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Number Theory

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A SURVEY ON PRACTICAL NUMBERS

Abstract. A positive integer m is said to be practical if every integer $n \in (1, m)$ is a sum of distinct positive divisors of m. In this paper we give an equivalent definition of practical number, and describe some arithmetical properties of practical numbers showing a remarkable analogy with primes. We give an improvement of the estimate of the gap between consecutive practical numbers and prove the existence of infinitely many practical numbers in suitable binary recurrence sequences, including the sequences of Fibonacci, Lucas and Pell.

1. Introduction

A positive integer m is said to be *practical* (see [11]) if every n with $1 \le n \le m$ is a sum of distinct positive divisors of m. Several authors dealt with some aspects of the theory of practical numbers. P. Erdős [3] in 1950 announced that practical numbers have zero asymptotic density. B. M. Stewart [12] proved the following structure theorem: an integer $m \ge 2$, $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, with primes $p_1 < p_2 < \cdots < p_k$ and integers $\alpha_i \ge 1$, is practical if and only if $p_1 = 2$ and, for $i = 2, 3, \ldots, k$,

$$p_i \le \sigma\bigl(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{i-1}^{\alpha_{i-1}}\bigr) + 1$$

where $\sigma(n)$ denotes the sum of the positive divisors of n.

Let $P(x)$ be the counting function of practical numbers:

$$P(x) = \sum_{\substack{m \le x \\ m \text{ practical}}} 1 .$$

M. Hausman and H. N. Shapiro [5] showed in 1984 that

$$P(x) \ll \frac{x}{(\log x)^{\beta}}$$

for any $\beta < \frac{1}{2}(1 - 1/\log 2)^2 \simeq 0.0979$. M. Margenstern ([6], [7]) proved that

$$P(x) \gg \frac{x}{\exp\left\{\frac{1}{2\log 2}(\log\log x)^2 + 3\log\log x\right\}}.$$

G. Tenenbaum ([13], [14]) improved the above upper and lower bounds as follows:

$$\frac{x}{\log x}(\log\log x)^{-5/3-\varepsilon} \ll_\varepsilon P(x) \ll \frac{x}{\log x}\log\log x\ \log\log\log x.$$

Moreover, Margenstern conjectured that

$$P(x) \sim \lambda\,\frac{x}{\log x}$$

with $\lambda \simeq 1.341$, in analogy with the asymptotic behavior of primes.

The author [8] recently proved two Goldbach-type conjectures for practical numbers first stated in [6]: (i) every even positive integer is a sum of two practical numbers; (ii) there exist infinitely many practical numbers m such that $m-2$ and $m+2$ are also practical.

The purpose of the present paper is to survey some of the above results and to give some new contributions to the theory of practical numbers.

Sierpiński [10] and Stewart [12] independently remarked that a positive integer m is practical if and only if every integer n with $1 \le n \le \sigma(m)$ is a sum of distinct positive divisors of m. Here we give an alternative proof of this equivalence.

We also give an improved version of [8, Lemma 2], which yields a slightly simpler proof of the Goldbach-type result (i) mentioned above.

We study the gap between consecutive practical numbers, improving upon a result of Hausman and Shapiro [5].

Finally we prove that some binary recurrence sequences, including the classical sequences of Fibonacci, Lucas and Pell, contain infinitely many practical numbers. We incidentally note that it is unknown whether the Fibonacci sequence $\{1, 1, 2, 3, 5, \ldots\}$ and the Lucas sequence $\{1, 3, 4, 7, 11, \ldots\}$ contain infinitely many prime numbers. Dubner and Keller [2] recently announced the primality of some "titanic" (i.e. having more than 1000 digits) Fibonacci and Lucas numbers, such as F_{9311}, F_{5387}, L_{14449}, L_{7741}, L_{5851}, L_{4793}, L_{4787}.

