Published in Travaux de logique 17, 146-168, 2004
which should be used for any reference to this work

J.-Y.Beziau, A.Costa-Leite and A.Facchini (eds)
Aspects of Universal Logic, 146-168

On Modulated Logics for 'Generally': Some Metamathematical Issues[1]

Sheila R.M.Veloso and Paulo A.S.Veloso [2]

Abstract

Logics for 'generally' are intended to express assertions with some vague notions, such as 'generally', by means of new generalised quantifiers, and to reason about them. Here, we review such logical systems and examine some issues about them: axiomatisation, behaviour of the quantifiers, as well as deductive and expressive powers.

Keywords: Vague notions, generally, several, many, most, logics for vague notions, generalised quantifiers, families of sets, metamathematics, axiomatisation, oppositions, inference, expressivity.

Outline

[1]Research reported herein has received partial financial support from the Brazilian agencies FAPERJ (E-26/152.395/2002 and E-26/131.180/2003) and CNPq (471608/03-3 and Locia Project). The authors gratefully acknowledge helpful conversations with Walter Carnielli and Mario Benevides as well as CNPq grants 522979/96-3 (to SRMV) and 301378/2003-7 (to PASV).

[2]Progr. Eng. Sist. Comput., COPPE, Universidade Federal do Rio de Janeiro, Caixa Postal 68511, 21945-970 Rio de Janeiro, RJ, Brasil

1 Introduction

In this paper, we review some logics for 'generally' and examine some issues about them: axiomatisations, behaviour of the quantifiers, as well as deductive and expressive powers.

Logics for 'generally' are intended to express assertions with some vague notions, such as 'generally', by means of new generalised quantifiers, and to reason about them (important issues in qualitative reasoning). The primary motivation is a precise treatment of some vague notions (such as 'generally', 'several', 'many', 'most', etc.), which appear often in ordinary language and in some branches of science.[3]

This paper is structured as follows. The next section provides some motivations and ideas underlying logics for 'generally'. In section 3 we examine logical systems for expressing and reasoning about assertions involving (some versions of) 'generally', with their syntax and semantics as well as sound and complete axiomatisations. In section 4 we examine some metamathematical properties of these logics for 'generally', comparing them to classical first-order logic: deductive and expressive powers, and the behaviour of the new quantifier. Section 5 contains some concluding remarks about on-going and related work.

2 Preliminaries

We will now review some motivations and ideas underlying logics for 'generally'.

[3]We would like to have logics for some vague notions, much as one has logics embodying some mathematical notions ([B+F'85], p. 3).

2.1 Basic ideas

We first examine motivations underlying logics for 'generally'

Assertions and arguments involving some vague notions appear often, both in ordinary language and in some branches of science, where "modifiers", such as 'generally', 'rarely', 'several', 'few', 'many', 'most', 'typical', 'generic', etc., occur. For instance, one frequently encounters assertions such as "Many bodies expand when heated", "Most birds fly" and "Few metals are liquid under ordinary conditions".[4] The assertions "Whoever likes sports watches Sports-TV" and "Boys generally like sports" appear to lead to "Boys generally watch Sports-TV". Such qualitative arguments involving these vague notions appear to be quite widespread.[5]

Considering a universe of birds, we can express some assertions within classical first-order logic.[6]; but, what about vague assertions like "Birds generally fly"? We wish to express such assertions and reason about them in a formal manner; so we need precise meanings for these vague notions. Now, the intended meaning of "objects generally have property φ" can be given directly as "the set of objects having φ is important", or in terms of the set of exceptions as "the set of objects failing to have φ is negligible".[7]

2.2 Families for 'rarely' and 'generally'

We will now indicate how some notions of 'generally' and 'rarely' can be described by means of families of important and negligible sets. We actually have various notions of 'generally' and 'rarely', but some of them may be expected to share properties, which can be used to characterise these vague notions by means of the corresponding families of important and negligible sets [Vel'99, Vel'02]. To describe the important and negligible subsets, we may use common properties of their families $\mathcal{K}$ and $\mathcal{N}$. For instance, the above argument about boys and sports seems correct because of the intuitive feeling that if a set L has several objects and $L \subseteq T$, then set T will also have several objects: the family $\mathcal{K}$ of important sets (those having

[4]Such notions may also be useful in reporting experimental set-ups and results. More elaborate expressions involving 'propensity' are often used as well: a physician may say that a patient's background indicates a certain propensity, making him (or her) prone to some ailments.

[5]A medical doctor usually prescribes a treatment considering it appropriate to a typical patient with such symptoms.

[6]For instance, "All birds fly" and "Some birds fly" by $\forall v F(v)$ and $\exists v F(v)$.

[7]One may understand "Eagles generally fly" as "The flying eagles form an important set" or "The non-flying eagles form a negligible set".

several objects) is closed under supersets. Families corresponding to other notions, such as (very) few, may be closed under union or intersection.[8]

For non-triviality, the family $\mathcal{K}$ of important subsets of a universe V should be proper ($\emptyset \notin \mathcal{K}$ and $V \in \mathcal{K}$). Some interesting classes of such families of important subsets are: up-closed (closed under supersets), lattices (closed under union and intersection), filters (closed under supersets and intersection), and ultrafilters (maximal filters).[9]

3 Logics for 'generally'

We shall now examine how one can set up logical systems for expressing and reasoning about assertions involving (some versions of) 'generally'. The goal is having logics for some vague notions, much as we have "logics embodying mathematical concepts" [B+F'85]. In this section we briefly review some of these logics: syntax, semantics and axiomatics as well as soundness and completeness.

Our logics for 'generally' add to classical first-order logic [End'72, Sho'67] a (non-standard) generalised quantifier, with intended interpretation "forming an important set of objects of the universe of discourse" [Gra'99, V+C'01].[10]

3.1 Syntax of ∇

The syntax of our logics is obtained by extending the usual first-order syntax by the new quantifier ∇.

Given a signature ρ, we let $\mathrm{L}(\rho)$ be the usual first-order language (with equality $\approx$) of signature ρ. We will use $\mathrm{L}^{\nabla}(\rho)$ for the extension of $\mathrm{L}(\rho)$ by the new operator ∇. The formulae of language $\mathrm{L}^{\nabla}(\rho)$ are built by

[8] If one accepts the assertions "Few naturals are below fifteen" and "Few naturals divide twelve", then one would probably accept also the assertions: "Few naturals are below fifteen <u>and</u> even" and "Few naturals are below fifteen <u>or</u> divide twelve".

[9] For instance, the sets having more than, say, 70 % of the elements form an up-closed family (corresponding to a notion of 'several'); both the finite unions of intervals of the reals and the cofinite open subsets of an infinite topological space form lattices (corresponding to notions of 'many'); the subsets including a given set as well as the cofinite subsets of an infinite universe form filters (corresponding to notions of 'most'); and the subsets having a given element form an ultrafilter. The dual classes of such families of negligible subsets are: down-closed (closed under subsets), lattices (closed under $\cup$ and $\cap$), ideals (closed under subsets and $\cup$), and maximal (prime) ideals.

[10] With such new quantifiers we can handle assertions, such as "Birds generally fly" and "Metals generally are solid", as well as properties like "animals generally fear x".

the usual formation rules and a new variable-binding formation rule giving *generalised formulae* :

for each variable v, if φ is a formula in $L^{\nabla}(\rho)$ then so is $\nabla v\varphi$.

Other syntactic notions, such as *substitution* ($\varphi[v/t]$ or $\varphi(t)$) and substitutable, can be easily adapted.

As a example, consider a signature ρ with binary predicate L (on persons). If we let $L(x, y)$ stand for 'x loves y', then $\forall x \nabla y L(x, y)$ expresses "everybody loves people in general", $\exists x \nabla y L(x, y)$ expresses "somebody loves people in general" and "people generally love each other" can be expressed by $\nabla x \nabla y L(x, y)$. If $L(x, y)$ stands for 'y is taller than x', then "people are generally taller than x" can be expressed by the formula $\nabla y L(x, y)$.

3.2 Semantics of 'generally'

The semantic interpretation for 'generally' is provided by enriching first-order structures with families of subsets and extending the definition of satisfaction to ∇. For this purpose, we resort to modulated structures.

A *modulated structure* $\mathcal{A}^{\mathcal{K}} = \langle \mathcal{A}, \mathcal{K} \rangle$ for signature ρ consists of a usual structure $\mathcal{A}$ for signature ρ together with a *complex*: a proper family $\mathcal{K}$ of subsets of the universe A of $\mathcal{A}$.

