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A New Path to the Logicist Construction of Numbers

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> Any theory can always be applied to infinitely many systems of basic elements. David Hilbert

1. A reductionist programme?

Since Frege, Whitehead and Russell, logicism has been widely described as a programme for the *reduction* of mathematics – centrally arithmetic – to logic. If the goal seems clear and can be summerized by the short claim that arithmetic is nothing but logic, it is nevertheless far from being easy to understand accurately what such a reduction is and what is its significance. Basically, the reduction of a theory to another is a technical notion: a theory S_1 is said to be reducible to a theory S_2, when the axioms of S_1 can be derived in S_2 by means of explicit definitions of the primitive terms of S_1 in the language of S_2. This of course does not mean that S_1 is nothing but S_2, but only that S_1 can be *interpreted* in S_2. However, the significance of such a technical reduction is not inconsiderable. It shows first that if S_2 is consistent, then S_1 is also consistent. It also shows that certain entities we can construct in S_2 can *play the rôle* of the objects of S_1, even if it is not a guarantee that these two groups of objects are simply identical.

The kind of reduction logicists usualy have in mind is something much stronger than this technical kind. In the case of arithmetic, the definition of "number" is supposed to grasp the very notion of number. So the definition must be *materially adequate* (in the sense Tarski uses these words in his papers on the concept of truth). In classical logicism, the definition is thus preceded by a philosophical investigation about the nature of numbers. But as Russell pointed out in 1903, logicist definitions are quite paradoxical: on the one hand they "are nothing but statements of symbolic abbreviations", but on the other hand "in the development of a subject, they always require a very large amount of thought, and often embody some of the greatest achievements of analysis" (1903: 63). Even if theoretically insignificant, introducing only a convenient abbreviation, the definition of the term "number" is supposed to have the value of an *explanation* of what a number is, that is to say the value of a real definition of *number*. Now, in order to show that statements of arithmetic are not synthetic, as Kant claimed, but purely analytical ones, the theory in which the definition is stated must be a purely analytic one. The aim of the reduction was actually for original logicists to provide arithmetic with an epistemic foundation: our mathematical knowledge would be secured if numbers can be shown to be logical entities, the properties of which only depend on basic logical laws.

Nevertheless, this form of logicism was a failure: the authors of the *Principia Mathematica* were forced to enlarge their logical basis with three non-logical axioms and Frege faced with contradiction. The only way for logicism was then the search of the weakest addition to pure logic allowing the reduction of Peano-Dedekind arithmetic (PA), while preserving the epistemic component of the original programme. We know today, first from C. Parsons (1965), but also from C. Wright (1983) and G. Boolos (1987), that Russell's paradox was not the death sentence of the whole of Frege's foundation of mathematics: what is now called Frege's Theorem – the proof that the fundamental laws of arithmetic can be derived from second-order logic through the (explicit) definition of three terms and the addition of a single proper axiom, namely Hume's Principle (HP)[1] – is considered as an extremly interesting result for the philosophy of mathematics. But, C. Wright's and B. Hale's claim that Frege's Theorem is still a form of *logicism* is highly controversial. HP is a *proper* axiom and there is no way to use an argument such as Russell's one concerning his axiom of infinity – i.e. that it can be considered as an hypothesis we must assume when entering the field of mathematics – for HP introduces into the system a (non-logical?) proper term (*the cardinal number of*). Though not purely logical truths, according to Wright, the laws of arithmetic are still shown to be analytic or purely conceptual by Frege's Theorem. The reason being put forward by neo-Fregeans is that HP is (or more precisely, involves) an explanation in logical terms of our concept of (cardinal) number. HP is then considered by these authors as an *implicit definition* of that concept, stated into the language of second-order logic. HP is considered by neo-Fregeans as an analytic truth, for – as they say – when stipulating HP as true[2], the process is meaning-constituting[3]. What I understand is that the addition of HP to a first-order logic axiomatic system provides an implicit definition giving the new proper term a *definite* meaning, the logical analysis of which allows to show that cardinal (and then natural) numbers exist as objects with the attended properties.

But the problems with implicit definitions abound. Their addition – as with any proper axiom – modify the whole system and can even lead to contradiction. Relative to a certain goal, they can be too strong, too weak, or even both. Contrary to Frege's Basic Law V (which is exactly of the same shape), HP is certainly consistent with second-

[1] HP can be formulated as $(\forall F)(\forall G)(\mathrm{N}x{:}Fx = \mathrm{N}x{:}Gx \equiv F \infty G)$, where '$\mathrm{N}x{:}Fx$' expresses "the cardinal number of *F*" and '∞' the relation of equinumerosity i.e. the existence of a one-one correspondance between the objects falling under *F* and those falling under *G*.

