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Logic and Arithmetic

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Since there are non-sortal predicates Frege's attempt to derive Arithmetic from Logic stumbles at its very first step. There are properties without a number, so the contingency of that condition shows Frege's definition of zero is not obtainable from Logic. But Frege made a crucial mistake about concepts more generally which must be remedied, before we can be clear about those specific concepts which are numbers.

1

Is the concept of being a prime number a function which takes the value True when applied to the argument 5? Frege thought the predicate 'is a prime number' was a denoting phrase, and denoted a concept, i.e. something which is 'unsaturated'. But while the predicate is incomplete, and so unsaturated – simply because it is just a part of a sentence – it does not denote anything at all. The concept of being a prime number is denoted not by the predicate but by its nominalisation, and so it is 'saturated' and cannot be a function – it is an object. Cocchiarella wrote the referring phrase 'λxFx', and the associated predicate 'λxFx( )' (Cocchiarella 1987, 83); in Kneale's terminology (Kneale & Kneale 1962, 602), if the predicate is 'F' then the referential '§xFx' refers to the concept. Refined discriminations between referring phrases to concepts and open predicates thus enable us to oppose Frege's thought that concepts are categorically distinct from objects. When we are talking about concepts we are nominalising the associated predicates, and it is those open predicates which are distinct from referring phrases to objects.

Frege was plainly not too clear about some of these discriminations. For he believed that concepts were not objects, and specifically that numbers were not concepts but objects. He believed that numbers were not concepts but objects, even though he could formulate no clear way of separating the objects which were numbers from other objects, like Julius Caesar. In fact, Julius Caesar, the concept horse, the number of the planets, etc. are all objects, but they are objects of different orders, and only the first may be presented independently of predications. It was Frege's identification of numbers with objects rather than concepts, however, which supported the specific reasoning which led him to separate objects from concepts, since he thought of objects as 'saturated' while concepts were not. So we must inspect his reasons for that identification. Frege said (Geach & Black 1952, 24-5):

> The two parts into which a mathematical expression is thus split up, the sign of the argument, and the expression of the function, are dissimilar; for the argument is a number, a whole complete in itself, as the function is not... For instance, if I say 'the function $2.x^3 + x$', x must not be considered as belonging to the function; this letter only serves to indicate the kind of supplementation that is needed; it enables one to recognise the places where the sign for the argument must go in.

But, by Frege's own grammatical criterion, the expression 'the function $2.x^3 + x$', being a definite description, ought to denote an object, even though, in his representation of it, it contains some gaps, and so is unsaturated. Moreover, although what might fill the gaps, namely numbers like the number 7, arise in arithmetical statements such as '7 + 5 = 12', this merely uses '7' as a substantive, and Frege elsewhere recognised that the numerals, in their adjectival use, were parts of second-order predicates, so that these terms also can form parts of incomplete expressions. The phrase 'there are (exactly) seven', for instance, needs a substantive added to it, such as 'horses', to make a complete thought. What Frege did not fully appreciate, therefore, was that corresponding to the referential and descriptive uses of numerals, there are complete and incomplete expressions with all predicates.

Thus, following Cocchiarella, there is the functional expression '$\lambda x(2.x^3 + x)(\ )$', which is not a referential phrase, and so does not denote any object at all, and there is the definite description 'the operation of doubling the cube of a number and adding it to that number', i.e. '$\lambda x(2.x^3 + x)$', which contains no gaps, and therefore refers to an object – a mathematical operation, which is one kind of abstract object. Being 7 in number ($\lambda Q(7x)Qx$) is another abstract object: it is that property of discrete and distinctive things of having a correlation with the non-zero numerals up to 'seven', while in the predication 'The Ps are 7 in number', i.e. '(7x)Px', or its equivalent '$\lambda Q(7x)Qx(P)$', the same property is not referred to but expressed. The natural numbers themselves therefore satisfy Hume's Principle, i.e.

$$Nx{:}Fx=Nx{:}Gx \equiv (\exists R)(R \text{ is } 1\text{-}1.R(F, G)).$$

2

We can now proceed to look more closely at one central consequence of the above definition of number – the fact that things with a number must be discrete. The point about discreteness is crucially involved in, amongst other things, the proof that Julius Caesar is not a number.

According to Wright (Wright 1983, 11), there were three specific considerations which were involved in Frege's judgement that numbers were objects. One was the use of definite descriptions like 'the number of the planets'. Another was the currency of numerical identities, like '5 + 7 = 12'. The remaining consideration Frege appealed to was the contrast between, for instance, 'the number of planets is 9' (Nx:Px = 9) and 'there are exactly 9 planets' ((9x)Px). Only the former represents '9' as a singular referential term, and so Frege took it to be the basis of his formal analysis of Arithmetic. But, as we shall now see, it is the predicative form which has priority, and it is that fact which also shows that numbers, while still objects, are categorically distinct from objects like Julius Caesar, since they then cannot be known independently of predications. By contrast, one does not need to know someone is Julius Caesar before one can be acquainted with him.