We extend the usual definition of *satisfaction* of a formula φ in a structure under an assignment $s : V \to \mathsf{A}$ to variables as follows

for a formula $\nabla v\varphi$, we define

$$\mathcal{A}^{\mathcal{K}} \models \nabla z\varphi[\, s] \text{ iff } \{b \in \mathsf{A} : \mathcal{A}^{\mathcal{K}} \models \varphi[\, s(v \mapsto b)]\} \text{ belongs to the complex } \mathcal{K}.$$

where, as usual, $s(v \mapsto b)$ is the assignment agreeing with s on every variable but v and $s(v \mapsto b)(v) = b$.[11]

Satisfaction of a formula hinges only on the realisations assigned to its symbols.[12]

A convenient notion is that of extension with respect to a variable: the *v-extension* of formula φ under assignment s is the set $\mathcal{A}^{\mathcal{K}}[\varphi(\, s|v)] := \{b \in \mathsf{A} : \mathcal{A}^{\mathcal{K}} \models \varphi[\, s(v \mapsto b)]\}$.[13] With this notation, satisfaction of a generalised

[11]Thus, the propositional connectives as well as the classical quantifiers $\forall$ and $\exists$ will keep their familiar interpretations.

[12]Thus, satisfaction for first-order formulae (without ∇) does not depend on the complex: for a formula φ of $L(\rho)$, we have $\mathcal{A}^{\mathcal{K}} \models \varphi[\, s]$ iff $\mathcal{A} \models \varphi[\, s]$. We can also use the familiar notation $\mathcal{A}^{\mathcal{K}} \models \varphi[\underline{a}]$ for an assignment $\underline{a}$ to the free variables of formula φ.

[13]We similarly have the extension $\mathcal{A}^{\mathcal{K}}[\varphi(\underline{a}|v)] := \{b \in \mathsf{A} : \mathcal{A}^{\mathcal{K}} \models \varphi[\underline{a}, b]\}$.

formula becomes $\mathcal{A}^{\mathcal{K}} \models \varphi[\, s]$ iff the extension $\mathcal{A}^{\mathcal{K}}[\varphi(\, s|v)]$ belongs to the to the complex $\mathcal{K}$.[14] Other semantic notions, such as reduct and model ($\mathcal{A}^{\mathcal{K}} \models \Gamma$) are as usual.

We will modulate our structures by their complexes: a class of complexes will be called a *module.* The *basic module* $\underline{\mathcal{B}}$ consists of the proper complexes. We will be mainly interested in some classes of proper complexes: the modules $\underline{\mathcal{C}}$, $\underline{\mathcal{L}}$, $\underline{\mathcal{F}}$, and $\underline{\mathcal{U}}$ consisting of the proper up-closed complexes, of the lattices, of the filters and of the ultrafilters, respectively.[15] This gives rise to notions of *modulated consequence* as expected: consequence under module $\underline{\mathcal{M}}$ is defined by $\Gamma \models^{\underline{\mathcal{M}}} \tau$ iff $\mathcal{A}^{\mathcal{K}} \models \tau$, for every model $\mathcal{A}^{\mathcal{K}} \models \Gamma$ with $\mathcal{K}$ in $\underline{\mathcal{M}}$[16], likewise for *(modulated) validity.*

3.3 Axiomatics for 'generally'

We now formulate deductive systems for our logics for 'generally' by adding schemata to a calculus for classical first-order logic [Vel'98].

To set up our deductive systems for logics of 'generally', we take a sound and complete deductive calculus for classical first-order logic, with Modus Ponens (MP) as the sole inference rule (as in [End'72]), and extend its set Φ of axiom schemata by adding a set Φ_M of new axiom schemata (coding properties of the module), to form an axiomatisation for 'generally'.[17] We find convenient to divide our schemata into groups, namely

- syntactic schemata: related to invariance under syntax;
- common schemata: fundamental to the notions of 'generally';
- specific schemata: shared only by some versions of 'generally'.

The syntactic schemata aim to capture the idea that satisfaction hinges only on extension of a formula, and not on its syntactic form.

A syntactic schema handles extensionality: formulae with the same extension must be indistinguishable under 'generally'.

$$[\leftrightarrow \nabla] \quad \forall z(\psi \leftrightarrow \theta) \rightarrow (\nabla z\psi \leftrightarrow \nabla z\theta)$$

[14]Notice that $\mathcal{A}^{\mathcal{K}} \models \exists v\varphi[\, s]$ iff the extension $\mathcal{A}^{\mathcal{K}}[\varphi(\, s|v)]$ belongs to the family $\wp(\mathsf{A}) - \{\emptyset\}$ of the non-empty subsets of A.

[15]These modules are clearly related, e.g. $\underline{\mathcal{C}} \subseteq \underline{\mathcal{F}} \subseteq \underline{\mathcal{U}}$.

[16]For the module $\underline{\mathcal{F}}$ of filters: $\Gamma \models^{\underline{\mathcal{F}}} \tau$ iff $\mathcal{A}^{\mathcal{K}} \models \tau$, for every filter model $\mathcal{A}^{\mathcal{F}} \models \Gamma$. These modulated consequences are related and others can be similarly introduced.

[17]These schemata depend on the signature ρ, but we will prefer to use the simpler notations Φ and Φ_M rather than $\Phi(\rho)$ and $\Phi_M(\rho)$.

Another syntactic schema covers, in a similar manner, alphabetic variants.

$$[\nabla\nu] \quad \nabla v\varphi \leftrightarrow \nabla w\varphi[v/w] \quad \text{for a new variable } w$$

The *syntactic schemata* consist of these two schemata:

$$\Phi_I := [\leftrightarrow \nabla] \cup [\nabla\nu]$$

The common schemata code properties of the proper complexes.

$$\begin{array}{lll} [\forall\nabla] & \forall v\varphi \rightarrow \nabla v\varphi & [V \in \mathcal{K}] \\ [\nabla\exists] & \nabla v\varphi \rightarrow \exists v\varphi & [\emptyset \notin \mathcal{K}] \end{array}$$

The *basic axiomatisation* extends the syntactic schemata by these two common schemata:

$$\Phi_B := \Phi_I \cup [\forall\nabla] \cup [\nabla\exists].$$

The specific schemata code closure properties of special modules.

$$\begin{array}{lll} [\rightarrow \nabla] & \forall v(\psi \rightarrow \theta) \rightarrow (\nabla v\psi \rightarrow \nabla v\theta) & [\text{up-closure}] \\ [\nabla\vee] & (\nabla v\psi \wedge \nabla v\theta) \rightarrow \nabla v(\psi \vee \theta) & [\cup\text{-closure}] \\ [\nabla\wedge] & (\nabla v\psi \wedge \nabla v\theta) \rightarrow \nabla v(\psi \wedge \theta) & [\cap\text{-closure}] \\ [\neg\nabla] & \neg\nabla v\varphi \rightarrow \nabla v\neg\varphi & [\text{prime}] \end{array}$$

We thus have some specific axiomatisations as follows.

- Up-closed logic: $\Phi_C := \Phi_B \cup [\rightarrow \nabla]$[18]
- Lattice logic: $\Phi_L := \Phi_B \cup [\nabla\vee] \cup [\nabla\wedge]$
- Filter logic: $\Phi_F := \Phi_C \cup [\nabla\wedge]$[19]
- Ultrafilter logic: $\Phi_U := \Phi_F \cup [\neg\nabla]$[20]

Now, each one of these axiomatisations for 'generally' gives a *derivability relation* $\vdash^M$, axiomatised by $\Phi^M := \Phi_B \cup \Phi_M$. Derivations are first-order derivations from the schemata

$$\Gamma \vdash^M \tau \text{ iff } \Gamma \cup \Phi_F \vdash \tau \quad (\vdash^\star)$$

In fact, each set $\Xi \subseteq \Phi_F$ of axioms for 'generally' gives a derivability relation $\vdash^\Xi$, axiomatised by $\Phi^\Xi := \Phi \cup \Phi_B \cup \Xi$.

[18]In up-closed logic we have $\vdash^C (\nabla v\psi \vee \nabla v\theta) \rightarrow \nabla v(\psi \vee \theta)$ and $\vdash^C \nabla v(\psi \wedge \theta) \rightarrow (\nabla v\psi \wedge \nabla v\theta)$ (by $[\rightarrow \nabla]$).

