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UNPROVABILITY AND UNDEFINABILITY

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Dedication: for Robert McNaughton
logician, teacher, friend

1.0. There are many refinements, generalizations, and extensions of the Gödel unprovability results and of the Tarski undefinability results. This paper focuses on one core result of Gödel (along with the main lemma used in its proof) and on one core result of Tarski. Both results can be taken to concern *one* of the many formalizations of *one* of the many axiomatic formulations of the body of knowledge known as number theory (*number science* would be a more apt name). Each result has been understood as establishing a limitation on the classical axiomatic method of organizing scientific knowledge about the natural numbers. This paper will consider the question of the premises necessary to extend to *all* axiomatizations of number theory the core results established for *one* axiomatization. In order to clarify the significance, the concreteness, and the objective validity of these two results, it is necessary to remind ourselves of certain relevant mathematical, ontological, epistemological, and historical details.

Number theory, the science of whole numbers, can be conceived of in many ways; what exactly the various conceptions have in common is hard to say, it might be an important philosophical or mathematical question. Fortunately, for our purposes it does not seem to be needed. Here number theory is taken to be the science of natural numbers beginning with zero. But many choices could have been made; we could have started with one, or included the negative integers, or even the rational numbers. Number theory originated independently (as far as is known) in several different parts of the world: China, India, and Greece are often mentioned. Number theory existed as a science long before it was organized axiomatically, regardless of whether we regard one of the Greek treatments in the third century B.C. as a genuine axiomatization or whether we regard one of the nineteenth century treatments (e.g., Dedekind's or Peano's) as the first axiomatization.

Long before any axiomatic organization took place there was an accumulation of *theorems*, propositions known to be true. No square number is twice a square number. No square number is three times a square number. For any given number *n* exceeding two, the sum of the first *n* consecutive odd numbers is the square of *n*. For any given number n exceeding two the sum of the first consecutive powers of two starting with one is one less than the next power of two. For any given number *n*, the sum of the first *n* consecutive even numbers beginning with two is the product of *n* with *n* plus one. The proof of Gödel's unprovability results shows an astounding mastery of number theory; people have trouble following Gödel's proof because they don't realize that they need to learn number theory not only to understand what the result is but also to follow the thinking that yields the result. Tarski was also a master of number theory as is evident from his later work.

From its origins before the fifth century the body of number theoretic theorems continued to grow. Alongside the body of theorems the Greek mathematicians were careful to note *hypotheses*, propositions that were not yet known to be true and not yet known to be false. A person's mastery of number theory is measured as much by the hypotheses that he knows of and can discuss as it is by the theorems that he knows and can prove. The Greeks knew that every number is exceeded by a prime, i.e., that no matter how far out the sequence of numbers one counts more prime numbers will be found. Several similar propositions were hypotheses that challenged the Greek mathematicians: every number is exceeded by a prime whose second successor is prime, every number is exceeded by a perfect number. These two propositions are still open problems (Shanks 1962: 2, 30).

As far as we know rudimentary axiomatizations of parts of the body of then-known mathematics were attempted more than a century before Euclid who flourished around 300 B.C. But for centuries before the rudimentary axiomatizations, Greek mathematical activity embodied several practices that can be seen in retrospect to be essential to the axiomatic method. It should be no surprise that the components of the axiomatic method had been in use before being combined into the method.

One of the most important preaxiomatic activities was that of *deduction*, i.e., determining or establishing that one given proposition (the conclusion) is a logical consequence of a given set of propositions (the premise-set) or of a given proposition (the premise). Deduction was used in several ways. It was the main component of the *hypothetical method* of testing hypotheses by deducing from them consequences that are simpler and more amenable to various forms of direct verification or falsification. Hypothetical applications of deduction, i.e., the deducing of consequences from a premise-set that includes one or more propositions not known to be true, are mentioned by Plato in several places. At one point Plato seems to give Socrates credit for some sort of innovation connected with the hypothetical method. Two forms of deduction were recognized. Direct deduction unpacks the information contained in a set of premises step-by-step without adding anything for purposes of reasoning; this turns out to be the natural way to proceed in many cases where the premises and the conclusion are all affirmative. Indirect deduction shows that a given conclusion follows logically from a given premise-set by deducing a contradiction from the result of augmenting the premise-set by adding the contradictory opposite of the conclusion. The basic idea behind indirect deduction is that in order for a given conclusion to be implied by a given premise-set it is sufficient for the contradictory opposite of that conclusion to be contradicted by the premise set. Indirect deduction is often quite convenient when the conclusion is negative, e.g., to deduce «Zero does not divide itself» from «No number that divides itself is divided by the number zero» or «No number that the number zero divides is a number that divides itself».

In addition to its use in the hypothetical method, deduction was also used in connection with generating paradoxes and in connection with demonstration (Corcoran 1989). Some paradoxes result from deducing a proposition believed to be false from premises all believed to be true.

