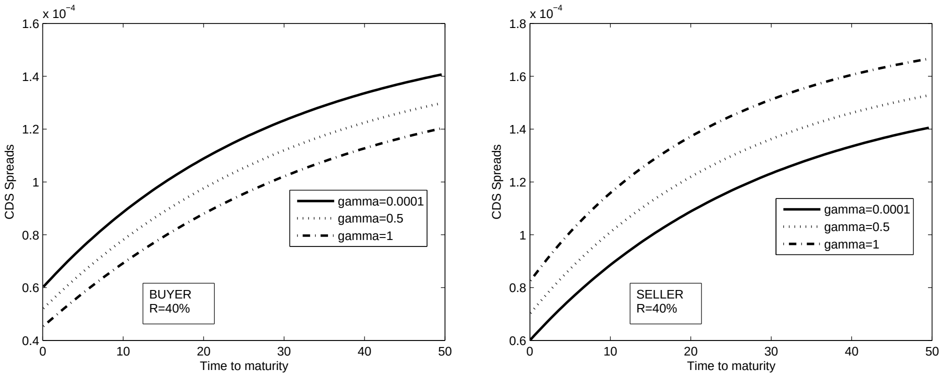


Figure 4.6: Buyer and seller's CDS spread curve, $r = 0.03$, $\alpha = 0.034$, $\bar{\lambda} = 0.00043$, $\phi = 0.014$, $\lambda_0 = 0.0001$

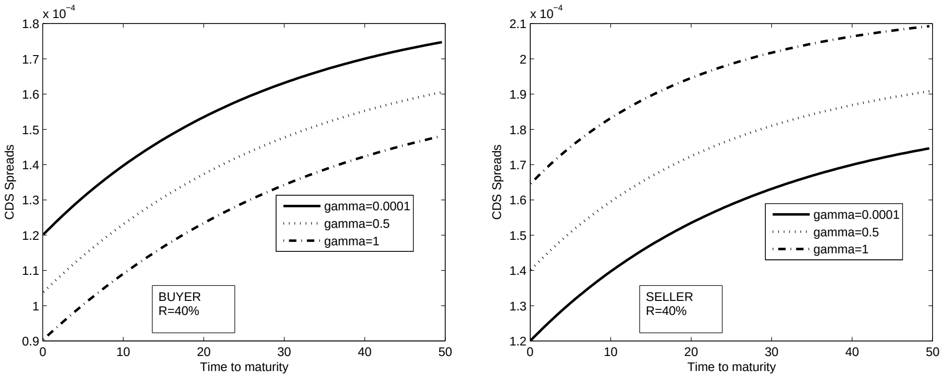


Figure 4.7: Buyer and seller's CDS spread curve, $r = 0.03$, $\alpha = 0.034$, $\bar{\lambda} = 0.00043$, $\phi = 0.014$, $\lambda_0 = 0.0002$

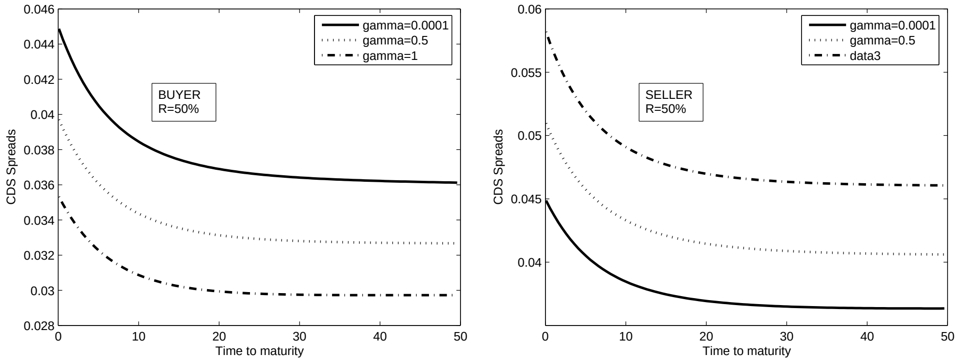


Figure 4.8: Buyer and seller's CDS spread curve, $r = 0.03$, $\alpha = 0.2060$, $\phi = 0.0303$, $\bar{\lambda} = 0.0646$, $\lambda_0 = 0.08$

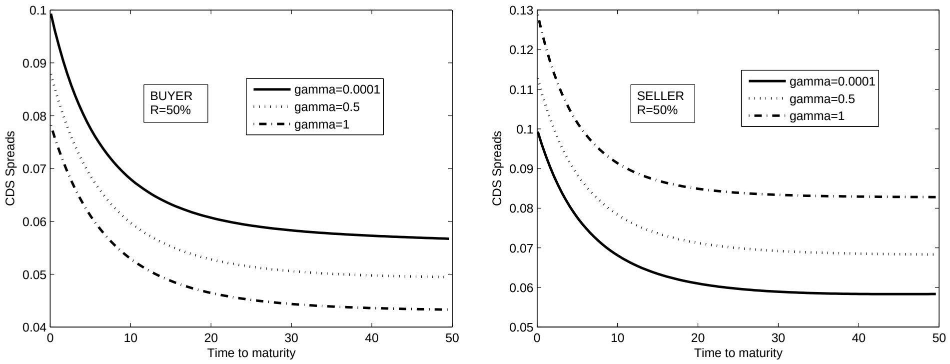


Figure 4.9: Buyer and seller's CDS spread curve, $r = 0.03$, $\alpha = 0.2060$, $\phi = 0.0303$, $\bar{\lambda} = 0.0646$, $\lambda_0 = 0.2$

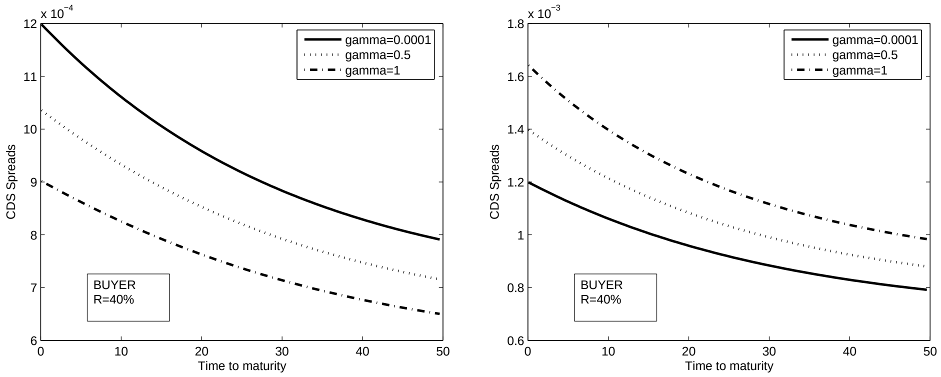


Figure 4.10: Buyer and seller's CDS spread curve, $r = 0.03$, $\alpha = 0.034$, $\bar{\lambda} = 0.00043$, $\phi = 0.014$, $\lambda_0 = 0.002$

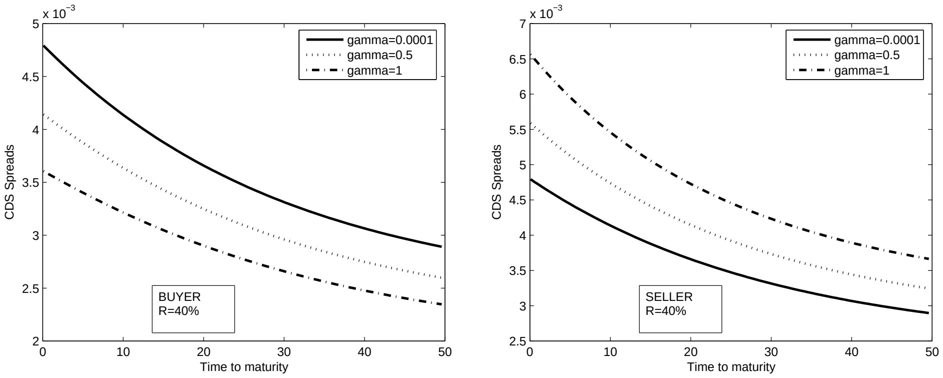


Figure 4.11: Buyer and seller's CDS spread curve, $r = 0.03$, $\alpha = 0.034$, $\bar{\lambda} = 0.00043$, $\phi = 0.014$, $\lambda_0 = 0.008$

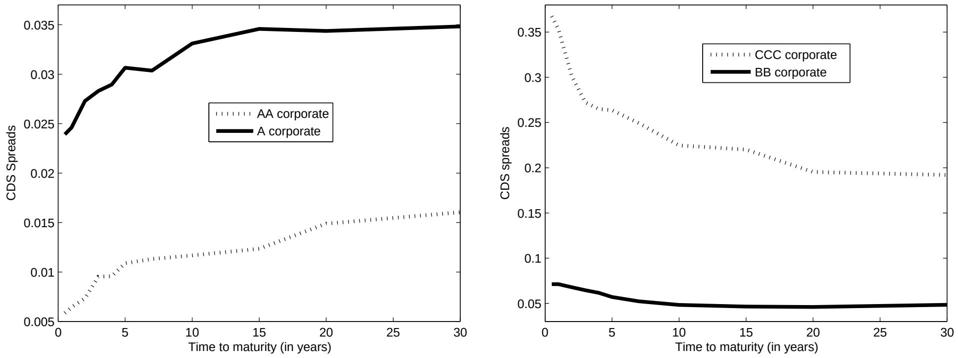


Figure 4.12: Spreads curve for 4 companies in the CDS market in October 1, 2008: Recovery rate 40%, sector Financials, country U.S