2. An arithmetical result

In this section we give an equivalent definition of practical number. We begin with the following lemma:

LEMMA 1. *Let m be a positive integer, and let $d_1 = 1 < d_2 < \cdots < d_r = m$ be the positive divisors of m. Let d_h be the least divisor such that $d_h \geq \sqrt{m}$. Then $d_1 + d_2 + \cdots + d_{h-1} + 1 \leq m$.*

Proof. The lemma is true for $m = 1, 2, 3, 4$. Let $m > 4$; since $d_{h-1} < \sqrt{m}$ we have

$$\begin{aligned} d_1 + d_2 + \cdots + d_{h-1} + 1 &\leq 1 + 2 + 3 + \cdots + [\sqrt{m}] + 1 \\ &= \frac{[\sqrt{m}]([\sqrt{m}] + 1)}{2} + 1 \\ &\leq \frac{\sqrt{m}\,(\sqrt{m} + 1)}{2} + 1 \\ &< m. \quad \blacksquare \end{aligned}$$

LEMMA 2. (MARGENSTERN) *Let m be a positive integer, and let $d_1, \ldots, d_h, \ldots, d_r$ be as in Lemma 1. Then m is such that every n with $1 \leq n \leq \sigma(m)$ is a sum of distinct positive divisors of m, if and only if $d_{j+1} \leq d_1 + \cdots + d_j + 1$ for every $j = 1, \ldots, h-1$.*

Proof. For the proof see Margenstern's paper [7]. $\blacksquare$

PROPOSITION 3. *A positive integer m is practical if and only if every n with $1 \leq n \leq \sigma(m)$ is a sum of distinct positive divisors of m.*

Proof. Since $\sigma(m) \geq m$, if m is such that every n with $1 \leq n \leq \sigma(m)$ is a sum of distinct positive divisors of m, *a fortiori* m is a practical number.

Let m be practical, i.e. every n with $1 \leq n \leq m$ is a sum of distinct positive divisors of m. Let $d_1, \ldots, d_h, \ldots, d_r$ be as in the preceding lemmas. For any j satisfying $1 \leq j \leq h-1$ we have $d_1 + \cdots + d_j + 1 \leq m$ by Lemma 1. Hence $d_1 + \cdots + d_j + 1$ is a sum of distinct divisors of m, of which at least one must be $\geq d_{j+1}$. It follows that $d_{j+1} \leq d_1 + \cdots + d_j + 1$, whence, by Lemma 2, every n with $1 \leq n \leq \sigma(m)$ is a sum of distinct positive divisors of m. $\blacksquare$

3. The Goldbach problem for practical numbers

In this section we prove that every even positive integer is a sum of two practical numbers.

LEMMA 4. *If m is a practical number and n is an integer such that $1 \leq n \leq \sigma(m) + 1$, then mn is a practical number. In particular, for $1 \leq n \leq 2m$, mn is practical.*

Proof. The first assertion easily follows from Stewart's structure theorem; see also [7, p. 6]. Since $m-1$ is a sum of distinct divisors of m, we have $m+(m-1) \leq \sigma(m)$, i.e. $2m \leq \sigma(m)+1$, and this proves the second assertion. ∎

The author [8, Lemma 2] proved that if m and $m+2$ are practical numbers then every even integer $2n \in \left[m^2, 3m^2\right]$ is a sum of two practical numbers. This can be improved as follows:

LEMMA 5. *If m and $m+2$ are two practical numbers, then every even integer $2n$ with $\frac{1}{2}m^2 \leq 2n \leq \frac{7}{2}m^2$ is a sum of two practical numbers.*

Proof. We split up the interval $\left[\frac{1}{2}m^2,\, \frac{7}{2}m^2\right]$ into the union of three subintervals:

(i) $\left[\frac{1}{2}m^2,\, m^2\right[$;

(ii) $\left[m^2,\, 3m^2\right]$;

(iii) $\left]3m^2,\, \frac{7}{2}m^2\right]$.

(i) If $m=2$, the only even number contained in the interval $\left[\frac{1}{2}m^2,\, m^2\right[$ is 2, which is a sum of two practical numbers $(2=1+1)$. Suppose $m>2$ and let $2n \in \left[\frac{1}{2}m^2,\, m^2\right[$. If $2n=\frac{1}{2}m^2$ or $2n=\frac{1}{2}m^2+m$, we use the decompositions

$$\frac{1}{2}m^2 = m\left(\tfrac{1}{2}m-1\right)+m,$$
$$\frac{1}{2}m^2+m = m\left(\tfrac{1}{2}m-1\right)+2m.$$

Otherwise we can represent $2n$ as $\frac{1}{2}m^2+km+2j$ with $0 \leq k < \frac{1}{2}m$, $1 \leq j \leq \frac{1}{2}m$, $(k,j) \neq \left(0, \frac{1}{2}m\right)$. Then

$$2n = \tfrac{1}{2}m^2+km+2j = m\left(\tfrac{1}{2}m+k-j\right)+(m+2)\,j.$$

By Lemma 4, $2n$ is a sum of two practical numbers.