Another preaxiomatic activity was *demonstration*, using propositions already known to be true as a basis on which to show that a given hypothesis actually is true. For example, once the reader knows that for every number *n* the sum of the first *n* consecutive odd numbers is the square of *n*, other similar theorems can be proved. It is easy to see that the sum of the first *n* even numbers beginning with two can be obtained by adding one to each of the first *n* consecutive odd numbers and then adding the results. This shows that the sum of the first *n* even numbers beginning with two is the sum of *n* with the square of *n*, which is the same as the product of *n* with its own successor. Legend attributes to Thales the introduction into mathematical practice of the activity of demonstration. Whether Thales is too early or too late, it is nevertheless clear that demonstration had become well-entrenched before the middle of the sixth century, over a century before the earliest axiomatization, and over two centuries before anyone systematically studied axiomatization, or demonstration, or even deduction. All three of these activities were first systematically investigated by Aristotle who constructed a general theory of all three, a theory whose most general aspects are widely accepted today, and form an essential context for the core results due to Gödel and Tarski. It is surprising at first to discover that the axiomatizations, demonstrations, and deductions constructed and discussed by Gödel and Tarski in connection with the core results are much more in keeping with Aristotle's theories as presented in the *Prior Analytics* and *Posterior Analytics* than are those found in Euclid's *Elements* or even in Hilbert's *Foundations of Geometry* (Corcoran 1974).

In a sense Aristotle's *truth-and-consequence conception of demonstration* is his most profound contribution to this field of investigation, more because than in spite of the fact that it has so pervaded modern thinking as to become an inconspicuous part of the background, rather than a central part of the foreground. Aristotle's view was that a proposition is demonstrated, proved to be true, by showing that it is a *logical consequence* of propositions already known to be *true*. For Aristotle a demonstration begins with a set of propositions already known to be true and continues with a chain of reasoning that shows that the conclusion follows logically. Aristotle was perfectly clear that truth depends on *matter* as opposed to form; that a proposition is true or false in virtue of its subject matter. In fact Tarski traces his own correspondence theory of truth to Aristotle. At the same time Aristotle realized that logical consequence depends on *form* as opposed to matter. In order for a given proposition to be a logical consequence of a given set of propositions it is necessary and sufficient for the same to hold in every formally similar case (Corcoran 1974: 105). In so far as truth involves matter and consequence involves form, demonstration is a function of matter and form.

Aristotle also developed a conception of the method by which one given proposition is *shown* to be a logical consequence of a given set of propositions and he fully realized that this consequence aspect of demonstration is totally separable from the truth aspect. Knowledge of truth requires reference to matter. Knowledge of logical consequence requires reference to form. According to Aristotle, there are *immediate inferences* that can be made by humans. These correspond to what are today called logical rules, rules of inference, rules of deduction.

Inference of «Two is an even prime» from the two premises «Two is even» and «Two is a prime» is immediate as is inference of «Some prime is even» from the single premise «Two is an even prime». But inference of «Some prime is even» from the two premises «Two is even» and «Two is a prime» would have been considered by Aristotle to be non-immediate, i.e., mediated. Aristotle's *immediate-inference-chaining conception of deduction* is that in order to show that a given proposition is logically implied by a given set of propositions it is necessary and sufficient to construct a chain of immediate inferences, *either* a *direct deduction* which begins with the given set and ends with the given proposition *or* an indirect deduction which begins with the given set augmented by the contradictory opposite of the given proposition and ending with a proposition which is the contradictory opposite of one of the preceding. The fine details are not important. What is of crucial importance is that a demonstration is conceived of as a deduction whose premises are known to be true and that a deduction is conceived of as a chaining of immediate inferences.

This clears the way for formalization of deduction and thus for formalization of proof and of the axiomatic method. Aristotle's proofs are deductions and the deductions are chainings of propositions linked by formal (as opposed to material) inferences. Thus once the premises have been established it is no longer necessary (indeed no longer possible) to be involved with the subject matter in order to complete a proof. In particular, a deduction in arithmetic contains no arithmetic constructions and no computations, and a deduction in geometry contains no geometrical ruler-and-compass constructions and no superpositions. It should be added that the Aristotelian prohibition of constructions, diagrams, intuitions, etc. from *deduction* (i.e., from the deducing of consequences from premise-sets) was never intended to apply to the process of *induction* by which the premises are established nondemonstratively. For more on establishing the «ultimate» premises see the 1980 article by Jaakko Hintikka «Aristotelian Induction», which is reviewed in *Mathematical Reviews* (1982) 82m:00016.

We can use the expression *formal deduction* to indicate a deduction which proceeds from the initial premise-set to the conclusion by a series of formal steps, i.e., by steps which have to do solely with the logical forms of the propositions involved and which in particular are in no essential way dependent on knowledge of the matter. In this sense, Aristotle's immediate-inference-chaining conception is a theory of formal deduction as is the theory of deduction which is the basis of the *underlying logic* of the axiomatization of number theory constructed by Gödel. It is in this connection that Gödel's axiomatization fulfills the Aristotelian requirements to a much greater extent than does Euclid's axiomatization or indeed perhaps any axiomatization constructed before the modern period.

Aristotle's theory of the axiomatic organization of the sciences is presented in the *Posterior Analytics* which presupposes the theory of proof and the theory of deduction found in the *Prior Analytics*, but it goes beyond those two theories in several respects. In the first place, it recognizes the need to determine from among all of the concepts involved in the given science a small set of concepts (the *principle* concepts) in terms of which the other concepts may be defined. In addition, it recognizes the need to determine from among all of the theorems of the science a small set of propositions (the *principle* propositions) which, when augmented by suitable definitions, is sufficiently rich to serve as a premise-set from which all of the theorems of the science may be *recovered* by logical deduction. It went without saying that the set of principle concepts and the set of principle propositions were both *finite*; the science to be axiomatized was already in existence as a human construction (admittedly based on an objectively existent universe of discourse).