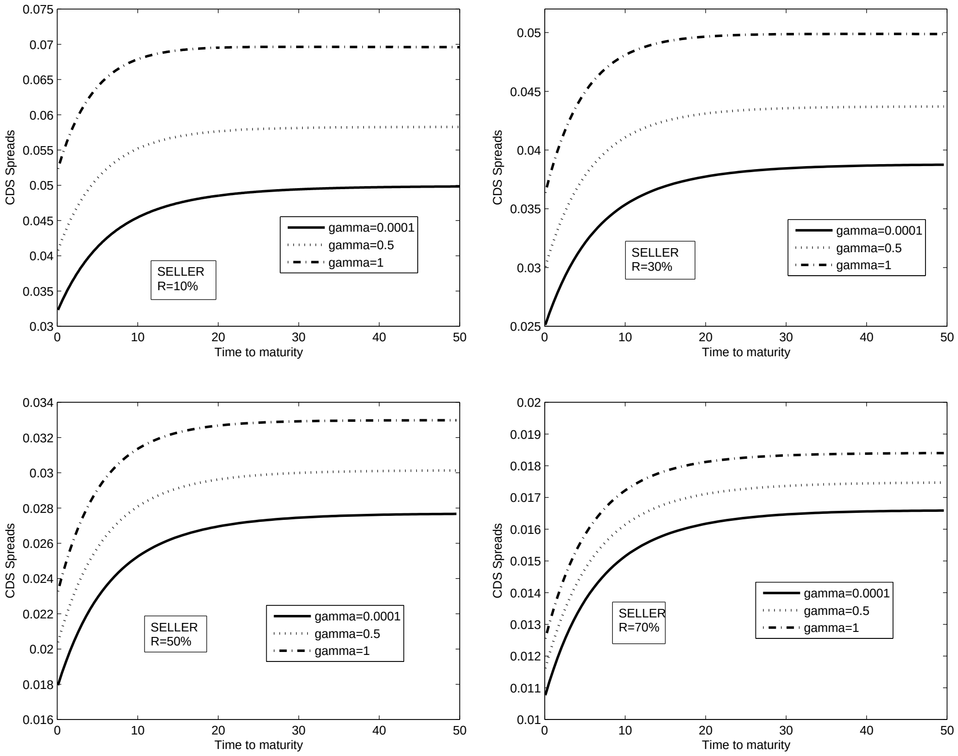


Figure 4.13: Sensitivity of the ask spread curves to the recovery rate, $r = 0.03$, $\alpha = 0.2060$, $\phi = 0.0303$, $\bar{\lambda} = 0.0646$, $\lambda_0 = 0.0356$

Chapter 5

Calibration of the Indifference Credit Default Swap Model

In this chapter, we focus in the calibration of the indifference CDS model from the protection buyer's view. Our goal is to propose a methodology to estimate the parameters of the model and to apply it to the CDS market data. Two classes of parameters are estimated, the physical default intensities and the constant absolute risk aversion coefficient. As described in the first chapter, the classical model of CDS estimates in general the default intensity under the risk neutral measure that does not reflect the physical probabilities of observing the default. The physical default intensity is useful in application such as the value-at-risk estimation, the calculation of the bank capital under the Basel II, etc. The constant absolute risk aversion γ is important for economists as it plays an important part in measuring the curvature of the utility function of the investor.

The estimation approach uses the nonlinear regression and proceeds in two steps. In a first step, an estimate of the short-term limit of CDS spread for a whole class of firms grouped by credit ratings is obtained. Two models are considered: in the model I, the quotes CDS spread for the shortest maturity (6 months) are used as proxy for the limit and in the model II, Nelson and Siegel (1987) model is fitted to the observed term stucture of CDS to obtain the short-term limit. These values are then used to estimate the dynamic of physical default intensities. In a second step, relying on the default intensities constructed and using the observations of CDS spreads and risk-free interest rate, we estimate by nonlinear least squares the parameters of CIR-intensity model and the investor's risk aversion parameter.

5.1 Estimation Approach

In the previous chapter, to evaluate the indifference CDS spread it is assumed a constant risk free interest rate and therefore a flat yield curve. Of course, the riskless interest rate is not constant, and bonds of different maturities have different yields. In this chapter for the purpose of calibration of the CDS model, we relax the assumption of constant interest rate required in the previous chapter. The risk free interest rate is taken as the US-treasury zero-coupon yield maturing closest the CDS contract expiration.
Let s_t be the bid CDS spread at time t for the maturity T and r_t the risk free interest rate at time t. For $\tau > t$, we have from (4.17)

$$\int_t^T e^{\gamma\left(\frac{s_t}{r_t} - \frac{s_t e^{-r_t(u-t)}}{r_t} - (1-R)e^{-r_t(u-t)}\right)} (C(u-t) + H(u-t)\lambda_t) e^{-B(u-t)\lambda_t} du \\ + e^{\frac{\gamma s_t}{r_t}\left(1 - e^{-r_t(T-t)}\right)} A(T-t) e^{-B(T-t)\lambda_t} = 1 \quad (5.1)$$

with

$$d\lambda_t = \alpha(\bar{\lambda} - \lambda_t)dt + \phi\sqrt{\lambda_t}dW_t$$

$$A(u-t) = \left\{ \frac{2\xi e^{(\alpha+\xi)(\frac{u-t}{2})}}{2\xi + (\alpha+\xi)(e^{\xi(u-t)} - 1)} \right\}^{\frac{2\alpha\bar{\lambda}}{\phi^2}}$$

$$B(u-t) = \frac{2(e^{\xi(u-t)} - 1)}{2\xi + (\alpha+\xi)(e^{\xi(u-t)} - 1)}$$

$$\xi = \sqrt{\alpha^2 + 2\phi^2}$$

$$C(u-t) = \frac{\alpha\bar{\lambda}}{\xi}\left(e^{\xi(u-t)} - 1\right) e^{\left(\alpha\bar{\lambda}(\alpha+\xi)\frac{(u-t)}{\phi^2}\right)} \left(\frac{1-\kappa}{1-\kappa e^{\xi(u-t)}} \right)^{\frac{2\alpha\bar{\lambda}}{\phi^2}+1}$$

$$H(u-t) = e^{\left((\alpha\bar{\lambda}(\alpha+\xi)+\xi\phi^2)\frac{(u-t)}{\phi^2}\right)} \left(\frac{1-\kappa}{1-\kappa e^{\xi(u-t)}} \right)^{\frac{2\alpha\bar{\lambda}}{\phi^2}+2}$$

and

$$\kappa = \frac{\alpha + \xi}{\alpha - \xi}$$

The equation (5.1) gives the mapping

$$1 = f(x_t, T; R, \beta) \tag{5.2}$$

$x_t = (t; s_t, r_t, \lambda_t)$ where λ_t is the value of the default intensity at time t; R is the recovery rate, $\beta = (\gamma, \alpha, \bar{\lambda}, \phi)'$ ($'$ is the transpose) and f is nonlinear function in β. We recall γ is the constant absolute risk aversion parameter of the protection buyer, $(\alpha, \bar{\lambda}, \phi)'$ is the CIR default intensity parameters. We are interested in the inverse problem i.e. given data x_t and for fixed maturity T and recovery rate R, find β such as

$$1 = f(x_t, \beta), \quad t \in \{1, 2, ..., n\} \tag{5.3}$$

Here n is the number of observations in the sample.
As we are dealing with the time series of default intensity, since we do not expect a perfect fit, we add an error term in (5.3) and we obtain the following nonlinear regression:

$$1 = f(x_t, \beta) + \epsilon_t \tag{5.4}$$

$\epsilon = (\epsilon_1, ..., \epsilon_n)'$ is the error vector with $E(\epsilon) = 0$ and covariance matrix $\Sigma_\epsilon = \sigma^2 I_n$ where I_n is the $n \times n$ identity matrix. In the model (5.4), $x_t = (t; s_t, r_t, \lambda_t)$ are the predictor variables and $\beta = (\gamma, \alpha, \bar{\lambda}, \phi)'$ is a vector of unknown parameters. Introducing an error term is justified since default intensities are not observed and are usually estimated from a sample of credit spreads or CDS spreads using bootstrapping or Nelson-Siegel techniques. This is usually done at the cost of some approximation errors.

For the estimation of the parameters β in the model (5.4), we follow two steps. First, based on the term structure of CDS spread for each time t, we estimate the default intensity λ_t. In the second step we use the estimates λ_t, the observations of CDS spreads and risk free interest rates to find by least squares the constant parameters γ, α, $\bar{\lambda}$ and ϕ.

Estimation approach for the time series of the default intensity λ

Let c_t be the short-term limit of the bid CDS spread at time t. From (4.9),

$$c_t = \lim_{T \to t} s_t = \frac{1 - e^{-(1-R)\gamma}}{\gamma} \lambda_t \tag{5.5}$$

We linearize the expression (5.5) with the approximation $e^x \cong 1+x$. Such an approximation is accurate only if x is in the neighborhood of zero, which is the case in the present context. In fact as mentioned in (1.2), the studies of Babcock et al. (1993), Ukhov (2002) and Guiso and Paiella (2008) show that the estimated constant absolute risk aversion γ is near zero and therefore it should be the same for $(1-R)\gamma$ since $0 \leq R < 1$. Using this approximation,

$$c_t \cong (1-R)\lambda_t \tag{5.6}$$

From (5.6), the default intensity λ_t are extracted from the short-term CDS spreads c_t. Two models are considered to find c_t. In the model I, we use the data of quoted CDS spread with the shortest maturity as proxy of c_t. In the model II, Nelson and Siegel (1987) model is fitted to the observed term stucture of CDS at each observation date t. The limit of the fitted curves when the maturity T goes to t gives rise to c_t.

Estimation approach for the parameters β

The estimated intensities are inserted in (5.4) and using the data of CDS spreads with an arbitrarily maturity T ($T = 5$ years for example) and spot rate for maturity T, we calibrate the model by minimization of sum of squared errors. The algorithm of the calibration is the following:
Let $t < u_1 < u_2 < ... < u_p = T$ are the premium payment dates for the CDS contract, $\Delta u_j = u_j - u_{j-1}$ ($u_0 = t$, usually $\Delta u_j = \Delta u = 0.25$) and $\beta = (\gamma, \alpha, \bar{\lambda}, \phi)'$ is defined in the parameter space $U_\beta \subset \mathbb{R}^4$.

