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Conjugacy growth series of groups

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Abstract

In this thesis we study the conjugacy growth series of several group constructions in terms of the standard and the conjugacy growth series of the building groups, with a specific generating set. This includes (1) groups of the form $G \wr L$ when L admits a Cayley graph that is a tree, (2) graph products, (3) a specific free product of $\mathbb{Z} * \mathbb{Z}$ with amalgamation over $\mathbb{Z}$, and (4) some HNN-extensions of graph products over isomorphic subgraph products. For all the groups mentioned we prove that the radius of convergence of the conjugacy growth series is the same as the radius of convergence of the standard growth series. We give an explicit formula for the conjugacy growth series of the groups $G \wr \mathbb{Z}$, $G \wr (C_2 * C_2)$, of the graph products, of a specific free product of $\mathbb{Z} * \mathbb{Z}$ with amalgamation over $\mathbb{Z}$, of the HNN-extension of graph products over isomorphic subgraph products based on disjoint subgraphs, and for an HNN-extension of a group of the form $H * H$ over itself by swapping the factor groups. We also prove at the end that for two infinite cardinals κ_1 and κ_2 with $\kappa_1 < \kappa_2$, there exists a group of cardinality κ_2, with κ_1 for the cardinality of its set of conjugacy classes.

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Key words: Geometric group theory, Combinatorial group theory, Conjugacy growth series, Asymptotic properties of groups, Wreath products, Formal languages, Graph products, HNN-extensions.

Résumé

Dans cette thèse nous étudions les séries de croissance de conjugaison de plusieurs groupes construits à partir d'autres groupes, en fonction des séries de croissance standard et de conjugaison des groupes de base, pour un système générateur spécifique. Ceci inclut (1) les groupes de la forme $G \wr L$ quand L admet un graphe de Cayley qui est un arbre (2) les produits graphés (3) un produit libre particulier de la forme $\mathbb{Z} * \mathbb{Z}$ avec amalgamation sur $\mathbb{Z}$, et (4) des extensions HNN de produits graphés sur des sous-produits graphés isomorphes. Pour tous ces groupes mentionnés, on prouve que le rayon de convergence de la série de croissance de conjugaison est le même que celui de la série de croissance standard. Nous donnons une formule explicite pour la série de croissance de conjugaison des groupes $G \wr \mathbb{Z}$, $G \wr (C_2 * C_2)$, de produits graphés, d'un produit libre particulier de la forme $\mathbb{Z} * \mathbb{Z}$ avec amalgamation sur $\mathbb{Z}$, d'extensions HNN de produits graphés sur des sous produits graphés isomorphes basés sur de sous-graphes disjoints, et pour une extension HNN de la forme $H * H$ sur lui-même en intervertissant les facteurs de groupes. Nous prouvons aussi à la fin de ce document que pour deux cardinaux infinis κ_1 et κ_2 avec $\kappa_1 < \kappa_2$, il existe un groupe de cardinalité κ_2, avec κ_1 pour la cardinalité de son ensemble de classes de conjugaison.

Mots clés: Théorie géométrique des groupes, Théorie combinatoire des groupes, Séries de croissance de conjugaison, Propriétés asymptotiques des groupes, Produits en couronne, Langages formels, Produits graphés, Extensions HNN.

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Contents

Chapter 1

Introduction

The topic of this thesis belongs to the broad area of combinatorial and geometric group theory. *Combinatorial group theory* traces its origin to the work of Dehn, Poincare, Tietze and others in the beginning of the 20th century and it revolves around studying groups based on their presentations, given by a set of generators and a set of relations [CM82]. Towards the end of the 20th century the focus shifted to *geometric group theory* [dlH00] by viewing groups as metric spaces: the group elements can be seen as vertices of a graph, where two elements are connected by an edge if one can obtain one from the other by multiplication on the left by a generator. This graph is called the Cayley graph of the group with respect to the given presentation, and if the generating set is symmetric (closed under taking inverses), then the Cayley graph admits the graph metric and hence one can view any group as a metric space on which various actions can be studied.

More precisely, this work is concerned with conjugacy growth in discrete groups. This is related to standard growth of groups, which has been extensively studied and is defined as follows. For every finitely generated group, an element can be written as a word on the generators, and choosing a shortest possible word for representing the element gives rise to the notion of length of a group element, that is, the minimal number of letters of a word that represents the element. Given a positive integer n, a natural question that arises is how many elements of length n there are in our group. This gives rise to the notion of the *standard growth function* $\mathfrak{S}_Z(n)$ of a group G with respect to a generating set Z, which is well defined if the generating set is finite. The data given by the growth function can be collected into the *standard growth series* $\sigma_{(G,Z)}(z)$ of the group G, that is, the complex power series taking the values of the standard growth function as coefficients. The advantage of considering the full standard growth series instead its coefficients separately is, for example, that the standard growth series of a direct, free or certain amalgamated products and HNN extensions can be expressed as algebraic expressions in the standard growth series of the initial groups with respect to appropriate generating sets. The standard growth function and the standard growth series of a given group depend on the generating set. There are two aspects to consider here: the combinatorial one and the asymptotic one. Changing the generating set does not change the asymptotic behaviour of the standard growth function because changing the generating set gives rise to a quasi-equivalent standard growth function. Moreover the standard growth function of a group is quasi-equivalent to the standard growth function of any of its finite index subgroup. However, changing the generating set might alter the algebraic properties of the growth series [Sto96].

We can interpret the standard growth series of a group as coming from an equivalence relation on the language of all words over the group generators, where two words are equivalent if they represent the same group element. More generally, any equivalence relation on the given group can be applied to the language of all words over the generators to obtain a new kind of growth series for the group. For example, any group admits conjugation as equivalence relation, hence it is natural to consider the equivalence relation on the set of words over the generators defined as ‘two words are equivalent if and only if they represent conjugate elements’. This leads us to the concept of conjugacy growth function and conjugacy growth series.

The conjugacy growth function of a finitely generated group G records the number of conjugacy classes with a minimal length representative on the sphere of radius n in the Cayley graph of G, for all $n \geq 0$. As for standard growth, this naturally leads to the *conjugacy growth series* $\tilde{\sigma}_{(G,Z)}(z)$, that is, the complex power series with the conjugacy growth values as coefficients. The conjugacy growth function and the conjugacy growth series were studied for several kind of groups.

The conjugacy growth function of free groups was studied in [Riv04] and [Riv10], and sharp approximations were computed in [Coo05]. Bounds for the conjugacy growth function depending on the standard growth rate for hyperbolic groups were given by Coornaert and Knieper in [CK02] and [CK04]. For solvable groups the paper [BdC10] shows that finitely generated solvable groups which are not virtually nilpotent have exponential conjugacy growth function. Linear groups were considered in [BdCLM13], where the authors show that finitely generated non-virtually-nilpotent subgroups of GL_d have uniform exponential conjugacy growth. In the paper [AC17] the authors show that the conjugacy growth series of a non-elementary hyperbolic group is transcendental. In [Fin14] the author gives a lower bound for the conjugacy growth function of certain branch groups, among them the Grigorchuk group; this bound is a function of intermediate growth. In [GS10] conjugacy growth for several classes of groups was discussed: among them, the Baumslag–Solitar group $BS(1, n)$ and the Houghton group $S_\infty \rtimes \mathbb{Z}$, for which the conjugacy growth function is exponential. In the paper [BdlH16] the authors give conjugacy growth series of some infinitely generated groups; this is possible because even if the standard growth series is not defined for this kind of groups, there are only finitely many conjugacy classes of a given length.

Historically, one of the initial motivations for counting conjugacy classes of a given length came from counting closed geodesics of bounded length in compact Riemannian manifolds (see [Mar69] for example).

As for standard growth in groups, one has to ask how conjugacy growth depends on the generating set. Similarly to the standard growth function, changing the generating set gives rise to a quasi-isometric conjugacy growth function. However, something very different happens for the conjugacy growth function if we pass to finite index subgroups. While the asymptotic behaviour of the standard growth is preserved by taking finite index subgroups, this is absolutely not the case for conjugacy growth function. The paper [HO13] produces a finitely generated group of exponential conjugacy growth with a subgroup of index two with only 2 conjugacy classes. This implies in particular that the kind of asymptotic behaviour of the conjugacy growth is not a quasi-isometric invariant and hence that there will definitely not be an analogue of Gromov’s Theorem for conjugacy growth function.

We remark that not every non-decreasing function represents the standard growth function of a group. However, [HO13] shows that any non-decreasing function bounded by an exponential can be realized as the conjugacy growth function of a group. Nevertheless, the groups constructed by Osin and Hull in [HO13] are infinitely presented and we do not know if such unintuitive behaviors can occur in finitely presented groups.

1.1 Results and organization of the thesis

In this thesis we compute the conjugacy growth series of groups built from other groups. In each instance there is a canonical generating set coming from the generating sets of the other groups, and we express the conjugacy growth series of our groups in terms of the standard and conjugacy growth series of those of the building groups. Most of the work in this thesis is of a combinatorial nature, with the goal to obtain an exact formula for the conjugacy growth series of a group with respect to a suitable generating set. We did not cover different generating sets for the same group and hence we cannot claim to produce group invariants. At the core of the thesis is showing how counting conjugacy classes is equivalent to counting nice combinatorial objects, such as 'necklaces' made up of 'pearls', which are particular words build up to cyclic permutations.

Related to computing conjugacy growth series, we study conjugacy growth from an asymptotic point of view by considering the conjugacy growth rate of the group, that is, the inverse of the radius of convergence of the conjugacy growth series. In [GS10] it was conjectured that for every amenable group of exponential growth, the conjugacy growth function is exponential, and that an amenable group with polynomial conjugacy growth function is virtually nilpotent. Besides amenable groups, further investigation shows that the above conjecture can be strengthened and holds for many other classes of groups: that is, with the notable exception of the 'monster' groups of Osin [Osi10] and Ivanov [Ol'91, Theorem 41.2] that have a finite number of conjugacy classes but exponential standard growth, or the group in [HO13] of exponential growth with two conjugacy classes, for the groups we consider not only is the conjugacy growth function exponential when the standard growth function is exponential, but the two functions have the same growth rate.

In Chapter 2 we introduce the necessary prerequisites for growth series and conjugacy growth series in finitely generated groups. In Section 2.3 we give some basic constructions such as free products with amalgamation and HNN-extensions over admissible subgroups, and we give the formula to compute the standard growth series. Our main tool in this thesis to compute the conjugacy growth series of a group is the *Necklaces series* NLS of a language, which is explained and computed in Section 2.4.

In Chapter 3 we investigate conjugacy growth from both a formal and an asymptotic point of view in wreath products of the form $G \wr L$, where L is a group which admits a Cayley graph that is a tree. We consider a natural generating set of $G \wr L$ built out of the standard generating sets of G and L (as defined in (3.1)). The computations in this chapter rely on describing a set of minimal length conjugacy representatives, and lead to some explicit conjugacy growth series formulas for the groups $G \wr \mathbb{Z}$ and $G \wr (C_2 * C_2)$ in Section 3.4, as well as an asymptotic estimate of the conjugacy growth of the Lamplighter group L_2 that proves that its conjugacy growth series is transcendental over $\mathbb{Q}(z)$ in Proposition 3.21.

One of our main tools comes from the paper [Par92], where the author expresses the standard growth series of a group $G \wr L$, when L admits a tree as its Cayley graph, in terms of the standard growth series of G. In order to compute the conjugacy growth in a group one needs to know when different elements are conjugate, and we use the criteria for solving the conjugacy problem developed in the paper [Mat66]. We need to make distinction between the conjugacy classes having cursor position of infinite order, finite order, or the trivial element. In Section 3.3 we describe the conjugacy growth series of these three types of conjugacy classes, and in Section 3.4 we give explicit formulas for some examples. When L admits a Cayley graph which is an infinite tree and not a line, we were unable to obtain a general formula for the conjugacy growth series; the difficulty resides in counting orbits of finite sub-trees in the Cayley graph of L containing the origin, under the left action of L.

One of our results in Section 3.3.1, Corollary 3.17, shows that the radius of convergence of the conjugacy growth series of $G \wr L$ is the same as the radius of convergence of the standard growth series of $G \wr L$, for all groups where L has a tree as its Cayley graph. This kind of groups are infinitely presented in general (except if L is finite and G finitely presented).
The other main results of this Chapter are Propositions 3.15, 3.18 and 3.19, which together describe the conjugacy growth series of a group $G \wr L$, for all groups where L has a tree as its Cayley graph.

Let $L = \mathbb{Z}$ or $L = C_2 * C_2$, and let $\vec{Y}$ be the canonical generating set of $G \wr L$ build up in (3.1) on page 19, and let $\phi(r)$ be the Euler's totient function of r.
The following formulas can be found in Section 3.4:

$$\tilde{\sigma}_{(G\wr\mathbb{Z},\vec{Y})}(z) = 2\sum_{r\geq 1}\frac{\phi(r)}{r}\sum_{s\geq 1}\frac{z^{rs}\sigma_{(G,Y)}(z^r)^s}{s} + \tilde{\sigma}_{(G,Y)}(z) + \frac{z^2(\tilde{\sigma}_{(G,Y)}(z)-1)^2}{1-z^2\tilde{\sigma}_{(G,Y)}(z)}, \text{ and}$$

$$\begin{aligned}\tilde{\sigma}_{(G\wr(C_2*C_2),\vec{Y})}(z) = {} & 2\sum_{r\geq 1}\frac{\phi(r)}{r}\sum_{s\geq 1} z^{2rs}\sigma_{(G,Y)}(z^r)^{2s} + 2z\tilde{\sigma}_{(G,Y)}(z)\frac{1-z^2}{1-z^2\tilde{\sigma}_{(G,Y)}(z)} \\ & + \tilde{\sigma}_{(G,Y)}(z) + z^2\left(1+z^2\tilde{\sigma}_{(G,Y)}(z)\right)\frac{(\tilde{\sigma}_{(G,Y)}(z)-1)^2}{1-z^4\tilde{\sigma}_{(G,Y)}(z)^2} + z^2\frac{(\tilde{\sigma}_{(G,Y)}(z^2)-1)}{1-z^4\tilde{\sigma}_{(G,Y)}(z^2)}.\end{aligned}$$

We point out that conjugacy growth series for some wreath products have been recently computed in [BdlH16], but the groups studied there are infinitely generated (unless they are finite), and different methods apply. One can also find the papers [Loc16] and [Wag16] for some conjugacy growth series of other infinitely generated wreath products.

Chapter 3 gave rise the preprint [Mer17], submitted for publication.

In Chapter 4 we study graph products. The graph product construction is a natural generalization of both direct and free products. Given a finite simplicial graph Γ with a group attached to each vertex, the associated *graph product* is the group generated by the vertex groups with the added relations that elements of groups attached to adjacent vertices commute. Right-angled Artin groups (also known as graph groups) and right-angled Coxeter groups arise in this way, as the graph products of infinite cyclic groups and cyclic groups of order 2 respectively, and have been widely studied. Graph products were introduced by Green in her PhD thesis [Gre90] where, in particular, a normal form was developed, and their growth series have been computed by Chiswell and others (for RAAGs), given the growth series of the vertex groups.

The Chiswell formula for the standard growth series has an elegant form in terms of the standard growth series of the building groups and also in terms of the subgraphs. We give an analogous formula for the conjugacy growth series of a graph product in Section 4.2:

Theorem 4.14
The formula for the conjugacy growth series of the graph product G_V is given by

$$\tilde{\sigma}_V(z) = \sum_{\Delta\in\mathsf{Clq}(V)}\prod_{v\in\Delta}(\tilde{\sigma}_v(z)-1)\Psi(\mathsf{Ct}(\Delta))(z),$$

where Ψ is defined on page 53 and on page 75.

An immediate consequence of this result is Theorem 4.16, which shows that the radius of convergence of a graph product is the same as the radius of convergence of its standard growth series, provided that the graph is not complete or that all the building groups satisfy this property. Theorem 4.14 is based on Proposition 4.12, which gives the formula for the conjugacy growth series of a graph product in terms of the conjugacy growth series of some chosen vertex and some sub-graphs products, as well as Proposition 4.6, which gives an explicit formula in terms of a full graph, for a function from a graph to a ring that is defined in term of induction on some sub-graphs.

In Section 4.3 we use the formula of Theorem 4.14 to compute the conjugacy growth series for some particular graphs such as the 3 by 3 bipartite graph, a line, a polygon and a house (on 5 vertices).

Chapter 4 is part of a preprint with Laura Ciobanu and Susan Hermiller [CHM17], to be submitted for publication.

In Chapter 5 we study further constructions of groups and their conjugacy growth series. While a graph product can be seen as a particular kind of free product with amalgamation, we wanted to also give the conjugacy growth series of a free product with amalgamation that is not coming from a graph product. In Section 5.1.1 of Chapter 5 we consider the free product with amalgamation

$$\begin{tikzcd}[column sep=small] G(n,m) := \mathbb{Z} & * & \mathbb{Z}, = \langle\{x,x^{-1},y,y^{-1}\}\,|\,x^m=y^m\rangle \quad \text{with } n,m\in\mathbb{N}\setminus\{0\}. \\ & \mathbb{Z} \arrow[ul, hook', "\times m"] \arrow[ur, hook, "\times n"'] & \end{tikzcd}$$

Corollary 5.2
The conjugacy growth series of $G(n,m)$ with respect to the generating set $\{x, x^m, x^{-m}, y, y^n, y^{-n}\}$ is given by

$$\tilde{\sigma}_{(G(n,m),\{x,x^m,x^{-m},y,y^n,y^{-n}\})}(z) = \frac{1+z}{1-z}\left(1+\mathsf{NLS}\left(z^2\frac{(1-z^{m-1})(1-z^{n-1})}{(1-z)^2}\right)\right).$$

In particular, the radius of convergence of $\tilde{\sigma}_{(G(n,m),\{x,x^m,x^{-m},y,y^n,y^{-n}\})}(z)$ is the same the radius of convergence of $\sigma_{(G(n,m),\{x,x^m,x^{-m},y,y^n,y^{-n}\})}(z)$.

In Chapter 5 we also study HNN-extensions of some graph products. In a graph product, if the underlying graph admits two isomorphic subgraphs V_1, V_{-1} with an isomorphism $\bar{\cdot}: V_1 \to V_{-1}$ such that for every $v \in V_1$, the vertex group $G_{\{v\}}$ is isomorphic to the vertex group $G_{\{\overline{v}\}}$, then the graph product G_V admits two isomorphic subgraph products. Moreover with our choice of generating sets, these two isomorphic subgraph products are admissible in the full graph product. Hence it is natural to study the HNN-extension over these two subgraph products. We study this kind of construction in Section 5.2, and prove the following:

Corollary 5.5
Let G_V be a graph product, with two isomorphic subgraph products G_{V_1} and $G_{V_{-1}}$. If for each of the groups $G_{\{v\}}$ $v \in V$ the radius of convergence of its standard growth series is the same as the radius of convergence of its conjugacy growth series, then the same holds for the HNN-extension of G_V over G_{V_1} and $G_{V_{-1}}$.

In Section 5.2.1 we give an explicit formula for the conjugacy growth series of the HNN-extension of a graph product when the two isomorphic subgraph products are based on disjoint subgraphs V_1 and V_{-1}, in terms of the conjugacy growth series of the subgraph product (hence by Theorem 4.14 one can have an explicit formula). Let

$$G = \langle X_V \cup \{t, t^{-1}\}\,|\, tht^{-1} = \overline{h},\ \forall h \in G_{V_1}\rangle,$$

where $\bar{\cdot}: G_{V_1} \to G_{V_{-1}}$ is the induced isomorphism between the two subgraph products.

Proposition 5.6
Let $S := X_V \cup \{t, t^{-1}\}$. The conjugacy growth series of G with respect to the generating set S is given by

$$\begin{aligned}\tilde{\sigma}_{(G,S)}(z) = {} & \tilde{\sigma}_V(z) - \tilde{\sigma}_{V_1}(z) + 1 + \frac{2z}{1-z} + \mathsf{NLS}\left(\frac{2z}{1-z}\left(\frac{\sigma_V(z)}{\sigma_{V_1}(z)} - 1\right)\right) \\ & + \sum_{\emptyset \neq V' \subset V_1} \tilde{\sigma}^{\mathcal{M}}_{V'}(z)\mathsf{NLS}\left(z^2\left(\frac{\sigma_{\mathsf{Ct}(\overline{V'})}(z)}{\sigma_{V_1 \cap \mathsf{Ct}(\overline{V'})}(z)} - 1\right)\left(\frac{\sigma_{\mathsf{Ct}(V')}(z)}{\sigma_{V_{-1} \cap \mathsf{Ct}(V')}(z)} - 1\right)\right),\end{aligned}$$

where $\tilde{\sigma}^{\mathcal{M}}_{V'}(z)$ refers to the conjugacy growth series of the elements having a minimal representative with support exactly V' (see Section 4.1.1 and Lemma 4.10), and NLS *refers to the Necklaces series, which is defined in Section 2.4.*

We conclude with an example of an HNN-extension of a graph product, Proposition 5.8 in Section 5.2.2.2, which gives the conjugacy growth series of the HNN-extension of $H * H$ over itself which swaps the factor groups. This last example cannot be solved by Proposition 5.6. We finish the chapter by pointing out the difficulties encountered when trying to obtain a general formula for any HNN-extension of graph products over subgraph products.

We would like to point out that our choice of generating sets built from others generating set have the properties to be minimal (this means that removing one generator yields the set to not be a generating set anymore), assuming the building generating sets are. Also, most of our generating sets are symmetric, except for the generating set in Section 5.1.1, where it would not have been possible to have admissible subgroups with a symmetric generating set.

One surprising fact we observed is that, except in the case of direct products, the conjugacy growth series of the groups we built from others depends not only on their conjugacy growth series, but also on their standard growth series.

In Appendix A we prove the following Theorem:

Theorem A.1
Let κ_1, κ_2 be two infinite cardinals such that $\kappa_1 < \kappa_2$. Then there exists a group G such that $\sharp G = \kappa_2$ and $\sharp G_\sim = \kappa_1$.

To finish, let us mention the following even if it not really related to this thesis. We would like to explain why conjugation is natural in a formal sense. In category theory, (see [ML98][Chapter 1] for background on this) a group can be seen as a category with only one object and where the morphisms are the elements of the group (the composition law being the multiplication of the group). With this point of view a functor between 2 such categories is equivalent to a homomorphism of groups. The advantage of this approach is that between two functors one can consider the natural transformations. The following happens. If G_1, G_2 are two groups seen as categories as explained above, and if $f, h : G_1 \to G_2$ are two homomorphisms seen as functors, then there is a natural transformation $\alpha : f \Rightarrow h$ if and only if there is an element $g_\alpha \in G_2$ such that for every $g \in G_1$, $f(g) = g_\alpha^{-1} h(g) g_\alpha$. More material about categories can be found in [AHS06].

A table of some notation currently used can be found at page 75.

Chapter 2

Preliminaries

In this section we introduce all the background necessary for defining the conjugacy growth series of a group. As mentioned in the introduction, this is a complex power series where each coefficient records the number of conjugacy classes of a given length; hence we introduce complex power series, and their basic properties in Section 2.1. Section 2.2 defines the standard and conjugacy growth functions, as well as the standard and conjugacy growth series of a group, with respect to a given generating set. In Section 2.3 we define admissible subgroups and show some basic properties. This is a very nice property for the computation of standard growth series because if a group admits an admissible subgroup, then the standard growth series of the full group admits the standard growth series of its admissible subgroup as factor. We then present how to express the standard growth series in free products with amalgamation and HNN-extensions over admissible subgroups. Since we need to consider quotient languages under certain group actions, we introduce a tool in Section 2.4 that allows us to compute the growth series of a language that consists of words from another language, up to cyclic permutation.

2.1 Growth series and languages

We first recall some basic facts about power series in complex analysis (see for example the reference [Con78, Chapter III Section 1]). We denote the open disc of radius $r > 0$ centered at $c \in \mathbb{C}$ by $\mathscr{D}(c, r) := \{z \in \mathbb{C} : |z - c| < r\}$, and define a *complex power series* as a function $f : \mathscr{D}(0, r) \to \mathbb{C}$ of the form $f(z) = \sum_{j=0}^{\infty} a_j z^j$, where $a_j \in \mathbb{C}$ for all j. We express the fact that a_j is the *coefficient* of z^j by writing

$$[z^j]f(z) := a_j.$$

The *radius of convergence* $\mathcal{RC}(f)$ of f can be defined as

$$\mathcal{RC}(f) = \sup\{r \in \mathbb{R} : f(z) \text{ converges } \forall\, z \in \mathscr{D}(0, r)\},$$

or equivalently as

$$\mathcal{RC}(f) = \frac{1}{\limsup_{j\to\infty} \sqrt[j]{|a_j|}}.$$

If $\mathcal{RC}(f) > 0$, then f is defined at every point in the open disc $\mathscr{D}(0, \mathcal{RC}(f))$ and f converges absolutely. In general, $f(z)$ is not defined for z with $|z| = \mathcal{RC}(f)$.

The following proposition will be used later on for the computation of the radius of convergence of some growth series.

Proposition 2.1
Let $f \neq 0$ be a complex power series such that $\mathcal{RC}(f) > 0$, each coefficient $[z^j]f(z)$ is a non-negative number, and $f(0) = 0$. Then there exists a unique positive number $t > 0$ such that $f(t) = 1$ and

$$t = \inf\{|z| \,:\, |f(z)| = 1\} = \sup\{r > 0 \,:\, |f(z)| \leq 1, \forall z \in \mathscr{D}(0, r)\},$$

and the infimum and supremum are attained.

Proof. Let $f = \sum_{n=1}^{\infty} a_n z^n$, where $a_n \in \mathbb{N}$ for all n. On the interval $[0, \mathcal{RC}(f)[$ the function f is strictly increasing and continuous, so there exists a unique $t \in \mathbb{R}^{>0}$ such that $f(t) = 1$. Now for any $|z| \leq t$ the following holds:

$$|f(z)| = |\sum_{n=1}^{\infty} a_n z^n| \leq \sum_{n=1}^{\infty} a_n |z^n| \leq \sum_{n=1}^{\infty} a_n t^n = f(t) = 1.$$

☐

Definition
Let X be a set. We call the elements of X *letters* and write X^* for the set of words over X. A subset $L \subseteq X^*$ is called a *language*, and the empty word is denoted by ϵ.

Let L be a language and $\simeq$ be an equivalence relation on L. For $l \in L$, we write

$$[l]_{\simeq} := \{l' \in L \,:\, l' \simeq l\}$$

for the *equivalence class* of l, and

$$L \big/_{\simeq} := \{[l]_{\simeq} \,:\, l \in L\}$$

for the *quotient language of* L *by* $\simeq$.

Notation
If X is a set, we write $\sharp X$ for its cardinal.

We now associate a complex power series to a language.

Definition
For $w \in X^*$, the *length* of w is the number of letters in w, we write it as $|w|$. If $l = (l_1, \dots l_n) \in {X^*}^n$, the *length* of l is defined to be $|(l_1, \dots, l_n)| := \sum_{j=1}^{n} |l_j|$.

Let L be a language such that for every $m \in \mathbb{N}$ the set $\{l \in L \,:\, |l| = m\}$ is finite, (this is for example the case if the underlying alphabet is finite). Then we define the *growth series of* L, $F_L(z)$, to be the complex power series given by

$$[z^m]F_L(z) := \sharp\{l \in L \,:\, |l| = m\}.$$

Notation
Let L be a language and $\simeq$ an equivalence relation on L. For $[l]_{\simeq} \in L \big/_{\simeq}$ we define the *length* of $[l]_{\simeq}$ to be

$$|[l]_{\simeq}| := \min\{|l'| \,:\, l' \in [l]_{\simeq}\}.$$

If L and $\simeq$ are such that for every $m \in \mathbb{N}$ the set $\{[l]_{\simeq} \in L \,:\, |[l]_{\simeq}| = m\}$ is finite, we define the *growth series of* $L \big/_{\simeq}$, $F_{L \big/_{\simeq}}(z)$, to be the complex power series given by

$$[z^m]F_{L \big/_{\simeq}}(z) := \sharp\{[l]_{\simeq} \in L \,:\, |[l]_{\simeq}| = m\}.$$

For the practical and explicit computations of $F_{L\big/\simeq}(z)$, there are several approaches; one approach is to find exactly one representative w in each equivalence class for the relation $\simeq$ on L that satisfies $|w| = |[w]_{\simeq}|$, in such a way that we can easily compute the growth series of the language formed by these representative words. Another approach is to see the equivalence classes for the relation $\simeq$ as the orbits of a group acting on a language for which we know the growth series. In this last case we can apply Burnside's Lemma to obtain information about the growth series $F_{L\big/\simeq}(z)$.

Lemma 2.2 (Burnside, (see [Fro87] for the original proof))
Let G be a finite group acting on a finite set S. The number of different orbits in S under the action of G is given by

$$\sharp\, {}^{S}\big/_{G} = \frac{1}{\sharp G}\sum_{g\in G} \sharp\mathrm{Fix}(g),$$

where $\mathrm{Fix}(g)$ *denotes the set of elements of S fixed by g.*

2.2 Standard and conjugacy growth series of a finitely generated group

Let H be a group with finite symmetric generating set Z. For $w \in Z^*$ we write $\overline{w}$ for the corresponding element in H, so $\overline{\epsilon} = e$ for the trivial element of H. For $u, v \in Z^*$, we define $u \equiv v$ to hold if and only if $\overline{u} = \overline{v}$. The *length* of $h \in H$ relative to Z is $|h|_Z := \min\{|w| \,:\, w \in Z^*,\, \overline{w} = h\}$, and for $i \in \mathbb{N}$, we write

$$\mathfrak{S}_Z(i) := \{h \in H \,:\, |h|_Z = i\}$$

for the *sphere of radius* i in H. The growth series

$$\sigma_{(H,Z)}(z) := F_{Z^*\big/\equiv}(z),$$

for which $[z^i]\sigma_{(H,Z)}(z) = \sharp\mathfrak{S}_Z(i)$, is called the *standard growth series of* H (relative to Z).

Let us assume that no generator in Z is trivial, and no two different generators represent the same element. We define the *Cayley graph* $\mathsf{Cay}(H, Z)$ of H, with respect to the generating set Z to be the simple graph with vertex set H where two vertices h_1, h_2 are connected by an edge if and only if there exists $q \in Z$ such that $h_1 = h_2\overline{q}$.

We write $\sim_H$ (or just $\sim$ if the context of H is clear) for the equivalence relation on H given by conjugation and we write $H_\sim$ for the set of conjugacy classes of H. For $h \in H$, we write $[h] \in H_\sim$ for the conjugacy class of h. Let $\eqsim$ be the equivalence relation on Z^* defined by $w \eqsim v$ if and only if $\overline{w} \sim \overline{v}$. For $h \in H$ we write

$$|h|_{\sim,Z} := \min\{|w|_Z \,:\, w \in Z^*,\, \overline{w} \in [h]\}$$

for the *conjugacy length* of h relative to Z, i.e. the minimum length of an element in the conjugacy class of h. Hence $|[h]|_Z := |h|_{\sim,Z}$ is also well defined. Note that $|[h]|_Z = |[w]_{\eqsim}|$ for the $[w]_{\eqsim} \in {}^{Z^*}\big/_{\eqsim}$ such that $[\overline{w}]_\sim = [h]_\sim$. We write

$$\min([h])_Z := \{h' \in [h] \,:\, |h'|_Z = |h|_{\sim,Z}\}$$

for the subset of $[h]$ consisting of elements of minimal length.

The growth series

$$\tilde{\sigma}_{(H,Z)}(z) := F_{Z^* \big/ \approx}(z),$$

for which $[z^i]\sigma_{(H,Z)}(z) = \sharp\{[h]_\sim \in H_\sim \,:\, |[h]|_Z = i\}$, is called the *conjugacy growth series of* H (relative to Z).

As mentioned in Section 2.1, we would like to compute the growth series of the quotient languages $Z^* \big/ \equiv$ and $Z^* \big/ \approx$ by considering the languages formed by exactly one minimal length representative word in Z^* in each equivalence class. This leads us to consider the following languages. We use the notation from [CHHR16]:

$$\begin{aligned} \mathsf{Geo}(H,Z) &:= \{w \in Z^* \,:\, |w| = |\overline{w}|_Z\} \quad \text{is } \textit{the geodesic language}, \\ \mathsf{ConjGeo}(H,Z) &:= \{w \in Z^* \,:\, |w| = |\overline{w}|_{\sim,Z}\} \quad \text{is } \textit{the conjugacy geodesic language}. \end{aligned}$$

So $\mathsf{ConjGeo}(H,Z) = \{w \in \mathsf{Geo}(H,G) \,:\, \overline{w} \in \min([\overline{w}])_Z\}$.

We choose a *geodesic normal form*, that is, a subset $\mathsf{GeoNorm}$ of $\mathsf{Geo}(H,Z)$ that contains a geodesic representative word for every element of H. Then the growth series $\sigma_{(H,Z)}(z)$ is equal to the growth series $F_{\mathsf{GeoNorm}}(z)$.

Definition

Let H be a group with generating set Z. A subset $\mathsf{ConjGeoNorm} \subseteq \mathsf{ConjGeo}(H,Z)$ such that the map

$$\begin{tikzcd} \mathsf{ConjGeoNorm} \arrow[r] & H_\sim \\ w \arrow[r, mapsto] & {[\overline{w}]} \end{tikzcd}$$

is a bijection is called a set of *conjugacy normal forms.* The elements of the set $\{\overline{w} \,:\, w \in \mathsf{ConjGeoNorm}\}$ will be called the *conjugacy representatives.* This means that for every conjugacy class we choose an element of smallest length in this class, and associate to it a geodesic word.

Thus the growth series $\tilde{\sigma}_{(H,Z)}(z)$ is equal to the growth series $F_{\mathsf{ConjGeoNorm}}(z)$.

Whenever the conjugacy growth series of a group will be computed, we will explain which conjugacy normal form and conjugacy representatives we have chosen.

