

Interactions of metaphysical and epistemic concepts

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Abstract

This work sets out the results of research on topics at the intersection of logic and philosophy. It shows how *methods for combining logics* can be *applied* to the study of *epistemology* and *metaphysics*. In a broader perspective, it investigates interactions of modal concepts in order to create a bridge between metaphysics and epistemology. The first study of such interactions offers an analysis of a paradox in the limits of knowledge from the viewpoint of fusions and products of modal logics. We next examine the formal combination of an epistemic logic of knowledge with a contingency logic in order to determine whether contingent propositions can be known. The third example investigates a combination of logics for belief and contingency logics to determine in what sense the world is an object of belief. The work also studies contradictions in the context of modal operators and their relations with paraconsistent logics. In the final section, some general approaches to modalities are studied and some directions are presented. The concept of *paraconsistentization* of theories and logics, as well the idea of *philosophical categorification* are presented from an abstract viewpoint. The work presents some tools to deal with modalities and proposes some concepts for exploring problems in philosophy and logic.

Mot clés en français: combinaisons, connaissance, croyance, contingence, paraconsistentization, categorification philosophique

Mot clés en anglais: combinations, knowledge, belief, contingency, paraconsistentization, philosophical categorification

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Introduction

“All men by nature desire to know. An indication of this is the delight we take in our senses; for even apart from their usefulness they are loved for themselves...” (ARISTOTLE in [2])

This book deals with concepts. It is a study on the *interactions* of metaphysical and epistemic modalities (i.e a particular family of concepts), and the relations between the notions of *knowledge* and *belief*, and the further connection of these concepts with the notions of possibility and contingency. Thus it deals with one aspect of the relations between metaphysics and epistemology. Below the reader find some explanations of its main objectives and purposes. At the end of this opening section, a general theory of the combination of logics is proposed. This serves to introduce the main topic of the work.

Interactions

This thesis started as an attempt to defend skepticism. That the world is not an object of knowledge but just the object of belief is a familiar philosophical thesis. This work aims to provide logical foundations for an account of the interactions of different modalities. The study of such interactions starts in old problems such as induction, the contingency of the world, and the epistemic status of agents in relation to such contingency. Inconsistencies in the borders of knowledge and belief are investigated, as is a universal logical framework for dealing with modalities. The concept of *interaction*, which appears in the title of the work, refers here to a logical formalism in which non-interdefinable modalities appear in the same formula. The interactions of two or more modalities can appear in different ways. People studying the combination of logics have employed the notion of interaction especially in the analysis of implications (See for instance the work [44]). But an interaction can also be viewed as a formula in which different non-interdefinable modalities appear.

To characterize an interaction is straightforward. Take a simple non-combined model and a particular formula. If it is possible to analyze the truth-condition of the formula in the model, then no interaction is involved. But if at a given stage the model *halts*, and some formulas are not explicitly assessed in the sense that we are not able to determine their truth-conditions, then we have a case of interaction. An example is provided by the analysis of $\diamond K$ in single Kripke model for \diamond . At a given stage in analyzing the meaning (or truth-condition) of K , the model will *halt*, because it does not recognize the symbol K . This problem has been pointed out by GABBAY (See D. GABBAY in [43]) and we call it *the fibring problem*. Such interactions are very useful while modelling interconnections of concepts, according to logicians working in the field of combining logics. GABBAY ET AL state that:

“... realistic applications of modal logics in computer science, artificial intelligence, philosophy and other disciplines usually require a number of *interacting* modal operators. It is not sufficient to model time, space, belief, terminology, action, etc., independently of each other. What we actually need is semantically wellfounded composed logics, which leads us to models of many (at least two) dimensions.”
(GABBAY ET AL in [44])

The basic idea of this work consists in modelling connections between metaphysical and epistemic notions using interactions of modal concepts. Such interactions indeed appear everytime we have a complex formula involving concepts of different natures.

Applications

The applications of logic in epistemology and metaphysics have a very long history. Consider just the example of ontology. The attempt to develop a metaphysics on the basis of the assumption that logical forms describe the structure of (at least a particular fragment of) reality goes back to the earliest philosophers. But ARISTOTLE is usually considered the first philosopher to have applied logical tools to the analysis of ontological and epistemic questions in a way which was entirely systematic. His account of the syllogism formed a crucial step towards a general logical theory of deduction. At the very beginning of [1], he clearly stipulates his purposes:

“We must first state the subject of our inquiry and the faculty to which it belongs: its subject is demonstration and the faculty that carries it out demonstrative science. We must next define a premiss, a

term, and a syllogism, and the nature of a perfect and of an imperfect syllogism; and after that, the inclusion or noninclusion of one term in another as in a whole, and what we mean by predicating one term of all, or none, of another.” (ARISTOTLE in [1])

As is well known, ARISTOTLE contributed to almost all areas of science of his time, from biology to logic. In the above passage, a very powerful guiding idea is presented: to develop, on the basis of a general logical theory (the science of demonstration) an analysis of the attribution of properties to any given entity. This theory is intended to answer metaphysical questions about the existence and properties, because by means of a precise theory of demonstration, it should be possible to determine what kinds of property hold of what kinds of object. The theory of deduction developed by ARISTOTLE continues to play a fundamental role in well-developed portions of the subject, notably what we now term first-order logic; and the influence of an Aristotelian perspective can undoubtedly still be traced in many topics which form the subject matter of contemporary logical inquiry, notably the logical treatment of quantifiers and modalities, amongst other topics.

It is well known that for a long period, the technical apparatus of logical analysis remained substantially arrested in its development at the level achieved by ARISTOTLE, and the manner in which logic was used in the formulation and solution of metaphysical questions remained fixed in the Aristotelian mould. This situation did not change fundamentally until de 17th Century. At that period, LEIBNIZ set out to formalize his metaphysics in such a way that the solution of metaphysical questions should be reduced to the same kind of procedure seen in mathematical computation. The ideal analysis would be on a level where to do metaphysics could be reduced to computation:

“If controversies were to arise there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, and say to each other: Let us calculate.” (as LEIBNIZ is cited and translated by B. RUSSELL in [96])

The attempt to put metaphysics in a mathematical form leads us view LEIBNIZ as the founder of formal metaphysics, the branch of metaphysics which applies mathematical tools in metaphysics. His work occupied a central position in modern philosophy, and he had illuminating theories on modalities. He analyzed some basic ontological concepts such as those of necessity and possibility in a way which provides continuing and precise

motivation for the treatment of modalities pursued in contemporary philosophy:

“There are also two kinds of truths, those of reasoning and those of fact. Truths of reasoning are necessary and their opposite is impossible: truths of fact are contingent and their opposite is possible. When a truth is necessary, its reason can be found by analysis, resolving it into more simple ideas and truths, until we come to those which are primary.” (LEIBNIZ in [68])

This is the source of modal logic, the logic designed to express mathematically the notions of possibility and necessity. From the time of ARISTOTLE till the era of BOOLE and FREGE, logic as a science had undergone no substantial development in its technical apparatus. KANT even stated in [60] the following proposition about the situation of logic in his time:

“That logic has already, from the earliest times, proceeded upon this sure path is evidenced by the fact that since Aristotle it has not required to retrace a single step, unless, indeed, we care to count as improvements the removal of certain needless subtleties or the clearer exposition of its recognised teaching, features which concern the elegance rather than the certainty of the science. It is remarkable also that to the present day this logic has not been able to advance a single step, and is thus to all appearance a closed and completed body of doctrine.” (KANT in [60])

However, in the last Century, logic has developed such important new tools and concepts as to have become almost different in its structure and in its understanding of its own subject matter from KANT’s “closed and completed body of doctrine”. Such new “technologies” have, amongst their many other consequences, radically affected the character of logic as a tool of philosophical enquiry. Indeed some subject matters which were recognized as portions of philosophical enquiry have become so chiefly through the developments of recent mathematical and logical tools. The way in which logical investigations are viewed as relating to other broad areas of philosophical enquiry has undergone diversification, and the traditional Aristotelian project of tracing the logical shapes of metaphysical and epistemological issues has assumed new forms, given the new logical tools available. The impact of such developments on our understanding of logic as tool of other enquiries comprises such topics as Gödel’s incompleteness theorem, and the development of epistemic logic, modal logic and indeed all aspects and branches of logical systems. The logical tools deployed in the recent literature are

often more sophisticated than those available at the beginning of the 20th Century. This obviously allows an enlarged comprehension of problems at certain levels of technical analysis: in much the same way that a physician understands physiology and metabolism better nowadays than in the time of ARISTOTLE, and is hence able to cure patients who would have died in former times.

In the 20th Century, the applications of logic to philosophy gained in importance with the works of analytical philosophers - notably FREGE, WITTGENSTEIN and RUSSELL. Nowadays such applications are everywhere in the treatment of issues in epistemology and metaphysics as one can check at any mainstream journals such as *Analysis* and *The Journal of Philosophical Logic*. It is true that the logical approach to philosophical investigation is not the only one. There are some philosophers who prefer to use different tools for their philosophical projects. Such tools can be poems, metaphors and other resources drawn from literature, music and the arts. It is important to recognize that the logical approach to philosophy is not the sole possible one, and other kinds of philosophical projects are also possible. To reject such diversity is to reject any vision of philosophy as an adventure of thought based on openness to critical thinking, to ignore a wide range of possibilities and stay into the field of dogmatism.

This book stands in the tradition which applies logical tools to formalize and clarify philosophical concepts and problems. Today, there are many researchers offering competing labels for this tradition in philosophy, but indeed all these names, in the end, express much the same thing (formal philosophy, exact philosophy, analytic philosophy, scientific philosophy and particular variations: formal epistemology, formal ontology, etc). What all these labels point to is the use of rigorous scientific methods in philosophy. The name one gives to this approach is irrelevant. This work takes advantage of some recent developments in logic. Its purpose is to consider formal methods for combining logics and the relationship of those methods to some basic metaphysical and epistemological problems. Briefly stated, the most general purpose of this book is to show how and why these methods for combining logics are useful in philosophy.

Purposes and goals

The literature on methods of combining logics is not very large. There are some articles and books on particular methods, but, up to now, no introductory text on the subject. The situation is even worse when one examines the applications of these methods in computer science and philosophy. For

applications in computer science, there are some attempts to apply such methods while dealing with software engineering and knowledge databases (see for instance the recent book [44]), but a literature on the use of these methods in epistemology and metaphysics scarcely exists, with the exception of some work such as that developed by VAN BENTHEM and SARENAC in [99]. What follows is an attempt to show how such methods can also be applied to philosophical issues.

The basic goal is to present a plurality of concepts, methods and techniques that can be used by philosophers, logicians and even mathematicians in order to analyze, clarify and solve certain basic problems. The work starts from the consideration of a particular problem - one nowadays called FITCH's paradox, which is one of the fundamental problems of formal epistemology. Afterwards, turning to a more abstract and complicated concept, we concentrate attention on some methods of combining logical notions such as combinations of different modalities and also generating a two-dimensional concept based on the notion of product. We try to show how our strategy elucidates problems related to the limits of knowledge and belief. The following is a summary of the main ideas and topics which are treated :

1. A general theory of the combination of logics;
2. Applications of products and fusions of modal logics in logics of knowledge, belief and contingency; A related analysis of skepticism;
3. Paraconsistentization of logics: the general theory of paraconsistent logics;
4. Philosophical categorification; arguments showing that category theory is important for logical and philosophical research;

Although not all these ideas are themselves part of the main subject of this work, they are introduced in carrying out its main aims. These tools are important, for instance, in explaining the way in which the study of the formal combination of logics is relevant to any modelling of the limits of knowledge and also to rescuing a version of the verification principle from the difficulties centred on a basic epistemic paradox. We intend to offer a philosophical motivation for the methods used in the formal combination of logics and show that these methods are important tools for philosophers dealing with the formalization and analysis of propositions in natural languages.

Suppose, for instance, that a philosopher is trying to understand the verification principle:

All true propositions can be known.

Given that there are two non-interdefinable modalities in the statement (i.e. possibility and knowledge), it follows that just a combined modal epistemic logic should be able to give a clear formalization of the verification principle. But a modal epistemic logic is a combined logic generated by a fusion of modal logics and, therefore, a study of how fusions affects the notions involved is important to understand the principle. There are many others such cases. Collecting examples from epistemology and metaphysics, we argue that methods such as the fusion of modal logics proposed in [63] and [41], the method of forming a product of Kripke frames proposed in [43] and by [101], the temporalization proposed in [45] and fibring [43] are useful logical tools for the study of metaphysical and epistemological issues. In this way we seek to demonstrate the general thesis of this book, which consists in showing how widely the issue of formal combination of logics is relevant to the analysis, clarification and solution of issues connecting epistemology and metaphysics. A brief descriptive outline of the book now follows.

In this *introduction*, the reader find some explanation and historical remarks on the relations between logic and philosophy. We particularly draw attention to the relationship between methods for combining logics and issues in epistemology and metaphysics. The reader also find a general theory of the combination of logics based on the idea of Universal Logic. Also, one particular example of application of methods for combining logics to issues in metaphysics is briefly illustrated and discussed.

In *chapter one*, the reader find a very detailed discussion of the first philosophical problem analyzed in this book: Fitch's paradox. This is a kind of counterintuitive reasoning which occurs in some modal logics in connection with the thesis that "All truths can be known". The problem is that adding the verification principle to some logics forces us to conclude that "All truths are known", thus collapsing the distinction between knowledge and truth. In chapter one we show what is the right language and logic in which to formulate Fitch's paradox. In a setting generated by the application of fusions of modal logics, it is possible to find a model which falsifies Fitch's paradox. We show that, formulated in the correct framework, the paradox is no longer so paradoxical or problematic. We show how the formal study of fusions of logics is useful for alleviating, if not dissolving the paradox. This is an example of how the formal study of such fusions touches on issues in

metaphysics and epistemology. A more complicated method for combining logics (i.e products of modal logic) is used to model interactions between the concepts of possibility and knowledge in a higher-dimensional frame. It is used to define a two-dimensional modality which we label *knowability modality*. The logics for this modality are the logics of the limits of knowledge. We study some syntactical and semantical aspects of such a modality and sketch an initial logic for it. We also investigate the relations between imagination and possible knowledge.

In *chapter two*, the relations between metaphysics and epistemology appear in analyzing the question “Is it possible to know the world?” We investigate a proposed general structure for the world based on the notion of proposition found in the writings of B. RUSSELL [97] and the early L. WITTEGENSTEIN [110]. It is showed how the problem of induction and the problem of the contingency of the world impose limits for all possible knowledge about the world expressible on the basis of such an approach. Using logics of contingency and epistemic logics, we generate a logic where it is possible for both sets of operators in the respective logics to interact in such a way that an answer to the problem can be outlined. In this chapter, therefore, we combine via fusions two distinct formalisms, one for knowledge and the other for contingency, in order to examine the consequences for an account of the mappings between mental states and the world.

In *chapter three*, the concept of belief and its combinations with the concepts of knowledge and contingency are the target of investigation. Building on the framework developed in chapter two, we try to show in what sense it is possible to believe that the world is or will be in a particular state. The tool for such an investigation is once again the study of the formal fusion of logical systems. We fuse logics of belief with logics of knowledge and logics of contingency in order to analyze a problem according to which knowledge and belief are the same concept. Answers to the problems “Is it possible to know the world?” as well “What are the limits of knowledge?” are defined from the point of view provided by such tools and such a framework.

In *chapter four*, the existence of contradictions in the scope of modalities, and the so-called *logical skepticism*, lead us to investigate the relations between inconsistencies and combined modalities. To analyze this we are led to create a new method allowing us to generate paraconsistent logical systems from logics which are not paraconsistent *ab initio*. This method is labelled *paraconsistentization* of logics and it is applied to the basic logic obtained in the first chapter. In this chapter, we also examine questions relating contingency and contradictions. It attempts to study ways of combining modal concepts in the context of contradictory theories.

In *chapter five*, the project called *philosophical categorification* is introduced. It is illustrated through the example of fusions and products of modal logics. This chapter shows how to represent some methods for combining logics, logical operators and their applications by universal constructions in a given category. This method of universal (categorical) constructions is used to introduce a systematic approach to ontological questions motivated by some of the themes examined in previous chapters. We also show in what sense some new branches of logic such as n -opposition theory, n -dimensional modal logics and new conceptual perspectives on logical structure, such as the structuralist theory of logics can contribute to a general account of modalities. This chapter examines modalities, and their combinations, from a general, abstract viewpoint.

The names of two general modalities appear in the title of each chapter (except the last one). It should be clear to the reader that those modalities are used as examples of combinations. For instance, if in the title “knowledge” and “contingency” appear, it means that the relations and combinations between both concepts are the object of investigation and both are investigated through the study of formal languages in which they are primitive notions.

Now for some advertisements: If a player sets out to play *chess* using a different set of rules from his adversary, whatever they end up doing, it will not be to play a game of chess in any accepted sense. This document is for all those who accept the rules stated at the outset as defining the game. By this remark, I intend to remind the reader that this book is a work on concepts, and that to investigate these we use some fixed conceptions of possibility, knowledge, imagination, contingency etc. In order to communicate, the reader should accept the notions as defined here. Only in this way will he or she grasp the main results. It is not claimed that the definitions offered here are the only possible definitions, or the best definitions. But for the purposes of reading and understanding this document it is important to be clear what they are. It is not offered as an investigation of what certain concepts really are or should be. It is offered as an investigation of how things would be, if the reader grants a definition of certain concepts from the outset. In particular the words “knowledge” and “belief” are sometimes used informally in this work and sometimes in the context of a formal theory of the logic of those notions. Thus the reader must be alert for distinctions in usage somewhat analogous to those between occurrences of terms at two levels of language: object-language and metalanguage. Sometimes one writes that one *knows* something, using knowledge in the sense it has in the metalanguage, not using knowledge as truth in all possible worlds. Sometimes the

usage does not mean what is stated is forever true, as the idealized notion of knowledge involved in other places in the text would imply. I also wish to underline that the purpose of this work is not to describe or reproduce the conceptions that some famous philosophers have held of modal concepts such, for example, as knowledge and belief. Sometimes an author is cited so as to illustrate or analyze difficulties concerning a given position. It should be borne in mind that the overall aim of the book is to provide an account of concepts using some general logical tools such as formal theories of the combination of logics. Such an approach, up to now, has been very largely unexplored.