2.0. It would have been entirely possible for Aristotle to have produced an axiomatization of number theory not differing essentially from the (non-formalized) axiomatization of number theory that was to be formalized by Gödel...had Aristotle thought of it. The universe of discourse, as already indicated, is the set of natural numbers: 0, 1, 2, 3,.... The principle concepts are the concepts «zero», «successor» and «number». And the principle propositions, which are called the *Gödel Axioms*, the *Peano-Gödel Axioms* or even (anachronistically) the *Peano Axioms*, are as follows.

The Peano-Gödel Axiom Set

A1 No number has zero for its successor.
A2 Every two distinct numbers have respectively distinct successors.

A3 Every property that belongs to zero and to the successor of every number it belongs to belongs also to every number.

The first axiom, sometimes called *the zero axiom* amounts to the proposition that given any number *n*, the successor of *n* is not zero. The second axiom, sometimes called *the successor axiom*, amounts to the proposition that the successor function is one-to-one: given a number *n*, for any number *m*, if *m* is not *n*, then the successor of *m* is not the successor of *n*. By the way, arithmetic tradition dictates that 'equals' is used in the sense of the *is* of identity, but since this tradition does not extend to geometry it is sometimes misleading, especially in foundational discussions. Tarski's 1946 *Introduction to Logic* includes an interesting elaboration of this point. The third axiom is the principle of mathematical induction which became an integral part of modern thought as a result of the work of Blaise Pascal.

Contrary to the impression one gathers from various sources, principle concepts are not *ipso facto* undefinable in any ordinary sense of the word and, of course, it would be absurd to think that by dint of use as an expression for a principle concept, a previously meaningful expression (e.g., 'zero', 'successor', 'number') somehow becomes meaningless or somehow loses its status as constant and becomes a variable. The literature of this subject is replete with passages that betray lapses in precision, in objectivity, and in common sense (Keyser 1922: 54-59; Russell 1918: 5-9). In preformalization terminology it is said that a concept, e.g., the concept of zero, is definable in a given axiomatization if and only if a proposition giving a defining (necessary and sufficient) condition for it is provable (deducible from the axioms). For example, in order for «zero» to be definable in the above preformalized axiomatization, which is called *PGA* (Peano-Gödel Axiomatization), it is sufficient for the following proposition to be provable as a theorem.

DZ In order for a given number to be zero it is necessary and sufficient for the given number to not be the successor of any number.

The property of not being *the* successor of any number, or more briefly, the property of not being *a* successor, i.e., the property of being a non-successor, is a defining property of zero (Tarski 1944: 345). Proposition DZ amounts to the proposition that the property of being zero is *co-extensive* with the property of not being a successor. Thus DZ is logically equivalent to the proposition that the property of being zero and the property of being a successor are mutually exclusive and jointly exhaustive of the numbers. The proposition that they are mutually exclusive, i.e., «No number both is zero and is a successor (sc. of a number)» can be seen to be of course logically equivalent to the first axiom and is thus *a fortiori* provable as a theorem. The proposition that they are jointly exhaustive of the numbers, «Every number either is zero or is a successor», is not implied by the first axiom nor even by the first two axioms, as was known by Dedekind (1888), but it can be deduced from the third axiom alone. This joint exhaustiveness proposition can be construed as a universal proposition with a disjunctive predicate: «Every number has the property of being either zero or a successor». To see that this proposition follows from the principle of mathematical induction (axiom three) alone it is sufficient to follow a chain of individually trivial steps of reasoning. In the first place, it is clearly *tautologous* that zero has the property of either being zero or a successor because this follows *a fortiori* from the tautology that zero has the property of being zero, which in turn follows immediately from the tautologous identity that zero is zero.

It is likewise easy to see that it is tautologous that every successor of a number having this disjunctive property also has the property, i.e., that for any given number *n*, if *n* is either zero or a successor then the successor of *n* is either zero or a successor. Notice that the consequent of the imbedded conditional is itself tautologous. The two propositions just established form respectively the so-called basis step and induction step for an «application» of mathematical induction yielding the required conclusion--that every number is either zero or a successor (sc. of a number). Let us display the three theorems just obtained and label them.

T1 No number is both zero and a successor.
T2 Every number is either zero or a successor.
T3 Every number is such that it is zero if and only if it is a non-successor.

The third theorem amounts to the proposition DZ; the two are equivalent definitions of zero. As we have seen, then, «zero» is definable in the Peano-Gödel Axiomatization in terms of the concepts of (immediate) successor and (natural) number. This shows, as indicated above, that being a principle concept of an axiomatization does not *ipso facto* exclude being a definable concept of the same axiomatization. The real interest in definability lies of course with non-principle concepts.

The concept of one is defined by the property of being the successor of zero and by the property of being the successor of a non-successor. The concept of two is defined by the property of being the successor of one, and so forth for as many such concepts as one may consider.

DI One is the successor of zero.
DII Two is the successor of one.

The concept of precedence which is often said to give the natural ordering of the numbers is now known to be definable in terms of the principle concepts of the Peano-Gödel Axiomatization as a result of stunningly brilliant insights independently achieved by Dedekind (1888) and by Frege (1879). Here is the background of the insight. Let *p* be a number and let *q* be a number. In order for *p* to precede *q* it is sufficient for the successor of *p* to be *q* itself or for the successor of the successor of *p* to be *q* itself, or and so on. Dedekind and Frege discovered how to give logically rigorous expression to «and so on», i.e., to the definition of precedence pointed out by the above background remark.