[2] I must confess I cannot understand what "stipulating HP *as* true" really means, for I see only two possibilities regarding HP: either it is a proposition or an open formula with 'N(-)' as a free variable. In the former case, 'N(-)' already has a meaning in the language in which HP is considered and then the truth value of HP does not depend at all on our *stipulation*. In the latter case (open formula), it cannot have (or receive) any truth value. The only way I can understand "I stipulate HP as true" is: "let me *consider* 'N(-)' with one of *these* meaning*s* (if any!) which are adequate for HP to be understood as a true proposition".

[3] Cf. for example Ebert & Rossberg in this volume.

order logic. But, as a *definition* of a *single* proper term, it is too strong, for it also modifies the logical constants with which the term to be defined is explained. Due to its impredicative character, HP excludes interpretations in a finite domain of individuals. In other terms it involves an axiom of infinity. From a proof-theoretical point of view, expressions of the form

$$(\exists x_1)(\exists x_2)\ldots(\exists x_n)(x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge \ldots \wedge x_1 \neq x_n \wedge x_2 \neq x_3 \wedge x_2 \neq x_4 \wedge \ldots \wedge x_2 \neq x_n \wedge \ldots \wedge x_{n-1} \neq x_n) \quad [\text{with } n \geq 2]$$

which are not theorems of logic and do not contain the new term, are logical consequences of the "definition"[4].

On the other hand, HP is also too weak for being a *logicist* definition of cardinal number, for it does not warrant that only one kind of entity fall under the concept. In Hilbertian terms, HP does not warrant the existence of only one "system of objects" satisfying the "definition". The so-called Ceasar problem is a consequence of this weakness. Saying, with neo-Fregeans, that the open sentences of the form "$y = \mathrm{N}x{:}Fx$" are only satisfied by objects falling under the identity conditions expressed by HP is not enough, for we cannot exclude that there is no (non-standard) "system of objects" including Julius Ceasar and satisfying HP however (even if Julius Ceasar clearly does not fall under the identity conditions which follows from *our* intended interpretation of 'N-:-').

At the end, in spite of its elegance and impressive economical character (just a single short and intuitive proper axiom with only one primitive term, the others being introduced by explicit definitions), it

[4] In Łukasiewicz's vocabulary, such a definition is said to be *creative*, for it allows the derivation of expressions which are independant from the axioms and do not include any occurrence of the defined term (Łukasiewicz: 1928). Important and difficult questions remain open concerning the power of creative definitions and the way to fix the limite between definitions and expressions which are too strong to be considered simply as definitions. As an obvious example of the latter case, consider the propositional expression E: $(\sim p \supset \sim q) \supset ((\sim p \supset q) \supset p)$. In spite of apparences, E cannot be used as an implicit definition of negation in a system which only contains the intuitionist version of conditional. Of course *E explains* the new symbol '$\sim$' (negation), but it also strongly modify the meaning of the conditional '$\supset$', for it allows the derivation of Peirce Law $((p \supset q) \supset p) \supset p$, which only contains '$\supset$' and is not a theorem of intuitionist logic. On creative definitions, see (Joray: 2005, 2006).

is not obvious whether Frege Arithmetic (FA = second-order logic + HP) is more *logicist* than Peano-Dedekind Arithmetic (PA). After all, Peano's axioms also constitute (taken together) an implicit definition of "zero", "number" and "successor", explaining the related concepts in terms of pure logic. Moreover, the unsolved Ceasar problem shows that FA also invites the criticism Russell opposed to PA:

> We want our numbers to be such as can be used for counting common objects, and this requires that our numbers should have a *definite* meaning, not merely that they should have certain formal properties. (Russell 1921: 10).

Following R. Heck (1997, 2000), I will say that the reduction of a theory to another one – for example PA to FA – does not signify that the reduced theory *itself* is derivable in the other one, but only that the former is *interpretable* in the latter. The very object of an axiomatic mathematical theory is what is called a structure by mathematicians. Of course, every axiomatisation is *guided* by a certain intended or pre-interpretation. For example, Greek geometry was certainly guided by the consideration of the construction of concrete figures[5] and arithmetic by the common practice of counting concrete objects.

Strictly speaking, the very mathematical notion of number cannot be defined from logic or from any other theory, for it would make numbers having properties we are not ready to recognize as arithmetical ones[6]. Mathematicians' natural numbers cannot *be* such things as extensions or other logical objects abstracted from concepts; they cannot *be* classes of classes, or certain kind of ZF sets, neither – as suggested in (Simons: to appear) – properties of multitudes. In all these cases, numbers would have mathematicaly irrelevant properties, expressed by propositions which do not belong to arithmetic. What I

[5] As an example, the implicit presence of such a concrete interpretation becomes apparent since the demonstration of the first proposition of Euclide's *Elements*, where the existence of a point of intersection between two circles can only be infered from the observation of the diagram associated with the reasonning. The relation to common practice is also visible in the first three "demands", the expression of which is obviously inspired by the use of the ruler and the pair of compasses.