First step: We calculate the integral in (5.1) by the trapezoidal rule, that is

$$f(x_t, \beta) \approx \Delta u \left[\frac{g(t,t) + g(t,T)}{2} \sum_{j=1}^{p-1} g\left(t, t + j\Delta u\right) \right] + h(x_t, \beta) \tag{5.7}$$

with

$$g(t,u) = e^{\gamma\left(\frac{s_t}{r_t} - \frac{s_t e^{-r_t(u-t)}}{r_t} - (1-R)e^{-r_t(u-t)}\right)} (C(u-t) + H(u-t)\lambda_t) e^{-B(u-t)\lambda_t}$$

and

$$h(x_t, \beta) = e^{\frac{\gamma s_t}{r_t}\left(1 - e^{-r_t(T-t)}\right)} A(T-t) e^{-B(T-t)\lambda_t}$$

Second step: We solve the optimization problems

$$\hat{\beta} = \arg\ \min_{\beta \in U_\beta} \sum_{t=1}^{n} (1 - f(x_t, \beta))^2 \tag{5.8}$$

subject to the constraint

$$\beta > 0$$

However there is not enough market data available to uniquely determine the value of β which minimises (5.8). Moreover the minimum value of a functional like (5.8) typically does not depend continuously on the data. Consequently the problem of determining β in this way is ill-posed. To understand what an ill-posed problem is we need to know the definition of a well posed problem. (Historically the definition is due to Hadamard).
A problem

$$F(x) = y,\ x \in X,\ y \in Y$$

is said to be well posed if

- For every $y \in Y$, there exists $x \in X$ such that $F(x) = y$
- For every $y \in Y$, there exists at most one $x \in X$ such that $F(x) = y$
- The solution x depends continuously on the data y

A problem is said to be ill posed if it is not well posed. Ill-posed problems are often dealt with using regularisation strategies which means the strategies convert an ill-posed problem into a reasonable well-posed problem. There are many approaches to constructing regularization strategies. Probably the best know is due to Tikhonov. (See Appendix F or Tikhonov (1963) for the original discussion). For the estimation of constant parameters in (5.8), the modified Gauss-Newton iterations for regularization are applied for minimizing the sum of squared residuals. The methodology is presented in the next section.

5.2 Nonlinear Regression

The basic idea of nonlinear regression is the same as that of linear regression, namely to relate a response y to a vector of predictor variables $(x_1, ..., x_k)$.

Nonlinear regression is characterized by the fact that the prediction equation depends nonlinearly on one or more unknown parameters. Whereas linear regression is often used for building a purely empirical model, nonlinear regression usually arises when there are physical reasons for believing that the relationship between the response and the predictors follows a particular functional form. A nonlinear regression model has the form

$$y_i = f(x_i, \beta) + \epsilon_i \;\; i = 1, ..., n \tag{5.9}$$

where the y_i are responses, f is a known function of a (row) vector of predictors $x_i = (x_{i1}, ..., x_{ik})$ and the vector of parameters $\beta = (\beta_1, ..., \beta_p)'$. The errors ϵ_i are usually assumed to be uncorrelated with mean zero and constant variance σ^2.

The unknown parameter vector β in the nonlinear regression model is estimated from the data by minimizing a suitable goodness-of-fit expression with respect to β. The most popular criterion is the sum of squared residuals

$$S(\beta) = \sum_i^n (y_i - f(x_i, \beta))^2$$

and estimation based on this criterion is known as nonlinear least squares. Differentiating $S(\beta)$,

$$\frac{\partial S(\beta)}{\partial \beta} = -2 \sum_i^n (y_i - f(x_i, \beta)) \frac{\partial f(x_i, \beta)}{\partial \beta}$$

Setting the partial derivatives to 0 produces estimating equations for the regression coefficients. Because these equations are in general nonlinear, they require solution using numerical optimization.

If the errors ϵ_i follow a normal distribution with mean 0 and variance σ^2, the likelihood of the nonlinear linear regression model is

$$L(\beta, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp \left\{ -\frac{\sum_i^n (y_i - f(x_i, \beta))^2}{2\sigma^2} \right\}$$

The likelihood is maximized when the sum of squared residuals $S(\beta)$ is minimized. Then the least squares estimator for β is also the maximum likelihood estimator.

The definition of nonlinearity relates to the unknown parameters and not to the relationship between the covariates and the response. For example the quadratic regression model

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \epsilon$$

is considered to be linear rather than nonlinear because the regression function is linear in the parameters β_j and the model can be estimated by using classical linear regression methods. Practical introductions to nonlinear regression including many data examples are given by Ratkowski (1983) and Bates and Watts (1988).

Except in a few isolated cases, nonlinear regression estimates must be computed by iteration using optimization methods to minimize the goodness-of-fit expression. The most popular iterative technique is the Gauss-Newton algorithm

5.2.1 Gauss-Newton Algorithm

If the function f is continuously differentiable in β, then it can be linearized locally as

$$f(x, \beta) = f(x, \beta^0) + X^0(\beta - \beta^0) \tag{5.10}$$

where $f(x, \beta)$ is a $n \times 1$ vector with elements $f(x_i, \beta)$ and X^0 is the $n \times p$ gradient matrix with elements $\frac{\partial f(x_i,\beta^0)}{\partial \beta_j}$ $, 1 \leq j \leq p$. This leads to the Gauss-Newton algorithm for estimating β:

$$\beta^1 = \beta^0 + (X^{0'} X^0)^{-1} X^{0'} e^0$$

where e^0 is the vector of working residuals $y_i - f(x_i, \beta^0)$.

Given β^w, set

$$\beta^{w+1} = \beta^w + (X^{w'} X^w)^{-1} X^{w'} e^w \tag{5.11}$$

and repeat until convergence. The Gauss-Newton algorithm increments the working estimate β at each iteration by an amount equal to the coefficients from the linear regression of the current residuals e on the current gradient matrix X. If the errors ϵ_i are independent and normally distributed, then the Gauss-Newton algorithm is an application of Fisher's method of scoring for obtaining maximum likelihood estimators. If X is of full column rank in a neighborhood of the least squares solution, then it can be shown that the Gauss-Newton algorithm will converge to the solution from a sufficiently good starting value. There is no guarantee, though, that the algorithm will converge from values further from the solution. However, it is possible to modify the Gauss-Newton algorithm in order to secure convergence. There are two ways in which a smaller step can be taken: line-search methods and Levenberg-Marquardt damping.

In the line-search algorithm, the sum of squared residuals will be reduced by the step from β^w to $\beta^w + \alpha(X^{w'}X^w)^{-1}X^{w'}e^w$, $\alpha > 0$, where α is sufficiently small. The line-search method consists of using one-dimensional optimization techniques to minimize the sum of squares with respect to α at each iteration. This method reduces the sum of squares at every iteration and is therefore guaranteed to converge unless rounding error intervenes.

In the Levenberg-Marquardt damping, the sum of squares will be reduced by the step from β^w to $\beta^w + (X^{w'}X^w + \lambda D)^{-1}X^{w'}e^w$, if λ is sufficiently large. The matrix D is usually chosen to be either the diagonal part of $X^{w'}X^w$ or the identity matrix. In practice, λ is increased as necessary to ensure a reduction in the sum of squares at each iteration, and is otherwise decreased as the algorithm converges to the solution. In other words, the Levenberg-Marquardt damping is equivalent to the Tikhonov regularization applied to the linearized problem

$$y - f(x, \beta^w) = X^w(\beta - \beta^w) \tag{5.12}$$

When $\lambda \to 0$, the Levenberg-Marquardt damping is reduced to the standard Gauss-Newton method. For more details about these algorithms, see Jennrich (1969).

5.2.2 Statistical Inference

Suppose that the ϵ_i are uncorrelated with mean zero and variance σ^2. Then the least squares estimators $\hat{\beta}$ are asymptotically normal with mean β and covariance matrix $\sigma^2(X'X)^{-1}$ where X is the gradient matrix. The variance σ^2 is usually estimated by

$$s^2 = \frac{1}{n-p}\sum_{i}^{n}(y_i - f(x_i, \hat{\beta}))^2 \tag{5.13}$$

Standard errors and confidence intervals for the parameters can be obtained from the estimated covariance matrix $s^2(X'X)^{-1}$ where X evaluated at $\beta =$

$\hat{\beta}$. In practice, the linear approximations that the standard errors and confidence intervals are based on can be quite poor especially when the sample size n is small and/or if the model is very nonlinear with respect to one or more of the parameters. Bates and Watts (1988) state: "We hasten to warn the reader that linear approximation regions can be extremely misleading". In addition to the concerns about nonlinearity in the model, there is another reason the linear approximation method may result in incorrect confidence regions. To calculate the gradient matrix X, people usually approximates it by the finite difference method. The use of finite differences for derivative gives very poor approximations of derivatives. Since the nonlinear regression modeling technique contains sources of errors that are largely unavoidable, residuals are often not close to 0, and finite difference derivatives should be used with caution. Thus, standard errors and therefore hypothese tests about the parameters computed with the linear approximation method may be compromised by highly nonlinear models and small data sets.

Hypotheses about the parameters can also be tested using F-statistics obtained from the extra sum of squares between the unrestricted and the restricted models. Suppose for example that

$$f(x, \beta) = \beta_1 \exp(-\beta_2 x) + \beta_3 \exp(-\beta_4 x)$$

and we wish to test $H_0 : \beta_4 = \beta_{4,0}$ against $H_1 : \beta_4 \neq \beta_{4,0}$, Let SS_1 be the residual sum of squares of the restricted model (i.e. with $\beta_4 = \beta_{4,0}$) and SS_2 the residual sum of squares of the unrestricted model. Then

$$F = \frac{SS_1 - SS_2}{s^2} \tag{5.14}$$

follows approximately an F-distribution on 1 and $n - p$ degrees of freedom, with here $p = 4$. This is closely analogous to the corresponding F-distribution result for linear regression. Tests and confidence regions based on the residual sum of squares are generally more reliable than tests or confidence intervals based on standard errors. Additionally, the F-statistics method does not require the approximation of derivatives, so this source of error is avoided. If the ϵ_i follow a normal distribution then tests based on F-statistics are equivalent to tests based on likelihood ratios.