[19]In filter logic we have $\vdash^F \nabla v\neg\varphi \rightarrow \neg\nabla v\varphi$ (by $[\nabla\vee]$ and $[\nabla\exists]$) and ∇ distributes over $\wedge$: $\vdash^F \nabla v(\psi \wedge \theta) \leftrightarrow (\nabla v\psi \wedge \nabla v\theta)$ (by $[\nabla\wedge]$ and $[\rightarrow \nabla]$).

[20]In ultrafilter logic, ∇ commutes with negation ($\vdash^U \neg\nabla v\varphi \leftrightarrow \nabla v\neg\varphi$) and distributes over the binary propositional connectives (we have, for instance, $\vdash^U \nabla v(\psi\vee\theta) \leftrightarrow (\nabla v\psi\vee \nabla v\theta)$). We thus have prenex normal form.

3.4 Soundness and completeness

We shall now establish soundness and completeness of our deductive systems for the corresponding logics for 'generally'.

Soundness ($\vdash^M \subseteq \models^{\underline{\mathcal{M}}}$) is easy to establish as usual: the axioms in each axiomatisation Φ^M code properties of the complexes in the module $\underline{\mathcal{M}}$.

Completeness ($\models^{\underline{\mathcal{M}}} \subseteq \vdash^M$) is not so immediate, but, we can extend Henkin's familiar method of witnesses [Hen'49, Sho'67, C+K'73, End'72]. The crucial point here is obtaining an appropriate complex, which we can do by using the witnesses. We proceed to outline how this can be done [Vel'98, V+C'01]. To fix ideas, we will focus on filter logic and later indicate how to adapt these ideas to the other cases.

Consider a set Γ of sentences of $L^\nabla(\rho)$ that is filter-consistent: $\Gamma \not\vdash^F \bot$. We will show how to obtain a filter model $\mathcal{H}^{\mathcal{K}_\Sigma} \models \Gamma$ (with cardinality at most that of $L^\nabla(\rho)$).

We first extend set $\Gamma \subseteq L^\nabla(\rho)$ to a maximally consistent set Σ with witnesses for the existential sentences of $L^\nabla(\rho \cup C)$ in set C of new constants (with cardinality $|C| \leq |L^\nabla(\rho)|$).[21] We form the canonical structure $\mathcal{H}$, for signature $\rho \cup C$ as usual.[22]

We provide a complex, by considering the formulae of $L^\nabla(\rho \cup C)$, having a single variable free, as follows. For each formula φ of $L^\nabla(\rho \cup C)$ with single free variable v, let $\Sigma[\varphi|v] := \{c^{\mathcal{H}} \in \mathsf{H} : \varphi[v/c] \in \Sigma\}$, and form the family $\Sigma_\nabla := \{\Sigma[\varphi|v] \subseteq \mathsf{H} : \nabla v\varphi \in \Sigma\}$.[23] By our axioms, this family Σ_∇ is proper and has the finite intersection property.[24] So, its closure $\mathcal{K}_\Sigma := \Sigma_\nabla^{\supseteq}$ under supersets is a filter, with the property $\Sigma[\varphi|v] \in \Sigma_\nabla$ iff $\Sigma[\varphi|v] \in \mathcal{K}_\Sigma$.[25] We can now show, by induction $\mathcal{H}^{\mathcal{K}_\Sigma} \models \tau$ iff $\tau \in \Sigma$, for each sentence τ of $L^\nabla(\rho \cup C)$.[26]

---

[21]The properties of conservative extensions by the addition of witnesses and Lindenbaum extensions for our deductive systems can be established as in classical first-order logic, by relying on the connection ($\vdash^*$) in 4.3.

[22]The canonical structure $\mathcal{H}$ has universe $\mathsf{H} := C/\sim^\Sigma$, where $c' \sim^\Sigma c$" iff $\Sigma \vdash Fc' \approx c$".

[23]One can view $\Sigma[\varphi|v]$ as the set v-represented within Σ by formula φ and Σ_∇ as the family of provably important represented subsets.

[24]Family Σ_∇ is proper by the basic schemata [$\forall\nabla$] and [$\nabla\exists$], and its closure under finite intersection follows from the schemata [$\nabla\wedge$] and [$\nabla\nu$].

[25]Notice that family Σ_∇ is not closed under arbitrary supersets, but this extension $\mathcal{K}_\Sigma \supseteq \Sigma_\nabla$ adds no definable subset: property $\Sigma[\varphi|v] \in \Sigma_\nabla$ iff $\Sigma[\varphi|v] \in \mathcal{K}_\Sigma$ follows from the schemata [$\rightarrow \nabla$] and [$\nabla\nu$].

[26]The inductive step for the new quantifier ∇, namely $\mathcal{H}^{\mathcal{K}_\Sigma} \models \nabla v\varphi$ iff $\nabla v\varphi \in \Sigma$, follows from the crucial property $\Sigma[\varphi|v] \in \Sigma_\nabla$ iff $\Sigma[\varphi|v] \in \mathcal{K}_\Sigma$ of the complex $\mathcal{K}_\Sigma$. The inductive steps for the propositional connectives as well as for the classical quantifiers $\forall$

We thus have the desired result: a Löwenheim-Skolem Theorem.

Theorem. Löwenheim-Skolem Theorem (for filter logic) Each filter-consistent set of sentences of $L^{\nabla}(\rho)$ has a filter model with cardinality at most $|L^{\nabla}(\rho)|$.

We now indicate how to adapt these ideas to our other logics.

- For up-closed logic, we use the same construction.[27]
- For basic and lattice logics, we take $\mathcal{K}_{\Sigma} := \Sigma_{\nabla}$.[28]
- For ultrafilter logic, we extend family Σ_{∇} to an ultrafilter.[29]

4 Metamathematics of 'generally'

Our logics for 'generally' extend classical first-order logic. We have sound and complete deductive systems for these logics. As usual, such a result transfers the finitary character of derivability to the compactness of the corresponding semantic consequence. Thus, our extensions of classical first-order logic by generalised quantifiers have compactness.

We shall now examine some other metamathematical properties of these extensions of classical first-order logic by generalised quantifiers. We will take a closer look at these extensions comparing them to classical first-order logic.

4.1 Behaviour of quantifiers

We will first examine the behaviour of quantifiers in our logics for 'generally'. We wish to compare them to classical first-order logic, pointing out similarities and contrasts.

---

and $\exists$ are as in Henkin's proof.

[27] Here, family Σ_{∇} will be proper and we take $\mathcal{K}_{\Sigma}$ to be its closure under supersets. The crucial property $\Sigma[\varphi|v] \in \Sigma_{\nabla}$ iff $\Sigma[\varphi|v] \in \mathcal{K}_{\Sigma}$ follows from the schemata $[\rightarrow\nabla]$ and $[\nabla\nu]$.

[28] Family Σ_{∇} already gives an appropriate complex.

[29] As family Σ_{∇} has the finite intersection property, it can be extended to a proper ultrafilter $\mathcal{K}_{\Sigma}$. The property $\Sigma[\varphi|v] \in \Sigma_{\nabla}$ iff $\Sigma[\varphi|v] \in \mathcal{K}_{\Sigma}$ now follows from the schema $[\neg\nabla]$.

In our extensions of classical first-order logic, the behaviour of the classical quantifiers $\forall$ and $\exists$ remain the same, but what about the new quantifier ∇? We know that ∇ is intermediate between $\forall$ and $\exists$, in terms of behaviour[30], and we feel intuitively that it is closer to the universal quantifier[31].

Oppositions of quantifiers

We will now compare leading classical and generalised quantifiers.

First, we do not have instantiation for ∇.[32]

We now wish to examine some opposition relations between classical and generalised quantifiers.[33]

Consider the classical square of oppositions, involving affirmative and negative, universal and particular assertions, as well as the relations of contrary, subcontrary and contradictory.[34] We wish to consider analogue connections involving also generalised quantifiers.

First, we have to make room for ∇, placing it in between $\forall$ and $\exists$. This transforms the usual square of oppositions into a hexagon (see figure 1).

