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GROUNDWATER FLOW IN HETEROGENEOUS, ANISOTROPIC FRACTURED MEDIA: A SIMPLE TWO-DIMENSIONAL ELECTRIC ANALOG

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Abstract: Freeze and Witherspoon[1,2]) have simulated regional groundwater flow in two- and three-dimensional heterogeneous and anisotropic groundwater basins, using mathematical models. The principal directions of the permeability tensor were always parallel to the network axes (i.e. horizontal and vertical).
The study of the relations between fracture geometry and permeability tensor in fractured media[3]) led us to simulate any desired permeability tensor by a certain fracture geometry. Changing the fracture geometry from place to place, we may simulate any heterogeneous, anisotropic permeability field in a groundwater basin. In a two-dimensional basin, the potential field is easily determined by an electric analog made of conducting paper.

1. Fracture geometry and permeability tensor

As we have shown[3]), the permeability tensor can be calculated from fissure geometry, assuming the following hypothesis:
H1: flow obeys Darcy's law in the fissures
H2: fissures are plane and continuous in an "elementary" volume
H3: hydraulic conductivity is isotropic in the plane of the fissure
H4: flow is saturated.

Figure 1 gives a two-dimensional diagram for the computations, but gener-

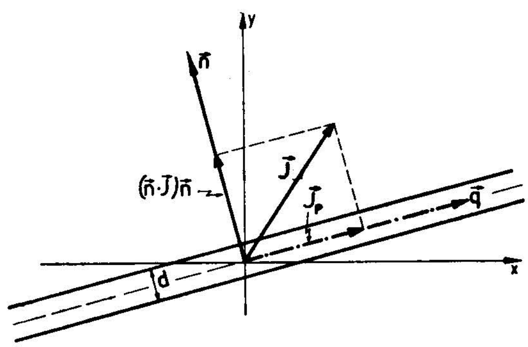

Fig. 1. Two-dimensional diagram for the computation of permeability tensor from fracture geometry.

alization for three-dimensional cases is immediate. In the generalized Darcy's law the permeability tensor K transforms linearly the gradient vector $\boldsymbol{J}$ to the discharge vector $\boldsymbol{q}$:

$$\boldsymbol{q} = K\boldsymbol{J}.$$

In one fissure the discharge vector $\boldsymbol{q}$ is given by

$$\boldsymbol{q} = \frac{g}{12\nu}\, \mathrm{d}^3 \cdot \boldsymbol{J}_\mathrm{p}$$

g = acceleration due to gravity
ν = kinematic viscosity of water
d = aperture of the fracture
$\boldsymbol{J}_\mathrm{p}$ = projection of $\boldsymbol{J}$ in the plane of fracture

If $\boldsymbol{n}$ is the normal of the fracture, $\boldsymbol{n}\otimes\boldsymbol{n}$ is the tensor product of $\boldsymbol{n}$ by itself, and I is the identity matrix, then we write for $\boldsymbol{J}_\mathrm{p}$:

$$\boldsymbol{J}_\mathrm{p} = \boldsymbol{J} - (\boldsymbol{J}\cdot\boldsymbol{n})\,\boldsymbol{n} = [I - \boldsymbol{n}\otimes\boldsymbol{n}]\cdot\boldsymbol{J}$$

and K must be, from Darcy's law:

$$K = \frac{g}{12\nu}\, d^3 [I - \boldsymbol{n}\otimes\boldsymbol{n}].$$

If we have N fracture systems, in each system there being f fractures per meter, then the tensor K is given by:

$$K = \sum_{i=1}^{N} K_i = \frac{g}{12\nu} \sum_{i=1}^{N} f_i d_i^3 [I - \boldsymbol{n}_i \otimes \boldsymbol{n}_i].$$

Statistical variability of K can be calculated either by the method of Snow [4]) or by the variance tensor of the orientation of fissures [5]).

If we are dealing with permeability ratios rather than with absolute values of the permeabilities, we write for K:

$$K = \sum_{i=1}^{N} f_i d_i^3 [I - \boldsymbol{n}_i \otimes \boldsymbol{n}_i]$$

K, f_i, and d_i being dimensionless and K depending only on the geometrical parameters of the fissuration.

2. Permeability tensors simulated by fracture geometry

We present the method in two dimensions, but generalization for three-dimensional cases is immediate, assuming we are interested only in permeability ratios. Let us assume a basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ in a two-dimensional ordinary vector space, and in this space we find two sets of mutually orthogonal fractures with normals in the direction of the $\boldsymbol{e}_1$ and the $\boldsymbol{e}_2$ axes:

$$\boldsymbol{n}_1 = [0 \quad 1]' \qquad \text{and} \quad \boldsymbol{n}_2 = [1 \quad 0]'.$$

If we choose $f_1 = 1, f_2 = 1$, $d_1^3 = 1$ and $d_2^3 = 1$, then we have for K

$$K = \sum K_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is the expression of an *isotropic* permeability. When $f_1 \neq f_2$ and/or $d_1^3 \neq d_2^3$, the matrix K is no longer the identity matrix and the permeability becomes *anisotropic*.