(ii) For the interval $\left[m^2,\, 3m^2\right]$ see [8, Lemma 2].

(iii) If $m=2$, the only even number contained in the interval $\left]3m^2,\, \frac{7}{2}m^2\right]$ is 14, which is a sum of two practical numbers $(14=6+8)$. Suppose $m>2$ and let $2n \in \left]3m^2,\, \frac{7}{2}m^2\right]$. We can represent $2n$ as $\frac{7}{2}m^2-km+2j$ with $1 \leq k \leq \frac{1}{2}m$, $1 \leq j \leq \frac{1}{2}m$. Then

$$2n = \tfrac{7}{2}m^2-km+2j = m(2m-k-j-3)+(m+2)\left(\tfrac{3}{2}m+j\right),$$

which is a sum of two practical numbers by Lemma 4. ∎

THEOREM 6. *Every even positive integer is a sum of two practical numbers.*

Proof. Since $(2,4)$, $(4,6)$, $(6,8)$ are pairs of twin practical numbers, by Lemma 5 every $2n \leq 126$ is a sum of two practical numbers. Suppose we have a sequence $\{m_n\}$ such that

(i) $m_1 = 16$

and for every n

(ii) m_n is practical

(iii) $m_n + 2$ is practical

(iv) $1 < m_{n+1}/m_n < \sqrt{7}$.

Since, by (iv), the intervals $\left[\frac{1}{2}m_n^2, \frac{7}{2}m_n^2\right]$ and $\left[\frac{1}{2}m_{n+1}^2, \frac{7}{2}m_{n+1}^2\right]$ overlap, every even positive integer $2n \geq 128$ is a sum of two practical numbers by Lemma 5. We shall construct a sequence $\{m_n\}$ satisfying (i), (ii), (iii) and a condition slightly stronger than (iv), i.e. $1 < m_{n+1}/m_n < 2$.

Let $S_0 = \{16, 30, 54, 88, 160\}$. For every $r \in S_0$, r and $r+2$ are practical numbers. Denote $S_0 = \{r_{0,1}, r_{0,2}, \ldots, r_{0,5}\}$ with $r_{0,1} < r_{0,2} < \cdots < r_{0,5}$. Note that $r_{0,i} < 2r_{0,i-1}$ $(i = 2,3,4,5)$ and $r_{0,5} = \frac{1}{2}r_{0,1}^2 + 2r_{0,1}$. Let $h_0 = 5$ and, for $k = 1, 2, \ldots$, define

$$\begin{aligned} S_k &= \left\{\tfrac{1}{2}r_{k-1,i}^2 + 2r_{k-1,i}\ ,\ r_{k-1,i}^2 + 3r_{k-1,i} \;\middle|\; i = 1,2,\ldots,h_{k-1}\right\} \\ &= \{r_{k,1}, r_{k,2}, \ldots, r_{k,h_k}\} \end{aligned}$$

with $r_{k,1} < r_{k,2} < \cdots < r_{k,h_k}$. Further let $S = \bigcup_{k=0}^{\infty} S_k$. If we write $S = \{m_n\}$, with $m_n < m_{n+1}$ for every n, one can see that $\{m_n\}$ satisfies (i), (ii), (iii) and $m_{n+1} < 2m_n$. The proof of this is similar to the argument given in [8, Theorem 1]. ■

4. k-tuples of twin practical numbers

It is easy to find infinitely many pairs $(m, m+2)$ of twin practical numbers (see the proof of Theorem 6 above). The following was conjectured in [6] and [7]:

THEOREM 7. *There exist infinitely many practical numbers m such that $m-2$ and $m+2$ are also practical.*

Proof. For the proof see [8, Theorem 2]. ■

It is shown in [7] that for any even $m > 2$, at least one of m, $m+2$, $m+4$, $m+6$ is not practical. However, we state the following

CONJECTURE 8. *There exist infinitely many 5-tuples of practical numbers of the form $(m-6, m-2, m, m+2, m+6)$.*