The priority of the predicative form arises because the foundation for the theory of number is to be found in appropriate definitions of quantificational expressions like '(nx)Fx' (i.e. 'there are exactly n Fs'), from which expressions like 'εm(mx)Fx=n' (i.e. the number of Fs is n) follow quite straightforwardly. For '(nx)Fx' entails '(∃m)(mx)Fx', by existential generalisation, and so '([εm(mx)Fx]y)Fy', by the epsilon definition of the existential quantifier, which equates '(∃x)Px' with 'PεxPx'. One can then derive 'εm(mx)Fx=n' because of the uniqueness of the exact numerical quantifier. The reverse entailment crucially does not hold, however, because of the numerical indeterminacy of non-sortals: one can have 'εm(mx)Fx=n' without the epsilon term numbering the Fs, i.e. without '([εm(mx)Fx]y)Fy', since there may not be any Fs, but merely some F. The numerical identity then can still arise, but only through the arbitrary specification of a value for the epsilon term, in a case where ¬(∃m)(mx)Fx, i.e. where the predicate 'F' is not count, and so does not discriminate discrete things. Such a case is when 'F' is a mass term, and there is only an amount of stuff, in which case 'the number of Fs' must be non-attributive. Thus we do not have the epsilon equivalence '(nx)Fx ≡ εm(mx)Fx=n', but merely a one-way implication '(nx)Fx ⊃ εm(mx)Fx=n'. Certainly one can have the iota equivalence '(nx)Fx ≡ ιm(mx)Fx=n', but this does not involve an individual term on the right hand side, since it is the same as '(nx)Fx ≡ (∃m)((mx)Fx.m=n)'.

The crucial difference between epsilon terms and iota terms is that epsilon terms are complete terms for individuals, unlike iota terms, which are incomplete terms, as this last point shows. That means epsilon terms may be non-descriptive, and so can formalise Millian names; in fact they are the logically proper names Russell hypothesised, but did not have a symbolism for. The epsilon definition of the existential quantifier means that '¬(∃x)Px' equates with '¬PεxPx', so in the present case, even if there is no number of Fs

(¬(∃m)(mx)Fx), still 'εm(mx)Fx' will refer, although then, like 'The Morning Star', for instance, its reference will be given accidentally. Thus, just as Venus is not a star, although 'The Morning Star' conventionally refers to it, so 'the number of F' when 'F' denotes some stuff, does not refer to a number which numbers discrete things. One specific consequence of the possibility of such deceptive, Millian 'number names', which dramatises the matter, is that their arbitrary reference might well be, on occasion, a physical object – for instance even Julius Caesar.

3

How can the basis for the theory of number lie in numerical quantification? In fact David Bostock essayed a deduction of Arithmetic from Logic in this quantificational style (see also, more recently, Agustin Rayo 2002). Bostock defined the numerical quantifiers in the Fregean fashion, and used a generalised theory of quantification, applicable to the numerical place in such expressions as '(nx)Fx', to deduce Peano's Postulates, with certain further assumptions.

Bostock was much more appreciative of the difference between numbers and amounts than other logicians (see Bostock 1974, and Bostock 1979, respectively, as a whole). But he nevertheless did not appreciate the above points about the differences between count and mass terms. For, right at the start, he tried to define a weak numerical quantifier with '(∃1x)Fx ≡ (∃x)Fx', and a strong numerical quantifier with '(0x)Fx ≡ ¬(∃x)Fx', (Bostock 1974, 9-10). So the given foundation for Bostock's deduction was unsafe – as unsafe, as we shall see, as Frege's. If 'F' is a mass term, then '(∃x)Fx' and '¬(∃x)Fx' simply read 'there is some F' and 'there is no F', and even '(∃x)(Fx.(∃y)(Fy.y≠x))' merely reads 'there is some F, and some more F'. In none of these cases, therefore, do numbers or pluralisation enter the content. There are guaranteed to be Fs, however, if there are two or more Fs, so there is no difficulty with the strong numerical quantifiers above 1, and a construction of Arithmetic in Bostock's style remains possible. If ‘$\underline{n}$’ ranges from 2 upwards then F is count iff $(\exists\underline{n})(\underline{n}x)Fx$ or $(\exists x)(y)(Fy \equiv y=x) . M(\exists\underline{n})(\underline{n}x)Fx$ or $\neg(\exists x)Fx.M(\exists\underline{n})(\underline{n}x)Fx$, and abbreviating the latter disjuncts to ‘$(1x)Fx$’, and ‘$(0x)Fx$’, respectively gives us the simplified definition that ‘F’ is count iff $(\exists n)(nx)Fx$, where ‘n’ ranges from 0 upwards. That means that Bostock’s definitions above will hold only on the supposition that $(\exists n)(nx)Fx$. The restriction in the case of the number 1 is required not just because of the possibility of mass terms, but also proper names, since ‘is Peter’, for instance, will hold without ‘is one Peter’ holding. The restriction with the number zero is required because for non-sortals there may be no F without there being zero Fs.