DP In order for one given number to precede a second given number it is necessary and sufficient for the second to have every property belonging to the successor of the first and to the successor of any number to which it belongs.

Once the definition of precedence and a few other standard concepts of number theory had been constructed, it began to seem plausible to suspect that, without exception, absolutely every concept then-employed by number-theorists going back through Gauss, Fermat, Pascal and so on to Diophantus and the other early arithmeticians could indeed be defined in terms of the principle concepts of the Peano-Gödel Axiomatization. As a result of his own experience constructing definitions together with his encyclopedic knowledge of number theory, Peano was able to verify that every concept already in use in number theory in fact admitted of a definition in terms of zero, successor, and number. No comparable result had ever been achieved in the whole history of mathematics; a comprehensive conceptual basis of a science had been isolated.

Does this mean that every numerical concept is definable in terms of zero, successor, and number? Certainly not! Because of the finite complexity characteristic of the properties constructible from the principle concepts it was clear that the class of «definable» properties is countably infinite, i.e., that the class of «definable» properties is the same size as the set of natural numbers. In contrast, as a result of the work of Cantor it was known that there are uncountably many numerical properties, i.e., that there are more numerical properties than there are definable properties. Thus many, perhaps we should say "most", properties are not definable in terms of any given finite set of concepts and in particular in terms of the concepts of zero, successor, and number.

3.0. These results, positive and negative, carry over to the formalization of the Peano-Gödel Axiomatization which is the real topic of this paper. Before the formalization is considered it remains to discuss provability and unprovability in the unformalized or preformalized setting. Before turning to that however it is important to notice a difference between what it means to say that a given principle concept is definable in an axiomatization in terms of other principle concepts, on the one hand, and, on the other hand, to say that a given non-principle concept is definable in terms of the principle concepts. This important point, often overlooked, helps to avoid confusion. In order for a given principle concept to be definable in terms of other principle concepts in a given axiomatization it is necessary and sufficient for a proposition of a certain kind to be provable as a theorem in that axiomatization, more explicitly, for a proposition in the form of a definition of the given concept in terms of the other concepts to be provable from the axioms. Now in the case of the non-principle concept, proving such a proposition is out of the question because the propositions involving non-principle concepts in an essential way are not even in the running, so to speak. When it is said that a non-principle concept of number theory is definable in terms of «zero», «successor» and «number» what is meant is that a proposition in a suitable definitional form is a theorem of number theory itself, the preaxiomatic science, the science being axiomatized.

In accordance with the thinking involved in the above point there are two ways of treating the addition of definitions to an axiomatization. For many purposes it is technically convenient to regard the definitions being added not as «real» definitions of non-principle concepts but rather as «nominal» («stipulative» or «abbreviational») definitions of «new» words (regarded as antecedently meaningless). For example, on this approach in the above Peano-Gödel Axiomatization, DI would be understood as stipulating that the word 'one' is to be used as an abbreviation of the term 'the successor of zero'. In this way, the only numerical concepts ever occurring in theorems of this axiomatization are those of zero, successor, and number. This is the *definitions-as-abbreviations* approach.

The other approach takes the «new» words with their standard meanings. The sentences expressing the added definitions therefore express genuine propositions involving the «new» concepts and the principle concepts already in use. Thus, this approach takes the added definition to be a new axiom expanding the set of principle concepts of the axiomatization. This is the *definitions-as-axioms* approach favored by Tarski and the California logicians (Suppes 1957: 153).

In this paper the abbreviational approach to definition is adopted except where explicit indications to the contrary are given. This means that we take the propositions (whether true or false) relevant to the Peano-Gödel Axiomatization to be those whose universe of discourse is the class of numbers (the extension of the concept of number) and whose non-logical (material, mathematical) concepts other than «number» are only «zero» and «(immediate) successor». The *sentence* 'The successor of one is two' thus does not express a proposition containing the concepts «one» and «two» but rather is an abbreviation of the sentence 'The successor of the successor of zero is the successor of the successor of zero' which expresses a tautology. The proposition normally expressed by the sentence 'The successor of one is two' is not implied by the Peano-Gödel Axiom-Set, any more than is «The predecessor of Johnson was Kennedy» (cf. Keyser 1922: 57).

4.0. Once the class of propositions (true and false) relevant to the axiomatization is settled we can turn to the question of provability (in the axiomatization): is every (relevant) proposition of number theory provable in this axiomatization? In other words, is every proposition involving no concepts other than «zero», «successor» and «number» deducible from the PG Axiom-set? In still other words is the axiomatization as a whole *comprehensively complete*? This question concerns the entire axiomatization, its material aspect as well as its formal aspect, the mathematics *per se* as well as the logic *per se*, the axiom-set *and* the underlying logic. The question breaks down into two parts: one about the axiom-set alone and one about the logic alone. The first is whether the PG Axiom-set is rich enough in information to contain as logical consequences absolutely every relevant true proposition. The second is whether the human potentiality to perform logical deduction is rich enough to be able to deduce from a given premise-set each and every proposition that is a logical consequence of the premise-set. If the answer to the first question is affirmative then the axiom-set is *implication complete* in the sense that every relevant proposition is either implied or contradicted by the axiom-set itself. An implicationally complete set of propositions is so rich in information content that it implies every (relevant) proposition whose negation it does not imply and it implies the negation of every relevant proposition that is not one of its implications. If the answer to the second question is affirmative then the system of deductions of the underlying logic («our logic») is *deduction complete*.