5. Gaps between practical numbers

Here we give an estimate of the gap between consecutive practical numbers. The same problem for primes has been extensively studied. If $\{p_n\}$ is the sequence of primes, R. C. Baker and G. Harman [1] recently proved that

$$p_{n+1} - p_n \ll p_n^{0.535},$$

the exponent 0.535 being of course replaced by $\frac{1}{2} + \varepsilon$ under the Riemann Hypothesis. If $\{s_n\}$ is the sequence of practical numbers, Hausman and Shapiro [5] proved that

$$s_{n+1} - s_n \le 2s_n^{1/2}.$$

We can improve this inequality as follows:

THEOREM 9. *Let $\{s_n\}$ be the sequence of practical numbers and let $A > 4e^{-\gamma/2}$, where γ is the Euler-Mascheroni constant. For any sufficiently large n we have*

$$s_{n+1} - s_n < A\,\frac{s_n^{1/2}}{(\log\log s_n)^{1/2}}.$$

Proof. Let $\delta > 0$ and $c < e^\gamma$ be such that $4c^{-1/2}(1+\delta)(1-\delta)^{-1/2} < A$. Let $N_k = \prod_{p \le e^k} p^k$, where p denotes a prime. By [4, §22.9] we have

$$\lim_{k\to\infty} \frac{\sigma(N_k)}{N_k \log\log N_k} = e^\gamma. \tag{1}$$

For every k, let $m^{(k)}$ be any integer such that $N_{k-1} | m^{(k)}$, $m^{(k)} | N_k$. It is easy to see, by induction on k, that N_k is practical for all $k \ge 1$, and if $k \ge 3$ then $m^{(k)}$ is also practical. To prove this, note that $N_1 = 2$ and $N_2 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2$ are practical, and $m^{(k)}/N_{k-1}$ is a product of primes not exceeding e^k. Since $e^k \le 2N_{k-1}$ for $k \ge 3$, $m^{(k)}$ and hence N_k are practical by repeated application of Lemma 4.

Since $n|m$ easily implies $\sigma(n)/n \le \sigma(m)/m$, we get

$$\frac{\sigma(N_{k-1})}{N_{k-1}\log\log N_k} \le \frac{\sigma\big(m^{(k)}\big)}{m^{(k)}\log\log m^{(k)}} \le \frac{\sigma(N_k)}{N_k \log\log N_{k-1}}.$$

Clearly

$$\log\log N_{k-1} \sim \log\log N_k,$$

whence, by (1),

$$\lim_{k\to\infty} \frac{\sigma\big(m^{(k)}\big)}{m^{(k)}\log\log m^{(k)}} = e^\gamma.$$

Thus there exists an integer k_0 such that for any $k \ge k_0$

$$\min_{\substack{m \\ N_{k-1}|m \\ m|N_k}} \frac{\sigma(m)}{m \log\log m} > c. \tag{2}$$

Let s_n be a practical number such that $s_n > c\,N_{k_0}^2 \log\log N_{k_0}$ and let κ be the least positive integer such that

$$N_\kappa \ge \frac{s_n}{c\,N_\kappa \log\log N_\kappa}.$$

Further, let

$$m_1^{(\kappa)} = N_{\kappa-1} < m_2^{(\kappa)} < \cdots < m_\lambda^{(\kappa)} = N_\kappa$$

be all the integers satisfying $N_{\kappa-1}|m_i^{(\kappa)},\ m_i^{(\kappa)}|N_\kappa,$ and let ν be such that

$$m_\nu^{(\kappa)} < \frac{s_n}{c\,m_\nu^{(\kappa)} \log\log m_\nu^{(\kappa)}} \tag{3}$$

and

$$m_{\nu+1}^{(\kappa)} \ge \frac{s_n}{c\,m_{\nu+1}^{(\kappa)} \log\log m_{\nu+1}^{(\kappa)}}. \tag{4}$$