[6] So is, for example, in FA, the property of zero expressed by the formula "$(\exists F)(\mathrm{N}x{:}Fx = 0)$", or in ZF: "$(\forall E)(0 \subsetneq E)$".

am ready to call "natural number" in mathematics is actually only one of these abstract and general entities which stricly satisfy all the theorems of arithmetic *and no other*.

In this perspective, what a logicist approach to a mathematical theory can provide is only what I will call a *picture* of this theory, that is to say a specific interpretation in which purely logical objects or constructions can play the rôle of mathematical notions (numbers, in particular). Of course, the very existence of such a *purely logical picture* is far from being pointless for the philosophy of mathematics. As a (technical) reduction, it gives of course a relative proof of consistency. But it especially provides an *objective and conceptual path* to arithmetical knowledge. A logicist picture gives such a secured epistemic justification for it allows to replace by a conceptual construction the intuitive content and naïve notions which, in the development of their practice, lead mathematicians step by step to the axiomatic caracterisation of their theory – in Russell's terms, the picture provides a *logical analysis* of the intuitive notions. The route which is thus constructed is epistemicaly secured, for it consists in propositions the truth of which depends only on logic.

The possibility of reinterpretations is today widely recognised by logicians and mathematicians as an essential advantage of axiomatic theories and Russell's above mentionned criticism was clearly overtaken by further developments of formal sciences. Nonetheless, his requierments – that our numbers can be used for counting common objects and that they have a definite meaning – are perfectly relevant relative to the *picture* of arithmetic logicists want to elaborate. In order to provide the kind of justification I have just described, the picture must be *materialy adequate* – it must present an adequate analysis of the naïve notion of number we use when counting concrete objects. On the other hand, it must also be *definite in meaning*. This requiers the meaning of the defined terms of the construction to be *fully determined* by the logical constants, excluding reinterpretations. For this reason, the use of any *implicit* definition should be prohibited in this conception of logicism.

Frege's requierment that only logical constants occure in his Basic Laws is not followed by neo-Fregeans. FA is undoubtly a very nice theory to which arithmetic can be reduced. Nevertheless, it is neither arithmetic *itself* (as PA is), nor it is a good logicist *picture* of arithmetic, for it does not exclude reinterpretations of the proper term introduced by HP. For the latter condition to be satisfied, only explicit definitions must be used in the construction.

In the following pages, I am going to show that a valuable logicist picture of arithmetic can be constructed from logic using only explicit definitions. This will be done without introducing extensions of concepts or classes – even as incomplete symbols or way of speaking – but on the ground of Stanisław Leśniewki's logical notion of name, which unable to express in the logical language pluralities of things – like plural terms do in natural language.

2. A logic of names

When we assert a numerical statement like "there are five continents", according to Frege, we are ascribing a certain property to the concept *being a continent*. For Russell and Whitehead, it is to the class of continents that the property is ascribed. But which property? Certainely not that of *being five*. Of course, neither the concept, nor the class can simply be said to be five. Before being analysed by means of the logical relation of equinumerosity, the property in question can only be described as *having five objects falling under it* (for the concept), or *being a member of it* (for the class). On the other hand, *being five* is obviously not a property of the objects themselves: the continents *are* five, but none of them *is* five. According to P. Simons (to appear), *being five* is a property of the "multitude" of continents, a notion he says to be akin to Husserl's "Vielheit" or Russell's "class as many". But where is the expected solution? Like with "the concept of continent" or "the class of continents", "the multitude of continents" is obviously a singular term. The ordinary fact that we can use a single word or a single expression in order to refer to several objects seems to be mysterious for logicians as long as we do not postulate the existence of a single intermediate entity which has the (still mysterious?) virtue to gather together the things in question. In this direction, set theory is the most achieved solution and I am sceptical about the possibility to do better with the introduction of an other kind of abstract "multiple" entity for the definition of numbers.

The idea underlying the logical picture of numbers to be presented hereafter is much more unsophisticated. Without trying to *explain* the one-many link between expressions and objects, one just *observes* that ordinary language involves expressions or words which are used to refer sometimes to a single thing (for example, "Cairo" or "the capital of Egypt"), sometimes to several things ("The African capitals", "horses") and also sometimes to none of them ("the capital of Africa", "Ulysses", "round circles"). We all know – at least when speaking – that our words or expressions exist. What we do not know so surely is whether these words or expressions really refer to objects as we expect; pure logic cannot inform us about the existence of these references[7]. In the logical picture, the idea is to interprete numbers as *certain semantic properties of names*. So to say, zero will be depicted as the property of a name to be empty, one as the property of a name to be singular and three as the property of a name to refer to three things.