This hexagon of oppositions has interesting interpretations in terms of corroboration and refutation: generalised sentences are harder to corroborate than universal ones and harder to refute than existential ones.[35]

---

[30]The common schemata [$\forall\nabla$] and [$\nabla\exists$] give $\vdash^{B} \forall v\varphi \rightarrow \nabla v\varphi$ and $\vdash^{B} \nabla v\varphi \rightarrow \exists v\varphi$. The converse implications are not valid ($\not\models^{\mathcal{M}} \nabla v\varphi \rightarrow \forall v\varphi$ and $\not\models^{\mathcal{M}} \exists v\varphi \rightarrow \nabla v\varphi$): either one would trivialise the new generalised quantifier, collapsing ∇ to $\forall$ or to $\exists$.

[31]One may feel the generalised quantifier ∇ to be closer to $\forall$ because of the intuitive interpretation "all, but negligibly few exceptions" for 'many', 'most', etc. One can define a dual generalised quantifier for 'rarely', closer to $\exists$.

[32]Indeed, $\nabla v\varphi$ does not yield $\varphi[v/t]$ (neither is the converse inference correct: $\varphi[v/t]$ does not yield $\nabla v\varphi$).

[33]Some square-of-opposition relations among 'few', 'many', and 'most' have been analysed [Pet'79].

[34]Contrary assertions cannot be both true, subcontrary assertions cannot be both false, and contradictory assertions cannot be both true nor false. The classical square of oppositions is as follows (where the diagonally opposed assertions are contradictory)

	Affirmative		Negative
Universal	$\forall v\varphi$	$\overset{contrar.}{\leftrightarrow}$	$\forall v\neg\varphi$
	$\Downarrow$		$\Downarrow$
Particular	$\exists v\varphi$	$\overset{subcontrar.}{\leftrightarrow}$	$\exists v\neg\varphi$

[35]Thus, generalised sentences fail to present a clear asymmetry between corroboration and refutation, of importance to some views of Popper (cf. [Pop'34], [Pop'75]).

Let us now take a closer look at the above hexagon of oppositions. We still have some further generalised assertions to place, namely those corresponding to $\neg\nabla\upsilon\varphi$ and $\neg\nabla\upsilon\neg\varphi$.[36] We can locate them by relying on equivalences concerning the behaviour of the classical quantifiers $\forall$ and $\exists$ under negation.[37] This transforms the above hexagon of oppositions into an octagon (see figure 2).[38]

This octagon displays oppositions holding in basic logic. In stronger logics for 'generally' one has some more information. For instance, we have some more oppositions in the octagon for filter logic (we have as contraries

[36] Indeed, the unary modalities ∇ and $\neg$ generate the four modalities with ∇: ∇, $\nabla\neg$, $\neg\nabla$ and $\neg\nabla\neg$.

[37] We have $\vdash \neg\exists\upsilon\varphi \leftrightarrow \forall\upsilon\neg\varphi$ and $\vdash \neg\forall\upsilon\varphi \leftrightarrow \exists\upsilon\neg\varphi$.

[38] The schemata [$\nabla\exists$] and [$\forall\nabla$] give $\vdash^B \neg\exists\upsilon\varphi \to \neg\nabla\upsilon\varphi$ and $\vdash^B \neg\nabla\upsilon\varphi \to \neg\forall\upsilon\varphi$.

	Affirmative		Negative
Universal	$\forall\upsilon\varphi$	*contrar.* $\leftrightarrow$	$\forall\upsilon\neg\varphi$
	$\Downarrow$		$\Downarrow$
Generalised	$\nabla\upsilon\varphi$		$\nabla\upsilon\neg\varphi$
	$\Downarrow$		$\Downarrow$
Particular	$\exists\upsilon\varphi$	*subcontrar.* $\leftrightarrow$	$\exists\upsilon\neg\varphi$

Table 1: Hexagon of oppositions in basic logic

		Affirmative		Negative		
$\neg\exists\upsilon\neg\varphi$	$\leftrightarrow$	$\forall\upsilon\varphi$	*contrar.* $\leftrightarrow$	$\forall\upsilon\neg\varphi$	$\leftrightarrow$	$\neg\exists\upsilon\varphi$
		$\Downarrow$				$\Downarrow$
		$\nabla\upsilon\varphi$	*contradict.* $\leftrightarrow$			$\neg\nabla\upsilon\varphi$
$\Downarrow$				$\Downarrow$		$\Downarrow$
$\neg\nabla\upsilon\neg\varphi$			*contradict.* $\leftrightarrow$	$\nabla\upsilon\neg\varphi$		
$\Downarrow$		$\Downarrow$		$\Downarrow$		
$\neg\forall\upsilon\neg\varphi$	$\leftrightarrow$	$\exists\upsilon\varphi$	*subcontrar.* $\leftrightarrow$	$\exists\upsilon\neg\varphi$	$\leftrightarrow$	$\neg\forall\upsilon\varphi$

Table 2: Octagon of oppositions in basic logic

$\neg\nabla v\neg\varphi$ and $\nabla v\varphi$ and as sub-contraries $\nabla v\neg\varphi$ and $\neg\nabla v\varphi$).[39] This octagon reduces back to a hexagon in the case of ultrafilter logic (because of the equivalences $\vdash^U \neg\nabla v\varphi \leftrightarrow \nabla v\neg\varphi$ and $\vdash^U \neg\nabla v\neg\varphi \leftrightarrow \nabla v\varphi$).[40]

Classical and generalised quantifiers

We now wish to examine how adjacent classical and generalised quantifiers interact. Having compared leading classical and generalised quantifiers, we now wish to examine some interactions among them.

We first consider some transfer principles where the behaviour of the new generalised quantifier is similar to that of the classical ones. In classical first-order logic we have the transfer principle $\vdash \exists u\forall z\varphi \rightarrow \forall z\exists u\varphi$. Since ∇ is between $\forall$ and $\exists$, one might expect some similar transfer principles for ∇.[41] Indeed, we can see that we have the transfer principles for ∇ in up-closed logic:

- $\vdash^C \nabla u\forall z\varphi \rightarrow \forall z\nabla u\varphi$
- $\vdash^C \exists u\nabla z\varphi \rightarrow \nabla z\exists u\varphi$.

---

[39]The octagon for filter logic is follows.

$$\begin{array}{ccccccccc}
 & & \text{Affirmative} & & \text{Negative} & & \\
\neg\exists v\neg\varphi & \leftrightarrow & \forall v\varphi & \overset{contrar.}{\leftrightarrow} & \forall v\neg\varphi & \leftrightarrow & \neg\exists v\varphi \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\neg\nabla v\neg\varphi & \overset{contrar.}{\leftrightarrow} & \nabla v\varphi & \overset{contradict.}{\leftrightarrow} & \nabla v\neg\varphi & \overset{subcontrar.}{\leftrightarrow} & \neg\nabla v\varphi \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\neg\forall v\neg\varphi & \leftrightarrow & \exists v\varphi & \overset{subcontrar.}{\leftrightarrow} & \exists v\neg\varphi & \leftrightarrow & \neg\forall v\varphi
\end{array}$$

In filter logic, formulae $\neg\nabla v\neg\varphi$ and $\nabla v\varphi$ are contraries because of the schemata $[\nabla\wedge]$ and $[\nabla\exists]$.

[40]The hexagon for ultrafilter logic is as follows.

$$\begin{array}{ccccccccc}
 & & \text{Affirmative} & & \text{Negative} & & \\
\neg\exists v\neg\varphi & \leftrightarrow & \forall v\varphi & \overset{contrar.}{\leftrightarrow} & \forall v\neg\varphi & \leftrightarrow & \neg\exists v\varphi \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\neg\nabla v\neg\varphi & \leftrightarrow & \nabla v\varphi & \overset{contradict.}{\leftrightarrow} & \nabla v\neg\varphi & \leftrightarrow & \neg\nabla v\varphi \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\neg\forall v\neg\varphi & \leftrightarrow & \exists v\varphi & \overset{subcontrar.}{\leftrightarrow} & \exists v\neg\varphi & \leftrightarrow & \neg\forall v\varphi
\end{array}$$

In ultrafilter logic, we have the equivalence $\vdash^U \neg\nabla v\varphi \leftrightarrow \nabla v\neg\varphi$ (because of $[\neg\nabla]$, $[\nabla\wedge]$ and $[\nabla\exists]$).

[41]In basic logic, $\forall u\forall z\varphi$ yields $\forall u\nabla z\varphi$, $\nabla u\forall z\varphi$ and $\nabla u\nabla z\varphi$ (by $[\forall\nabla]$), and $\nabla u\nabla z\varphi$ yields $\exists u\nabla z\varphi$, $\nabla u\exists z\varphi$ and $\exists u\exists z\varphi$ (by $[\nabla\exists]$).