Let ϑ and τ be defined by $m_\nu^{(\kappa)} = \vartheta N_{\kappa-1},\ N_\kappa = \tau m_\nu^{(\kappa)}$. Clearly $\tau > 1$. Let p'' be the least prime factor of τ, and let p' be the greatest prime $< p''$ (if $p'' = 2$, we let $p' = 1$). By Bertrand's postulate we have $p'' \le 2p'$. Since $N_\kappa = \vartheta\tau N_{\kappa-1}$, we have

$$\vartheta\tau = \left(\prod_{p \le e^{\kappa-1}} p\right)\left(\prod_{e^{\kappa-1} < p \le e^\kappa} p^\kappa\right),$$

whence $p'|\vartheta\tau,\ p'|\vartheta$, and $p'|m_\nu^{(\kappa)}$. Therefore

$$p'' \cdot \frac{m_\nu^{(\kappa)}}{p'} = p'' \cdot \frac{\vartheta}{p'} \cdot N_{\kappa-1}$$

is a multiple of $N_{\kappa-1}$. Moreover

$$N_\kappa = \tau m_\nu^{(\kappa)} = p' \cdot \frac{\tau}{p''} \cdot p'' \cdot \frac{m_\nu^{(\kappa)}}{p'}$$

is a multiple of $p'' m_\nu^{(\kappa)}/p'$. Hence

$$p'' \cdot \frac{m_\nu^{(\kappa)}}{p'} = m_i^{(\kappa)}$$

for some $i > \nu$, since $p'' > p'$. It follows that

$$m_{\nu+1}^{(\kappa)} \le p'' \cdot \frac{m_\nu^{(\kappa)}}{p'} \le 2m_\nu^{(\kappa)}. \tag{5}$$

Let $q = \left[s_n/m_{\nu+1}^{(\kappa)}\right] + 1$. By (2) and (4) we have

$$\begin{aligned} q &\le \frac{s_n}{m_{\nu+1}^{(\kappa)}} + 1 \\ &\le c\, m_{\nu+1}^{(\kappa)} \log\log m_{\nu+1}^{(\kappa)} + 1 \\ &< \sigma\left(m_{\nu+1}^{(\kappa)}\right) + 1, \end{aligned}$$

whence, by Lemma 4, $r = q\, m_{\nu+1}^{(\kappa)}$ is a practical number. Further

$$r - s_n = m_{\nu+1}^{(\kappa)} \left(\left[\frac{s_n}{m_{\nu+1}^{(\kappa)}} \right] + 1 \right) - s_n > 0,$$

whence, by (3) and (5),

$$\begin{aligned} s_{n+1} - s_n &\le r - s_n \\ &= m_{\nu+1}^{(\kappa)} \left(1 - \left\{ \frac{s_n}{m_{\nu+1}^{(\kappa)}} \right\} \right) \\ &< 2 \frac{s_n}{c\, m_\nu^{(\kappa)} \log\log m_\nu^{(\kappa)}}. \end{aligned}$$

For any $\varepsilon > 0$ and any sufficiently large n we have, by (3), (4) and (5),

$$m_{\nu+1}^{(\kappa)\,2} \ge \frac{s_n}{c \log\log m_{\nu+1}^{(\kappa)}} \ge s_n^{1-\varepsilon} \tag{6}$$

and

$$\begin{aligned} s_{n+1} - s_n &< 2 \frac{m_{\nu+1}^{(\kappa)}}{m_\nu^{(\kappa)}} \cdot \frac{s_n}{c\, m_{\nu+1}^{(\kappa)} \log\log m_\nu^{(\kappa)}} \\ &\le 4 \frac{c^{1/2} \left(\log\log m_{\nu+1}^{(\kappa)} \right)^{1/2}}{s_n^{1/2}} \cdot \frac{s_n}{c \log\log m_\nu^{(\kappa)}} \qquad (7) \\ &\le 4c^{-1/2}(1+\delta) \frac{s_n^{1/2}}{\left(\log\log m_\nu^{(\kappa)} \right)^{1/2}}. \end{aligned}$$

Since, by (5) and (6),

$$m_\nu^{(\kappa)} = \frac{m_\nu^{(\kappa)}}{m_{\nu+1}^{(\kappa)}} m_{\nu+1}^{(\kappa)} \ge \tfrac{1}{2} s_n^{(1-\varepsilon)/2},$$

we get

$$\log\log m_\nu^{(\kappa)} \ge \log \left(\frac{1-\varepsilon}{2} \log s_n - \log 2 \right) \ge (1-\delta) \log\log s_n,$$

whence, by (7),

$$s_{n+1} - s_n < A \frac{s_n^{1/2}}{(\log\log s_n)^{1/2}}. \qquad \blacksquare$$

REMARK. By Gronwall's theorem [4, Theorem 323] we have

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n \log\log n} = e^{\gamma},$$

which justifies the choice of the sequence N_k in our proof of Theorem 9.