5.3 Data and Results

5.3.1 Data

Our dataset consists of 22 observations of daily single-name bid CDS spreads covering the period from September 15th, 2008 to October 15th, 2008 for all corporates traded in various currencies and for maturities 6m, 1y, 2y, 3y, 4y, 5y, 7y, 10y, 15y, 20y, 30y. The data are downloaded from Markit, the industry standard provider in credit markets. We concentrate on the CDS spread data covering the period after the failure of Lehman Brothers to show how the estimation procedure works in practice, and how the results of estimation reflect the decrease of the global stock market during this period. For each day and for each obligor there is also a recovery rate reported that we use later in our analysis. Additional information like sector, rating and country are reported as well.

For each day and for each rating class, we calculate the average of the CDS spreads traded in US dollars, from the financials sector and for the recovery rate of 40%. We repeat this process for all observation dates and we get 22 observations of daily aggregated term structures of CDS spreads for credit classes AA, BBB, B. The interest rates r_t are constructed using the US-treasury zero yield data with maturity 5 years.

The sample of 22 observations is a bit limited regarding the highly nonlinearity of the indifference CDS model and therefore the hypothesis tests based on standard error or condidence interval must be analyzed with this caveat in mind. More precisely, the F-statistic is used to test the hypotheses about the parameters and this statistic should be more reliable since it does not require the gradient matrix. However the F-statistic rejects for all rating classes, the null hypothesis of a zero-valued parameter ϕ because the residual sum of squares of the restricted model goes to infinity. The calibration's algorithms and the statistical inference are implemented in Matlab and the function nlinfit is used for the Levenberg-Marquardt algorithm for nonlinear least square.

For the choice of starting values for the intensity's parameters, the estimates of Denault et al. (2009) for various rating classes are used. For the rating class AA, $\alpha_0 = 0.007$, $\bar{\lambda}_0 = 0.002$, $\phi_0 = 0.03$; for the rating class BBB, $\alpha_0 = 0.006$, $\bar{\lambda}_0 = 0.014$, $\phi_0 = 0.07$ and for the rating class B, $\alpha_0 = 0.02$, $\bar{\lambda}_0 = 0.42$, $\phi_0 = 0.24$. For the starting value of risk aversion coefficient, we take $\gamma_0 = 0.019$ as confirmed by Guiso and Paiella (2008) in (1.2).

5.3.2 Estimation Results of the Calibration Using the Default Intensities Extracted from the Model I

In this section, we use the CDS spread data with the shortest maturity as a proxy of the short-term limit c_t. Suppose that observations of the CDS spread s_t at time t are given for different maturities $T_1 < T_2 < ... < T_m$. The default intensity at time t is then given by

$$\lambda_t = \frac{s_t(T_1)}{1 - R}$$

In our data set, the shortest CDS spreads data available are those with the maturity for 6 months and there are used to generate the dynamic default intensities. We also use the observations of 5-years CDS spread to estimate the constant parameters because they are the most liquid and the most common credit derivatives in recent years. The tables 5.1, 5.2 and 5.3 present for the rating classes AA, BBB and B, the summary statistics of the time series for 6-months and 5-years CDS spreads. The analysis of the tables shows that the mean of the CDS spread increases with the maturity for the class AA and decreases with the maturity for classes BBB and B. This is expected since it was shown in the previous chapter a upward sloping curve for the high quality issuer and an inverted curve for the low quality issuer. In general, the standard deviation for the CDS spreads are small and increase when the credit quality decreases. Also the 6-months spreads are observed to be more volatile than the 5-years spreads in all credit classes and this is expected since the 5-years spreads are more liquid.

The tables 5.4, 5.5 and 5.6 present the least square estimates for the square-root intensity and the constant absolute risk aversion using the data of the CDS spreads and the spot rates. The tables report respectively for each class, the parameter estimates, the asymptotic standard error, the asymptotic condidence interval and the F-statistic based in difference of residuals sum of square. The estimates for γ, the risk aversion parameter are positive and increasing as credit quality is decreasing. First, the positivity of the estimated γ implies that during the subprime crisis the protection buyers of the CDS contract are risk averse. Second, the fact that the risk aversion parameter changes with the rating classes signifies that various categories of protection buyers invest in the CDS market. Those who are more risk averse prefer to buy the protection against the default of a low credit quality firm. Consequently, the liquidity of the CDS contract for the low credit quality firm will dry up. This is explained by the fact that, as the risk aversion increases, the protection buyers will seek to pay less the CDS spreads and the protection sellers will seek to demand more. In this situation the bid-ask spread will be high. Generally, this parameter is not estimated with a high precision. At the 5% level of significance, the F-test and the standard error suggest that the null hypothesis of a zero-valued parameter cannot be rejected for the class BBB. The mean reversion parameter α is estimated to be small for all credit classes and is increasing as credit quality is decreasing. It means that the speed of adjustment of the default intensity is high as the credit quality decreases. At the 5% level, the F-test suggests that α is significantly different from zero for all credit classes, whereas from the standard error α is not significant for the rating class B. The estimates for $\bar{\lambda}$, the long run average, are significant for all credit classes and the magnitudes also increase as credit quality decreases. This is expected and implies that the instantaneous rate of default is increasing as the credit quality is decreasing. The standard deviation parameter for the CIR intensity, ϕ, is small and significant in all cases except B as suggested by the standard error. The small parameter estimates for ϕ also imply that the default intensity is always positive since $\hat{\phi}^2 < 2\hat{\alpha}\hat{\bar{\lambda}}$.

The estimated time series for the unobserved intensities are plotted in figure 5.1. The graphs show a peak of the estimated intensities on September 17, 2008 (3 in the graph) for the credit class AA and on September 18, 2008 for the credit classes BBB and B. An increase for all rating classes is also observed in the period from September 26, 2008 (10 in the graph) to October 10, 2008 (20 in the graph). In this period, the firms rated BBB showed a peak in estimated intensities on October 2nd, 2008 (14 in the graph) whereas a peak was estimated on October 10, 2008 for the firms rated B. These high intensity periods are due to the decrease of the global stock market which is affected by the subprime mortgage crisis. Specially, the peaks around September 17, 2008 may be related to the annunciation of the banckruptcy of Lehman Brothers on September 15, 2008.

Figure 5.2 shows on October 14, 2008 for each class, the default probability $DP(t,T)$ curves implied by the estimated parameters.

$$DP(t,T) = 1 - A(T-t)e^{-B(T-t)\lambda_t} \tag{5.15}$$

As we expect the default probability increases when the credit quality decreases; the upward sloping shape signifies that the forecast of the credit quality of the reference entity over longer maturities is less certain.

Statistics	Maturity 6M	Maturity 5Y
Mean	0.0088	0.0125
Median	0.0081	0.0121
Std	0.0025	0.0017
Max	0.0128	0.0160
Min	0.0059	0.0104

Table 5.1: Summary Statistics of Aggregated CDS spreads for the rating AA

Statistics	Maturity 6M	Maturity 5Y
Mean	0.0459	0.0434
Median	0.0449	0.0429
Std	0.0114	0.0051
Max	0.0680	0.0532
Min	0.0327	0.0367

Table 5.2: Summary Statistics of Aggregated CDS spreads for the rating BBB

Statistics	Maturity 6M	Maturity 5Y
Mean	0.1353	0.12
Median	0.1250	0.1119
Std	0.0302	0.0247
Max	0.2026	0.1645
Min	0.0974	0.0921

Table 5.3: Summary Statistics of Aggregated CDS spreads for the rating B

Parameter	Value	Standard error	Confidence Interval 95%		F-test
			Lower	Upper	
γ	$2.63.10^{-5}$	$7.88.10^{-6}$	$9.74.10^{-6}$	$4.29.10^{-5}$	126.13
α	0.0158	$1.97.10^{-4}$	0.0153	0.0162	89.48
$\bar{\lambda}$	0.0135	$3.53.10^{-4}$	0.0127	0.0142	45.57
ϕ	0.0136	$4.96.10^{-4}$	0.0126	0.0147	-

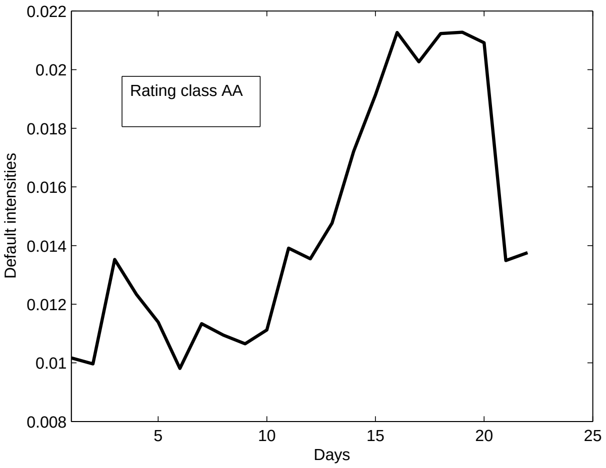
Table 5.4: Parameter estimates and statistical properties for *AA* class: Model I

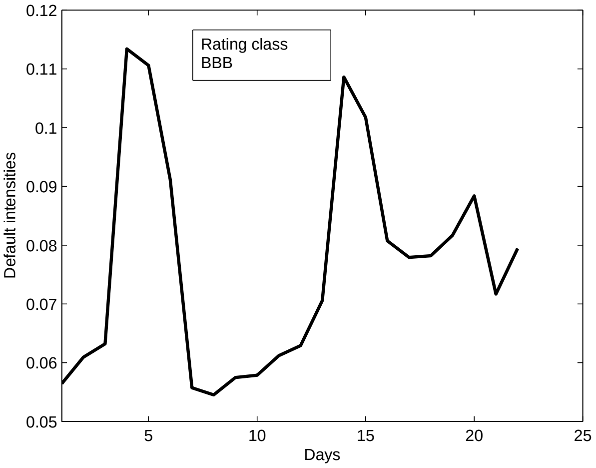
Parameter	Value	Standard error	Confidence Interval 95%		F-test
			Lower	Upper	
γ	$1.73.10^{-4}$	$4.16.10^{-4}$	$-7.02.10^{-4}$	0.0010	3.51
α	0.0950	0.0015	0.0919	0.0981	52.2
$\bar{\lambda}$	0.0607	0.0010	0.0586	0.0629	199.3
ϕ	0.0531	0.0014	0.0502	0.0560	-