Primitive functor epsilon

Leśniewski's calculus of names[8] – called "Ontology" – is grounded on such basic observations. It is constructed as an expansion of a quantified propositional calculus – called "Protothetic" – through the addition of a single axiom. This axiom introduces variables for names

[7] Strictly speaking, it's a defect of standard predicate logic that it allows to show there exist at least one object in the universe.

[8] For a systematic presentation of Leśniewski's logic, see Miéville (1984, 2001-04) and the papers in Srzednicki & Rickey (1984).

(I will use here the first small latin letters *a*, *b*, *c*, ...), and a copula: a binary nominal relator called *epsilon*. The axiom is the following[9]:

AxOnto:
$$\lfloor ab \rfloor \lceil a\varepsilon b \;.{\equiv}.\; \lfloor \exists c \rfloor \lceil c\varepsilon a \rceil \wedge \lfloor cd \rfloor \lceil (c\varepsilon a \wedge d\varepsilon a) \supset c\varepsilon d \rceil \wedge \lfloor c \rfloor \lceil c\varepsilon a \supset c\varepsilon b \rceil \rceil$$

The left hand side of the biconditional of this universal expression '$a\varepsilon b$' is the general form of a singular proposition. It can be read as "*a* is *b*", or more precisely "the only object denoted by '*a*' is also denoted by '*b*'". In so doing, I am considering the right hand side as expressing the truth conditions of the singular proposition in the following way:

'$a\varepsilon b$' is true iff

1. something is denoted by '*a*';
2. '*a*' does not denote more than a single object;
3. what is denoted by '*a*' is also denoted by '*b*'.

In other words, '$a\varepsilon b$' is truly asserted iff '*a*' stands for a singular name (not empty and not plural) and '*b*' for a singular or plural name which denotes (possibly among others) the object denoted by '*a*'.

Among the inference rules, there are of course rules governing the use of quantifiers, which are subject to the standard principles. It has nevertheless to be noticed that the usual objectual or referential interpretation of quantifiers is not adequate. As a name can be singular, plural, but also empty, it is possible to express with a quantifier that there is a name which is empty.

Definition rules

As a very important peculiarity, the logic of names also includes rules for stating explicit definitions of two kinds. Instead of stating definitions in the metalanguage – like in the *Principia*, using the unspecified symbol '$=_{df}$' and introducing only convenient abbreviations – Leśniewski uses his primitive logical constants for expressing the equivalence relation between the *definiendum* (*Dum*)

[9] Where the universal and particular quantifiers are expressed by the respective forms "$\lfloor v \rfloor \lceil$ E $\rceil$" and "$\lfloor \exists v \rfloor \lceil$ E $\rceil$".

and the *definiens* (*Diens*). The first rule allows the introduction of propositional constants or functors, the second one, the introduction of nominal constants or functors. The logical equivalence is thus expressed by one of the two forms:

$\lfloor v_1\, v_2\, v_3 \ldots \rfloor \lceil\, Dum \equiv Diens \,\rceil$ Def$_S$ (propositional rule)

$\lfloor a\, v_1\, v_2\, v_3 \ldots \rfloor \lceil\, a\varepsilon Dum \equiv Diens \,\rceil$ Def$_N$ (nominal rule)

where 1. the left and right hand sides of the biconditional involve the same (free) variables; 2. *Diens* is a formula with only primitive or already defined symbols; 3. in the case of Def$_N$, *Diens* must be such that the name '*a*' is expressed to be a singular term[10] and 4. *Dum* is of the following form, where # is the symbol to be defined and no symbol occure more than once:

Dum: $\#\,(v_1\, v_2 \ldots)(\, v_i\, v_{i+1} \ldots) \ldots (\, v_j\, v_{j+1} \ldots v_n)$

As we will see below, the general form of *Dum* relates to three possiblities. First, there can be no variable in *Dum*. The defined symbol is then either a constant proposition (with Def$_S$), or a constant name (with Def$_N$), like in the following examples:

D1. $\top \equiv \lfloor p \rfloor \lceil\, p \equiv p \,\rceil$ Def$_S$ ($\top$: constant *true*)

D2. $\lfloor a \rfloor \lceil\, a\varepsilon\Lambda \equiv (a\varepsilon a \wedge \sim (a\varepsilon a)) \,\rceil$ Def$_N$ (Λ : empty name)