Admitted we want to create the permeability tensor

$$K_e = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}$$

then we choose either $f_1 = 10$, $f_2 = 1$, $d_1^3 = 1$, and $d_2^3 = 1$, or $f_1 = 1$, $f_2 = 1$, $d_1^3 = 10$, and $d_2^3 = 1$.

Naturally, there is an infinity of solutions and we choose only the simplest of them.

If two sets of fractures are orthogonal, the principal directions of the permeability tensor (i.e. the eigenvectors of the matrix K_e) are always parallel to the normals of the two sets. Consequently, any desired orientation of the principal directions can be obtained by simple rotation of the two mutually orthogonal sets of fractures. The matrix K_e is easily rewritten in its new basis $\{\boldsymbol{g}_1, \boldsymbol{g}_2\}$ after the rotation, if we know the angle of rotation α:

$$K_g = AK_eA^{-1} \quad \text{where} \quad A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

We may conclude that any permeability tensor can be simulated by two mutually orthogonal sets of fractures (or by three mutually orthogonal sets, in a three-dimensional space) if we choose adequately the parameters f_i and d_i^3, as well as the orientation of the normals $\boldsymbol{n}_i$. Naturally it is also always possible to simulate a permeability tensor by more than two sets of fractures, especially in the case of known real fracture geometry. Now, the simulation of a heterogeneous, anisotropic permeability field is easy enough: we divide the ground-water basin into regions with different permeability tensors and we admit the existence, in each region, of the corresponding fracture geometry.

3. Fracture geometry simulated by conducting paper

Take the fracture parameters $\boldsymbol{n}_i$, f_i and d_i^3 corresponding to a certain permeability tensor. Fractures are simulated by conducting stripes drawn on non-conducting paper (for example: graphite stripes drawn on fine-grained, transparent tracing paper). The stripes are drawn perpendicular to the normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. If the stripes are of the same width, then the number of stripes per unit "cross-section" is f_i and the conductivity of a stripe per unit length is d_i^3.

In practice, there is some difficulty in obtaining the same conductivity and the same width of stripes through the whole groundwater basin, but after drawing the model it is always possible to measure the conductivity of each segment and to calculate the variability of the product $f_i d_i^3$.

4. Potential field, flow lines, and flow systems: a theoretical example

A theoretical example is presented on Fig. 2. The groundwater basin is divided into regions designated as 1, 2, or 3, each region having its own

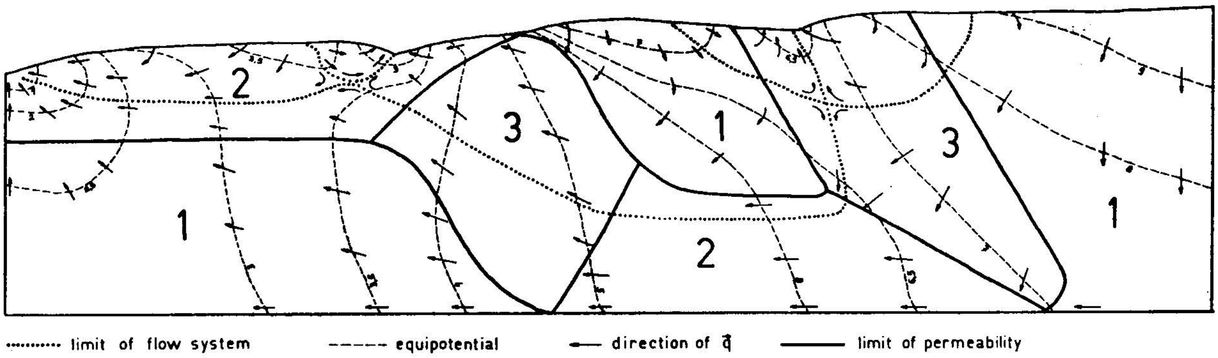

Fig. 2. Regional groundwater flow in a two-dimensional heterogeneous, and anisotropic ground-water basin.

anisotropic permeability, represented by the following tensors:

$$K_1 = \begin{bmatrix} 1.250 & -0.433 \\ -0.433 & 0.750 \end{bmatrix} \quad \text{for the regions 1}$$

$$K_2 = \begin{bmatrix} 3.25 & -0.43 \\ -0.43 & 0.75 \end{bmatrix} \quad \text{for the regions 2}$$

$$K_3 = \begin{bmatrix} 1.75 & -1.30 \\ -1.30 & 2.25 \end{bmatrix} \quad \text{for the regions 3}$$

The "permeability ellipses", constructed with the square roots of the principal permeabilities, are shown on Fig. 3 for K_1, K_2, and K_3.