Example 2.2.1 (Fundamental group of the Klein bottle)

Let G be the group given by $G = \langle \{a, a^{-1}, b, b^{-1}\} \,|\, bab^{-1} = a^{-1} \rangle$. Since for every $k, l \in \mathbb{Z}$, $b^l a^k = a^{(-1)^l k} b^l$, one sees that every element of G can be uniquely written as $a^n b^m$. So the standard growth series of G with respect to $\{a, a^{-1}, b, b^{-1}\}$ is the same as the standard growth series of $\mathbb{Z}^2$ with respect to the usual generating set, which is

$$\sigma_{(G,\{a,a^{-1},b,b^{-1}\})}(z) = \frac{(1+z)^2}{(1-z)^2}.$$

Since for every $i, n \in \mathbb{Z}$, $ba^i b^n b^{-1} = a^{-i} b^n$, and $aa^i b^n a^{-1} = a^{1+i-(-1)^n} b^n$, one deduces that we can take as conjugacy representative set the following union

$$\{a^i b^n \,:\, i \in \mathbb{N}, n \text{ even}\} \cup \{a^i b^n \,:\, i \in \{0,1\},\, n \text{ odd}\}.$$

The growth series of the first set in this union is

$$\frac{1}{1-z}\frac{1+z^2}{1-z^2},$$

while the growth series of the second set in this union is

$$(1+z)\frac{2z}{1-z^2}.$$

Hence summing all we find

$$\tilde{\sigma}_{(G,\{a,a^{-1},b,b^{-1}\})}(z) = \frac{1+2z+z^2-2z^3}{(1-z)^2(1+z)}.$$

The ratio of the number of conjugacy classes of length m over the number of elements of length m

$$\frac{[z^m]\tilde{\sigma}_{(G,\{a,a^{-1},b,b^{-1}\})}(z)}{[z^m]\sigma_{(G,\{a,a^{-1},b,b^{-1}\})}(z)}$$

is

$$\frac{[z^m]\tilde{\sigma}_{(G,\{a,a^{-1},b,b^{-1}\})}(z)}{[z^m]\sigma_{(G,\{a,a^{-1},b,b^{-1}\})}(z)} = \begin{cases} \frac{m+3}{4m} & \text{if } m \text{ is even} \\ \frac{m+2}{4m} & \text{if } m \text{ is odd}. \end{cases}$$

Hence when $m \to \infty$,

$$\lim_{m\to\infty} \frac{[z^m]\tilde{\sigma}_{(G,\{a,a^{-1},b,b^{-1}\})}(z)}{[z^m]\sigma_{(G,\{a,a^{-1},b,b^{-1}\})}(z)} = \frac{1}{4}.$$

Remark 2.3
Note that the notion of conjugacy growth series can be defined for any group H generated by Z, provided that the set $\{[h]_\sim \in H_\sim : |[h]|_{\sim,Z} = i\}$ is finite, for every i, even if the set Z is not finite. In the paper [BdlH16] the authors considered some infinitely generated groups for which the latter sets are finite, and computed the conjugacy growth series for those groups (although the standard growth series is not well-defined).

2.3 Admissible subgroup

The notion of admissible inclusion was first introduced by Alonso in [Alo91]. He used this to express the growth series of some amalgamated products. Later Chiswell considered also admissible inclusions to express the growth series of some HNN-extensions in [Chi94b]. We introduce the notion of admissible inclusion and then the notion of admissible subgroups.

Definition ([Alo91])
Let A be a group with generating set S_A and let G be a group with generating set S. Let $\alpha : A \to G$ be a monomorphism such that $\alpha(S_A) \subset S$. We say that α is an *admissible inclusion of the pairs* $(A, S_A) \to (G, S)$ is there exists a right transversal T of $\alpha(A)$ in G such that for all $a \in A$ and for all $t \in T$ we have

$$|\alpha(a)t|_S = |\alpha(a)|_{S_A} + |t|_S.$$

If the inclusion monomorphism is obvious and we can see the group A as a subgroup then we can also consider the concept of admissible subgroup.

Definition ([Man12])
Let G be a group, H be a subgroup of G, X be a generating set of G and Y be a generating set of H. We say that H is *admissible in G with respect to the pair (X, Y)* if $Y \subset X$ and there exists a right transversal U of H in G such that, if $g = hu$, with $g \in G$, $h \in H$ and $u \in U$, then

$$|g|_X = |h|_Y + |u|_X. \tag{2.1}$$

In this case we say that U is an *admissible* right transversal of H in G.

In this document we will make no distinction between admissible subgroup or admissible inclusion when the inclusion is obvious.

Remark 2.4
With the notation of the preceding definition, the relation $\sigma_{(G,X)} = \sigma_{(H,Y)}\sigma_{(U,X)}$ holds, where $\sigma_{(U,X)}$ denotes the growth series of the elements of U with respect to X. This implies in particular that the function $\frac{\sigma_{(G,X)}}{\sigma_{(H,Y)}}$ is a power series with natural coefficients.

Proposition 2.5
Let G be a group, H be a subgroup of G, X be a generating set of G and Y be a generating set of H. Assume that H is admissible in G with respect to the pair (X, Y). Then the following hold.

1) The representative of the coset H is 1.

2) For all $h \in H$, $|h|_Y = |h|_X$.

3) The right transversal is unique.

4) Every element $t \in U$ is the unique element of Ht of minimal length with respect to X.

Proof. 1) Take $h = 1$ in the equality (2.1).

2) Take $t = 1$ in the equality (2.1).

3) Assume there exist U and U', two different admissible right transversals of H in G, and let $t \in U$ and $t' \in U'$ be such that $Ht = Ht'$. Let $h \in H$ be such that $t = ht'$, ($t' = h^{-1}t$). The fact that U is admissible implies that $|t'|_X = |h^{-1}|_X + |t|_X$ and the fact that U' is admissible implies that $|t|_X = |h|_X + |t'|_X$. Hence $|t'|_X = |h^{-1}|_X + |h|_X + |t'|_X$, and hence $|h| = |h^{-1}| = 0$, which implies that $t = t'$. Thus we find that $U = U'$.

4) Every element g in Ht, $g \neq t$ can be written as $g = ht$ with $h \neq 1$. Hence (2.1) gives $|g|_X > |t|_X$.

□

In fact Alonso proves in [Alo91][Proposition 2] a result that implies 1) and 2) in the last proposition.

Example 2.5.1 (Examples (2) and (3) in [Alo91])
The canonical inclusion $(G_1, S_1) \to (G_1 \times G_2, S_1 \sqcup S_2)$ and the canonical inclusion $(G_1, S_1) \to (G_1 * G_2, S_1 \sqcup S_2)$ are admissible.

A generalization of these two previous examples will be explained in Section 4.1.

Example 2.5.2 (Hypothesis of Lemma 1 in Section 3 of [Chi94b])
Let $G = \langle \{x, x^{-1}, y, y^{-1}\} \mid y = x^n \rangle \cong \mathbb{Z}$ for $n \in \mathbb{N}$ and let $H = \langle \{y, y^{-1}\} | \rangle$. Then H is admissible in G with respect to the pair $(\{x, x^{-1}, y, y^{-1}\}, \{y, y^{-1}\})$ if and only if n is odd.

The next two sections will motivate the consideration of admissible subgroups.

2.3.1 Free products with amalgamation

Let G_1 be a group with generating set S_1, G_2 a group with generating set S_2, and A a group with generating set S_A that is a subgroup of both G_1 and G_2 . Let us consider the free product of G_1 and G_2 with amalgamation over A,

$$G := G_1 *_A G_2.$$

In this case it is clear that $S := S_1 \cup S_2 \cup S_A$ is a generating set of G but there is a priori no obvious way to compute $\sigma_{(G,S)}(z)$ in terms of $\sigma_{(G_1,S_1)}(z)$, $\sigma_{(G_2,S_2)}(z)$ and $\sigma_{(G_A,S_A)}(z)$.

Let U be a right transversal of A in G_1 and let V be a right transversal of A in G_2.

Lemma 2.6 (Normal form Theorem for Free Products with Amalgamation [LS77] Theorem 2.6)
Every element of G can be uniquely written as

$$g = av_0u_1v_1 \cdots u_nv_nu_{n+1},$$

where $a \in A$, $n \geq 0$, $v_0 \in V$, $u_{n+1} \in U$ and for $i \in \{1, \ldots, n\}$, $u_i \in U \setminus \{e\}$ and $v_i \in V \setminus \{e\}$.

In the last expression cancellations are possible in general; however, the notion of admissibility guarantees the fact that in the normal form no cancellation occurs.

Lemma 2.7 (Proof of Proposition 14.2 in [Man12])
Assume A is admissible in G_1 with respect to the pair (S_1, S_A), and is admissible in G_2 with respect to the pair (S_2, S_A). Then, with the generating set S, the length of an element $g \in G$ written in normal form

$$g = av_0u_1v_1 \cdots u_nv_nu_{n+1}$$

obtained by Lemma 2.6, is given by

$$|g|_S = |a|_{S_A} + |v_0|_{S_2} + |u_1|_{S_1} + |v_1|_{S_2} + \ldots + |u_n|_{S_1} + |v_n|_{S_2} + |u_{n+1}|_{S_1}.$$

This Lemma implies:

Proposition 2.8 ([Alo91], [Lew91])
Assume A is admissible in G_1 with respect to the pair (S_1, S_A), and is admissible in G_2 with respect to the pair (S_2, S_A). Then

$$\sigma_{(G,S)}(z) = \sigma_{(A,S_A)}(z) \frac{\sigma_{(U,S_1)}(z)\sigma_{(V,S_2)}(z)}{1 - (\sigma_{(U,S_1)}(z) - 1)(\sigma_{(V,S_2)}(z) - 1)},$$

which is equivalent to

$$\frac{1}{\sigma_{(G,S)}(z)} = \frac{1}{\sigma_{(G_1,S_1)}(z)} + \frac{1}{\sigma_{(G_2,S_2)}(z)} - \frac{1}{\sigma_{(A,S_A)}(z)}.$$

Alonso first proved this Proposition in [Alo91] using different methods, namely Lemaire's Theorem (5.1.4) and its Corollary [[Lem74], Lemme (5.1.10)].

2.3.2 HNN-extensions

Notation
Let $i \in \mathbb{Z} \setminus \{0\}$. We define

$$\operatorname{sgn}(i) = \begin{cases} +1 & \text{if } i > 0, \\ -1 & \text{if } i < 0. \end{cases}$$

Let H be a group with generating set S_H and $\alpha_1, \alpha_{-1} : H \to B$ be two monomorphisms where B is a group with generating set S_B. We consider the HNN-extension

$$G := \langle t, B \mid t\alpha_1(h)t^{-1} = \alpha_{-1}(h), \forall h \in H \rangle.$$

Clearly the set

$$S =: S_B \cup \alpha_1(S_H) \cup \alpha_{-1}(S_H) \cup \{t, t^{-1}\}$$

is a generating set of G.

Lemma 2.9 (Normal form Theorem for HNN-extension [LS77] Theorem IV.2.1)
Let $G = \langle t, B \mid t\alpha_1(h)t^{-1} = \alpha_{-1}(h), \forall h \in H \rangle$ be an HNN-extension as above and let T_1, T_{-1} be right transversals of $\alpha_1(H)$, $\alpha_{-1}(H)$ respectively. Then any element $g \in G$ be be uniquely written as

$$g = bt^{n_1}w_1t^{n_2}w_2 \ldots t^{n_m}w_m,$$

where $m \in \mathbb{N}$, $b \in B$, $n_i \in \mathbb{Z} \setminus 0$, $w_i \in T_{\operatorname{sgn}(n_i)}$ for $i \in \{1 \ldots, m\}$ and $w_i \neq \{e\}$ for $i \in \{1, \ldots, m-1\}$.

In the last expression there can be some cancellation in general, but the notion of admissibility guarantees that in the normal form as above no cancellation occurs.

Lemma 2.10 (Proof of Proposition 14.3 in [Man12])
Let $G = \langle t, B \mid t\alpha_1(h)t^{-1} = \alpha_{-1}(h), \forall h \in H \rangle$ be an HNN-extension such that the monomorphisms α_1 and α_{-1} are admissible inclusions $(H, S_H) \to (B, S_B)$. Let $S := S_B \cup \{t, t^{-1}\}$ be the generating set of G. Then an element $g \in G$ written in normal form (as in Lemma 2.9)

$$g = bt^{n_1}w_1t^{n_2}w_2 \cdots t^{n_m}w_m,$$

is geodesic with respect to S and

$$|g|_S = |b|_{S_B} + |n_1| + |w_1|_{S_B} + |n_2| + |w_2|_{S_B} + \cdots + |n_m| + |w_m|_{S_B}.$$

This Lemma implies:

Proposition 2.11 ([Lew91], [Chi94b])
Let $G = \langle t, B \mid t\alpha_1(h)t^{-1} = \alpha_{-1}(h), \forall h \in H \rangle$ be an HNN-extension such that the monomorphisms α_1 and α_{-1} are admissible inclusion $(H, S_H) \to (B, S_B)$. Let S_H, S_B be the generating sets of H, B respectively and let $S := S_B \cup \{t, t^{-1}\}$ be the generating set of G. Then the standard growth series of G with respect to S is given by

$$\frac{1}{\sigma_{(G;S)}(z)} = \frac{1}{\sigma_{(B,S_B)}(z)} - \frac{2z}{1+z} \cdot \frac{1}{\sigma_{(H,S_H)}(z)},$$

which is equivalent to

$$\sigma_{(G;S)}(z) = \sigma_{(B,S_B)}(z) \cdot \frac{1}{1 + z - 2z\frac{\sigma_{(B,S_B)}(z)}{\sigma_{(H,S_H)}(z)}}.$$

In particular

$$\mathcal{RC}(\sigma_{(G;S)}(z)) = \min\left\{\mathcal{RC}(\sigma_{(B,S_B)}(z)), \inf\{|z| \,:\, 1 + z - 2z\frac{\sigma_{(B,S_B)}(z)}{\sigma_{(H,S_H)}(z)} = 0\}\right\}.$$

2.4 Necklaces set associated to a language

Let A be a finite alphabet and L be a language over A. For $n > 0$, we write $L^n := \underbrace{L \times \cdots \times L}_{n \text{ times}}$. For an element $(l_1, \ldots, l_n) \in L^n$, the elements l_j for $j \in \{1, \ldots, n\}$ are called the *components*, and the *l*ength of this n-tuple is defined to be $|(l_1, \ldots, l_n)| := \sum_{j=1}^{n} |l_j|$.

Let $C_n := \mathbb{Z}/n\mathbb{Z}$. The group C_n acts on L^n by $g \cdot (u_1, ..., u_n) := (u_{1+g}, ..., u_{n+g})$ for all $g \in C_n$ and $u_1, ..., u_n \in L$, where the index $i + g$ of u_{i+g} is taken modulo n; that is, elements of C_n cyclically permute the entries of tuples in L^n. Let L^n/C_n denote the quotient by this action, and define the set of *necklaces over L* as

$$\mathsf{Necklaces}(L) := \bigcup_{n=1}^{\infty} L^n \Big/ C_n .$$

Since the length of an element in L^n is preserved by cyclic permutation of its components, we extend the definition of length on L^n to $\mathsf{Necklaces}(L)$.

Next we collect several identities among the strict growth series for languages. Given $u \in L$, let $\mathsf{diag}(u)$ denote the diagonal element $\mathsf{diag}_n(u) := (u, u, ..., u)$ in $L^{\,n}$. Similarly, for $v = (v_1, ..., v_d) \in L^d$ and $m \in \mathbb{N}$, let $\mathsf{diag}_m(v)$ denote the element $\mathsf{diag}_m(v) := (v_1, ..., v_d, ..., v_1, ..., v_d)$ of L^{md}.

Lemma 2.12
Let L and be a language and let $1 \leq n \in \mathbb{N}$. *Then the following hold.*

1. $F_{L^n}(z) = \big(F_L(z)\big)^n$.
2. $F_{\{\mathsf{diag}(u)\,:\,u \in L\}}(z) = F_L(z^n)$.
3. $F_{\{\mathsf{diag}_m(u)\,:\,u \in L^d\}}(z) = F_{L^d}(z^m)$.
4. $[z^m](F_L(z))^d = [z^{nm}](F_L(z^n))^d$.

The following gives a computation of the strict growth series $F_{\mathsf{Necklaces}(L)}(z)$ from $F_L(z)$.

Proposition 2.13
The growth series of the set of necklaces over L is

$$F_{\mathsf{Necklaces}(L)}(z) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{\phi(k)}{kl} \big(F_L(z^k)\big)^l .$$

Proof. Since for $n \neq n'$ the sets L^n/C_n and $L^{n'}/C_{n'}$ are disjoint, we have $F_{\mathsf{Necklaces}(L)}(z) = \sum_{n=1}^{\infty} F_{L^n/C_n}(z)$.

For every $n, m \in \mathbb{N}$, the set $S^n(m) := \{w \in L^n \ :\ |w| = m\}$ is invariant under the cyclic permutation action of C_n on L^n. Then the coefficient $[z^m]F_{L^n/C_n}(z)$ is the number of orbits in $S^n(m)$ under the action of C_n. For each $g \in C_n$, let $\mathrm{Fix}_{S^n(m)}(g)$ denote the set of elements of $S^n(m)$ that are fixed by the action of g. Using Burnside's Lemma we find

$$[z^m]F_{L^n/C_n}(z) = \frac{1}{n} \sum_{g \in C_n} \sharp\mathrm{Fix}_{S^n(m)}(g) = \frac{1}{n} \sum_{d|n} \sum_{\substack{1 \leq g \leq n \\ (g,n)=d}} \sharp\mathrm{Fix}_{S^n(m)}(g).$$

In fact whenever $d|n$, $1 \leq g \leq n$, $(g,n) = d$, and $w \in L^n$, then $w \in \mathrm{Fix}_{S^n(m)}(g)$ if and only if $w = \mathsf{diag}_{\frac{n}{d}}(v)$ for some $v \in L^d$ with $|v| = \frac{md}{n}$. In this case that $(g,n) = d$, then

$$\sharp\mathrm{Fix}_{S^n(m)}(g) = [z^{\frac{md}{n}}]\big(F_L(z)\big)^d = [z^m]\big(F_L(z^{\frac{n}{d}})\big)^d$$

where the latter equality is given by part 4 of Lemma 2.12. Therefore we find

$$F_{L^n/C_n}(z) = \frac{1}{n}\sum_{d|n} \sharp\{1 \leq g \leq n,\, (g,n) = d\}\big(F_L(z^{\frac{n}{d}})\big)^d = \frac{1}{n}\sum_{d|n}\phi(\frac{n}{d})\big(F_L(z^{\frac{n}{d}})\big)^d.$$

Finally,

$$F_{\mathsf{Necklaces}(L)}(z) = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{d|n}\phi(\frac{n}{d})\big(F_L(z^{\frac{n}{d}})\big)^d = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\phi(k)}{kl}\big(F_L(z^k)\big)^l.$$

☐

Note that if the language L contains the empty word and $F_L(0) \geq 1$ then $F_{\mathsf{Necklaces}(L)}(z)$ is nowhere defined. Thus for the remainder of the paper, every language L for which we consider the series $F_{\mathsf{Necklaces}(L)}(z)$ is assumed to satisfy $F_L(0) = 0$.

Remark 2.14
For every $m \in \mathbb{N}$, the following holds.

$$[z^m]\sum_{n=1}^{\infty}\frac{\big(F_L(z)\big)^n}{n} \leq [z^m]F_{\mathsf{Necklaces}(L)}(z) \leq [z^m]\underbrace{\sum_{n=1}^{\infty}\big(F_L(z)\big)^n}_{\frac{F_L(z)}{1-F_L(z)}}.$$

Corollary 2.15
The radius of convergence of $F_{\mathsf{Necklaces}(L)}(z)$ is given by

$$\mathcal{RC}(F_{\mathsf{Necklaces}(L)}(z)) = \min\{|z| \,:\, z \in \mathbb{C},\, |F_L(z)| = 1\},$$

which is by Proposition 2.1 the positive number t such that $F_L(t) = 1$.

Proof. Remark 2.14 implies that

$$\mathcal{RC}\left(\sum_{n=1}^{\infty}\frac{\big(F_L(z)\big)^n}{n}\right) \geq \mathcal{RC}(F_{\mathsf{Necklaces}(L)}(z)) \geq \mathcal{RC}\left(\sum_{n=1}^{\infty}\big(F_L(z)\big)^n\right)$$

The radius of convergence of the geometric series $\sum_{n>0} z^n$ is 1, and so the series $\sum_{n=1}^{\infty}\big(F_L(z)\big)^n$ converges for all z satisfying $|F_L(z)| < 1$ and diverges for all z such that $|F_L(z)| > 1$. Hence

$$\begin{aligned}\mathcal{RC}\left(\sum_{n=1}^{\infty}\big(F_L(z)\big)^n\right) &= \sup\{r > 0 \,:\, |F_L(z)| \leq 1,\, \forall z \in \mathscr{D}(0,r)\} \\ &= \min\{|z| \,:\, z \in \mathbb{C},\, |F_L(z)| = 1\}.\end{aligned} \tag{2.2}$$

Therefore it suffices to prove that

$$\mathcal{RC}\left(\sum_{n=1}^{\infty}\frac{\big(F_L(z)\big)^n}{n}\right) = \mathcal{RC}\left(\sum_{n=1}^{\infty}\big(F_L(z)\big)^n\right).$$

Note that because $F_L(0) = 0$,

$$[z^m]\sum_{n=1}^{\infty} F_L(z)^n = [z^m]\sum_{n=1}^{m} F_L(z)^n = [z^m]\frac{1-F_L(z)^{m+1}}{1-F_L(z)},$$

and

$$[z^m]\sum_{n=1}^{\infty} \frac{F_L(z)^n}{n} = [z^m]\sum_{n=1}^{m} \frac{F_L(z)^n}{n} \geq [z^m]\sum_{n=1}^{m} \frac{F_L(z)^n}{m} = \frac{1}{m}[z^m]\frac{1-F_L(z)^{m+1}}{1-F_L(z)}.$$

Hence

$$\begin{aligned}\frac{1}{\mathcal{RC}\left(\sum_{n=1}^{\infty}\frac{F_L(z)^n}{n}\right)} &= \limsup_{m\to\infty} \sqrt[m]{[z^m]\sum_{n=1}^{\infty}\frac{F_L(z)^n}{n}} \geq \limsup_{m\to\infty} \sqrt[m]{\frac{1}{m}[z^m]\frac{1-F_L(z)^{m+1}}{1-F_L(z)}} \\ &= \limsup_{m\to\infty} \sqrt[m]{[z^m]\frac{1-F_L(z)^{m+1}}{1-F_L(z)}} = \limsup_{m\to\infty} \sqrt[m]{[z^m]\sum_{n=1}^{\infty} F_L(z)^n} \\ &= \frac{1}{\mathcal{RC}\left(\sum_{n=1}^{\infty} F_L(z)^n\right)}.\end{aligned}$$

Therefore

$$\mathcal{RC}\left(\sum_{n=1}^{\infty}\frac{\left(F_L(z)\right)^n}{n}\right) \leq \mathcal{RC}\left(\sum_{n=1}^{\infty}\left(F_L(z)\right)^n\right).$$

□

Example 2.15.1
Let $L = \{c_1, \ldots, c_p\}$ be a finite subset of A; that is, $|c_i| = 1$ for all i. Then $F_L(z) = pz$ and the set L can be viewed as a set of colors. In this case Proposition 2.13 says that

$$F_{\mathsf{Necklaces}(L)}(z) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\phi(k)}{kl}p^l z^{kl}.$$

The coefficient of z^m in this series is the number of necklaces that we can make with m pearls, all with a color in L.

This example leads us to the following definition.

Definition
For any complex power series f with integer coefficients satisfying $[z^0]f(z) = 0$, let

$$\mathsf{NLS}(f)(z) := \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\phi(k)}{kl}\left(f(z^k)\right)^l = \sum_{k=1}^{\infty}\frac{-\phi(k)}{k}\mathrm{Log}(1-f(z^k)).$$

For a complex power series f with integer coefficients such that $[z^0]f(z) = 0$ and $\mathcal{RC}(f) > 0$, let

$$R := \sup\{r \,:\, |f(z^k)| < 1, \forall k \in \mathbb{N}\setminus\{0\}\,\forall z \in \mathscr{D}(0,r)\} > 0.$$

Then $\mathcal{RC}(\mathsf{NLS}(f)) = R$. If f is the growth series of a language (without the empty word), then by Corollary 2.15, R is the unique positive number t such that $f(t) = 1$.

The series $\mathsf{NLS}(f)$ is defined for any complex power series f, not just those that are the growth series of a language, to streamline notation later in the paper when growth series are decomposed into combinations of other series. In particular we apply the following lemma.

Lemma 2.16
Let f, g be two complex power series with positive radius of convergence, natural number coefficients and constant term equal to 0. Then

$$\mathsf{NLS}(f)(z) + \mathsf{NLS}(g)(z) = \mathsf{NLS}(f + g - fg)(z),$$

for all $z \in \mathbb{C}$ such that $\mathsf{NLS}(f)(z)$, $\mathsf{NLS}(g)(z)$ and $\mathsf{NLS}(f + g - fg)$ are defined.
More generally, for $f_1, \dots, f_r$ with positive radius of convergence, natural number coefficients and constant term equal to 0,

$$\sum_{l=1}^{r} \mathsf{NLS}(f_l)(z) = \mathsf{NLS}\left(\sum_{l=1}^{r}(-1)^{r-1} \sum_{J\subset\{1,\dots,r\}\,:\,|J|=l} \prod_{j\in J} f_j\right)(z),$$

for all z such that all $\mathsf{NLS}(f_l)(z)$ and $\mathsf{NLS}\left(\sum_{l=1}^{r}(-1)^{r-1}\sum_{J\subset\{1,\dots,r\}\,:\,|J|=l}\prod_{j\in J} f_j(z)\right)$ are defined.

Proof. Let R be the radius of convergence of $\mathsf{NLS}(f) + \mathsf{NLS}(g)$; that is,

$$R = \sup\{r > 0 \,:\, \max(|f(z)|, |g(z)|) < 1 \,\forall z \in \mathscr{D}(0, r)\}$$

by applying (2.2) to both $N(f)$ and $N(g)$.

For $z \in \mathscr{D}(0, R)$ the series $\mathsf{NLS}(f)(z)$ and $\mathsf{NLS}(g)(z)$ converge uniformly, hence we can write

$$\begin{aligned}
\mathsf{NLS}(f)(z) + \mathsf{NLS}(g) =& \sum_{k=1}^{\infty} \frac{-\phi(k)}{k}\left(\mathrm{Log}(1 - f(z^k)) + \mathrm{Log}(1 - g(z^k))\right)\\
=& \sum_{k=1}^{\infty} \frac{-\phi(k)}{k}\left(\mathrm{Log}(1 - (f(z^k) + g(z^k) - f(z^k)g(z^k)))\right)\\
=& \sum_{k=1}^{\infty} \frac{-\phi(k)}{k}\left(\mathrm{Log}(1 - (f + g - fg)(z^k))\right)\\
=& \mathsf{NLS}(f + g - fg)(z),
\end{aligned}$$

where the second equality holds only if $\log(1 - (f(z^k) + g(z^k) - f(z^k)g(z^k))$ is defined for every k and only if the series converges.

The second part of the lemma is proved by induction on r. $\square$

Chapter 3

Conjugacy growth series of some wreath products

Here we investigate conjugacy growth series in wreath products of the form $G \wr L$, where L is a group which admits a Cayley graph that is a tree. We consider a natural generating set of $G \wr L$ built out of the standard generating sets of G and L (as defined in (3.1)). One of our main tools comes from the paper [Par92], where the author expresses the standard growth series of a group $G \wr L$, when L admits a tree as its Cayley graph, in terms of the standard growth series of G. In order to compute the conjugacy growth in a group one needs to know when different elements are conjugate, and we use the criteria for solving the conjugacy problem developed in the paper [Mat66].

3.1 Wreath products

We now fix a group G with symmetric generating set Y and neutral element e, and a group L with symmetric generating set X and neutral element e'.

Notation
Let I be a non-empty set. For $\eta \in \bigoplus_{i\in I} G$ we write $\eta(i)$ for the i^{th} component of η, and if moreover I is a group and $x \in I$, we define $\eta^x \in \bigoplus_{i\in I} G$ by $\eta^x(i) = \eta(x^{-1}i)$, and say that η^x is the *left translate* of η by x. We write $\mathrm{Supp}(\eta) = \{i \in I \,:\, \eta(i) \neq e\}$ for the *support* of η.

Definition
Let G and L be two groups. The *wreath product of G by L*, written $G \wr L$, is defined as

$$G \wr L := \bigoplus_{i\in L} G \rtimes L,$$

where for $(\eta, m), (\theta, n) \in G \wr L$,

$$(\eta, m)(\theta, n) = (\eta\theta^m, mn).$$

For $(\eta, n) \in G \wr L$ we say that n is the *cursor position.* For $g \in G$, let $\vec{g} \in \bigoplus_{i\in L} G$ be such that $\vec{g}(e') = g$ and $\vec{g}(i) = e$ for $i \neq e'$. The neutral element of $G \wr L$ is then denoted by $(\vec{e}, e')$, and for any $(\eta, m) \in G \wr L$, $(\eta, m)^{-1} = ((\eta^{m^{-1}})^{-1}, m^{-1})$.

The following set generates $G \wr L$:

$$\vec{Y} := \{(\vec{e}, x) \,:\, x \in X\} \cup \{(\vec{y}, e') \,:\, y \in Y\}. \tag{3.1}$$

Remark 3.1
One can interpret the generating set $\vec{Y}$ as follows. If $\mathsf{Cay}(L, X)$ denotes the Cayley graph of L with generating set X, one may imagine that on each vertex v of $\mathsf{Cay}(L, X)$ there is a copy of the group G. Hence an element $(\eta, b) \in G \wr L$ can be viewed as

- a finite subset V of the set of vertices of $\mathsf{Cay}(L, X)$,
- for each $v \in V$, an element $g_v \in G \setminus \{e\}$ such that $g_v = \eta(v)$, and
- a vertex v' of $\mathsf{Cay}(L, X)$ corresponding to the element b.

With this interpretation, the elements of the set $\{(\vec{e}, x) : x \in X\}$ induce moves of the cursor along the graph $\mathsf{Cay}(L, X)$, while the elements of $\{(\vec{y}, 0) : y \in Y\}$ can be seen as a generating set of the copy of G at the vertex of $\mathsf{Cay}(L, X)$ corresponding to the current cursor position.

Proposition 3.2 ([Par92]:Section 1 or [Man12]:Section 14.2)
The length of an element (η, b) (relative to $\vec{Y}$) is the sum of the following two values:

1. *The length of a minimal walk on $\mathsf{Cay}(L, X)$ that starts at the origin, visits every vertex v for which $\eta(v) \neq e$ and ends at b,*
2. *the sum of the lengths $|\eta(v)|_Y$ for $v \in L$.*

Thus computing element length in a wreath product involves finding the walks of minimal length in $\mathsf{Cay}(L, X)$, a very difficult computational problem related to the traveling salesman problem. For this reason Parry ([Par92]) only considers the case where the Cayley graph of L is a tree.

3.1.1 Description of the conjugacy classes in a wreath product

We follow the approach of [Mat66], which gives a sufficient and necessary condition for two elements in $G \wr L$ to be conjugate. In [Mat66] the wreath product is defined via right action, so since in this thesis the wreath product is defined via left action, we modify the notation and statements in [Mat66] accordingly.

Notation
(1) For $(\eta, b) \in G \wr L$, let

$$\pi_G((\eta, b)) := \eta(b).$$

be the entry in η at the cursor position.
(2) Let $b \in L$ be an element and let $\langle b \rangle t$ be a right coset of the subgroup $\langle b \rangle$ generated by b. We define the map $\pi_{\langle b \rangle t} : \bigoplus_{l \in L} G \to G \cup G_{\sim}$ by

$$\pi_{\langle b \rangle t}(\eta) := \begin{cases} \left[\prod_{k=0}^{K-1} \eta(b^{-k}t)\right] & \text{if } b \text{ is of finite order } K \\ \prod_{k=-\infty}^{\infty} \eta(b^{-k}t) & \text{else}. \end{cases}$$

Note that this is well defined since the support of η is finite.
If $\pi_{\langle b \rangle t}(\eta) = [e] \in G_{\sim}$, we just write it as $\pi_{\langle b \rangle t}(\eta) = e$. Now let us follow the approach of Jane Matthews in [Mat66] with our notation. Assume that (η, b) is conjugate to (η', b') in $G \wr L$. This means that there is $(\theta, d) \in G \wr L$ such that

$$(\eta', b') = (\theta, d)^{-1}(\eta, b)(\theta, d) = ((\theta^{d^{-1}})^{-1} \eta^{d^{-1}} \theta^{d^{-1}b}, d^{-1}bd).$$

In other words

$$b' = d^{-1}bd \qquad \text{and} \qquad \eta'^{d} = \theta^{-1}\eta\theta^{b}.$$

Fixing a right transversal T of $\langle b\rangle$, this is equivalent to saying that for every $k \in \mathbb{Z}$ and every $t \in T$

$$b' = d^{-1}bd \qquad \text{and} \qquad \eta'^{d}(b^{-k}t) = \theta^{-1}(b^{-k}t)\eta(b^{-k}t)\theta(b^{-(k+1)}t). \tag{3.2}$$

Hence taking the product over all values of k in the equality (3.2) gives the following.