6. Binary recurrence sequences

Let P, Q be non-zero integers; a pair of Lucas sequences $\{u_n(P,Q)\}$, $\{v_n(P,Q)\}$ is a pair of binary recurrence sequences defined as

$$\begin{cases} u_0(P,Q) = 0 \\ u_1(P,Q) = 1 \\ u_n(P,Q) = P\,u_{n-1}(P,Q) - Q\,u_{n-2}(P,Q) \quad \text{for } n \geq 2 \end{cases}$$

and

$$\begin{cases} v_0(P,Q) = 2 \\ v_1(P,Q) = P \\ v_n(P,Q) = P\,v_{n-1}(P,Q) - Q\,v_{n-2}(P,Q) \quad \text{for } n \geq 2. \end{cases}$$

The sequence $\{u_n(P,Q)\}$ is also called a *fundamental Lucas sequence* and $\{v_n(P,Q)\}$ its *companion sequence*.

Suppose $P^2 - 4Q \neq 0$ and let α, β be the distinct roots of the polynomial

$$x^2 - Px + Q.$$

We have

$$u_n(P,Q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$v_n(P,Q) = \alpha^n + \beta^n.$$

Using a shorter notation, we shall write u_n and v_n instead of $u_n(P,Q)$ and $v_n(P,Q)$. For $(P,Q) = (1,-1)$, u_n and v_n are the sequence of Fibonacci numbers and the sequence of Lucas numbers, respectively; for $(P,Q) = (2,-1)$, u_n is the sequence of Pell numbers [9, p. 56].

THEOREM 10. *Let $\{u_n(P,Q)\}$ be a fundamental Lucas sequence. If $P^2 - 4Q > 0$ and $PQ + P$ is even, then the sequence $\{|u_n(P,Q)|\}$ contains infinitely many practical numbers.*

Proof. We shall prove that, for sufficiently large k, $|u_{3\cdot 2^k}|$ is a practical number. Let $\{v_n\}$ be the companion sequence of $\{u_n\}$. Since $u_{2m} = u_m v_m$ for every m, we have, for $k > 0$,

$$u_{3\cdot 2^k} = u_3 \cdot \prod_{h=0}^{k-1} v_{3\cdot 2^h}.$$

Also, $P^2 - 4Q > 0$ implies $u_3 = P^2 - Q > 0$. Note that $v_3 = P(P^2 - 3Q)$, whence $\operatorname{sgn} v_3 = \operatorname{sgn} P$. Since $P^2 - 4Q > 0$, we have $\alpha, \beta \in \mathbb{R}$, whence $v_n = \alpha^n + \beta^n$ is positive for n even. Therefore

$$|u_{3\cdot 2^k}| = u_3\, |v_3| \cdot \prod_{h=1}^{k-1} v_{3\cdot 2^h}.$$

Since $PQ + P$ is even, v_{3m} is even for all m. Denoting $v'_{3m} = v_{3m}/2$, we have

$$|u_{3\cdot 2^k}| = 2^k\, u_3\, |v'_3| \cdot \prod_{h=1}^{k-1} v'_{3\cdot 2^h}.$$

Let $2^{k+1} \geq \max\{u_3, |v'_3|\}$, and define $u_j^* = 2^k\, u_3\, |v'_3| \cdot \prod_{h=1}^{j-1} v'_{3\cdot 2^h}$. We show, by induction on j, that u_j^* is practical for $j = 1, \ldots, k$. For $j = 1$ this follows from Lemma 4 applied twice, since 2^k is practical and $u_3, |v'_3| \leq 2^{k+1}$. Let $1 \leq j \leq k-1$, and assume that u_j^* is practical. We have

$$u_j^* = 2^{k-j}\, |u_{3\cdot 2^j}|$$

and

$$u_{j+1}^* = u_j^*\, v'_{3\cdot 2^j},$$

where

$$v'_{3\cdot 2^j} = \tfrac{1}{2} v_{3\cdot 2^j} = \tfrac{1}{2}\left(\alpha^{2^j} + \beta^{2^j}\right)\left(\alpha^{2^{j+1}} - \alpha^{2^j}\beta^{2^j} + \beta^{2^{j+1}}\right).$$

