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Resonance fluorescence spectrum of intense amplitude modulated laser light†

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Abstract. The spectrum of light scattered by a two-level atom in an intense laser beam with amplitude modulated light is calculated. The spectrum is obtained by evaluating the Fourier transform of the autocorrelation function of the dipole moment. The modulation introduces new components in both the coherent and incoherent parts of the frequency distribution. The total intensity when plotted as a function of the average Rabi frequency exhibits parametric resonances.

1. Introduction

The spectral distribution of light scattered by a two-level atom in an intense laser beam of constant amplitude has been the subject of numerous investigations (Mollow 1969, Stroud 1971, Cohen-Tannoudji 1977 and references therein). The spectrum consists of a coherent part (Rayleigh scattering) at the laser frequency ω_L and a three-peaked incoherent part which is symmetric about ω_L. The splitting depends on both the Rabi frequency and the detuning of the laser with respect to the atomic transition. At low laser amplitude, the coherent scattering is dominant, but it becomes negligibly small compared with the incoherent scattering when the laser amplitude is large enough to saturate the transition.

The response of a two-level atom in an amplitude modulated beam has been considered, but only as far as the atomic inversion is concerned (Armstrong and Feneuille 1975, Feneuille *et al* 1976, McClean and Swain 1976, Thomann 1976). In this paper, the effect of an amplitude modulated laser beam on the scattered spectrum is calculated. The laser field is treated classically and the spectrum is calculated by evaluating the Fourier transform of the autocorrelation function of the dipole moment. The laser frequency is taken to be equal to the atomic frequency.

The coherent and incoherent spectra are both affected by the modulation. They exhibit a resonant behaviour where $\Omega_0+m\omega_1=0$ (Ω_0 is the average Rabi frequency, ω_1 is the modulation frequency and m is an integer). In addition, new frequencies appear in the coherent spectrum at $\omega=\omega_L\pm m\omega_1$, and in the incoherent spectrum at $\omega=\omega_L\pm\Omega_0\pm m\omega_1$. The presence of these new components and their relative intensities are interpreted in the context of parametric resonances. Some properties of the incoherent spectrum can also be derived from the dressed-atom picture.

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2. Calculation of the scattered intensity

The spectral distribution of light scattered by a two-level atom is proportional to the Fourier transform of the correlation function of the atomic dipole moment (Mollow 1969, Cohen-Tannoudji 1977):

$$I(\omega) \sim \frac{1}{T} 2 \operatorname{Re}\left(\int_0^T \mathrm{d}t \int_0^t \mathrm{d}t' \langle \sigma_+(t) \sigma_-(t') \rangle \exp[-\mathrm{i}(\omega - \omega_\mathrm{L})(t - t')] \right). \tag{1}$$

The time T is the integrating time of the detector. Its maximum is determined by the interval during which the light from a given atom reaches the detector. The operators $\sigma_+(t)$ and $\sigma_-(t')$ are the atomic raising and lowering operators. Their expectation values $\langle \sigma_+(t) \rangle$ and $\langle \sigma_-(t') \rangle$ are solutions of the optical Bloch equations (Mollow 1969, Allen and Eberly 1975) in a frame rotating at the laser frequency ω_L. When the laser is tuned to the atomic transition (ω_0), the optical Bloch equations are of the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \sigma_\pm(t) \rangle = -\tfrac{1}{2}\gamma \langle \sigma_\pm(t) \rangle \mp \tfrac{1}{2}\mathrm{i}\Omega(t) \langle \sigma_z(t) \rangle \tag{2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \sigma_z(t) \rangle = \mathrm{i}\Omega(t)(\langle \sigma_-(t) \rangle - \langle \sigma_+(t) \rangle) - \gamma(\langle \sigma_z(t) \rangle + 1). \tag{2b}$$

The expectation value of $\sigma_z(t)$ is equal to the difference between the upper-state and lower-state populations (atomic inversion), γ is the spontaneous decay rate of the upper level and $\Omega(t)$ is the time-dependent Rabi frequency, which has the following form:

$$\Omega(t) = \Omega_0(1 + a \cos \omega_1 t). \tag{3}$$

In equation (3) Ω_0 is the average Rabi frequency, a is the modulation depth and ω_1 the modulation frequency.

Approximate solutions of equations (2a) and (2b), valid for strong driving fields at resonance ($\Omega^2(t) \gg \frac{1}{16}\gamma^2$, $\omega_\mathrm{L} = \omega_0$), have been calculated previously (Thomann 1976) and are given in the appendix.