D3. $\lfloor a \rfloor \lceil\, a\varepsilon \mathrm{V} \equiv a\varepsilon a \,\rceil$ Def$_N$ (V : universal name)

In the second case, the variables of *Dum* occure in a single pair of parentheses. The defined symbol is then a functor:

D4. $\lfloor ab \rfloor \lceil\, \doteq\{ab\} \equiv (a\varepsilon b \wedge b\varepsilon a) \,\rceil$ Def$_S$
($\doteq\{ab\}$: *a* is the same object as *b*)

D5. $\lfloor ab \rfloor \lceil\, =\{ab\} \equiv \lfloor c \rfloor \lceil\, c\varepsilon a \equiv c\varepsilon b \,\rceil \,\rceil$ Def$_S$
($=\{ab\}$: *a* and *b* are identical or have the same reference(s))

D6. $\lfloor a \rfloor \lceil\, 0\{a\} \equiv \sim \lfloor \exists b \rfloor \lceil b\varepsilon a \rceil \,\rceil$ Def$_S$ ($0\{a\}$: *a* is empty)

[10] In fact, this inelegant condition becomes superfluous if we impose the nominal definitions to be stated in the following form: $\lfloor a\, v_1\, v_2\, v_3 \ldots \rfloor \lceil\, a\varepsilon Dum \equiv (a\varepsilon a \wedge Diens) \,\rceil$.

D7. $\lfloor a \rfloor \lceil 1\{a\} \equiv a \varepsilon a \rceil$ Def$_S$ (1{a} : a is singular)

D8. $\lfloor \varphi\psi \rfloor \lceil \approx[\varphi\psi] \equiv \lfloor a \rfloor \lceil \varphi\{a\} \equiv \psi\{a\} \rceil \rceil$ Def$_S$
($\approx[\varphi\psi]$: φ and ψ are coextensive or satisfied by the same name(s))

D9. $\lfloor abc \rfloor \lceil a\varepsilon(b.c) \equiv (a\varepsilon b \wedge a\varepsilon c) \rceil$ Def$_N$ (. : nom. intersection)

D10. $\lfloor abc \rfloor \lceil a\varepsilon(b+c) \equiv (a\varepsilon b \vee a\varepsilon c) \rceil$ Def$_N$ (+ : nominal union)

D11. $\lfloor abc \rfloor \lceil a\varepsilon(b-c) \equiv (a\varepsilon b \wedge \sim (a\varepsilon c)) \rceil$ Def$_N$ (– : nom. complement)

In the last case, the variables of *Dum* split up into more than one pair of parentheses. The symbol to be defined is thus a multi-link or parametric functor, i.e. a functor forming functor:

D12. $\lfloor ab \rfloor \lceil =\langle a \rangle \{b\} \equiv =\{ab\} \rceil$ Def$_S$
(parametric nominal identity; $=\langle a \rangle$: denoting like a)

D13. $\lfloor ab \rfloor \lceil \varepsilon\langle b \rangle \{a\} \equiv a\varepsilon b \rceil$ Def$_S$
(parametric *epsilon*; $\varepsilon\langle b \rangle$: being one of the b's)

I will not go here into the proofs of the formal properties of the above introduced symbols. Let me just underline the aspects of the definition rules which are central for the understanding of the definition of numbers in the next section.

First, it has to be noticed that the definition rules allow to introduce symbols of categories which are not previously available in the language. This is particulary obvious with the introduction of multi-link or parametric functors. In D12, for example, the parametric nominal identity is introduced on the basis of the usual identity binary relation. According to D5, '$=\{ab\}$' means "the names 'a' and 'b' denote the same things (if any)". D12 introduces an other linguistic possibility to express the same content: first the symbol for the parametric identity is applied to 'a' and the result '$=\langle a \rangle$' expresses the nominal property "denoting-(exactly)-the-a's"; this property can be applied to a name 'b', obtaining thus '$=\langle a \rangle\{b\}$' which expresses that 'b' denotes (exactly) the a's. This is very akin to a λ-abstraction and '$=\langle a \rangle$' is in Leśniewski's language what in λ-notation would be expressed by '$\lambda b.[=\{ab\}]$'.

Secondly, Leśniewski's system is such that the definition of a symbol of a new category allows the use of variables of that category and also the binding of these variables by quantifiers. See for example the use of the bind nominal-property-variables φ and ψ in D8, which depends on the introduction of the nominal-property-category in D6 (definition of '0{-}', the nominal property of emptiness).