Lemma 3.3 ([Mat66] Lemma 3.2)
In $G \wr L$, the elements (η, b) and (η', b') are conjugate if and only if there exists $(\theta, d) \in G \wr L$ such that, $b' = d^{-1}bd$ and for any right transversal T of $\langle b\rangle$, for any $t \in T$, for all $u \in \mathbb{Z}$ and all $v \in \mathbb{N}$, we have

$$\prod_{k=u}^{u+v} \eta'^{d}(b^{-k}t) = \theta^{-1}(b^{-u}t)\left(\prod_{k=u}^{u+v} \eta(b^{-k}t)\right)\theta(b^{-(u+v+1)}t). \tag{3.3}$$

Lemma 3.4 ([Mat66] Proposition 3.5)
Let (η, b) and (η', b') be in $G \wr L$ with b of infinite order. Then (η, b) and (η', b') are conjugate if and only if there exists $d \in L$ such that $b' = d^{-1}bd$ and for any right coset $\langle b\rangle t$

$$\pi_{\langle b\rangle t}(\eta) = \pi_{\langle b\rangle t}(\eta'^{d}). \tag{3.4}$$

Proof. Suppose first that (η, b) and (η', b') are conjugate. Since $\langle b\rangle$ is infinite and $\mathrm{Supp}(\theta)$, $\mathrm{Supp}(\eta)$ and $\mathrm{Supp}(\eta')$ are finite, there exist $u \in \mathbb{Z}$ and $v \in \mathbb{N}$ such that $\theta^{-1}(b^{-u}t) = \theta(b^{-u-v-1}t) = e$ and $\pi_{\langle b\rangle t}(\eta) = \prod_{k=u}^{u+v} \eta(b^{-k}t)$ and $\pi_{\langle b\rangle t}(\eta'^{d}) = \prod_{k=u}^{u+v} \eta'^{d}(b^{-k}t)$. This gives, via (3.3), the equality (3.4).

For the other direction, assume there exists $d \in L$ such that $b' = d^{-1}bd$ and for any right coset

$$\pi_{\langle b\rangle t}(\eta) = \pi_{\langle b\rangle t}(\eta'^{d}).$$

Let T be a right transversal of $\langle b\rangle$. Define $\theta \in \bigoplus_{i\in L} G$ by

$$\theta(b^{-k}t) := \prod_{l=k}^{\infty} \eta(b^{-l}t)\left(\prod_{l=k}^{\infty} \eta'^{d}(b^{-l}t)\right)^{-1},$$

for every $t \in T$ and every $k \in \mathbb{Z}$, and verify that (3.2) is satisfied.
Note that the hypothesis $\pi_{\langle b\rangle t}(\eta) = \pi_{\langle b\rangle t}(\eta'^{d})$ assures that $\mathrm{Supp}(\theta)$ is finite. ☐

Example 3.4.1
Here is an illustration of Lemma 3.4 in $G \wr \mathbb{Z}$. Let $\mathbb{Z} = \langle \{a, a^{-1}\} | \rangle$. We represent an element $\xi \in \bigoplus_{i\in\mathbb{Z}} G$ as a bi-infinite line on which for all i such that $\xi(a^i) \neq e$ there is a symbol $\underset{i}{g}$ meaning $\xi(a^i) = g$. Let $(\eta, a^3) \in G \wr \mathbb{Z}$ be such that η is represented by the following line

$$\cdots\cdots\ \underset{-2}{g_1} \;\rule[0.5ex]{3em}{0.4pt}\; \underset{-1}{g_2} \;\rule[0.5ex]{3em}{0.4pt}\; \underset{0}{g_3} \;\rule[0.5ex]{3em}{0.4pt}\; \underset{1}{g_4} \;\rule[0.5ex]{3em}{0.4pt}\; \underset{2}{g_5} \;\rule[0.5ex]{3em}{0.4pt}\; \underset{3}{g_6} \;\rule[0.5ex]{3em}{0.4pt}\; \underset{4}{g_7}\ \cdots\cdots \quad .$$

Taking $T = \{e', a, a^2\}$ as right transversal of $\langle a^3\rangle$, we find

$$\pi_{\langle a^3\rangle e'}(\eta) = g_6g_3, \quad \pi_{\langle a^3\rangle a}(\eta) = g_7g_4g_1, \quad \pi_{\langle a^3\rangle a^2}(\eta) = g_5g_2.$$

Lemma 3.4 shows that (η, a^3) is conjugate in $G \wr \mathbb{Z}$ to (ζ, a^3), where ζ is represented as:

$$\cdots\cdots\ \underset{-2}{e} \;\rule[0.5ex]{3em}{0.4pt}\; \underset{-1}{e} \;\rule[0.5ex]{3em}{0.4pt}\; \underset{0}{g_6g_3} \;\rule[0.5ex]{3em}{0.4pt}\; \underset{1}{g_7g_4g_1} \;\rule[0.5ex]{3em}{0.4pt}\; \underset{2}{g_5g_2} \;\rule[0.5ex]{3em}{0.4pt}\; \underset{3}{e} \;\rule[0.5ex]{3em}{0.4pt}\; \underset{4}{e}\ \cdots\cdots \quad .$$

The conjugator is (θ, e'), where θ is represented as:

$$\cdots\cdots\ \underset{-2}{g_1} \text{ ——— } \underset{-1}{g_2} \text{ ——— } \underset{0}{g_6^{-1}} \text{ ——— } \underset{1}{g_7^{-1}} \text{ ——— } \underset{2}{e} \text{ ——— } \underset{3}{e} \text{ ——— } \underset{4}{e}\ \cdots\cdots \quad .$$

Lemma 3.5 ([Mat66] Proposition 3.6)
Let (η, b) and (η', b') be in $G \wr L$ with b of finite order K. Then (η, b) and (η', b') are conjugate if and only if there exists $d \in L$ such that $b' = d^{-1}bd$ and for every right coset $\langle b\rangle t$, $\pi_{\langle b\rangle t}(\eta) = \pi_{\langle b\rangle t}(\eta'^d)$.

Proof. Let T be a right transversal of $\langle b\rangle$ in L.
Suppose $\pi_{\langle b\rangle t}(\eta) = \pi_{\langle b\rangle t}(\eta'^d)$. Since $b^K = e$, it suffices to take $u = 0$ and $v = K - 1$ in Lemma 3.3 to show that (η, b) and (η', b') are conjugate.

Now assume there exists $d \in L$ such that $b' = d^{-1}bd$ and for any $t \in T$ there exists $\alpha_t \in G$ such that

$$\prod_{k=0}^{K-1} \eta'^d(b^{-k}t) = \alpha_t \prod_{k=0}^{K-1} \eta(b^{-k}t))\alpha_t^{-1}.$$

Define $\theta \in \bigoplus_{i \in L} G$ by

$$\theta(b^{-k}t) = \left(\prod_{l=k}^{K-1} \eta(b^{-l}t)\right) \alpha_t^{-1} \left(\prod_{l=k}^{K-1} \eta'^d(b^{-l}t)\right)^{-1},$$

for every $t \in T$ and every $k \in \{0, \ldots, K-1\}$, and verify that (3.2) is satisfied. ☐

Corollary 3.6
Let $(\eta, b) \in G \wr L$, let T be a right transversal of $\langle b\rangle$ in L and let $\{t_1, \ldots, t_k\} \subset T$ be the set of elements of T such that $\pi_{\langle b\rangle t_j}(\eta) \neq e$ for $j \in \{1, \ldots, k\}$. Then (η, b) is conjugate to an element (η', b) for which $\mathrm{Supp}(\eta') \cap \langle b\rangle t_j$ *is a singleton $\{l_j\}$ for each $j \in \{1, \ldots, k\}$. Moreover,*

A) $\eta'(l_j) = \pi_{\langle b\rangle t_j}(\eta)$ if b is of infinite order, and

B) $\eta'(l_j) \in \pi_{\langle b\rangle t_j}(\eta)$ if b is of finite order.

Proof. This is an immediate consequence of Lemmas 3.4 or 3.5. Indeed, consider Lemma 3.4 if b is of infinite order, and Lemma 3.5 if b is of finite order. In both cases choose $d = e'$. Now for every $j \in \{1, \ldots, k\}$ choose some $l_j \in \langle b\rangle t_j$, and define η' as $\eta'(l_j) = \pi_{\langle b\rangle t_j}(\eta)$ if b is of infinite order, $\eta'(l_j) \in \pi_{\langle b\rangle t_j}(\eta)$ if b is of finite order, and $\eta'(l) = e$ if $l \notin \{l_1, \ldots, l_k\}$.

Then (η', b) satisfies the hypotheses of Lemmas 3.4 or 3.5, and η' satisfies the corollary. ☐

3.2 Conjugacy class length and conjugacy growth series in $G \wr L$

From now on we assume that the generating sets X and Y of L and G, respectively, are finite. This section explains how one can shorten an element in $G \wr L$ by conjugation until it is of minimal length in its conjugacy class. We use Lemmas 3.4 or 3.5, depending on the order of the cursor position; since the order, finite or infinite, influences the computations, we make the following distinction.

The elements (η, b) in $G \wr L$ with b of infinite order are called of *type* A. The words w over a generating set of $G \wr L$ satisfying $\overline{w}$ is of type A are called of *type* A, and a conjugacy class of an element of type A is also called of *type* A. We denote by $(G \wr L)^A$ the set of elements of $G \wr L$ of type A and by $(G \wr L)^A_\sim$ the set of conjugacy classes of type A. The contribution to the conjugacy growth series of $G \wr L$ of the elements of type A will be denoted by $\tilde{\sigma}^A_{(G\wr L,\vec{Y})}$.

The elements (η, b) in $G \wr L$ with b of finite order are called of *type* B, and similarly use type B for all other items mentioned above. Hence

$$\tilde{\sigma}_{(G\wr L,\vec{Y})} = \tilde{\sigma}^A_{(G\wr L,\vec{Y})} + \tilde{\sigma}^B_{(G\wr L,\vec{Y})}.$$

Definition
Let $(\eta, b) \in G \wr L$, let T be a right transversal of the subgroup $\langle b \rangle$ of L and let $t_1, \ldots, t_k \in T$ be the elements of T such that $\pi_{\langle b \rangle t_j}(\eta) \neq e$ for $j \in \{1, \ldots, k\}$.

1. An *optimal coset walk for* (η, b) is a walk of smallest possible length on $\mathsf{Cay}(L, X)$ starting at the origin, visiting some point $l_j \in \langle b \rangle t_j$ for all $1 \leq j \leq k$ and ending at b.

2. An *optimal conjugacy walk for* (η, b) is an optimal coset walk for $(\eta^{d^{-1}}, d^{-1}bd)$ that is of minimal length among all optimal coset walks for $(\eta^{d'^{-1}}, d'^{-1}bd')$ with $d' \in L$.

Note that if $b = e'$ then an optimal coset walk for (η, b) is equivalent to a minimal walk on $\mathsf{Cay}(L, X)$ as in Proposition 3.2.

Lemma 3.7
Let $(\eta, b) \in G \wr L$ and let $\langle b \rangle t_1, \ldots, \langle b \rangle t_k$ be the right cosets of $\langle b \rangle$ in L such that $\pi_{\langle b \rangle t_i}(\eta) \neq e$ for $i \in \{1, \ldots, k\}$. Then there is a conjugacy representative $(\zeta, d^{-1}bd)$ of $[(\eta, b)]$ with the following properties.

1. *An optimal coset walk for $(\zeta, d^{-1}bd)$ is an optimal conjugacy walk for (η, b).*

2. *If such a walk intersects $d^{-1}\langle b \rangle t_i$ at $d^{-1}b^{a_i}t_i$, where $a_i \in \mathbb{Z}$, $i \in \{1, \ldots, k\}$, then* $\mathrm{Supp}(\zeta) = \{d^{-1}b^{a_1}t_1, \ldots, d^{-1}b^{a_k}t_k\}$ *and*

   A) $\zeta(d^{-1}b^{a_i}t_i) = \pi_{\langle b \rangle t_i}(\eta)$ if b is of infinite order,

   B) $\zeta(d^{-1}b^{a_i}t_i) \in \min(\pi_{\langle b \rangle t_i}(\eta))_Y$ if b is of finite order.

Proof. For all $i \in \{1, \ldots, k\}$ let $l_i \in \langle b \rangle t_i$ be such that an optimal coset walk for (η, b) passes through l_i. Corollary 3.6 implies that (η, b) is conjugate to an element (ξ, b) of smaller or equal length that satisfies $\mathrm{Supp}(\xi) = \{l_1, \ldots, l_k\}$, and for all $i \in \{1, \ldots, k\}$, if (i) b is of infinite order, then $\xi(l_i) = \pi_{\langle b \rangle t_i}(\eta)$, and if (ii) b is of finite order, then $\xi(l_i) \in \min(\pi_{\langle b \rangle t_i}(\eta))_Y$.

In this case the length of (ξ, b) is equal to the sum of

1. the length of an optimal coset walk of (η, b) on $\mathsf{Cay}(L, X)$, and

2. $\sum_{i=1}^k |\pi_{\langle b \rangle t_i}(\eta)|_Y$.

A consequence of Lemmas 3.4 and 3.5 is that inside the conjugacy class of (η, b) in $G \wr L$ no element (ζ, c) has the value $\sum_{l \in \mathrm{Supp}(\zeta)} |\zeta(l)|_Y$ strictly smaller then $\sum_{l \in \mathrm{Supp}(\xi)} |\xi(l)|_Y$.

Now we can also translate the support of ξ along $\mathsf{Cay}(L, X)$ and modify the cursor position in order to minimize the walk on $\mathsf{Cay}(L, X)$. More precisely, taking a suitable $d \in L$ and conjugating (ξ, b) by $(\vec{e}, d)$ we find the element $(\zeta, d^{-1}bd) := (\vec{e}, d)^{-1}(\xi, b)(\vec{e}, d) = (\xi^{d^{-1}}, d^{-1}bd)$. Hence this ζ satisfies $\zeta^d = \xi$ and $\mathrm{Supp}(\zeta) = \{d^{-1}l_1, \ldots, d^{-1}l_k\}$.

□

Corollary 3.8
The conjugacy length of (η, b) is equal to the sum of

1. *the length of an optimal conjugacy walk for (η, b), and*
2. $\sum_{i=1}^{k} |\pi_{\langle b\rangle t_i}(\eta)|_Y$.

Example 3.8.1
Let (η, a^3) be as in Example 3.4.1. We choose $T = \{e', a, a^2\}$ as the right transversal of $\langle a^3\rangle$ in $\mathbb{Z}$, and let $t_1 := e'$, $t_2 = a$ and $t_3 := a^2$ be the elements of T such that $\pi_{\langle a^3\rangle t_i}(\eta) \neq e$. Since $\mathbb{Z}$ is abelian and the direct walk on $\mathsf{Cay}(\mathbb{Z}, \{a, a^{-1}\})$ from e' to a^3 visits every coset $\langle a^3\rangle t_j$, this is the only optimal coset walk for (η, a^3).

The optimal conjugacy walk for (η, a^3) starts at e' and creates g_6g_3, goes to a and creates $g_7g_4g_1$, then goes to a^2 and creates g_5g_2, and ends up at a^3. This is an optimal walk for the element (ζ, a^3) for which $\mathrm{Supp}(\zeta) \subset \{e', a, a^2\}$ as in Example 3.4.1. In this case the d and the a_i's in Lemma 3.7 are all trivial and $l_1 = e$, $l_2 = a$ and $l_3 = a^2$. The length of the walk on $\mathsf{Cay}(\mathbb{Z}, \{a, a^{-1}\})$ is 3 and the sum the length of the components is $|g_6g_3|_Y + |g_7g_4g_1|_Y + |g_5g_2|_Y$, hence $|[(\eta, a^3)]|_{\vec{Y}} = 3 + |g_6g_3|_Y + |g_7g_4g_1|_Y + |g_5g_2|_Y$. Note that (η, b^3) is also conjugate to the element $(\zeta', a^3) \in \min([(\eta, a^3)])$ where ζ' is represented as

$$\cdots\cdots \underset{-2}{e} \text{ ——— } \underset{-1}{e} \text{ ——— } \underset{0}{e} \text{ ——— } \underset{1}{g_7g_4g_1} \text{ ——— } \underset{2}{g_5g_2} \text{ ——— } \underset{3}{g_6g_3} \text{ ——— } \underset{4}{e} \cdots\cdots \quad .$$

In this case the conjugator is (θ', e'), where θ' is represented as:

$$\cdots\cdots \underset{-2}{g_1} \text{ ——— } \underset{-1}{g_2} \text{ ——— } \underset{0}{g_3} \text{ ——— } \underset{1}{g_7^{-1}} \text{ ——— } \underset{2}{e} \text{ ——— } \underset{3}{e} \text{ ——— } \underset{4}{e} \cdots\cdots \quad .$$

Thus finding the conjugacy length of an element involves finding a conjugate of the cursor position which produces an optimal walk on $\mathsf{Cay}(L, X)$. We will see in the next section that when $\mathsf{Cay}(L, X)$ is a tree, a $(\zeta, d^{-1}bd)$ as in Lemma 3.7 can be chosen such that $d^{-1}bd$ is minimal in its conjugacy class (in L), hence cyclically reduced if b is of infinite order.

3.2.1 The case when the Cayley graph of L is a tree

From now on, we assume that $\mathsf{Cay}(L, X)$ is a tree. This is precisely the case when L admits the presentation

$$L = \langle a_1^{\pm}, \ldots, a_M^{\pm}, b_1, \ldots, b_N \mid b_1^2, \ldots, b_N^2\rangle.$$

Hence we assume that $X = \{a_1, a_1^{-1}, \ldots, a_M, a_M^{-1}, b_1, \ldots, b_N\}$ and each generator b_j is equal to its inverse. This implies that $\mathsf{Cay}(L, X)$ is the $(2M + N)$-regular tree.

In this case $\mathsf{GeoNorm}(L, X) = \mathsf{Geo}(L, X)$ is the set of freely reduced words on X. For simplicity we will make no distinction between elements of L and freely reduced words, and by a word on X, we will mean freely reduced word.

An element of L is minimal in its conjugacy class in L if and only if it is cyclically reduced or equal to one of the torsion generators b_j for some $j \in \{1, \ldots, N\}$ or is trivial, so the set of conjugacy geodesics $\mathsf{ConjGeo}(L, X)$ is given by

$$\mathsf{ConjGeo}(L, X) = \mathsf{ConjGeo}(L, X)^A \cup \{b_1, \ldots, b_N\} \cup \{\epsilon\},$$

where $\mathsf{ConjGeo}(L, X)^A$ denotes the set of cyclically reduced words, whose growth series computation is similar to that in [Riv10, Theorem 1.1].

Lemma 3.9 (see Theorem 1.1 [Riv10])
In the group L as above the number of cyclically reduced words of length $k > 0$ is

$$(2M + N - 1)^k + (-1)^k (M + N - 1) + M,$$

and the associated growth series written $F_{\mathsf{ConjGeo}(L,X)^A}(z)$ is given by

$$F_{\mathsf{ConjGeo}(L,X)^A}(z) = \frac{1}{1 - (2M + N - 1)z} + \frac{(2M + N)z^2 - (N - 1)z - 1}{1 - z^2}.$$

The proof is similar to the proof of Theorem 1.1 [Riv10]. In [Riv10], Rivin counts the number of cyclically reduced words of length k in a free group by counting the number of closed paths of length k in a graph having the generators (and their inverses) as vertices, where each vertex is connected to all the others vertices except for its inverse. He then computes the trace of the k^{th} power of the adjacency matrix. We generalize his result by also adding a vertex for each generator of order 2 that is connected to all the vertices excepted itself. We compute the trace of the k^{th} power of the adjacency matrix in the same way as Rivin.

Remark 3.10
If $2M + N \geq 2$, then

$$\mathcal{RC}(F_{\mathsf{ConjGeo}(L,X)^A}(z)) = \frac{1}{2M + N - 1}.$$

Now let us adapt Lemma 3.7 to this kind of group.

Lemma 3.11
Let $(\eta, b) \in G \wr L$ be such that $b \neq e'$. Then there exists $(\zeta, c) \in \min([(\eta, b)])_{\tilde{Y}}$ such that either c is of infinite order and cyclically reduced, or c is one of the torsion generators b_j of L for some $j \in \{1, \ldots, N\}$.

Proof. Assume that b is not cyclically reduced and $b \neq b_j$, for any $j \in \{1, \ldots, N\}$. We consider the two cases, when b is of infinite order, and when b is of finite order, separately. In both cases Lemma 3.7 shows that (η, b) is conjugate to an element (η', b) of the same length for which $\mathrm{Supp}(\eta')$ intersects each right coset of $\langle b \rangle$ at most once and an optimal walk for this element is an optimal coset walk. Hence up to conjugating (η, b), one can assume that (η, b) is of this form. We consider a particular suitable coset walk for (η, b) and then after conjugation we show that (η, b) is conjugate to an element $(\zeta, c) \in \min([(\eta, b)])_{\tilde{Y}}$ with c as claimed.

A) If b is of infinite order it can be written as $\overline{w}^{-1}\overline{cw}$, where $w = w_1 \ldots w_n$ and $c = c_1 \ldots c_k$ are in X^* and freely reduced, and $w_1 \neq c_1 \neq {c_k}^{-1} \neq w_1$. For simplicity of notation, we write w for $\overline{w}$ and c for $\overline{c}$.

   Let us consider an optimal walk on $\mathsf{Cay}(L, X)$ starting at the origin, visiting each right coset $\langle b \rangle t$ of $\langle b \rangle$ such that $\pi_{\langle b \rangle t}(\eta) \neq e$, and ending at b. For $r = \{0, \ldots, k - 1\}$ let T_r be the set of (freely reduced) words in X that do not begin with c_r^{-1} or c_{r+1}, where $c_0 := c_k$. Then we can assume that the optimal walk on $\mathsf{Cay}(L, X)$ starts at the origin, goes inside $w^{-1}T_0$ and comes back to w^{-1}, then goes to $w^{-1}c_1$ and inside $w^{-1}c_1T_1$, comes back to $w^{-1}c_1$, until it reaches $w^{-1}c_1 \cdots c_{k-1}$, goes inside $w^{-1}c_1 \cdots c_{k-1}T_{k-1}$, comes back to $w^{-1}c_1 \cdots c_{k-1}$ and finally stops at $w^{-1}cw$. In other words, we assume

$$\mathrm{Supp}(\eta) \subset w^{-1}T_0 \cup w^{-1}c_1T_1 \cup \ldots \cup w^{-1}c_1 \cdots c_{k-1}T_{k-1},$$

   and that the walk on $\mathsf{Cay}(L, X)$ is as Figure 3.1:

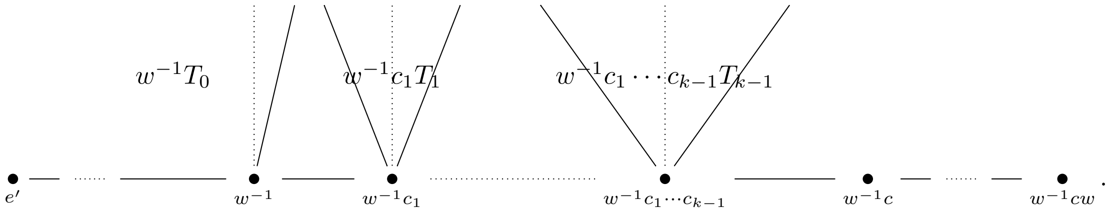


Figure 3.1: Illustration of $(\eta, w^{-1}cw)$

For $r \in \{0, \ldots, k-1\}$, let l_r be the length of the walk in $w^{-1}c_1 \cdots c_r T_r$. This is 2 times the number of edges in the tree spanned by $\mathrm{Supp}(\eta) \cap w^{-1}c_1 \cdots c_r T_r$ for $r \in \{1, \ldots, k-1\}$ and at most 2 times the number of edges in the tree spanned by $\mathrm{Supp}(\eta) \cap w^{-1}T_0 + 2n$ for $r = 0$. Note that l_r can be equal to 0. Then the length of the walk on $\mathsf{Cay}(L, X)$ is $\sum_{r=0}^{k-1} l_r + 2n + k$.

Now let us conjugate (η, b) by $(\vec{e}, w)$ to obtain (ζ, c). In this case $\zeta^{w^{-1}} = \eta$ and hence $\mathrm{Supp}(\zeta) = w\mathrm{Supp}(\eta)$. Then an optimal walk on $\mathsf{Cay}(L, X)$ starting at the origin, visiting every element of $\mathrm{Supp}(\zeta)$ and ending at c has the following trajectory: it first goes inside T_0, then comes back to e', then goes to c_1, inside c_1T_1, comes back to c_1, and so on until it reaches $c_1 \cdots c_{k-1}$, walks inside $c_1 \cdots c_{k-1}T_{k-1}$, comes back to $c_1 \cdots c_{k-1}$ and finally stops at c, as shown in Figure 3.2:

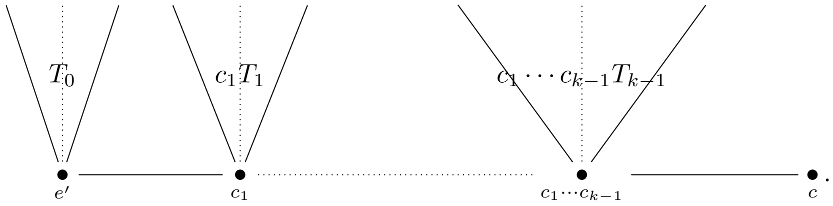


Figure 3.2: Illustration of (ζ, c)

This is an optimal coset walk for (ζ, c). Therefore the length of the walk on $\mathsf{Cay}(L, X)$ is less or equal than $\sum_{r=0}^{k-1} l_r + k + 2n$. On the other hand, the value $\sum_{v\in\mathrm{Supp}(\zeta)} |\zeta(v)|_Y$ is the same as the value $\sum_{v\in\mathrm{Supp}(\eta)} |\eta(v)|_Y$. Therefore the length of (ζ, c) is less or equal than the length of (η, b) as claimed.

B) If b is of finite order it can be written as $\overline{w}^{-1}\overline{b_j}\overline{w}$ for some $j \in \{1, \ldots, N\}$, where $w = w_1 \cdots w_n$ is in X^* and freely reduced and with $w_1 \neq b_j$.

Let T be the set of (freely reduced) words in X that do not begin with b_j. Then as before we can assume that the optimal walk on $\mathsf{Cay}(L, X)$ starts at the origin, goes inside $w^{-1}T$ and comes back to w^{-1} then goes to $w^{-1}b_j$ and goes inside $w^{-1}b_jT$, comes back to $w^{-1}b_j$ and finally stops at $w^{-1}b_jw$. In other words we assume

$$\mathrm{Supp}(\eta) \subset w^{-1}T,$$

and that the walk on $\mathsf{Cay}(L,X)$ is as represented in Figure 3.3:

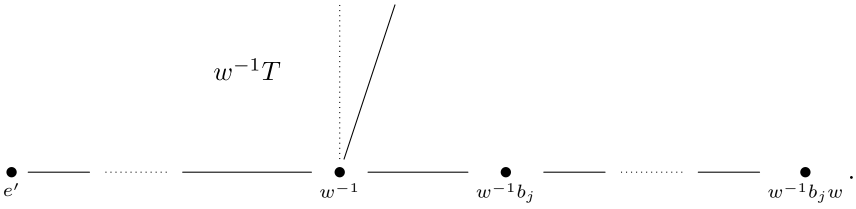


Figure 3.3: Illustration of $(\eta, w^{-1}b_jw)$

In the same way as before (η, b) is conjugate to a (ζ, b_j) with $\mathrm{Supp}(\zeta) = w\mathrm{Supp}(\eta)$, as represented in Figure 3.4;

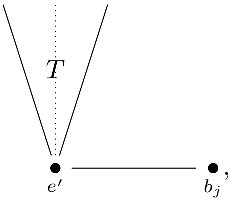


Figure 3.4: Illustration of (ζ, b_j)

and the length of (ζ, b_j) is smaller or equal than the length of (η, b), since $\sum_{v\in\mathrm{Supp}(\zeta)} |[\zeta(v)]|_Y$ is the same as the value $\sum_{v\in\mathrm{Supp}(\eta)} |[\eta(v)]|_Y$. Therefore we also find (ζ, b_j) as claimed.

This proves the lemma. □

The previous proof also shows the following.

Corollary 3.12
Let $(\eta, c) \in G \wr L$ be an element with $c \neq e'$ and such that $(\eta, c) \in \min([(\eta, c)])_{\vec{Y}}$. Using the same notation as above the following holds.

A) If $c = c_1 \cdots c_k$ is cyclically reduced and we choose a conjugacy representatives (η, c) that satisfies

$$\mathrm{Supp}(\eta) \subset T_0 \cup c_1T_1 \cup \ldots \cup c_1 \cdots c_{k-1}T_{k-1},$$

then such an element is uniquely determined up to cyclic permutation of c and the components of η in the trees T_r.

B) If $c = b_j$ for some $j \in \{1, \ldots, N\}$ and for every conjugacy classes in G we choose a unique conjugacy representative, then there is a unique conjugacy representative (η, b_j) that satisfies

$$\mathrm{Supp}(\eta) \subset T.$$

Now one needs an analogue to Lemma 3.11 for elements having trivial cursor position.

Lemma 3.13
Let $(\eta, e') \in G \wr L$ be non-trivial and such that $(\eta, e') \in \min([(\eta, e')])_{\vec{Y}}$. Then the finite tree in $\mathsf{Cay}(L, X)$ spanned by $\mathrm{Supp}(\eta)$ contains the vertex e'.

Moreover, two elements $(\eta, e') \in \min([(\eta, e')])_{\vec{Y}}$ and $(\eta', e') \in \min([(\eta', e')])_{\vec{Y}}$ are conjugate if and only if there exists $d \in L$ such that for every $l \in L$, $\eta'(l)$ and $\eta^d(l)$ are conjugate in G.

Proof. This is an immediate consequence of Lemma 3.7. ☐

3.3 The conjugacy growth series when the Cayley graph of L is a tree

In this section, we compute the contribution to $\tilde{\sigma}^A_{(G\wr L, \vec{Y})}(z)$ of the elements (η, c), with $|c|_X = k$, and the contribution to $\tilde{\sigma}^B_{(G\wr L, \vec{Y})}(z)$ of the elements (η, c), with $|c|_X = 1$ or 0.

For the conjugacy classes of elements (η, c), with $|c|_X = k > 0$, we consider a cyclically reduced element $c = c_1 \cdots c_k \in L$, and (with the notation in the proof of Lemma 3.11) we count the contribution of all the possible η satisfying the condition

$$\mathrm{Supp}(\eta) \subset T_0 \cup c_1 T_1 \cup \ldots \cup c_1 \cdots c_{k-1} T_{k-1}.$$

To do this we must consider the contribution of the possible walks along the subtrees T_0, $c_1 T_1$, ..., $c_1 \cdots c_{k-1} T_{k-1}$. We will follow the approach of Parry from [Par92], where he computed the standard growth series of $G \wr L$, and use his notation.

Notation
The intersection of the trajectory of the walk with each of the trees T_0, $c_1 T_1$, ..., $c_1 \cdots c_{k-1} T_{k-1}$ is a finite tree T. For such a tree T define a vertex to be a leaf if it has valence 1 or 0 (when the tree contains just one vertex). Choose $v \in X \setminus \{c_1, c_k^{-1}\}$. Let $\mathscr{T}$ be the set of all finite subtrees containing e' but no other vertex adjacent to e' than v, and define

$$F_{\mathscr{T}}(x, y) = \sum_{i,j \geq 0} a_{ij} x^i y^j$$

to be the formal complex power series with 2 variables with

$$a_{ij} := \sharp\{T \in \mathscr{T} \, : \, T \text{ contains } i \text{ non-leaves and } j \text{ leaves other than } e'\}.$$

Parry observed that $F_{\mathscr{T}}(x, y)$ does not depend on the edge $\{e', v\}$, but only on the degree of the regular tree (in this case $2M + N$).

Lemma 3.14 ([Par92]:Lemma 3.1)
If the degree of the tree is D, then $F_{\mathscr{T}}(x, y)$ satisfies

$$F_{\mathscr{T}}(x, y) = 1 + y - x + x F_{\mathscr{T}}(x, y)^{D-1}.$$

In particular, if $D = 1$ then $F_{\mathscr{T}}(x, y) = 1 + y$, and if $D = 2$ then $F_{\mathscr{T}}(x, y) = 1 + y\frac{1}{1-x}$. He first computes the contribution to the growth series of a walk starting at the origin with support in T, for a $T \in \mathscr{T}$, creating the non-trivial elements of G at the leaves of T and the other elements of G in the non-leaves of T and then coming back to the origin.

Let the group L have presentation $\langle a_1^{\pm}, \ldots, a_M^{\pm}, b_1, \ldots, b_N \,|\, b_1^2, \ldots, b_N^2 \rangle$ and symmetric generating set $X = \{a_1, a_1^{-1}, \ldots, a_M, a_M^{-1}, b_1, \ldots, b_N\}$ and assume the valence of the tree $\mathsf{Cay}(L, X)$

is at least 2, in other words $2M + N \geq 2$. We keep the notation in the proof of Lemma 3.11 and Section 3.2, and we recall that $F_{\mathscr{T}}(x, y)$ refers to the 2-variable complex power series with natural number coefficients that satisfies the relation

$$F_{\mathscr{T}}(x, y) = 1 + y - x + xF_{\mathscr{T}}(x, y)^{2M+N-1}.$$

3.3.1 Contribution of conjugacy classes with cursor of infinite order

The goal of this section is to prove Proposition 3.15. Recall that $\tilde{\sigma}^{A}_{(G \wr L, \vec{Y})}(z)$ denotes the contribution of the conjugacy classes of type A (i.e. with cursors of infinite order) to the conjugacy growth series of $G \wr L$.