The two-time expectation value in equation (1) can be rewritten by expressing $\sigma_\pm$ as the sum of its expectation value $\langle \sigma_\pm \rangle$ and its fluctuation $\delta\sigma_\pm$ ($\langle \delta\sigma_\pm \rangle = 0$ by definition):

$$\sigma_\pm = \langle \sigma_\pm \rangle + \delta\sigma_\pm. \tag{4}$$

The scattered intensity thus has two components. The contribution of the average motion of the dipole gives the coherent part of the spectrum

$$I_\mathrm{coh}(\omega) \sim \frac{1}{T} 2 \operatorname{Re}\left(\int_0^T \mathrm{d}t \int_0^t \mathrm{d}t' \langle \sigma_+(t) \rangle \langle \sigma_-(t') \rangle \exp[-\mathrm{i}(\omega - \omega_\mathrm{L})(t - t')] \right) \tag{5a}$$

whereas the contribution of the fluctuations gives the incoherent part of the spectrum (Cohen-Tannoudji 1977)

$$I_\mathrm{inc}(\omega) \sim \frac{1}{T} 2 \operatorname{Re}\left(\int_0^T \mathrm{d}t \int_0^t \mathrm{d}t' \, \langle \delta\sigma_+(t) \delta\sigma_-(t') \rangle \exp[-\mathrm{i}(\omega - \omega_\mathrm{L})(t - t')] \right). \tag{5b}$$

The terminology 'coherent–incoherent' (Mollow 1969) is preferred here to the 'elastic–inelastic' one (Cohen-Tannoudji 1977) for reasons which will be clear later.

By integrating equations (5a) and (5b) over the frequency ω one obtains, respectively, the coherent and incoherent contributions to the total scattered intensity. Since ω only appears in the exponential, this integration gives $\delta(t-t')$ and thus the two-time correlation function reduces to a single-time expectation value. If one assumes that the transients in the atomic evolution have died out before the atom enters the observation region one obtains the following results:

$$I_{\text{coh}} \sim \frac{1}{T}\int_0^T |\langle\sigma_+(t)\rangle|^2\,\mathrm{d}t \tag{6a}$$

$$I_{\text{inc}} \sim \frac{1}{T}\int_0^T [\tfrac{1}{2}(1+\langle\sigma_z(t)\rangle) - |\langle\sigma_+(t)\rangle|^2]\,\mathrm{d}t \tag{6b}$$

$$I_{\text{tot}} = I_{\text{coh}} + I_{\text{inc}} \sim \frac{1}{T}\int_0^T \tfrac{1}{2}(1+\langle\sigma_z(t)\rangle)\,\mathrm{d}t. \tag{6c}$$

The explicit expressions for $\langle\sigma_+(t)\rangle$ and $\langle\sigma_z(t)\rangle$ are given in the appendix. The scattered intensities in equations (6a)–(6c) are formally identical to the corresponding ones for a constant laser amplitude. The modulation of the Rabi frequency, however, introduces two new features. First, the steady-state atomic dipole moment ($\langle\sigma_+(t)\rangle$) and inversion ($\langle\sigma_z(t)\rangle$) are now time-dependent and contain oscillations at all the harmonics of the modulation frequency ω_1. Second, the coherent intensity no longer decreases steadily to zero when Ω_0 increases toward saturating values, as is the case with a constant laser amplitude. (Compare in figure 1 the full curves ($a=1$) with the broken curves ($a=0$).) When the incident laser amplitude is modulated, the coherent intensity shows maxima when the average Rabi frequency Ω_0 is a multiple of the modulation frequency (see figure 1). This is another manifestation of parametric resonances, which have already been observed in the total scattered intensity (Thomann 1976, 1980). The mechanism underlying these resonances can easily be understood by considering an

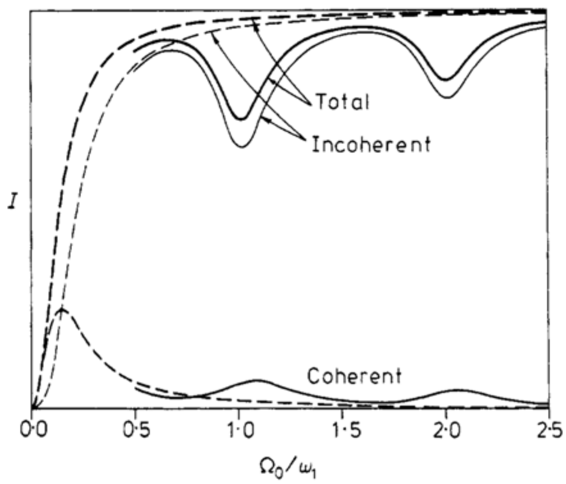

Figure 1. Intensity of light scattered by a two-level atom in a strong laser beam versus the average Rabi frequency. The full curves refer to the modulated case $\Omega(t)=\Omega_0(1+a\cos\omega_1 t)$ with $a=1$; the broken curves refer to the unmodulated case ($a=0$). The coherent and incoherent contributions are shown together with the total intensity (equations (6a), (6b) and (6c) respectively, with $T \gg \omega_1^{-1}, \gamma^{-1}$). The decay rate is $\gamma = 0{\cdot}2\omega_1$.