Table 5.5: Parameter estimates and statistical properties for BBB class: Model I

Parameter	Value	Standard error	Confidence Interval 95%		F-test
			Lower	Upper	
γ	0.0015	0.0005	0.0004	0.0026	69.43
α	0.1306	0.0934	-0.0657	0.3268	99.7
$\bar{\lambda}$	0.1980	0.0413	0.1112	0.2848	188.38
ϕ	0.0035	5.2258	-10.9758	10.9828	-

Table 5.6: Parameter estimates and statistical properties for B class: Model I





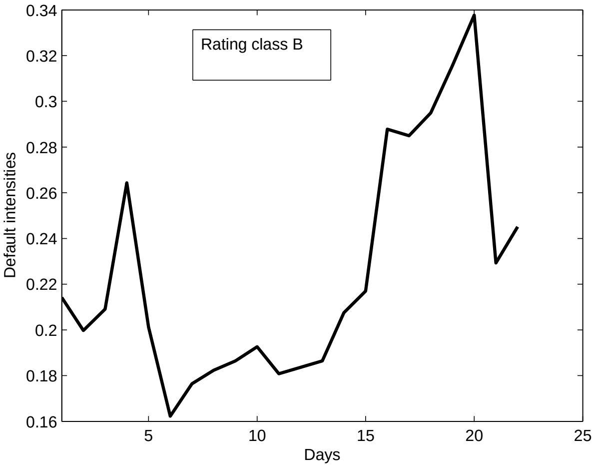


Figure 5.1: Implied Default Intensities from the model I

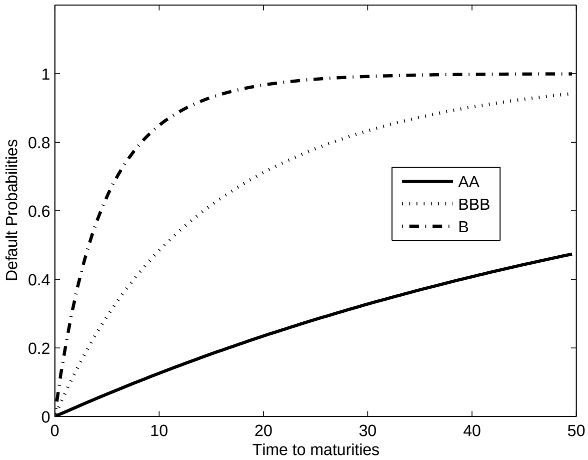


Figure 5.2: Term structure of Default Probability on 14.10.2008: Model I

5.3.3 Estimation Results of the Calibration Using the Default Intensities Extracted from the Model II

To get a smooth CDS spread curve, we suppose the CDS spread at time t for maturity T, $s_t(T)$, is a Nelson and Siegel (1987) function

$$s_t(T) = \beta_0 + (\beta_1 + \beta_2)\left(\frac{1-\exp(-\frac{m}{\tau})}{\frac{m}{\tau}}\right) - \beta_2 \exp\left(-\frac{m}{\tau}\right) \qquad (5.16)$$

where $m = T - t$.

This function can generate many different curve shapes. The parameter β_0 is the long term mean of the CDS spread. Parameter β_1 is the deviation from the mean, with $\beta_1 > 0$ implying a downward sloping and $\beta_1 < 0$ implying an upward sloping term structure. In addition, the reversion rate toward the long-term mean is negatively related to τ. The parameter β_2 is responsible for generating humps when it is different from zero. Bluhm et al. (2003) argue against using humps as this may lead to overfitting problems. We therefore assume that $\beta_2 = 0$ and therefore $s_t(T)$ has the following form.

$$s_t(T) = \beta_0 + \beta_1\left(\frac{1-\exp(-\frac{m}{\tau})}{\frac{m}{\tau}}\right) \qquad (5.17)$$

where $m = T - t$.

Using the term structure for the CDS spread at each time t, we estimate the parameters β_0, β_1 and τ using the Levenberg-Marquardt algorithm for nonlinear least square. For example, the figures 5.3, 5.4 and 5.5 show for the rating classes AA, BBB, B, the spread curves from fitting the Nelson-Siegel model and the quotes spreads on October 10th, 2008. From the fitted curve, we extract the short-term limit CDS spread c_t and the equation (5.6) gives the estimate of the default intensity λ_t. To find the short-term limit c_t, one takes the time to maturity of one day for the short-term horizon; that is $m = \frac{1}{250}$ is replaced in the fitted model. We repeat this process for all 22 observations of CDS spread curve to get 22 estimates for time series of λ_t.

The tables 5.7, 5.8 and 5.9 present the estimation results for the risk aversion coefficient and the CIR intensity parameters. The estimated parameters are qualitatively similar to those of the model I as the magnitude of all estimated parameters is increasing when credit quality is decreasing. In contrast to the model I, the F-test rejects the null hypothesis of zero value for all parameters whatever the rating classes AA, BBB, B. The results are similar for the standard error test except the rating class BBB for which the estimated risk aversion coefficient is not significant. Therefore, the statistical properties of the estimates are more supportive for the Nelson-Siegel interpolation approach. This may be explained by the fact that the data of 6-months CDS spreads which are only used to estimate the default intensity in the model I, are less liquid. When compared with the parameter estimates in the model I, the estimates obtained for the intensity parameters are generally higher for the classes BBB and B. These parameter estimates also imply positive default intensity since $\hat{\phi}^2 < 2\hat{\alpha}\hat{\bar{\lambda}}$.

The estimated time series for the unobserved intensities are plotted in figure 5.6. Generally, the estimated intensities show similar patterns to those obtained for the model I except that in the model II the peaks of the default intensities are higher for the classes BBB and B. Figure 5.7 shows on October 14, 2008 the default probability $DP(t,T)$ curves implied by the estimated parameters for each credit classes. The patterns are similar to those obtained in the case of the model I.

We also evaluate and compare the relative performances of the model I and II in estimating the time series of the default intensity. This is done by analyzing the root mean squared error (RMSE) and the mean absolute error (MAE) from the calibration of the indifference CDS model. The reason of doing that is the estimates of the default intensity are used as inputs in the calibration of the indifference CDS model. Thus a small RMSE and MAE would indicate a good performance of the model. The RMSE and the MAE are calculated from the following formulas

$$RMSE = \sqrt{\frac{1}{n}\sum_{t=1}^{n}\epsilon_t^2}$$

and

$$MAE = \frac{1}{n}\sum_{t=1}^{n}|\epsilon_t|$$

Results of the RMSE and the MAE using the models I and II are presented in 5.10 and 5.11. The results show that the model I presents the smallest RMSE and MAE for the rating class AA. This implies that the performance of the model I in estimating the default intensities for the class AA is better than that of the model II. In contrast, the smallest values of RMSE and MAE for the classes BBB and B are more supportive for the model II. In summary, both statistics support the Nelson-Siegel interpolation as the best model to generate the time series of the default intensity.

Finally, table 5.12 presents the average of physical default intensities of models I and II as well as the average of physical default intensity of Hull et al. (2005). In fact, Hull et al. (2005) estimated the real-world default intensity from statistics on average cumulative default rates published by Moody's between 1993 and 2003. As expected, the models I and II indicate an increase of the average of default intensities during the period following the failure of Lehman Brother.

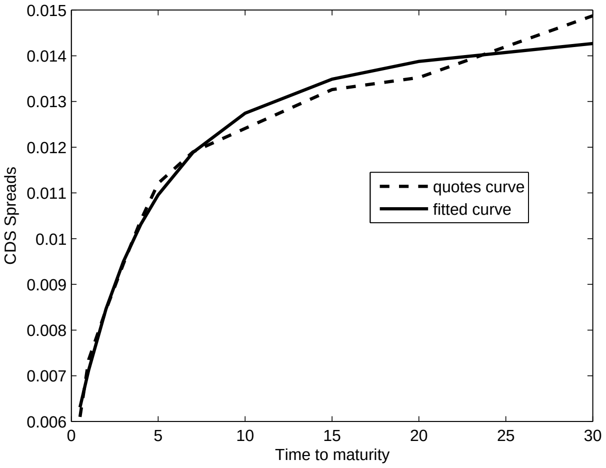


Figure 5.3: AA-rated observed CDS spreads and Nelson-Siegel fitted curve for October 10th, 2008.

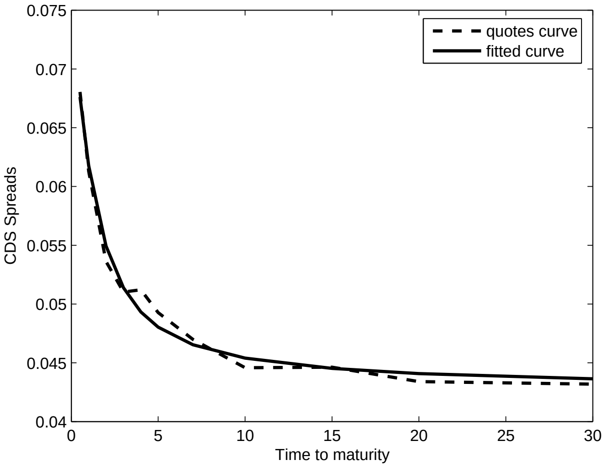


Figure 5.4: BBB-rated observed CDS spreads and Nelson-Siegel fitted curve for October 10th, 2008.