Proposition 3.15
The formula for $\tilde{\sigma}^{A}_{(G \wr L, \vec{Y})}(z)$ *is*

$$\tilde{\sigma}^{A}_{(G \wr L, \vec{Y})}(z) = \sum_{r \geq 1} \frac{\phi(r)}{r} \sum_{s \geq 1} \frac{(2M + N - 1)^s + (-1)^s (M + N - 1) + M}{s} (F_E(z^r))^s, \tag{3.5}$$

where $\phi(r)$ *denotes the Euler's totient function of* r *and where*

$$F_E(z) := z\sigma_{(G,Y)}(z) \left(F_{\mathscr{T}}(z^2 \sigma_{(G,Y)}(z), z^2(\sigma_{(G,Y)}(z) - 1)) \right)^{2M+N-2}.$$

We construct a conjugacy geodesic normal form of type A as follows. For every cyclically reduced word c of length k we associate a geodesic normal form representing all the elements (η, c) where the support of η is represented in Figure 3.5 (see Corollary 3.12 A):

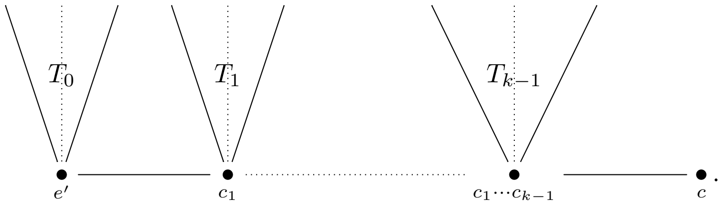


Figure 3.5: Illustration of (η, c)

For a given c of length k, each such geodesic normal form can be seen as the concatenation of *elementary blocks* (η_r, c_{r+1}), where $\mathrm{Supp}(\eta_r) \subset T_r$, for $r \in \{0, \ldots, k-1\}$, as represented in Figure 3.6:

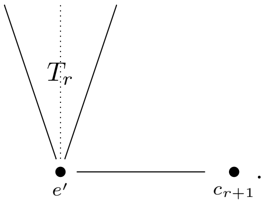


Figure 3.6: Illustration of (η_r, c_{r+1})

For such a concatenation, each vertex of each T_r will be in a different coset of $\langle c\rangle$. This is why any value of G can be associated to this vertex.
For a given c_{r+1}, the growth series of the geodesic normal forms representing the set of elementary blocks $\mathrm{Supp}(\eta_r) \subset T_r$ does not depend on r but only on the degree of the tree and on the growth series of G. This leads us to introduce the following.

Notation
For $x_1, x_2 \in X$ such that $x_1 \neq x_2$, we define

$$E_{x_1,x_2} = \{(\eta, x_2) \in G \wr L \,:\, \mathrm{Supp}(\eta) \subset \{\text{ elements in } L \text{ that do not begin with } x_1 \text{ or } x_2\}\}\,. \tag{3.6}$$

Lemma 3.16
The contribution to the standard growth series of the elements of E_{x_1,x_2} is given by

$$F_E(z) := z\sigma_{(G,Y)}(z)\left(F_{\mathscr{T}}(z^2\sigma_{(G,Y)}(z), z^2(\sigma_{(G,Y)}(z)-1))\right)^{2M+N-2}. \tag{3.7}$$

Proof. Let us compute the contribution of a walk that starts at e', goes along all directions except x_1 and x_2, comes back to e' and ends at x_1. Consider one direction x among the $2M + N - 2$. If the trajectory of this walk is a tree with i non-leaves and j leaves (other than e'), then the length of this walk on $\mathsf{Cay}(L, X)$ is $2(i + j)$, hence $z^{2(i+j)}$ is a factor of the series. On the other hand, any element in G can be associated to a non-leaf, while any non-trivial element in G can be associated to a leaf (different from e'). Therefore $\sigma_{(G,Y)}(z)^i$ and $(\sigma_{(G,Y)}(z) - 1)^j$ are also factors of the series. Considering all the possible walks in direction x gives $F_{\mathscr{T}}(z^2\sigma_{(G,Y)}(z), z^2(\sigma_{(G,Y)}(z)-1))$. Now considering all the $2M+N-2$ possible choices for x, we find $\left(F_{\mathscr{T}}(z^2\sigma_{(G,Y)}(z), z^2(\sigma_{(G,Y)}(z)-1))\right)^{2M+N-2}$. Finally, the value of G at e' is arbitrary, so $\sigma_{(G,Y)}(z)$ appears again, and the walk from e' to x_1 produces the factor z. ☐

Now, considering the language of all such geodesic normal forms for the cyclically reduced words of length k will give rise to a new language; within this language we define two words to be equivalent if and only if they differ by a cyclic permutation of the k elementary blocks. Then the conjugacy geodesic normal forms of type A will be the union, over all $k \in \mathbb{N}$, of the languages above, up to the cyclic permutation equivalence. So let us define

$$\mathsf{ConjRep}^A(G \wr L) := \bigcup_{\substack{c_1\cdots c_k \in L,\, k\geq 1\\ \text{cyclically reduced}}} E_{c_k^{-1},c_1}E_{c_1^{-1},c_2}\cdots E_{c_{k-1}^{-1},c_k}, \tag{3.8}$$

and

$$\mathsf{ConjRep}_k{}^A(G \wr L) := \bigcup_{\substack{c_1\cdots c_k \in L\\ \text{cyclically reduced}}} E_{c_k^{-1},c_1}E_{c_1^{-1},c_2}\cdots E_{c_{k-1}^{-1},c_k}. \tag{3.9}$$

It follows that the contribution to the standard growth series $\sigma_{G\wr L,\vec{Y}}(z)$ of the elements in $\mathsf{ConjRep}^A(G \wr L)$ is

$$F_{\mathsf{ConjRep}^A(G\wr L)}(z) = F_{\mathsf{ConjGeo}(L,X)^A}(F_E(z)).$$

Using Lemma 3.9 one deduces that

$$F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z) = \left((2M+N-1)^k + (-1)^k(M+N-1) + M\right)(F_E(z))^k, \tag{3.10}$$

and

$$F_{\mathsf{ConjRep}^A(G\wr L)}(z)=\sum_{k\geq 1}F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z).$$

Proof of Proposition 3.15. All the elements of $\mathsf{ConjRep}^A(G\wr L)$ are minimal in their conjugacy class, every element of type A in $G\wr L$ is conjugate to an element in $\mathsf{ConjRep}^A(G\wr L)$, and two elements in $\mathsf{ConjRep}^A(G\wr L)$ are conjugate if and only if they differ by a cyclic permutation of the E_{x_1,x_2} components.

For every $k\geq 1$, the cyclic group C_k acts on the set

$$\mathsf{ConjRep}_k{}^A(G\wr L)=\bigcup_{\substack{c_1\cdots c_k\in L\\ \text{cyclically reduced}}} E_{c_k^{-1},c_1}E_{c_1^{-1},c_2}\cdots E_{c_{k-1}^{-1},c_k}$$

by cyclic permutation.

Since for $k\neq k'$, $\mathsf{ConjRep}_k{}^A(G\wr L)\Big/C_k\cap\mathsf{ConjRep}_{k'}{}^A(G\wr L)\Big/C_{k'}=\emptyset$,

$$\tilde{\sigma}^A_{G\wr L,\vec{Y}}(z)=\sum_{k\geq 1}F_{\mathsf{ConjRep}_k{}^A(G\wr L)\Big/C_k}(z).$$

For every $k>1$ and every m, the action of C_k on $\mathsf{ConjRep}_k{}^A(G\wr L)$ preserves

$$S^k(m):=\Big\{U\in\mathsf{ConjRep}_k{}^A(G\wr L)\,:\,|U|_{\vec{Y}}=m\Big\}.$$

So $[z^m]F_{\mathsf{ConjRep}_k{}^A(G\wr L)\Big/C_k}(z)$ is the number of orbits in $S^k(m)$ under the action of C_k. Using Burnside's Lemma we find

$$[z^m]F_{\mathsf{ConjRep}_k{}^A(G\wr L)\Big/C_k}(z)=\frac{1}{k}\sum_{r\in C_k}\sharp\mathrm{Fix}(r)=\frac{1}{k}\sum_{d|k}\sum_{\substack{1\leq r\leq k\\(r,k)=d}}\sharp\mathrm{Fix}(r),$$

where $\mathrm{Fix}(r)$ denotes the set of elements of $S^k(m)$ fixed by the class of r in C_k. In fact, for $d|k$, $1\leq r\leq k$ and $(r,k)=d$, $U\in\mathrm{Fix}(r)$ if and only if $U=W^{\frac{k}{d}}$ for some $W\in\mathsf{ConjRep}_d{}^A(G\wr L)$ and hence $|W|_{\vec{Y}}=\frac{md}{k}$. So

$$\sharp\mathrm{Fix}(r)=[z^{\frac{md}{k}}]F_{\mathsf{ConjRep}_d{}^A(G\wr L)}(z)=[z^m]F_{\mathsf{ConjRep}_d{}^A(G\wr L)}(z^{\frac{k}{d}}).$$

Therefore we find

$$F_{\mathsf{ConjRep}_k{}^A(G\wr L)\Big/C_k}(z)=\frac{1}{k}\sum_{d|k}\phi(\frac{k}{d})F_{\mathsf{ConjRep}_d{}^A(G\wr L)}(z^{\frac{k}{d}}),$$

and summing over all possible values of k gives

$$\tilde{\sigma}^A_{(G\wr L,\vec{Y})}(z)=\sum_{k\geq 1}\sum_{d|k}\frac{1}{k}\phi(\frac{k}{d})F_{\mathsf{ConjRep}_d{}^A(G\wr L)}(z^{\frac{k}{d}})=\sum_{r\geq 1}\sum_{s\geq 1}\frac{\phi(r)}{rs}F_{\mathsf{ConjRep}_s{}^A(G\wr L)}(z^r).$$

By (3.10) we obtain the result. □

Corollary 3.17
If L is infinite, then the radius of convergence of the conjugacy growth series $\tilde{\sigma}_{(G\wr L,\vec{Y})}(z)$ is the same as the radius of convergence of the standard growth series $\sigma_{(G\wr L,\vec{Y})}(z)$.

Proof. Firstly, it is clear that $\mathcal{RC}(\tilde{\sigma}^{A}_{(G\wr L,\vec{Y})}(z)) \geq \mathcal{RC}(\tilde{\sigma}_{(G\wr L,\vec{Y})}(z)) \geq \mathcal{RC}(\sigma_{(G\wr L,\vec{Y})}(z))$, so it remains to prove that $\mathcal{RC}(\tilde{\sigma}^{A}_{(G\wr L,\vec{Y})}(z)) \leq \mathcal{RC}(\sigma_{(G\wr L,\vec{Y})}(z))$. In [[Par92], Theorem 4.1], it is proved that $\mathcal{RC}(\sigma_{(G\wr L,\vec{Y})}(z))$ is the smallest positive value t such that

$$(2M + N - 1)(F_E(t)) = 1.$$

Proposition 2.1 and Remark 3.10 show that $\mathcal{RC}\left(F_{\mathsf{ConjGeo}(L,X)^A}(F_E(z))\right)$ is also equal to the smallest positive value t such that

$$(2M + N - 1)(F_E(t)) = 1.$$

Claim: For every $m \in \mathbb{N}$, the following holds:

$$[z^m]\sum_{k\geq 1}\frac{F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)}{k} \leq [z^m]\tilde{\sigma}^{A}_{(G\wr L,\vec{Y})}(z) \leq [z^m]F_{\mathsf{ConjGeo}(L,X)^A}(F_E(z)).$$

And hence

$$\mathcal{RC}\left(\sum_{k\geq 1}\frac{F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)}{k}\right) \geq \mathcal{RC}(\tilde{\sigma}^{A}_{(G\wr L,\vec{Y})}(z)) \geq \mathcal{RC}\left(F_{\mathsf{ConjGeo}(L,X)^A}(F_E(z))\right).$$

Proof of the claim: The first inequality is because for each k there are at most k different elements in $\mathsf{ConjRep}^A_k(G \wr L)$ that represent the same conjugacy class (due to cyclic permutation of order k). While the second inequality follows from

$$\begin{aligned}
[z^m]\tilde{\sigma}^{A}_{(G\wr L,\vec{Y})}(z) &\leq [z^m]\sum_{k\geq 1}F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)\\
&= [z^m]F_{\mathsf{ConjRep}^A(G\wr L)}(z)\\
&= [z^m]F_{\mathsf{ConjGeo}(L,X)^A}(F_E(z)).
\end{aligned}$$

Therefore with the claim, it suffices to prove that

$$\mathcal{RC}\left(\sum_{k\geq 1}\frac{F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)}{k}\right) \leq \mathcal{RC}\left(\sum_{k\geq 1}F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)\right).$$

Note that because $F_E(0) = 0$, for every $m \in \mathbb{N}$ we have

$$[z^m]\sum_{k\geq 1}F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z) = [z^m]\sum_{k=1}^{m}F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)$$

and

$$[z^m]\sum_{k\geq 1}\frac{F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)}{k} = [z^m]\sum_{k=1}^{m}\frac{F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)}{k}.$$

Hence

$$\begin{aligned}[z^m]\sum_{k\geq 1}\frac{F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)}{k} =& [z^m]\sum_{k=1}^{m}\frac{F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)}{k}\\ \geq & [z^m]\frac{1}{m}\sum_{k=1}^{m}F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z) = \frac{1}{m}[z^m]\sum_{k\geq 1}F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z).\end{aligned}$$

Therefore passing to limsup we find

$$\begin{aligned}\limsup_{m\to\infty}\sqrt[m]{[z^m]\sum_{k\geq 1}\frac{F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)}{k}} &\geq \limsup_{m\to\infty}\sqrt[m]{\frac{1}{m}[z^m]\sum_{k\geq 1}F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)}\\ &= \limsup_{m\to\infty}\sqrt[m]{\sum_{k\geq 1}F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)}.\end{aligned}$$

This proves that

$$\mathcal{RC}\left(\sum_{k\geq 1}\frac{F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)}{k}\right) \leq \mathcal{RC}\left(\sum_{k\geq 1}F_{\mathsf{ConjRep}_k{}^A(G\wr L)}(z)\right),$$

and finalizes the proof.

□

3.3.2 Contribution of conjugacy classes with cursor of finite order

Recall that $\tilde{\sigma}^{B}_{(G\wr L,\vec{Y})}(z)$ denotes the contribution to the conjugacy growth series of $G\wr L$ of the conjugacy classes of type B (i.e. with cursors of finite order).

We will write $\tilde{\sigma}^{B\neq e'}_{(G\wr L,\vec{Y})}(z)$ for the contribution to $\tilde{\sigma}^{B}_{(G\wr L,\vec{Y})}(z)$ of the conjugacy classes of elements (η, b), where $b\neq e'$, and $\tilde{\sigma}^{e'}_{(G\wr L,\vec{Y})}(z)$ for the contribution to $\tilde{\sigma}^{B}_{(G\wr L,\vec{Y})}(z)$ of the conjugacy classes of elements of the form (η, e'). Hence $\tilde{\sigma}^{B}_{(G\wr L,\vec{Y})}(z) = \tilde{\sigma}^{B\neq e'}_{(G\wr L,\vec{Y})}(z)+\tilde{\sigma}^{e'}_{(G\wr L,\vec{Y})}(z)$. We consider these two cases separately.

3.3.2.1 Contribution of conjugacy classes with torsion cursor

Proposition 3.18
The growth series $\tilde{\sigma}^{B\neq e'}_{(G\wr L,\vec{Y})}(z)$ *is given by*

$$\tilde{\sigma}^{B\neq e'}_{(G\wr L,\vec{Y})}(z) = Nz\tilde{\sigma}_{(G,Y)}(z)\left(F_{\mathscr{T}}(z^2\tilde{\sigma}_{(G,Y)}(z), z^2(\tilde{\sigma}_{(G,Y)}(z)-1))\right)^{2M+N-1}. \tag{3.11}$$

Proof. Let (η, b) be an element in $G\wr L$, where $b\in L$ is a torsion element. By Lemma 3.11 one can assume that $b=b_j$ for some $j\in\{1,\dots,N\}$. Let T be the right transversal of $\langle b_j\rangle$ in L consisting of the set of (freely reduced) words in L that do not begin with b_j. By Corollary 3.12 B) one can assume that $\mathrm{Supp}(\eta)\subset T$. Then an element (η', b) with $\mathrm{Supp}(\eta')\subset T$ is conjugate to (η, b) in $G\wr L$ if and only if for every $l\in L$, $\eta'(l)$ is conjugate to $\eta(l)$ in G.

Hence we have to consider the walks on $\mathsf{Cay}(L, X)$ that go first in the $2M+N-1$ directions other than b, create the non-trivial conjugacy classes in G of η, come back to the origin, and then finish at b without creating any element at this position. So the contribution to $\tilde{\sigma}^{B^{\neq e'}}_{(G\wr L,\vec{Y})}(z)$ of the conjugacy classes of the elements (η, b_j) for a given $j \in \{1,\dots,N\}$ is given by

$$z\tilde{\sigma}_{(G,Y)}(z)\left(F_{\mathscr{T}}(z^2\tilde{\sigma}_{(G,Y)}(z), z^2(\tilde{\sigma}_{(G,Y)}(z)-1))\right)^{2M+N-1}.$$

The term $\left(F_{\mathscr{T}}(z^2\sigma_{(G,Y)}(z), z^2(\sigma_{(G,Y)}(z)-1))\right)^{2M+N-1}$ counts the contribution of the walks starting at the origin, visiting every vertex of $\mathrm{Supp}(\eta)$ (in the $2M+N-1$ directions away from b_j), creating the non-trivial conjugacy classes in G, and coming back to e'. The term $\tilde{\sigma}_{(G,Y)}$ counts the contribution at the component e', and z accounts for the step from e' to b. Therefore summing over all the j's, we find the result. ☐

3.3.2.2 Contribution of conjugacy classes with trivial cursor

We now explain how to compute $\tilde{\sigma}^{e'}_{(G\wr L,\vec{Y})}(z)$. Lemma 3.13 shows that a non-trivial element (η, e') is minimal in its conjugacy class if and only if the tree spanned by $\mathrm{Supp}(\eta)$ contains the origin e' and all the components of η are minimal in their conjugacy class (in G). Moreover, two such elements (η, e'), (η', e') are conjugate if and only if there exists $d \in L$ such that for every $l \in L$, $\eta'(l)$ is conjugate (in G) to $\eta^d(l)$.

Therefore we have to consider the following. For a finite subset $L' \subset L$, we write $\mathrm{Span}(L')$ for the set of vertices of the tree spanned by L', i.e. the smallest subset of L containing L' that forms a tree in $\mathsf{Cay}(L, X)$. Let Υ be the set

$$\Upsilon := \{L' \subset L \,:\, \sharp L' < \infty,\, e' \in L',\, L' = \mathrm{Span}(L')\}.$$

Although $L' \in \Upsilon$ denotes a set of vertices, we call it a tree, denote by $\mathscr{L}(L')$ its set of leaves (vertices of degree ≤ 1), and similarly write $\mathscr{L}^c(L') := L' \setminus \mathscr{L}(L')$ for its set of non-leaves. Now let

$$\Omega := \bigcup_{L' \in \Upsilon} \{\eta : L' \to G \,:\, \forall l \in \mathscr{L}(L'),\, \eta(l) \neq e\}.$$

The set Ω represents the collection of the labeled finite trees containing the origin with the leaves having non-trivial label. For an element $\eta : L' \to G$ in Ω we define a weight $|\eta|_{\Omega,\sim} \in \mathbb{N} \setminus \{0\}$ to be

$$\eta|_{\Omega,\sim} := 2\sharp\{\text{ edges of the tree spanned by } L'\} + \sum_{l \in L'} |\eta(l)|_{\sim,Y}.$$

Two functions $\eta_1 : L_1 \to G$ and $\eta_2 : L_2 \to G$ in Ω are called equivalent if there exists $d \in L$ such that $L_1 = dL_2$ and for every $l \in L_1$, $\eta_1(l)$ is conjugate in G to $\eta_2(d^{-1}l)$. We denote this equivalence by $\sim$. Note that the weight $|\cdot|_{\Omega,\sim}$ is preserved under this equivalence. Then there is a bijection between the set $\Omega \big/ \sim$ and the set of non-trivial conjugacy classes $[(\eta, e')]$ in $G \wr L$, which for every n restricts to the set of elements of $\Omega \big/ \sim$ of weight n and the set non-trivial conjugacy classes $[(\eta, e')]$ of conjugacy length n, i.e the following diagram commutes :

$$\begin{tikzcd}
\Omega \big/ \sim \arrow[rr, leftrightarrow] \arrow[dr, "|\cdot|_{\Omega,\sim}"'] & \circlearrowright & \{[(\eta, e')] \in (G \wr L)_\sim \,:\, \eta \neq \vec{e}\} \arrow[dl, "|\cdot|_{\sim,G\wr L}"] \\
& \mathbb{N} \setminus \{0\}. &
\end{tikzcd}$$

One restricts the relation $\sim$ to Υ by defining that $L_1 \sim L_2$ if there exists $d \in L$ such that $L_1 = dL_2$. Let $\widetilde{\Upsilon} \subset \Upsilon$ be a set of representatives of the elements in Υ with respect to $\sim$.

For any $L' \in \widetilde{\Upsilon}$, the subgroup of L fixing L' by left multiplication is finite, since L' is finite, hence this subgroup is either trivial or isomorphic to C_2 (generated by a conjugate of some b_j); in the former case we call L' *symmetric*, and in the latter *asymmetric*. Note that since an element L' of $\widetilde{\Upsilon}$ contains e', it is symmetric if and only if there exist $L_1 \subset L'$, $h \in L_1$ and $j \in \{1, \ldots, N\}$ such that $L' = L_1 \sqcup hb_jh^{-1}L_1$. In this case L_1, h and j are unique, and $\langle hb_jh^{-1} \rangle$ is the subgroup of L fixing L' by left multiplication; the element h is the vertex where L_1 connects to $hb_jh^{-1}L_1$, and the following holds:

$$\begin{cases} \sharp\mathscr{L}(L_1 \sqcup hb_jh^{-1}L_1) = 2(\sharp\mathscr{L}(L_1) - 1) \quad \text{and} \quad \sharp\mathscr{L}^c(L_1 \sqcup hb_jh^{-1}L_1) = 2(\sharp\mathscr{L}^c(L_1) + 1) \\ \qquad\qquad\qquad\qquad \text{if } L_1 \neq \{e'\}, \text{ while} \\ \qquad\qquad \sharp\mathscr{L}(\{e, b_j\} = 2 \quad \text{and} \quad \sharp\mathscr{L}^c(\{e, b_j\}) = 0 \\ \qquad\qquad\qquad\qquad \text{for } j \in \{1, \ldots, M\} \end{cases} \tag{3.12}$$

Note that the number of edges in the tree spanned by $L' \in \widetilde{\Upsilon}$ is $\sharp L' - 1$.

By Burnside's Lemma we have the following.

Proposition 3.19
The contribution to conjugacy growth series of the conjugacy classes of the elements with trivial cursor is given by

$$\begin{aligned} \tilde{\sigma}^{e'}_{(G\wr L,\vec{Y})}(z) = 1 &+ \sum_{\text{asymmetric } L' \in \widetilde{\Upsilon}} z^{2(\sharp L'-1)} \big(\tilde{\sigma}_{(G,Y)}(z) - 1\big)^{\sharp\mathscr{L}(L')} \tilde{\sigma}_{(G,Y)}(z)^{\sharp\mathscr{L}^c(L')} \\ &+ \frac{1}{2} \sum_{\text{symmetric } L' = L_1 \sqcup hb_jh^{-1}L_1 \in \widetilde{\Upsilon}} z^{2(\sharp L'-1)} \big(\tilde{\sigma}_{(G,Y)}(z) - 1\big)^{\sharp\mathscr{L}(L')} \tilde{\sigma}_{(G,Y)}(z)^{\sharp\mathscr{L}^c(L')} \\ &+ \frac{1}{2} M z^2 (\tilde{\sigma}_{(G,Y)}(z^2) - 1) \\ &+ \frac{1}{2} \sum_{\substack{\text{symmetric } L' = L_1 \sqcup hb_jh^{-1}L_1 \in \widetilde{\Upsilon} \\ L_1 \neq \{e\}}} z^{2(\sharp L'-1)} \big(\tilde{\sigma}_{(G,Y)}(z^2) - 1\big)^{\sharp\mathscr{L}(L_1)-1} \tilde{\sigma}_{(G,Y)}(z^2)^{\sharp\mathscr{L}^c(L_1)+1}. \end{aligned}$$

The first term, 1, is the contribution of the trivial conjugacy class. The second term comes from the contribution of the asymmetric trees in $\widetilde{\Upsilon}$, the term in the second line comes from the contribution of the symmetric trees in $\widetilde{\Upsilon}$ fixed by the identity, and the terms in the third and the forth line come from the contribution of the symmetric trees in $\widetilde{\Upsilon}$ fixed by the elements of order 2. According to (3.12) we distinguish the L_1's being $\{e\}$ or not.

Now the difficulty of computing $\tilde{\sigma}^{e'}_{(G\wr L,\vec{Y})}(z)$ resides in finding a suitable set $\widetilde{\Upsilon}$.

If L is free then there exists an explicit left order $<$ on L [Šun13], and then for a given $L' \in \Upsilon$ there exists a unique $dL' \in \Upsilon$ (for $d \in L$) such that every element of dL' is greater than or equal to e' with respect to $<$.

Lemma 3.20 ([Šun13]:Theorem 1.2)
If $L = \langle a_1, \ldots, a_M | \rangle$ is a free group then a left order $>$ on L can be defined by

$$w > e \iff \begin{cases} \sharp\{a_j a_i^{-1} \text{in } w,\, j > i\} > \sharp\{a_j^{-1} a_i \text{ in } w,\, j > i\} \\ \qquad\qquad\qquad \text{or} \\ \sharp\{a_j a_i^{-1} \text{ in } w,\, j > i\} = \sharp\{a_j^{-1} a_i \text{ in } w, j > i\} \\ \qquad \text{and } w \text{ ends with } a_i,\, i \in \{1, \ldots, M\}\,. \end{cases}$$

So if L is free one can define $\widetilde{\Upsilon}$ as

$$\widetilde{\Upsilon} := \{L' \in \Upsilon \, : \, \forall l \in L' \setminus \{e'\},\, l > e\},$$

and no element of $\widetilde{\Upsilon}$ will be symmetric (from our last definition). It is a classic result that free groups are left-orderable, dating all the way to [Vin49], and we explicitly mention the left order given by Šunić above because of its simplicity and with the hope that it can exploited to find formulas for the conjugacy growth series in this setting.

Although we were unable to find a general formula for the conjugacy series of groups of the form $G \wr L$, where $\mathsf{Cay}(L, X)$ is a tree, we observe that $\tilde{\sigma}^{e'}_{(G\wr L,\vec{Y})}(z)$ is a complex power series with natural coefficients in the variables $z\tilde{\sigma}_{(G,Y)}(z)$, $z(\tilde{\sigma}_{(G,Y)}(z) - 1)$, $z^2\tilde{\sigma}_{(G,Y)}(z^2)$ and $z^2(\tilde{\sigma}_{(G,Y)}(z^2) - 1)$, or just $z^2\tilde{\sigma}_{(G,Y)}(z)$ and $z^2(\tilde{\sigma}_{(G,Y)}(z) - 1)$ when L is a free group.

3.4 Examples

In this section we give explicit computations of the conjugacy growth series for several examples.

Example 3.20.1 (Conjugacy growth series of $G \wr C_2$)
Let $C_2 = \langle \{b\} | b^2 \rangle$. Since in C_2 there are no elements of infinite order, $\tilde{\sigma}^{A}_{(G\wr C_2,\vec{Y})}(z) = 0$. The series $\tilde{\sigma}^{B^{\neq e'}}_{(G\wr C_2,\vec{Y})}(z)$ is

$$\tilde{\sigma}^{B^{\neq e'}}_{(G\wr C_2,\vec{Y})}(z) = z\tilde{\sigma}_{(G,Y)}(z).$$

Now the series $\tilde{\sigma}^{e'}_{(G\wr L,\vec{Y})}(z)$ is the sum of one term counting the contribution of the classes of elements (η, e') when $\mathrm{Supp}(\eta)$ is a singleton or the empty set, which is equal to $\tilde{\sigma}_{(G,Y)}(z)$, and the contribution of the conjugacy classes of elements (η, e') with $\mathrm{Supp}(\eta) = C_2$. Such a conjugacy class is unique up to permutation by C_2 of the components of η and up to conjugation in G. So using Burnside's Lemma we find

$$\tilde{\sigma}^{e'}_{(G\wr C_2,\vec{Y})}(z) = \tilde{\sigma}_{(G,Y)}(z) + \frac{z^2}{2}\left((\tilde{\sigma}_{(G,Y)}(z) - 1)^2 + \tilde{\sigma}_{(G,Y)}(z^2) - 1 \right).$$

Therefore

$$\tilde{\sigma}_{(G\wr C_2),\vec{Y}}(z) = \tilde{\sigma}_{(G,Y)}(z) + z\tilde{\sigma}_{(G,Y)}(z) + \frac{z^2}{2}\left((\tilde{\sigma}_{(G,Y)}(z) - 1)^2 + \tilde{\sigma}_{(G,Y)}(z^2) - 1 \right).$$

Note that the radius of convergence of $\tilde{\sigma}_{G\wr C_2,\vec{Y}}(z)$ is the same as the radius of convergence of $\tilde{\sigma}_{(G,Y)}(z)$.

Example 3.20.2 (Conjugacy growth series of $G \wr \mathbb{Z}$)
Let $\mathbb{Z}$ be equipped with the standard generating set. Using Formula (3.5) in Section 3.3.1 with $M = 1$ and $N = 0$ we find

$$\tilde{\sigma}^{A}_{(G\wr\mathbb{Z},\vec{Y})}(z) = 2\sum_{r\geq 1} \frac{\phi(r)}{r} \sum_{s\geq 1} \frac{z^{rs}\sigma_{(G,Y)}(z^r)^s}{s}.$$

Since in $\mathbb{Z}$ there are no torsion elements, $\tilde{\sigma}^{B^{\neq e'}}_{(G\wr\mathbb{Z},\vec{Y})}(z) = 0$. Now for $\tilde{\sigma}^{e'}_{(G\wr\mathbb{Z},\vec{Y})}(z)$ notice that if $\mathrm{Supp}(\eta) \neq \emptyset$, then there exists a unique translate η' of η (i.e. $\eta' = \eta^d$ for a certain $d \in \mathbb{Z}$) such that $\mathrm{Supp}(\eta') \subset \mathbb{N}$ and $\eta'(e') \neq e$. Hence we must count for every subset $\{0, \ldots, n\} \subset \mathbb{N}$, the contribution of the classes of elements $(\eta, 0)$ with $\mathrm{Supp}(\eta) \subset \{0, \ldots, n\}$, $\eta(0) \neq e$ and $\eta(n) \neq e$. We finally find

$$\tilde{\sigma}^{e'}_{(G \wr \mathbb{Z}, \vec{Y})}(z) = \tilde{\sigma}_{(G,Y)}(z) + z^2(\tilde{\sigma}_{(G,Y)}(z) - 1)^2 \sum_{k=0}^{\infty} \left(z^2 \tilde{\sigma}_{(G,Y)}(z)\right)^k = \tilde{\sigma}_{(G,Y)}(z) + \frac{z^2(\tilde{\sigma}_{(G,Y)}(z) - 1)^2}{1 - z^2 \tilde{\sigma}_{(G,Y)}(z)}.$$

Therefore

$$\tilde{\sigma}_{(G \wr \mathbb{Z}, \vec{Y})}(z) = 2 \sum_{r \geq 1} \frac{\phi(r)}{r} \sum_{s \geq 1} \frac{z^{rs} \sigma_{(G,Y)}(z^r)^s}{s} + \tilde{\sigma}_{(G,Y)}(z) + \frac{z^2(\tilde{\sigma}_{(G,Y)}(z) - 1)^2}{1 - z^2 \tilde{\sigma}_{(G,Y)}(z)}.$$

The radius of convergence of $\tilde{\sigma}_{(G \wr \mathbb{Z}, \vec{Y})}(z)$ is the unique positive value t such that $t\sigma_{(G,Y)}(t) = 1$.

Example 3.20.3 (Conjugacy growth series of the Lamplighter group $C_2 \wr \mathbb{Z}$)
Let $G = C_2$ be the group with 2 elements. Using the formula above we find

$$\tilde{\sigma}_{(C_2 \wr \mathbb{Z}, \vec{Y})}(z) = 2 \sum_{r \geq 1} \frac{\phi(r)}{r} \sum_{s \geq 1} \frac{z^{rs}(1 + z^r)^s}{s} + 1 + z + \frac{z^4}{1 - z^2(1 + z)}.$$

We purpose to give an asymptotic estimate of the conjugacy growth $[z^m]\tilde{\sigma}_{(C_2 \wr \mathbb{Z}, \vec{Y})}(z)$. We say that two functions $f, g : \mathbb{N} \to \mathbb{N}$ are asymptotically equivalent, that we write $f \sim_{\text{as}} g$ if

$$\lim_{m \to \infty} \frac{f(m)}{g(m)} = 1.$$

Since

$$\mathcal{RC}\left(2 \sum_{r \geq 1} \frac{\phi(r)}{r} \sum_{s \geq 1} \frac{z^{rs}(1 + z^r)^s}{s}\right) < \mathcal{RC}\left(1 + z + \frac{z^4}{1 - z^2(1 + z)}\right)$$

it follows that

$$[z^m]\tilde{\sigma}_{(C_2 \wr \mathbb{Z}, \vec{Y})}(z) \sim_{\text{as}} [z^m] 2 \sum_{r \geq 1} \frac{\phi(r)}{r} \sum_{s \geq 1} \frac{z^{rs}(1 + z^r)^s}{s}.$$

We have the following.

$$\begin{aligned}
[z^m]2\sum_{r\geq 1}\frac{\phi(r)}{r}\sum_{s\geq 1}\frac{z^{rs}(1+z^r)^s}{s} &= 2\sum_{\substack{(r,s,k)\in\mathbb{N}^*\times\mathbb{N}^*\times\mathbb{N}\\ r(s+k)=m,\, k\leq s}}\frac{\phi(r)}{rs}\binom{s}{k}\\
&= 2\sum_{r|m}\frac{\phi(r)}{r}\sum_{s=\frac{m}{2r}}^{\frac{m}{r}}\frac{1}{s}\binom{s}{\frac{m}{r}-s}\\
&\overset{\text{wolframalpha}}{=} 2\sum_{r|m}\frac{\phi(r)}{m}\left(\left(\frac{-1+\sqrt{5}}{2}\right)^{\frac{m}{r}}+\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{m}{r}}\right)\\
&= 2\frac{1}{m}\left(\left(\frac{-1+\sqrt{5}}{2}\right)^{m}+\left(\frac{1+\sqrt{5}}{2}\right)^{m}\right)\\
&+2\sum_{\substack{r|m\\ r<m}}\frac{\phi(r)}{m}\left(\left(\frac{-1+\sqrt{5}}{2}\right)^{\frac{m}{r}}+\left(\frac{1+\sqrt{5}}{2}\right)^{\frac{m}{r}}\right)\\
&\sim_{\text{as}} \frac{2}{m}\left(\left(\frac{-1+\sqrt{5}}{2}\right)^{m}+\left(\frac{1+\sqrt{5}}{2}\right)^{m}\right)\\
&\sim_{\text{as}} \frac{2}{m}\left(\frac{1+\sqrt{5}}{2}\right)^{m}.
\end{aligned}$$

Therefore

$$[z^m]\tilde{\sigma}_{(C_2\wr\mathbb{Z},\vec{Y})}(z)\sim_{\text{as}}\frac{2}{m}\left(\frac{1+\sqrt{5}}{2}\right)^{m}. \tag{3.13}$$

Proposition 3.21
The conjugacy growth series $\tilde{\sigma}_{(C_2\wr\mathbb{Z},\vec{Y})}(z)$ of the Lamplighter group is transcendental on $\mathbb{Q}(z)$.