extreme case for the modulated field, namely, a periodic sequence of optical pulses separated by a time interval $\tau = 2\pi\omega_1^{-1}$ ($\omega_1 \gg \gamma$) and having each an area

$$A = \int_0^\tau \Omega(t)\,\mathrm{d}t = 2\pi\frac{\Omega_0}{\omega_1}. \tag{7}$$

If Ω_0 is large enough to saturate the transition but the area A of the pulse is different from $2\pi n$, an atom starting in its ground state will be left at the end of each pulse in some different superposition of its two states. Spontaneous emission, which is the dominant process between the pulses, will destroy the coherence of the superposition and after a few pulses the atom will settle near zero inversion and zero dipole moment. If, however, the area is equal to $2\pi n$, the atom undergoes a sequence of $2\pi n$ pulses, returning to the ground state at the end of each pulse. Because the nutation is slower between the pulses, the average inversion $\langle\sigma_z\rangle$ is negative although the frequency Ω_0 is large enough to saturate the transition. In addition to the decrease of inversion when the parametric resonance condition is met, the dipole moment associated with the sustained optical nutation is responsible for the increase of the coherently scattered radiation.

3. The coherent spectrum

The spectral distribution of the coherently scattered light is easily obtained by introducing in equation (5a) the steady-state solution for $\langle\sigma_+(t)\rangle$ (equation (A.8b)). By taking the limit $T\to\infty$, one obtains the following expression for the coherent spectrum:

$$I_{\mathrm{coh}}(\omega) \sim \frac{\gamma^2}{16} 2\pi \sum_{p=-\infty}^{\infty} Y_p Y_p^* \delta(\omega - \omega_{\mathrm{L}} + p\omega_1)$$

with

$$Y_p = \sum_{m=-\infty}^{\infty} J_m\left(\frac{a\,\Omega_0}{\omega_1}\right)\left(\frac{J_{m-p}(a\,\Omega_0/\omega_1)}{\frac{3}{4}\gamma - \mathrm{i}(\Omega_0 + m\omega_1)} - \frac{J_{m+p}(a\,\Omega_0/\omega_1)}{\frac{3}{4}\gamma + \mathrm{i}(\Omega_0 + m\omega_1)}\right) \tag{8}$$

where J_m is a Bessel function of integer order.

In addition to the standard Rayleigh peak at the laser frequency ω_{L}, the coherent spectrum contains sidebands, symmetrically placed about ω_{L} and separated by the modulation frequency ω_1. Their position is independent of Ω_0. Figure 2 shows the coherent spectrum for increasing values of the average Rabi frequency. The relative intensities vary in a fairly complex way with Ω_0 but the global enhancement of coherent scattering when $\Omega_0 + m\omega_1 = 0$ is clearly visible.

4. The incoherent spectrum

The calculation of the incoherent spectrum requires the knowledge of the two-time correlation function $\langle\delta\sigma_+(t)\delta\sigma_-(t')\rangle$. It is useful to introduce the following notation for the three correlation functions

$$c_\pm(t, t') = \langle\delta\sigma_\pm(t)\delta\sigma_-(t')\rangle \tag{9a}$$

$$c_z(t, t') = \langle\delta\sigma_z(t)\delta\sigma_-(t')\rangle. \tag{9b}$$

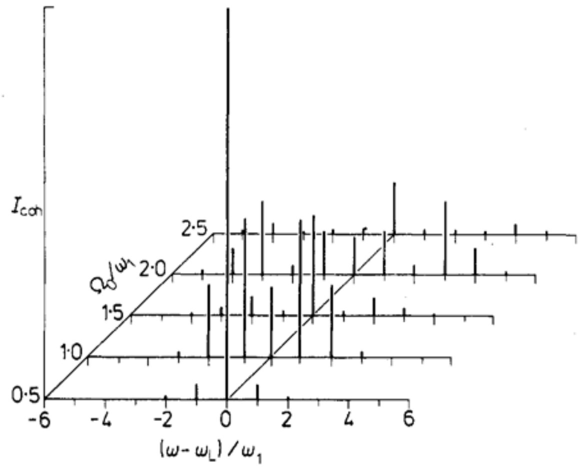

Figure 2. Coherent part of the spectrum for increasing values of the average Rabi frequency ($a = 1$). For comparison the spectrum in the unmodulated case (not shown) consists of a single peak at $\omega = \omega_L$, of decreasing intensity with Ω_0 (see lower broken curve in figure 1). The decay rate is $\gamma = 0{\cdot}2\omega_1$.