A very general remark before beginning : what follows is a piece of research into the means of combination of formalisms intended themselves to capture/permit the analysis of underlying philosophical positions. It is extremely important to remember that *logic and formal methods in general are not criteria for, let alone, any kind of guarantee of having sound notions. But at the same time, if a notion does not respect logical standards, it is probably not a sound notion.*

I have tried to construct the work so that the reader will find in each chapter:

1. a treatment of one combination of modalities;
2. a contribution to philosophical debate;
3. one *surprise*.

This research is a consequence of my studies in universal logic, the combination of logics, paraconsistency, category theory, modal logic, epistemology and ontology, and can be viewed in general as an attempt to apply mathematical constructions to philosophy. After the conclusion there is a section called “History and Interview” which is mainly dedicated to biographical information. One part of this material is an interview for the book *Masses of Formal Philosophy* edited by V. HENDRICKS and J. SYMONS. Since this research has been sponsored by the **Swiss National Science Foundation**, at the end of the document, the reader find a short history of the work. *De plus*, a conclusion setting out its main achievements and a full bibliography. In the way of technical material of use in the evaluation of each chapter, the reader should understand minimal and simple concepts on the following subjects:

1. Universal logic;

2. Modal logics and paraconsistent logics;
3. Combining logics;
4. Category theory;

I think that there is still much work to be done and some points to be investigated in fuller detail, but I hope I have at least clearly marked out several promising new directions of research.

Towards a general theory of the combination of logics

The purpose of this section ¹ is to present some general concepts and problems related to the combination of logics ². A brief history of the combination of logics is presented with the aim of finding unity in the methods for combining logics. Some general notions are analyzed (for instance, methods entail methods) - as well as some problems: the paradox of the combination of logics and the collapsing problem. Despite the existence of different methods for combining logics, is it possible to come up with a universal and general approach able to unify the subject? I propose the *powerful method problem* and argue that a positive answer to this question can also be seen as an initial clue towards a general theory on the combination of logics.

Logics and its combinations

Questions about the nature of logical systems and logics are controversial and there is no global agreement related to the following problems: (A) What is a logical system? (B) What is a logic? Despite the plurality of answers, it is possible to *assume* that a logic is a pair composed by a set of

¹This section with some modifications was published in the book *Aspects of Universal Logic*, Travaux de Logique, Neuchâtel, 2004.

²I decided to write this section while I was reading the tutorial given by C. CALEIRO in the *First World Congress and School on Universal Logic*. CALEIRO says the following: “We adopt a methodological abstract viewpoint that is concerned with general universal mechanisms for combining logics. Rather than focusing on the specific details of the combination of particular logics, we aim at rigorously defining a logic combination mechanism at the adequate level of abstraction and then establishing meaningful transference results that may be used in many situations. The typical questions to be asked and answered are: (i) When does it make sense to combine two given logics and what is the result? (ii) If two logics with property P are combined does the resulting logic inherit the property P?”

propositions, indeed an algebra of propositions, and a consequence relation (i.e. logical consequence) on this set³. Not only are there many definitions of logic but also there is not just only one logic. The twentieth Century has shown that logics are like some kind of computer virus, in the sense that each day a new one appears. To understand all these varieties of logics, it was urgent to find a way to unify this multiplicity. In this sense, universal logic was created in order to study general properties of logical systems (see BEZIAU in [7] and [6]). Universal logic is a general theory of logical systems motivated basically by the proliferation of the logical systems available in the logical land. Universal logicians try to develop abstract tools which can be used to understand a plurality of logics from an abstract viewpoint instead of investigating a particular logic.⁴ In the same way that there are different logics, there are also many methods to combine logics: temporalization, synchronization, fusion, product, fibring etc. There is a vast literature about each one of these methods. They are useful in the sense that they allow us to better understand complex statements in natural languages as well as helping us understand problems related to technical fields like computer science (see [102]). Indeed, combining logics is not an easy task, assuming that there are many definitions of the logical systems, as well as of different logics, and that there are many different presentations of the same logic: sequent calculus, natural deduction, tableaux, different semantics, etc. For this reason, it is urgent to develop a general theory on the combination of logics, able to answer, at least, one basic question: What is the definition of the combination of logics? Such a theory must also be able to answer the following questions: How to unify all these methods for combining logics? What general properties are inherent in all of them? Is there a general theory about methods for combining logics? This section gives a clue about how to answer these questions applying a universal approach to the combination of logics.

A brief history of the combination of logics

There are many different perspectives on combination. One may want to combine two given logics, combining all their operators or one may want add a particular feature to a given logic, for example: temporalization - adding

³A consequence relation without restriction means that it does not need to respect the Tarskian axioms for logical consequence.

⁴It seems to me that universal logic is a kind of platonism, given that it accepts that logics, despite of their different manifestations, have something in common, a kind of essence. It is the same idea but now applied to logics.

the concept of time to a given logic - fuzzification - adding a fuzzy character to a logic - paraconsistentization of logics - given a logic, how to obtain the paraconsistent counterpart of this logic. There are also different kinds of decomposition of a given logic into fragments such as for example possible-translation semantics proposed by W.CARNIELLI in [27] and studied in [29.2]. At the beginning, combination of logics appeared in the environment of modal logic and, therefore, many methods were especially created to model on the one hand combination of Kripke structures, on the other hand combination of axiomatic systems, although nowadays they are applied to a great variety of model-theoretical and proof-theoretical notions. The ideal would be to look for a universal conception of logical structure and thus to define an abstract method for combining logics independently of any particular conceptions of a logic.

The simplest method introduced to combine modal logics was fusion (see [63], [41], [43] and [44]). Semantically speaking, fusion consists in putting together without interferences two Kripke semantics, that is to say putting side by side the accessibility relations. Fusion of Hilbert proof systems also consists just of putting together rules and axioms. The basic idea underlying fusion is the combination of the languages of the two logics, done in a natural way, the rest follows straightforwardly. Fusion generally preserves soundness, completeness, finite model property etc.

Another method, much more complicated, is product. It was first used by SHEHTMAN in [101] to introduce a two-dimensional modal logic. Product defines the notion of dimension in logics.⁵ Fusions and products have been very useful in modal logics⁶ and in issues related to the modelling of philosophical concepts. A difficult and interesting task is to determine the proof-theoretical counterpart of a product of models (see [44] and also [43])⁷, i.e. which method for combining proof-theoretical systems preserves completeness for product of semantics.

Even fibring, the most famous method, was developed in the context of modal logics. According to D. GABBAY, who proposed fibring, this mechanism allows us to associate to each possible world a model using a fibring function. Fibring is a method used to combine logics while evaluating a formula which has an operator that cannot be recognized in a particular language. Fibring is considered the most powerful method of combining logics because it allows, in one of its variations, interactions between lan-

⁵Not just modal logics, but logics in general.

⁶A nice presentation of multi-dimensional modal logics, and of how it is obtained by products, is presented in [75] and [44].

⁷For many cases it is known.

guages. GABBAY uses an idea of fibring which is based on the idea of a fibring function, and on complex models with a fibring function - however, nowadays there are different definitions of fibring which are strongly related to different conceptions of logic (See for example the works of the Portuguese school and the importance of selecting the right level of abstraction or the working universe of logics.) To determine whether metalogical notions as soundness and completeness are preserved by fibring depends strongly on a given conception of logic and fibring (see [26]).

Combining logics

A logician who combines logics performs a task similar to that of chemists, but instead of atoms and molecules, the logicians deal with languages, models and logics. The chemist has tools to realize the process of combining substances and afterwards he or she separates them. And the logicians also have their own methods and techniques for combining logics and decomposing logics. When logicians decide to combine logics they have to select the correct level of abstraction related to the nature of logics. It does not make sense to combine logics without stating, first, what is assumed as a logic. To explain the problem let me mention the works developed by the Portuguese school, especially [26]. The simplest type of logic considered is called a consequence system, which is a pair composed of a set and a consequence operator obeying TARSKI's axiom. However, this kind of consequence system does not constitute the right level of abstraction to combine logics, because the structure of the formulas is not defined, but it can be useful to introduce the combination of logics in a high level of abstraction. The next natural step is to determine the structure of the formulas and to work with a structural consequence operator in the sense of LOS and SUSZKO. The Portuguese School go further by defining the notion of deductive systems, which is a structural consequence operator together with a set of inference rules (see [26]). Then they define the fibring of deductive systems. On the other hand they also introduce the concept of interpretation systems in order to combine semantics. An interpretation system is a structural consequence operator together with a set of interpretation structures, i.e. models (check [26]). They then define fibring of interpretation systems. To define what a logical system is, they make use of both notions. From [26] we learn that: logical systems are obtained by putting together deductive systems and interpretation systems in order to create a nice environment to talk about soundness and completeness. Therefore, logical systems are a good

level of abstraction to begin to define particular combination methods between logics. Almost all results obtained in the combination of logics depend strongly on particular conceptions of the logical systems and of the methods for combining logics.⁸ In the same way that a theory of truth should be able to answer the question about what truth is and what criteria are used to determine when a proposition is true, a general theory on the combination of logics should be able to answer the question about what combination is and what the procedure is for combining logics.⁹ There are many different methods of combination: fusion, product, fibring, synchronization, etc, each one applying to some particular proof systems, semantical structures, logical structures incorporating or not proof or semantical features. However, up to now there is no general clear definition of what combination of logics is, independent of circumstances. GABBAY in [43] says that the combination of two logics is the least conservative extension defined on the combined language, but as CALEIRO noted that two logics may not have a common conservative extension. BEZIAU gave a simple example: the combination of a logic with only a classical negation and a logic with only an intuitionistic implication. In the combined logic, the intuitionistic implication becomes classical, therefore we do not have a conservative extension.

General questions about combination of logics

Instead of investigating particular conceptions of logics and particular methods for combining logics, we can inquire what are the general problems related to the combination of logics.

Preservation of properties

The first, and one of the most important and popular problems in the combination of logics, is the question about the *preservation of properties* or the *transfer theorems*. Preservation of properties is very normal in mathematical theories. It is also called invariance results. Just as an example, in the case of modal logics, the question is to know whether modal satisfaction is preserved if we apply some operations to our models (check for instance in [17] the topic on bisimulations). This question can be stated as follows: given two logics each having the property P, which methods of combination of

⁸I mentioned the article [26] just as an example, but these techniques appear in all the works of the Portuguese school, see for example Caleiro's PhD thesis [20].

⁹CARNIELLI and CONIGLIO have tried to give a categorial definition of combining logics in [28].

these two logics preserve this property, i.e. produce a combined logic which also satisfies P? This property can be truth-functionality, completeness, the finite model property, etc.¹⁰

To give an answer to the question above we should have information about the nature of the logics considered and about the nature of the method used in the combination. This problem - let us call it the *preserving properties problem* - is a good example of a universal question related to the combination of logics. As it is known, the process of combining logics can be realized in two different directions. The first one is the combination, literally, of logics. The second one is the decomposition of logics. The same general problem above applies also in the last case. Given a complex logic, if it is decomposed, how goes the preservation of its properties for its fragments?

Categorical Representation

Instead of investigating particular objects and particular transformations between these objects, the pure categorist studies general constructions with categories. Logical systems can be viewed as objects of categories. In this sense it is possible to construct categories of deductive systems, of interpretation systems and of logic systems (see [26]). An interesting problem that arises in questions about the combination of logics is the categorical representation of a given method. For example, in [26] it is proved that fibring is a kind of universal construction in a particular category. The second general problem related to the combination of logics is the *categorical representation problem* and can be stated as follows: Given a method for combining logics, does this method have a universal construction in a category?

Methods entail Methods

The third problem related to our universal approach to the combination of logics is the *methods entail methods problem*. This problem is a clue in the direction of finding a powerful and universal method for combining logics. When one selects a particular method for combining logics, would there be another method that could be generated by this method? To illustrate this problem, note that in [23] the authors show a plurality of relations between fibring, synchronization and parameterization and show how we can deduce one from another. This is a clear example of the methods entail methods

¹⁰Concerning the preservation of properties (soundness, completeness etc) related to particular methods I recommend [23, 24, 43, 63].

problem, which can be stated as: Given a method for combining logics, does this method imply other methods?

Paradoxes related to the combination of logics

Although the fact that combining logics allows us to get more powerful logics, there are some problems related to the basic concepts in the subject.

The collapsing paradox

GABBAY has pointed out (see [43]) that the fibration of two logics leads to collapse. For example the fibring of classical implication and intuitionistic implication logic leads to collapses into classical implication. Logicians are developing many variations of fibring trying to solve this problem (see for instance [100] and [25]).

The copulation paradox

BEZIAU has pointed out a interesting problem arising with combination of logics (see [10]) . It deals with combination of truth-tables. If one puts together in a natural way the standard semantics of conjunction and the standard semantics of disjunction, one gets a logic in which distributivity holds between conjunction and disjunction, so the combined logic produced by this combination of semantics is not the least conservative extension of the logic of conjunction and the logic of disjunction. BEZIAU calls this phenomenon by the suggestive name *copulation paradox*, because the conjunction and the disjunction are interacting and producing a new property.

Universals

A very important task in the combination of logics is to find the right level of abstraction for logical structures. Logicians usually prefer to speak about logical structures where it is possible to express syntactical and semantical properties of the logics like, for instance, soundness and completeness. After deciding about the best way to express what a logic is, it is then possible to define operations between these logics as, for example, fibring. In this sense, it is possible to define many varieties of fibring: fibring of deductive systems, interpretation systems, logical systems presentation and so on (see [26]). Methods for combining logics are tools which can be used by the universal

logician to find again a unity in logic. The *powerful method problem* is the intersection of the three general problems presented in this section:

- Is it possible to find a universal method able to show that most of all known methods are particular cases?
- Is it possible to give a categorial characterization of this method?
- What are the properties preserved by this method?

A positive answer to the above questions would probably be a paradise for those who are working in the combination of logics. Many people think that fibring would be a solution to this problem, but there are some problems with fibring such as the collapsing problem. This problem leads many logic combinatorics to propose new kinds of fibring as for example [25] and [100]. Despite these proposals, we do not have any guarantee that the powerful method problem is already solved.

In order to combine logics it should first be clear what a logic is. However, there is not a unique answer to this question. The most that one can do is suppose that a logic is something in particular and see what follows from the supposition. It is reasonable enough to depart from a conception of logic which permits us to speak about the two sides of a logical system: its syntax and semantics. Afterwards, we have to be able to determine exactly what are the properties of an abstract method for combining logics. Using a method for combining logics, we can enrich our languages and consequently our logics, being able thus to better understand formal languages and their applications ¹¹.

PostScript ¹²

It seems that a general theory of the combination of logics is still far away from being achieved, although GABBAY and others think that it is achieved by the theory of fibring. The applications of formalisms for particular cases are already revolutionizing the subject. We now have a variety of clear and precisely defined particular methods for combining logics. We know what fusion, products, temporalization, fuzzification and fibring are, but we still lack a general definition of combination of logics independent of particular methods. This is indeed the main task of a general theory of the combination of logics. In order to propose a general definition of the

¹¹End of the article

¹²Postscript on the article *Towards a general theory of the combination of logics*.

combination of logics, we have first to study exactly the content of a formal system and then to determine what are the minimal principles inherent in all the methods hitherto proposed for combining logics. The first point to note is that without concrete examples of logics it is impossible to combine logics. We therefore need, first of all, to know exactly what a logic is. As explained, without a good definition of logic, it is not possible to define means of combining logics.

One might think that the most general principles relating to all logical systems are those principles usually called: non-contradiction, excluded middle, bivalence and identity. But this is called in question by a study of the history of logic and a consideration of the many systems of non-classical logics. Some who hold to a strong pluralist view of logical methods and concepts claim that these logics show that there are *no* universal logical principles (see some arguments also in BEZIAU [8]), since one can point to good logical systems (satisfying soundness and completeness conditions) in which the usual candidates for such principles are no longer valid. However, we claim that there are still some universal logical principles shared by classical and non-classical logics. To identify and explore such principles is one of the chief tasks of universal logic.

It is a striking fact that the study of Universal Logic should have led us to discern new logical principles. The first logical principle to which all logical systems conform is not one that can be stated in the object language of a given logic, but is something more basic. I call this the *principle of existence of a language*: all logics, classical or non-classical, are constructed in a given formal language, which is a kind of algebra generated by a set of symbols and the set of operations defined between such symbols. The principle of the existence of a language is a consequence of two more basic principles: *the existence of symbols* and *the existence of operations between symbols*. The expressions created by successive applications of such operations will generate the language of a given logic. Given the general definition of logics, one could argue that there are logics composed by the empty set of propositions, but here we take the position that the empty language is also a language.