In 1904 Oswald Veblen presented an unformalized axiomatization of Euclidean geometry using as principle concepts «point» and «between». He went on to show that his axiom-set is *categorical*, i.e., that any two reinterpretations of the non-logical constants (used to express the principle concepts) will be isomorphic to each other if they both satisfy the axiom-set. The only reason for mentioning categoricity here is that, because of the nature of logical form of propositions, every categorical axiom-set is implication complete. In Veblen's words, every relevant proposition is either implied by his axiom-set or is contradicted by his axiom-set. This means, as already indicated, that his axiom-set is as rich in information as could be desired--no true proposition could be irredundantly added (Corcoran 1980, 1981).

Veblen went on to raise the questions of whether the axiomatization as a whole (geometry plus logic) is comprehensively complete and he realized that his implication completeness result reduces the question of comprehensive completeness to the question of deduction completeness of the underlying unformalized logic. Veblen specially raised the question of whether for each proposition logically implied by his axiom-set there exists a finite chain of immediate inferences certifying the implication. More specifically, Veblen said that each relevant true proposition is implied by his axiom-set «even were it not deducible from the axioms by a finite number of syllogisms» (Veblen *1904*, 346n).

The Veblen implication completeness result concerning geometry is relevant to the concerns of this paper even though this paper focuses exclusively on axiomatizations of arithmetic. The Veblen result illustrates the fact that the distinction between implication and deduction had been made long before the work of Gödel and Tarski. It also illustrates the fact that deep and fascinating results concerning an *unformalized* axiomatization had been achieved. Veblen's axioms were stated in natural language augmented with a few mathematical symbols; no formal language was used much less described and nothing was said about the rules for making deductions--except that each deduction was a finite chain of immediate inferences. Moreover, in view of the fact that Veblen gave a proof about an axiomatization *per se* it is necessary for him to have used premises that were *about* the axiomatization and that were known to be true and it was necessary for him to have shown by a deduction that his conclusion was a consequence of his premises. Discussion of Veblen's premises would detract from the main theme of this paper.

Perhaps the most important reason for mentioning the Veblen Implication Completeness Result for geometry is to provide a parallel to the fact that the analogous result (or at least all hard mathematical work needed for it) had already been achieved for arithmetic as an outcome of the work of Dedekind and Cantor. And this is even before Veblen. In particular, this early work, which must have been known to Gödel and which certainly was known to Tarski, makes it clear that *the Peano-Gödel Axiom-set is categorical and therefore implicationally complete.*

There is another comparison which is well worth a small digression. As we saw above, it was found that every arithmetical concept employed by number theorists turns out to be definable in terms of «zero», «succession» and «number», the principle concepts of the Peano-Gödel Axiomatization. Veblen reported an analogous result for the principle concepts of his own axiomatization of geometry. In particular, he claimed that he proved that line congruence (or line equality) is definable in terms of «point» and «between». However, in the 1920s Tarski who was somewhat of an expert on definition, found fallacies in

Veblen's proof and then went on to prove himself, with the help of Lindenbaum, that the concept of line congruence (as in «line AB is congruent to line CD») is not definable in terms of the Veblen principle concepts. Thus there is a striking disanalogy between the Peano-Gödel Axiomatization and the Veblen Axiomatization. See Corcoran 1986 for a few more details on this curious episode in the history of the theory of definition.

5.0. In a sense Gödel's core result was designed to illuminate the nature of the axiomatic or deductive method. More particularly, again in a sense, it was designed to illuminate the nature of the Peano-Gödel Axiomatization and, thereby, to the extent that the Peano-Gödel Axiomatization is representative of axiomatizations of number theory, it was designed to make a statement about the scope and limitations of the axiomatic method. The method that Gödel used to establish his result was the standard method of mathematical proof--the truth-and-consequence method of proof: beginning with premises known to be true show by means of chaining of immediate inferences that the conclusion is logically implied by the premise-set.

Gödel's first step was to replace the vague and informal axiomatization by a mathematically precise axiomatization about which premises could be established. This first step itself involved three constructions. First, to model or express or represent the class of relevant propositions Gödel constructed a mathematically precise formalized language *L* of formal sentences about which various theorems could be established and in particular to which number theory could be applied. Second, to model or represent the class of deductions of the underlying logic of the PG Axiomatization Gödel constructed a mathematically precise system *D* of deductions, extended discourses, likewise amenable to treatment as mathematical objects. Third, to model or represent the mathematics or matter or content of the PG Axiomatization, Gödel constructed a set *G* of sentences of the language *L*.

In order to specify the language *L*, i.e., the set of sentences of the language *L*, Gödel presented a *sentential grammar*, which specified an alphabet (or vocabulary) of meaningless (but potentially symbolic) characters, a set of simple strings of characters as a basis and a set of rules of generation or transformation for constructing complex strings from simple strings. In order to specify the system D of deductions, Gödel presented a *discourse grammar* by specifying another set of rules for generating «complex deductions» from «simple deductions» which were simply concatenations of premises. For more details on sentential grammars and discourse grammars see my 1971 series of articles «Discourse Grammars and the Structure of Mathematical Reasoning». The ideas for the «logical structure» of the formal sentences and for the «logical structure» of the formal deductions was taken with modifications from *Principia Mathematica* which itself was taken from the underlying logics presupposed in mathematical practice including the work mentioned above by Veblen, Peano and Dedekind. Once a person sees the connection between the fully formalized underlying logic in the 1931 Gödel paper and the unformalized writings of Veblen, Dedekind and others it is amazing how easily the latter admits of being «translated into formalized form». Indeed, were it not the case that the Gödel Axiomatization had and has the appearance of being a formalization of *preexisting nonformalized mathematical practice*, no one would have had the motivation to attach to it the importance that it has attracted.