A set of differential equations for $c_\pm(t, t')$ and $c_z(t, t')$ can be obtained as a consequence of the quantum regression theorem (Cohen-Tannoudji 1977, Lax 1968). The assumption underlying the quantum regression theorem is that the correlation time of the radiation field fluctuations is negligibly small compared with the correlation time of the average motion of the dipole. This does not impose any practical limitation onto the strength or modulation frequency of the applied laser field. The time dependence of $c_\pm(t, t')$ and $c_z(t, t')$ is thus governed by the homogeneous part of the Bloch equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}c_\pm(t, t') = -\tfrac{1}{2}\gamma c_\pm(t, t') \mp \tfrac{1}{2}\mathrm{i}\Omega(t)c_z(t, t') \tag{10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}c_z(t, t') = \mathrm{i}\Omega(t)(c_-(t, t') - c_+(t, t')) - \gamma c_z(t, t'). \tag{10b}$$

The correlation functions can be expressed in terms of the transient solutions of the optical Bloch equations with a modulated Rabi frequency (see equations (A.4)–(A.6) in the appendix). In particular, the time dependence of $c_+(t, t')$ is given by

$$\begin{aligned} c_+(t, t') = {} & \tfrac{1}{2}(c_+(t', t') + c_-(t', t')) \exp(-\tfrac{1}{2}\gamma(t - t')) \\ & + \tfrac{1}{4}(c_+(t', t') - c_-(t', t') - c_z(t', t')) \exp(\mathrm{i}f_+(t, t')) \\ & + \tfrac{1}{4}(c_+(t', t') - c_-(t', t') + c_z(t', t')) \exp(\mathrm{i}f_-(t, t')) \end{aligned} \tag{11}$$

with

$$f_\pm(t, t') = (\tfrac{3}{4}\mathrm{i}\gamma \pm \Omega_0)(t - t') \pm (a\Omega_0/\omega_1)(\sin\omega_1 t - \sin\omega_1 t'). \tag{12}$$

The two-time correlation function is thus expressed in terms of single-time correlation functions and the known eigenfrequencies of the homogeneous Bloch equations. The evaluation of the single-time correlation functions requires the knowledge of the state of the atom at time t':

$$c_+(t',t') = \langle \delta\sigma_+(t')\delta\sigma_-(t')\rangle = \mathrm{Tr}(\delta\sigma_+\delta\sigma_-\rho(t')). \tag{13}$$

For large values of t', $\rho(t')$ is expressed in terms of the long-term solutions of the Bloch equations as

$$\rho(t') = \tfrac{1}{2}(1+\langle\sigma_z(t')\rangle\sigma_z + 2\langle\sigma_+(t')\rangle\sigma_- + 2\langle\sigma_-(t')\rangle\sigma_+). \tag{14}$$

By using the properties of the Pauli matrices one can rewrite equation (11). The approximate result is

$$\begin{aligned} c_+(t,t') = {} & \tfrac{1}{4}(1+\langle\sigma_z(t')\rangle)\exp(-\tfrac{1}{2}\gamma(t-t')) \\ & + \tfrac{1}{8}[1+\langle\sigma_z(t')\rangle - 4|\langle\sigma_+(t')\rangle|^2 - 2\langle\sigma_+(t')\rangle(1+\langle\sigma_z(t')\rangle)]\exp(\mathrm{i}f_+(t,t')) \\ & + \tfrac{1}{8}[1+\langle\sigma_z(t')\rangle - 4|\langle\sigma_+(t')\rangle|^2 + 2\langle\sigma_+(t')\rangle(1+\langle\sigma_z(t')\rangle)]\exp(\mathrm{i}f_-(t,t')). \end{aligned} \tag{15}$$