The second basic logical principle is the *existence a consequence relation*. For a language to be such as to permit the expression of any kind of logic, we have to know how to construct inferences in it. This is secured by the second principle. There is no logic without a consequence relation. Even an empty consequence relation is a consequence relation. To generate a logic is to specify a way to obtain some conclusions from a given set of premises, that is, to say how to infer in a language. And note that the rules of inference

can take different forms depending of the kind of presentation of a logic ¹³.

One might think that the *existence of a proof-theory* and the *existence of a model theory* are also basic universal logical principles. But this is a problematic claim. One can characterize a logic just at an abstract level, without specifying its proof-theory and model theory and therefore I do not include such principles as universal logical principles: one could find instances of logics purely syntactically motivated, or purely semantically motivated, but one could not find a logic without a language and a consequence relation. So what are the universal logical principles which all logical systems must satisfy? We know that there are at least two. Are there others? Unfortunately, up to now we are not able to answer this question. The search for further universal logical principles is still open. ¹⁴

Combining logics: applications

Modal logics and modal concepts have in the recent past been the subject of a large and growing literature. Single boxes and diamonds (the operators of propositional modal logic) have been studied from a wide variety of perspectives. On the one hand, people studying the purely mathematical aspect of modal notions and the logical constructions in which they feature, usually prefer not to emphasize the modal character of the concepts involved, such as necessity, possibility, and knowledge. They prefer to study the mathematical behaviour of such operators from an independent, purely formal viewpoint. On the other hand, philosophers usually prefer to study these operators as representations of metaphysical and epistemological concepts. However, it is quite possible to give a mathematical treatment of modalities even from the philosophical point of view.

Multimodal systems containing different modal operators are very important from the philosophical standpoint. The interest in them goes back at least to the work of DANA SCOTT. There D. SCOTT wrote (As cited by HENDRICKS and SYMONS in [53]):

“Here is what I consider one of the biggest mistakes of all in modal

¹³The two principles above (i.e. existence of language and existence of consequence relation) are not sufficient conditions for having a logic, but they are necessary conditions. This means that both principles together will not generate a logic, but without such principles there is no logic.

¹⁴Considering a logic as a structure composed by a set of formulas and a consequence relation generates the following interesting philosophical question: Is the empty logic a logic? The empty logic is a structure where the set of formulas is empty and the consequence relation is also empty: $L_\emptyset = \langle \emptyset; \emptyset \rangle$.

logic: concentration on a system with just one modal operator. The only way to have any philosophically significant results in deontic logic or epistemic logic is to combine these operators with: Tense operators (otherwise how can you formulate principles of change?); the logical operators (otherwise how can you compare the relative with the absolute?); the operators like historical or physical necessity (otherwise how can you relate the agent to his environment?); and so on and so on.”

SCOTT was right. Combining modalities is indeed one of the keys to understand the links between epistemology and ontology. What DANA SCOTT did not tell us was how to improve our multimodal systems using the theory of combination of logics. More recently, M. KRACHT and F. WOLTER state the following:

“Given that in many applications of modal logic one modality is not sufficient, the lack of general results is acutely felt by the *users* of modal logics...” (M. KRACHT and F. WOLTER in [63])

Multimodal logics have been studied for a long time. One can find many multimodal systems in the literature. GABBAY in [43] (a book devoted to the study of fibring), showed how many multimodal logics in the literature are indeed special cases of his methodology (Check [43]). Many people have sought to apply this kind of modal logic in the investigation of philosophical concepts, but only a few noted the ways this kind of system can be generated by special methods of combining logical structures, such as fusions, products, temporalization etc. Just recently an entire book studying in a mathematical fashion the basic results on combination of logics and its applications (not philosophical applications) has been proposed (See [44])¹⁵.

GABBAY explained in particular detail the problem of combining logics, and the fibring problem. Complex formalisms are used where single formalisms are not able to capture all the richness needed for applications. The fibring problem is involved wherever we find a complicated modality which cannot be analyzed in single frames. This book studies combined modalities from a philosophical perspective. The basic results are those involving cases where modalities are studied in relation to other modalities.

¹⁵Some authors as for example K.FINE and G. SCHURZ proposed one possible philosophical application using multimodal logics, concerning especially deontic logics and erotetic logics. The examples are given in the article [41]. RABINOWICZ and SEGERBERG also applied products in order to investigate philosophical problems in [92]. In some sense, they are precursors of the work developed here concerning applications of methods for combining logics in philosophy.

There are many modal languages containing just one basic modal operator. These are not the focus of interest considered here. We are chiefly interested in modal languages with more than one modal operator. If in a given modal language, one can define the modality Q using negation and another modality V , we are still in the scope of monomodal logics. The cases where we have more than one non-interdefinable modality are those of interest for our enterprise. The basic cases considered here are those where the modal languages contains the following types of operator:

Table 1: Interactions of metaphysical and epistemic concepts

possibility	knowledge
knowledge	contingency
belief	knowledge
belief	contingency

Each line above mentions one particular interaction. We are interested in studying the different relations involving such interactions that can be obtained in bimodal languages. Eventually, n -modal languages could be also considered.

A mistake commonly found in the philosophical literature is the confusion between multimodal logics and n -dimensional, or multi-dimensional, modal logics. The first of these refers to a modal formalism which has more than one modal operator which is not definable in the language. A n -dimensional modal logic is a modal logic in which the notion of satisfaction between worlds and formulas is not defined at single points, but just in pairs, n -order structures etc. One can have a multimodal logic which is at the same time a n -dimensional modal logic. Generally, multimodal logics are generated by fusions of modal logics while n -dimensional modal logics are generated by products. Given a multimodal logic it is always possible to introduce the concept of dimension on it. However, given a n -dimensional modal logic, it is not straightforward to introduce a new modality, although it is possible. Below two important definitions of the theory of combination of logics which are used in the text. The first one is that of fusion which departs from the notion of generalized KRIPKE frames. The notion of fusion appears at least in the works [41] and [63], and it is studied in detail in [43] and [44]. The definition of fusion can be found in [43] and [44]. Given two generalized KRIPKE frames:

$$F_1 = \langle W, R_1, \dots, R_n \rangle$$

$$F_2 = \langle W, P_1, \dots, P_n \rangle$$

Then, a *fusion of generalized frames* is defined as

$$F_1 \oplus F_2 = \langle W, R_1, \dots, R_n, P_1, \dots, P_n \rangle$$

According to the theory of combination of logics, fusions have two sides, a proof theory and a model theory. From the proof theoretical point of view, fusions of axiomatic systems consists in putting all axioms and inference rules together, a fusion of languages consists in putting all connectives together in a bigger structure. Some of the most famous results are related to preservation properties. Many results have been obtained in the literature: fusions of modal logics preserve completeness, finite model property and decidability. Fusions introduces multimodalities (many modalities) in logics.

Another important definition is that of product. According to logicians working in the combination of logics, in the same way that fusions can be done in both sides of a logical system, products also can be done. As it is well-known, the idea of products is to define the concept of dimension in logical systems. In this sense, semantically, products are defined using the notion of generalized KRIPKE frame (see [43] and [44]). Given two generalized KRIPKE frames:

$$\begin{aligned} F_1 &= \langle W, R_1, \dots, R_n \rangle \\ F_2 &= \langle S, P_1, \dots, P_n \rangle \end{aligned}$$

Then, a *simple product of generalized frames* is defined as

$$F_1 \times F_2 = \langle W \times S, R'_1, \dots, R'_n, P'_1, \dots, P'_n \rangle$$

such that:

$$\begin{aligned} R'_i &= \{ \langle \langle x, y \rangle, \langle z, y \rangle \rangle : xR_iz, y \in S \} \\ P'_i &= \{ \langle \langle x, y \rangle, \langle x, z \rangle \rangle : yP_iz, x \in W \} \end{aligned}$$

According to the literature on combining logics (see [43] and [44]), the notion of product can also be applied to classes of frames and logics. The notion of product from the syntactical viewpoint is complicated. It consists in a fusion extended with two new properties: Church-Rosser property and commutativity. In the above definition one can verify that 2-dimensions are playing an important role. One could also define a n -dimensional product. Products are much more complicated than fusions and the metatheorems

for products are not so clear. General affirmations such: “Products preserve completeness” cannot be obtained in the same way of fusions. Each particular product has to be proved independently.

The methodology employed here is based on the use of logical tools to analyze philosophical problems. However, given what often happens in philosophy, I am afraid that some serious logicians could misunderstand the proposals set out here. Therefore, a brief clarification of my viewpoint on the uses of logic in philosophy should be given here. It is not difficult to find bad uses of logic in philosophy. This commonly happens when a philosopher misunderstands some logical concepts and criteria. The use of a logical formalism in an incorrect or imprecise way is one of the surest ways to generate confusion in philosophy. It is important that when logic is not correctly used, it is better not to use it at all. To employ a metaphor : a patient goes to the doctor. The doctor diagnoses his or her illness and prescribes some medicaments. The medicament helps the patient recover. But if used incorrectly, ignoring the instructions on the prescription, deterioration or even death can result. The same situation holds with the applications of logic. Given a philosophical problem, logic can play a role both in diagnosis and treatment, but if one uses logic in the wrong way then the situation may be made much worse. But how to use the logical medicament? Given a problem stated in natural language, the first step is to formalize this problem inside a given formal language. If it is an argument, the second thing to do is to determine in what kind of system some given problematic deduction should be carried out. The problem should also be assessed semantically. Sometimes, some philosophers seem to think that simply putting formalism into their text will make it clearer or more rigorous. This is not necessarily the case. On one occasion I read an article offering a new study of a familiar paradox. The author formalized the paradox using a new operator. Later, he introduced three further new operators and claimed that, in doing this, the paradox would be blocked. But the author did not clearly specify the rules that each operator was supposed to respect or at least should respect. The author also did not state the truth-conditions of the operators and gave no explanation of how each operator had been introduced into the language. I wondered what could be the reasons for someone using logical expressions in this way without any clear account of the criteria governing their use. BEZIAU in [8] has drawn attention to the manner in which some philosophers are prone to use mathematical concepts in a metaphorical way, in the same way of the work of continental philosophers. I think there is no problem in general principle in doing philosophy using metaphors. It is possible to develop interesting ideas and lines of argument, and find in-

interesting results using this kind of strategy. As pointed out by BEZIAU, one curious fact is that some philosophers consider themselves superior in this respect to others philosophers whose writings are located within the so-called continental tradition. He says that the fact that someone knows how to draw a truth-table does not guarantee the soundness of their prior logical understanding of the concepts they employ.¹⁶

How then to use logic in philosophy? Consider a philosopher who seeks to prove the proposition “Contingent propositions cannot be known”. If (s)he wants to use logic to understand the proposition, the first thing to be done is find a modal language in which to formalize the statement. She or he then formalizes the statement using the symbolic resources of her or his chosen formal language. Having found the formal statement which corresponds to the proposition in natural language, she or he has to determine if there is a logical system where the proposition is an axiom or a theorem, or the converse. If it is neither a theorem nor an axiom, but a formula which allows a counterexample in some model, then (s)he has reasons to reject the truth of the proposition. However, we also have to ask what would happen if the formula were simply added to a given formalism. Would it be possible to find a logical system validating the formula? There are many instances of such a procedure; the reader will find illustrations of such a method in the following chapters. The important thing to note is that the simple fact of using logical tools in philosophy is no guarantee of sound work. What is rather required is the rigorous examination of philosophical concepts from the logical viewpoint. By a logic, I mean a formal system having two aspects: a syntax and a semantics. The syntactical aspect of a logic appears in the kind of presentation one decides to give: Hilbert-style, natural deduction, sequent calculus etc. The semantical aspect of a logic appears with the notion of a model. Semantics is related to truth, and to the truth-conditions of propositions expressed in a given language. When there is communication between both aspects, we say that a logic is complete. A fuller discussion of how logical criteria should be used to determine whether a philosophical position is sound will be presented in chapter two.

To summarize very briefly: This work is a study on the interactions of modalities. Such interactions are studied in order to trace (some aspects of) the relations between metaphysics and epistemology. Some formal tools are used in order to give a more precise conceptual clarification of those aspects. This work is, therefore, a study on the interactions of concepts. Such interactions are studied to establish a bridge between metaphysics and

¹⁶In personal communication.

epistemology. Some tools are used in order to realize such a task.

What the reader WILL NOT FIND here

In the present era, it seems there is a new logical system born every day. In this book, we intend to concentrate our work on the most general aspects of logics, although sometimes illustrations will depend on a given particular logic, where this allows the better understanding of a specific problem. However, the reader will not find in this thesis results such as “A sequent calculus for the logic XYZ”, “A new natural deduction system for the logic XYZ”, “A new completeness proof of the system XYZ”, “Some properties of the system XYZ” and so forth. I am not claiming that such results do not deserve detailed study; only that such detailed study is out of place in regard to the aims and purposes of this work. However I do claim that there are strategies for the presentation of results, designed to ramp up their appearance of novelty. And these strategies come to assume greater and greater importance. Trivial results are sometimes announced as deep discoveries. Truly deep and unifying ideas are scarce. I am fortunate to have worked on the ideas in this book under a supervisor who is acutely alert to this tendency, and strongly opposed to it. The focus of attention in this work is not on novel results or the proofs in logics with particular properties but on the search for truly unifying and clarifying *ideas* and *concepts*. We do not subscribe to that ideology whereby a work is assessed by the criterion of how many new “results” it can claim to present.

The systems studied in this book are investigated, in the best case, up to metatheorems such as completeness and the finite model property. But computational and algebraic problems of the combined systems are not touched. Such an investigation would undoubtedly attract able computer scientists. Nor is this a work which can be offered for use as a main text in courses on philosophical logic, although we hope it could usefully feature in the secondary bibliography. Technical details are avoided in the body of the text. Nor does this document provide a comprehensive representation of basic concepts. The reader is assumed to be familiar with simple notions and metatheorems of classical propositional logic, and some familiarity with modal logic, paraconsistent logic and category theory is also desirable, though not a necessary condition. As to the philosophical ideas, a basic knowledge of some standard definitions of skepticism is important. Familiarity with the basic literature on the problem of induction and with the work of HUME, KANT, WITTGENSTEIN and with principal works in

the tradition of analytic philosophy would certainly be a positive aspect to anyone reading it.

To conclude, I think that a work of philosophy is never finished , but always in some sense a work in progress. Unfortunately, I am not able to provide the reader with an account of all the innermost laws of the universe, but I hope to offer some useful weaponry that will assist in protecting ourselves against the consequences of its flux.

The structure/diagram below can help prevent the reader from becoming lost in the land of modal concepts (each number represents a chapter):



Chapter 1

Plug-ins: possibility and knowledge

“If I know an object I also know all its possible occurrences in states of affairs. (Every one of these possibilities must be part of the nature of the object.) A new possibility cannot be discovered later. (WITTEGENSTEIN, 2.0123, in [110])

Some mathematicians and computer scientists have the tendency to believe that modal logic is just about relational structures (i.e. structures composed by a set and relations on this set ¹. Check for instance (BLACKBURN ET AL in [17]). This is just one possible way to understand modal logic and, therefore, it does not imply that modal logic can be reduced to such a conception. Conceiving modal logics as “a tool for working with relational structures” (see [17]) allows logicians, especially mathematically-oriented logicians, to unify a lot of different objects under the same label. However, for philosophical applications such a definition is not entirely adequate because it is not able to capture single philosophical aspects of concepts. Modal logic cannot be reduced to the study of Kripke semantics; nor can it be reduced to the research on what modalities such as possibility and necessity are. Indeed, one can find many definitions of modal logic in the literature. Some of the most important modal logicians have a lot of different conceptions of modal logic. At the very beginning of HUGHES and CRESSWELL in [56], one finds the following remarks:

“Modal logic is the logic of necessity and possibility, of ‘must be’ and ‘may be’. By this is meant that it considers not only truth and

¹A modified version of this section appeared in the *Logica Yearbook 2006*.

falsity applied to what is or is not so as things actually stand, but considers what would be so if things were different. If we think of how things are as the actual world then we may think of how things might have been as how things are in an alternative, non-actual but possible, state of affairs - or possible world.” (HUGHES and CRESSWELL in [56])

The above conception is clearly not founded in the *relational structures slogan*, but in a much more passionate account of modal logic. Such a conception can make someone think therefore that modal logic is not about the real world, but just about fictional worlds, because modal logicians are talking about *possible worlds* or *possible situations* and such entities are not the real world, although the real world is also a possible world. Nobody knows exactly what possible worlds are or if they really exist. Questions about the ontological status of possible worlds have been studied in the literature for a long time. D.LEWIS in [70] is one of the most famous philosophers who argues that possible worlds have an existence in the same way that the real world has. Such a conception generates an interesting philosophical discussion. Accepting the actual world as a constant realization of possible worlds (or possible worlds becoming real by updating reality), it follows that some possible worlds, those which become real, have an ontological status and then really exist, given that they are the actual world.

Other definition of modal logic which is proposed by some main authorities in the area, such as for example CHAGROV and ZAKHARYASCHEV, is (in [21]):

“Modal logic is a branch of mathematical logic studying mathematical models of correct reasoning which involves various kinds of necessity-like and possibility-like operators. (CHAGROV and ZAKHARYASCHEV in [21])

It seems that both definitions above of modal logic were targets of criticism especially developed by those people working on the relational structures slogan. BLACKBURN, DE RIJKE and VENEMA in [17] state the following:

“One still encounters with annoying frequency the view that modal logic amounts to rather simple-minded uses of two operators \diamond and \square . The view has been out of date at least since the late 1960s...” (BLACKBURN ET AL in [17])

Such a comment attempts to establish a new approach to modal logic. Even if the relational structures are so fundamental to modal logic, there

is no guarantee that in the future new ways to study modalities will not change the way actual researchers on modal logic think about their subject.