This power of definition rules makes Ontology a strong analytical tool, but it has important consequences that I cannot present here in detail. As constants of any semantic category can be defined step by step, the system cannot be said to be of a determined order. Only specific definitional developpements of the axiomatic basis can be said to be of such or such order. On the other hand, it is clear that the formation of expressions cannot be specified as usual, by a previously defined set of well formed formulae. One of Leśniewski's main achievments, in the field of formal languages, was his ability to elaborate completly explicit semantic and syntactic constraints in order to impose extensionality at each level and to avoid ambiguity in the potentially infinite process of definition[11]. This paper cannot present and even make use of this full apparatus. Local conventions like differences in the kind of letters and the use of different sorts of parentheses are sufficient for its purpose.

4. The definition of numbers[12]

In the present logical construction, natural numbers are going to be depicted as cardinal properties of finite names (names which denote only a finite quantity of objects). Before going into the definition of the general notion of natural number, let me consider how any particular natural number can be defined. Zero and one have already been introduced by definitions D6 and D7:

[11] More on this issue in Gessler's paper in this volume p. 68*ff* and, for a full presentation, see Miéville (1984 or 2001-04).

[12] For the full presentation of the following logicist construction, with proofs and technical details, see Gessler, Joray, Degrange (2005: 73-137). The construction is partially inspired from Canty (1967).

D6. $\lfloor a \rfloor \lceil 0\{a\} \equiv \sim \lfloor \exists b \rfloor \lceil b \varepsilon a \rceil \rceil$ $\quad$ Def_S (zero)

D7. $\lfloor a \rfloor \lceil 1\{a\} \equiv a \varepsilon a \rceil$ $\quad$ Def_S (one)

Now, two and three can be defined in the following way:

D14. $\lfloor a \rfloor \lceil 2\{a\} \equiv \lfloor \exists b \rfloor \lceil b \varepsilon a \wedge 1\{a - b\} \rceil \rceil$ $\quad$ Def_S (two)

D15. $\lfloor a \rfloor \lceil 3\{a\} \equiv \lfloor \exists b \rfloor \lceil b \varepsilon a \wedge 2\{a - b\} \rceil \rceil$ $\quad$ Def_S (three)

The idea is very simple: in order to define the successor *n'* of a previously defined natural number *n*, one have to state that a name *a* has the number *n'* iff a name which denotes exactly the *a*'s exepted one of them has the number *n*. This gives rise to the general definition of the *successor* of a nominal property:

D16. $\lfloor \varphi\, a \rfloor \lceil \text{S}\backslash \varphi / \{a\} \equiv \lfloor \exists b \rfloor \lceil b \varepsilon a \wedge \varphi\{a - b\} \rceil \rceil$ $\quad$ Def_S (successor)

From D16, it is obvious that a symbol $\underline{n}$ for any natural number $n > 0$ can be introduced with a definition of the following form, where 'S\–/' is iterarted *n* times:

$\lfloor a \rfloor \lceil \underline{n}\{a\} \equiv \text{S}\backslash\, \text{S}\backslash \ldots \text{S}\backslash 0/ \ldots // \{a\} \rceil$ $\quad$ Def_S

From these definitions of particular numbers, the point is now to generalize. As in other logicist programmes this is done with the relation of equinumerosity. This relation obtains between two names when their references are in a one-one correspondance. More precisely, we need first the definitions of one-one relations, and of the domain and counter-domain of a relation:

D17. $\lfloor R \rfloor \lceil \text{OneOne}[R] \equiv$
$\lfloor abc \rfloor \lceil ((R\{ac\} \wedge R\{bc\}) \vee (R\{ca\} \wedge R\{cb\})) \supset \doteq\{ab\} \rceil \rceil$ Def_S

D18. $\lfloor aR \rfloor \lceil a \,\varepsilon\, \text{Dom}<R> \equiv (a \varepsilon a \wedge \lfloor \exists b \rfloor \lceil R\{ab\} \rceil) \rceil$ $\quad$ Def_N

D19. $\lfloor aR \rfloor \lceil a \,\varepsilon\, \text{Cdom}<R> \equiv (a \varepsilon a \wedge \lfloor \exists b \rfloor \lceil R\{ba\} \rceil) \rceil$ $\quad$ Def_N

Notice that the use of '$\doteq\{ab\}$' (*a* and *b* denote the same object) instead of '$=\{ab\}$' (*a* and *b* have the same references) at the end of D17 restricts the notion of one-one relation to relations which obtain only between *singular* names. Then a one-one relation expresses always a correspondance between objects[13]. Now comes the definition of the nominal relation of *equinumerosity*:

$$\text{D20. } \lfloor ab \rfloor \lceil a \infty b \equiv \lfloor \exists R \rfloor \lceil (\text{OneOne}[R] \wedge \text{Dom}<R> = a \wedge \text{Cdom}<R> = b \rceil \rceil \quad \text{Def}_S$$