In order to obtain the incoherent spectrum equation (15) has to be inserted into equation (5b). The result is too complicated to be given in detail. The following approximate result is obtained by dropping the contribution of the product terms ($|\langle\sigma_+(t')\rangle|^2$ and $\langle\sigma_+(t')\rangle\langle\sigma_z(t')\rangle$ in equation (15)), which makes the structure of the solution more apparent without introducing any significant errors. In the limit of long averaging times ($T\to\infty$) one obtains

$$\begin{aligned} I_{\mathrm{inc}}(\omega) \sim {} & \frac{\frac{1}{4}\gamma}{(\frac{1}{2}\gamma)^2+(\omega-\omega_{\mathrm{L}})^2}\left(1-\frac{3}{4}\sum_{m=-\infty}^{\infty} J_m^2\left(\frac{a\Omega_0}{\omega_1}\right)\frac{\gamma^2}{(\frac{3}{4}\gamma)^2+(\Omega_0+m\omega_1)^2}\right) \\ & + \left[\frac{3\gamma}{16}\sum_{m=-\infty}^{\infty}\frac{J_m^2(a\Omega_0/\omega_1)}{(\frac{3}{4}\gamma)^2+(\omega-\omega_{\mathrm{L}}-\Omega_0-m\omega_1)^2}\right. \\ & \times\left(1-\frac{4}{3}\,\frac{(\frac{3}{4}\gamma)^2+(\Omega_0+m\omega_1)(\omega-\omega_{\mathrm{L}}-\Omega_0-m\omega_1)}{(\frac{3}{4}\gamma)^2+(\Omega_0+m\omega_1)^2}\right) \\ & \left. + (\omega-\omega_{\mathrm{L}})\to-(\omega-\omega_{\mathrm{L}})\right]. \end{aligned} \tag{16}$$

It should be stressed that this result is only valid for laser amplitudes large enough to saturate the transition ($\Omega(t)\gg\gamma$). When the modulation depth goes to zero, the well known result for a constant laser amplitude (Mollow 1969) is recovered. In this case the incoherent spectrum contains three peaks. The central one is at $\omega=\omega_{\mathrm{L}}$ and the two sidebands are at $\omega=\omega_{\mathrm{L}}\pm\Omega_0$. The main effect of the modulation is that the two sidebands are now accompanied by their own sidebands, located at $\omega=\omega_{\mathrm{L}}\pm\Omega_0\pm m\omega_1$, whereas no additional sidebands are generated at frequencies $\omega=\omega_{\mathrm{L}}\pm m\omega_1$. This can be justified by examining the origin of the different peaks in the incoherent spectrum. The width and the position of the central peak are determined by the transient behaviour of $\langle\sigma_x\rangle=\langle\sigma_+\rangle+\langle\sigma_-\rangle$, which is independent of the applied field (equation (2a), added to its complex conjugate, becomes $\mathrm{d}\langle\sigma_x(t)\rangle/\mathrm{d}t=-\frac{1}{2}\gamma\langle\sigma_x(t)\rangle$ and decouples from the other equations). On the other hand, the width and position of the sidebands are determined by the transient behaviour of $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$, which is modulated at the Rabi frequency. When the Rabi frequency itself is modulated, the decay of $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$ also contains the frequencies $\Omega_0\pm p\omega_1$ (see equations (A.6) and (A.7) in the appendix).

A set of spectra corresponding to increasing values of Ω_0 (the modulation depth a is kept constant), is shown in figure 3. For comparison the corresponding spectra for the unmodulated case ($a = 0$) are shown in figure 4.

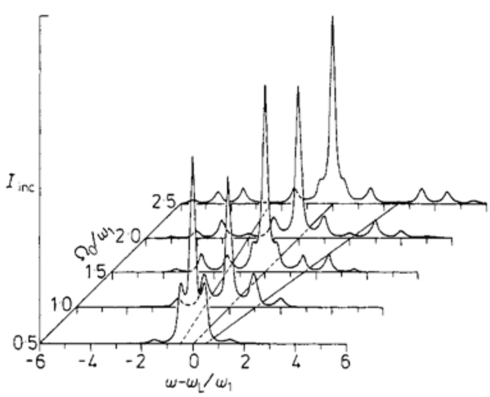

Figure 3. Incoherent part of the spectrum for increasing values of the average Rabi frequency (modulated laser amplitude, $a = 1$). The positions of the three peaks corresponding to the unmodulated case are connected by broken lines. The decay rate is $\gamma = 0{\cdot}2\omega_1$.