These transfer principles are easily interpreted and derived.[42] The converse transfers fail, as we will see shortly.

We now consider a case where the behaviour of the new generalised quantifier contrasts with that of the classical ones. In first-order logic, the classical quantifiers commute.[43] In contrast, iterated generalised quantifiers do not commute: $\nabla u \nabla z \varphi$ does not yield $\nabla z \nabla u \varphi$.

A single example can serve to establish the three failures just mentioned: the naturals with order $<$ and 'generally' meaning cofinite. Expand the structure $\mathcal{N} = \langle \mathbb{N}, < \rangle$ by a Fréchet ultrafilter $\mathcal{U}$: having no finite subset. We then see that structure $\mathcal{N}^{\mathcal{U}}$ satisfies none of

- $\forall u \nabla z u < z \rightarrow \nabla z \forall u u < z$,
- $\nabla u \exists z u < z \rightarrow \exists z \nabla u u < z$,
- $\nabla u \nabla z u < z \rightarrow \nabla z \nabla u u < z$.[44]

4.2 Deductive power

We shall now examine the deductive power of our logics for 'generally'. We will show that we have conservative extensions of classical first-order logic and that we can reduce some simple cases of consequences to classical conditions.

Concerning deductive powers, our extensions of classical first-order logic have increasing deductive powers.[45]

Conservative extension

We will first show that the addition of the new generalised quantifiers produces conservative extensions of classical first-order logic.

[42]The behaviour of ∇ is reminiscent of that of $\exists$ in the first transfer (over $\forall$), and of that of $\forall$ in the second transfer (over $\exists$). These transfer principles follow from schema $[\rightarrow \nabla]$. If some u is related to several z's (via φ), then several z's are related to some u's: $\models^{C} \exists u \nabla z \varphi \rightarrow \nabla z \exists u \varphi$ (as $\mathcal{A}^{\mathcal{K}}[\varphi(\underline{a}, b|z)] \subseteq \mathcal{A}^{\mathcal{K}}[\exists u \varphi(\underline{a}|z)]$). The dual transfer $\vdash^{C} \nabla u \forall z \varphi \rightarrow \forall z \nabla u \varphi$ follows from schema $[\rightarrow \nabla]$ as $\vdash \forall u(\forall z \varphi \rightarrow \varphi)$.

[43]We have $\vdash \forall u \forall z \varphi \leftrightarrow \forall z \forall u \varphi$ and $\vdash \exists u \exists z \varphi \leftrightarrow \exists z \exists u \varphi$.

[44]Indeed, we see that $\mathcal{N}^{\mathcal{U}}$ satisfies $\forall u \nabla z u < z$ (for every $m \in \mathbb{N}$: $\{n \in \mathbb{N} : m < n\}$ is cofinite), so also $\nabla u \nabla z u < z$ and $\nabla u \exists z u < z$; but $\mathcal{N}^{\mathcal{U}}$ fails to satisfy $\exists z \nabla u u < z$ (for no $n \in \mathbb{N}$: $\{m \in \mathbb{N} : m < n\}$ is cofinite), whence $\mathcal{N}^{\mathcal{U}} \not\models \nabla z \nabla u u < z$ and $\mathcal{N}^{\mathcal{U}} \not\models \nabla z \forall u u < z$. This shows that these formulae are not valid in any modulated logic with intermediate module $\mathcal{M}$: $\underline{\mathcal{B}} \subseteq \underline{\mathcal{M}} \subseteq \underline{\mathcal{U}}$.

[45]The increasing deductive powers can be seen by considering the schemata in our axiomatisations (cf. 4.3 in section 4).

It is easy to see that we have conservative extensions of classical first-order logic. [46]

Theorem. Conservative extension (of classical first-order logic) Consider a set $\Xi \subseteq \Phi_U$ of axioms for 'generally'. Given a set of sentences $\Delta \cup \tau \subseteq \mathrm{L}(\rho)$: $\Delta \vdash^{\Xi} \tau$ iff $\Delta \vdash \tau$.[47]

A pleasing consequence of having conservative extensions is that one can reuse classical first-order reasoning.

For instance, imagine that $P(d)$ and $M(d)$ stand for 'd is pleasant' and 'd is mild', respectively, and consider a classical first-order theory Δ (about days) where $\Delta \vdash \forall v[P(v) \leftrightarrow M(v)]$ ["pleasant days are mild days"]. We can then infer $\Delta \vdash^{B} \nabla v \neg P(v) \leftrightarrow \nabla v \neg M(v)$ ["days are generally unpleasant iff they are generally not mild"].[48]

Inference of simple generalised formulae

We will now examine some simple cases of consequences that reduce to classical conditions. The first step beyond first-order adds a single initial ∇. We shall examine some cases of inference and refutation of such formulae with a single initial ∇. We already know that, for classical formulae (without ∇), our logics for 'generally' have the same deductive power as classical first-order logic, but what about formulae with ∇? We will show that, for some formulae (with a single initial ∇), we can reduce inference and refutation to classical first-order conditions.

A *positive generalised formula* is one of the form $\nabla v\varphi$, where φ has no ∇. A *negative generalised formula* is the negation of a positive generalised formula: of the form $\neg\nabla v\varphi$, where φ has no ∇. The *simple generalised formulae* consist of the positive and negative generalised formulae. We shall examine some cases of inference and refutation of such formulae with a single (initial) ∇.

We first show that the absence of generalised information reduces inference and refutation to classical first-order logic.

As an example, consider a consistent purely first-order theory Δ, with three axioms expressing "Mercury is not solid", "Mercury is not the only

---

[46]for classical formulae (without ∇), our ∇-axioms add no extra deductive power.

[47]Every first-order structure $\mathcal{A}$ can be expanded to an ultrafilter structure $\mathcal{A}^{\mathcal{U}} = \langle \mathcal{A}, \mathcal{U} \rangle$ satisfying the same first-order sentences.

[48]This assertion follows from the syntactic schema $[\leftrightarrow \nabla]$.

metal" and "Every metal, other than mercury, is solid". In this case, we cannot decide whether "metals generally are solid", as we will see.

The next result gives conditions for inferring or refuting simple generalised formulae from a purely first-order theory.

Theorem. Simple generalised consequences of first-order theory Consider a set $\Xi \subseteq \Phi_U$ of axioms for 'generally'. Given a set of sentences $\Delta \subseteq L(\rho)$ and a formula $\varphi \in L(\rho)$, we have the following conditions.

I : $\Delta \vdash^{\Xi} \nabla v\varphi$ iff $\Delta \vdash \forall v\varphi$

R : $\Delta \vdash^{\Xi} \neg\nabla v\varphi$ iff $\Delta \vdash \neg\exists v\varphi$

Proof. For I: the universe is the only set in every complex gives $(\Rightarrow)$ and $[\forall\nabla]$ yields $(\Leftarrow)$. For R: each nonempty set is in some complex gives $(\Rightarrow)$ and $[\nabla\exists]$ yields $(\Leftarrow)$.[49]

This result explains the preceding example.[50]

We now examine the effect of adding a single simple generalised formula to a purely first-order theory. The first-order formulae that are consequences of such extensions have similar first-order characterisations.[51]

Proposition. First-order consequences of extension by simple generalised formula Consider a set $\Xi \subseteq \Phi_U$ of axioms for 'generally'. Given a set of sentences $\Delta \subseteq L(\rho)$ and formulae ψ and θ of $L(\rho)$, we have the following conditions.

$+$: $\Delta \cup \{\nabla v\psi\} \vdash^{\Xi} \theta$ iff $\Delta \cup \{\exists v\psi\} \vdash \theta$

[49]For (I$\Rightarrow$): if $\Delta \nvdash \forall v\varphi$, then some model $\mathcal{M} \models \Delta$ can be expanded by an ultrafilter $\mathcal{U}$ with $\mathcal{M}[\varphi(\ s|v)] \notin \mathcal{U}$, and thus $\mathcal{M}^{\mathcal{U}} \not\models \nabla v\varphi[\ s]$. Similarly for (R$\Rightarrow$): if $\Delta \nvdash \neg\exists v\varphi$, then some model $\mathcal{M} \models \Delta$ can be expanded by an ultrafilter $\mathcal{U}$ with $\mathcal{M}[\varphi(\ s|v)] \in \mathcal{U}$, thus $\mathcal{M}^{\mathcal{U}} \not\models \neg\nabla v\varphi[\ s]$.