The cardinality of a name is thus very simply expressed by the introduction of the parametric version of '∞':

$$\text{D21. } \lfloor ab \rfloor \lceil \infty\langle a \rangle \{b\} \equiv a \infty b \rceil \qquad \text{Def}_S$$

By the abstraction of '*b*' in '$a \infty b$', one obtains the complex functor '$\infty\langle a \rangle$', which expresses the nominal property "denoting as many objects as *a*" or "having the cardinality of *a*". '$\infty\langle - \rangle$' is then a multi-link functor which gives the cardinal property of the name to which it is applied. As numbers in this construction are properties of names, it is natural to read '$\infty\langle a \rangle$' as "the cardinal number of *a*"[14] and the following theorem, which is easy to derive from D21, as the Leśniewskian version of Hume's Principle:

$$\lfloor ab \rfloor \lceil \infty\langle a \rangle \approx \infty\langle b \rangle \;\equiv\; a \infty b \rceil$$

Contrary to the Fregean version, the left hand side does not express an identity between singular names, but an identity between nominal functors. Leśniewskian versions of Fregean "abstraction principles" are strictly predicative – the value of the arguments of the identity sign in the left hand side is not among the possible values of the variables in the right hand side. This has important consequences on the present construction. First, Leśniewskian versions never lead to contradiction. In particular, the analogue of Frege's Basic Law V is perfectly harmless and can be easily infered from D12:

$$\lfloor ab \rfloor \lceil =\langle a \rangle \approx =\langle b \rangle \;\equiv\; a = b \rceil$$

[13] Of course a relation can also obtain between plural and even empty names, but when it only obtains between singular names, it corresponds to a relation between the objects denoted by the names in question. On this issue, very specific to Ontology, see Joray (1999: 187-190).

[14] This is of course only a *façon de parler*, for '$\infty\langle a \rangle$' is not the name of an object, but a symbol for a function. In natural languages, nominalization is a very usefull way to state dependant or incomplete meanings as objects of the discourse. In the present context, it remains harmless as long as it does not come with a reification of the dependant meaning. On this issue, see N. Gessler's paper in this volume.

Secondly, the fact that abstraction's results are not designated as objects preserves the ontological neutrality of logic. Theorems of Leśniewski's calculus are logicaly true in the sense they are true in all domains, included the empty one. As we will see further, a consequence of this is that there will be no way to avoid the addition of an axiom of infinity for the derivation of all Peano's propositions.

From D21, the general definition of *cardinal number* can be stated as:

D22. $\lfloor \varphi \rfloor \lceil \mathrm{Cn}[\varphi] \equiv \lfloor \exists a \rfloor \lceil \infty\langle a \rangle \approx \varphi \rceil \rceil$ $\quad$ Def_S

Now, in order to specify which cardinal numbers are natural numbers the definition of finite names is required. Like in Frege's *Grundlagen*, this will be done using the notion of inductivity: a name is said to be *finite* or *inductive* if it has all the properties of the empty name that are preserved by the addition of a single denotation:

D23. $\lfloor a \rfloor \lceil \mathrm{Ind}\{a\} \equiv$
$\lfloor \varphi \rfloor \lceil (\varphi\{\Lambda\} \wedge \lfloor bc \rfloor \lceil (\varphi\{c\} \wedge 1\{b\}) \supset \varphi\{c{+}b\} \rceil) \supset \varphi\{a\} \rceil \rceil$

From this, *natural numbers* can be caracterised as the cardinal numbers of finite names:

D24. $\lfloor \varphi \rfloor \lceil \mathrm{Nn}[\varphi] \equiv (\mathrm{Cn}[\varphi] \wedge \lfloor a \rfloor \lceil \varphi\{a\} \supset \mathrm{Ind}\{a\} \rceil) \rceil$ $\quad$ Def_S

D6, D16 and D24 are the respective definitions in Leśniewski's Ontology of Peano's primitive terms *zero*, *successor* and *number*. It has been demonstrated that Peano's propositions I, IV and V are derivable from these definitions in pure Ontology

P_I $\quad$ $\mathrm{Nn}[0]$
(zero is a number)

P_{IV} $\quad$ $\lfloor \varphi \rfloor \lceil \mathrm{Nn}[\varphi] \supset \mathrm{S}\backslash \varphi / \neq 0 \rceil$
(zero is not the successor of a number)