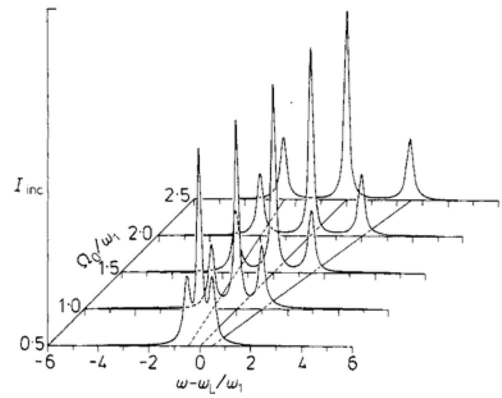

Figure 4. Same as figure 3 but for an unmodulated laser amplitude ($a = 0$, normal dynamic Stark effect).

5. Dressed-atom description of the incoherent spectrum

The dressed-atom picture provides a straightforward way of deriving the main properties of the incoherent spectrum in the case of a laser beam of constant amplitude. The idea is to neglect, as a first step, the effect of spontaneous emission and find the eigenstates of the atom dressed by the laser field. At resonance ($\omega_L = \omega_0$), the energy spectrum consists of pairs of degenerate levels separated by the laser frequency (a typical pair of degenerated states would be the atomic excited state with $(n-1)$ photons and the ground state with n photons). The laser–atom coupling removes the degeneracy and each pair of levels becomes a doublet with an energy separation equal to $\hbar\Omega_0$. When the decay rate of the excited atomic state is much smaller than the Rabi frequency, the effect of spontaneous emission can be treated as a perturbation and the intensities radiated at different frequencies can be evaluated using Fermi's golden rule. The intensity of a transition between any two dressed states is proportional to the square of the dipole matrix element between these states.

This model predicts the correct frequencies in the spectrum and the correct area under the corresponding peaks. However, it is difficult to generalise this procedure to a modulated field where the dressing photons belong to three different optical modes instead of only one. The dressed-atom picture, however, can still be used in a slightly modified way.

The starting point is the semiclassical Hamiltonian which is used to derive the Bloch equations ($\hbar = 1$). At resonance ($\omega_L = \omega_0$) we have

$$H = \tfrac{1}{2}\omega_0\sigma_z + \tfrac{1}{2}\Omega(t)(\sigma_+ \, e^{-i\omega_0 t} + \sigma_- \, e^{i\omega_0 t}). \tag{17}$$

The time dependence at the optical frequencies is first transformed away by going to the interaction representation:

$$H_{\mathrm{I}} = \exp(\tfrac{1}{2}\mathrm{i}\omega_0\sigma_z t)H\exp(-\tfrac{1}{2}\mathrm{i}\omega_0\sigma_z t) - \tfrac{1}{2}\omega_0\sigma_z$$
$$= \tfrac{1}{2}\Omega_0\sigma_x + \tfrac{1}{2}a\,\Omega_0\cos\omega_1 t\,\sigma_x. \tag{18}$$

The remaining time dependence is now removed by quantising the RF field of frequency ω_1. The new Hamiltonian, which replaces H_{I}, is

$$H_{\mathrm{IQ}} = \underbrace{\tfrac{1}{2}\Omega_0\sigma_x + b^{\dagger}b\omega_1}_{H_{\mathrm{IQ1}}} + \underbrace{\tfrac{1}{2}\lambda(b+b^{\dagger})\sigma_x}_{H_{\mathrm{IQ2}}} \tag{19}$$

with b the annihilation operator for the RF field and

$$\lambda = a\,\Omega_0/2\sqrt{n}. \tag{20}$$

The eigenstates of H_{IQ1} are $|n\rangle|\epsilon\rangle_x = |n, \epsilon\rangle$, where $|\epsilon\rangle_x$ denotes the eigenstates of $\sigma_x(\epsilon = \pm 1;\ |\epsilon\rangle_x = \sqrt{\tfrac{1}{2}}(|+\rangle + \epsilon|-\rangle))$. The corresponding energies are

$$E(n, \epsilon) = \tfrac{1}{2}\epsilon\,\Omega_0 + n\omega_1. \tag{21}$$

An exact diagonalisation of H_{IQ} has been given by Polonsky and Cohen-Tannoudji (1965). The eigenstates of H_{IQ} are

$$|\bar{n}_\epsilon, \epsilon\rangle = |\epsilon\rangle_x \sum_{q=-\infty}^{\infty} J_q\left(\epsilon\frac{a\,\Omega_0}{2\omega_1}\right)|n-q\rangle \tag{22}$$

and their energies are equal to the unperturbed energies except for a negligible shift:

$$E(\bar{n}_\epsilon, \epsilon) = \tfrac{1}{2}\epsilon\,\Omega_0 + n\omega_1 - \lambda^2/4\omega_1 \simeq \tfrac{1}{2}\epsilon\,\Omega_0 + n\omega_1. \tag{23}$$