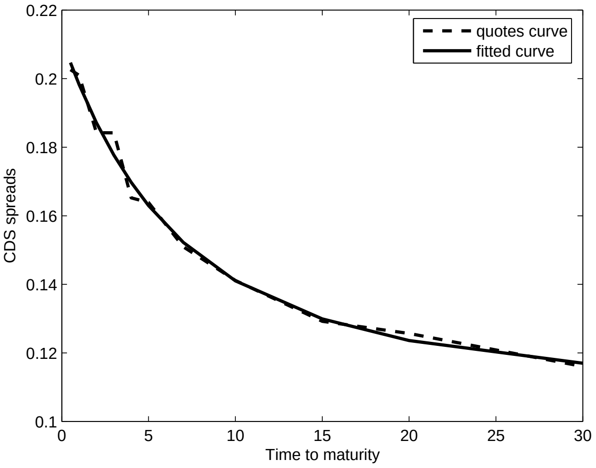


Figure 5.5: B-rated observed CDS spreads and Nelson-Siegel fitted curve for October 10th, 2008.

Parameter	Value	Standard error	Confidence Interval 95%		F-test
			Lower	Upper	
γ	$4.72.10^{-5}$	$1.36.10^{-6}$	$4.43.10^{-5}$	$5.01.10^{-5}$	10.73
α	0.0153	$2.85.10^{-4}$	0.0147	0.0159	53.86
$\bar{\lambda}$	0.0125	$6.49.10^{-4}$	0.0111	0.0138	22.59
ϕ	0.0123	0.0010	0.0102	0.0145	-

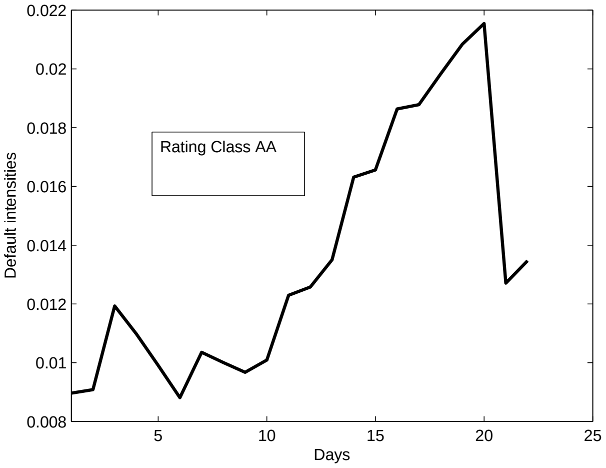
Table 5.7: Parameter estimates and statistical properties for *AA* class: Model II

Parameter	Value	Standard error	Confidence Interval 95%		F-test
			Lower	Upper	
γ	$1.88.10^{-4}$	$1.07.10^{-4}$	$-3.84.10^{-4}$	$4.14.10^{-4}$	104.69
α	0.0964	0.0021	0.0920	0.1008	22.3
$\bar{\lambda}$	0.0631	0.0030	0.0567	0.0694	57.4
ϕ	0.0550	0.0043	0.0461	0.0640	-

Table 5.8: Parameter estimates and statistical properties for BBB class: Model II

Parameter	Value	Standard error	Confidence Interval 95%		F-test
			Lower	Upper	
γ	0.0007	$2.90.10^{-4}$	0.0001	0.0013	257.90
α	0.3413	0.0305	0.2772	0.4055	18.84
$\bar{\lambda}$	0.1935	0.0084	0.1759	0.2112	108
ϕ	0.0781	0.0102	0.0568	0.0995	-

Table 5.9: Parameter estimates and statistical properties for B class: Model II



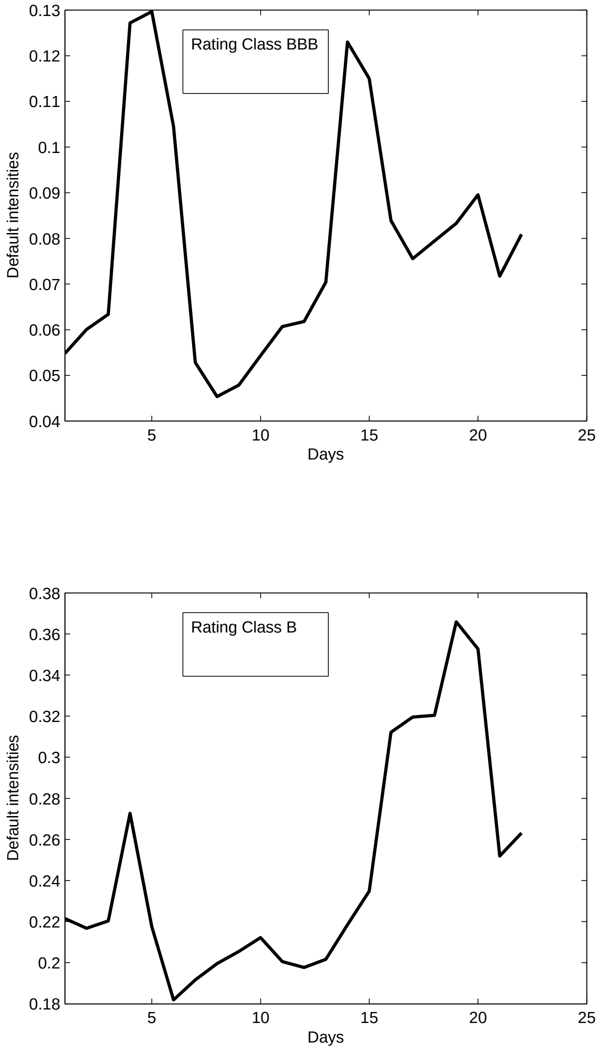


Figure 5.6: Implied Intensities from the Nelson-Siegel interpolation

Measures	AA	BBB	B
RMSE	$7.61.10^{-10}$	$4.47.10^{-7}$	$4.72.10^{-5}$
MAE	$5.89.10^{-10}$	$3.28.10^{-7}$	$3.78.10^{-5}$

Table 5.10: RMSE and MAE for the model I

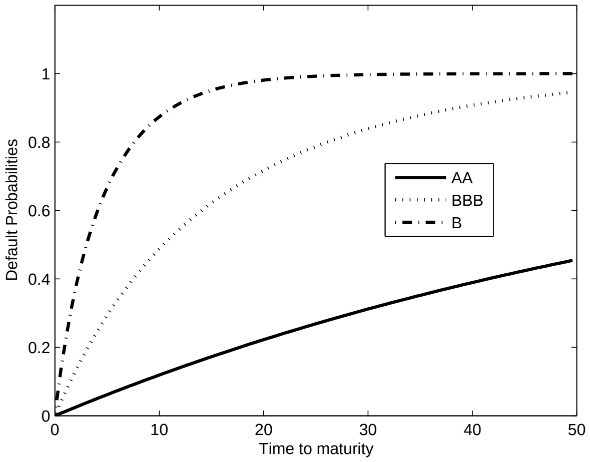


Figure 5.7: Term structure of Default Probability on 14.10.2008 : Model II

Measures	AA	BBB	B
RMSE	$1.25.10^{-9}$	$1.54.10^{-7}$	$2.28.10^{-6}$
MAE	$1.08.10^{-9}$	$1.21.10^{-7}$	$1.70.10^{-6}$

Table 5.11: RMSE and MAE for the model II

Credit classes	Hull et al. (2005)	Model I	Model II
AA	6	146	135
BBB	47	766	789
B	749	2254	2445

Table 5.12: Average physical default intensities in basis points

Chapter 6

Conclusion

The classical reduced models for credit risk were modified by embedding them in the framework of the utility based valuation. One reason of doing this, as mentioned in the introduction, is that the default intensity is a source of uncertainty that is not traded and the equivalent martingale measure is not unique. This leads to work with the utility indifference valuation in the context of defaultable instruments. In this situation a new hybrid model for credit risk is introduced since besides the uncertainty of the intensity, another correlated diffusion process which comes from the firm's stock price is used to drive the default instrument. In the first stage, we make the assumption as Sircar and Zariphopoulou (2007) that after the default event, the investor receives full pre-default market value of his stock and then invests the total wealth in the money market. In this context, two news parameters are introduced in the model, namely the risk aversion parameter of the investor and the correlation coefficient between the intensity and the stock price, which results in the nonlinearity of the pricing rule. The Hamilton-Jacobi-Bellman equations (HJB) verified by the values functions are derived, and are reduced to the nonlinear reaction-diffusion equations. The nonlinear partial differential equations were solved by the implicit finite difference scheme. The effects of the intensity, the risk aversion parameter and the correlation coefficient on the credit spread curves when analyzed. In the second stage, the independence between the intensity process and the firm's stock price process is assumed. This assumption greatly simplifies the modelling of defaultable bonds and the CDS spreads because the correlation coefficient between the stock price and the intensity process disappears from the model. Closed formula of the yield-spreads are obtained and numerical algorithms are developped for the CDS spreads. The asymptotic behavior of the spread curves as well as the impacts of the default intensity, the risk aversion and the recovery rate are also analyzed. Finally, the convergence of the indifference pricing model to the classical reduced model in the context of defaultable bond and CDS spreads are shown.

The estimation of physical CIR-intensity process and the constant absolute risk aversion is investigated. In a first step, the time series of unobserved default intensities are estimated from the short-term limit of the CDS spreads. In the second step, one estimates the intensity parameters and the risk aversion coefficient from the CDS model. The estimation shows consistent results with the mortgage subprime crisis.

Any attempt to improve the performance of the model should address the simplifying assumptions used here. First, the recovery of pre-market value of the stock in the event of default is problematic since in practice there is some loss or jumps in the stock price when the default occurs. One may extend to include some jumps in the stock price without the usual independence assumption between the firm's default point process and state variables; this will be done in further research. We suppose a constant risk free rate and therefore a non-dependence between the risk free rate and the default intensity. In a previous study, Garcia and Ginderen (2003) mention that there is a high correlation between the short-term risk free rate and the default intensity. One can include the stochastic interest rate to derive the indifference price for the defaultable bonds and the Credit Default Swap, but at high complexity and the calibration of the model may be difficult.