The permeability tensor K_1 is simulated by two sets of fractures: by one set of horizontal fractures and by another set of fractures inclined 60° to the horizontal. The geometrical parameters are:

$$\boldsymbol{n}_1 = [0 \quad 1]' \qquad f_1 = 1 \quad d_1^3 = 1$$
$$\boldsymbol{n}_2 = [0.866 \quad 0.500]' \quad f_2 = 1 \quad d_2^3 = 1.$$

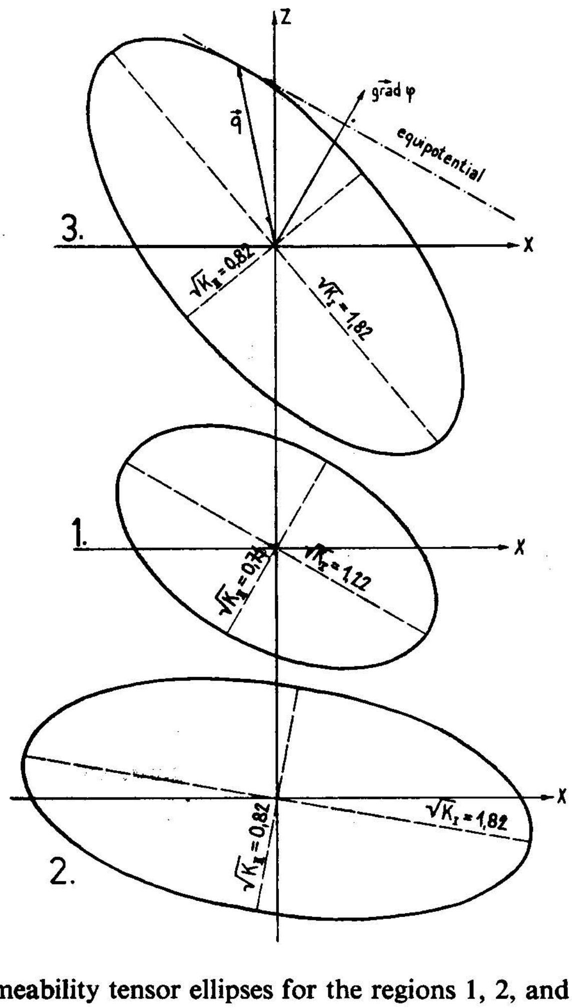

Fig. 3. Permeability tensor ellipses for the regions 1, 2, and 3, of Fig. 2.

This fracture geometry is simulated by horizontal and 60° inclined graphited stripes of equal conductivity, of equal width, and of equal spacing.

The permeability tensor K_2 is created by the same sets of fractures, and we change only the frequencies: $f_1 = 3$ and $f_2 = 1$. In the electric analog the frequency of the horizontal stripes is three times the frequency of the inclined stripes.

For the permeability tensor K_3 we choose the same sets of fractures as before but with frequencies $f_1 = 1$ and $f_2 = 3$. In the electric analog the frequency of the inclined stripes is three times the frequency of the horizontal stripes.

After having drawn the model and having defined the boundary conditions, we determine the potential field and equipotentials in exactly the same way as for an ordinary electric analog made in teledeltos paper[6]).

As the discharge vectors $\boldsymbol{q}$ are not, generally, perpendicular to the equipotential lines, their direction must be calculated from Darcy's law ($\boldsymbol{q} = K\,\boldsymbol{J}$) or must be constructed graphically. Liakopoulos[7]) proposed the use of the permeability ellipse, drawn with the inverse square-roots of the principal permeabilities, for the determination of $\boldsymbol{q}$ from the direction of $\boldsymbol{J}$. However, the vector field $\boldsymbol{J}$ is not directly available and we preferred to use the permeability ellipse drawn with the square roots of the principal permeabilities, which permits the determination of the direction of $\boldsymbol{q}$ directly from the equipotential lines (see ellipse 3 of Fig. 3):

- translate the ellipse until an equipotential becomes tangent to it
- the radius-vector of the tangent point gives the direction of flow.

Knowing the singular points of the potential field and the direction of the flow lines, we can delineate the flow systems as defined by Toth[8]): a flow system is a set of flow lines in which any two flow lines adjacent at one point of the flow region remain adjacent through the whole region: they can be intersected anywhere by an uninterrupted surface across which flow takes place in one directions only. The results are presented on Figure 2.

This simple electric analog gives us the possibility of going further into the study of flow in fractured media: we can now drop the hypothesis H2 of paragraph 1, and we can study the effect of irregularly developed and irregularly connected fractures on the potential field, on the permeability field, and on the flow systems. The results will be presented in a paper under preparation.

5. Conclusion

1. Each permeability tensor can be simulated by a certain fracture geometry.
2. In a two-dimensional space, each fracture geometry can be simulated by a very simple electric analog using conducting stripes painted on no-conducting paper. Consequently we can simulate groundwater flow in a heterogeneous, anisotropic permeability field by the means of a very simple electric analog.
3. This electric analog permits us to study the effect of irregularly developed and irregularly connected fractures on the potential field, on the permeability field, and on the flow systems.

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