Proof. This follows from the estimate (3.13) and [Fla87, Theorem D]. ☐

Example 3.21.1 (Conjugacy growth series of $G\wr(C_2 * C_2)$)
Let $C_2 * C_2 = \langle\{b_1, b_2\}|b_1^2, b_2^2\rangle$. Using Formula (3.5) in Section 3.3.1 with $M = 0$ and $N = 2$ we find

$$\tilde{\sigma}^{A}_{(G\wr(C_2*C_2),\vec{Y})}(z)=\sum_{r\geq 1}\frac{\phi(r)}{r}\sum_{s\geq 1}\frac{z^{2rs}\sigma_{(G,Y)}(z^r)^{2s}}{s}.$$

Using Formula (3.11) in Section 3.3.2.1 with $M = 0$ and $N = 2$, we find

$$\tilde{\sigma}^{B^{\neq e'}}_{(G\wr(C_2*C_2),\vec{Y})}(z)=2z\tilde{\sigma}_{(G,Y)}(z)\frac{1-z^2}{1-z^2\tilde{\sigma}_{(G,Y)}(z)}.$$

For the computation of $\tilde{\sigma}^{e'}_{G\wr(C_2*C_2)}(z)$ we proceed as follows. Let us consider the set Υ as in Section 3.3.2.2. Every such element spans a line. Let $L' \in \Upsilon$ and let r be the diameter of L'. There are two ways according to the parity of r, to choose a representative L' under the relation given by the left action of L.

- If r is even there exists a unique $\tilde{L} \subset \{$words that do not begin with $b_2\}$ that is a left translate of L' in Υ. The subgroup of L fixing $\tilde{L}$ by left multiplication is trivial. Hence the contribution of all such elements is

$$\begin{aligned}&\tilde{\sigma}_{(G,Y)}(z) - 1 + \big(\tilde{\sigma}_{(G,Y)}(z) - 1\big)^2 \sum_{0<r \text{ even}} z^{2r}\tilde{\sigma}_{(G,Y)}(z)^{r-1} \\ &= \tilde{\sigma}_{(G,Y)}(z) + \big(\tilde{\sigma}_{(G,Y)}(z) - 1\big)^2 \frac{z^4\tilde{\sigma}_{(G,Y)}(z)}{1 - z^4\tilde{\sigma}_{(G,Y)}(z)^2}\,.\end{aligned}$$

- If r is odd then exactly one of the following happens:

  i) There exists a unique $\tilde{L} \subset \{\text{words that do not begin with } b_2\}$ that is a left translate of L' in Υ, or

  ii) there exists a unique $\tilde{L} \subset \{b_2\} \cup \{\text{words that do not begin with } b_2\}$ that is a left translate of L' in Υ.

  In both cases the subgroup of L fixing $\tilde{L}$ by left multiplication is C_2. Hence each of the cases i) and ii) gives rise to

$$\begin{aligned}&\frac{1}{2}\left((\tilde{\sigma}_{(G,Y)}(z) - 1)^2 \sum_{0<r \text{ odd}} z^{2r}\tilde{\sigma}_{(G,Y)}(z)^{r-1} + (\tilde{\sigma}_{(G,Y)}(z^2) - 1) \sum_{0<r \text{ odd}} z^{2r}\tilde{\sigma}_{(G,Y)}(z^2)^{\frac{r-1}{2}} \right) \\ &= \frac{z^2}{2}\left(\frac{(\tilde{\sigma}_{(G,Y)}(z) - 1)^2}{1 - z^4\tilde{\sigma}_{(G,Y)}(z)^2} + \frac{\tilde{\sigma}_{(G,Y)}(z^2) - 1}{1 - z^4\tilde{\sigma}_{(G,Y)}(z^2)} \right).\end{aligned}$$

Therefore summing the contribution of the even and odd r's and the contribution of the trivial conjugacy class we find

$$\tilde{\sigma}^{e'}_{(G\wr(C_2*C_2),\vec{Y})}(z) = \tilde{\sigma}_{(G,Y)}(z) + z^2\left(1 + z^2\tilde{\sigma}_{(G,Y)}(z)\right) \frac{(\tilde{\sigma}_{(G,Y)}(z) - 1)^2}{1 - z^4\tilde{\sigma}_{(G,Y)}(z)^2} + z^2\frac{(\tilde{\sigma}_{(G,Y)}(z^2) - 1)}{1 - z^4\tilde{\sigma}_{(G,Y)}(z^2)}.$$

Finally

$$\begin{aligned}\tilde{\sigma}_{(G\wr(C_2*C_2),\vec{Y})}(z) =\,& 2\sum_{r\geq 1}\frac{\phi(r)}{r}\sum_{s\geq 1} z^{2rs}\sigma_{(G,Y)}(z^r)^{2s} + 2z\tilde{\sigma}_{(G,Y)}(z)\frac{1 - z^2}{1 - z^2\tilde{\sigma}_{(G,Y)}(z)} \\ &+ \tilde{\sigma}_{(G,Y)}(z) + z^2\left(1 + z^2\tilde{\sigma}_{(G,Y)}(z)\right) \frac{(\tilde{\sigma}_{(G,Y)}(z) - 1)^2}{1 - z^4\tilde{\sigma}_{(G,Y)}(z)^2} + z^2\frac{(\tilde{\sigma}_{(G,Y)}(z^2) - 1)}{1 - z^4\tilde{\sigma}_{(G,Y)}(z^2)}.\end{aligned}$$

Chapter 4

Conjugacy growth series of graph products

This chapter gives an explicit formula for the conjugacy growth series of a graph product in terms of the standard and the conjugacy growth series of the vertex groups, and also proves that the radius of convergence of the conjugacy growth series is the same as that of the standard growth series. Graph products generalize direct and free products of groups. The first systematic study was done by Green in her thesis [Gre90]. In Section 4.1 we give the definition of a graph product and some basic properties, and in Section 4.1.1 we give a tool concerning some functions from a finite simple graph to a ring, which will be used to obtain a formula for the conjugacy growth series that is not based on induction on the subgraph-products. Section 4.2 gives the explicit formula of the conjugacy growth series of a graph product as well as its radius of convergence. Finally Section 4.3 gives explicit computations of conjugacy growth series for several examples of graph products based on simple graphs.

4.1 Graph products

Let $\Gamma = (V, E)$ be a simple graph, that is, a non-oriented graph without loops or multiple edges, with V for the set of vertices and E for the set of edges. For each vertex v of Γ, let G_v be a group. The *graph product* denoted by G_V *of the groups* G_v *with respect to* Γ is defined to be the quotient of their free product by the normal closure of the relators $[g_v, g_w]$ for all $g_v \in G_v$, $g_w \in G_w$ for which $\{v, w\}$ is an edge of Γ.

Given a graph product group G over a graph $\Gamma = (V, E)$ and any subset $V' \subseteq V$, the *subgraph product* associated to V' is the subgroup $G_{V'} := \langle G_v \mid v \in V' \rangle$ of G. By [Gre90, Proposition 3.31] (see also the discussion in [CH14, Section 3]), $G_{V'}$ is isomorphic to the graph product of the G_v ($v \in V'$) on the induced subgraph of Γ with vertex set V'. Note that $G_V = G$ and $G_\emptyset$ is the trivial group.

For any nonempty subset $V' \subseteq V$, the *centralizing set* $\mathsf{Ct}(V')$ of V' denotes the set of all vertices of Γ whose associated vertex groups commute with the subgraph product $G_{V'}$ (and hence the set of vertices that are adjacent to all of the vertices in V'). That is, for any vertex $v \in V$ the set

$$\mathsf{Ct}(v) := \{w \in V \,:\, \{v, w\} \in E\}$$

is the set of neighbours of v; for a subset $V' \subset V$, we have

$$\mathsf{Ct}(V') := \bigcap_{v \in V'} \mathsf{Ct}(v);$$

hence $\mathsf{Ct}(\emptyset) = V$. The set of *cliques* of V is defined to be

$$\mathsf{Clq}(V) := \{A \subset V \ : \forall a \in A, \mathsf{Ct}(a) = A \setminus \{a\}\};$$

that is, the elements of $\mathsf{Clq}(V)$ are the subsets V' of V satisfying the property that all of the vertex groups associated to vertices in V' commute. For any pair of subsets $V_1, V_2 \subset V$, we write $V_1 \approx V_2$ whenever $V_1 \subset \mathsf{Ct}(V_2)$ (and hence the subgraph products G_{V_1} and G_{V_2} commute in G_V); note that this is equivalent to $V_2 \subset \mathsf{Ct}(V_1)$ and that this implies $V_1 \cap V_2 = \emptyset$. If $V_1 \approx V_2$ we say that V_1 and V_2 are *connected*.

Each graph product over a graph with more than one vertex can be decomposed as an amalgamated product of graph products of groups over the graph product of an appropriate centralizing set.

Lemma 4.1 ([Gre90], [Chi94a])
Let G_V be a graph product of groups, and let $v \in V$. Using the inclusion maps from $G_{\mathsf{Ct}(v)}$ into both $G_{V \setminus \{v\}}$ and $G_{\mathsf{Ct}(v)} \times G_v$ (as the subgroup $G_{\mathsf{Ct}(v)} \times \{\epsilon\}$), the group G_V can be decomposed as the amalgamated product

$$G_V = G_{V \setminus \{v\}} \ *_{G_{\mathsf{Ct}(v)}} \ (G_{\mathsf{Ct}(v)} \times G_v).$$

Any nonidentity element in a graph product can be written as a product $g_1 \cdots g_l$ for some $l > 1$, where each g_i is a nontrivial element of a vertex group G_{v_i}. By [Gre90, Theorem 3.9] (also in [CH14, Section 3]), one can get from any such expression of minimal length to any other by swapping the order in the expression of elements g_i, g_{i+1} from commuting vertex groups. Hence every minimal length expression for an element $g \in G_V$ over the generating set $Y_V := \cup_{v \in V} G_v \setminus \{\epsilon\}$ has the same length l, which is called the *syllable length* of g, and involves the same set $\{g_1, g_2, \dots g_l\}$ of vertex group elements, with the same multiplicities, called the *syllables* of g. Whenever $g_1 \cdots g_l$ is a minimal length expression for g over Y and $0 \leq i \leq n$, we call the product $g_1 \cdots g_i$ a *left divisor* of g, and the product $g_{i+1} \cdots g_n$ a *right divisor* of g.

Definition
Let $g \in G_V$. The *support* of g is the set

$$\mathrm{Supp}(g) := \bigcap_{V' \subset V \text{ and } g \in G_{V'}} V'.$$

The support of g can also be realized as the set of all vertices v for which a nontrivial element in the associated vertex group G_v appears in a geodesic word representative of g over Y_V [CH14, Proposition 3.3].

Suppose that every vertex group G_v of the graph product has a symmetric generating set X_v. For each $V' \subseteq V$, let

$$X_{V'} := \cup_{v \in V'} X_v;$$

then $X_{V'}$ is a symmetric generating set for $G_{V'}$.

4.1.1 Möbius-type inversion formulas

Definition
Let $\Gamma = (V, E)$ be a finite simple graph, and let $V' \subset V$. We say that V' is *indecomposable* if it cannot be written as

$$V' = V'_1 \sqcup V'_2$$

with $V'_1, V'_2 \subset V$ both non-empty and with $V'_1 \approx V'_2$. If V' can be written in this way we say that it is *decomposable.*

Definition
Let $n \in \mathbb{N}$. An element $\{V_1, \dots, V_n\} \subset \mathscr{P}(V)$ is called a *commuting family of order* n in V if

1. Each V_i is non-empty and indecomposable, and
2. for $i, j \in \{1, \dots, n\}$ with $i \neq j$, $V_i \approx V_j$.

Note that a commuting family in V is not necessarily a partition of V, because its union can be strictly contained in V. However, for any two V_i and V_j, $i \neq j$, as above, we have $V_i \cap V_j = \emptyset$.

The next statement shows that there is a unique commuting family that is also a partition for V.

Lemma 4.2
Let $\Gamma = (V, E)$ *be a finite simple graph. Then there exists a unique number* n *and non-empty indecomposable subsets* $V_1, \dots, V_n \subset V$ *such that*

$$V = \bigsqcup_{i=1}^{n} V_i$$

and for all $i, j \in \{1, \dots, n\}$ *with* $i \neq j$, $V_i \approx V_j$.

Proof. We prove this by induction on the number of vertices, where the base case with V empty or a singleton is immediate. So let us assume that the result holds for every simple graph with $< N$ vertices and let us show that it also holds for a graph with N vertices. Let v be a vertex in V. By the induction hypothesis we can write $V \setminus \{v\}$ uniquely as

$$V \setminus \{v\} = \bigsqcup_{j=1}^{m} V'_j,$$

with the V'_j's non-empty indecomposable and connected to each other. Partition now this decomposition into

$$\{A_1, \dots, A_{m_1}\} := \{V'_1, \dots, V'_m\} \cap \mathscr{P}(\mathsf{Ct}(v))$$

and

$$\{B_1, \dots, B_{m_2}\} := \{V'_1, \dots, V'_m\} \setminus \{A_1, \dots, A_{m_1}\}.$$

The set

$$W := \{v\} \cup \bigcup_{j=1}^{m_2} B_j$$

is indecomposable because v does not commute with the B_j's. The family

$$A_1, \dots, A_{m_1}, W$$

is a family of non-empty indecomposable subset of V which are connected to each other; if there were another decomposition of V, that would contradict the fact that $\{v\} \cup \bigcup_{j=1}^{m_2} B_j$ is indecomposable., or would contradict the induction hypothesis. $\square$

Now let $\mathsf{ComFam}_n(V) \subset \mathscr{P}(\mathscr{P}(V))$ be the set of all commuting families of order n in V, and let

$$\mathsf{ComFam}(V) := \bigcup_{n \geq 0} \mathsf{ComFam}_n(V).$$

Note that $\mathsf{ComFam}_0(V) = \{\emptyset\}$.

Corollary 4.3
For every $V' \subset V$ there exists a unique $\tilde{D} \in \mathsf{ComFam}(V)$ such that

$$V' = \bigsqcup_{D \in \tilde{D}} D.$$

Proof. This follows by Lemma 4.2 by considering the unique commuting family that is also a partition of V', and contains no elements of $V \setminus V'$. ☐

Now we recall the Möbius inversion principle in a form that will be convenient to us.

Theorem 4.4 (Section 2.1 of [Sta86])
Let $\mathcal{R}$ be an infinite commutative ring with unit and let $f, g : \mathscr{P}(V) \to \mathcal{R}$ be two functions. Then

$$f(W) := \sum_{W' \subseteq W} (-1)^{|W|-|W'|} g(W').$$

if and only if

$$g(W) = \sum_{W' \subseteq W} f(W').$$

We write

$$g^{\mathcal{M}} = \sum_{W' \subseteq W} (-1)^{|W|-|W'|} g(W'),$$

so $f = g^{\mathcal{M}}$, and say that f and g are *Möbius inverses* of each other.

Let $\Phi : \mathscr{P}(V) \to \mathcal{R}$ is a function that satisfies the properties

a) $\Phi(V') = 0$ if $V' \in \mathsf{Clq}(V)$, and

b) $\Phi(V_1 \cup V_2) = \Phi(V_1) + \Phi(V_2)$ for V_1, V_2 such that $V_1 \approx V_2$.

Lemma 4.5
If $W \subseteq V$ is decomposable then $\Phi^{\mathcal{M}}(W) = 0$.

Proof. If $W = V_1 \sqcup V_2$ with $V_1, V_2 \neq \emptyset$ and $V_1 \approx V_2$, then $W' \subseteq W$ can be decomposed as $W' = V_1' \sqcup V_2'$, where $V_i' = W' \cap V_i$; so $V_1' \approx V_2'$ and V_i' possibly empty. Based on this observation we note that the coefficient in $\Phi^{\mathcal{M}}(W)$ of some $V_1' \subseteq V_1$ is the binomial expansion of $(1-1)^{|V_2|} = 0$, and similarly for $V_2' \subseteq V_2$ the coefficient is $(1-1)^{|V_1|} = 0$.

☐

We now apply the inversion principle to a function $\Psi : \mathscr{P}(V) \to \mathcal{R}$ defined recursively by $\Psi(\emptyset) = 1$, and for $v \in V$,

$$\Psi(V) = \Psi(V \setminus \{v\}) + \sum_{S \subset \mathsf{Ct}(v)} \Phi^{\mathcal{M}}(S)\big(\Phi(\mathsf{Ct}(S)) - \Phi(\mathsf{Ct}(S) \setminus \{v\})\big).$$

Proposition 4.6
The formula for $\Psi(V)$ *is given by*

$$\Psi(V) = \sum_{n=0}^{\infty} \sum_{\{A_1,\ldots,A_n\}\in \mathsf{ComFam}_n(V)} \prod_{j=1}^{n} \Phi^{\mathcal{M}}(A_j).$$

Proof. We prove this by induction on the number of vertices in V.

If $V=\emptyset$, then the formula holds because $\mathsf{ComFam}_0(\emptyset)=\{\emptyset\}$ and $\mathsf{ComFam}_n(\emptyset)=\emptyset$ for $n>0$, so this is a product over an empty family, and is equal to 1.

Let us assume that the result holds for any graph with $< N$ vertices, and show that it also holds for a graph with N vertices. Pick $v \in V$. Then for every $n \in \mathbb{N}$, $\mathsf{ComFam}_n(V)$ is the disjoint union of the two following sets:

$$\mathsf{ComFam}_n(V \setminus \{v\}) \quad \text{and}$$

$$\underbrace{\left\{ \{A_1,\ldots,A_{n-1},B\} \; : \; \begin{array}{r} \{A_1,\ldots,A_{n-1}\} \in \mathsf{ComFam}_{n-1}(\mathsf{Ct}(\{v\})) \\ \{B\} \in \mathsf{ComFam}_1(\mathsf{Ct}(A_1 \cup \ldots \cup A_{n-1})) \\ v \in B \end{array} \right\}}_{=:\tau_n}.$$

For $W \subset V$, let

$$P_n(W) := \sum_{\{A_1,\ldots,A_n\}\in \mathsf{ComFam}_n(W)} \prod_{j=1}^{n} \Phi^{\mathcal{M}}(A_j).$$

By induction $\Psi(V\setminus\{v\}) = \sum_{n\geq 0} P_n(V\setminus\{v\})$, hence it suffices to prove that for every $n>0$,

$$\begin{aligned} &\sum_{\{A_1,\ldots,A_{n-1},B\}\in\tau_n} \Phi^{\mathcal{M}}(A_1)\cdots\Phi^{\mathcal{M}}(A_{n-1})\Phi^{\mathcal{M}}(B) \\ &= \sum_{S\subset \mathsf{Ct}(v)} \underbrace{\left(P_{n-1}(S) + \sum_{S' \subsetneq S} (-1)^{|S|-|S'|} P_{n-1}(S') \right)}_{P^{\mathcal{M}}_{n-1}} \big(\Phi(\mathsf{Ct}(S)) - \Phi(\mathsf{Ct}(S)\setminus\{v\})\big). \end{aligned}$$

For $S \subset \mathsf{Ct}(V)$, Theorem 4.4 and Remark 4.5 imply that

$$\Phi(\mathsf{Ct}(S)) - \Phi(\mathsf{Ct}(S)\setminus\{v\}) = \sum_{\substack{B\subset \mathsf{Ct}(S) \\ v\in B}} \Phi^{\mathcal{M}}(B) = \sum_{\substack{\{B\}\in \mathsf{ComFam}_1(\mathsf{Ct}(S)) \\ v\in B}} \Phi^{\mathcal{M}}(B). \tag{4.1}$$

Claim: Let $W \subseteq V$. Then

$$\begin{aligned} &P_{n-1}(W) + \sum_{W' \subsetneq W} (-1)^{|W|-|W'|} P_{n-1}(W') \\ &= \begin{cases} \prod_{j=1}^{n-1} \Phi^{\mathcal{M}}(A_j) & \text{if } W = \bigcup_{j=1}^{n-1} A_j \text{ for } \{A_1,\ldots,A_{n-1}\} \in \mathsf{ComFam}_{n-1}(W) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Proof of the claim: Let $r < |W|$ and let $\{A_1,\ldots,A_{n-1}\} \in \mathsf{ComFam}_{n-1}(W)$ be such that $|\bigcup_{j=1}^{n-1} A_j| = r$. In the sum

$$P_{n-1}(W) + (-1)^{|W|} \sum_{k=0}^{|W|-1} (-1)^k \sum_{W'\subset W \,:\, |W'|=k} P_{n-1}(W') = P^{\mathcal{M}}_{n-1}(W)$$

the term $\prod_{j=1}^{n-1} \Phi^{\mathcal{M}}(A_j)$ will appear for every subset of W that contains $A_1 \cup \ldots \cup A_{n-1}$, and thus will have the coefficient equal to

$$\begin{aligned} &\sum_{k=r}^{|W|} (-1)^{|W|-k} \binom{|W|-r}{k-r} \\ &= (-1)^{|W|+r} \sum_{\ell=0}^{|W|-r} (-1)^{\ell} \binom{|W|-r}{\ell} \\ &= (-1)^{|W|+r} (1+(-1))^{|W|-r} = 0. \end{aligned}$$

On the other hand, if $W = \bigcup_{j=1}^{n-1} A_j$ for $\{A_1, \ldots, A_{n-1}\} \in \mathsf{ComFam}_{n-1}(W)$, then the term $\prod_{j=1}^{n-1} \Phi^{\mathcal{M}}(A_j)$ appears once. This proves the claim.

Now we have that

$$\begin{aligned} &\sum_{S \subset \mathsf{Ct}(v)} \left(P_{n-1}(S) + \sum_{S' \subsetneq S} (-1)^{|S|-|S'|} P_{n-1}(S') \right) \left(\Phi(\mathsf{Ct}(S)) - \Phi(\mathsf{Ct}(S) \setminus \{v\}) \right) \\ &\overset{(4.1)}{=} \sum_{S \subset \mathsf{Ct}(v)} \left(P_{n-1}(S) + \sum_{S' \subsetneq S} (-1)^{|S|-|S'|} P_{n-1}(S') \right) \left(\sum_{\substack{\{B\} \in \mathsf{ComFam}_1(\mathsf{Ct}(S)) \\ v \in B}} \Phi^{\mathcal{M}}(B) \right) \\ &\overset{\text{Claim}}{=} \sum_{\{A_1, \ldots, A_{n-1}\} \in \mathsf{ComFam}_{n-1}(\mathsf{Ct}(\{v\}))} \prod_{j=1}^{n-1} \Phi^{\mathcal{M}}(A_j) \left(\sum_{\substack{\{B\} \in \mathsf{ComFam}_1(\mathsf{Ct}(\bigcup_{j=1}^{n-1} A_j)) \\ v \in B}} \Phi^{\mathcal{M}}(B) \right) \\ &= \sum_{\{A_1, \ldots, A_{n-1}, B\} \in \tau_n} \Phi^{\mathcal{M}}(A_1) \cdots \Phi^{\mathcal{M}}(A_{n-1}) \Phi^{\mathcal{M}}(B). \end{aligned}$$

□

4.2 The conjugacy growth of a graph product

In this section we will compute the conjugacy growth series of a graph product based on Chiswell's computation of the standard growth series of a graph product in [Chi94a].

We begin with a discussion of the admissibility of the subgroup $G_{\mathsf{Ct}(v)}$ in the two factor groups in Lemma 4.1. Given groups $G_i = \langle X_i \rangle$ for $i = 1, 2$, it follows directly from Definition 2.3 that each group G_i is admissible in the direct product group $G_1 \times G_2$ with respect to the pair of generating sets $(X_i, X_1 \cup X_2)$. Any subgraph product is also an admissible subgroup in a graph product (see [Chi94a], [Man12, Proposition 14.4]); we consider the admissible transversal which we use later in this section. Combining these results shows that the amalgamating subgroup is admissible in both of the factor groups in Lemma 4.1.

Applying Proposition 2.8 to the amalgamated product in Lemma 4.1, using the admissibility of direct factors of a direct product yields Corollary 4.7; the formulas for the growth series in this Corollary were also obtained by Chiswell in [Chi94a, Proof of Proposition 1].

Notation
Let G_V be a graph product and assume that every vertex group G_v has a symmetric generating set X_v. For each $V' \subseteq V$, let $X_{V'} := \cup_{v \in V'} X_v$ and write

$$\sigma_{V'}(z) := \sigma_{(G_{V'}, X_{V'})}(z), \text{ and } \tilde{\sigma}_{V'}(z) := \tilde{\sigma}_{(G_{V'}, X_{V'})}(z). \tag{4.2}$$

Corollary 4.7
Let G_V be a graph product group over a graph with vertex set V, let $v \in V$ be a vertex, and for each $v' \in V$ let $X_{v'}$ be a symmetric generating set for the vertex group $G_{v'}$. Let $U = U_{G_{\mathsf{Ct}(v)} \backslash G_{V \backslash \{v\}}}$ be the admissible right transversal for $G_{\mathsf{Ct}(v)}$ in $G_{V \backslash \{v\}}$ and let $\sigma_U(z)$ be the growth series of the elements of U with respect to X_V. Using Notation (4.2), then

$$\begin{aligned}\sigma_V(z) &= \frac{\sigma_{\mathsf{Ct}(v)}(z)\sigma_{V\backslash\{v\}}(z)\sigma_{\{v\}}(z)}{\sigma_{\mathsf{Ct}(v)}(z)\sigma_{\{v\}}(z) + \sigma_{V\backslash\{v\}}(z) - \sigma_{V\backslash\{v\}}(z)\sigma_{\{v\}}(z)} \\ &= \sigma_{\mathsf{Ct}(v)}(z)\frac{\sigma_U(z)\sigma_{\{v\}}(z)}{\sigma_{\{v\}}(z) + \sigma_U(z) - \sigma_U(z)\sigma_{\{v\}}(z)}.\end{aligned}$$

Moreover, the radius of convergence of the spherical growth series for G_V satisfies

$$\mathcal{RC}(\sigma_V(z)) = \min\{\mathcal{RC}(\sigma_{\mathsf{Ct}(v)}(z)), \inf\{|z| \,:\, \sigma_{\{v\}}(z) + \sigma_U(z) - \sigma_U(z)\sigma_{\{v\}}(z) = 0\}\}.$$

Proof. If $V = \mathsf{Ct}(v) \cup \{v\}$ then $G_V = G_v \times G_{\mathsf{Ct}(v)}$ and $U = \{1\}$. From Remark 2.4, the spherical growth series of a direct product of groups is the product of the spherical growth series of the factors, and so in this case we have $\sigma_U(z) = 1$ and $\sigma_V(z) = \sigma_{\{v\}}(z)\sigma_{\mathsf{Ct}(v)}(z)$, as required.

Next assume that $V \neq \mathsf{Ct}(v) \cup \{v\}$ and so $U_{\mathsf{Ct}(v) \backslash (V \backslash \{v\})} \neq \{1\}$. Note from Remark 2.4 that $\sigma_{V\backslash\{v\}}(z) = \sigma_{\mathsf{Ct}(v)}(z)\sigma_U(z)$. We take the admissible transversal for $G_{\mathsf{Ct}(v)}$ in the direct product $G_{\mathsf{Ct}(v)} \times G_v$ to be the set $\{1\} \times G_v$. Lemmas 4.1 and Proposition 2.8 give the required equalities between the spherical growth series. Since the radius of convergence of the product of two functions is the minimum of the radius of convergence of the two functions, we obtain the claim about $\mathcal{RC}(\sigma_V(z))$ as well. ☐

Note that the amalgamated product decomposition in Lemma 4.1 involves a factor group $G_{V\backslash\{v\}}$ and amalgamating group $G_{\mathsf{Ct}(v)}$ that are graph products over graphs whose vertex sets are proper subsets of V, along with a direct product of two graph products on proper vertex subsets. Thus this decomposition process and Corollary 4.7 can be iterated to obtain Chiswell's formula [Chi94a, Proposition 1](Proposition 4.8) for the growth series $\sigma_V(z)$ of the graph product in terms of the growth series $\sigma_{\{v\}}(z)$ of each of the vertex groups.

Proposition 4.8 ([Chi94a] Proposition 1)
The growth series of the graph product $\Gamma(V, E)$ is given by

$$\sigma_V(z) = \frac{1}{\sum_{\Delta \in \mathsf{Clq}(V)} \prod_{i \in \Delta} \left(\frac{1}{\sigma_{\{i\}}(z)} - 1\right)}.$$

In particular

$$\mathcal{RC}(\sigma_V(z)) = \inf\left\{|r| : \sum_{\Delta \in \mathsf{Clq}(V)} \prod_{i \in \Delta} \left(\frac{1}{\sigma_{\{i\}}(r)} - 1\right) = 0\right\}.$$

Suppose, as in the above proof, that the underlying graph is not complete and there is $v \in V$ that is not connected to all the other vertices. In order to be able to compute the conjugacy growth series of G_V we need to understand the conjugacy classes of the elements

$$bu_1c_1 \cdots u_nc_n, \quad (b \in G_{\mathsf{Ct}(v)} \setminus \{1\},\, c_i \in G_{\{v\}} \setminus \{1\},\, u_i \in U \setminus \{1\},\, n > 0),$$

where U is the right admissible transversal of $G_{\mathsf{Ct}(v)}$ in $G_{V\setminus\{v\}}$. The transversal U can be seen as the set of words of elements in the G_w's, with $w \in V \setminus \{v\}$, that cannot be written with a non trivial prefix in $G_{\mathsf{Ct}(v)}$.

The next proposition determines when two such elements are conjugate.

Proposition 4.9
Let $v \in V$ be a vertex for which $\{v\} \cup \mathsf{Ct}(v) \subsetneq V$ and $g \in G_V$ be such that $g \notin G_{V\setminus\{v\}} \cup (G_{\mathsf{Ct}(v)} \times G_v)$. Let $U = \widehat{U}_{G_{\mathsf{Ct}(v)} \setminus G_{V\setminus\{v\}}}$ be the admissible transversal for $G_{\mathsf{Ct}(v)}$ in $G_{V\setminus\{v\}}$ with respect to $(X_{\mathsf{Ct}(v)}, X_{V\setminus\{v\}})$.

1. *There exists an element $\tilde{g}$ of minimal length in $[g]_{\sim,G_V}$ of the form*
$$\tilde{g} = \tilde{b}\tilde{u}_1\tilde{c}_1 \cdots \tilde{u}_n\tilde{c}_n, \tag{4.3}$$
where $n > 0$, $\tilde{b}$ is of minimal length in $[\tilde{b}]_{\sim,G_{\mathsf{Ct}(v)}}$, $\tilde{u}_i \in U \setminus \{1\}$, $\tilde{c}_i \in G_v \setminus \{1\}$, and $[\tilde{b}, \tilde{u}_i] = 1$ for all i.

2. *Two elements $\tilde{b}\tilde{u}_1\tilde{c}_1 \cdots \tilde{u}_n\tilde{c}_n$ and $\tilde{b}'\tilde{u}'_1\tilde{c}'_1 \cdots \tilde{u}'_{n'}\tilde{c}'_{n'}$ of the form (4.3) (minimal in their conjugacy class) are conjugate if and only if*
   - *(i) $\tilde{b}$ and $\tilde{b}'$ are conjugate in $G_{\mathsf{Ct}(v)}$,*
   - *(ii) $n = n'$,*
   - *(iii) $\tilde{u}_1\tilde{c}_1 \cdots \tilde{u}_n\tilde{c}_n$ and $\tilde{u}'_1\tilde{c}'_1 \cdots \tilde{u}'_{n'}\tilde{c}'_{n'}$ are cyclically conjugate.*

Proof. Let $g \in G_V$ be such that $g \notin G_{V\setminus\{v\}} \cup (G_{\mathsf{Ct}(v)} \times G_v)$. Then g has the form $g = bu_1c_1 \cdots u_nc_n$, where $b \in G_{\mathsf{Ct}(v)} \setminus \{1\}$, $c_i \in G_v \setminus \{1\}$, $u_i \in U \setminus \{1\}$, $n > 0$, by Lemma 2.6.