While the use of exponential utility is convenient for mathematical tractability, it assumes that the investor has constant absolute risk aversion, which empirically is not true in most cases. One very unsatisfying property of exponential utility is the fact that in the standard Merton investment problem, the amount invested in the risky asset is a constant and independent of the investor's intial wealth. In the case of power utility $u(x) = \frac{x^\gamma}{\gamma}$ and logarithmic utility $u(x) = \log(x)$, it is not the amount that is constant, but the fraction of total wealth invested in each tradable asset. This result is intuitively more appealing. Moreover, with the exponential utility function, indifference prices are independent of the initial wealth. In reality, prices should depend on the wealth of the company, as emphasized by Rouge and El Karoui (2000).

Appendix A

Ito's Formula for jump-diffusion processes

This appendix states Ito's Formula, allowing for jumps, and including some background properties of semimartingales. A standard source is Protter (2004). We first establish some preliminary definitions. We fix a complete probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t : t > 0\}$ satisfying the usual conditions:

- For all t, $\mathcal{F}_t$ contains all the null set of $\mathcal{F}$.
- For all t, $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s$, a property called right-continuity.

A function $X : [0, \infty) \to \mathbb{R}$ is left continuous if, for all t, we have $X_t = \lim_{s\uparrow t} X_s$; the process has left limits if $X_{t^-} = \lim_{s\uparrow t} X_s$ exists; and finally the process is right-continuous if $X_t = \lim_{s\downarrow t} X_s$. The jump ΔX of X at time t is $\Delta X_t = X_t - X_{t^-}$. The class of processes that are right-continuous with left limits is called RCLL, or sometimes " cadlag," for "continue à droite, limitée à gauche."

Under the usual conditions, we can without loss of generality for our applications assume that a martingale has sample paths that are almost surely right-continuous with left limits. See, for example, Protter (2004), page 8.

Lemma 1. *Suppose Q is equivalent to P with density ξ. Then an adapted process Y that is right-continuous with left limits is a Q-martingale if and only if ξY is a P-martingale.*

A process X is a finite-variation process if $X = U - V$, where U and V are right-continuous increasing adapted processes with left limits. For example, X is finite-variation if $X_t = \int_0^t \delta_s ds$, where δ is an adapted process such that the integral exists. The next lemma is a variant of Itô's Formula.

Lemma 2. *Suppose X is a finite-variation process and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then*

$$f(X_t) = f(X_0) + \int_{0^+}^t f'(X_{s^-})dX_s + \sum_{0<s\leq t} [f(X_s) - f(X_{s^-}) - f'(X_{s^-})\Delta X_s]$$

Like our next version of Itô's Formula, this can be found, for example, in Protter (2004), page 71.
A semimartingale is a process of the form $V+M$, where V is a finite variation process and M is a local martingale.

Lemma 3. *Suppose X and Y are semimartingales and at least one of them is a finite-variation process. Let Z=XY. Then Z is a semimartingale and*

$$dZ_t = X_{t^-}dY_t + Y_{t^-}dX_t + \Delta X_t \Delta Y_t \tag{A.1}$$

We now extend the last two lemmas. From this point, B denotes a standard Brownian motion in $\mathbb{R}^d$

Lemma 4. *Suppose $X = M + A$, where A is a finite variation process in $\mathbb{R}^d$ and $M_t = \int_0^t \sigma_u dB_u$ is in $\mathbb{R}^d$, where B is a standard Brownian motion in $\mathbb{R}^d$ and σ is an $\mathbb{R}^{d\times d}$ progressively-measurable adapted process with $\int_0^t \|\sigma_s\|^2 ds < \infty$ almost surely for all t. Suppose $f : \mathbb{R} \to \mathbb{R}$ twice continuously differentiable. Then*

$$\begin{aligned} f(X_t) &= f(X_0) + \int_{0^+}^t \nabla f(X_{s^-})dX_s + \frac{1}{2}\sum_{i,j}\int_0^t \partial^2_{i,j} f(X_s)\left(\sigma_s\sigma_s^{'}\right)_{i,j} ds \\ &+ \sum_{0<s\leq t} [f(X_s) - f(X_{s^-}) - \nabla f(X_{s^-})\Delta X_s] \end{aligned}$$

where $\Delta X_t = X_t - X_{t^-}$ is the jump of X at time t and $(\nabla f(x))_i = \frac{\partial f(x)}{\partial x_i}$, $\partial^2_{i,j} f(x) = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$

Lemma 5. *Suppose* $dX_t = dA_t + \sigma_t dB_t$ *and* $dY_t = dC_t + v_t dB_t$ *where* B *is a standard Brownian motion in* $\mathbb{R}^d$, *and where* A *and* C *is a finite variation processes, and* σ *and* v *are progressively measurable processes in* $\mathbb{R}^d$ *such that* $\int_0^t \sigma_s \cdot \sigma_s ds$ *and* $\int_0^t v_s \cdot v_s ds$ *are finite almost surely for all* t. *Let* $Z{=}XY$. *Then* Z *is a semimartingale and*

$$dZ_t = X_{t^-} dY_t + Y_{t^-} dX_t + \Delta X_t \Delta Y_t + \sigma_t \cdot v_t dt \tag{A.2}$$

Appendix B

Variational Results

We present some bounds for the value function M.

Proposition 3. *The value function M satisfies*

$$-e^{-\gamma x} \leq M(t,x,y) \leq -e^{-\gamma x - \frac{\mu^2}{2\sigma^2}(T-t)}, \tag{B.1}$$

Proof. To etablish (B.1), we first observe that the function $\bar{M}(t,x,y) = -e^{-\gamma x}$ verifies

$$\sup_\pi \left\{ \frac{E_t\left[d\bar{M}(t,x,y)\right]}{dt} \right\} = \frac{1}{2} e^{-\gamma x} \frac{\mu^2}{\sigma^2} \geq 0$$

and then is a subsolution of the HJB (3.15). Moreover $\bar{M}(T,x,y) = M(T,x,y)$. The lower bound then follows from the comparison principle.
Similarly, testing the function

$$\bar{M}(t,x,y) = -e^{-\gamma x - \frac{\mu^2}{2\sigma^2}(T-t)}$$

yields

$$\sup_\pi \left\{ \frac{E_t\left[d\bar{M}(t,x,y)\right]}{dt} \right\} = \lambda(y) e^{-\gamma x} \left(e^{-\frac{\mu^2}{2\sigma^2}(T-t)} - 1 \right) \leq 0$$

Therefore $\bar{M}$ is a supersolution, with $\bar{M}(T,x,y) = M(T,x,y)$, and the upper bound follows. $\square$

Appendix C

Proof of formula (3.39)

$$F(0,\lambda,T) = E(e^{-\int_0^T \lambda_s ds} | \lambda_0 = \lambda)$$

Cox et al. (1985) show that F satisfies the partial differential equation:

$$\frac{\phi^2}{2}\lambda F_{\lambda\lambda} + \alpha(\bar{\lambda} - \lambda_t)F_\lambda - \lambda F - F_T = 0 \tag{C.1}$$

subject to the boundary condition $F(0,\lambda,0) = 1$. Factorize $F(0,\lambda,t)$ as $F(0,\lambda,t) = A(t)e^{-B(t)\lambda}$ and substituting into (C.1), $A(t)$ and $B(t)$ satisfy the following equations,

$$B' + \frac{\phi^2}{2}B^2 + \alpha B - 1 = 0 \tag{C.2}$$

$$A' = \alpha\bar{\lambda}AB \tag{C.3}$$

subject to $A(0) = 1$ and $B(0) = 0$. The equation (C.2) is the well known Riccati equation. It can now be solved by introducing another dependent variable u such that

$$B = \frac{2}{\phi^2 u} \cdot \frac{\partial u}{\partial t} \tag{C.4}$$

Substituting (C.4) into (C.2) leads to the following homogeneous ordinary differential equation

$$\frac{\partial^2 u}{\partial t^2} + \alpha\frac{\partial u}{\partial t} - \frac{\phi^2 u}{2} = 0 \tag{C.5}$$

Solving (C.5), we finally get (3.41) and (3.42). The expression (3.40) is obtained by direct integration of (C.3).

Appendix D

Proof of theorem 4 and 5

We denote $E_t(\cdot) = E(\cdot|X_t = x, Y_t = y)$. For the bid CDS spreads, the value function at time t is

$$
\begin{aligned}
&H_b(t,x,y) = \sup_{\pi\in\mathcal{A}_{t,T}} E_t\left(-e^{-\gamma\left(X_T - s_b\int_t^T e^{-r(v-t)}1_{\{\tau_t>v\}}dv + (1-R)e^{-r(\tau_t-t)}1_{\{\tau_t\le T\}}\right)}\right)\\
&= M(t,x)\times E\left(e^{\gamma\left(s_b\int_t^T e^{-r(v-t)}1_{\{\tau_t>v\}}dv - (1-R)e^{-r(\tau_t-t)}1_{\{\tau_t\le T\}}\right)}|Y_t = y\right)
\end{aligned}
$$

and this is justified by the assumption (6).