It is important to realize that the underlying logic that Gödel took in formalized form for the axiomatization of number theory that was to become so famous was a full unrestricted predicate logic in which the principle of mathematical induction can be expressed in full by means of one sentence, as is the case of course in normal unformalized mathematical discourse. It is also interesting to point out that before Gödel undertook the work that was to lead to his unprovability result involving «the» unrestricted predicate logic (i.e., higher order logic) he had proved the deduction completeness of «the» restricted predicate logic (i.e., lower predicate logic, first order logic). See Shapiro *1990*.

The axiom-set, properly so-called, of the Gödel Axiomatization is the set G consisting of three sentences from the formal language *L* which are obtained by «translating» or «formalizing» the Peano-Gödel Axiom Set. Thus *G* contains formal analogues of the zero axiom, the one-one axiom and the axiom of mathematical induction. This completes the specification of the Gödel Axiomatization which is denoted *GA*.

As the logic of the Gödel Axiomatization was taken from *Principia Mathematica* which at the time was the most modern codification of the formal aspect of traditional demonstrative practice in mathematics, the content of the Gödel Axiomatization was taken from the Peano work, which represented the most modern codification of the contentual aspect of the body of number-theoretic knowledge tracing back to ancient Asian and European cultures.

By the way the propositions that are called 'logical axioms' in an underlying logic are really not axioms in the mathematical sense but rather are zero-place rules of inference. A *two-place* rule of inference such as *modus ponens* or *barbara* lengthens a deduction on the basis of *two* lines already in the deduction. A *one-place* rule of inference such as *universal instantiation* or any of the *conversion* rules (e.g., going from «zero is not the successor of one» to «one is not the successor of zero») lengthen a deduction on the basis of *one* line already present. A zero-place rule such as *self-identity* (e.g., enter «zero is zero», «one is one», etc.) lengthens a deduction without preconditions. Overlooking this point has led to some tragic confusions as pointed out by Blumberg (1967, esp. 24-25) and others. In this paper 'axiom' is used in the sense of subject-matter axiom. Regardless of what they are called, zero-place rules or «logical axioms» are part of the underlying logic having to do with the formal aspect of demonstration whereas axioms properly so-called are part of the content of an axiomatization.

The Gödel Axiomatization, *GA*, of number theory consists of an underlying logic (with language *L* and deduction system *D*) together with the axiom-set *G* whose non-logical constants are the two characters *'O'* and *'s'*. The first of these, *'O'*, is an individual constant intended to denote the individual number zero and the second, *'S'*, is a one-place function constant intended to denote the one-place successor function. Thus *'sO'* the concatenation of *'s'* with *'O'*, is a string of length two which is intended to serve as a name of the number one: *'sO'* formalizes 'the successor of zero'. In the language *L* there are individual variables intended to range over the intended universe of discourse which is the class of natural numbers. There are property variables intended to range over the class of numerical properties (or, in extension, the class of sets of natural numbers), and there are property-property variables that range over the class of property properties (such as the property of belonging to countably many numbers which itself belongs to the property of being odd, the property of being even, etc.). There is no limit to the *order* of variables. As could be deduced from what has already been said, it was clear at the time that this axiomatization was sufficient for formalization of all of the then-known number theory together with the so-called arithmetization of analysis including integral and differential calculus.

Gödel's Core Result is that the *Gödel Axiomatization is not comprehensively complete*, i.e., that the combination of the axiom-set *G* with the underlying logic (involving the language *L* and the deduction system *D*) is not comprehensive enough to include for every sentence of *L* either a proof of the sentence itself or a proof of the negation of the sentence. This means that when we look at the set of all formal deductions in the underlying logic there will be a formal sentence *f* whose deductions and «antideductions» all use premises other than the three axioms in *G*. By a deduction of *f* is meant a deduction whose conclusion is *f* and by an antideduction of *f* is meant a deduction whose conclusion is the negation of *f*. Thus the Gödel Unprovability Result is not to the effect that some true *proposition* is not provable (by humans) but rather that some formal sentence has neither a formal deduction nor a formal antideduction using exclusively premises in *G*. From the point of view of the axiomatization *GA* «deductions» are certain concatenations of formal sentences, viz. those concatenations in *D*, and proofs are deductions whose premises are all in *G*: D provides the formal aspect of demonstration, *G* provides the material aspect or content.

There are two ways of looking at the class of formal deductions in *D*: we can choose a certain set *S* of sentences and look at the class of deductions each of whose premise-sets are subsets of *S* and we can choose a sentence *f* and look at the class of deductions having *f* or the negation of *f* as conclusion. There is the premise-oriented approach and the conclusion oriented-approach.

The conclusion-oriented approach was taken above when we noticed that the Gödel Unprovability Result implies that there exists a sentence *f* in the language *L* whose deductions all involve premises other than simply those in *G* and whose negation has the same property of being deducible only by deductions using premises other than those in G. Such sentences may be said to be *nonconcludable in GA*; Gödel called them 'undecidable' but this word has an entirely different use today as we will have occasion to notice below. To take the premise-oriented approach we can call a premise-set *inconclusive with respect to a deduction system* if there exists a sentence such that neither it nor its negation is the conclusion of a deduction in that system. From this point of view the Gödel Unprovability Result is simply that the axiom-set *G* is inconclusive with respect to the deduction system *D*.