[50]In the preceding example, the axioms of Δ are $\neg S(h)$, $\exists v \neg v \approx h$ and $\forall v(\neg v \approx h \rightarrow S(v))$. Then, $\Delta \nvdash^{\Xi} \nabla v S(v)$ (since $\Delta \nvdash \forall v S(v)$) and $\Delta \nvdash^{\Xi} \neg\nabla v S(v)$ (since $\Delta \nvdash \neg\exists v\ S(v)$).

[51]To illustrate the next result, consider a purely first-order theory Δ about birds. Imagine that we wish to know what kind of support the belief "birds generally fly" (i. e. $\nabla v F(v)$) may provide to the first-order question θ: "birds have wings". If one accepts $\nabla v F(v)$, then one can conclude θ iff θ follows from Δ and the existential assertion $\exists v F(z)$: "some bird flies". If, on the other hand, one accepts the negation $\neg\nabla v F(v)$, then one can conclude θ iff θ follows from Δ and the existential assertion $\exists v \neg F(v)$: "some bird does not fly".

$- \ : \Delta \cup \{\neg \nabla v \psi\} \vdash^{\Xi} \theta$ iff $\Delta \cup \{\exists v \neg \psi\} \vdash \theta$

Proof. The conditions follow from the preceding result by contraposition.[52] The conditions seen so far apply to basic logic and its extensions. They

can be summarised and interpreted as follows. We already know that ∇ is between $\forall$ and $\exists$; now, a single initial ∇ will behave as either extreme: as $\forall$ (in the case of conclusion) or as $\exists$ (in the case of hypothesis). By examining more closely the expressive power of the generalised quantifier in 5.3, we will be able to see that such reduction of consequences to classical logic is restricted to simple generalised formulae, failing for other, more complex, formulae. We will now examine a more general case: sufficient and necessary conditions for inferring simple generalised formulae from the extension of a purely first-order theory by a single simple generalised formula.

Proposition. Simple generalised consequences of simple generalised extension Given a set $\Xi \subseteq \Phi_U$ of axioms for 'generally', consider a set of sentences Δ of $L(\rho)$ and positive generalised formulae $\nabla v \psi$ and $\nabla v \theta$ of $L^{\nabla}(\rho)$.

$++ \ : \Delta \vdash \exists v \psi \to \forall v \theta \Rightarrow \Delta \cup \{\nabla v \psi\} \vdash^{\Xi} \nabla v \theta \Rightarrow \Delta \vdash \forall v \psi \to \exists v \theta$

$+- \ : \Delta \vdash \exists v \psi \to \forall v \neg \theta \Rightarrow \Delta \cup \{\nabla v \psi\} \vdash^{\Xi} \neg \nabla v \theta \Rightarrow \Delta \vdash \forall v \psi \to \exists v \neg \theta$

$-+ \ : \Delta \vdash \exists v \neg \psi \to \forall v \theta \Rightarrow \Delta \cup \{\neg \nabla v \psi\} \vdash^{\Xi} \nabla v \theta \Rightarrow \Delta \vdash \forall v \neg \psi \to \exists v \theta$

$-- \ : \Delta \vdash \exists v \neg \psi \to \forall v \neg \theta \Rightarrow \Delta \cup \{\neg \nabla v \psi\} \vdash^{\Xi} \neg \nabla v \theta \Rightarrow \Delta \vdash \forall v \neg \psi \to \exists v \neg \theta$

Proof. Sufficiency follows from [$\forall\nabla$] and [$\nabla\exists$]. For necessity: the universe is in every complex and the empty set is in no complex.[53]

The conditions considered this far apply to basic logic and its extensions. We will now examine necessary and sufficient conditions for inferring simple generalised formulae from the extension of a purely first-order theory by a single simple generalised formula. These conditions hold for specific extensions of basic logic.[54]

[52] By contraposition: condition (R) yields $(+)$ and $(-)$ follows from (I).

[53] Sufficiency of $(++)$: if $\Delta \vdash \exists v \psi \to \forall v \theta$, then $\Delta \cup \{\nabla v \psi\} \vdash^{\Xi} \forall v \theta$ (by [$\nabla\exists$]), whence $\Delta \cup \{\nabla v \psi\} \vdash^{\Xi} \nabla v \theta$ (by [$\forall\nabla$]). Necessity of $(++)$: if $\Delta \nvdash \forall v \psi \to \exists v \theta$, then some model $\mathcal{M} \models \Delta$ can be expanded by an ultrafilter $\mathcal{U}$ so that $\mathcal{M}^{\mathcal{U}} \models \nabla v \psi[\, s]$ but $\mathcal{M}^{\mathcal{U}} \not\models \nabla v \theta[\, s]$, whence $\Delta \cup \{\nabla v \psi\} \nvdash^{\Xi} \nabla v \theta$. The other cases are similar.

[54] To illustrate the next result, consider purely first-order information Δ about workers in a plant. Assume that one observes that "workers generally are careless": $\nabla v C(v)$. One can then conclude (in up-closed logic) that "workers generally are accident prone" (i. e. $\nabla v A(v)$) iff Δ entails the universal assertion $\forall v[C(v) \to A(v)]$: "all careless workers are accident prone".

Theorem. Behaviour of simple generalised formulae (in specific logics) Given a set of sentences $\Delta \subseteq \mathrm{L}(\rho)$, consider positive generalised formulae $\nabla v\psi$ and $\nabla v\theta$ of $\mathrm{L}^{\nabla}(\rho)$. We then have the following conditions.[55]

$+I$: $\Delta \cup \{\nabla v\psi\} \vdash^{C} \nabla v\theta$ iff $\Delta \vdash \forall v(\psi \to \theta)$

$+R$: $\Delta \cup \{\nabla v\psi\} \vdash^{F} \neg\nabla v\theta$ iff $\Delta \vdash \neg\exists v(\psi \wedge \theta)$

$-I$: $\Delta \cup \{\neg\nabla v\psi\} \vdash^{U} \nabla v\theta$ iff $\Delta \vdash \forall v(\psi \vee \theta)$

$-R$: $\Delta \cup \{\neg\nabla v\psi\} \vdash^{C} \neg\nabla v\theta$ iff $\Delta \vdash \neg\exists v(\neg\psi \wedge \theta)$

Proof. In each case, specific schemata yield ($\Leftarrow$), for ($\Rightarrow$): if the first-order condition fails, some model of Δ can be expanded to an appropriate ultrafilter model falsifying the generalised inference.[56]

Let us now examine the case of extending a purely first-order theory by a set of simple generalised formulae. For this case, we also have necessary and sufficient conditions for inferring simple generalised formulae in specific logics for 'generally'.

Corollary. Behaviour of extensions by simple generalised formulae Given a set of sentences $\Delta \subseteq \mathrm{L}(\rho)$, consider a positive generalised formula $\nabla w\varphi$ of $\mathrm{L}^{\nabla}(\rho)$ and a set $\Gamma_{+} \subseteq \mathrm{L}^{\nabla}(\rho)$ of positive generalised formulae.

{+} Then $\Delta\cup\Gamma_{+} \vdash^{F} \nabla w\varphi$ iff, for some finite subset $\{\nabla u_1\psi_1, \ldots, \nabla u_m\psi_m\}$ of Γ_{+} and a new variable z,

$$\Delta \vdash \forall z[(\psi_1[u_1/z] \wedge \ldots \wedge \psi_m[u_m/z]) \to \varphi[w/z]].$$

{+−} Given also a set $\Gamma_{-} \subseteq \mathrm{L}^{\nabla}(\rho)$ of negative generalised formulae, we have $\Delta\cup\Gamma_{+}\cup\Gamma_{-} \vdash^{U} \nabla w\varphi$ iff, for some finite subsets $\{\nabla u_1\psi_1, \ldots, \nabla u_m\psi_m\}$ of Γ_{+} and $\{\neg\nabla v_1\theta_1, \ldots, \neg\nabla v_n\theta_n\}$ of Γ_{-}, and a new variable z,

$$\Delta \vdash \forall z[(\psi_1[u_1/z] \wedge \ldots \wedge \psi_m[u_m/z] \wedge \neg\theta_1[v_1/z] \wedge \ldots \wedge \neg\theta_n[v_n/z]) \to \varphi[w/z]].$$

Proof. Compactness gives the finite subsets and specific schemata reduce the finite case to the preceding result ($+I$).[57]

---

[55] In each case, the first-order condition is necessary in basic logic.