Why are there guaranteed to be Fs if not just $(\exists x)(Fx.(\exists y)(Fy.y\neq x))$, but $(\exists x)(Fx.(\exists y)(Fy.y\neq x.(z)(Fz \supset z=x \vee z=y)))$? Consider two rings of gold (or two atoms of water, say). Since these are both gold (water) there is clearly a third object which is also gold (water), namely the mereological sum of the previous two objects. So if there are <u>just</u> two ($(2x)Fx$), then the predicate must be count. The case of 1, as it is standardly formalised ($(\exists x)(y)(Fy \equiv y=x)$), still allows the predicate to be a mass term, since if there is just one atom of water, then while portions of that atom are not water themselves, and so only the whole atom is water, that whole atom is still ‘some water’ and not ‘one water’. In a somewhat similar manner, although 2 is the only even prime it is still ‘an’ even prime, not ‘one’ even prime. So always the possibility of there being two items is required before we can start to count with a term. The point even holds when there can be nothing of the kind in question. For if we could say there were no round squares we could rightly say there were zero round squares. But in fact there is merely no round square, from which it does not follow there is zero round square.

4

We can now finally see how the restricted definition of zero, which emerges from such considerations about plurality, undermines entirely all Fregean, and Neo-Fregean attempts to derive Arithmetic from Logic. In the above terms, Frege presumed that '(∃n)(nx)Fx' held for all predicates, and the leading Neo-Fregean Crispin Wright is notable for being amongst the first to publicise the fact that this is not so. But this criticism has more radical consequences than Wright realised for the development of Arithmetic using Hume's Principle.

Boolos and Wright, with others, have demonstrated how most of Frege's development of Arithmetic can be obtained from Hume's Principle, starting from Frege's definition of zero as the number of things which are not self-identical (Nx:x≠x). But in this extensive, and now very elaborate discussion, no question has been raised about whether '¬(∃x)(x≠x)', entails 'Nx:x≠x = 0'. If the negative existence statement entails the numerical statement, then Nx:x≠x must be determinate, and that is contingent on 'x≠x' being a sortal predicate, as Wright has admitted. But what unit is determined by non-self-identity? No argument for there being one has been given, either by Wright, or by any one else within this tradition. Indeed, at one time it was simply presumed that all predicates were sortal. But Wright has recently given a proof that self-identity is not a sortal concept (Hale & Wright 2001, 315). As a result (as Wright himself explicitly realised earlier, see Wright 1983, 187), argument is needed to show that non-self-identity is a sortal concept. On the above definition it clearly is not. Much ink has been spilled debating whether Hume's Principle is analytic, and so whether the Arithmetic taken to be derivable from it can, or cannot, be properly described as a part of Logic. But if Logic does not discriminate between sortal and non-sortal concepts, then there is no way to get from it the other crucial element in Frege's generation of the number series: its starting point.

We say 'the number of Fs is n' and can do so whether n is 0, 1, 2, or more; but only count nouns pluralise in the appropriate way. Mass terms sometimes appear pluralised, but not in the same sense: in 'there are several champagnes', for instance, we are speaking about glasses of champagne, maybe, or varieties of champagne. In English we can say 'it is F' rather than 'it is an F', but nothing corresponding to this is to be found in Frege's language. Maybe there is no martini in a glass. Does that mean the number of martini in the glass is zero? It does not. There is no such thing as a number, i.e. a number of Fs, in this case. There might have been some F, rather than no F, and in both cases as much F as G, but the required plural case, and so the possibility of a number, and the same number, just does not arise. Could we not simply introduce a count noun, and talk about the number of Cs which are F instead? Certainly if there is no gold then the number of ingots which are gold is zero. Wright has discussed this matter more than most, and he has admitted: 'to number the instances of some non-sortal concept is intelligible only if relativised to a sortal' (Wright 1983, 3; see also Hale & Wright 2001, 315, 387). But the necessary distinction between substantives and adjectives is just what is lacking. Maybe '$\neg(\exists x)Fx$', is equivalent to '(C)(Nx: (Cx.Fx) = 0)', where 'C' ranges over count terms, but one cannot say this in Frege's concept-script.

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