6.0. As has been emphasized above the core result by Gödel, the Gödel Unprovability Result, is about a formalized axiomatization constructed by Gödel in order to study questions concerning axiomatization of number theory. Can we conclude from the Gödel Unprovability Result that the unformalized Peano-Gödel Axiomatization is not comprehensively complete? Can we conclude from the Gödel Unprovability Result that no comprehensively complete axiomatization of number theory is possible?

Before inquiring into the implications of the Gödel Unprovability Result it might be useful to inquire into its subject-matter to see whether we can locate more precisely the deficiency in the axiomatization (whose components include the formal language *L*, the deduction system *D*, and the axiom-set *G*). Could it be that the formalized language *L* contains beyond the formal sentences that can be used to express ordinary number-theoretic propositions such as those whose truth-values were determined by classical number-theorists other formal sentences which are artifacts of the construction and which do not correspond to any genuine propositions? This question, and variants of it, have led to interesting developments even though a deeper acquaintance with number theory and logic lead conclusively to a negative answer: there is no trouble with the formal language *L* and, in particular, it is not the case that bloating of the language with bizarre artifactual sentences is the cause of the trouble with the axiomatization.

The next component to be considered is the axiom-set *G*: could it be that *G* is deficient in information, i.e., that there are relevant propositions in number theory not implied by the set of propositions expressed by *G*? This question can likewise confidently and conclusively be answered in the negative as a result of work done years earlier by Dedekind and others. The fact that the Peano-Gödel Axiom-set is categorical and therefore *implication complete* carries over without difficulty to the formalized analogue. The formalized Gödel Axiom-set *G* is implication complete in the relevant sense, i.e., given a formal sentence *f* in *L*, either *f* is a logical consequence of *G* or the negation of *f* is a logical consequence of *G*. In view of the Gödel core result, this means that there is a sentence *f* in *L* which is logically implied by *G* but not deducible from *G* by means of a deduction in *D*. Thus some logical consequence of a set of sentences is not deducible from that set using deductions in *D*. In traditional terminology there are logically valid arguments which can not be seen to be valid by a deduction constructed in accordance with the rules of deduction that Gödel chose for his axiomatization. The deduction system of the Gödel Axiomatization is *deduction incomplete*.

Now the underlying logic of the Gödel Axiomatization contains as a sublogic a logic of first order, a logic of second order, a logic of third order, etc. Gödel had already shown the year earlier that the first-order sublogic of this unrestricted underlying logic *is* deduction complete. Thus, the deduction incompleteness must be found at one of the higher levels. Which one?

The Main Lemma of the Gödel core result, i.e., the lemma that culminates the chaining of reasoning and that yields more or less immediately the Unprovability Result, identifies by name a sentence, call it *g*, which expresses a proposition that can be proved to be true on the basis of premises that can be known to be true by *reflecting* on the axiomatization itself but which can not be known to be true by means of a formal proof concluding with *g*. This sentence g is a first-order sentence. The axiom-set *G*, known to imply *g*, is second-order. Thus, deduction incompleteness of the underlying logic enters at the lowest place compatible with known results: *Gödel's Second-Order Logic is deduction incomplete*.

Moreover, since the axiom-set *G*, which implies *g* but from which *g* is not deducible using Gödel's deduction system is finite, there is a single sentence logically equivalent to *G*, viz. the conjunction of the first axiom with the conjunction of the second with the third. Thus the conditional of this conjunction with the sentence *g* is a tautology which is not deducible in *D* from the null set and thus not «logically provable». This means that *D* is not complete in the weak sense. The above argumentation is due to Henkin (1950: 81).

As mentioned above Gödel's proof of his Main Lemma depends on applying number theory itself to the investigation of strings of initially meaningless characters; strings that once interpreted express propositions and proofs in number theory. The key to this application of number theory is based on the fact that the strings in question admit of being enumerated S0, S1, S2,...and thus each string is assigned a natural number. Once this is done then it is possible to imagine the formal sentences to be about the strings enumerated and not just about the numbers themselves. Tarski and Gödel thought of this strategy independently (Tarski *1983*: 278).

In the course of his proof Gödel proved that the property of being a number of a provable sentence admits of being defined in terms of zero, successor and number by a certain formula *PR(x)* which has exactly the variable *x* free. In other words, Gödel produced a formula *PR(x)* and then showed that for every number *n*, *n* satisfies the formula *PR(x)* under the intended interpretation if and only if *n* is the number of a string that occurs as the conclusion of a deduction whose premises are strings (sentences) in *G*, the Gödel Axiom-set. Tarski, as is well-known, showed how to define the string-theoretic property of being a sentence of a given formalized language expressing a true proposition and in particular (in effect) for *L* interpreted in the intended way. Then Tarski continued in a different direction to get what is called here *Tarski's Core Result*, viz. that the number-theoretic property of being a number of a string which expresses a true proposition is not definable on the basis of zero, successor and number, in other words no matter which formula *F(x)* from language *L* is considered, under the intended interpretation there is a number *n* which satisfies *F(x)* but which is the number of a «false sentence» or there is a number *n* which does not satisfy *F(x)* but which is the number of a «true sentence». In short, Tarski showed that any formula *F(x)* considered as a «definition» of «number of a sentence of L expressing a true proposition» is either too narrow or too broad.