P_V $\quad$ $\lfloor P \rfloor \lceil (P[0] \wedge \lfloor \varphi \rfloor \lceil (\mathrm{Nn}[\varphi] \wedge P[\varphi]) \supset P[\mathrm{S}\backslash \varphi /] \rceil) \supset$
$\lfloor \psi \rfloor \lceil \mathrm{Nn}[\psi] \supset P[\psi] \rceil \rceil$
(mathematical induction)

the remaining two propositions being derivable in infinite Ontology:

P_{II} $\lfloor \varphi \rfloor \lceil \mathrm{Nn}[\varphi] \supset \mathrm{Nn}[S\backslash\varphi/] \rceil$
(the successor of a number is a number)

P_{III} $\lfloor \varphi\psi \rfloor \lceil (\mathrm{Nn}[\varphi] \wedge \mathrm{Nn}[\psi]) \supset (S\backslash\varphi/ = S\backslash\psi/ \supset \varphi = \psi) \rceil$
(different numbers have different successors)

It would be inappropriate, in such a presentation, to give the proofs which are long and have already been published with all the technical details in (Gessler, Joray, Degrange 2005: 75-137).

In a way quite similar to what is done in the *Principia Mathematica*, it is now possible to explicitly define addition and multiplication[15]. A full picture of Peano Arithmetic is thus constructed in a third-order development of infinite Ontology: a system of pure logic with the addition of an axiom of infinity[16].

Just notice that the dependance of Peano propositions vis-à-vis the single non-logical axiom is not exactly like in the *Principia*, for not only P_{III}, but also P_{II} (the successor of a number is a number) requires the existence of infinitely many objects. This is due to the fact that P_{II} cannot be read here as "ambiguous as to type", avoiding the very artificial meaning of P_{II} in the *Principia*: *for every number* n, *there is a type* t *in which the successor of* n *(in fact the analogue of* n *for* t*) is a number*.

As it has been shown by Nadine Gessler[17], type (or categorial) ambiguity is not needed to warrant the unity of all the higher-degree arithmetics which can be developed in Ontology. Anyhow, since what is to be constructed is not arithmetic *itself*, but a logical *picture* of it – an interpretation of general arithmetic in a system of certain definite

[15] Cf. Joray (2002).

[16] As Dedekind's finitude and inductivity are only equivalent with the principle of choice, the axiom of infinity in the 2005 publication (using Dedekind's notion) is too strong and can be replaced by a formula expressing there exists a name which is not inductive:
$\lfloor \exists a \rfloor \lceil \sim \mathrm{Ind}\{a\} \rceil$
which is an abbreviation of the official axiom which must be stated without any defined term:
$\lfloor \exists \varphi \rfloor \lceil \lfloor a \rfloor \lceil \sim \lfloor \exists b \rfloor \lceil b\varepsilon a \rceil \supset \varphi\{a\} \rceil \wedge \lfloor abc \rfloor \lceil (a\varepsilon a \wedge \varphi\{b\} \wedge \lfloor d \rfloor \lceil d\varepsilon c \equiv (d\varepsilon b \vee d\varepsilon a) \rceil) \supset \varphi\{c\} \rceil \wedge$
$\sim \lfloor a \rfloor \lceil \varphi\{a\} \rceil \rceil$.

[17] Cf. Gessler, Joray, Degrange (2005: 9-36).

logical entities – the classical question of the unity of the type hierarchy of arithmetics becomes almost superfluous.

5. Conclusion

Even if the above presented construction constitutes an interpretation of Peano's general arithmetic and a way to reduce it to Leśniewski's logical system, the mathematical theory remains independant from logic. Such an approach does not provide any argument for the claim that arithmetic would be a part of logic or for an answer to ontological questions about the nature or the essence of numbers.

It is nevertheless a logicist approach for it shows the possibility to reach arithmetical knowledge in the realm of logical entities and logical laws. Each natural number is depicted through the explicit definition of a purely logical constant. On the other hand, Peano's propositions can be obtained under the assumption that the universe is not finite. This is not arithmetic itself, which is more general, but a picture of it, where arithmetic is applied to definite logical constants. But the definition of these constants is not ad hoc, for it provides a logical analysis of the naïve notions involved in the act of counting concrete objects.

Neither in common counting, nor in any application of pure arithmetic, the assumption that there will always be enough available objects for the successor of a given number to be different from the number in question implies any ontological commitment concerning the nature of the real world. The axiom of infinity is not an empirical statement concerning the world, but an hypothesis specifying the kind of idealization through which we apply arithmetic to specific concrete situations.

The given logical picture does not inform us about the ontology of abstract numbers. Neither it explains in which sense arithmetical sentences can be said to be true. Nevertheless, providing a conceptual content which guides us to Peano's axioms, it gives an analytic justification for the adoption of these axioms as forming the basis for the coherent and applicable theory of pure mathematics we know.

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