Note that

$$\alpha^{2^j} + \beta^{2^j} = v_{2^j}$$

and

$$\alpha^{2^{j+1}} - \alpha^{2^j}\beta^{2^j} + \beta^{2^{j+1}} = v_{2^{j+1}} - Q^{2^j}$$

are positive integers (not both odd). In order to prove that u_{j+1}^* is practical, by Lemma 4 applied twice it suffices to show that

$$M = \max\left\{\alpha^{2^j} + \beta^{2^j},\ \alpha^{2^{j+1}} - \alpha^{2^j}\beta^{2^j} + \beta^{2^{j+1}}\right\} < u_j^*.$$

Since $x + y \leq x^2 - xy + y^2 + 1$ for all $x, y \in \mathbb{R}$, we have

$$\begin{aligned} M &\leq \alpha^{2^{j+1}} - \alpha^{2^j}\beta^{2^j} + \beta^{2^{j+1}} + 1 = v_{2^{j+1}} - Q^{2^j} + 1 \\ &< v_{2^{j+1}} + Q^{2^j} = \alpha^{2^{j+1}} + \alpha^{2^j}\beta^{2^j} + \beta^{2^{j+1}}. \end{aligned}$$

From $P^2 - 4Q > 0$ and $P = \alpha + \beta \neq 0$ it follows that $\alpha \neq \pm\beta$. Therefore

$$u_{2^j} = \frac{\alpha^{2^j} - \beta^{2^j}}{\alpha - \beta} \neq 0,$$

i.e. $|u_{2^j}| \geq 1$. Hence

$$\begin{aligned} M &\leq |u_{2^j}| \left(\alpha^{2^{j+1}} + \alpha^{2^j}\beta^{2^j} + \beta^{2^{j+1}}\right) \\ &= \left|\frac{\alpha^{3\cdot 2^j} - \beta^{3\cdot 2^j}}{\alpha - \beta}\right| = |u_{3\cdot 2^j}| < 2^{k-j}\,|u_{3\cdot 2^j}| = u_j^*. \qquad \blacksquare \end{aligned}$$

THEOREM 11. *Let $\{v_n(P,Q)\}$ be a companion Lucas sequence with $Q = -1$ and $P > 0$. If there exists a positive integer t such that v_{35t} is practical, then $\{v_n\}$ contains infinitely many practical numbers.*

Proof. We shall prove by induction that, for every $k \geq 0$, $v_{3^k 35t}$ is practical. For $k = 0$ this is true by assumption. Suppose that $v_{3^k 35t}$ is practical for some k. Since $v_n = \alpha^n + \beta^n$, where α and β are the roots of the polynomial $x^2 - Px + Q$, we have

$$v_{3^{k+1}35t} = v_{3^k 35t}\left(\alpha^{3^k 70t} - \alpha^{3^k 35t}\beta^{3^k 35t} + \beta^{3^k 70t}\right).$$

Define

$$\Phi_d(x,y) = \begin{cases} x^{\varphi(d)}\phi_d(y/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = y = 0 \\ y^{\varphi(d)}\phi_d(x/y) & \text{if } y \neq 0, \end{cases}$$

where ϕ_d is the d-th cyclotomic polynomial and φ is the Euler totient function. Note that $x^{\varphi(d)}\phi_d(y/x) = y^{\varphi(d)}\phi_d(x/y)$ if $x \neq 0$ and $y \neq 0$.

Since $x^{70} - x^{35}y^{35} + y^{70} = \Phi_6(x,y)\,\Phi_{30}(x,y)\,\Phi_{42}(x,y)\,\Phi_{210}(x,y)$, we have

$$v_{3^{k+1}35t} = v_{3^k 35t}\,\Phi_6\big(\alpha^{3^k t},\beta^{3^k t}\big)\,\Phi_{30}\big(\alpha^{3^k t},\beta^{3^k t}\big)\,\Phi_{42}\big(\alpha^{3^k t},\beta^{3^k t}\big)\,\Phi_{210}\big(\alpha^{3^k t},\beta^{3^k t}\big).$$