The time dependence and magnitude of the dipole moment for a transition between any two eigenstates $|\bar{n}_\epsilon, \epsilon\rangle$ and $|\bar{n}'_{\epsilon'}, \epsilon'\rangle$ of H_{IQ} is then given by the matrix element of $\sigma_+(t)$:

$$\langle\sigma_+(t)\rangle = \langle\bar{n}_\epsilon, \epsilon|\exp(\mathrm{i}Et)\exp(\tfrac{1}{2}\mathrm{i}\omega_0\sigma_z t)\sigma_+\exp(-\tfrac{1}{2}\mathrm{i}\omega_0\sigma_z t)\exp(-\mathrm{i}E't)|\bar{n}'_{\epsilon'}, \epsilon'\rangle. \tag{24}$$

By using equation (22) and introducing the explicit time dependence of the eigenstates of H_{IQ}, whose energies are given by equation (23), one obtains

$$\langle\sigma_+(t)\rangle = \tfrac{1}{2}\epsilon'\exp\{\mathrm{i}[\omega_0 + (n-n')\omega_1 + \tfrac{1}{2}(\epsilon-\epsilon')\Omega_0]t\}\sum_{q=-\infty}^{\infty} J_q\left(\epsilon\frac{a\,\Omega_0}{2\omega_1}\right)J_{q+n'-n}\left(\epsilon'\frac{a\,\Omega_0}{2\omega_1}\right). \tag{25}$$

Equation (25) can be simplified by means of the following identity (Abramowitz and Stegun 1964)

$$J_p(\alpha \pm \beta) = \sum_q J_{p\mp q}(\alpha)J_q(\beta). \tag{26}$$

We finally obtain

$$\langle\sigma_+(t)\rangle \sim \begin{cases} \tfrac{1}{2}\epsilon\exp(\mathrm{i}\omega_0 t) & (\epsilon' = \epsilon) \\ -\tfrac{1}{2}\epsilon J_m\left(\epsilon\dfrac{a\,\Omega_0}{\omega_1}\right)\exp[\mathrm{i}(\omega_0 + m\omega_1 + \epsilon\,\Omega_0)t] & (\epsilon' = -\epsilon). \end{cases} \tag{27}$$

The frequencies of the scattered light are ω_0 or $\omega_0 \pm \Omega_0 + m\omega_1$ depending on whether the transition takes place between states belonging to the same multiplicity ($\epsilon = \epsilon'$) or to different ones ($\epsilon = -\epsilon'$). The relative intensities, or more precisely the relative areas under the various peaks of the spectrum, are obtained by squaring the modulus of the matrix elements of equation (27). Noting that, for any value of α, $\Sigma_{m=-\infty}^{\infty} J_m^2(\alpha) = 1$, one immediately gets the result that the area under the central peak is equal to the sum of the areas under all the sidebands.

6. Conclusion

We have calculated the spectrum of the light scattered by a two-level atom in an intense, resonant and amplitude modulated laser beam. The results differ from the ones for a constant laser amplitude in two aspects: as a function of Ω_0 the total coherent and incoherent parts of the spectrum exhibit parametric resonances in the same way as the atomic dipole moment and atomic inversion, to which they are closely connected. In addition, the modulation of the Rabi frequency introduces new components in both the coherent and incoherent parts of the spectrum.

The generation of sidebands in the coherent part of the spectrum can be seen as a manifestation of the non-linear polarisability of the atom in the presence of three phase-locked modes of the optical radiation field. The enhancement of the coherent sideband intensities when the parametric resonance condition is met appears as a consequence of the phase coherence between the three modes.

The dressed-atom picture can be used to derive the general features of the incoherent part of the spectrum. The width of the various components, however, is more directly obtained from the dynamic properties of the autocorrelation function of the dipole moment, which also explains why only the side peaks in the unmodulated AC Stark effect give rise to additional sidebands in the modulated case.