Tarski himself realized sometime later that his core result is actually a lemma that yields the Gödel core result more or less immediately. One premise in the argumentation is that every sentence provable as a theorem in the Gödel Axiomatization expresses a true proposition and thus that every number satisfying *PR*(x) has the property *TR* of being the number of a sentence expressing a truth. But there is no formula that expresses *TR*. Thus the set of numbers satisfying *PR(x)* is not the extension of *TR*. Thus some number having *TR* does not satisfy *PR(x)*. Thus some sentence expressing a true proportion is not provable in *GA*.

Gödel's core result was originally obtained from a lemma that identifies a sentence that can not be known to express a true proposition of number theory by a formal proof in *GA* but which is known to be true by *reflecting* on the axiomatization itself. Likewise Tarski's core result identifies a numerical (or number-theoretic) property which can not be defined in terms of the principle concepts of Gödel's axiomation of number theory (and hence can not be defined in terms of the normally used concepts of number theory) but which can be defined in terms of concepts that arise naturally in *reflecting* on the axiomatization.

Gödel's proof is deduction of his core result from premises which are known to be true. If it should ever be determined that even one of these premises is not known to be true by the community of investigators then Gödel's deduction will no longer be regarded as a proof of its conclusion. Among these premises will be found the definition of truth discovered by Tarski a few years later and the basic laws of string theory also set forth later. Some of these points are discussed in the 1974 paper «String Theory» by Corcoran, Frank and Maloney.

7.0. In conclusion allow me to say a few words about what would be involved in generalizing Gödel's proof so that, instead of proving that the Gödel Axiomatization of number theory is not comprehensively complete, we would prove that no axiomatization of number theory is possible. In order to do this we will have to discover several things about axiomatizations which are either not known at all or else which have not been explained clearly enough to warrant wide acceptance. One overarching need is for specification of the string-theoretic (syntactic) property that a «system» must have in order to be suitable to serve as an axiomatization. In particular, if a formalized language L is in fact suitable for use in an axiomatization of a science, what string-theoretic condition does it satisfy in virtue of its being suitable for such use? In other words, what is the *axiomatic-language-suitability condition*? Again, if a formalized system of «derivations» is in fact suitable for use as the deduction system in the underlying logic of an axiomatization, what string-theoretic property does it have in virtue of this suitability? In other words, what is the *deduction-system-suitability condition*? Finally, if a set of formal sentences is suitable for use as an axiom-set, then what string-theoretic property does it have in virtue of its suitability? In other words, what is the *axiom-set suitability condition?*

Gödel seems to have changed his mind about these issues between the time he announced his core result in *1930* and the time he published it in *1931*. Gödel's views differ from those of Tarski *1969* and from the still different views expressed in Church *1956* and Corcoran *1969* and *1973*. The variety of views on just the axiom-set-suitability condition is remarkable and representative. It is clear from surveying the literature that before 1930 the idea that formal languages must consist in finite strings over a finite alphabet was virtually universal and there was likewise universal or near-universal agreement that an axiom-set must be finite. See for example the revealing remark in Presburger's 1930 paper which was based on work supervised by Tarski (Presburger *1930*, English 229, German 95). The idea of infinitely many axioms seemed to many mathematicians and logicians to be contrary to the intuitive concept of axiomatization and it still seems so. Gödel *1930* seems to condone the idea of infinitely many axioms as long as they are given in such a way that for any given number it is possible to prove as a theorem about the axiomatization a proposition to the effect that that number is (or is not) a number of an axiom whenever it in fact is (or is not) such a number. Tarski *1969* explicitly accepts an axiom-set whose set of numbers is definable by means of the principle concepts of the axiomatization. Today the overwhelming majority of mathematical logicians would easily accept as an axiom-set a set of formal sentences which is decidable in the modern sense, i.e., a set for which there exists an algorithmic procedure for determining of an arbitrary string whether or not it is in the proposed axiom-set (Mendelson *1987*: 28, 165). It is important to note that being decidable does not mean being known to be decidable; there are many decidable sets for which no algorithm has been constructed or for which no constructed algorithm is known to «work». The condition of definability (proposed by Tarski) and the condition of decidability (accepted by Mendelson and others) seems so remote from the traditional axiomatic method that further explanation may be demanded--and yet none had been offered, at least not by Tarski or Mendelson. This is one small indication that investigation into the foundations of the axiomatic method continues to challenge philosophers, logicians and mathematicians.

I hope that I have succeeded in showing how Aristotle's truth-and-consequence conception of proof and his immediate-inference-chaining conception of deduction paved the way for investigation of the phenomenon of axiomatization which led to formalized axiomatizations and to the studies of the inherent-limitations of formalized axiomatizations. In addition, it was my intention to show that the same investigations by Gödel and Tarski that led to the inherent limitation results also produced insights into the nature of axiomatization which made possible realization of potential for increased rigor already implicit in the traditional conception of axiomatization. Perhaps most importantly, I wanted to illustrate that in establishing results about the axiomatic method Gödel and Tarski both applied the truth-and-consequence conception of proof which was part of the basis of the axiomatic method. And finally, I hope that I showed that just as the activity of identifying the premises in proofs of number-theoretic propositions leads to axiomatization and to insights into the foundations of number theory, the activity of identifying the premises of the Gödel proof led to discoveries *and* the activity being carried out today of identifying and articulating the premises of the proofs by Gödel, Tarski and others continues to produce advances in understanding.

The most rigorous mathematical analogue of the traditional notion of axiomatization used above is the notion of *formalism* developed by Tarski and Givant in *1987*.

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