Note that, since $Q = -1$,

$$\begin{aligned} \Phi_6\big(\alpha^{3^k t},\beta^{3^k t}\big) &= v_{3^k 2t} - (-1)^t \\ \Phi_{30}\big(\alpha^{3^k t},\beta^{3^k t}\big) &= v_{3^k 8t} + (-1)^t v_{3^k 6t} - (-1)^t v_{3^k 2t} - 1 \\ \Phi_{42}\big(\alpha^{3^k t},\beta^{3^k t}\big) &= v_{3^k 12t} + (-1)^t v_{3^k 10t} - (-1)^t v_{3^k 6t} - v_{3^k 4t} + 1 \\ \Phi_{210}\big(\alpha^{3^k t},\beta^{3^k t}\big) &= v_{3^k 48t} - (-1)^t v_{3^k 46t} + v_{3^k 44t} + (-1)^t v_{3^k 38t} - v_{3^k 36t} \\ &\quad + 2(-1)^t v_{3^k 34t} - v_{3^k 32t} + (-1)^t v_{3^k 30t} + v_{3^k 24t} - (-1)^t v_{3^k 22t} \\ &\quad + v_{3^k 20t} - (-1)^t v_{3^k 18t} + v_{3^k 16t} - (-1)^t v_{3^k 14t} - v_{3^k 8t} - v_{3^k 4t} - 1. \end{aligned}$$

Since $P > 0$ and $Q = -1$, for every $n > 0$ we have $v_n < v_{n+1}$, whence

$$0 < v_{3^k 2t} - 1 \le \Phi_6(\alpha^{3^k t}, \beta^{3^k t}) \le v_{3^k 2t} + 1 < v_{3^k 35t},$$

$$0 < v_{3^k 8t} - v_{3^k 6t} + v_{3^k 2t} - 1 \le \Phi_{30}(\alpha^{3^k t}, \beta^{3^k t}) \le v_{3^k 8t} + v_{3^k 6t} < v_{3^k 35t},$$

$$0 < v_{3^k 12t} - v_{3^k 10t} + v_{3^k 6t} - v_{3^k 4t} \le \Phi_{42}(\alpha^{3^k t}, \beta^{3^k t}) \le v_{3^k 12t} + v_{3^k 10t} + 1 < v_{3^k 35t}.$$

Since $v_{3^{k+1}35t}$, $v_{3^k 35t}$, $\Phi_6(\alpha^{3^k t}, \beta^{3^k t})$, $\Phi_{30}(\alpha^{3^k t}, \beta^{3^k t})$, $\Phi_{42}(\alpha^{3^k t}, \beta^{3^k t})$ are positive integers, we have $\Phi_{210}(\alpha^{3^k t}, \beta^{3^k t}) > 0$, and it is easy to show that $\Phi_{210}(\alpha^{3^k t}, \beta^{3^k t}) < 2v_{3^k 48t}$. By Lemma 4, we have that

$$m = v_{3^k 35t}\, \Phi_6(\alpha^{3^k t}, \beta^{3^k t})\, \Phi_{30}(\alpha^{3^k t}, \beta^{3^k t})\, \Phi_{42}(\alpha^{3^k t}, \beta^{3^k t})$$

is a practical number. Since $v_{3^{k+1}35t} = m\, \Phi_{210}(\alpha^{3^k t}, \beta^{3^k t})$, to complete the proof it suffices to show that $2v_{3^k 48t} \le 2m$, and this can be proved by straightforward and tedious calculations that we omit. ∎

The Fibonacci sequence $\{u_n(1, -1)\}$ and the Pell sequence $\{u_n(2, -1)\}$ satisfy the assumptions of Theorem 10. Since $L_{630} = v_{35\cdot 18}(1, -1)$ is a practical number, the Lucas sequence $\{v_n(1, -1)\}$ satisfies the assumptions of Theorem 11. Therefore there exist infinitely many practical Fibonacci, Pell and Lucas numbers.

It is interesting to note that the first practical Fibonacci numbers are F_3, F_6, F_{12}, F_{24}, F_{30}, F_{36}, F_{42}, F_{48}, which, except for F_3, have practical subscripts. It is well known that every prime Fibonacci number, except for F_4, has a prime subscript [4], but there exist some practical Fibonacci numbers with non-practical subscripts. The least such number is F_{444}. In fact, $444 = 2^2 \cdot 3 \cdot 37$ is not practical, but

$$\begin{aligned} F_{444} = {} & 2^4 \cdot 3^2 \cdot 73 \cdot 149 \cdot 443 \cdot 2221 \cdot 4441 \cdot 11987 \cdot 1121101 \cdot 54018521 \cdot 55927129 \\ & \cdot 6870470209 \cdot 8336942267 \cdot 81143477963 \cdot 1459000305513721 \end{aligned}$$

is a practical number.

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