Modal logic is interesting for philosophers because it is related to the metaphysical status of objects and with the content of an agent's mental states. In this sense, modal logic is the study of different existential dimensions of objects and the relations between such objects. For example, in the case of propositional logics, the objects to be considered are propositions and its different existential dimensions are expressed by modalities. Given a hierarchy of possibility operators, each one would be responsible for a given existential dimension of a given proposition. Modal logic, therefore, is a form of research that is concerned with the different ontological dimensions of objects and shows how to manipulate such dimensions. In this thesis, such different dimensions of objects are investigated in order to show how to apply combined modal logics in philosophy. Modal logic is helpful because it is a tool to clarify the analysis of philosophical concepts. As we saw in the introduction, combining logics plays an important role in philosophical issues because there are some statements containing non-interdefinable operators which cannot be formalized using single modal logic (i.e modal logic with just one modal operator). The philosopher usually constructs and finds complicated propositions containing at the same time different modal notions. The first example is that of the verification principle, which can be stated as: "All true propositions can be known." Recently, such principle often appears in discussions about realism and anti-realism. In the verification principle, one can find two non-interdefinable modalities: possibility and knowledge. Therefore, a very simple modal logic is not able to formalize such a sentence. (A detailed study to this problem is proposed in this chapter). The second example, the one treated in the next chapter, is that of skepticism about the world. The statement "Contingent propositions are not known." involves two non-interdefinable modalities: contingency and knowledge. It is difficult to find works studying in detail how to combine contingency and knowledge. Therefore, attempts to study skepticism fail while formalizing the statement. The philosophical statement above is studied from the viewpoint of combined logical systems. Indeed, the method of fusion is applied to provide an example of formalization. It is showed in what sense such complex formalisms can help in the understanding and formalization of statements linking metaphysics and epistemology. We study and consider modal logic, as well other mathematical tools, especially because we think that such tools are able to help in the analysis of philosophical concepts. Such tools isolated just as simple mathematical abstractions are not interesting for our approach, given that we are especially involved in

applications.

The general theory of modalities still awaits some basic developments, considering that up to now it is not clear what modalities are and what properties do modalities possess. There are a lot of different modalities and each modality is a particular way to modify a proposition updating its dimensional content. Given a proposition ϕ , one can always introduce to it a modality. One could create, for instance, $\diamond\phi$ (the possibility of ϕ) or $K\phi$ (the knowledge of ϕ). Such modalities allow the construction of expressions of the form (ϕ is possible) or (ϕ is known). The intuition and the study of modalities is important to understand other dimensions and properties of the actual world. Although introducing modalities in a given proposition allows statements of the above form, nothing can be said, from the viewpoint of non-combined modal systems, when multiple modalities are interacting in a proposition. The interactions between two different families of modalities, those which are called *metaphysical* (or alethic) modalities and those which are called *epistemic* modalities are examined.

While studying metaphysical and epistemic modalities there is also an attempt to provide explanations in metaphysics and epistemology, respectively. The study of formal concepts can help in the analysis and understanding of philosophical areas. However, it is important to note how such formal tools and concepts are limited. VAN BENTHEM in [107] argued that:

“Here is the worst that can happen. Some atlases of philosophical logic even copy philosophical geography (epistemic logic, deontic logic, alethic modal logic), leading to a bad copy of a bad map of reality.”
(VAN BENTHEM in [107])

This statement seems to contain the key to discovering what is the exact role of formal methods in philosophy. What VAN BENTHEM is arguing for is that formal tools can help, but cannot give an entire understanding of philosophical areas. And there is no doubt that sometimes a formal approach to some philosophical notion can even be a caricature of how to proceed. VAN BENTHEM’s claim is correct. It is not reasonable to think that a formal study of metaphysical concepts will examine entirely or even replace a more realistic and intuitive approach, because many aspects of concepts cannot be formalized inside logical systems. In this sense, it is a mistake to think that a formal approach to a given concept can give a complete account to the whole of a given philosophical area. It seems that there will always be a controversy or problem. His metaphor is interesting to show in what sense we have, from one side, a reality and from the other side a language.

Consider the modality of possibility, which is usually formalized as \diamond . Does this modality corresponds to what possibility is? It is hard to say. Formal tools help in the clarification and partial description of what a concept is, but they are never able to describe the totality of the concept. What is interesting is that some philosophical revolutions can be reached by a formal approach to modalities. The example of KRIPKE's necessary *a posteriori* truths in [65]. Without a formal analysis of some basic philosophical concepts, it would not be possible for KRIPKE to find this kind of truth (In the last chapter, KRIPKE's argument is described in some detail). Such kind of result, make us think about the exact role of logic in philosophy. : from one side, logic cannot eliminate all philosophical problems and it cannot give a total description of a given concept. But from the other side, the use of logic helps in order to create some models of reality.

The basic modalities studied in this work are those of possibility, necessity, contingency, knowledge and belief. These modalities are in some sense studied isolated, but what interest to our approach is a study of the interactions of metaphysical and epistemic modalities. Before beginning the first concrete combination, it is important to clarify what is a metaphysical and what is an epistemic modality.

Metaphysical (or alethic) modalities are those related to the general structure of reality. The name metaphysical reflects this content. A metaphysical modality is one which is not directly related to the actual world, but with some potential configuration of this world (i.e. ways that the world could be), that which is called possible worlds, or possible situations. The alethic suggests that the notion of truth appears in these modalities. However, it is important to say that the name alethic is not a appropriate one, because it makes somebody think that just alethic modalities deal with the notion of truth and, therefore, perhaps epistemic modalities do not have anything to teach us about truth. This is false, and for this reason it is not appropriate to use the name alethic. The more general name *metaphysical modalities* reflects exactly the kind of thing that we are looking for: modalities which describe a potential configuration of reality. According to our definition of modal logic, metaphysical modalities manifests the dimensions of concepts. The name *metaphysics* has a long tradition in philosophy, and clearly its scope is larger then the scope of metaphysical modalities. I will not discuss here in detail what metaphysics is, but the idea of metaphysical modalities is a kind of mirror of some particular map of metaphysics. Note that we are not arguing that a study of metaphysical modalities will replace a study of other aspects of metaphysics, but we are arguing that a study of metaphysical modalities can help in clarifying some aspects of metaphysics.

Epistemic modalities are not directly related to the general structure of reality, but rather with the cognitive status that an agent can have about the world. The name *epistemic* suggests, of course, that some relation between the agents and the world are considered. Epistemic modalities are also related to the concept of knowledge in the sense that while analyzing the truth-conditions of propositions, the notion of truth plays an important role. The relations between epistemic modalities and epistemology are present while we are trying to define what is knowledge of a given proposition. However, a study of epistemic modalities is not committed to the study of how agents know a given proposition, but especially with formal and mathematical properties of knowledge. The abstract properties of knowledge, and some other epistemic modality such as belief are clarified inside a logical approach to modalities. Again, such approach will not finish all questions in epistemology, but just give a general light in the subject.

As we said, we are not interested in studying metaphysical and epistemic modalities separatedly, although we present some basic properties of such concepts while isolated, but in studying the relations and the *interactions* of such modalities. Clearly, brief clarifications of isolated modalities are important before studying its interplays with other modalities, but what interest us here are statements of the form, for instance: $\Diamond K\phi$, $K\phi \rightarrow B\phi$, $\nabla K\phi$ (∇ means *contingency*). These statements reflect propositions where two different families of modalities are interacting in just one combined statement. A study on the interactions of metaphysical and epistemic modalities is important in order to show how to connect metaphysics and epistemology. VON WRIGHT once said in [108]:

“Modal logic has turned out to be of singular value to the philosopher’s conceptual analysis. There are several families of concepts, the members of which exhibit in their mutual relations the same formal pattern as the modalities.” (G.H.VON WRIGHT in [108])

Such applications of modal logics in modelling and simulating philosophical concepts have been explored for a long time and it is very difficult nowadays to state something which has not been stated by some philosopher of modal logic while considering modal logics with just one single operator without combined modalities. Thus, I would replace in the above quotation the expression “modal logic” by “combined modal logic”. The reasons of such replacement are expressed in this book.

The main tool which we use here to understand the interplays between metaphysical and epistemic modalities is KRIPKE semantics, and some methods for combining KRIPKE semantics such as fusions and products

of KRIPKE semantics. However, it is important to say that this is not the unique account to modalities. KRIPKE semantics is a powerful tool, but not the only tool one can find. At the beginning, modal logic has been especially related to many-valued logic. BEZIAU has an interesting article explaining some philosophical aspects of many-valued logic approach to modal logic:

“Today many people identify Kripke semantics with modal logic. Typically a book called “modal logic” nowadays is a book about Kripke semantics (cf. e.g. the recent book by [Blackburn et al (2001)]). But modal logic can be developed using other kinds of semantics and Kripke semantics can be used to deal with many different logics and it is totally absurd to call all of these logics “modal logics” . Kripke semantics are also often called “possible worlds semantics”, however this is quite misleading because the crucial feature of these semantics is not the concept of possible world but the relation of accessibility. Possible worlds can easily be eliminated from the definition of Kripke semantics and then the accessibility relation is defined directly between the bivaluations. For this reason it seems better to use the terminology “relational semantics” ”. (See [11])

It is important to note that KRIPKE structure is just a kind or a particular manifestation of relational structure. All KRIPKE structures are relational structures, but there are some relational structures which are not KRIPKE structures. I will not discuss the details of BEZIAU’s idea, but it is important to state that the many-valued environment probably also can help in the research about the essence of modal logic. What is interesting is to know why such approach has been almost eliminated in recent discussions.

Here we are interested in a particular kind of relational structures: KRIPKE structures. Unfortunately, we will not be able to use many-valued logics, with the exception of some arguments using SUSZKO’s inference proving that all logics can be reduced to a bivalent logic (in the last chapter we will examine better such point). Intuitively and *grosso modo*, given a KRIPKE model, a given actual world w , possibility of a given proposition means that there is another world w' related to w such that the proposition which is true in w is also true in w' . If the structure is reflexive, then it suffices that the proposition be true in w . Necessity means truth in all possible worlds, and contingency means that both the proposition and its negations hold in some possible, but different, worlds. Knowledge is the counterpart of necessity, but now considering single KRIPKE models for epistemic modalities.

In the next section the reader find the first interaction of modalities which is studied in this research. The relations between possibility and knowledge which appear in the verification principle and consequently in a very simple

paradox called FITCH's paradox, are studied and investigated. A kind of subject that the reader will not find here concerns detailed comments about other solutions to the paradox. This has been done in different ways and using multiple strategies, but an approach using combined logics (fusions) seems to be the first time ². The reader can easily check references for a detailed study on different approaches to the paradox. Other subject is that of different forms of realism and anti-realism. This point will not be studied here because this discussion is still missing some basic conclusions. What we are interested now is to discuss in what sense there is a paradox. In the following sections of this chapter, the reader will find a particular approach to FITCH's paradox using the technology of combining logics. In order to introduce the problem, take the modality \diamond and also the modality K . There are two possible combinations:

1. $\diamond K\phi$ (can be known, knowability, possible knowledge)
2. $K\diamond\phi$ (knowledge of possibilities)

Next section we are interested in the first case, that of where the knowledge operator is directly influenced by the possibility modality. Interactions between \diamond and K can also appear in different forms:

1. $\diamond\phi \rightarrow K\phi$ (possibility implies knowledge)
2. $K\phi \rightarrow \diamond\phi$ (knowledge implies possibility)

But such interactions will just be considered later. Now, the main purpose is to propose a re-interpretation of FITCH's paradox and of the knowability modality from the viewpoint of fusions and products of modal logics.

1.1 Fusions of modal logics and Fitch's paradox

This section³ shows that although FITCH's paradox has been extremely widely studied, up to now no correct formalization of the problem has been proposed. The purpose of this section is to present the paradox from the

²RABINOWICZ and SEGERBERG in [92] used the notion of product, but it does not seem that they were arguing that a method for combining logics played an important role in solving FITCH's paradox.

³A modified version of this section appeared in the *Croatian Journal of Philosophy*, Vol.VI, No.17, 2006. Thanks to Carlos Caleiro for comments on combining logics and thanks to Joe Salerno for useful e-mails on Fitch's theorems.

viewpoint of combining logics. It is argued that the correct minimal logic to state the paradox is composed by a fusion of modal frames, and a fusion of modal languages and logics.

1.1.1 Church or Fitch?

What is nowadays known as FITCH's paradox appeared in 1963 in the famous *Journal of Symbolic Logic* (FITCH in [42]). However, FITCH states that the argument was discovered rather by an anonymous referee (ibid.). J. SALERNO recently launched an investigation to uncover this anonymous referee by examining some correspondences between E. NAGEL and A. CHURCH (SALERNO in [98]). He suggests that CHURCH was the referee:

“In 1945 Church refereed a paper written by Fitch; the author of the report was anonymous to Fitch; and Fitch's paper was (at least, at this stage) not being accepted for publication. If this was the paper in question and there were no other referees on the job, then it would seem that Church was the anonymous referee who conveyed the knowability result to Fitch in 1945.” (SALERNO in [98])

Despite the problem examined by J. SALERNO showing that CHURCH was indeed the author of the paradox, FITCH's name is used to identify the problem which shows that if we accept that “All truths are knowable” then we should also accept that “All truths are known”. The purpose of this section is to examine the nature of FITCH's paradox of knowability from the viewpoint of combining logics. In order to introduce the paradox, the verification principle and the collapse principle are investigated showing how FITCH's paradox connects both. The language of the paradox is not that of simple normal modal logics, but a language generated by fusion of two modal languages: that of normal modal logic with \diamond and that of epistemic logic with modality K . Also, it is shown what is the logic of the paradox (i.e. what is the logic used within FITCH's argument) by examining in detail FITCH's theorems. A countermodel is proposed in order to avoid FITCH's paradox.

1.1.2 The verification principle

There are different ways to state the verification principle, for example:

- All true propositions can be known (i.e. are knowable);

- If a proposition is true, then it can be known (i.e. it is knowable);

The “it can be known” and “knowable” have different interpretations. Long time ago the verification principle has been stated by David HUME:

“When we entertain, therefore, any suspicion that a philosophical term is employed without any meaning or idea (as is but too frequent), we need but enquire, from what impression is that supposed idea derived?” (HUME in [58])

Of course, HUME’s investigation of what impression corresponds to a given philosophical concept (or idea) is a discrete manifestation of the principle of verification. Afterwards, The Vienna Circle also defended the verification principle to determine when propositions are meaningless ⁴. This principle has also been called by some authors “knowability principle” and it is certainly a central step towards an understanding of the knowability paradox. H. PUTNAM in [90] criticized the principle of verification arguing that it cannot itself be verified, and is thus meaningless ⁵. T. WILLIAMSON in [115] calls this principle weak verificationism and argues that it deals with the limits of knowledge, given that it states exactly up to what point it is possible to know. KANT, in his *Critique of Pure Reason* [60], investigated how/what human beings can know and what are the limits of knowledge. The principle of verification is also an answer to KANT’s problems and, obviously, to the antinomies of pure reason ⁶. Formally, we can drop quantifiers and announce this principle by the following statement in a fusion of the languages of modal logic **K** and the epistemic version of **KT**(**T***):

$$(VP) \phi \rightarrow \diamond K \phi$$

⁴There are two levels of verification: verifiability of truth and verifiability of meaning. The verification principle treated here refers to the verifiability of truth, but it can also be understood, up to a certain level, as verifiability of meaning. However, it is important to state that both are different. If verifiability of truth is equivalent to verifiability of meaning, then a proposition is meaningful if and only if it is true. But this is not correct, assuming that many false propositions can be accepted as meaningful propositions.

⁵PUTNAM is considering verifiability of meaning.

⁶Although KANT is not a verificationist, it is interesting to show how the verification principle is connected even to his philosophy. It is well-known the relations between the verification principle and the philosophers of The Vienna Circle, but it is not evident how such a principle is related to KANT’s philosophy. Considering that KANT intends to discover what are the limits of knowledge, and what can be known, the verification principle touches his philosophy, given that a verificationist should answer: True propositions can be known.

B. Brogaard and J. Salerno in [19] argued that knowability is factive because it implies truth ($\Diamond K\phi \rightarrow \phi$). If knowability is factive, and if the verification principle holds, then knowability is equivalent to truth, what is clearly incorrect. R. COOK in [30] also has an interesting and intuitive argument showing why knowability cannot be factive. The verification principle is the main statement of FITCH's paradox, because from it FITCH deduces the collapse principle.

1.1.3 The collapse principle

There are different ways to state the collapse principle:

- All true propositions are known;
- If a proposition is true, then it is known;

Formally, we can eliminate quantifiers and announce this principle by the following statement in the epistemic version of the modal logic $\mathbf{KT}(\mathbf{T}^*)$:

$$(CP) \phi \rightarrow K\phi$$

The collapse principle, also called strong verificationism by T. WILLIAMSON, is a problematic statement because if it is true, then it implies that: There are omniscient agents who are able to know all true propositions. The collapse principle is obviously equivalent to the following: $\neg(\phi \wedge \neg K\phi)$

The concept of knowledge collapses with the concept of truth. From the intuitive viewpoint, the collapse principle states that true propositions are always known, which is not reasonable, given that there are many propositions which are true and that we do not know. From the logical viewpoint, the collapse principle states that the concept of knowledge and the concept of truth are the same, and then there is no need for epistemic logics, because they collapse with classical propositional logic, given the verification principle.