By ([LS77], Theorem 2.8) any conjugate of g in G_V can be obtained by cyclically permuting the syllables in g and then conjugating by an element of $G_{\mathsf{Ct}(v)}$. Note that for any $u \in U$ and $b \in G_{\mathsf{Ct}(v)}$ either $ub = bu$ or $ub \in U$. Suppose $[u, b] \neq 1$; then we can write b as $b = b'b''$, where $b', b'' \in G_{\mathsf{Ct}(v)}$, $[b', u] = 1$ and $[b'', u] \neq 1$. We then get

$$ub = b'ub'' = b'u',$$

where $u' \in U$. Together with the fact that b commutes with all the c_i's, we see that after a cyclic permutation and shuffles of b to the left we get

$$bu_1c_1 \cdots u_nc_n \sim u_1c_1 \cdots u_nc_nb = \beta u'_1c_1u'_2c_2 \cdots u'_nc_n,$$

where $\beta \in G_{\mathsf{Ct}(v)}$, $u'_i \in U \setminus \{1\}$ are the result of merging u_i with the subword of b which does not commute with u_i, and $[\beta, u_i] = 1$ for all $i \in \{1, \ldots, n\}$. Additionally, $|\beta u'_1c_1u'_2c_2 \cdots u'_nc_n| \leq |bu_1c_1 \cdots u_nc_n|$ and $|\beta| \leq |b|$ with $|\beta| = |b|$ if and only if $\beta = b$.

After further cyclic permutations and shuffles to the left of β we get a conjugate $\hat{g} = \hat{b}\tilde{u}_1c_1 \cdots \tilde{u}_nc_n$ of g of minimal length among the cyclic conjugates of g, where $\hat{b} \in G_{\mathsf{Ct}(v)}$, $\tilde{u}_i \in U$

and $[\hat{b}, \tilde{u}_i] = 1$ for all i. This last element is uniquely determined up to cyclic permutation of the syllables $\tilde{u}_i c_i$, i.e. the only other conjugates of this form (among the cyclic conjugates) are

$$\hat{b}\tilde{u}_2 c_2 \cdots \tilde{u}_n c_n \tilde{u}_1 c_1,\ \hat{b}\tilde{u}_3 c_3 \cdots \tilde{u}_n c_n \tilde{u}_1 c_1 \tilde{u}_2 c_2,\ \ldots,\ \hat{b}\tilde{u}_n c_n \tilde{u}_1 c_1 \cdots \tilde{u}_{n-1} c_{n-1}.$$

Since $[\hat{b}, \tilde{u}_i] = 1$ for all $i \in \{1, \ldots, n\}$, we get $\mathrm{Supp}(\hat{b}) \subset \mathsf{Ct}(\mathrm{Supp}(\tilde{u}_1 \tilde{u}_2 \cdots \tilde{u}_n))$.

Finally, to obtain the minimal conjugacy representative, we conjugate $\hat{g} := \hat{b}\tilde{u}_1 c_1 \cdots \tilde{u}_n c_n$ by an element of $G_{\mathsf{Ct}(v)}$; we can not shorten $\hat{g}$ if we conjugate by an element w with $w \notin \mathrm{Supp}(\hat{b})$. So pick $d \in G_{\mathsf{Ct}(v)}$ with $d \in \mathrm{Supp}(\tilde{b})$. In this case d commutes with all the $\tilde{u}_i$ (and by definition all the c_i), so

$$d\hat{g}d^{-1} = d\hat{b}d^{-1}\tilde{u}_1 c_1 \cdots \tilde{u}_n c_n,$$

which is the same as conjugating $\hat{b}$ by d. Thus $\hat{b}\tilde{u}_1 c_1 \cdots \tilde{u}_n c_n$ is conjugate to an element $\tilde{b}\tilde{u}_1 c_1 \cdots \tilde{u}_n c_n \in \min([g]_{\sim, G_V})$, where $\tilde{b} \in \min([\hat{b}]_{\sim, G_{\mathsf{Ct}(v)}})$ and commutes with all the $\tilde{u}_i$.

This completes the proof.

□

We are now ready to give an explicit formula for $\tilde{\sigma}_V$ based on the formulas for $\tilde{\sigma}_{V'}$ and $\sigma_{V'}$, where $V' \subsetneq V$. By Proposition 4.9, the contribution of the elements $g \in G_V \setminus (G_{V\setminus\{v\}} \cup (G_{\mathsf{Ct}(v)} \times G_v))$ to the conjugacy growth series of G_V will be the sum over every subset $V' \subset \mathsf{Ct}(v)$ of the conjugacy growth series of elements of $G_{\mathsf{Ct}(v)}$ having V' as support, times the Necklaces series of the growth series counting products of non-trivial elements of G_v and U having support included in $\mathsf{Ct}(V')$.

Lemma 4.10 is an immediate application of the inclusion-exclusion principle.

Lemma 4.10
Let G_S be a graph product ($S \subset V$). The contribution to $\tilde{\sigma}_V(z)$ of the conjugacy classes having a minimal representative with support exactly S is given by

$$\tilde{\sigma}_S(z) + (-1)^{|S|} \sum_{k=0}^{|S|-1} (-1)^k \sum_{S' \subset S \,:\, |S'| = k} \tilde{\sigma}_{S'}(z) = \tilde{\sigma}_S^{\mathcal{M}}(z).$$

Lemma 4.11 follows the same counting argument as in Remark 2.4.

Lemma 4.11
Let $S \subset \mathsf{Ct}(v)$. The growth series of elements u of U with $\mathrm{Supp}(u) \subseteq \mathsf{Ct}(S)$ *is given by*

$$\frac{\sigma_{\mathsf{Ct}(S)\setminus\{v\}}(z)}{\sigma_{\mathsf{Ct}(v)\cap\mathsf{Ct}(S)}(z)}.$$

The following provides the formula for the conjugacy growth series of a graph product, taking some vertex v as reference.

Proposition 4.12
Let G_V be a graph product and $v \in V$. The conjugacy growth series of G_V is given by

$$\begin{aligned}\tilde{\sigma}_V(z) = {} & \tilde{\sigma}_{V\setminus\{v\}}(z) + (\tilde{\sigma}_{\{v\}}(z) - 1)\tilde{\sigma}_{\mathsf{Ct}(v)}(z) \\ & + \sum_{S \subset \mathsf{Ct}(v)} \tilde{\sigma}_S^{\mathcal{M}}(z) \mathsf{NLS}\left((\sigma_{\{v\}}(z) - 1) \left(\frac{\sigma_{\mathsf{Ct}(S)\setminus\{v\}}(z)}{\sigma_{\mathsf{Ct}(v)\cap\mathsf{Ct}(S)}(z)} - 1 \right) \right).\end{aligned}$$

Proof. By Lemma 4.1 we can decompose the group $G = G_V$ as the disjoint union

$$G = G_{V\setminus\{v\}} \bigsqcup (G_{\mathsf{Ct}(v)} \times (G_v \setminus \{1\})) \bigsqcup (G \setminus (G_{V\setminus\{v\}} \cup (G_{\mathsf{Ct}(v)} \times G_v))).$$

Let $V' \subseteq V$. Notice that if $w \in \mathsf{ConjGeoNorm}(G_V, X_V) \cap X^*_{V'}$ then $w \in \mathsf{ConjGeoNorm}(G_{V'}, X_{V'})$. Hence $\tilde{\sigma}_G(z)$ is the sum of the contributions of the conjugacy classes in the three sets above, that is, $\tilde{\sigma}_G(z)$ is the sum of $\tilde{\sigma}_{V\setminus\{v\}}(z)$, $\tilde{\sigma}_{\mathsf{Ct}(v)}(z)(\tilde{\sigma}_{\{v\}}(z) - 1)$ and the series corresponding to the conjugacy representatives of elements in $G \setminus (G_{V\setminus\{v\}} \cup (G_{\mathsf{Ct}(v)} \times G_v))$.

Proposition 4.9 implies that a set of minimal conjugacy representatives of elements in $G \setminus (G_{V\setminus\{v\}} \cup (G_{\mathsf{Ct}(v)} \times G_v))$ can be written as the disjoint union

$$\bigsqcup_{S \subset \mathsf{Ct}(v)} \{b \in G_{\mathsf{Ct}(v)} \,:\, \mathrm{Supp}(b) = S\} \times \mathsf{Necklaces}(U'C'),$$

where $U' = \{u \in U \setminus \{1\} \,:\, \mathrm{Supp}(u) \subset \mathsf{Ct}(S)\}$ and $C' = \{c \in G_v \setminus \{1\}\}$.

The growth series of $\{b \in G_{\mathsf{Ct}(v)} \,:\, \mathrm{Supp}(b) = S\}$ is given by Lemma 4.10. The growth of U' is given by Lemma 4.11. So applying the formula for necklaces in Proposition 2.13 we find that the contribution to $\tilde{\sigma}_G$ of the conjugacy representatives of elements in $G \setminus (G_{V\setminus\{v\}} \cup (G_{\mathsf{Ct}(v)} \times G_v))$ is

$$\sum_{S \subset \mathsf{Ct}(v)} \tilde{\sigma}^{\mathcal{M}}_S(z) \mathsf{NLS}\left((\sigma_{\{v\}}(z) - 1) \left(\frac{\sigma_{\mathsf{Ct}(S)\setminus\{v\}}(z)}{\sigma_{\mathsf{Ct}(v)\cap\mathsf{Ct}(S)}(z)} - 1 \right) \right).$$

□

Notation

Here we introduce a function which quantifies the contribution to the conjugacy growth series of elements whose syllables come from non-commuting vertices. Let $V' \subset V$. We define

$$\mathsf{NC}(V')(z) := \mathsf{NLS}\left(1 - \prod_{i \in V'} \sigma_{\{i\}}(z) \sum_{\Delta \in \mathsf{Clq}(V')} \prod_{j \in \Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \right).$$

Lemma 4.13

The following hold.

1. *If* $V' \in \mathsf{Clq}(V)$, *then* $\mathsf{NC}(V')(z) = 0$.
2. *If* $v \in V$ *and* $V' \subset \mathsf{Ct}(v)$, *then* $\mathsf{NC}(V' \cup \{v\})(z) = \mathsf{NC}(V')(z)$.
3. *If* $v \in V$ *and* $V' \subset V \setminus \{v\}$, *then*

$$\mathsf{NC}(V')(z) + \mathsf{NLS}\left((\sigma_{\{v\}}(z) - 1) \left(\frac{\sigma_{V'}(z)}{\sigma_{\mathsf{Ct}(v)\cap V'}(z)} - 1 \right) \right) = \mathsf{NC}(V' \cup \{v'\})(z). \tag{4.4}$$

4. *If* $V_1, V_2 \subset V$ *satisfy* $V_1 \approx V_2$, *then*

$$\mathsf{NC}(V_1)(z) + \mathsf{NC}(V_2)(z) = \mathsf{NC}(V_1 \cup V_2)(z).$$

Proof. 1. If $\Delta \in \mathsf{Clq}(V)$ then

$$\sum_{\Delta' \in \mathsf{Clq}(\Delta)} \prod_{i \in \Delta'} \left(\frac{1}{\sigma_{\{i\}}(z)} - 1 \right) = \frac{1}{\prod_{i \in \Delta} \sigma_{\{i\}}(z)}. \tag{4.5}$$

We prove (4.5) by induction on $|\Delta|$. If $\Delta = \emptyset$, then the result is clear, so let us assume that the result holds for every strict subset of Δ and show that it also holds for Δ. Let $v \in \Delta$. We have

$$\begin{aligned}
&\sum_{\Delta' \in \mathsf{Clq}(\Delta)} \prod_{i \in \Delta'} \left(\frac{1}{\sigma_{\{i\}}(z)} - 1 \right) \\
&= \sum_{\Delta' \in \mathsf{Clq}(\Delta)\,:\, v \in \Delta'} \prod_{i \in \Delta'} \left(\frac{1}{\sigma_{\{i\}}(z)} - 1 \right) + \sum_{\Delta' \in \mathsf{Clq}(\Delta \setminus \{v\})} \prod_{i \in \Delta'} \left(\frac{1}{\sigma_{\{i\}}(z)} - 1 \right) \\
&\left(\frac{1}{\sigma_{\{v\}}(z)} - 1 \right) \sum_{\Delta' \in \mathsf{Clq}(\Delta \setminus \{v\})} \prod_{i \in \Delta'} \left(\frac{1}{\sigma_{\{i\}}(z)} - 1 \right) + \sum_{\Delta' \in \mathsf{Clq}(\Delta \setminus \{v\})} \prod_{i \in \Delta'} \left(\frac{1}{\sigma_{\{i\}}(z)} - 1 \right) \\
&= \frac{1}{\sigma_{\{v\}}(z)} \sum_{\Delta' \in \mathsf{Clq}(\Delta \setminus \{v\})} \prod_{i \in \Delta'} \left(\frac{1}{\sigma_{\{i\}}(z)} - 1 \right) \\
&\overset{\text{hyp.}}{=} \frac{1}{\sigma_{\{v\}}(z)} \frac{1}{\prod_{i \in \Delta \setminus \{v\}} \sigma_{\{i\}}(z)} = \frac{1}{\prod_{i \in \Delta} \sigma_{\{i\}}(z)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathsf{NC}(\Delta)(z) =& \mathsf{NLS}\left(1 - \prod_{i \in \Delta} \sigma_{\{i\}}(z) \sum_{\Delta' \in \mathsf{Clq}(\Delta)} \prod_{j \in \Delta'} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \right) \\
&= \mathsf{NLS}\left(1 - \prod_{i \in \Delta} \sigma_{\{i\}}(z) \frac{1}{\prod_{j \in \Delta} \sigma_{\{j\}}(z)} \right) = \mathsf{NLS}(0) = 0.
\end{aligned}$$

2. Note that if $V' \subset \mathsf{Ct}(v)$, then $v \notin V'$. We have

$$\begin{aligned}
&\prod_{i \in V' \cup \{v\}} \sigma_{\{i\}}(z) \sum_{\Delta \in \mathsf{Clq}(V' \cup \{v\})} \prod_{j \in \Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \\
&= \prod_{i \in V' \cup \{v\}} \sigma_{\{i\}}(z) \left(\sum_{\Delta \in \mathsf{Clq}(V' \cup \{v\})\,:\, v \in \Delta} \prod_{j \in \Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) + \sum_{\Delta \in \mathsf{Clq}(V')} \prod_{j \in \Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \right) \\
&= \prod_{i \in V' \cup \{v\}} \sigma_{\{i\}}(z) \left(\left(\frac{1}{\sigma_{\{v\}}(z)} - 1 \right) \sum_{\Delta \in \mathsf{Clq}(V')} \prod_{j \in \Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) + \sum_{\Delta \in \mathsf{Clq}(V')} \prod_{j \in \Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \right) \\
&= \prod_{i \in V' \cup \{v\}} \tilde{\sigma}_{\{i\}} \left(\frac{1}{\sigma_{\{v\}}(z)} \sum_{\Delta \in \mathsf{Clq}(V')} \prod_{j \in \Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \right) \\
&= \prod_{i \in V'} \tilde{\sigma}_{\{i\}} \sum_{\Delta \in \mathsf{Clq}(V')} \prod_{j \in \Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right).
\end{aligned}$$

Therefore

$$\mathsf{NC}(V' \cup \{v\})(z) = \mathsf{NC}(V')(z).$$

3. We use the formulas from Lemma 2.16 and Corollary 4.8:

$$\begin{aligned}
&\mathsf{NC}(V')(z) + \mathsf{NLS}\left((\sigma_{\{v\}}(z) - 1)(\frac{\sigma_{V'}(z)(z)}{\sigma_{\mathsf{Ct}(v)\cap V'}(z)} - 1) \right) \\
&= \mathsf{NLS}\left(1 - \prod_{i\in V'} \sigma_{\{i\}}(z) \sum_{\Delta\in\mathsf{Clq}(V')} \prod_{j\in\Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \right) + \mathsf{NLS}\left((\sigma_{\{v\}}(z) - 1)\left(\frac{\sigma_{V'}(z)(z)}{\sigma_{\mathsf{Ct}(v)\cap V'}(z)} - 1 \right) \right) \\
&= \mathsf{NLS}\left(1 - \prod_{i\in V'} \sigma_{\{i\}}(z) \sum_{\Delta\in\mathsf{Clq}(V')} \prod_{j\in\Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \left(\sigma_{\{v\}}(z) + \frac{\sigma_{V'}(z)(z)}{\sigma_{\mathsf{Ct}(v)\cap V'}(z)} - \sigma_{\{v\}}(z)\frac{\sigma_{V'}(z)(z)}{\sigma_{\mathsf{Ct}(v)\cap V'}(z)} \right) \right).
\end{aligned}$$

Now

$$\begin{aligned}
&\left(\sigma_{\{v\}}(z) + \frac{\sigma_{V'}(z)}{\sigma_{\mathsf{Ct}(v)\cap V'}(z)} - \sigma_{\{v\}}(z)\frac{\sigma_{V'}(z)}{\sigma_{\mathsf{Ct}(v)\cap V'}(z)} \right) \\
&= \sigma_{\{v\}}(z)\left(1 + \left(\frac{1}{\sigma_{\{v\}}(z)} - 1 \right) \frac{\sigma_{V'}(z)}{\sigma_{\mathsf{Ct}(v)\cap V'}(z)} \right) \\
&\overset{\text{Proposition 4.8}}{=} \sigma_{\{v\}}(z)\left(1 + \left(\frac{1}{\sigma_{\{v\}}(z)} - 1 \right) \frac{\sum_{\Delta\in\mathsf{Clq}(V'\cap\mathsf{Ct}(v))} \prod_{j\in\Delta}\left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right)}{\sum_{\Delta\in\mathsf{Clq}(V')} \prod_{j\in\Delta}\left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right)} \right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\prod_{i\in V'} \sigma_{\{i\}}(z) \sum_{\Delta\in\mathsf{Clq}(V')} \prod_{j\in\Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \left(\sigma_{\{v\}}(z) + \frac{\sigma_{V'}(z)}{\sigma_{\mathsf{Ct}(v)\cap V'}(z)} - \sigma_{\{v\}}(z)\frac{\sigma_{V'}(z)}{\sigma_{\mathsf{Ct}(v)\cap V'}(z)} \right) \\
&= \prod_{i\in V'\cup\{v\}} \sigma_{\{i\}}(z) \left(\sum_{\Delta\in\mathsf{Clq}(V')} \prod_{j\in\Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) + \left(\frac{1}{\sigma_{\{v\}}(z)} - 1 \right) \sum_{\Delta\in\mathsf{Clq}(V'\cap\mathsf{Ct}(v))} \prod_{j\in\Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \right) \\
&= \prod_{i\in V'\cup\{v\}} \sigma_{\{i\}}(z) \left(\sum_{\Delta\in\mathsf{Clq}(V')} \prod_{j\in\Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) + \sum_{\Delta\in\mathsf{Clq}(V'\cup\{v\})\,:\,v\in\Delta} \prod_{j\in\Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \right) \\
&= \prod_{i\in V'\cup\{v\}} \sigma_{\{i\}}(z) \left(\sum_{\Delta\in\mathsf{Clq}(V'\cup\{v\})} \prod_{j\in\Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathsf{NC}(V')(z) + \mathsf{NLS}\left((\sigma_{\{v\}}(z) - 1)(\frac{\sigma_{V'}(z)}{\sigma_{\mathsf{Ct}(v)\cap V'}(z)} - 1) \right) \\
&= \mathsf{NLS}\left(1 - \prod_{i\in V'\cup\{v\}} \sigma_{\{i\}}(z) \left(\sum_{\Delta\in\mathsf{Clq}(V'\cup\{v\})} \prod_{j\in\Delta} \left(\frac{1}{\sigma_{\{j\}}(z)} - 1 \right) \right) \right) \\
&= \mathsf{NC}(V'\cup\{v\})(z).
\end{aligned}$$

4. If $V_1 \approx V_2$ then

$$\mathsf{Clq}(V_1 \cup V_2) = \{\Delta_1 \cup \Delta_2 \,:\, \Delta_1 \in \mathsf{Clq}(V_1),\, \Delta_2 \in \mathsf{Clq}(V_2)\}.$$

Therefore

$$\begin{aligned}
&\mathsf{NC}(V_1 \cup V_2)(z)\\
&= \mathsf{NLS}\left(1 - \prod_{i\in V_1\cup V_2} \sigma_{\{i\}}(z) \sum_{\Delta\in \mathsf{Clq}(V_1\cup V_2)} \prod_{j\in\Delta}\left(\frac{1}{\sigma_{\{j\}}(z)} - 1\right)\right)\\
&= \mathsf{NLS}\left(1 - \prod_{i\in V_1} \sigma_{\{i\}}(z) \left(\sum_{\Delta\in \mathsf{Clq}(V_1)} \prod_{j\in\Delta}\left(\frac{1}{\sigma_{\{j\}}(z)} - 1\right)\right) \prod_{i\in V_2} \sigma_{\{i\}}(z) \left(\sum_{\Delta\in \mathsf{Clq}(V_2)} \prod_{j\in\Delta}\left(\frac{1}{\sigma_{\{j\}}(z)} - 1\right)\right)\right)\\
&\overset{\text{Lemma 2.16}}{=} \mathsf{NLS}\left(1 - \prod_{i\in V_1} \sigma_{\{i\}}(z) \sum_{\Delta\in \mathsf{Clq}(V_1)} \prod_{j\in\Delta}\left(\frac{1}{\sigma_{\{j\}}(z)} - 1\right)\right)\\
&\qquad + \mathsf{NLS}\left(1 - \prod_{i\in V_2} \sigma_{\{i\}}(z) \sum_{\Delta\in \mathsf{Clq}(V_2)} \prod_{j\in\Delta}\left(\frac{1}{\sigma_{\{j\}}(z)} - 1\right)\right)\\
&= \mathsf{NC}(V_1)(z) + \mathsf{NC}(V_2)(z).
\end{aligned}$$

□

We introduce some more notation, according to Section 4.1.1.

Notation
Let $W \subset V$. We define

$$\alpha(W)(z) := \mathsf{NC}^{\mathcal{M}}(W)(z) = \sum_{W'\subseteq W} (-1)^{|W|-|W'|}\mathsf{NC}(W')(z)$$

and

$$\Psi(W)(z) := \sum_{n=0}^{\infty} \sum_{\{A_1,\ldots,A_n\}\in \mathsf{ComFam}_n(W)} \prod_{j=1}^{n} \alpha(A_j)(z).$$

In particular if $\Delta \in \mathsf{Clq}(V)$ then $\Psi(\Delta)(z) = 1$.

Theorem 4.14
The formula for the conjugacy growth series of the graph product G_V is given by

$$\tilde{\sigma}_V(z) = \sum_{\Delta\in\mathsf{Clq}(V)} \prod_{v\in\Delta} (\tilde{\sigma}_v(z) - 1)\Psi(\mathsf{Ct}(\Delta))(z).$$

Proof. First we notice that $\tilde{\sigma}_V(z)$ is a polynomial on the variables

$$\prod_{v\in\Delta} (\tilde{\sigma}_v(z) - 1) \text{ for } \Delta \in \mathsf{Clq}(V), \text{ and } \mathsf{NC}(W)(z) \text{ for } W \subseteq V.$$

This follows by induction on the number of vertices in V. If $V = \emptyset$ then $\tilde{\sigma}_V(z) = 1$. Assume it holds for every subgraph of V; then this is also true for $\tilde{\sigma}_V(z)$, since by Proposition 4.12

$$\begin{aligned}
\tilde{\sigma}_V(z) =& \tilde{\sigma}_{V\setminus\{v\}}(z) + (\tilde{\sigma}_{\{v\}}(z) - 1)\tilde{\sigma}_{\mathsf{Ct}(v)}(z)\\
&+ \sum_{S\subset \mathsf{Ct}(v)} \tilde{\sigma}_S^{\mathcal{M}}(z)\mathsf{NLS}\left((\sigma_{\{v\}}(z) - 1)\left(\frac{\sigma_{\mathsf{Ct}(S)\setminus\{v\}}(z)}{\sigma_{\mathsf{Ct}(v)\cap\mathsf{Ct}(S)}(z)} - 1\right)\right).
\end{aligned}$$

By (4.4), when $S \subset \mathsf{Ct}(v)$

$$\mathsf{NLS}(\sigma_{\{v\}}(z) - 1)\left(\frac{\sigma_{\mathsf{Ct}(S)\setminus\{v\}}(z)}{\sigma_{\mathsf{Ct}(v)\cap\mathsf{Ct}(S)}(z)} - 1\right) = \mathsf{NC}(\mathsf{Ct}(S)) - \mathsf{NC}(\mathsf{Ct}(S) \setminus \{v\}).$$

Now we show that, for each $\Delta \in \mathsf{Clq}(V)$, the coefficient of variable $\prod_{v\in\Delta}(\tilde{\sigma}_v(z) - 1)$ in the polynomial $\tilde{\sigma}_V(z)$ is indeed $\Psi(\mathsf{Ct}(\Delta))(z)$.

Let $\tilde{\Psi}_\Delta(V)(z)$ be the coefficient of $\prod_{v\in\Delta}(\tilde{\sigma}_v(z)-1)$ in $\tilde{\sigma}_V(z)$. We first show that $\tilde{\Psi}_\Delta(V)(z) = \tilde{\Psi}_\emptyset(\mathsf{Ct}(\Delta))(z)$ by induction on the number of vertices in Δ.
If $\Delta = \emptyset$ then this is true by definition. So let us assume that $\tilde{\Psi}_{\Delta'}(V)(z) = \tilde{\Psi}_\emptyset(\mathsf{Ct}(\Delta'))(z)$ for any $\Delta' \subset \Delta$ and any graph V, and let us show that this holds for Δ. Pick $w \in \Delta$. If we apply Proposition 4.12 with respect to w, then the term $\prod_{v\in\Delta}(\tilde{\sigma}_v(z) - 1)\tilde{\Psi}_\Delta(V)(z)$ appears only in $(\tilde{\sigma}_{\{w\}}(z) - 1)\tilde{\sigma}_{\mathsf{Ct}(w)}(z)$ and hence

$$\tilde{\Psi}_\Delta(V)(z) = \tilde{\Psi}_{\Delta\setminus\{w\}}(V \setminus \{w\})(z) \overset{\text{hyp.}}{=} \tilde{\Psi}_\emptyset(\mathsf{Ct}(\Delta \setminus \{w\}) \setminus \{w\})(z) = \tilde{\Psi}_\emptyset(\mathsf{Ct}(\Delta))(z).$$

Therefore for any $\Delta \in \mathsf{Clq}(V)$, $\tilde{\Psi}_\Delta(V)(z)$ is the coefficient of 1 (corresponding to the empty clique) in the conjugacy growth series $\tilde{\sigma}_{\mathsf{Ct}(\Delta)}(z)$. So it suffices to prove that $\tilde{\Psi}_\emptyset(V)(z) = \Psi(V)(z)$. It is immediate that $\tilde{\Psi}_\emptyset(\emptyset)(z) = 1$. By Proposition 4.12 we have that $\tilde{\Psi}_\emptyset(V)(z)$ satisfies the relation because for $v \in V$,

$$\tilde{\Psi}_\emptyset(V)(z) = \tilde{\Psi}_\emptyset(V \setminus \{v\})(z) + \sum_{S\subset\mathsf{Ct}(v)} \tilde{\Psi}_\emptyset^{\mathcal{M}}(S)(z)\big(\mathsf{NC}(\mathsf{Ct}(S))(z) - \mathsf{NC}(\mathsf{Ct}(S) \setminus \{v\})(z)\big).$$

Therefore $\tilde{\Psi}_\emptyset(z)$ satisfies the hypothesis of Proposition 4.6 with $\Phi = \mathsf{NC}$, so this proves that $\tilde{\Psi}_\emptyset(V)(z) = \Psi(V)(z)$, and finally

$$\tilde{\Psi}_\Delta(V)(z) = \Psi(\mathsf{Ct}(\Delta))(z).$$

□

Example 4.14.1
If all the vertex group are $\mathbb{Z}\big/_{2\mathbb{Z}}$, in other words if it a right angled Coxeter group, then we find

$$\tilde{\sigma}_V(z) = \sum_{\Delta\in\mathsf{Clq}(V)} z^{\sharp\Delta} \sum_{n=0}^{\infty} \sum_{\{A_1,\dots,A_n\}\in\mathsf{ComFam}_n(\mathsf{Ct}(\Delta))} \prod_{j=1}^{n} \alpha(A_j)(z),$$

with

$$\alpha(A_j)(z) = \sum_{W'\subseteq A_j} (-1)^{|A_j|-|W'|}\mathsf{NLS}\left(1 - \sum_{\Delta'\in\mathsf{Clq}(W')} (-1)^{\sharp\Delta'} z^{\sharp\Delta'}(1+z)^{\sharp W'-\sharp\Delta'}\right).$$

Example 4.14.2
If all the vertex group are $\mathbb{Z}$, in other words if it a right angled Artin group, then we find

$$\tilde{\sigma}_V(z) = \sum_{\Delta\in\mathsf{Clq}(V)} \left(\frac{2z}{1-z}\right)^{\sharp\Delta} \sum_{n=0}^{\infty} \sum_{\{A_1,\dots,A_n\}\in\mathsf{ComFam}_n(\mathsf{Ct}(\Delta))} \prod_{j=1}^{n} \alpha(A_j)(z),$$

with

$$\alpha(A_j)(z) = \sum_{W'\subseteq A_j} (-1)^{|A_j|-|W'|}\mathsf{NLS}\left(1 - \frac{1}{(1-z)^{\sharp W'}} \sum_{\Delta'\in\mathsf{Clq}(W')} (-2z)^{\sharp\Delta'}(1+z)^{\sharp W'-\sharp\Delta'}\right).$$

Remark 4.15
The formula for $\Psi(V)(z)$ is:

$$\Psi(V)(z) = 1 + \mathsf{NC}(V)(z) + \sum_{n=2}^{\infty} \sum_{\{A_1,\ldots,A_n\}\in\mathsf{ComFam}_n(V)} \prod_{j=1}^{n} \alpha(A_j)(z).$$

The first term is explained by the fact that $\mathsf{ComFam}_0(V) = \{\emptyset\}$, and its coefficient is

$$\sum_{\{A_1,\ldots,A_n\}\in\mathsf{ComFam}_0(V)} \prod_{j=1}^{n} \alpha(A_j)(z) = 1;$$

the second term follows by Remark 4.5 and Remark 4.4, which give

$$\sum_{\{A\}\in\mathsf{ComFam}_1(V)} \alpha(A)(z) = \sum_{A\subset V} \alpha(A)(z) = \mathsf{NC}(V)(z).$$

Moreover, for $A \in \mathsf{ComFam}_1(V)$, if A is a singleton then $\alpha(A) = 0$. So in practice, when one enumerates $\mathsf{ComFam}_n(\mathsf{Ct}(\Delta))$ for $\Delta \in \mathsf{Clq}(V)$ and $n \geq 2$, one only needs to consider the elements $\{A_1, \ldots, A_n\}$ for which all the A_j's contain at least 2 elements.

Theorem 4.16
Let G_V be a graph product and assume that for each vertex $v \in V$ the standard growth series radius of convergence $\mathcal{RC}(\sigma_v)$ equals the conjugacy growth series radius of convergence $\mathcal{RC}(\tilde{\sigma}_v)$. Then

$$\mathcal{RC}(\sigma_V(z)) = \mathcal{RC}(\tilde{\sigma}_V(z)).$$

In other words, the conjugacy growth rate of G_V is the same as the standard growth rate of G_V.

Proof. First note that Proposition 4.8 implies that

$$\begin{aligned}\mathcal{RC}(\sigma_V(z)) &= \inf\left\{|r| : \sum_{\Delta\in\mathsf{Clq}(V)} \prod_{i\in\Delta}\left(\frac{1}{\sigma_{\{i\}}(r)} - 1\right) = 0\right\}\\ &= \min\left(\inf\left\{|r| : \prod_{j\in V}\sigma_{\{j\}}(r) \sum_{\Delta\in\mathsf{Clq}(V)} \prod_{i\in\Delta}\left(\frac{1}{\sigma_{\{i\}}(r)} - 1\right) = 0\right\}, \mathcal{RC}\left(\prod_{j\in V}\sigma_{\{j\}}(z)\right)\right).\end{aligned}$$

It is clear that

$$\mathcal{RC}(\tilde{\sigma}_V(z)) \geq \mathcal{RC}(\sigma_V(z)),$$

hence it suffices to prove that

$$\mathcal{RC}(\tilde{\sigma}_V(z)) \leq \mathcal{RC}(\sigma_V(z)).$$

If the graph is complete then the graph product is just the direct product of the vertex groups and the result follows immediately from the hypothesis. So let us assume that the graph is not complete. In this case the term $\mathsf{NC}(V)$ is not trivial. Theorem 4.14 and Remark 4.15 imply that

$$\mathcal{RC}(\tilde{\sigma}_V(z)) \leq \min\left(\mathcal{RC}\left(\mathsf{NC}(V)(z)\right), \mathcal{RC}(\{\tilde{\sigma}_{\{v\}}(z)\})\right),$$

for each $v \in V$. This is because $\tilde{\sigma}_V(z)$ contains the term $\mathsf{NC}(V)(z)$, coming from $\Psi(V)$, and also contains the term $(\sigma_{\{v\}}(z) - 1)\Psi(\mathsf{Ct}(\{v\}))$ (which contains the term $(\sigma_{\{v\}}(z) - 1)$). Hence

$$\mathcal{RC}(\tilde{\sigma}_V(z)) \leq \min\left(\mathcal{RC}\left(\mathsf{NC}(V)(z)\right), \mathcal{RC}(\prod_{v\in V}\tilde{\sigma}_{\{v\}}(z))\right).$$

And the hypothesis implies that

$$\mathcal{RC}(\prod_{v\in V}\tilde{\sigma}_{\{v\}}(z)) = \mathcal{RC}(\prod_{v\in V}\sigma_{\{v\}}(z)).$$

Therefore

$$\mathcal{RC}(\tilde{\sigma}_V(z)) \leq \min\left(\mathcal{RC}\left(\mathsf{NC}(V)(z)\right), \mathcal{RC}(\prod_{v\in V}\sigma_{\{v\}}(z))\right).$$

Finally the definition of NC and Corollary 2.15 imply that

$$\mathcal{RC}\left(\mathsf{NC}(V)(z)\right) = \inf\left\{|r| : \prod_{j\in V}\sigma_{\{j\}}(r)\sum_{\Delta\in\mathsf{Clq}(V)}\prod_{i\in\Delta}\left(\frac{1}{\sigma_{\{i\}}(r)}-1\right)=0\right\}.$$

This gives the result.