$$
\begin{aligned}
& E\left(e^{\gamma\left(s_b\int_t^T e^{-r(v-t)}1_{\{\tau_t>v\}}dv - (1-R)e^{-r(\tau_t-t)}1_{\{\tau_t\le T\}}\right)}|Y_t = y\right)\\
= \;& E\left(E\left(e^{\gamma\left(s_b\int_t^T e^{-r(v-t)}1_{\{\tau_t>v\}}dv - (1-R)e^{-r(\tau_t-t)}1_{\{\tau_t\le T\}}\right)}|\mathcal{G}_\infty\vee\mathcal{H}_t\right)|Y_t = y\right)\\
= \;& 1_{\{\tau_t>t\}}E\left(\int_t^\infty e^{\gamma\left(s_b\int_t^T e^{-r(v-t)}1_{\{u>v\}}dv - (1-R)e^{-r(u-t)}1_{\{u\le T\}}\right)}\left(\lambda(Y_u)e^{-\int_t^u\lambda(Y_s)ds}\right)du|Y_t = y\right)\\
= \;& 1_{\{\tau_t>t\}}E\left(\int_t^T e^{\gamma\left(s_b\int_t^T e^{-r(v-t)}1_{\{u>v\}}dv - (1-R)e^{-r(u-t)}1_{\{u\le T\}}\right)}\left(\lambda(Y_u)e^{-\int_t^u\lambda(Y_s)ds}\right)du|Y_t = y\right)\\
+ \;& 1_{\{\tau_t>t\}}E\left(\int_T^\infty e^{\gamma\left(s_b\int_t^T e^{-r(v-t)}1_{\{u>v\}}dv - (1-R)e^{-r(u-t)}1_{\{u\le T\}}\right)}\left(\lambda(Y_u)e^{-\int_t^u\lambda(Y_s)ds}\right)du|Y_t = y\right)\\
= \;& 1_{\{\tau_t>t\}}E\left(\int_t^T e^{\gamma\left(s_b\int_t^u e^{-r(v-t)}dv - (1-R)e^{-r(u-t)}\right)}\left(\lambda(Y_u)e^{-\int_t^u\lambda(Y_s)ds}\right)du|Y_t = y\right)\\
+ \;& 1_{\{\tau_t>t\}}E\left(\int_T^\infty e^{\gamma\left(s_b\int_t^T e^{-r(v-t)}dv\right)}\left(\lambda(Y_u)e^{-\int_t^u\lambda(Y_s)ds}\right)du|Y_t = y\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 1_{\{\tau_t > t\}} E\left(\int_t^T e^{\gamma\left(\frac{s_b}{r} - \frac{s_b e^{-r(u-t)}}{r} - (1-R)e^{-r(u-t)}\right)} \left(\lambda(Y_u) e^{-\int_t^u \lambda(Y_s)ds}\right) du | Y_t = y\right) \\
&+ 1_{\{\tau_t > t\}} e^{\frac{\gamma s_b}{r}\left(1 - e^{-r(T-t)}\right)} E\left(\int_T^\infty \left(\lambda(Y_u) e^{-\int_t^u \lambda(Y_s)ds}\right) du | Y_t = y\right) \\
&= \int_t^T e^{\gamma\left(\frac{s_b}{r} - \frac{s_b e^{-r(u-t)}}{r} - (1-R)e^{-r(u-t)}\right)} 1_{\{\tau_t > t\}} E\left(\lambda(Y_u) e^{-\int_t^u \lambda(Y_s)ds} | Y_t = y\right) du \\
&+ e^{\frac{\gamma s_b}{r}\left(1 - e^{-r(T-t)}\right)} 1_{\{\tau_t > t\}} E\left(\int_T^\infty \left(\lambda(Y_u) e^{-\int_t^u \lambda(Y_s)ds}\right) du | Y_t = y\right) \\
&= \int_t^T e^{\gamma\left(\frac{s_b}{r} - \frac{s_b e^{-r(u-t)}}{r} - (1-R)e^{-r(u-t)}\right)} 1_{\{\tau_t > t\}} G(t, y, u) du \\
&+ e^{\frac{\gamma s_b}{r}\left(1 - e^{-r(T-t)}\right)} 1_{\{\tau_t > t\}} F(t, y, T)
\end{aligned}
$$

where $G(t, y, u) = E\left(\lambda(Y_u) e^{-\int_t^u \lambda(Y_s)ds} | Y_t = y\right)$ and $F(t, y, T) = E\left(e^{-\int_t^T \lambda(Y_s)ds} | Y_t = y\right)$. In fact, $1_{\{\tau_t > t\}} G(t, y, u)$ is the density of the default occuring at time u conditioned on $Y_t = y$ at time $t < u$ and $1_{\{\tau_t > t\}} F(t, y, T)$ is the probability of default not occuring in T years conditioned on $Y_t = y$ at time $t < T$. The bid CDS spread s_b at time 0 is such that $M(0, X_0) = H_b(0, X_0, Y_0)$ and s_b is the zero of the function

$$
\left\{
\begin{array}{rcl}
H(z) & = & \int_0^T e^{\gamma\left(\frac{z}{r} - \frac{z e^{-ru}}{r} - (1-R)e^{-ru}\right)} G(0, Y_0, u) du \\
 & + & e^{\frac{\gamma z}{r}\left(1 - e^{-rT}\right)} F(0, Y_0, T) - 1
\end{array}
\right.
\tag{D.1}
$$

For the ask CDS spreads, The value function at time t is

$$
\begin{aligned}
&H_s(t, x, y) = \sup_{\pi_t} E_t\left(-e^{-\gamma\left(X_T + s_s \int_t^T e^{-r(v-t)} 1_{\{\tau_t > v\}} dv - (1-R)e^{-r(\tau_t - t)} 1_{\{\tau_t \le T\}}\right)}\right) \\
&= M(t, x) \times E\left(e^{-\gamma\left(s_s \int_t^T e^{-r(v-t)} 1_{\{\tau_t > v\}} dv - (1-R)e^{-r(\tau_t - t)} 1_{\{\tau_t \le T\}}\right)} | Y_t = y\right)
\end{aligned}
$$

One can also show that

$$
\begin{aligned}
&E\left(e^{-\gamma\left(s_s \int_t^T e^{-r(v-t)} 1_{\{\tau_t > v\}} dv - (1-R)e^{-r(\tau_t - t)} 1_{\{\tau_t \le T\}}\right)} | Y_t = y\right) = \\
&= \int_t^T e^{-\gamma\left(\frac{s_s}{r} - \frac{s_s e^{-r(u-t)}}{r} - (1-R)e^{-r(u-t)}\right)} 1_{\{\tau_t > t\}} G(t, y, u) du \\
&+ e^{\frac{-\gamma s_s}{r}\left(1 - e^{-r(T-t)}\right)} 1_{\{\tau_t > t\}} F(t, y, T)
\end{aligned}
$$

The ask CDS spread s_s at inception time 0 is then the zero of the function,

$$\left\{\begin{array}{rcl} \tilde{H}(z) & = & \int_0^T e^{-\gamma\left(\frac{z}{r}-\frac{ze^{-ru}}{r}-(1-R)e^{-ru}\right)} G(0,Y_0,u)du \\ & + & e^{\frac{-\gamma z}{r}\left(1-e^{-rT}\right)} F(0,Y_0,T)-1 \end{array}\right. \tag{D.2}$$

To get the function in (4.7), we simply replace γ by $-\gamma$ in the expression of the function of (4.5).

Appendix E

Proof of formula (4.13)

Let

$$G(0,\lambda,t) = E\left(\lambda_t e^{-\int_0^t \lambda_s ds} \,|\lambda_0 = \lambda\right)$$

Duffie et al. (2000) show that F satisfies the partial differential equation:

$$\frac{\phi^2}{2}\lambda G_{\lambda\lambda} + \alpha(\bar{\lambda} - \lambda_t)G_\lambda - \lambda G - G_t = 0 \tag{E.1}$$

subject to the boundary condition $G(0,\lambda,0) = \lambda$. Factorize $G(0,\lambda,t)$ as $G(0,\lambda,t) = (C(t) + H(t)\lambda)e^{-B(t)\lambda}$ and substituting into (E.1), $C(t)$, $H(t)$ and $B(t)$ satisfy the following Ricatti equations

$$\begin{aligned} B' + \frac{\phi^2}{2}B^2 + \alpha B - 1 &= 0 \\ H' + H\left(\alpha\bar{\lambda} + \phi^2\right) + \alpha H &= 0 \\ C' + \alpha\bar{\lambda}C - \alpha\bar{\lambda}H &= 0 \end{aligned}$$

subject to the boundary conditions $B(0) = C(0) = 0$, and $H(0) = 1$. These equations can be solved as in Appendix A to get (4.14), (4.15) and (4.13).

Appendix F

Tikhonov Regularization for Ill-Posed Problems

In this appendix, we describe Tikhonov regularization for finding a stable approximate solution to a linear ill-posed problem represented in the form of an operator equation

$$Ax = y \tag{F.1}$$

where, instead of the exact data y, noisy data y_δ is available with

$$\|y - y_\delta\| \leq \delta \tag{F.2}$$

Here the operator A is a linear compact injective operator between Hilbert spaces X and Y. The solution x and data y belong to X and Y, respectively. The inner products in X and Y are denoted by $(.,.)_X$ and $(.,.)_Y$.
The problem (F.1) is ill-posed in the sense that the inverse A^{-1} of A exists but it is not continuous. Hence although, the problem (F.1) has a unique solution, solving directly will not give a right solution. Indeed, the linear operator A is so badly conditionned that any numerical attempt to directly solve (F.1) may fail.

In order to find a solution in stable manner, Tikhonov (1963) proposed to solve

$$x_\alpha = \arg\min_{w\in X} J_\alpha(w) = \|Aw - y_\delta\|_Y^2 + \alpha \|w\|_X^2 \tag{F.3}$$

where the regularization parameter α is found such that

$$\|Ax_\alpha - y_\delta\|_Y = \delta. \tag{F.4}$$

It can be easily shown that for every positive parameter α there exists a unique $x_\alpha \in X$ for which the functional J_α attains its minimal. Furthermore, it can be shown that there exists a positive value of α for which the condition (F.4) is satisfied.
The computation of the approximate solution x_α consists in solving the Euler equation corresponding to the fonctional J_α. This equation has the form

$$(A^*A + \alpha I)x_\alpha = A^*y_\delta, \tag{F.5}$$

where A^* is the adjoint operator of A and I is the identity operator. The regularization α satisfying the condition (F.4) can be optimally determined from L-curve method, GCV criterion, Morozov's discrepancy principle, etc. For more details about the choice of regularization parameter, see Morozov (1967) or Vainikko and Veretennikov (1986).

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