1.1.4 The nature of Fitch's paradox

There is a very famous problem related to the verification principle and, consequently, to the concept of knowability. It has been treated recently by a great variety of philosophers and logicians. FITCH's paradox (or the CHURCH-FITCH paradox, as J. SALERNO suggested) shows that the verification principle entails the collapse principle. There is a substantial amount of articles presenting the knowability paradox (BENTHEM in [106], MELIA

in [77], WANSING in [112], WILLIAMSON in [114], EDGINGTON in [40], BROGAARD and SALERNO in [19]). I examine FITCH's text itself and explore what is the exact language of the paradox as well its right logic, the minimal logic used to generate the problem.

The language of the paradox

In order to understand FITCH's paradox, let us assume that there is a set *PROP* of propositional variables and full classical propositional constructors $\{\neg, \wedge, \vee, \rightarrow\}$ expanded by two modal operators \diamond and K . The main argument is the following:

(FITCH, theorem 5 of [42]) "If there is some true proposition which nobody knows (or has known or will know) to be true, then there is a true proposition which nobody can know to be true."

The problem begins with the formalization of the argument. The full formal language of FITCH's theorem 5 should be second-order modal epistemic temporal logic, but this would make our work much too complicated, considering quantification over propositions, a temporal dimension and multi-agent systems. FITCH even states that "For the purposes of simplification, the element of time will be ignored..." and "we will often ignore the agent". Choosing another kind of language to express the problem, we can concentrate on the language defined as a fusion of a modal language and an epistemic language without quantifiers. FITCH's argument can be decomposed in the following way:

1. "There is some true proposition which nobody knows to be true."
2. "There is a true proposition which nobody can know to be true."

Consider two modal languages, one alethic $L1 = \langle \neg, \wedge, \vee, \rightarrow, \diamond \rangle$ and the other epistemic $L2 = \langle \neg, \wedge, \vee, \rightarrow, K \rangle$. The language of their fusion is defined as the union of all constructors $L1 \oplus L2 = \langle \neg, \wedge, \vee, \rightarrow, \diamond, K \rangle$; (see GABBAY in [43] for a study on fusion of modal languages). Only in a fusion of the languages it is possible to state and formalize the verification principle and, therefore, FITCH's paradox for epistemic modalities. The first clause states that there is at least one true proposition which is unknown:

$$\phi \wedge \neg K\phi$$

Indeed, the above formula, which can be formalized without a fusion of modal languages, is equivalent to the negation of the collapse principle, and it states that epistemic agents are not omniscient. The second clause states that there is at least one true proposition which is unknowable (i.e. which can not be known). Its formalization is the following:

$$\phi \wedge \neg \diamond K \phi$$

Clearly, if we have two non-interdefinable modal operators, a simple modal logic does not work while we are formalizing FITCH's paradox. The above formula is equivalent to the negation of the verification principle. FITCH's theorem 5, then, can be reduced to the following inference, assuming that there is always a consequence relation \vdash associated with an \rightarrow (and also assuming that the metatheorem of deduction holds):

$$(\phi \wedge \neg K \phi) \rightarrow (\phi \wedge \neg \diamond K \phi)$$

By contraposition, it follows that:

$$\neg(\phi \wedge \neg \diamond K \phi) \rightarrow \neg(\phi \wedge \neg K \phi)$$

And by the De Morgan rules together with the definition of classical \rightarrow , it follows that:

$$(\phi \rightarrow \diamond K \phi) \rightarrow (\phi \rightarrow K \phi)$$

So, FITCH's paradox deduces from the verification principle the thesis that all truths are known. This fact is clearly a problem at the intuitive level and at the logical level, because inside the logic of FITCH's reasoning there is the knowledge's axiom which states that:

$$K \phi \rightarrow \phi$$

Therefore, theorem 5 implies the collapse of knowledge with truth:

$$K \phi \leftrightarrow \phi$$

The minimal language to generate the paradox has to be a language powerful enough to express the combined modality $\diamond K$. Therefore, the first conclusion of this section is that we need a fusion of two modal languages in order to model the verification principle. Hence, the language of FITCH's paradox is a fusion of two modal languages: one alethic and other epistemic.

The logic of the paradox

How do we prove FITCH’s theorem 5? How do we prove that the verification principle entails the collapse principle? In [42], he intends to “provide a partial logical analysis of a few concepts that may be classified as value concepts, or as concepts that are closely related to value concepts”. Such concepts, according to FITCH, have some basic properties:

1. They are closed with respect to conjunction elimination;
2. They are closed with respect to conjunction introduction;
3. They are truth-classes.

The first clause states that if \star is a value concept, then it respects:

$$(1)\star(\phi \wedge \psi) \rightarrow (\star\phi \wedge \star\psi)$$

Such a principle holds in all extensions of the normal modal logic **K**. The second clause states that if \star is a value concept, then it respects:

$$(2)(\star\phi \wedge \star\psi) \rightarrow \star(\phi \wedge \psi)$$

And the third clause means that “A class of propositions will be said to be a truth class if (necessarily) every member of it is true”. Indeed, what FITCH wants to say is that the following law holds:

$$(3)(\star\phi \rightarrow \phi)$$

What is the minimal modal epistemic logic which respects (1)–(3)? B. BROGAARD and J. SALERNO in [19] state that “the logic of FITCH’s result is modest: a minimal, normal, modal logic and two very intuitive epistemic principles”. This is partially correct, because there is no normal modal logic with epistemic principles, but only a combined and complex modal logic with alethic and epistemic modalities ⁷. We cannot simply add epistemic principle to a modal logic. If we do this, it would not be possible to semantically judge formulas with K , given that our models would not be able to recognize a complex modality as $\diamond K$ (see the fibring problem in GABBAY

⁷Adding epistemic principles to a normal modal logic is not so trivial as it seems to be. When we add epistemic principles to a given normal modal logic, we are changing the content of the axiomatic system and, consequently, changing its semantics. If we proceed as the authors suggest, we will not be able to prove metatheorems such completeness, and we will not be able to determine how things are from the semantical viewpoint. In this sense, it is better to discuss about combined modal logics.

[43], and see GABBAY ET AL. in [44] for a detailed study on fusions of modal logics). H. WANSING in [112] states that the logic of the paradox is modal epistemic logic based on classical propositional logic. This is correct, but H. WANSING does not show exactly how to generate this combined logic. The same happens with J. VAN BENTHEM in [106]. Before presenting the right logic of the reasoning, let us observe that FITCH proves a very important theorem that is used to demonstrate the knowability paradox (see also BROGAARD and SALERNO [19] , and SALERNO in [98] for some remarks on FITCH's theorems and its history):

(FITCH, Theorem 1 of [42]) “If \star is a truth class which is closed with respect to conjunction elimination, then the proposition $(\phi \wedge \neg \star \phi)$, which asserts that ϕ is true but not a member of \star (where ϕ is any proposition), is itself necessarily not a member of \star .”

It means, formally, that it is the case:

$$\Box \neg \star (\phi \wedge \neg \star \phi)$$

Or:

$$\neg \Diamond \star (\phi \wedge \neg \star \phi)$$

The concept of knowledge is not only a truth class but also a notion which satisfies conjunction elimination ⁸. Therefore, as a result of FITCH's theorem 1, we have the same for the knowledge operator:

$$\neg \Diamond K(\phi \wedge \neg K\phi)$$

There is a non-constructive proof of theorem 1 (see FITCH [42]): Suppose, for *reductio ad absurdum*, that $(\phi \wedge \neg \star \phi)$ is a member of \star : $\star(\phi \wedge \neg \star \phi)$. Given clause (1), we know that \star distributes over ϕ and $(\neg \star \phi)$, therefore $\star(\phi)$ and $\star(\neg \star \phi)$. Given that \star is a truth class, it follows $\neg \star \phi$, which is a contradiction. FITCH uses theorem 1 to prove a very similar result, but now in a constructive way:

(FITCH, Theorem 2 of [42]) “If \star is a truth class which is closed with respect to conjunction elimination, and if ϕ is any true proposition which is not a member of \star , then the proposition, $\phi \wedge \neg K\phi$, is a true proposition which is necessarily not a member of \star .”

⁸Inference rules apply to propositions, and not concepts. When we say that the concept of knowledge satisfies conjunction elimination we state, indeed, that some propositions containing the notion of knowledge satisfy conjunction elimination.

FITCH's paradox is generated by putting together the instantiation of theorem 1 and, as said J. VAN BENTHEM in [106], "a clever substitution instance" of the verification principle:

$$(VP)(p \wedge \neg Kp) \rightarrow \diamond K(p \wedge \neg Kp)$$

Using an instance of FITCH's theorem 1:

$$(VP)\neg \diamond K(p \wedge \neg Kp)$$

which leads to:

$$(VP)\neg(p \wedge \neg Kp)$$

or, equivalently:

$$(VP)p \rightarrow Kp$$

So, by theorem 1 and contraposition, FITCH's theorem 5 can be proved. Therefore, propositions of the form $(p \wedge \neg Kp)$ are counterexamples to the verification principle because they cannot be known. As an instantiation of the verification principle, if the proposition $(p \wedge \neg Kp)$ is true, then it can be known. But there is a proof that this kind of proposition cannot be known. So, the proposition $(p \wedge \neg Kp)$ is false. Therefore, there are omniscient agents able to know every true proposition. Indeed, FITCH states in his real anonymous theorem 4 that "For each agent who is not omniscient, there is a true proposition which that agent cannot know". This is probably the foundation of the "clever substitution instance" mentioned by J.VAN BENTHEM. Note that in the above argument ϕ has been replaced by p because we are not dealing with propositional variables, but atomic propositions.

What is the right logic of the paradox? We know that to formulate (VP) we need two modalities. and therefore a fusion of modal languages. Hence, no simple modal logic can be used. But how do we discover what is the exact logic of the paradox? The answer is this: evaluating formulas containing $\diamond K$, we need at least fusions of KRIPKE frames or fibred models with fibring functions, (otherwise the fibring problem mentioned in GABBAY [43] would appear).

Thus, the axiomatic system of the logic of FITCH's paradox is the following: Take the propositional classical modal logic \mathbf{K} and the propositional epistemic modal logic \mathbf{T}^* (single-agent). The fusion of the axiomatic systems \mathbf{K} and \mathbf{T}^* is composed by the following set of modal axioms (for a complete study on fusions of modal logics check GABBAY ET AL in [44]):

1. $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$;
2. $(K(\phi \rightarrow \psi) \wedge K\phi) \rightarrow K\psi$;
3. $K\phi \rightarrow \phi$;

In order to obtain the logic of FITCH's paradox, we have to add the verification principle:

4. $\phi \rightarrow \Diamond K\phi$;

And three inference rules which came from each axiomatic system:

5. $\vdash \phi$ then $\vdash \Box\phi$;
6. $\vdash \phi$ then $\vdash K\phi$;
7. *MP*

Indeed, the right, minimal logic of FITCH's paradox is composed by the fusions of a modal and an epistemic languages and logics, and by the fusion of two KRIPKE models (one alethic and the other epistemic) plus the verification principle. In particular, for FITCH's paradox, the fusion is $\mathbf{K} \oplus \mathbf{T}^* \oplus (\phi \rightarrow \Diamond K\phi)$. One can check that those principles which are considered as the basic rules for generating FITCH's paradox are just consequences of the above axiomatic system. From the semantical viewpoint, the frames that should be used to model FITCH's reasoning are obtained by fusions of KRIPKE frames: $F_1 \oplus F_2 = \langle W, R, P \rangle$ where:

1. W is a set of possible worlds;
2. R is an accessibility relation for \Diamond ;
3. P is a reflexive accessibility relation for K .

$F_1 = \langle W, R \rangle$ is a frame for the alethic modal logic \mathbf{K} and $F_2 = \langle W, P \rangle$ is a modal frame for the epistemic modal logic \mathbf{T}^* . The resulting logic $\mathbf{K} \oplus \mathbf{T}^*$ has a complete and sound axiomatization because the fused logic preserves completeness (Check the results contained in [44]). But what about the logic $\mathbf{K} \oplus \mathbf{T}^* \oplus (\phi \rightarrow \Diamond K\phi)$? Is it also complete?

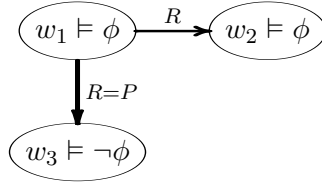
A countermodel to FITCH's paradox

If frames are fusions of the form $F_1 \oplus F_2 = \langle W, R, P \rangle$, then there is a countermodel based on the fused frame where FITCH's paradox does not hold. It is the following ⁹:

$M_1 \oplus M_2 = \langle W, R, P, V \rangle$, where:

1. $W = \{w_1, w_2, w_3\}$;
2. $w_1 R w_2, w_1 R w_3$;
3. $w_1 P w_3$ and P is reflexive;
4. $P \subseteq R$;
5. $V(\phi) = \{w_1, w_2\}$

In a picture, the above model is the following:



The formula $\vdash (\phi \rightarrow \Diamond K\phi) \rightarrow (\phi \rightarrow K\phi)$ is not valid in the model $M_1 \oplus M_2$. To check that $(\phi \rightarrow \Diamond K\phi)$ holds, consider the world w_1 . In $w_1 \models \phi$ and $w_2 \models \phi$. Given that w_2 is just accessible from itself, then $w_2 \models K\phi$. Therefore, $w_1 \models \Diamond K\phi$. To check that $\vdash (\phi \rightarrow K\phi)$ does not hold, consider that $w_1 \models \phi$. However, w_3 does not prove ϕ . Therefore, w_1 does not prove $(\phi \rightarrow K\phi)$. Hence, in w_1 the implication $\vdash (\phi \rightarrow \Diamond K\phi) \rightarrow (\phi \rightarrow K\phi)$ is not true, and then the deduction $\vdash (\phi \rightarrow \Diamond K\phi) \rightarrow (\phi \rightarrow K\phi)$ is not valid in the model $M_1 \oplus M_2$.

1.1.5 So many people

FITCH's paradox shows that the verification principle leads to the collapse principle. J. VAN BENTHEM said that there are basically two approaches to the paradox:

⁹Considering the notion of local countermodel, and not the notion of global countermodel, we do not write the reflexive relation.

“Some weaken the logic in the argument still further. This is like turning down the volume on your radio so as not to hear the bad news. You will not hear much good news either. Other remedies leave the logic untouched, but weaken the verification principle itself. This is like censoring the news: you hear things loud and clear, but they may not be so interesting.” (J. VAN BENTHEM in [106])

We can mention as an example of those who are turning down the volume: the article of H. WANSING in [112], where he proposes a paraconsistent constructive relevant modal epistemic logic with strong negation to avoid some inferences of the argument. Strategies which go in the same direction are COSTA-LEITE in [36] and CARNIELLI ET AL in [36.1], but the difference is that the later do not have all those ontological commitments of WANSING’s solution. Other approaches use intuitionistic logic to avoid FITCH’s argument (see WILLIAMSON in [115]). As examples of those who weaken the verification principle we can mention RESTALL in [93] and EDGINGTON in [40]. The last one adds one extra modal operator and applies it to reformulate the verification principle. Two similar approaches go in the same direction trying to reformulate the position defended by EDGINGTON. The first one is that of RABINOWICZ and SEGERBERG in [92]. The authors propose a way to combine actuality, possibility, and knowledge by using two-dimensional modal semantics. Another approach is that of LINDSTRÖM in [71]. The conclusion of this section is the following: the logic to formulate FITCH’s paradox is composed semantically by a fusion of KRIPKE frames and syntactically by a fusion of languages and logics plus the verification principle $\mathbf{K} \oplus \mathbf{T}^* \oplus \phi \rightarrow \diamond \mathbf{K}\phi$. As an open problem, how to prove that the logic $\mathbf{K} \oplus \mathbf{T}^* \oplus \phi \rightarrow \diamond \mathbf{K}\phi$ is complete? Or is it incomplete? Given that there is a countermodel able to falsify the deduction, probably the logic of FITCH’s reasoning is not a complete logic (soundness included). If FITCH’s paradox is formulated in the environment of fusion of modal logics, then there is a countermodel which validates the verification principle but falsify the collapse principle. Indeed, FITCH’s paradox occurs when there is not a distinction between the accessibility relation for \diamond and the accessibility relation for K ¹⁰.

¹⁰End of the article.

1.2 Other formulations

There are many philosophers trying to solve the paradox ¹¹. Almost all articles dealing with Fitch’s paradox have the same structure: they present the paradox, then they propose some comments on the literature, and a supposed solution to it. Here we will not repeat such variety of different solutions. I will not describe in detail all possible solutions to the paradox. Moreover, a general criticism that I have to almost all of them is that they are not using the correct framework to state the problem. Some authors, as J.SALERNO, are considering the paradox just in the scope of the so-called single normal modal logics ¹², and therefore they are not able to analyze correctly the problem which is, from the logical viewpoint, mainly a problem of combination of logics ¹³. The same happens with T.WILLIAMSON: he considered the logic of FITCH’s paradox as simple normal modal logic. Even FITCH, the supposed author of the paradox, did not know that his deduction was realized in a combined modal logic. This discovery was just possible after the development of tools such combining logics. I hope have shown how this technique play an important role in the paradox.

FITCH’s paradox, as stated in the last section, is a puzzle within combined modal systems. We analyzed the standard presentation of the paradox. Now, let consider other forms of the verification principle which induces the same consequences for the limits of knowledge. There is no doubt that the combined concept of “possible knowledge” deals with the limits of knowledge, because what can be known is manifested in the natural KRIPKE semantics for both operators.