□

4.3 Some examples

In this section we compute the conjugacy growth series of several graph products.

Let G_V be a graph product. For each $\Delta\in\mathsf{Clq}(V)$ we denote by

$$\mathsf{Ct}(\Delta) \rightsquigarrow \{A_1,\ldots,A_n\}\in\mathsf{ComFam}(\mathsf{Ct}(\Delta))$$

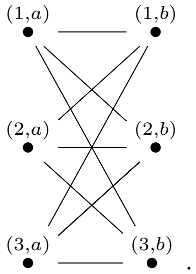
the assignment to Δ of those $\{A_1,\ldots,A_n\}\in\mathsf{ComFam}(\mathsf{Ct}(\Delta))$ for which $\prod_{j=1}^{n}\alpha(A_i)(z)\neq 0$.

Proposition 4.17 (Free products)
Let $\{(G_i,X_i)\}_{i=1}^{n}$ be a finite family of finitely generated groups with finite symmetric generating sets, and let $\sigma_i(z)$ (respectively $\tilde{\sigma_i}(z)$) be the associated standard (respectively conjugacy) growth series of G_i. The conjugacy growth series of $(G_1*\ldots*G_n, X_1\sqcup\ldots\sqcup X_n)$ is given by

$$\tilde{\sigma}_{(G_1*\ldots*G_n,X_1\sqcup\ldots\sqcup X_n)}(z) = \sum_{i=1}^{n}\tilde{\sigma_i}(z)-(n-1)+\mathsf{NC}(\{1,\ldots,n\})(z).$$

Proof. This is the graph product of the simple graph V with n vertices and no edges. We use Theorem 4.14. The only cliques are the empty set and the singleton. The centralizing set of the empty set is the full graph V, for which $\mathsf{ComFam}(V)=\{\emptyset\}\cup\{V\}$. The centralizing set of each singleton is the empty-set for which $\mathsf{ComFam}(\emptyset)=\{\emptyset\}$. □

Example 4.17.1
Exceptionally for this example we allow to call each vertex by a couple of symbols instead of just a natural number. Let Γ be the following 3 by 3 bipartite graph



.

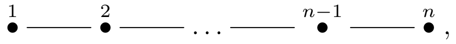
We adopt the convention that $\overline{a} = b$ and $\overline{b} = a$. We use Theorem 4.14. The cliques of Γ are the empty-set, the singleton, and the edges which are of the form $\{(i,a),(j,b)\}$ for $i,j \in \{1,2,3\}$. Hence Theorem 4.14 and Remark 4.15 give

$$\begin{aligned}\tilde{\sigma}_V(z)(z) = & \,1 + \mathsf{NC}(V)(z) + \mathsf{NC}(\{(1,a),(2,a),(3,a)\})(z)\mathsf{NC}\left(\{(1,b),(2,b),(3,b)\}\right)(z)\\ &+ \sum_{x\in\{a,b\}}\sum_{i\in\{1,2,3\}} (\tilde{\sigma}_{\{(i,x)\}}(z)-1)(1+\mathsf{NC}(\{(1,\overline{x}),(2,\overline{x}),(3,\overline{x})\})(z)\\ &+ \sum_{i\in\{1,2,3\}}\sum_{j\in\{1,2,3\}}(\tilde{\sigma}_{\{(i,a)\}}(z)-1)(\tilde{\sigma}_{\{(j,b)\}}(z)-1).\end{aligned}$$

If in particular all the vertex groups are ${}^{\mathbb{Z}}\big/_{2\mathbb{Z}}$ then

$$\tilde{\sigma}_V(z) = 1+6z+9z^2+6z\mathsf{NLS}(2z^2+3z^3)+\mathsf{NLS}(2z^2+3z^3)^2+\mathsf{NLS}\left((1+z)^4(-1+4z-4z^2)+1\right).$$

Example 4.17.2
Let Γ be the graph



consisting of a line with n vertices, for $n \geq 1$. We use Theorem 4.14. The cliques of Γ are the empty-set, the singleton, and the edges of the form $\{i,i+1\}$ for $i \in \{1,\dots,n-1\}$.

Hence according to Remark 4.15 we have to consider the following:

$$\begin{aligned}\mathsf{Ct}(\emptyset) = V &\rightsquigarrow \emptyset, \{\{1,\dots,n\}\},\\ \mathsf{Ct}(\{j\}) = &\begin{cases} \{2\} & & \emptyset & \text{for } j=1\\ \{j-1,j+1\} & \rightsquigarrow & \emptyset,\{\{j-1,j+1\}\} & \text{for } j\in\{2,\dots,n-1\}\\ \{n-1\} & & \emptyset & \text{for } j=n, \text{ and}\end{cases}\\ \mathsf{Ct}(\{i,i+1\}) = \emptyset &\rightsquigarrow \emptyset \text{ for } i \in \{1,\dots,n-1\}.\end{aligned}$$

By Theorem 4.14

$$\begin{aligned}\tilde{\sigma}_V(z) = & \,1 + \mathsf{NC}(\{1,\dots,n\})(z)\\ &+ (\tilde{\sigma}_{\{1\}}(z)-1) + \sum_{j=2}^{n-1}(\tilde{\sigma}_{\{j\}}(z)-1)\left(1+\mathsf{NC}(\{j-1,j+1\})(z)\right) + (\tilde{\sigma}_{\{n\}}(z)-1)\\ &+ \sum_{i=1}^{n-1}(\tilde{\sigma}_{\{i\}}(z)-1)(\tilde{\sigma}_{\{i+1\}}(z)-1).\end{aligned}$$

In this last formula, note that the two extremal points play a different role than the others.

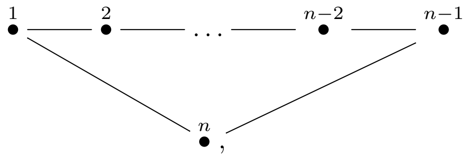
The radius of convergence of $\tilde{\sigma}_V(z)$ is the smallest positive value t such that

$$\prod_{i=1}^{n}\sigma_{\{i\}}(t)\left(1+\sum_{j=1}^{n}\left(\frac{1}{\sigma_{\{j\}}(t)}-1\right)+\sum_{j=1}^{n-1}\left(\frac{1}{\sigma_{\{j\}}(t)}-1\right)\left(\frac{1}{\sigma_{\{j+1\}}(t)}-1\right)\right) = 1.$$

If in particular all the vertex groups are ${}^{\mathbb{Z}}\big/_{2\mathbb{Z}}$ then

$$\tilde{\sigma}_V(z) = 1+nz+(n-1)z^2+(n-2)z\mathsf{NLS}(z^2)+\mathsf{NLS}\left((1+z)^{n-2}((n-2)z-1)+1\right).$$

Example 4.17.3
Let Γ be the graph



consisting of a polygonal on n vertices, for $n \geq 5$. We compute the conjugacy growth series using Theorem 4.14. Let $C_n := \mathbb{Z}/n\mathbb{Z}$. The cliques of Γ are, the empty-set, the singletons, and the edges of the form $\{i, i+1\}$ of $i \in C_n$. Hence according to Remark 4.15 we have to consider the following:

$$\begin{aligned} \mathsf{Ct}(\emptyset) = V &\rightsquigarrow \emptyset, \{\{1, \ldots, n\}\}, \\ \mathsf{Ct}(\{j\}) = \{\{j-1, j+1\}\} &\rightsquigarrow \emptyset, \{\{j-1, j+1\}\} \text{ for } j \in C_n, \text{ and} \\ \mathsf{Ct}(\{i, i+1\}) = \emptyset &\rightsquigarrow \emptyset \text{ for } i \in C_n. \end{aligned}$$

By Theorem 4.14

$$\begin{aligned} \tilde{\sigma}_V(z) = {} & 1 + \mathsf{NC}(\{1, \ldots, n\})(z) \\ & + \sum_{j \in C_n} (\tilde{\sigma}_{\{j\}}(z) - 1)\,(1 + \mathsf{NC}(\{j-1, j+1\})(z)) \\ & + \sum_{i \in C_n} (\tilde{\sigma}_{\{i\}}(z) - 1)(\tilde{\sigma}_{\{i+1\}}(z) - 1). \end{aligned}$$

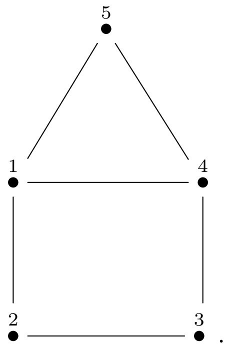
The radius of convergence of $\tilde{\sigma}_V(z)$ is the smallest positive value t such that

$$\prod_{i \in C_n} \sigma_{\{i\}}(t) \left(1 + \sum_{j \in C_n} \left(\frac{1}{\sigma_{\{j\}}(t)} - 1 \right) + \sum_{j \in C_n} \left(\frac{1}{\sigma_{\{j\}}(t)} - 1 \right) \left(\frac{1}{\sigma_{\{j+1\}}(t)} - 1 \right) \right) = 1.$$

If in particular all the vertex groups are $\mathbb{Z}/2\mathbb{Z}$ then

$$\tilde{\sigma}_V(z) = 1 + nz + nz^2 + nz\mathsf{NLS}(z^2) + \mathsf{NLS}\left((1+z)^{n-2}(-z^2 + (n-2)z - 1) + 1 \right).$$

Example 4.17.4
Let Γ be the following graph:



The cliques of Γ are the empty set, the singleton, the edges and the triangle on the top. Hence according to Remark 4.15 we have to consider the following.

$$\begin{aligned}
\mathsf{Ct}(\emptyset) = \{1,2,3,4,5\} &\rightsquigarrow \emptyset, \{\{1,2,3,4,5\}\}, \{\{1,3\},\{2,4\}\} \\
\mathsf{Ct}(\{1\}) = \{2,4,5\} &\rightsquigarrow \emptyset, \{\{2,4,5\}\} \\
\mathsf{Ct}(\{2\}) = \{1,3\} &\rightsquigarrow \emptyset, \{\{1,3\}\}\} \\
\mathsf{Ct}(\{3\}) = \{2,4\} &\rightsquigarrow \emptyset, \{\{2,4\}\} \\
\mathsf{Ct}(\{4\}) = \{1,3,5\} &\rightsquigarrow \emptyset, \{\{1,3,5\}\} \\
\mathsf{Ct}(\{5\}) = \{1,4\} &\rightsquigarrow \emptyset \\
\mathsf{Ct}(\{1,2\}) = \mathsf{Ct}(\{2,3\}) = \mathsf{Ct}(\{3,4\}) = \emptyset &\rightsquigarrow \emptyset \\
\mathsf{Ct}(\{1,4\}) = \{5\} &\rightsquigarrow \emptyset \\
\mathsf{Ct}(\{1,5\}) = \{4\} &\rightsquigarrow \emptyset \\
\mathsf{Ct}(\{4,5\}) = \{1\} &\rightsquigarrow \emptyset \\
\mathsf{Ct}(\{1,4,5\}) = \emptyset &\rightsquigarrow \emptyset.
\end{aligned}$$

By Theorem 4.14

$$\begin{aligned}
\tilde{\sigma}_{\{1,2,3,4,5\}}(z) = &\, 1 + \mathsf{NC}(\{1,2,3,4,5\})(z) + \mathsf{NC}(\{1,3\})(z)\mathsf{NC}(\{2,4\})(z) \\
&+ (\tilde{\sigma}_{\{1\}}(z) - 1)(1 + \mathsf{NC}(\{2,4,5\})(z)) + (\tilde{\sigma}_{\{2\}}(z) - 1)(1 + \mathsf{NC}(\{1,3\})(z)) \\
&+ (\tilde{\sigma}_{\{3\}}(z) - 1)(1 + \mathsf{NC}(\{2,4\})(z)) + (\tilde{\sigma}_{\{4\}}(z) - 1)(1 + \mathsf{NC}(\{1,3,5\})(z)) + (\tilde{\sigma}_{\{5\}}(z) - 1) \\
&+ (\tilde{\sigma}_{\{1\}}(z) - 1)(\tilde{\sigma}_{\{2\}}(z) - 1) + (\tilde{\sigma}_{\{2\}}(z) - 1)(\tilde{\sigma}_{\{3\}}(z) - 1) + (\tilde{\sigma}_{\{3\}}(z) - 1)(\tilde{\sigma}_{\{4\}}(z) - 1) \\
&+ (\tilde{\sigma}_{\{4\}}(z) - 1)(\tilde{\sigma}_{\{1\}}(z) - 1) + (\tilde{\sigma}_{\{1\}}(z) - 1)(\tilde{\sigma}_{\{5\}}(z) - 1) + (\tilde{\sigma}_{\{4\}}(z) - 1)(\tilde{\sigma}_{\{5\}}(z) - 1) \\
&+ (\tilde{\sigma}_{\{1\}}(z) - 1)(\tilde{\sigma}_{\{4\}}(z) - 1)(\tilde{\sigma}_{\{5\}}(z) - 1).
\end{aligned}$$

If in particular all the vertex groups are ${}^{\mathbb{Z}}\big/_{2\mathbb{Z}}$ then

$$\tilde{\sigma}_{\{1,2,3,4,5\}}(z) = 1 + 5z + 6z^2 + z^3 + 2z\mathsf{NLS}(z^2) + 2z\mathsf{NLS}(2z^2 + z^3) + \mathsf{NLS}(z^2)^2 + \mathsf{NLS}(4z^2 + 3z^3 - z^4 - z^5).$$

Chapter 5

Conjugacy growth series of some free products with amalgamation and HNN-extensions over admissible subgroups

In this Chapter we compute the conjugacy growth series of a specific free product with amalgamation in Section 5.1.1, and we study some HNN-extensions of graph products in Section 5.2. We give in Section 5.2.1 an explicit formula for the conjugacy growth series of the HNN-extensions of graph products when the two isomorphic subgraphs are disjoint. Then we give two examples in Section 5.2.2, where one is not among the case treated in Section 5.2.1. Finally we emphasize the difficulty to obtain an explicit formula for the conjugacy growth series of the HNN-extensions when the two isomorphic subgraphs are not disjoint in the general case. For all the groups treated in this Chapter, we prove that the radius of convergence of the conjugacy growth series is the same as the radius of convergence of the standard growth series.

5.1 A free product with amalgamation

5.1.1 Free products of $\mathbb{Z}$ by $\mathbb{Z}$ with amalgamation over $\mathbb{Z}$

The following is an example of an amalgamated free product over admissible subgroups due to Alonso [Alo91, (b) of §6] . Let

$$\begin{tikzcd} G(n,m) := & \mathbb{Z} & * & \mathbb{Z}. \\ & & \mathbb{Z} \arrow[ul, hook', "\times m"'] \arrow[ur, hook, "\times n"'] & \end{tikzcd}$$

Let $A := \mathbb{Z} \cong \langle \{w, w^{-1}\} | \rangle$. The standard growth series of A is

$$\sigma_{(A,\{w,w^{-1}\})}(z) = \frac{1+z}{1-z}.$$

Now let $G_1 \cong \mathbb{Z}$ with generating set $\{x, x^m, x^{-m}\}$, $G_2 \cong \mathbb{Z}$ with generating set $\{y, y^n, y^{-n}\}$, and let $\alpha_1 : A \to G_1$ and $\alpha_2 : A \to G_2$ be the homomorphisms given by

$$\alpha_1 : w \longmapsto x^m$$
$$\alpha_2 : w \longmapsto y^n.$$

Then $\alpha_1(A)$ is admissible in G_1 with respect to the pair $(\{x, x^m, x^{-m}\}, \{x^m, x^{-m}\})$ and $\alpha_2(A)$ is admissible in G_2 with respect to the pair $(\{y, y^n, y^{-n}\}, \{y^n, y^{-n}\})$. Let us assume that m and

n are positive. The admissible transversal T_1 of $\alpha_1(A)$ in G_1 is $\{e, x, x^2, \ldots, x^{m-1}\}$, with growth series $\sigma_{(T_1,\{x,x^m,x^{-m}\})}(z) = 1 + z + \ldots + z^{m-1}$, and the admissible transversal T_2 of $\alpha_2(A)$ in G_2 is $\{e, y, y^2, \ldots, y^{n-1}\}$, with growth series $\sigma_{(T_2,\{y,y^n,y^{-n}\})}(z) = 1 + z + \ldots + z^{n-1}$.

Since $G(n,m)$ is the amalgamated product of G_1 and G_2 over $\alpha_1(A) \cong \alpha_2(A)$, by Proposition 2.8 the standard growth series of $G(n,m)$ with respect to the given generating set is

$$\begin{aligned}
&\sigma_{(G(n,m),\{x,x^m,x^{-m},y,y^n,y^{-n}\})}(z)\\
&= \sigma_{(A,\{w,w^{-1}\})}(z)\frac{\sigma_{(T_1,\{x,x^m,x^{-m}\})}(z)\sigma_{(T_2,\{y,y^n,y^{-n}\})}(z)}{1-(\sigma_{(T_1,\{x,x^m,x^{-m}\})}(z)-1)(\sigma_{(T_2,\{y,y^n,y^{-n}\})}(z)-1)}\\
&= \frac{1+z}{1-z}\frac{(1-z^m)(1-z^n)}{(1-z)^2 - z^2(1-z^{m-1})(1-z^{n-1})}.
\end{aligned}$$

The radius of convergence of $\sigma_{(G(n,m),\{x,x^m,x^{-m},y,y^n,y^{-n}\})}(z)$ is the smallest positive value t such that $(1-t)^2 - t^2(1-t^{m-1})(1-t^{n-1}) = 0$, which is smaller than 1 if $n, m > 1$.

Now we discuss the conjugacy growth series of $G(n,m)$. Let A' be the image of A in $G(n,m)$. Let us consider an element $g \in G(n,m)$, written in the normal form according to Lemma 2.6 as

$$g = au_1v_1 \ldots u_kv_k,$$

where $a \in A'$, $k \in \mathbb{N}$, $u_i \in T_1$, $v_i \in T_2$, and $u_j, v_j \neq 1$ for $2 \leq j \leq k-1$. Since A' is in the center of $G(n,m)$, conjugation by elements of A' has no effect on g. On the other hand since there are no possible cancellations between elements of T_1 and elements of T_2, any pair u_iv_j or v_ju_i is already in geodesic normal form. It follows that the only way to shorten g by conjugation is by cyclic permutation. Therefore we find the following.

Lemma 5.1
Every element $g \in G(n,m)$ is conjugate to an element

$$au_1v_1 \ldots u_kv_k \in \min([g]),$$

where $a \in A'$, $k \in \mathbb{N}$, $u_i \in T_1 \setminus \{e\}$, $v_i \in T_2 \setminus \{e\}$. Moreover, two such elements

$$au_1v_1 \ldots u_kv_k \quad \text{and} \quad a'u'_1v'_1 \ldots u'_kv'_k$$

are conjugate if and only if $a = a'$ and $u_1v_1 \ldots u_kv_k$ is a cyclic permutation of $u'_1v'_1 \ldots u'_kv'_k$.

Corollary 5.2
The conjugacy growth series of $G(n,m)$ with respect to $\{x, x^m, x^{-m}, y, y^n, y^{-n}\}$ is given by

$$\begin{aligned}
&\tilde{\sigma}_{(G(n,m),\{x,x^m,x^{-m},y,y^n,y^{-n}\})}(z)\\
&= \sigma_{(A,\{w,w^{-1}\})}(z)\left(1 + \mathsf{NLS}\left((\sigma_{(T_1,\{x,x^m,x^{-m}\})}(z)-1)(\sigma_{(T_2,\{y,y^n,y^{-n}\})}(z)-1)\right)\right)\\
&= \frac{1+z}{1-z}\left(1 + \mathsf{NLS}\left(z^2\frac{(1-z^{m-1})(1-z^{n-1})}{(1-z)^2}\right)\right).
\end{aligned}$$

In particular, the radius of convergence of $\tilde{\sigma}_{(G(n,m),\{x,x^m,x^{-m},y,y^n,y^{-n}\})}(z)$ is the same the radius of convergence of $\sigma_{(G(n,m),\{x,x^m,x^{-m},y,y^n,y^{-n}\})}(z)$.

Remark 5.3
If m or n is negative, it suffices to replace it by its opposite in the growth series formulas.

5.2 Some HNN-extensions of graph products

We would like to consider some HNN-extensions of graph products over sub-graph products. We keep the same notation as in Chapter 4.

Definition
Let G_W, G_V be two graph products, for some finite simple graphs W and V. We say that $\iota : G_W \to G_V$ is an inclusion of graph products if

1. there is an injective map $\iota^{\mathscr{V}} : W \to V$ satisfying that if v is connected to v' then $\iota^{\mathscr{V}}(v)$ is connected to $\iota^{\mathscr{V}}(v)$, and
2. for every $v \in W$, there is an isomorphism $\iota_v : G_{\{v\}} \to G_{\{\iota^{\mathscr{V}}(v)\}}$.

If moreover the graph products are given with generating sets as in Chapter 4 and the isomorphisms ι_v's induce a bijection between the generating sets of $G_{\{v\}}$ and of $G_{\{\iota^V(v)\}}$ for every $v \in W$, then we say that ι is an *admissible inclusion of graph products.*

Each graph product is given with a generating set built as in Chapter 4. Let G_W and G_V be two graph products and let $\iota_1, \iota_{-1} : G_W \to G_V$ be two admissible inclusions of graph products. We know that ι_1, ι_{-1} are admissible inclusions in the sense of Section 2.3.
We consider the HNN-extension G of G_V over ι_1 and ι_{-1}:

$$G := \langle X_V \cup \{t, t^{-1}\} \,|\, t\iota_1(h)t^{-1} = \iota_{-1}(h),\, \forall h \in G_W\rangle.$$

Notation
Let $V_1 := \iota_1^{\mathscr{V}}(W)$, $V_{-1} := \iota_{-1}^{\mathscr{V}}(W)$ be the isomorphic sub-graphs of V, let $B := G_V$, let $H_1 := \iota_1(G_W)$, $H_{-1} := \iota_{-1}(G_W)$ be the isomorphic subgraph products of G_V and let T_1, T_{-1} be the admissible transversals of H_1 and of H_{-1} respectively, in B. Let $S := X_V \cup \{t, t^{-1}\}$ be the generating set of G, and let $\alpha_1 : H_1 \to H_{-1}$ and $\alpha_{-1} : H_{-1} \to H_1$ be the isomorphisms coming from ι_1 and ι_{-1} in such a way that

$$G = \langle X_V \cup \{t, t^{-1}\} \,|\, tht^{-1} = \alpha_1(h),\, \forall h \in H_1\rangle = \langle X_V \cup \{t, t^{-1}\} \,|\, t^{-1}ht = \alpha_{-1}(h),\, \forall h \in H_{-1}\rangle.$$

With this hypothesis,

$$\begin{aligned}
&\sigma_{(G_{V_1}, X_{V_1})}(z) = \sigma_{(G_{V_{-1}}, X_{V_{-1}})}(z) = \sigma_{(G_W, X_W)}(z),\\
&\tilde{\sigma}_{(G_{V_1}, X_{V_1})}(z) = \tilde{\sigma}_{(G_{V_{-1}}, X_{V_{-1}})}(z) = \tilde{\sigma}_{(G_W, X_W)}(z), \text{ and}\\
&\sigma_{(T_1, X_V)}(z) = \frac{\sigma_{(G_V, X_V)}(z)}{\sigma_{(G_{V_1}, X_{V_1})}(z)} = \frac{\sigma_{(G_V, X_V)}(z)}{\sigma_{(G_{V_{-1}}, X_{V_{-1}})}(z)} = \sigma_{(T_{-1}, X_V)}(z).
\end{aligned}$$

We know how to compute the standard growth series of G in terms of the $\sigma_{\{v\}}(z)$'s for $v \in V$ by Propositions 4.8 and 2.11.

Notation (and Explanation)
Let $h \in H_1 \cup H_{-1}$ and $j \in \mathbb{Z}\backslash\{0\}$. Suppose that for each $k \in \{0, \ldots, |j|-1\}$, $\alpha^k_{\mathrm{sgn}(j)}(h) \in H_{\mathrm{sgn}}(j)$. In this case we can write

$$t^j h = \alpha^j_{\mathrm{sgn}(j)}(h) t^j.$$

If h satisfies this hypothesis we note

$$\overset{j}{h} := \alpha^j_{\mathrm{sgn}(j)}(h).$$

The next Proposition will be useful to compute the conjugacy growth series of G.

Proposition 5.4
For each $g \in G$, there exists a unique element $\tilde{g} \in \min([g])_S$ which satisfies:

a) $\tilde{g} \in H_1 \cup H_{-1}$ up the equivalence relation $\sim_A$ on $H_1 \cup H_{-1}$ generated by, for $h, h' \in H_i$, $h \sim_A \alpha_i(h)$ and $h \sim_A h'$ if h and h' are conjugate in H_i, or

b) $\tilde{g} \in B \setminus (H_1 \cup H_{-1})$ up to conjugation in B, or

c) $\tilde{g}$ is in C, up to the equivalence $\sim_C$, where

$$C := \{ht^j \ : \ j \in \mathbb{Z} \setminus \{0\},\ \forall k \in \mathbb{N},\ \alpha^k_{\mathrm{sgn}(j)}(h) \in H_{\mathrm{sgn}(j)}\},$$

and the equivalence relation $\sim_C$ is generated by $ht^j \sim_C h't^j$ if $h' = \overset{j}{\overline{h}}$ or if there exists $c \in H_{\mathrm{sgn}(j)}$, with $\mathrm{Supp}(c) \subset \mathrm{Supp}(h)$ such that $h' = ch\,\overset{j}{\overline{c}}^{-1}$, or

d) $\tilde{g}$ is in P, up to the equivalence relation $\sim_P$, where

$$P = \left\{ ht^{j_1}w_1 \cdots t^{j_n}w_n \ : \ \begin{array}{c} h \in H_1 \cup H_{-1} \\ n \geq 1 \\ \forall i \in \{1, \ldots, n\},\ j_i \in \mathbb{Z} \setminus \{0\}, w_i \in T_{\mathrm{sgn}(i)} \setminus \{e\}, \\ \forall r \in \mathbb{N},\ \mathrm{Supp}(w_i) \approx \mathrm{Supp}\left(\overset{r(j_1 \ldots j_n)+j_{i+1}+\ldots+j_n}{\overline{h}}\right), \\ \forall k \in \{0, \ldots, |j_i| - 1\}, \overset{r(j_1 \ldots j_n)+k+j_{i+1}+\ldots+j_n}{\overline{h}} \in H_{\mathrm{sgn}(j_i)} \end{array} \right\},$$

and the equivalence relation $\sim_P$ is generated by

$$ht^{j_1}w_1 \cdots t^{j_n}w_n \sim_P \overset{j_l+\cdots+j_n}{\overline{h}}\ t^{j_l}w_l \cdots t^{j_n}w_n t^{j_1}w_1 \cdots t^{j_{l-1}}w_{l-1}, \forall l \in \{2, \ldots, n\}$$

and also by

$$ht^{j_1}w_1 \cdots t^{j_n}w_n \sim_P h't^{j_1}w_1 \cdots t^{j_n}w_n, \text{ if there exists } c \in G_{\mathrm{Supp}(h)} \text{ such that } h' = c^{-1}h\,\overset{j_1+\cdots+j_n}{\overline{c}}.$$

Proof. Every element of $b \in B$ can be uniquely written as $b = h_1u_1 = h_{-1}u_{-1}$ with $h_i \in H_i$, $u_i \in T_i$. The normal form of tb or $t^{-1}b$ as in Lemma 2.10 is

$$tb = th_1u_1 = \overset{1}{\overline{h_1}}\ tu_1, \quad t^{-1}b = t^{-1}h_{-1}u_{-1} = \overset{-1}{\overline{h_{-1}}}\ t^{-1}u_{-1}. \tag{5.1}$$

Let $g \in G$.

a) If $g \in H_1 \cup H_{-1}$, since conjugation by t or t^{-1} is a bijection between H_1 and H_{-1} that preserves the length, one can choose a conjugacy representative in H_1. On the other hand, to shorten this element by conjugation, one needs to conjugate it by an element of H_1.

b) If $g \in B \setminus (H_1 \cup H_{-1})$, it follows from (5.1) that conjugating g by t or t^{-1} will increase its length. Hence it admits a conjugate either in $B \setminus (H_1 \cup H_{-1})$ or in $H_1 \cup H_{-1}$, in which case we are are in situation a).

Now let us consider an element $g \in G \setminus B$ written in the geodesic normal form (Lemmas 2.9 and 2.10) as

$$g = bt^{j_1}w_1 \cdots t^{j_n}w_n,$$

with $b \in B$, $n \geq 1$, $j_i \in \mathbb{Z} \setminus \{0\}$, $w_i \in T_{\mathrm{sgn}(j_i)}$ for $i \in \{1, \ldots, n\}$, and with $w_i \neq e$ for $i \in \{1, \ldots, n-1\}$.

- Let $\mathcal{CRNF}$ (Circular rewrite in normal form) be the following procedure for an element g written in such normal form: The element g is conjugate to

  $$g_1 := t^{j_1}w_1 \cdots t^{j_n}w_nb.$$

  We write w_nb as $w_nb = h_nw'_n$ with $h \in H_{\mathrm{sgn}(j_n)}$ and $w'_n \in T_{\mathrm{sgn}(j_n)}$. Using the rules (5.1), $t^{j_n}w_nb = h_{n-1}t^{j_n}w'_n$, for some $h_{n-1} \in H_{-\mathrm{sgn}(j_n)}$ and $w'_n \in T_{\mathrm{sgn}(j_n)}$, so

  $$g_1 = t^{j_1}w_1 \cdots t^{j_{n-1}}w_{n-1}h_{n-1}t^{j_n}.$$

  If $w_{n-1}h_{n-1} = e$ we collapse $t^{j_{n-1}}$ and t^{j_n} and stop, otherwise we write $w_{n-1}h_{n-1}$ as $h'_{n-1}w'_{n-1}$ and use again the rules (5.1), and finally we obtain

  $$g_1 = ht^{j'_1}w'_1 \cdots t^{j'_{n'}}w_{n'} \quad \text{or } g_1 = hw'_1 \in B,$$

  written in normal form, with $h \in H \in H_{-\mathrm{sgn}(j'_1)}$, $n' \in \mathbb{N}$, for $i \in \{1, \ldots, n'\}$, $j'_i \in \mathbb{Z} \setminus \{0\}$, $w'_i \in T_{\mathrm{sgn}(t^{j'}{}_i)}$ and $w'_i \neq e$ for $i \in \{1, \ldots, n-1\}$. We write this as

  $$ht^{j'_1}w'_1 \cdots t^{j'_{n'}}w_{n'} = \mathcal{CRNF}(bt^{j_1}w_1 \cdots t^{j_n}w_n).$$

- At each step of the process $\mathcal{CRNF}$ the length decreases or stays the same, because of admissibility of H_1, H_{-1} in B. Hence

  $$|\mathcal{CRNF}(bt^{j_1}w_1 \cdots t^{j_n}w_n)|_S \leq |bt^{j_1}w_1 \cdots t^{j_n}w_n|_S.$$

  Therefore after applying $\mathcal{CRNF}$ a finite number of times to g we find an element g_2 such that $|g_2|_S = |\mathcal{CRNF}(g_2)|_S$. If $g_2 \in B$, then we are in the case a) or b); hence let us assume that $g_2 \notin B$. Also at each step the length of the prefix $h \in H_1 \cup H_{-1}$ decrease, hence after applying $\mathcal{CRNF}$ a finite number of times to g_2 one find g_3 such that for all $r \in \mathbb{N}$
  If

  $$g_3 = ht^{j_1}w_1 \cdots t^{j_n}w_n \quad \text{and} \quad \mathcal{CRNF}^r(g_2) = h't^{j'_1}w'_1 \cdots t^{j'_{n'}}w_{n'},$$

  then

  $$|g_3| = |\mathcal{CRNF}^r(g_3)| \quad \text{and} \quad |h| = |h'| \quad \text{and} \quad |t^{j_1}w_1 \cdots t^{j_n}w_n| = |t^{j'_1}w'_1 \cdots t^{j'_{n'}}w_{n'}|.$$

  This is possible only if

  $$n = n',\, \forall i \in \{1, \ldots, n\},\, j_i = j'_i,\, w_i = w'_i,$$

  and also if

  $$\forall i \in \{1, \ldots, n\}, \forall k \in \{0, \ldots, |j_i| - 1\}, \forall r \in \mathbb{N}$$
  $$[\frac{r(j_1 + \ldots j_n) + j_{i+1} + \ldots + j_n}{h}, w_i] = e \text{ and}$$
  $$\frac{r(j_1 + \ldots j_n) + k + j_{i+1} + \ldots + j_n}{h} \in H_{\mathrm{sgn}(j_i)}.$$

  Thus $h' = \frac{r(j_1 + \ldots j_n) + j_1 + \cdots + j_n}{h}$, for some $r \in \mathbb{N}$.

We now make the following distinction.

c) If $w_n = e$ and $n = 1$ then g_2 is in C.

d) if $w_n = e$ but $n > 1$ then in the procedure $\mathcal{CRNF}$ applied to g_3 there is a step where a conjugate of g_3 starts and ends with a power of t, and hence we can conjugate it in such a way to not end with a power of t. Hence up to renaming, one can assume that $w_n \neq e$, and therefore g_3 is in P.