Appendix 1. Transient solutions

We give the transient and steady-state solutions of the optical Bloch equations with modulated Rabi frequency. It is useful to rewrite equations (2a) and (2b) by separating the real and imaginary parts of $\langle\sigma_\pm(t)\rangle$. To simplify the notation, one can define

$$\tfrac{1}{2}(x(t) \pm \mathrm{i}y(t)) = \langle\sigma_\pm(t)\rangle \tag{A.1a}$$

$$z(t) = \langle\sigma_z(t)\rangle. \tag{A.1b}$$

The homogeneous part of equations (2a) and (2b) then becomes

$$\dot{x}(t) = -\tfrac{1}{2}\gamma x(t) \tag{A.2a}$$

$$\dot{y}(t) = -\tfrac{1}{2}\gamma y(t) - \Omega(t) z(t) \tag{A.2b}$$

$$\dot{z}(t) = \Omega(t) y(t) - \gamma z(t) \tag{A.2c}$$

with

$$\Omega(t) = \Omega_0(1 + a \cos \omega_1 t). \tag{A.3}$$

Equations (2a)–(2c) have three eigensolutions with three different eigenfrequencies. One can write the solutions as

$$\xi(t, t') = \xi(t') \exp(\mathrm{i}f_0(t, t')) \tag{A.3a}$$

$$\eta(t, t') = \eta(t') \exp(\mathrm{i}f_+(t, t')) \tag{A.3b}$$

$$\zeta(t, t') = \zeta(t') \exp(\mathrm{i}f_-(t, t')). \tag{A.3c}$$

Equations (A.3) describe the state of the atom at time t with initial conditions specified at time $t' < t$.

When the Rabi frequency is always much larger than the decay rate ($\Omega^2(t) \gg \frac{1}{16}\gamma^2$), the solutions take the following form:

$$\xi(t, t') = x(t') \exp(\mathrm{i}f_0(t, t')) = (\langle\sigma_+(t')\rangle + \langle\sigma_-(t')\rangle) \exp(\mathrm{i}f_0(t, t')) \tag{A.4a}$$

$$\begin{aligned}\eta(t, t') &= (\mathrm{i}y(t') - z(t')) \exp(\mathrm{i}f_+(t, t')) \\ &= (\langle\sigma_+(t')\rangle - \langle\sigma_-(t')\rangle - \langle\sigma_z(t')\rangle) \exp(\mathrm{i}f_+(t, t'))\end{aligned} \tag{A.4b}$$

$$\begin{aligned}\zeta(t, t') &= (\mathrm{i}y(t') + z(t')) \exp(\mathrm{i}f_-(t, t')) \\ &= (\langle\sigma_+(t')\rangle - \langle\sigma_-(t')\rangle + \langle\sigma_z(t')\rangle) \exp(\mathrm{i}f_-(t, t'))\end{aligned} \tag{A.4c}$$

with

$$f_0(t, t') = \tfrac{1}{2}\mathrm{i}\gamma(t - t') \tag{A.5}$$

$$f_\pm(t, t') = (\tfrac{3}{4}\mathrm{i}\gamma \pm \Omega_0)(t - t') \pm (a\Omega_0/\omega_1)(\sin\omega_1 t - \sin\omega_1 t'). \tag{A.6}$$

Appendix 2. Long-term solutions

By taking the constant pumping term in equation (2b) into account the steady-state solutions can be obtained from the transient solutions by a simple integration over t'. It is useful to replace $\exp(\mathrm{i}f_\pm(t, t'))$ by its Fourier development

$$\begin{aligned}\exp(\mathrm{i}f_\pm(t, t')) = \exp[(-\tfrac{3}{4}\gamma \pm \mathrm{i}\Omega_0)(t - t')] \sum_{k=-\infty}^{\infty} J_k\left(\frac{a\Omega_0}{\omega_1}\right) \\ \times \exp(\mathrm{i}k\omega_1 t) \sum_{n=-\infty}^{\infty} J_n\left(\frac{a\Omega_0}{\omega_1}\right) \exp(-\mathrm{i}n\omega_1 t').\end{aligned} \tag{A.7}$$

One obtains

$$z(t) = \langle\sigma_z(t)\rangle = -\gamma\,\mathrm{Re}\left(\sum_{m,p\,-\infty}^{\infty} \frac{J_{m-p}(a\Omega_0/\omega_1)J_m(a\Omega_0/\omega_1)\exp(-\mathrm{i}p\omega_1 t)}{\frac{3}{4}\gamma - \mathrm{i}(\Omega_0 + m\omega_1)}\right) \tag{A.8a}$$

$$y(t) = -2\mathrm{i}\langle\sigma_+(t)\rangle = \gamma\,\mathrm{Im}\left(\sum_{m,p\,-\infty}^{\infty} \frac{J_{m-p}(a\Omega_0/\omega_1)J_m(a\Omega_0/\omega_1)\exp(-\mathrm{i}p\omega_1 t)}{\frac{3}{4}\gamma - \mathrm{i}(\Omega_0 + m\omega_1)}\right) \tag{A.8b}$$

$$x(t) = 0. \tag{A.8c}$$

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