[56] For ($+I$): schema $[\to \nabla]$ yields ($\Leftarrow$); for ($\Rightarrow$), if $\Delta \nvdash \forall v(\psi \to \theta)$, then some model $\mathcal{M} \models \Delta$ can be expanded by an ultrafilter $\mathcal{U} \in \underline{C}$ so that $\mathcal{M}^{\mathcal{U}} \models \nabla v\psi[\,s]$ but $\mathcal{M}^{\mathcal{U}} \not\models \nabla v\theta[\,s]$, whence $\Delta \cup \{\nabla v\psi\} \nvdash^{C} \nabla v\theta$. The other cases are similar.

[57] For {+}: $\Delta \cup \{\nabla u_1\psi_1, \ldots, \nabla u_m\psi_m\} \vdash^{F} \nabla w\varphi$ iff [by filter schemata], for a new variable z, $\Delta \cup \{\nabla z(\psi_1[u_1/z] \wedge \ldots \wedge \psi_m[u_m/z])\} \vdash^{F} \nabla w\varphi[w/z]$ iff [by the preceding result ($+I$)] $\Delta \vdash \forall z[(\psi_1[u_1/z] \wedge \ldots \wedge \psi_m[u_m/z]) \to \varphi[w/z]]$. The case of {+−} is similar.

We also have conditions for refuting a simple generalised formula, [58]

4.3 Expressive power

We shall now consider the expressive power of our logics for 'generally'. We will examine some conditions for eliminating the new generalised quantifier from some simple formulae and then show that we have proper extensions of classical first-order logic.

One would expect our logics for 'generally' to be strictly more expressive than classical first-order logic.[59] We will see that interesting generalised formulae equivalent to purely first-order formulae are indeed quite rare.

Elimination of 'generally'

We will first examine some conditions for eliminating the new quantifier ∇ from simple generalised formulae.

As an example, consider a consistent purely first-order theory Δ about birds asserting that "Some birds fly" and "Some birds do not fly". In this case, we cannot express within Δ "Birds generally fly", as we will see.

The next result gives conditions for eliminating the single initial ∇ from a positive generalised formula within a purely first-order theory.

Proposition. Reduction of positive generalised formula to first-order Consider a set $\Xi \subseteq \Phi_U$ of axioms for 'generally'. Given a set of sentences $\Delta \subseteq \mathrm{L}(\rho)$ and a formula $\psi \in \mathrm{L}(\rho)$, there exists a formula $\theta \in \mathrm{L}(\rho)$ such that $\Delta \vdash^{\Xi} \nabla v\psi \leftrightarrow \theta$ iff $\Delta \vdash \exists v\psi \rightarrow \forall v\psi$.

Proof. The conditions follow from previous results: (I) and $(+)$ in 5.2.[60] Thus, one can eliminate the single initial ∇ from $\nabla v\psi$ only when formula ψ becomes trivialised: this explains the preceding example.[61]

[58] $\Delta \cup \Gamma_+ \vdash^F \neg\nabla w\varphi$ iff, for a finite subset $\{\nabla u_1\psi_1, \ldots, \nabla u_m\psi_m\} \subseteq \Gamma_+$ and a new variable z, $\Delta \vdash \forall z[(\psi_1[u_1/z]\wedge\ldots\wedge\psi_m[u_m/z]) \rightarrow \neg\varphi[w/z]]$ and $\Delta\cup\Gamma_+\cup\Gamma_- \vdash^U \neg\nabla w\varphi$ iff, for finite subsets $\{\nabla u_1\psi_1, \ldots, \nabla u_m\psi_m\} \subseteq \Gamma_+$ and $\{\neg\nabla v_1\theta_1, \ldots, \neg\nabla v_n\theta_n\} \subseteq \Gamma_-$ and a new variable z, $\Delta \vdash \forall z[(\psi_1[u_1/z]\wedge\ldots\wedge\psi_m[u_m/z]\wedge\neg\theta_1[v_1/z]\wedge\ldots\wedge\neg\theta_n[v_n/z])\neg\varphi[w/z]]$.

[59] We know that satisfaction of a formula with the generalised quantifier ∇ depends on the complex, which is not the case for purely first-order formulae. So, it is to be expected that some formulae with ∇ will not be equivalent to formulae without ∇.

[60] Each result gives one direction of the desired equivalence.

[61] In the preceding example, as consequences of consistent Δ we have $\exists vF(v)$ and $\exists v\neg F(v)$, so $\Delta \not\vdash \exists vF(v) \rightarrow \forall vF(v)$; thus the result shows that one cannot express $\nabla vF(v)$ without ∇.

Proper extension

We will now show the expressive power of our logics for 'generally' extends properly that of classical first-order logic.

As mentioned, one expects the expressive power of our logics for 'generally' to be strictly more than that of classical first-order logic. It remains to exhibit specific examples of formulae from which the new generalised quantifier cannot be eliminated.

We will exhibit a formula that cannot be expressed within classical first-order logic: the (single) ∇ of $\exists u \nabla zu \approx z$ cannot be eliminated.

Theorem. Non-eliminable ∇
Consider the sentence $\exists u \nabla zu \approx z$. Given a set $\Xi \subseteq \Phi_U$ of axioms for 'generally' and a set of sentences $\Delta \subseteq \mathrm{L}(\rho)$ with infinite models, there exists no sentence $\tau \in \mathrm{L}(\rho)$ such that $\Delta \vdash^{\Xi} \exists u \nabla zu \approx z \leftrightarrow \tau$.

Proof. This sentence expresses that the complex has a singleton, and an infinite universe has principal and non-principal ultrafilters.[62]

In particular, there is no sentence $\tau \in \mathrm{L}(\rho)$ such that $\emptyset \vdash^{\Xi} \exists u \nabla zu \approx z \leftrightarrow \tau$.

We mentioned (in 5.2) that the reduction of consequences to classical logic is restricted to simple generalised formulae. The above sentence provides examples where such reductions fail.

Let γ be the sentence $\exists u \nabla zu \approx z$. Consider the conditions in 5.2 for inferring a simple generalised formula from a purely first-order theory. In contrast to (I), we have no sentence τ of $\mathrm{L}(\rho)$, such that $\Delta \vdash^{\Xi} \gamma$ iff $\Delta \vdash \tau$, for every set $\Delta \subseteq \mathrm{L}(\rho)$ of sentences having infinite models (e. g. $\Delta := \emptyset$).[63]

5 Conclusion

Logics for 'generally' are intended to express assertions with some vague notions, such as 'generally', by means of new generalised quantifiers, and to reason about them. The primary motivation is logics for precise treatment

[62] Sentence $\exists u \nabla zu \approx z$ expresses that the filter is generated by a singleton, and an infinite universe has both principal and non-principal ultrafilters [B+S'71].

[63] Such a purely first-order sentence τ would provide an elimination of ∇ from $\exists u \nabla zu \approx z$ within the first-order theory Δ. Also, in contrast to (+), given a sentence σ of $\mathrm{L}(\rho)$, we have no sentence τ of $\mathrm{L}(\rho)$, such that $\Delta \cup \{\gamma\} \vdash^{\Xi} \sigma$ iff $\Delta \cup \{\tau\} \vdash \sigma$, for every set of sentences $\Delta \subseteq \mathrm{L}(\rho)$ such that $\Delta \cup \neg\{\sigma\}$ has infinite models.

of some vague notions (such as 'generally', 'several', 'many', 'most', etc.), which appear often in ordinary language and in some branches of science, much as one has logics embodying some mathematical notions [B+F'85].

Here, we have reviewed some logics for 'generally' and examined some metamathematical issues about them: axiomatisations, behaviour of the new quantifier, as well as deductive and expressive powers.

We have seen that our logics for 'generally' are proper conservative extensions of classical first-order logic with sound and complete deductive systems. Also, our logics are proper extensions of classical first-order logic with compactness and Löwenheim-Skolem properties. This feature may confer to our logics for 'generally' some independent model-theoretic interest.[64]

We also have considered some other metamathematical properties of our logics for 'generally'. We have compared them to classical first-order logic, with focus on the behaviour of the new generalised quantifiers, pointing out similarities and contrasts. We have examined extensions of the classical square of oppositions covering the new generalised quantifiers and some transfer principles involving classical and generalised quantifiers.