Consider, for instance, adding the following forms of the verification principle to the fused language and logic $\mathbf{K} \oplus \mathbf{T}^*$:

1. $\phi \rightarrow \Diamond K\phi$
2. $\Diamond\phi \rightarrow \Diamond K\phi$

¹¹Fitch’s paradox is important for us here because it contains a typical example of combined modalities. Discussing many different formulations of the paradox, and discussing the literature on the paradox in detail, would distract the reader, and would put us far away of our main targets. Thus, such extra formulations and comments are avoided.

¹²Given that the verification principle contains two non-interdefinable modalities, it follows that just a combined modal logic is able to formalize it. And a combined modal logic is generated by a precise method of combination. Generating the paradox without this methodology is a way to misunderstand the content of the paradox.

¹³But from the philosophical viewpoint, it is also a problem related to anti-realism and verificationism.

3. $\Box\phi \rightarrow \Diamond K\phi$
4. $K\phi \rightarrow \Diamond K\phi$
5. $\nabla\phi \rightarrow \Diamond K\phi$ (considering a different fusion)

Consider the substitution instance as the non-omniscient thesis. Then: Number (1), as showed in the last section, is a component of FITCH's paradox and then entails the collapse of knowledge an truth ¹⁴. But what can be said about the other forms of verification? Do they also entail the collapse principle?. Number (2) also entails the paradox since $\phi \rightarrow \Diamond\phi$ is a theorem of the logic $\mathbf{K} \oplus \mathbf{T}^*$ we get again the verification principle ¹⁵. Number (3) entails a weak collapse, because it implies that $\Diamond(\phi \rightarrow K\phi)$. Number (4) entails $K(\phi \rightarrow K\phi)$ and given reflexivity, one can deduce again the collapse. Number (5) is considered in the next chapter.

Why should us be just interested in one form of FITCH's paradox? How to avoid FITCH's argument for all these forms of the verification principle? This is a question out of the scope of this book, but it will be treated in another occasion. I will not propose here solutions to all these forms of FITCH's paradox, but just to point out that *there are* other forms. It would be interesting an investigation towards the substance of FITCH's deduction and the multiple variations one can propose of the verification principle (as those proposed above). Given the fusion of the modal logic \mathbf{K} and the epistemic logic \mathbf{T}^* , what would happen from the viewpoint of logic if we add each time a different kind of the verification principle presented above? This question is complicated, and it shows that FITCH's paradox can be even worse than we supposed.

1.2.1 Philosophical foundations of the paradox

It is true that FITCH's paradox is a logical paradox. Otherwise, it has a very strong philosophical content which deals with the limits of knowledge. This fact is reflected by the combined modality: $\Diamond K$. Using classical KRIPKE semantics, this modality states that there is a world where one can find the knowledge of a given proposition ϕ . The verification principle, as argued, suggests an answer to KANT's problem in his book [60], where he asked up to where it is possible to know. The verification principle shows that all true propositions can be known, and therefore the border between true and false propositions looks like as a characteristic of the limits of knowledge. We

¹⁴At least without restrictions on the accessibility relations.

¹⁵I owe this remark to Claudio Pizzi in personal communication.

just can know true propositions, and if we try to know a false proposition it will not be the case. Indeed, false propositions, as it will be explained later, cannot be object of knowledge but just object of belief. Assuming that a contradiction or an inconsistency is not a true proposition follows that we cannot know contradictions. But can we believe in contradictions? It seems a common place that we have contradictory beliefs. Many authors did this kind of remark. But contradictory knowledge is not so clear. An investigation of this problem is examined in chapter 4.

FITCH's paradox shows that although knowledge is defined as justified true belief, if the paradox is the case, then knowledge does not need to be a justified belief, but just a true proposition. This is the meaning of the collapsing principle according to which if a proposition is true then it is known. FITCH's paradox is, therefore, related to GETTIER's problem proposed in [47], because both are counterexamples to the traditional definition of knowledge. In the case of FITCH, if knowledge is defined as desired by PLATO, then it collapses with truth ¹⁶.

In the philosophical tradition, KANT is the most important philosopher who studied informally the limits of knowledge. In his book [60] he tried to show how to know and what are the limits of knowledge. KANT's book is the most important modern philosophical book on the limits of knowledge and related to the verification principle and FITCH's paradox. Since my first reading of the whole of his *Critique*, KANT always appeared to me as a very important point in philosophy, because he can be viewed as a central station where all philosophers before him arrived and a point of depart of many contemporary philosophers and also to the philosophy developed in this book. From one side, empiricism and rationalism find in KANT's philosophy a kind of synthesis of both conceptions. On the other hand, continental and analytical philosophers also find in KANT a point to begin their journey towards the understanding of language and knowledge. The reader should take a look in the following KANT's comments and see the presence of the verification principle and questions related to the limits of knowledge in some of his statements:

“... the principles of which it is making use transcend the limits of experience, they are no longer subject to any empirical test. The battle-field of these endless controversies is called metaphysics.”
(KANT in [60])

¹⁶In the article [47] GETTIER states that “Plato seems to be considering some such definition at Theaetetus 201, and perhaps accepting one at Meno 98.”. Indeed, he is doing a reference to the standard definition of *knowledge* as *justified true belief*, which is attributed to PLATO.

For KANT, what is beyond all possible experience is something which belongs to metaphysics. It is interesting to note that although KANT was not an empirist, he defended that what cannot be verified in the grounds of all possible knowledge is certainly something not entirely clear which belongs to metaphysics. Thus, we can state that the limits of knowledge according to KANT are the area of all possible knowledge. His book [60] can be viewed as book on the limits of knowledge and how it is indeed possible. It is not difficult to find in the mentioned book a lot of arguments trying to show how knowledge is possible. For this reason, KANT's philosophy is strictly related to the FITCH's paradox. Both are analyzing the limits of knowledge. KANT do this in an epistemological way, while FITCH studies such limits from an logical approach. In what follows, we study the problems of limits of knowledge from a two-dimensional perspective using products of modal frames and logics.

1.3 Products of modal logics and the limits of knowledge

The purpose of this section is to explain an attempt to define a new modality with philosophical interest by combining the basic modal ingredients of possibility and knowledge ¹⁷. The modality $\diamond K$ has a dimensional content which is studied in this section.

In this chapter we got already a result showing that FITCH's paradox does not follows in a combined logic with special conditions in the accessibility relations, but we want more. What we are going to do now is to explore new aspects of applications of combining logics. In the first chapter, we used fusions of modal languages, logics and semantics. Now, we analyze a more complicated case, that of products. What we want to do is to explore the dimensional character of the combined modality $\diamond K$. This section can be seen as an attempt to model knowability using products. In this sense, we can continue the applications of combining logics in philosophical issues, especially proposing a bridge between epistemology and metaphysics. The strategy here consists in using the same fused logic which is able to analyze FITCH's paradox but now considering a product. Products are generated from fusions by adding some interaction axioms and designing a two-dimensional semantics obtained by combination of KRIPKE frames. This combination is realized via product of modal frames so as to construct

¹⁷A very incomplete and preliminary version of this section has been published in the Proceeding of CombLog'04.

a knowability modality, which is a bidimensional constructor of arity one defined in a two-dimensional modal frame. In order to achieve our main purpose such construction should not be definable using \diamond and K operators. A semantical interpretation for the operator is suggested, as well as an axiomatic system able to account for inferences related to this new modality. The resulting system for knowability **LK** is supposed to be sound and complete with respect to its class of modal-epistemic product models, although such a technical problem is not investigated. A very interesting property of the logic **LK** is that it can be applied to reformulate the knowability principle according to which “All truths are knowable.” in such a way that FITCH’s knowability paradox will not be so dangerous even from the two-dimensional point of view.

According to [43], “as logic is being used more and more to formalise field problems in philosophy, language, artificial intelligence, logic programming and computer science, the kind of logics required become more and more complex.”. This is the idea. The concept of knowability, which has been a target of several articles related to the knowability paradox, studied in the framework of combined modalities is a typical example of application. As the reader should already know is that this paradox is a modal argument which shows that given the bimodal logic $\mathbf{K} \oplus \mathbf{T}^*$ extended with the verification principle $A \rightarrow \diamond KA$ it is possible to deduce $A \rightarrow KA$ causing the collapse of the concept of knowledge with the concept of truth, at least without special restrictions on the accessibility relations. Proposing a modal formalism able to account for the complex knowability modality which emerges in the presence of the verification principle entails a natural conjecture: using the knowability modality one can define a 2-dimensional modal version of the logic $\mathbf{K} \oplus \mathbf{T}^*$ where it is possible to add a bidimensional version of the verificationist principle without causing the collapse of the knowledge operator. This should be viewed as a plausible solution to the trouble. Up to now, there is no definitive argument in this direction using two-dimensional products, but just a general clue.

One basic assumption here is that the basic difference between the knowability modality and other usual modalities, in general terms, is that it is not possible to define entirely knowability in monodimensional modal reality. Therefore, a knowability modality requires a two-dimensional environment defined by means of monomodal frames. A very *ad hoc* way to save the verification principle against the consequences of FITCH’s paradox is introduced. Everybody formalizes the knowability principle using two modalities: possibility and knowledge(a kind of box operator). What is interesting is that some experiments can be done in the scope of combining logics. I propose a

different formalization of the principle, by means of what I call knowability operator. In a nutshell, the strategy consists in replacing in the formalization of FITCH’s argument two modalities by just one combined modality: knowability. Then, one could state the verification principle as:

$$\phi \rightarrow \Box\phi$$

And then no knowledge’s collapse would follow, as it is explained later. The above re-formulation of the verification principle is obviously introduced in the fusion generated in the last section. Even if we do not have grounds to state that FITCH’s paradox finished from our perspective, we can believe that a nice way to the paradox has been proposed without falling down the volume of the radio - the metaphor mentioned by J.VAN BENTHEM (Check previous section)

1.3.1 The logic for knowability

How to express the concept of knowability? Such a concept is an hybrid concept composed by two modalities: possibility and knowledge. There is no difference in stating that a given agent can know ϕ or stating that ϕ is knowable. Both have the same reference. A sketch of a logic for knowability is presented syntactically and semantically, some properties of this logic are studied, and a conjecture that the metatheoretical result of completeness holds. In order to introduce the main ideas, let me first determine an exact language to generate propositions containing the hybrid notion of knowability.

The language of the logics for knowability is generated by a set of propositions *PROP*, full classical operators and three unary modal operators: \Diamond , K and \Box . Such a language allows the construction of genuine expressions to describe the limits of knowledge, because knowability means “what can be known”. The expressions can be generated by recursion with special attention to the clause where the knowability appears: If ϕ is a well-formed formula, then $\Diamond\phi$, $K\phi$, $\Box\phi$ are well-formed formulas. Now we know how to produce linguistic constructions containing “knowability” as a primitive connective. But we do not know neither how to evaluate such constructions nor how to manipulate symbols containing the knowability modality. This is exactly what is done below.

The knowability operator

It is well-known that the operation of product when applied to singular modal frames, defines many-dimensional modal frames. Many-dimensional frames are used to interpret modal languages for multi-dimensional modal logics (see for instance [75] and [44]). In [43] and [44], there is the definition of product of modal frames and a general idea of how to generate modalities of higher dimension. A particular case of such definition is adequate for our account of knowability. From one point of view, the horizontal lines describe metaphysical worlds while the vertical lines represent the epistemic worlds. In the two-dimensional environment we have exactly “what can be known” while in each monodimensional reality we have “possibility” and “knowledge”, respectively. In this sense knowability is presented here as a n -dimensional construction for $n = 2$.

We argue that it is reasonable to introduce bidimensional modal semantics to formalize the knowability operator: using a concrete case of the definition of products, and taking two given frames:

$$\begin{aligned} F_1 &= \langle W, R \rangle \\ F_2 &= \langle S, P \rangle \end{aligned}$$

where $F_1 = \langle W, R \rangle$ is interpreted as an alethic frame such that W is a set of possible worlds and R is an accessibility relation between worlds, while $F_2 = \langle S, P \rangle$ is interpreted as an epistemic frame composed by a set S of epistemic states and a plausibility relation P between epistemic states. The *simple product of a modal and an epistemic frame* is defined as:

$$F_1 \times F_2 = \langle W \times S, R', P' \rangle$$

where:

$$\begin{aligned} \langle w, s \rangle R' \langle w', s' \rangle &\text{ iff } wRw' \text{ and } s = s' \\ \langle w, s \rangle P' \langle w', s' \rangle &\text{ iff } w = w' \text{ and } sPs' \end{aligned}$$

Just single-agent frames are considered here. The elements of $W \times S$ are called *modal-epistemic states* while R' and P' are called *two-dimensional accessibility relations*. The basic difference between alethic and epistemic frames is that the accessibility relation does not have the notion of agents indexed to it. In this sense it is impossible to use accessibility relations and plausibility relations in the same way. Using the intuition to model possibility and knowledge, we can formulate the following standard truth conditions:

(Two-dimensional possibility and knowledge)

$\langle w, s \rangle \models \Diamond \phi$ iff $\exists \langle w', s' \rangle$ such that if $\langle w, s \rangle R' \langle w', s' \rangle$ then $\langle w', s' \rangle \models \phi$;

$\langle w, s \rangle \models K\phi$ iff $\forall \langle w', s' \rangle$ such that if $\langle w, s \rangle P' \langle w', s' \rangle$ then $\langle w', s' \rangle \models \phi$;

Given a product of a modal and an epistemic frames, the formal semantics for the the knowability operator is:

(Semantics for the knowability operator)

$\langle w, s \rangle \models \Box \phi$ iff $\exists \langle w', s' \rangle$ such that if $\langle w, s \rangle R' \langle w', s' \rangle$ then $\langle w', s' \rangle \models \phi$ but also $\forall \langle w'', s'' \rangle$ such that if $\langle w', s' \rangle P' \langle w'', s'' \rangle$ then $\langle w'', s'' \rangle \models \phi$

Adding to the two above clauses the following bidimensional classical valuations to connectives:

$\langle w, s \rangle \models p$ iff $\langle w, s \rangle \in v(p)$, for p atomic.
 $\langle w, s \rangle \models \neg \phi$ iff $\langle w, s \rangle \not\models \phi$
 $\langle w, s \rangle \models \phi \wedge \psi$ iff $\langle w, s \rangle \models \phi$ and $\langle w, s \rangle \models \psi$
 $\langle w, s \rangle \models \phi \vee \psi$ iff $\langle w, s \rangle \models \phi$ or $\langle w, s \rangle \models \psi$
 $\langle w, s \rangle \models \phi \rightarrow \psi$ iff $\langle w, s \rangle \not\models \phi$ or $\langle w, s \rangle \models \psi$

These valuations are used to show that in each point $\langle w, s \rangle$ it is possible to reason classically. In this sense, we can restore classical logic inside a two-dimensional environment. One could even define notions of n -knowledge, and n -possibility.

The knowability operator is introduced here with the aims to modelling the concept of “it is knowable” without using two modalities, but using instead just one complex modality. This is a theoretical and economic gain, because we do not need two modalities to express the notion of *knowability*, but just one. We can understand therefore knowability as a primitive concept and design a logic for it.¹⁸ The next natural step is to find an ax-

¹⁸We had in previous versions of this topic, two kinds of knowability, one which we called *global* and the other called *local* knowability. The global and local notions of knowability should express the central property that agents may have different levels of possible knowledge, meaning that a proposition can be known in more than a single way. But the

iomatization characterizing the above semantical level of local knowability.

An axiomatic system for LK?

A particular problem which arises in questions about combining modalities is that exposed in [43] and [44]: how to find an axiomatic system for a class of frames? An axiomatic system is proposed here in order to axiomatize the logic for knowability. A very important remark: we do not argue that this axiomatic system is indeed the only one and a correct axiomatic system for knowability, but we propose here the general idea of how should be an axiomatization based in product of KRIPKE frames. The axiomatic system of the logic for knowability generated by the product of modal frames is composed by the fusion $\mathbf{K} \oplus \mathbf{T}^*$ which is composed by \mathbf{K} and its reflexive epistemic version \mathbf{T}^* plus extra axioms.

The *axiomatic system for the logic of knowability LK* is:

1. All tautologies of normal modal logic \mathbf{K} and \mathbf{T}^* (the fusion);
2. $\diamond K\phi \rightarrow \Box\phi$;
3. $\Box K\phi \rightarrow K\Box\phi$;
4. $\diamond K\phi \rightarrow K\diamond\phi$;
5. *Modus ponens*;
6. $\vdash \phi$ then $\vdash \Box\phi$;
7. $\vdash \phi$ then $\vdash K\phi$;

Completing the logic, the knowability principle could also be expressed by the following inference rule:

$$\frac{}{\vdash \phi \text{ implies } \vdash \Box\phi}$$

way products of Kripke frames generate the knowability modality forced us to give up the notion of global knowability. Nevertheless, it would be interesting investigating how such different levels of knowability would interact. For us, knowability means, therefore, local knowability. It is important to note that the global knowability modality could not be defined using 2-dimensional possibility and knowledge, but the same is not the case related to local knowability, which could be defined using both modalities. The basic difference between local and global knowability is that in the local case knowability just implies possible knowledge, while in the global case it is equivalent to possible knowledge.