Now if $\tilde{g} \notin B$, then the criterion [LS77, Theorem 2.5] says that any conjugate of $\tilde{g}$ can be obtained by a cyclic permutation of the blocks $t^{j_i} w_i$ and $\tilde{h}$, followed by conjugation by an element of $H_1 \cup H_{-1}$. But in the preceding procedure of finding a conjugate not in B, we already used cyclic permutation of the blocks $t^{j_i} w_i$, which gives a unique element of P up to $\mathcal{CRNF}$. Finally if we conjugate g_3 by an element $\hat{h} \in H_1 \cup H_{-1}$, with $\mathrm{Supp}(\hat{h}) \subset \mathrm{Supp}(h)$, $\hat{h}$ behaves as h in the $\mathcal{CRNF}$ procedure and hence h is transformed as $\hat{h}h \frac{j_1+\ldots+j_n}{\hat{h}^{-1}}$. This gives the equivalence relations on C and on P, and ends the proof.

□

Corollary 5.5
If for each of the groups $G_{\{v\}}$ the radius of convergence of its standard growth series is the same as the radius of convergence of its conjugacy growth series, then

$$\mathcal{RC}(\tilde{\sigma}_{(G,S)}(z)) = \mathcal{RC}(\sigma_{(G,S)}(z)).$$

Proof. According to Theorem 4.16 and Proposition 2.11, it remains to show that

$$\mathcal{RC}(\tilde{\sigma}_{(G,S)}(z)) \leq \inf \left\{ |z| \,:\, 1 + z - 2z \frac{\sigma_{(G_V, X_V)}(z)}{\sigma_{(G_{V_1}, X_{V_1})}(z)} = 0 \right\}.$$

By Proposition 2.1

$$\inf \left\{ |z| \,:\, 1 + z - 2z \frac{\sigma_{(G_V, X_V)}(z)}{\sigma_{(G_{V_1}, X_{V_1})}(z)} = 0 \right\} = \min \left\{ r \in \mathbb{R}_+ \,:\, 1 + r - 2r \frac{\sigma_{(G_V, X_V)}(r)}{\sigma_{(G_{V_1}, X_{V_1})}(r)} = 0 \right\}.$$

Let P_0 be the subset of P in Proposition 5.4 with elements having h trivial. That is,

$$P_0 = \{ t^{j_1} w_1 \cdots t^{j_n} w_n \,:\, n \geq 1, \forall i \in \{1, \ldots, n\}, j_i \in \mathbb{Z} \setminus \{0\}, w_i \in T_{\mathrm{sgn}(j_i)} \setminus \{e\} \}.$$

The equivalence relation $\sim_P$ restricts to P_0 and is just the cyclic permutation of the blocks $t^{j_i} w_i$. Let $\sigma_{P_0}(z)$ be the contribution to $\tilde{\sigma}_{(G,S)}(z)$ of the conjugacy classes of the elements in P_0. Then clearly

$$\mathcal{RC}(\tilde{\sigma}_{(G,S)}(z)) \leq \mathcal{RC}(\sigma_{P_0}(z)),$$

and $\sigma_{P_0}(z)$ is the Necklaces series associated to the language

$$\{ t^j w \; j > 0, \, w \in T_1 \setminus \{e\} \} \cup \{ t^j w \,:\, j < 0, \, w \in T_{-1} \setminus \{e\} \},$$

having growth series

$$2 \frac{z}{1-z} \left(\frac{\sigma_{(G_V, X_V)}(z)}{\sigma_{(G_{V_1}, X_{V_1})}(z)} - 1 \right).$$

Hence by Corollary 2.15, $\mathcal{RC}(\sigma_{P_0}(z))$ is the unique positive number r such that

$$2 \frac{r}{1-r} \left(\frac{\sigma_{(G_V, X_V)}(r)}{\sigma_{(G_{V_1}, X_{V_1})}(r)} - 1 \right) = 1 \iff 1 + r - 2r \frac{\sigma_{(G_V, X_V)}(r)}{\sigma_{(G_{V_1}, X_{V_1})}(r)} = 0.$$

□

5.2.1 The case when V_1 and V_{-1} are disjoint subgraphs

We assume the subgraphs V_1 and V_{-1} in V are disjoint. In this case $(H_1 \setminus \{e\}) \cap (H_{-1} \setminus \{e\}) = \emptyset$. In particular $H_1 \subset T_{-1}$ and $H_{-1} \subset T_1$.

Notation
For $h \in H_1 \cup H_{-1} \setminus \{e\}$, let $\operatorname{sgn}(h)$ be the value in $\{1, -1\}$ such that $h \in H_{\operatorname{sgn}(h)}$.
Since in this case it makes sense, by abuse of notation we write and extend $\bar{\cdot} : V_1 \leftrightarrow V_{-1}$ to $\bar{\cdot} : H_1 \leftrightarrow H_{-1}$ in the obvious way.

One can express the conjugacy growth series of G as follows:

Proposition 5.6
The conjugacy growth series of G with respect to the generating set S is given by

$$\begin{aligned}\tilde{\sigma}_{(G,S)}(z) = {} & \tilde{\sigma}_V(z) - \tilde{\sigma}_{V_1}(z) + 1 + \frac{2z}{1-z} + \mathsf{NLS}\left(\frac{2z}{1-z}\left(\frac{\sigma_V(z)}{\sigma_{V_1}(z)} - 1\right)\right) \\ & + \sum_{\emptyset \neq V' \subset V_1} \tilde{\sigma}^{\mathcal{M}}_{V'}(z) \mathsf{NLS}\left(z^2 \left(\frac{\sigma_{\mathsf{Ct}(\overline{V'})}(z)}{\sigma_{V_1 \cap \mathsf{Ct}(\overline{V'})}(z)} - 1\right)\left(\frac{\sigma_{\mathsf{Ct}(V')}(z)}{\sigma_{V_{-1} \cap \mathsf{Ct}(V')}(z)} - 1\right)\right),\end{aligned}$$

where $\tilde{\sigma}^{\mathcal{M}}_{V'}(z)$ refers to the conjugacy growth series of the elements having a minimal representative with support exactly V' (see Section 4.1.1 and Lemma 4.10).

Proof. We use Proposition 5.4.

a) The contribution to $\tilde{\sigma}_{(G,S)}(z)$ of the conjugacy classes of the elements in H_1 is $\tilde{\sigma}_{V_1}(z)$.

b) The contribution to $\tilde{\sigma}_{(G,S)}(z)$ of the conjugacy classes of the elements in $B \setminus (H_1 \cup H_{-1})$ is $\tilde{\sigma}_V(z) - 2\tilde{\sigma}_{V_1}(z) + 1$.

c) Since $H_1 \cap H_{-1} = \{e\}$, the set C from Proposition 5.4 reduces to

$$C = \{t^j \,:\, j \in \mathbb{Z} \setminus \{0\}\},$$

and $\sim_C$ is the identity. Hence the contribution to $\tilde{\sigma}_{(G,S)}(z)$ of the conjugacy classes of the elements in C is $\frac{2z}{1-z}$.

d) Now for the set P in Proposition 5.4, since $H_1 \cap H_{-1} = \{e\}$, one observes that if $g = ht^{j_1}w_1 \cdots t^{j_n}w_n \in P$ and $h \neq e$, then n is even and

$$\begin{aligned}&\forall i \in \{1, \ldots, n\},\ j_i = (-1)^i \operatorname{sgn}(h), \\ &\forall i \text{ even } \in \{1, \ldots, n\},\ \operatorname{Supp}(w_i) \subset \mathsf{Ct}(\operatorname{Supp}(h)), \\ &\forall i \text{ odd } \in \{1, \ldots, n\},\ \operatorname{Supp}(w_i) \subset \mathsf{Ct}(\operatorname{Supp}(\overline{h})).\end{aligned}$$

Therefore we have to consider the following sets. Let

$$P_0 := \left\{ t^{j_1}w_1 \cdots t^{j_n}w_n \,:\, \begin{array}{c} n \in \mathbb{N} \setminus \{0\} \\ \forall i \in \{1, \ldots, n\},\ j_i \in \mathbb{Z} \setminus \{0\},\ w_i \in T_{\operatorname{sgn}(i)} \setminus \{e\} \end{array} \right\},$$

and let

$$P_1 := \left\{ ht^{j_1}w_1\cdots t^{j_n}w_n \ : \ \begin{array}{c} n\in 2\mathbb{N}\setminus\{0\} \\ h\in H_1\cup H_{-1}\setminus\{e\} \\ \forall i\in\{1,\dots,n\},\, j_i=(-1)^i\mathrm{sgn}(h),\, w_i\in T_{\mathrm{sgn}(i)}\setminus\{e\} \\ \forall i \text{ even } \in\{1,\dots,n\},\, \mathrm{Supp}(w_i)\subset \mathsf{Ct}(\mathrm{Supp}(h)) \\ \forall i \text{ odd } \in\{1,\dots,n\},\, \mathrm{Supp}(w_i)\subset \mathsf{Ct}(\mathrm{Supp}(\overline{h})) \end{array} \right\}.$$

We separate P as this way because now we can specify what the relation $\sim_P$ is on P_0 and on P_1, and then we can compute separately the contribution of the conjugacy classes of the elements in P_0 and in P_1. The relation $\sim_P$ restricted to P_0 is the equivalence up to cyclic permutation of the blocks $t^{j_i}w_i$. The relation $\sim_P$ restricted to P_1 gives

$$ht^{j_1}w_1\cdots t^{j_n}w_n \sim_{P_1} \overline{h}t^{j_2}w_2\cdots t^{j_n}w_nt^{j_1}w_1.$$

Hence one can choose as conjugacy representatives of an the elements in P_1, the elements that begin with $h\in H_1\setminus\{e\}$, followed by t^{-1}, followed by $w_1\in T_{-1}\setminus\{e\}$, and ending with tw_n. In other words, we consider the set

$$P_1' := \left\{ ht^{j_1}w_1\cdots t^{j_n}w_n \ : \ \begin{array}{c} n\in 2\mathbb{N}\setminus\{0\} \\ h\in H_1\setminus\{e\} \\ \forall i\in\{1,\dots,n\},\, j_i=(-1)^i,\, w_i\in T_{\mathrm{sgn}(i)}\setminus\{e\} \\ \forall i \text{ even } \in\{1,\dots,n\},\, \mathrm{Supp}(w_i)\subset \mathsf{Ct}(\mathrm{Supp}(h)) \\ \forall i \text{ odd } \in\{1,\dots,n\},\, \mathrm{Supp}(w_i)\subset \mathsf{Ct}(\mathrm{Supp}(\overline{h})) \end{array} \right\}.$$

Then the relation $\sim_P$ restricted to P_1' is the equivalence given by the cyclic permutation of the $\frac{n}{2}$ blocks $t^{-1}w_itw_{i+1}$, and also by the equivalence

$$ht^{j_1}w_1\cdots t^{j_n}w_n \sim_{P_1'} h't^{j_1}w_1\cdots t^{j_n}w_n$$

if h and h' are conjugate in H_1 by an element $\hat{h}\in H_1$ with $\mathrm{Supp}(\hat{h})\subset\mathrm{Supp}(h)$.

The contribution to $\tilde{\sigma}_{(G,S)}(z)$ of the conjugacy classes of the elements in P_0 is $\mathsf{NLS}\left(\frac{2z}{1-z}\left(\frac{\sigma_V(z)}{\sigma_{V_1}(z)}-1\right)\right)$, because this is the Necklaces series of the language

$$\{t^jw \ : \ j>0,\, w\in T_1\setminus\{e\}\}\cup\{t^jw \ : \ j<0,\, w\in T_{-1}\setminus\{e\}\}$$

with growth series

$$2\frac{z}{1-z}\left(\frac{\sigma_V(z)}{\sigma_{V_1}(z)}-1\right).$$

For every $\emptyset\neq V'\subset V_1$ the contribution to $\tilde{\sigma}_{(G,S)}(z)$ of the conjugacy classes of the elements $ht^{-1}w_1\cdots tw_n\in P_1'$ with $\mathrm{Supp}(h)=V'$ is

$$\tilde{\sigma}^{\mathcal{M}}_{V'}(z)\mathsf{NLS}\left(z^2\left(\frac{\sigma_{\mathsf{Ct}(\overline{V'})}(z)}{\sigma_{V_1\cap\mathsf{Ct}(\overline{V'})}(z)}-1\right)\left(\frac{\sigma_{\mathsf{Ct}(V')}(z)}{\sigma_{V_{-1}\cap\mathsf{Ct}(V')}(z)}-1\right)\right).$$

This is because h commutes with $t^{-1}w_1\cdots tw_n\in P_1'$, so we take the product of the two contributions. Then $\tilde{\sigma}^{\mathcal{M}}_{V'}(z)$ is the contribution of the conjugacy classes of the $h\in H_1$ having support exactly V'. And finally the contribution of the classes of the $t^{-1}w_1\cdots tw_n$'s is the Necklaces series of the language

$$\{t^{-1}w_1tw_2 \ : \ w_1\in T_{-1}\setminus\{e\},\, w_2\in T_1\setminus\{e\}\},$$

with growth series

$$z^2\left(\frac{\sigma_{\mathsf{Ct}(\overline{V'})}(z)}{\sigma_{V_1\cap\mathsf{Ct}(\overline{V'})}(z)}-1\right)\left(\frac{\sigma_{\mathsf{Ct}(V')}(z)}{\sigma_{V_{-1}\cap\mathsf{Ct}(V')}(z)}-1\right).$$

Summing all we find the result. □

5.2.2 Some examples

5.2.2.1 HNN-extension of $H * H$ over H

Let H be a group with finite generating set S_H. Let S_{H_1}, $S_{H_{-1}}$ be two copies of S_H and let H_1, H_{-1} be two copies of H generated by S_{H_1}, $S_{H_{-1}}$ respectively. Let $B = H_1 * H_{-1}$ and let $\alpha_1 : H \to B$, $\alpha_{-1} : H \to B$ be the canonical inclusions. Let $S_1 := \alpha_1(S_H)$, $S_{-1} := \alpha_{-1}(S_H)$, then $S_B := S_1 \cup S_{-1}$ is a generating set of B and the inclusions $\alpha_1, \alpha_{-1} : (H, S_H) \to (B, S_B)$ are admissible.

Let

$$G := \langle S_B \cup \{t, t^{-1}\} \mid t\alpha_1(h)t^{-1} = \alpha_{-1}(h)\,\forall h \in H\rangle$$

be the HNN-extension of B formed by these two inclusions.
We are in the situation of Section 5.2.1.

Proposition 5.6 gives the following.

Proposition 5.7
The conjugacy growth series of G with respect to the generating set S is given by

$$\tilde{\sigma}_{(G,S)}(z) = \sigma_{(H,S_H)}(z) + \frac{2z}{1-z} + \mathsf{NLS}\left(\left(\sigma_{(H,S_H)}(z)-1\right)^2\right) + \mathsf{NLS}\left(\frac{2z}{1-z}\cdot\frac{\sigma_{(H,S_H)}(z)-1}{2-\sigma_{(H,S_H)}(z)}\right).$$

5.2.2.2 HNN-extension of $H * H$ over itself by swapping the components

Let H be a group with finite generating set S_H. Let H_l, H_r be two isomorphic copies of H generated by S_{H_l}, S_{H_r} two copies of S_H. We consider $B := H_l * H_r$ generated by $S_B := S_{H_l} \cup S_{H_r}$. Let $\bar{\cdot} : B \leftrightarrow B$ be the isomorphism induced by replacing each element of H_l by its copies in H_r and each element of H_r by its copies in H_l. Note that this notation make sense since $\bar{\cdot} : B \to B$ is an involution. We consider the HNN-extension G of B over B by the identity and $\bar{\cdot}$:

$$G := \langle S_B \cup \{t, t^{-1}\} \mid tbt^{-1} = \bar{b},\ \forall b \in B\rangle.$$

The two inclusions are clearly admissible and the corresponding transversals are trivial. Let $S := S_B \cup \{t, t^{-1}\}$. The standard growth series of G is given by

$$\sigma_{(G,S)}(z) = \sigma_{(B,S_B)}(z)\cdot\frac{1+z}{1-z}.$$

Notation
For $h \in H_l \cup H_r \setminus \{e\}$ let $\mathrm{Supp}(h) \in \{l, r\}$ be such that $h \in H_{\mathrm{Supp}(h)}$.

We are in the situation of Section 5.2, where $H_1 = H_{-1} = H_l * H_r \cong H * H$ and where $T_1 = T_{-1} = \{e\}$.
The conjugacy growth series $\tilde{\sigma}_{(G,S)}(z)$ of G is given by the following:

Proposition 5.8
The conjugacy growth series of G is given by

$$\begin{aligned}\tilde{\sigma}_{(G,S)}(z) =& \tilde{\sigma}_{(H,S_H)}(z) + \frac{2z}{1-z}\\ &+ (\tilde{\sigma}_{(H,S_H)}(z)-1)\frac{2z^2}{1-z^2} + (\sigma_{(H,S_H)}(z)-1)\frac{2z}{1-z^2}\\ &+ \sum_{n>1,\text{ odd}} \frac{1}{n}\sum_{d|n}\phi(\frac{n}{d})\big(\sigma_{(H,S_H)}(z^{\frac{n}{d}})-1\big)^d\frac{2z}{1-z^2}\\ &+ \sum_{n>0,\text{ even}} \frac{1}{n}\sum_{d|n}\phi(\frac{n}{d})\big(\sigma_{(H,S_H)}(z^{\frac{n}{d}})-1\big)^d\frac{1+z^2}{1-z^2}.\end{aligned}$$

Proof. We use Proposition 5.4.

a) The contribution to $\tilde{\sigma}_{(G,S)}(z)$ of the conjugacy classes of the elements in $H_l \cup H_r$ is $\tilde{\sigma}_{(H,S_H)}(z)$. This is because up to $\bar{\cdot}$ one can assume that the element is in H_l and then we can use conjugation in H_l to shorten it. We count the contribution to $\tilde{\sigma}_{(G,S)}(z)$ of the conjugacy classes having a representative of minimal length in $H_l * H_r \setminus (H_l \cup H_r)$ in c)iii).

b) Since $B \setminus (H_1 \cup H_{-1}) = \emptyset$ there is no contribution to $\tilde{\sigma}_{(G,S)}(z)$ of the conjugacy classes of the elements in $B \setminus (H_1 \cup H_{-1})$.

c) Now for the sets C as in Proposition 5.4 and $H_l * H_r \setminus (H_l \cup H_r)$, we make the distinction as follows. Let $g = ht^j \in C \cup H_l * H_r \setminus (H_l \cup H_r)$ (with $j \in \mathbb{Z}$).

   i) If $h = e$ then j can be any integer different from 0 and g is conjugacy minimal. Hence the contribution to $\tilde{\sigma}_{(G,S)}(z)$ of the conjugacy classes of such elements is $\frac{2z}{1-z}$.

   ii) If $h \neq e$ then h can be seen as a word in $H_l \cup H_r$ alternating the letters of H_l and the letters of H_r. Suppose at first that $h \in H_l \cup H_r \setminus \{e\}$. Then with the equivalence on C given by $\bar{\cdot}$ one can assume that $h \in H_l \setminus \{e\}$.
   If j is even then one can choose a conjugate of h in H_l to decreases the length and then it will be conjugacy minimal, hence the term $(\tilde{\sigma}_{(H,S_H)}(z)-1)\frac{2z^2}{1-z^2}$ appears in $\tilde{\sigma}_{(G,S)}(z)$ as the contribution of the conjugacy classes of the elements of the form $ht^j \in C$ with $h \in H_l \setminus \{e\}$, $h \in \min([h])_{S_l}$ and j even ($\neq \{0\}$).
   Now if j is odd then we cannot shorten h because then it will be a twisted conjugation on h, which is obtained by multiplying h to the left by a letter in one H_i and to the right by a letter in the other H_i, hence this increases the length of h. Hence the term $(\sigma_{(H,S_H)}(z)-1)\frac{2z}{1-z^2}$ appears in $\tilde{\sigma}_{(G,S)}(z)$ as the contribution of the conjugacy classes of the elements of the form $ht^j \in C$ with $h \in H_l \setminus \{e\}$ and j odd.

   iii) Now assume that $h \in H_l * H_r \setminus (H_l \cup H_r)$. The element h can be written as a word with at least 2 letters in $H_l \cup H_r \setminus \{e\}$ where the letters alternate between $H_l \setminus \{e\}$ and $H_r \setminus \{e\}$. With the equivalence given by $\bar{\cdot}$ one can assume that the first letter of h is in $H_l \setminus \{e\}$.

   Assume that j is even and that the last letter of h is in H_l. Then ht^j is conjugate to an element $h't^j$ where h' starts with a letter in H_l and ends with a letter in H_r (or is of smallest length). Hence for j even we have to consider the set

$$C_2 := \left\{x_1y_1\cdots x_ny_nt^j\ : j \text{ even}, n \in \mathbb{N}\setminus\{0\}, \forall i \in \{1,\ldots,n\},\ x_i \in H_l\setminus\{e\}, y_i \in H_r \setminus\{e\}\right\}$$

with the equivalence relation $\sim_{C_2}$ given by

$$x_1y_1\cdots x_ny_nt^j \sim_{C_2} \overline{y_nx_1y_1\cdots x_n}t^j \sim_{C_2} x_ny_nx_1y_1\cdots x_{n-1}y_{n-1}t^j \sim_{C_2} \cdots$$
$$\cdots \sim_{C_2} \overline{y_1x_2y_2\cdot x_ny_nx_1}t^j.$$

The contribution to $\tilde{\sigma}_{(G,S)}(z)$ of the conjugacy classes of the elements in C_2 is

$$\sum_{n>0,\text{ even}} \frac{1}{n}\sum_{d|n}\phi(\frac{n}{d})\big(\sigma_{(H,S_H)}(z^{\frac{n}{d}})-1\big)^d\frac{1+z^2}{1-z^2},$$

because this has the same growth series as the concatenation of the language formed by the necklaces with an even ($\neq 0$) number of pearls in $H\setminus\{e\}$ and the language $\{t^j\ :\ j \text{ even}\}$, (see the proof of Proposition 2.13).

Assume that j is odd and that the last letter of h is in H_r. Then ht^j is conjugate to an element $h't^j$ where h' starts with a letter in H_l and ends with a letter in H_l (or is of smallest length). Hence for j odd we have to consider the set

$$C_1 := \big\{x_1y_1\cdots x_ny_nx_{n+1}t^j\ : j \text{ odd}, n\in\mathbb{N}\setminus\{0\},\ \forall i\in\{1,\dots,n+1\},\ x_i\in H_l\setminus\{e\},\, y_i\in H_r\setminus\{e\}\big\}$$

with the equivalence relation $\sim_{C_1}$ given by

$$x_1y_1\cdots x_ny_nx_{n+1}t^j \sim_{C_1} x_{n+1}\overline{x_1y_1\cdots x_ny_n}t^j \sim_{C_1} \overline{y_nx_{n+1}}x_1y_1\cdots x_ny_nt^j \sim_{C_1} \cdots$$
$$\sim_{C_1} \overline{y_1\cdots x_ny_nx_{n+1}}x_1t^j.$$

The contribution to $\tilde{\sigma}_{(G,S)}(z)$ of the conjugacy classes of the elements in C_1 is

$$\sum_{n>1,\text{ odd}} \frac{1}{n}\sum_{d|n}\phi(\frac{n}{d})\big(\sigma_{(H,S_H)}(z^{\frac{n}{d}})-1\big)^d\frac{2z}{1-z^2},$$

for analogous reasons to C_2.

d) Since $T_1 = T_{-1} = \{e\}$, the set P is empty and hence gives no contribution to $\tilde{\sigma}_{(G,S)}(z)$.

Summing all we find the result. □

Remark 5.9 (About further generalizations)
Suppose we would like to generalize this example as follows: Assume that $V_1\cap V_{-1} = \{v_l, v_r\}$ and that $\overline{v_l} = v_r$, but that $V_1\cup V_{-1}\neq\{v_l,v_r\}$. In this case in the computation of $\tilde{\sigma}_{(G,S)}$, we would need the contribution of the conjugacy classes of the elements in the set P as in Proposition 5.4. As in the proof of Proposition 5.6 d), one observes that if $g = ht^{j_1}w_1\cdots t^{j_n}w_n\in P$ and $h\notin H_1\cap H_{-1}$, then n is even and

$$\forall i\in\{1,\dots,n\},\ j_i = (-1)^i\mathrm{sgn}(h),$$
$$\forall i \text{ even } \in\{1,\dots,n\},\ \mathrm{Supp}(w_i)\subset \mathsf{Ct}(\mathrm{Supp}(h)),$$
$$\forall i \text{ odd } \in\{1,\dots,n\},\ \mathrm{Supp}(w_i)\subset \mathsf{Ct}(\mathrm{Supp}(\overline{h})).$$

But if $h\in H_1\cap H_{-1}\setminus\{e\} = H_l * H_r\setminus\{e\}$, then for the same reason as in parts c) ii) and iii) in the proof of Proposition 5.8, one has to treat separately the case where $\sum_{i=}^{n} j_i$ is even and when it is odd. The difficulty hence is to compute the contributions of the conjugacy classes of such elements of P with $\sum_{i=}^{n} j_i$ even and with $\sum_{i=}^{n} j_i$ odd. Therefore, unfortunately at this point we have no way to obtain a general formula for the conjugacy growth series of all the HNN-extensions of graph products.

Appendix A

On the cardinality of the set of conjugacy classes of infinite groups

We show that if we do not impose that the group is finitely generated or even countable then strange things can happen. More precisely we prove the following:

Theorem A.1
Let κ_1, κ_2 be two infinite cardinals such that $\kappa_1 < \kappa_2$. Then there exists a group G such that $\sharp G = \kappa_2$ and $\sharp G_\sim = \kappa_1$.

Background on ordinal and cardinal can be find in [Jec03, Chapter 2 and 3]. To prove this Theorem we use the following Lemma:

Lemma A.2
Let κ_2 we an infinite cardinal, then there exists a group of cardinal κ_2 with exactly 2 conjugacy classes.

Proof. This proof is mostly inspirited from the proof of Theorem 3.3 in [LS77, Chapter 4]. We construct inductively a group $G_\alpha = \langle X_\alpha \,|\, R_\alpha \rangle$ for every ordinal α as follows:

i) Step 0: $X_0 = \{t, t^{-1}\}$, $R_0 := \emptyset$;

ii) Successor ordinal $\alpha + 1$: Let $\{(g_i, h_i), : i \in I_\alpha\}$ be the set of all ordered pairs of non-trivial elements of G_α, $X_{\alpha+1} := X_\alpha \cup \{t_i \, : \, i \in I_\alpha\}^{\pm}$, $R_{\alpha+1} := R_\alpha \cup \{t_i g_i t_i^{-1} = h_i, \, \forall i \in I_\alpha\}$;

iii) Limit ordinal β: $X_\beta := \bigcup_{\alpha<\beta} X_\alpha$, $R_\beta := \bigcup_{\alpha<\beta} R_\alpha$.

For every successor ordinal $\alpha + 1$, G_α embeds into $G_{\alpha+1}$ and so for every limit ordinal β and for every $\alpha < \beta$, G_α embeds into G_β. By construction for every limit ordinal β, all the non-trivial element of G_β are conjugate. Hence G_β contains 2 conjugacy classes if β is a limit ordinal. Moreover if β is a limit ordinal ($\neq 0$) then $\sharp G_\beta = \sharp \beta$. By the Choice axiom there exists a limit ordinal β' such that $\sharp \beta' = \kappa_2$. Hence the group $G_{\beta'}$ satisfies the wanted property. □

Proof of Theorem A.1. Let G_β be a group given by Lemma A.2 of cardinality κ_2 and with 2 conjugacy classes. Then the group

$$G := \bigoplus_{\kappa_1} G_\beta$$

satisfies the desired property, since κ_1 and κ_2 are infinite. □

Table of notation

Symbol	Meaning	Defined in
$\sharp$	Cardinal of a set	
$\mathscr{D}(c, r)$	Open disk in the complex plane centered on c of radius r	
$\mathcal{RC}$	Radius of convergence of a complex power series	
$[z^m]$	Coefficient of z^m in a complex power series	
$F_L(z)$	Growth series of the language L	
$\lvert y\rvert$	Absolute value y for a complex number or number of letters of y for a word	
$\phi(k)$	Euler's totient function of k, for $k \in \mathbb{N}$	
e	Neutral element of a group	
$\lvert g\rvert_S$	Length of an element of g seen in a group with respect the generating set S	
$\sigma_{(G,S)}(z)$	Standard growth series of the group G with respect to S	Section 2.2
$\tilde{\sigma}_{(G,S)}(z)$	Conjugacy growth series of the group G with respect to S	Section 2.2
$\min([h])_Z$	Set of elements minimal in the conjugacy class of h $\{h' \in [h] \,:\, \lvert h'\rvert_Z = \lvert h\rvert_{\sim,Z}\}$	
$\mathsf{Necklaces}$	Necklaces set of a language	
NLS	Necklaces series associated to a growth series	Section 2.4
$\mathrm{Supp}(\eta)$	Support of η, i.e: $\{i \in I \,:\, \eta(i) \neq e\}$ for $\eta \in \bigoplus_{i \in I} G$	Page 19
$\vec{Y}$	Canonical generating set for the wreath products	Page 19 (3.1)
$\tilde{\sigma}^{A}_{(G \wr L, \vec{Y})}(z)$	Contribution to $\tilde{\sigma}_{(G \wr L, \vec{Y})}(z)$ of the conjugacy classes of type A	Page 22
$\tilde{\sigma}^{B}_{(G \wr L, \vec{Y})}(z)$	Contribution to $\tilde{\sigma}_{(G \wr L, \vec{Y})}(z)$ of the conjugacy classes of type B	Page 22
$\tilde{\sigma}^{B \neq e'}_{(G \wr L, \vec{Y})}(z)$	Part of $\tilde{\sigma}^{B}_{(G \wr L, \vec{Y})}(z)$ of those with torsion cursor	Page 33
$\tilde{\sigma}^{e'}_{(G \wr L, \vec{Y})}(z)$	Part of $\tilde{\sigma}^{B}_{(G \wr L, \vec{Y})}(z)$ of those with trivial cursor	Page 33
$\mathscr{L}(L')$	Set of leafs of a tree L'	Section 3.3.2.2
$\mathscr{L}^c(L')$	Set of vertices of a tree L' that are not leafs	Section 3.3.2.2
$\widetilde{\Upsilon}$	Set of representative tree with origin under L action	Section 3.3.2.2
$\mathsf{Ct}(V')$	Centralizing set of a subgraph V'	Page 42
$\mathsf{Clq}(V)$	Set of cliques of a graph V	Page 42
$V_1 \approx V_2$	All vertices of V_1 are connected to all vertices of V_2	
$G_{V'}$	Graph product associated to V'	Page 41
$\sigma_{V'}(z)$, $\tilde{\sigma}_{V'}(z)$	Standard, conjugacy growth series of $G_{V'}$, respectively	Page 47
$\sigma_i(z)$, $\tilde{\sigma}_i(z)$	$\sigma_{\{i\}}(z)$, $\tilde{\sigma}_{\{i\}}(z)$ for i a vertex	
$g^{\mathcal{M}}$	Möbius inverses of g, i.e $\sum_{W' \subseteq W} (-1)^{\lvert W\rvert - \lvert W'\rvert} g(W')$	Section 4.1.1
$\mathsf{ComFam}_n(V)$	Set of all commuting families of order n in V	Section 4.1.1
$\mathsf{ComFam}(V)$	Set of all commuting families in V, i.e. $\bigcup_{n \geq 0} \mathsf{ComFam}_n(V)$	Section 4.1.1
$\mathrm{Supp}(g)$	Support of g, i.e: $\bigcap_{V' \subset V \text{ and } g \in G_{V'}} V'$ for $g \in G_V$	Page 42
NC	Necklaces series of some growth build from a subgraph products	Page 50

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July 2012: Master of Science MSc in mathematics, fundamental mathematics (EPFL)

2010: Bachelor of Science BSc in mathematics (EPFL)

2013-2017: PhD Student at Université de Neuchâtel

2012: Invited researcher at Mathematical Institute of Burgundy (November-December), collaboration with Professor Dolecki on convergences theory

2007-2012: Student at École Polytechnique Fédérale de Lausanne (EPFL), section mathematics

Selected talks

- **July 4 2016:** "Introduction to conjugacy growth" (presentation of Guba-Sapir's article), Neuchâtel
- **July 7 2016:** "Conjugacy growth in wreath products", Neuchâtel
- **November 10 2016:** "Conjugacy growth series of graph products", Neuchâtel
- **November 10 2015:** "Conjugacy growth series of some wreath products", Neuchâtel
- **October 7 2014:** "Growth series of some wreath products" (presentation of Parry's article), Neuchâtel
- **June 16 2015:** "Some Eilenberg Moore categories", Neuchâtel
- **May 22 2014:** "On automata groups", Neuchâtel

Preprints

- *Conjugacy growth series of some wreath products*, 2017 (submited), arXiv:1610.07868
- *Formal conjugacy growth in graph products*, with L. Ciobanu, S. Hermiller, 2017 (in preparation)
- *Tensors, preorders and ultraspaces: constructions stemming from the Vietoris spaces*, Master project supervised by Gavin Seal, 2012

Professional experience

2013-2017: Assistant-PhD for the courses of Algèbre linéaire, calcul differentiel et integrable, théorie de Galois, analyse complexe, théorie algébrique des nombres, analyse complexe in the University of Neuchâtel,

2013: Hired by DGEO (direction générale de l'enseignement obligatoire du canton de Vaud), Substitute mathematics teacher at the secondary school of Cugy

2009-2012: Student-assistant for the courses of Analyse II, Applications des maths, Physique, for the Cours de mathématiques spéciales, and Analyse for Microtechnics in EPFL,
and Analyse, Algèbre linéaire for Polymath

Extra-curricular activities

23 April 2016: Presentation of the Mathematics Institute of Neuchâtel for the open day of the Faculty of Science

2014-2017: Collaboration in gardening project at the University of Neuchâtel, construction of a permaculture area