We have considered inference of simple generalised formulae, examining simple cases of consequences that reduce to classical conditions. This has led to conditions for eliminating the new quantifier ∇ from simple generalised formulae. We have also established the expressive power of our logics for 'generally' extends properly that of classical first-order logic by exhibiting formulae that cannot be expressed within classical first-order logic: with non-eliminable ∇.

We thus have logics for reasoning precisely about some versions of 'generally'. These logics are conservative extensions of classical first-order logic, with which they share various properties. This family of logics is undergoing further investigation [V+C'01,V+V'01, V+V'01a, RHV'01, Vel'02, Vel'02a, Vel'02b, V+V'02, V+V'02a].[65] They appear to have some interesting connections with fuzzy logic as used in expert systems, natural language and empirical reasoning. Such connections suggest the possibility of other applications ([C+V, Vel'98, Vel'99, V+C'01]).

[64]The apparent conflict with Lindström's results ([Lin'66], [Bar'77]) is explained because we are using a non-standard notion of model (due to the complexes).

[65]These developments include proof methods and sorted versions (to express relative 'generally', since relativisation fails to express the intended meaning, due to properties of ∇ and $\to$, [C+V'97, V+C'01])

References

[Ant'97] ANTONIOU, G. *Nonmonotonic Reasoning.* Cambridge, MA: MIT Press, 1997.

[Bar'77] BARWISE, J. (ed.) *Handbook of Mathematical Logic.* Amsterdam: North-Holland, 1977.

[BC'81] BARWISE. J. and COOPER, R. Generalized quantifiers and natural language. *Linguistics and Philosophy* 4: 159–219, 1981.

[B+F'85] BARWISE. J. and FEFERMAN, S. (eds.) *Model-Theoretic Logics.* New York: Springer-Verlag, 1985.

[B+S'71] BELL, J. L. and SLOMSON, A. B. *Models and Ultraproducts: an introduction.* Amsterdam: North-Holland, 1971 (2nd rev. pr.).

[C+G'00] CARNIELLI, W. and GRÁCIO, M. C. G. Modulated logics and uncertain reasoning. In *Proc. Kurt Gödel Colloquium*, Barcelona, 2000.

[C+K'73] CHANG, C. C. and KEISLER, H. J. *Model Theory.* Amsterdam: North-Holland, 1973.

[C+S'94] CARNIELLI, W. A. and SETTE, A. M. Default operators. In *Abstr. Workshop on Logic, Language, Information and Computation*, Recife, 1994.

[C+V'77] CARNIELLI, W. A. and VELOSO, P. A. S. Ultrafilter logic and generic reasoning. In Gottlob, G., Leitsch, A. and Mundici, D. (eds.) *Computational Logic and Proof Theory* (LNCS 1289): 34–53, Berlin: Springer-Verlag , 1997.

[End'72] ENDERTON, H. B. *A Mathematical Introduction to Logic.* New York: Academic Press, 1972.

[Fuh'03] FUHRMANN, A. Some remarks on ultrafilter logic. *Studia Logica* 73: 197–207, 2003.

[Gra'99] GRÁCIO, M. C. G. *Modulated Logics and Reasoning under Uncertainty* (in Portuguese). D. Sc. dissertation, Unicamp, Campinas, Oct. 1999.

[Gra'03] GRÁCIO, M. C. G. Implications of modulated logics in a natural language fragment (in Portuguese). In *XIII Encontro Brasileiro de Lgica, Resumos*: 69–70, Campinas, 2003 {cabrini@marilia.unesp.br}.

[Hen'49] HENKIN, L. The completeness of the first-order functional calculus. *J. Symb. Logic* 14: 59–166, 1949.

[Kei'70] KEISLER, H. J. Logic with the quantifier 'there exist uncountably many'. *Annals of Math. Logic* 1: 1–93, 1970.

[Lin'66] LINDSTRÖM, P. On extensions of elementary logic. *Theoria* 35: 1–11, 1966.

[Mos'57] Mostowski, A. On a generalization of quantifiers. *Fund. Mathemat.* 44: 12–36, 1957.

[Pet'79] Peterson, P. L. On the logic of 'few', 'many', and 'most'. *Notre Dame J. Formal Logic* 20: 407–428, 1979.

[Pop'34] Popper, K. R. *Logik der Forschung.* Tbingen: J. C. B. Molir, 1934 (5. Aufl. 1973) [English translation *The Logic of Scientific Discovery.* New York: Basic Books, 1959 (6. edn. 1972)].

[Pop'75] Popper, K. R. *Objective Knowledge: an evolutionary approach.* Oxford: Clarendon, 1975.

[RHV'01] Rentería, C. J., Haeusler, E. H. and Veloso, P. A. S. NUL- natural deduction for ultrafilter logic. In *Natural Deduction 2001*, Rio de Janeiro, 2001 [cf. Dedução natural para lógica de ultrafiltros, Res. Rept. MCC 16/02, PUC-Rio, Rio de Janeiro, 2002].

[Rei'80] Reiter, R. A logic for default reasoning. *Artif. Intellig.* 13: 81–123, 1980.

[SCV'99] Sette, A. M., Carnielli, W. A. and Veloso, P. A. S. An alternative view of default reasoning and its logic. In Haeusler, E. H. and Pereira, L. C. (eds.) *Pratica: Proofs, Types and Categories*: 127–158, Rio de Janeiro: PUC-Rio, 1999.

[Sho'67] Shoenfield, J. R. *Mathematical Logic.* Reading: Addison-Wesley, 1967.

[Vel'98] Veloso, P. A. S. On ultrafilter logic as a logic for 'almost all' and 'generic' reasoning. Res. Rept. ES-488/98, COPPE- UFRJ, Rio de Janeiro, 1998.

[Vel'99] Veloso, P. A. S. On 'almost all' and some presuppositions. Manuscrito XXII: 469–505, 1999 [special issue: Pereira, L. C. P. D. and Wrigley, M. B. (eds.) *Logic, Language and Knowledge: essays in honour of Oswaldo Chateaubriand Filho].*

[Vel'00] Veloso, P. A. S. On some misconceptions about ultrafilter logic. *Bull. Sct. Logic* 29(1-2): 1–12, 2000.

[Vel'01] Veloso, P. A. S. On interpolation and modularity for ultrafilter logic. In Abe, J. M. and Silva Filho, J. I. (eds.) *Logic, Artificial Intelligence and Robotics - LAPTEC'2001*: 270–278, Amsterdam: IOS Press, 2001.

[Vel'02] Veloso, P. A. S. Issues in reasoning with 'generally' and 'rarely'. In Cupani, A. O. and Mortari, C. A. (eds.) *Linguagem e Filosofia - Anais do II Simpósio Internacional Principia*: 51–72, Florianópolis: UFSC - Núcleo de Epistemologia e Lógica (Coleção Rumos da Epistemologia, vol. 6), 2002.

[Vel'02b] VELOSO, P. A. S. On a logic for 'almost all' and 'generic' reasoning. *Manuscrito* XXV(1): 191–271, 2002.
[V+C'01] VELOSO, P. A. S. and CARNIELLI, W. A. Logics for qualitative reasoning. *CLE e-Prints* 1, 2001 {http://www.cle.unicamp.br/e-prints}.
[V+V'01] VELOSO, P. A. S. and VELOSO, S. R. M. On qualitative reasoning with 'most' and 'typical': a logical approach. In *XXI Congresso da Sociedade Brasileira de Computação*: 1122–1133, Fortaleza, CE, 2001.
[V+V'01a] VELOSO, S. R. M. and VELOSO, P. A. S. On a logical framework for 'generally'. In ABE, J. M. and SILVA FILHO, J. I. (eds.) *Logic, Artificial Intelligence and Robotics - LAPTEC'2001*: 279–286, Amsterdam: IOS Press, 2001.
[V+V'02] VELOSO, S. R. M. and VELOSO, P. A. S. Qualitative logic for 'generally'. In ARABNIA, H. R. and MUN, Y. (eds.) *Proc. IC-AI'02 - International Conferences on Artificial Intelligence (vol. III)*: 11246-1252, Las Vegas, Nev., 2002.
[V+V'02a] VELOSO, S. R. M. and VELOSO, P. A. S. On special functions and theorem proving in logics for 'generally'. In BITTENCOURT, G. and RAMALHO, G. L. (eds.) *Advances in Artificial Intelligence - 16th Brazilian Symp. Artif. Intellig.* (LNAI 2507): 1–10, Berlin: Springer-Verlag, 2002.