Clearly, in order to simulate FITCH’s paradox in its interior, we have to add the monomodal version of verification principle. But when (VP) is added to the above system, given that $\diamond K\phi \rightarrow \Box\phi$ it follows that $\phi \rightarrow \Box\phi$. In this sense, we do not need two modalities to model the verification principle. In order to understand metatheoretical properties, let’s begin by the following conjecture: **LK** is sound and complete with respect to its class of modal-epistemic product models. In the scope of transfer theorems among logics it is already known in the literature that, if L_1 and L_2 are canonical logics, then the semantical product is also canonical. Given that every canonical logic is KRIPKE-complete, and given that the product logic is canonical, the desired result follows immediately. If the product is obtained from logics with known properties, then a general proof is also obtained by transferring properties of the given logics. In our case, indeed, we propose a product of a metaphysical frame and an epistemological frame which is the product frame of **K** and T^* plus some extra axioms. But what would happen if we add the above manifestation of the verification principle?

From a two-dimensional point of view

Indeed, if we use two modalities to model knowability, this kind of inference is sometimes a problem. Anyway, using the knowability modality allow us to avoid the problems of FITCH’s reasoning by a small change in our formal language, for this reason I call this solution in this section as “syntactical solution”. Is it possible to find a two-dimensional model where all axioms are true, but the conclusion $\phi \rightarrow K\phi$ is not true? We do not have particular grounds to state that this can be done. Anyway, here we gave a general clue of how to proceed. The reader should be surprised. We do not investigated neither soundness nor completeness and, however, we are going to investigate properties of the logic **LK**. Indeed, what we are going to do is to show what holds from a quick examination from the syntactical point of view, and what would happen if the logic satisfies good criteria.

1. In the logic **LK**, $\diamond K$ implies the knowability modality but the converse is not the case. Therefore, knowability modality is weaker than $\diamond K$ and can not be reduced to it;
2. The verification principle becomes $\phi \rightarrow \Box\phi$ because everytime it is added to the product, by the transitivity of implication, one gets automatically its new version;

3. $\mathbf{LK} + \phi \rightarrow \Box\phi$ does not imply $\phi \rightarrow K\phi$, especially because $\Diamond K$ and \Box are not equivalent, and then the *salva veritate* between both does not hold; This is important because considering the problematic instance $p \wedge \neg Kp$, one would have $(p \wedge \neg Kp) \rightarrow \Box(p \wedge \neg Kp)$. But one would not have $\neg \Box(p \wedge \neg Kp)$;

We invite the reader to investigate metatheoretical properties of such logic for knowability. We are sure that many interesting results can be obtained.

1.3.2 Combined modality?

Methods for combining logics are an important tendency, as they are useful in the task of finding powerful logics able to map natural language (although the existence of the collapsing problem related to the most powerful mechanism for combining logics: fibring) . Such methods also have many applications in fields varying from philosophy to computer science. It is important to note that the process of combining, for example, two logical objects depends strongly on the nature of these objects. This means that we must start making a selection of a particular case of structure. In this sense, combining logics constitutes a chapter of something called universal logic in [7] and [6]. To illustrate these last three mysterious sentences, let me make reference to two articles: [24] and [26]. In [24], the authors show how to give categorical descriptions of methods, or mechanisms, for combining logics. But to realize this task, they must first choose a particular kind of structure: signatures, hilbertian calculi or interpretation systems for creating the categories *SIG*, *HIL* and *INT* in order to represent the mechanisms in categorical terms. The same happens in [26], but now in order to define synchronization, parameterization and fibring it is necessary first to choose a particular kind of structure: consequence systems, simple logic systems, powerset logic systems, etc. The proposal of a knowability operator and a solution to Fitch's paradox using fusions are clues that we should enter in the world of multi-dimensional modal logics and combination of logics to realize philosophical tasks.

A n -dimensional theory of modalities is important while we are trying to generate an abstract and formal account to logic. One point that should be explored, but it will not be here, is how n -opposition theory would be from the viewpoint of n -dimensional modalities (Anyway, in the last chapter such approach will be fastly investigated). A n -dimensional modal logic is

especially helpful in the analysis of flowings between certain concepts capturing dynamics notions. This make us think that an interesting approach to belief able to capture its flowings and dynamics should be, over all, based in the notion of dimension and products between doxastic states and time order (we will come back in this topic in chapter 3).

Even if we did not explore such combined notion in extreme detail, I hope have shown that products can be a very powerful tool while simulating the behavior of interactive operators. The operation of product can be introduced in a more intuitive way especially when horizontal lines mean the flowing of time. There are some attempts trying to apply products to define logics for space-time interactions as those developed in [44]. Such approaches are not directly related to what space and time are, but helps us in the task of understanding how to manipulate logics or mathematical formalisms containing both modalities at the same time. This is indeed the power of logic. Logic does not show what the real world is, but shows how to treat and undestand a given possible configuration of the world.

The problem of examining how to obtain products of particular modal logics with epistemic logics is another task, as it is the question of understanding further properties related to our construction. Results concerning families of logics for knowability are still under investigation, but the task seems to be promising.

1.4 Imagination and possible knowledge

In order to conclude this chapter, we briefly study the relations between the limits of knowledge expressed in the knowability modality, in combined modalities such as “possible knowledge” and the concept of imagination. One can think that imagination goes beyond the limits of knowledge. We examine such a perspective without the purpose of finding an ultimate answer to the question.

As far as we know, the first formal approach to the concept of *imagination* has been proposed by NIINILUOTO in [81]. He presented a formal system trying to formalize the notion of imagination using KRIPKE semantics:

“The logic of imagination in this sense is primarily a tool for understanding natural language. It can hardly be expected to solve any philosophical disputes about the true nature of imagining or any factual disagreements about the psychological laws that imagination in fact satisfies.” (ILKKA NIINILUOTO in [81])

Modal logic is a good tool in the construction of a mathematical model of what is imaginable (and also imagination) because there are some interactions between imagination and possibility, necessity and contingency. The standard symbol of imagination I proposed by NIINILUOTO formalizes the notion of *imagination*. As is well-known, the notion of possibility can be used in many different senses. One can think some propositions as physically possible, or metaphysically or logically possible. Following NIINILUOTO, we are also using the concept of *possibility* to mean logical possibility. In this sense, the connections between imagination and possibility are analyzed assuming our possibility as logical possibility. As is well-known, the difference between logical possibility and physical possibility is that the former cannot entail a logical contradiction or a logical trivialization while the later cannot violate a physical rule.

There are some simple intuitive connections between modalities and imagination. NIINILUOTO treats in his article especially the relation between logical possibility and imagination, and this is the first obvious connection between possibility and imagination. It seems intuitive accepting that everything we can imagine is possible, but there are some propositions which are imaginable but not possible. It is possible a reality with n -dimensions, but our imagination cannot go beyond three dimensions.¹⁹ We can imagine a lot of things, monsters with 10 heads etc, but we cannot imagine (represent) a simple monster in a 7-dimensional environment. Also, we are not able to imagine (i.e. to have an image of a contradiction in our mind. This makes us think that the rule “Imagination implies possibility” holds, and it is a rule in the standard logic of imagination proposed by NIINILUOTO, but the rule “Possibility implies imagination” does not hold. Imagination should be therefore weaker than possibility. Not so intuitive are the connections between necessity and imagination, and others modalities. NIINILUOTO briefly examines some other links between imagination and truth. It does not seem natural to state that everything which is true is imaginable. “Truth do not implies imagination” because there are propositions which are true but we cannot imagine. And “imagination implies truth” is also not the case, because there are many propositions which we can imagine but which are not true. The reader can see that ϕ is imagined and ϕ can be imagined are different because the second one appears in the context of the modality of possibility while the first one do not appear in the scope of a modality. But what about other concepts such as contingency

¹⁹Everybody who already studied linear algebra knows that objects in dimensions higher than 3 cannot be figure out by the mind.

and necessity? NIINILUOTO does not studied such other relations. There are many things which are necessary, in the mathematical sense of “there is a proof”, but we cannot imagine. If somebody proves that a n -dimensional space has a given property F , such a fact would be a necessity although we would not be able to think, conceive or imagine it. Therefore, the rule “Necessity implies imagination” should not hold and the rule “Imagination implies necessity” also should not hold because there are many things which we can imagine and which are not necessary. However, how to create a mathematical model for the notion of imagination? Is there a modal logic for imagination? It seems that modal logic has a closed connection with imagination. The subject of modal logic and imagination deserves a little bit of attention, given that the existence of logics dealing with imagination can clarify the relations between thought and reality, and the interactions between imagination and other concepts. What is the relation between the verification principle, possible knowledge and imagination? Imagination is a complex set of ideas. The expression could be used in an empirical fashion following the basic notion of D. HUME in [58]:

“Nothing is more free than the imagination of man; and though it cannot exceed that original stock of ideas furnished by the internal and external senses, it has unlimited power of mixing, compounding, separating, and dividing these ideas, in all the varieties of fiction and vision. It can feign a train of events, with all the appearance of reality, ascribe to them a particular time and place, conceive them as existent, and paint them out to itself with every circumstance, that belongs to any historical fact, which it believes with the greatest certainty. Wherein, therefore, consists the difference between such a fiction and belief? It lies not merely in any peculiar idea, which is annexed to such a conception as commands our assent, and which is wanting to every known fiction. For as the mind has authority over all its ideas, it could voluntarily annex this particular idea to any fiction, and consequently be able to believe whatever it pleases; contrary to what we find by daily experience. We can, in our conception, join the head of a man to the body of a horse; but it is not in our power to believe that such an animal has ever existed. (DAVID HUME in [58])

What can be imagined? What are the limits of imagination? Certainly, it is neither natural nor easy to answer, in an ultimate way, such questions. Imagination is a mental representation of a given entity which can or cannot exist. If the reader find the following situation: “Imagine a monster with 10 heads”, he/she is able to have an idea, a kind of mental representation of such objects, although it does not exist in reality. But if one tells to the reader “Imagine the monster X with 10 heads and at the same time imagine

the monster X without 10 heads” then (s)he would not be able to get an image of such object. It is a confusion to think that contradictions can be imagined. We can imagine whatever, since it is not a contradictory object, proposition, fact or belongs to other dimension. Imagination is never exact. The fact that we can have an idea of a certain object justify the belief that we can imagine such an object. It is very difficult to state what imagination is. We can, however, assume a very vague notion of imagination just to reason about it.

Consider a modal language L composed by the structure $\langle \wedge, \vee, \rightarrow, \neg, \diamond, I \rangle$, where I is introduced in the language of a modal logic in order to represent the *imagination operator*. Such operator should respect the following intuitive axiom (some of these are valid in NIINILUOTO’s logic of imagination):

$$I\phi \rightarrow \diamond\phi$$

Certainly, the converse should not hold:

$$\diamond\phi \rightarrow I\phi$$

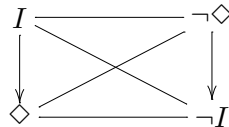
But not respects neither:

$$I\phi \rightarrow \square\phi$$

nor

$$\square\phi \rightarrow I\phi$$

In this sense, considering a order relation, what is the exact hierarchy of such modal operators? Again, what are the axioms for the imagination operator? How to create a relation between imagination and necessity? What is the truth-condition of such an operator? Is there a complete logic for imagination? If yes, is this logic alethic or epistemic? How would it be the square of imagination? Considering the logic of imagination where the implication $I\phi \rightarrow \diamond\phi$ holds, one configuration of the square of imagination would be:



Such a square reflects exactly the same relations of the square of oppositions: contradictories, contraries and sub-contraries propositions are exactly the same. Many other squares relating imagination and possibility could be drawn. What is important to conclude this chapter is that everything that can be imagined belongs to what is possible knowledge²⁰. However there are many knowable things which cannot be imagined. In this sense, we argued that the limits of knowledge exceed all possible imagination. The limits of knowledge are not, therefore, the limits of imagination (are the limits of imagination contradictions and higher-dimensions?). But where are the limits of knowledge? In order to investigate the limits of knowledge, in the next chapter we will analyze the notion of contingency and argue that the world is indeed contingent. Is contingency the limit of knowledge?

²⁰Questions concerning the logic of imagination (and also NIINILUOTO's logic of imagination) and its interactions will not be explored in detail in this section, but just in future work. The theme is extremely interesting and a detailed investigation of the subject would distract the reader. We leave, therefore, the topic for another occasion. The point has been mentioned here to explain in what sense it is connected with possible knowledge and its relations with other modal concepts.

Chapter 2

Interplays: knowledge and contingency

“Is it possible to know the future, what is going to be?” (G.H.VON WRIGHT in [109]).

“No dia seguinte ninguém morreu. O facto, por absolutamente contrário às normas da vida, causou nos espíritos uma perturbação enorme, efeito em todos os aspectos justificado, basta que nos lembremos de que não havia notícia nos quarenta volumes da história universal, nem ao menos um caso para amostra, de ter alguma vez ocorrido...” (JOSÉ SARAMAGO in *As Intermittências da Morte*)

In this chapter the metaphysical modality which is considered is the rich philosophically motivated modality of contingency¹. Sometimes, contingency can be defined in languages containing possibility, conjunction and negation, but sometimes contingency is a primitive connective. Contingency is a basic concept which reflects a property of reality: its flowing. Combining the metaphysical concept of contingency with the epistemological concept of knowledge allows us to investigate the extent to which certain epistemic attitudes are reasonable. This chapter, therefore, contains a study on the relations between the concepts of “truth”, “knowledge” and “contingency”. It is shown that the logic used to study this relationship has to be composed by a fusion of modal languages, axiomatic systems and Kripke models. Some philosophical problems such as the skeptical thesis are investigated. One of the limits of knowledge is examined using a combination of contingency

¹A traditional problem such as the *the future contingents* is treated by the end of this chapter using the ideas developed through the chapter.

logics and epistemic logics.

Take the modality ∇ which represents “contingency” and also the modality K which represents “knowledge”. There are two possible combinations considering propositional variables:

1. $\nabla K\phi$ (contingent knowledge)
2. $K\nabla\phi$ (knowledge of contingencies)

Take again the modality ∇ and also the modality K . There are two possible combinations considering an implication:

1. $\nabla\phi \rightarrow K\phi$ (Contingency implies knowledge)
2. $K\phi \rightarrow \nabla\phi$ (Knowledge implies contingency)

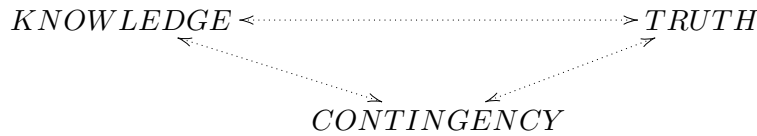
Such combinations of modalities allow expressions such as “Contingent propositions ϕ are known”, “The proposition ϕ is known and contingent”, “If a proposition is contingent, then it is known” and many others generated by combining concepts. Clearly, although other combinations can be considered, especially when both modalities appear in a conjunction (or disjunction), assume just the examples considered above.

As pointed out, philosophers usually find, while analyzing their concepts, arguments and problems, very complicated statements in natural languages. They can use logical models to better understand these statements, but sometimes very simple logical abstractions are not powerful enough to capture the richness of natural languages. Therefore, they need some logical structures able to formalize different aspects of a given concept in order to fill the gap. Consider, for instance, a philosopher who is formalizing a sentence which involves different concepts such as necessity, time and knowledge. A simple normal modal language would not be able to formalize sentences containing all these concepts. The philosopher should combine different formalisms in order to capture all aspects of the sentence. It is important, thus, to introduce other concepts in the basic propositional language. In this chapter, the case where “truth”, “knowledge” and “contingency” appear together in a complex modal logic is studied.² This is an interesting philosophically motivated logic because it is possible to analyze some metaphysical and epistemological positions in this framework. One of the

²The concept of truth is not directly considered in the combination, but it appears in the notion of atomic propositions. This means that when a proposition p is stated, what is said is that p is true. Therefore, the concept of truth is also examined.

motivations is to investigate a special skeptical thesis according to which “if a proposition ϕ is contingent, then ϕ cannot be known” and its variations. This kind of skepticism is studied in order to determine whether it is or not a powerful philosophical position from the logical viewpoint. By the end, a discussion of what is plausible or not from the logical viewpoint is examined in detail.

In the history of philosophy, the concept of truth appeared in the standard definition of knowledge attributed to PLATO as “justified true belief”, but it is difficult to find an example where the relations between the concepts of truth, knowledge and contingency are at the same time clarified. It is generally accepted that knowledge implies truth but the converse does not hold. However, the interactions between the three notions still need some explanation. These relations are captured by the following diagram:



Each arrow represents a possible relation of concepts. The main purpose of this chapter is to study formal systems obtained by combination of some given logics in order to formalize interactions between the three concepts above. In this sense, this chapter is an attempt to fulfill the arrows and discover the multiple relations between the concepts of truth, knowledge and contingency.

One could argue that contingency is not a primitive notion because it can be defined using possibility or necessity. But here the interest is to study the relationship between knowledge and contingency taking the last one as a primitive notion and possibility and necessity as definable concepts. It is in the fused language that the interplays between knowledge and contingency are better understood.

2.1 Logical skepticism

Many kinds of skepticism appeared in the history of philosophy. They can be divided into those skeptics who deny the existence of knowledge arguing that there is no possible knowledge, those who accept knowledge but assume that knowledge is just knowledge of some specific entities, and also

those according to which all reality is, in general, an illusion of our senses and, therefore, there is no external world. The skepticism considered here is a little bit different of the above conceptions and is essentially a very strong epistemological attitude because if it is true, it is not possible to know the world (i.e. propositions about the world). The skeptic considered here, however, cannot avoid the fact that there is a kind of knowledge, but just knowledge of formal truths (i.e truths which only can be known by deductive reasoning). This kind of skepticism has as foundation two arguments:

- (The contingency of the world) The world is contingent, because its configuration changes. This contingency is reflected especially by those propositions which are true in a given instant t and can be false in another instant t' such that t is different of t' . Therefore, given that the world is contingent, there is no knowledge of the world, but just knowledge of mathematical truths, assuming that what is known is expressed by the epistemic version of the necessitation rule;
- (The problem of induction) An inference is *inductive* when from true premises it is possible to infer a false conclusion. Inferences about the world are inductive, because conclusions about the world can be (could be) false, if they are consequences of true premises. In this sense, conclusions about state of affairs are not necessary, but probable to some degree. This is a slight manifestation of the problem of induction: given true premises, the conclusion could be false. The relation between premises and conclusion is never necessary, but probable. This is the essence of an inductive inference. The problem of induction states that there is no necessary knowledge about the world.

That the world is contingent is a very old philosophical position which can be traced at least since the pre-socratic fragments of HERACLITUS. His famous metaphor with fire and the statement according to which objects and facts are dynamics express the idea of *flux*, continuous change of reality³. This is essential to a conception related to the contingent nature of the world. However, that there is just knowledge of mathematical truths is a

³It is a surprising fact that the logical notion of contingency “it is possible ϕ and it is possible $\neg\phi$ ” has a strict connexion with pre-socratic philosophers. Stating that “All things are flowing ”is, indeed, to state that the world is contingent and therefore propositions about the world are also contingent, because states of affairs change their configurations, and true propositions become false and vice-versa. In this sense, it is important to mention HERACLITUS as the first philosopher who taught about the contingency of the world.

very artificial way to answer KANT's question "What can be known?". The skeptical thesis according to which there is just knowledge of *a priori* and formal statements appeared especially as a consequence of epistemic logic and the epistemic rule of necessitation, which states that if a proposition is proved to be true, then an agent knows such a proposition.

D. HUME can be considered the first philosopher who noted that there is no necessary connexion between facts. The "knowledge" that human beings are supposed to have of the world is, indeed, belief:

"The contrary of every matter of fact is still possible; because it can never imply a contradiction, and is conceived by the mind with the same facility and distinctness, as if ever so conformable to reality. That the sun will not rise tomorrow is no less intelligible a proposition, and implies no more contradiction than the affirmation, that it will rise. We should in vain, therefore, attempt to demonstrate its falsehood. Were it demonstratively false, it would imply a contradiction, and could never be distinctly conceived by the mind." (D. HUME in [58])

HUME's remark shows a relationship between the concepts of contingency and induction. The fact that there is the problem of induction is a consequence of the contingency of the world. If the world were not contingent, then it would be possible to know what would happen, and therefore the induction problem would not exist. It is interesting to note that D. HUME had a different conception of *possibility*: a proposition is possible if it does not imply any contradiction. Such a definition of possibility is not incompatible with the actual definition of possibility as truth in at least one possible world. To think that the world is contingent was not a privilege of greek or modern philosophers. From the point of view of more mathematically oriented philosophers such as WITTGENSTEIN, the contingency of the world appeared as a general and special premise. In his *Tractatus*, WITTGENSTEIN announces the contingency of the world by the following statements ⁴:

“(6.36311) It is an hypothesis that the sun will rise tomorrow: and this means that we do not know whether it will rise.”

⁴According to HUME (and also WITTGENSTEIN), we do not know whether the sun will rise tomorrow. It seems that WITTGENSTEIN studied in detail D.HUME, given that they use exactly the same arguments (Compare both propositions of the *Tractatus* with the above citation stated by HUME). Anyway, what both philosophers are arguing for is that assuming that the world changes its configuration, one could not know, with certainty, that something which happened, will happen in the future with the same regularity.

“(6.371) A necessity for one thing to happen because another has happened does not exist. There is only logical necessity.”
(WITTGENSTEIN in [110])”

The concept of “contingency”, and the fact that the world is contingent, are a common place from the analytical point of view and even from the continental point of view ⁵. Some philosophers as G.DELEUZE and F.GUATTARI also defended in [38] the *événement* as substantial part of reality but not responsible for the flowing of the world. More recently, A. BADIOU in [4] proposes axioms to formalize DELEUZE and GUATTARI’s proposal on contingency. It is interesting to check two of such axioms for what BADIOU calls “changement”:

“Le devenir-illimité devient l’événement lui-même.”

“L’événement est toujours ce qui vient de se passer, ce qui va se passer, mais jamais ce qui se passe.” (A. BADIOU in [4])

It is intriguing to note that both WITTGENSTEIN and BADIOU (representing DELEUZE) are stating the same thing, but the difference is that the former uses a clear way while the later uses a poetic way to announce contingency. Philosophers can use the contingency of the world as an argument against sciences in general. When a scientist is proud of his work, it suffices for a philosopher to argue that the world is contingent and therefore the results obtained by scientists could be false tomorrow. There is nothing necessary which *has been proved (empirically) scientifically*.

The problem of induction can be viewed as a result of the flowing of the world (its contingency). Given that the world is in flowing, no necessary conclusion about it can be reached. Many logicians and philosophers tried to solve the problem of induction finding necessities in the world. The way to proceed is to use a deductive system able to account for inductive inferences in a precise way. The idea would be to find an inductive system playing a role of a deductive system, in the sense that conclusions are necessary consequences of the premises and could not be different in that given closed formal system. Nevertheless, such a logic has not been found.

⁵The expression “continental” is used here to designate a kind of philosophy which does not make direct use of mathematical models in philosophy, but replace it for other tools such as metaphors, arts, musical theory, literature, poems etc. It is true that some known continental philosophers such as A. BADIOU indeed use mathematical tools in philosophical tasks, but this is not a rule.

Assuming the contingency of the world and the problem of induction, the skeptic considered here has respectable reasons to doubt about all possible knowledge about the world. Such a skeptic would even doubt that propositions such as “Water is H_2O ”, “The snow is white”, “The sun is a star” are necessary because they could be false if the world changes its configuration. Although skepticism is a very strong epistemological position, it is still a problematic theory (accepting or not some aspects of the old verification principle and other premises). Are there arguments against the kind of skeptic considered here?

Before continuing, it is important to clarify a theoretical strategy accepted here. One assumption of this work is that the content of a given proposition reflects, or is a mirror, of a given state of affairs of the world. This dogma is somewhat new, stated especially by those philosophers considered logical atomists such as B. RUSSELL and WITTGENSTEIN. For both, the world is composed by facts and each fact can make a proposition true or not. All true propositions, then, reflect the general structure of reality. Check for instance the propositions below (WITTGENSTEIN in [110])

(1.1) “The world is the totality of facts, not of things”.

(1.2) “The world divides into facts.”

For RUSSELL in [97], a fact is that thing which makes a given proposition true or false. Here, stating that a fact is contingent is a kind of circle, given that there is no necessary fact. Also, to state that the collection of all facts, i.e. the world, is contingent is also redundant, because the world, being a collection of facts, cannot be necessary.

This is not an unique way to investigate the general structure of the world, and indeed it would be fine discovering new ways to describe how such a structure is, as for example the approach using category theory. However, considering the scope of the present text, the principle according to which language is a mirror of reality plays a very important role. One basic distinction that is used here is that between atomic propositions (and molecular propositions) and the notion of logical form (what is called in modern logic *propositional variables*). This distinction is very important to the success of the main arguments. A logical form is an abstraction of the notion of atomic and molecular propositions. A basic conception of logical atomism is therefore assumed.

The distinction between language and the world is artificial. However, such a distinction is very helpful for philosophers because while they are not

able to study the world directly, a task which nowadays belongs to sciences, the philosopher can study the mirror of the world. This is a theoretical distinction which plays a very important role in proposing arguments.

Coming back to the contingency of the world, contingency can be viewed as a property of propositions and, therefore, as a property of the world, considering that language is a mirror of the general structure of the world. As a property of propositions, there are two kinds of contingencies: contingency as a metaproperty and modal contingency (this distinction is explained later). An interesting thing is that both sorts of linguistic contingency correspond to the same kind of contingency of the world. The distinction between contingent propositions and contingency of the world exists, but it is not so fundamental because both have the same ontological content. Assuming that language is a mirror of the world, a study of contingent propositions shows in what sense the world itself is also contingent and allows an understanding about the basic epistemic facts which can be described about the world. Thus, considering interactions between the concept of knowledge and the concept of contingency generates a connection between epistemology and metaphysics.

2.2 Combining knowledge and contingency

It is well known that the concept of knowledge is one of the most celebrated concepts in philosophical logic. In epistemology, knowledge plays a central role. Attempts to define knowledge are very old, since the works of PLATO. In epistemic logic, knowledge is the basic modality. Since the works of HINTIKKA in [54], there are a great variety of systems of knowledge, where those basic philosophical intuitions of knowledge are formalized and interpreted in formal languages. Knowledge is also an essential notion in technical areas such as artificial intelligence and software engineering. Although there have been many studies on the notion of knowledge, it seems that interactions between knowledge and other non-epistemic concepts have not been explored in detail from the viewpoint of combined logics. The relations between the concepts of truth, knowledge and contingency are not elucidated in the literature on modal logics, although many attempts trying to define necessity in non-contingency logics. Given that “knowledge” is not definable via “necessity”, it follows that “knowledge” is also not definable via “contingency”.

Considering the concept of “knowledge” in the context of contingency logics (i.e. a modal formalism to study the concept of “contingency”), one

basic question to be stated is: Is there knowledge of contingent propositions? Is it possible to know contingent propositions? To provide a solution to these problems using some formal tools such as epistemic logics, contingency logics and their combinations obtained by fusions of modal logics is a challenger task.

In this sense, two kinds of contingency are defined and one of them is combined with the concept of knowledge in order to analyze variations of skeptical arguments and interactions between knowledge and (non)-contingency: given that the world is contingent (i.e. propositions about the world are contingent), it follows that the world is not known (contingent propositions are not known). In order to realize such a task, combining modal logics of contingency and modal logics of knowledge is the main methodology to be used.

2.2.1 Two notions of contingency

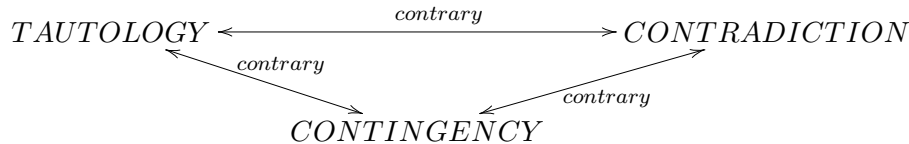
As pointed out at the beginning of this book, one can erroneously think that possibility and contingency are the same concept. This kind of confusion occurred a long time ago in the history of modal logic but recently it is rare to find philosophers or logicians committing the same mistake. The difference between both concepts is captured by modal logics in such a way that contingency implies possibility but the converse does not hold in all extensions of the normal modal logic **T**. Possibility and contingency are, therefore, two different concepts.

There are basically two notions of contingency: one out of date conception called here *metaproperty contingency* and another recent conception which is called *modal contingency*⁶. One is defined inside classical propositional logic, while the other in its extension to modal languages and logics.

Before formalized as a modal operator, and indeed this is the actual approach to contingency, the general idea of contingency was characterized inside classical propositional logic. Truth-tables present methods for testing decidability of logics. A given formula is a *tautology* if and only if it is true in all rows of the table. A given formula is a *contradiction* if and only if it is false in all rows of the table. And a given formula is a *contingency* if and only if there is at least one row where it is false and at least one row where it is true. Given this definition, tautologies and contradictions are duals in the sense that the negation of a tautology is a contradiction and vice-versa. However, the same duality did not appear for contingency because the negation of

⁶Such a distinction and the terminology *metaproperty contingency* is our own.

a contingency is still a contingency. Needless to say that nowadays people use the names *top particles* for tautologies. And one could think that by duality *bottom particles* are contradictions. One should note that bottom particles are not always contradictions, considering the scope of non-classical logics, as one can check reading the literature on paraconsistent logics. The duality between tops and bottoms is a natural consequence of that first mentioned duality between tautologies and contradictions. The discovery of the existence of logics where bottom particles are not always contradictions and that not all contradictions are bottom particles can be viewed as a very important revolution in logic. Using the terminology bottom/top, we do not have a special algebraic name for contingency. One could say that contingency now is the notion of a *formula* which is neither a bottom nor a top. All atomic propositions are contingent because when truth-values are assigned we have to select *all* possible values and it includes, therefore, at least one row where it is false and at least one row where it is true. (Just check that under normal conditions there is no atomic proposition which is always true in a given formal semantics.) The ARISTOTLE's square cannot be used to model the relations between metaproperties (tautologies, contradictions and contingencies), but we can use something which we call WITTGENSTEIN's triangle (the triangle of metaformulas), the situation is illustrated as follows:



As BEZIAU pointed out in [12], inspired in the works of L. HORN, considering the traditional square of oppositions there is no word for one corner of the square if the square is the square of modalities and the square of quantifiers. He suggests that the nameless corner could be viewed as a paraconsistent negation. It is also interesting to note: representing metaproperties in the triangle (of metaformulas) as tautologies, contradictions and contingencies, we find a new configuration avoiding the nameless corner of the square. The notion of contingency appears exactly in the relations between the A-corner(tautologies) and the E-corner(contradictions)⁷. A formula cannot be a tautology and a contradiction at the same time, and if the formula is

⁷The above triangle seems to be obvious, and it is founded in the intuitions WITTGEN-

neither a tautology nor a contradiction, then it is a contingency. A formula which is neither a tautology nor a contingency, is therefore a contradiction. A formula which is neither a contradiction nor a contingency is a tautology (For details and discussion on the relations between geometry and logic check MORETTI in [80] and BEZIAU in [12]. Check also the last chapter of this book.)

The actual notion of contingency uses a powerful tool of modal logic to model the notion of contingency. Given a Kripke model, the traditional approach to contingency states that a proposition is contingent if it is possible and also its negation is possible. In BLANCHE's hexagon contingency has a special place, but in the above configuration contingency occupies a different position (considering the place which it occupies in the triangle of metaformulas). What are the differences between contingency and possibility, and between two notions of contingency? The basic distinction between the two notions of contingency is, therefore, the following: metaproperty contingency is closed under negation because the negation of a metaproperty contingency is also a metaproperty contingency, while modal contingency is not closed under negation, because the negation of a modal contingency is not a modal contingency, but a modal non-contingency.

It can happen that other notions of contingency are used, but here just the two notions above are considered. So, in the first conception of metaproperty contingency, all propositions which are neither a tautology nor a contradiction (the term is used erroneously in general to designate bottom particles) are therefore contingencies. In the second conception, contingency is a modal operator including two levels of possibility.

ARISTOTLE can be considered the first one who used the notion of possibility in order to mean what nowadays is called contingency:

“...we next point out that the expression “to be possible” is used in two ways. In one it means to happen generally and fall short of necessity, e.g. man's turning grey or growing or decaying, or generally what naturally belongs to a thing...In another sense the expression means the indefinite, which can be both thus and not thus, e.g. an animal's walking or an earthquake's taking place while it is walking, or generally what happens by chance: for none of these inclines by nature in the one way more than in the opposite.” (ARISTOTLE in [1])

Indeed, the second use of the term possibility is what is considered as

STEIN had about contingency, contradictions and tautology (See [110]). We do not know whether somebody already related such notions with the triangle (or square).

contingency. ARISTOTLE’s theory of syllogismus was even able to give some answers to classical interpretations of metaphysical and epistemological modalities. The power of greek philosophers for describing problems in recent philosophy cannot be doubted. For this reason there is a very famous slogan in some philosophical circles according to which what could be said has already been said by some greek philosopher. Unfortunately, here we cannot explore details of ARISTOTLE’s contributions and other greek philosophers’ theory of modalities.

What is important to have in mind is that the notion of contingency that is used here is the contemporary concept of modal contingency. In this sense, there are families of contingency logics, as the reader will check below.

2.2.2 Contingency logics

“Contingency” here means modal contingency. And logics of contingency are extensions of classical propositional logic by adding in its language, axioms and models a new modal operator, and new axioms and models. It would also be interesting to investigate how to study the concept of contingency in classical propositional logic and work, instead of conceiving contingency as a modal operator, with a definable operator in classical propositional logic or in the sense of metaproperty contingency. It is also interesting to investigate the properties of this approach. However, now contingency logics are investigated in the second sense, where contingency is an extra modal operator added to the language of classical propositional logic. Contingency logics capture the exact sense of what modal contingency means.

The history of logics of contingency is not so rich and just a few authors studied in detail these logics. It is an interesting point why contingency logics were not target of extensive research in the same way that logics for possibility and necessity have been investigated in extreme detail. The basic reason is that being defined in logics containing traditional modal operators, the study of contingency logics has become neglected. Another reason of such *oubli* is due to the fact that it is not clear what are the advantages of assuming contingency as a primitive ⁸. Anyway, an approach considering contingency and non-contingency operators as primitives has been developed by some researchers.

Logics of (non-)contingency were proposed the first time by MONTGOMERY and ROUTLEY in [79] who studied the behavior of the operator ∇ which means “it is contingent that” and its dual Δ which means “it is

⁸A similar fact occurred with the Scheffer operator.

not contingent that” . The authors introduced such languages considering its philosophical relevance. The following passage can be found in [79]:

“Contingency and non-contingency bases for modal logics provide direct bases for various logical investigations of philosophical interest. For example: for logics of causation and causal implication, for certain theories of entailment, for syllogistic systems with only contingent propositions, and for theories of future contingents. Furthermore, by taking contingency as a primitive various new extensions of weak modal logics - extensions which include systems **S6-S8** and provide interpretations of philosophical interest for these neglected systems - are suggested. Contingency and non-contingency are also of some formal interest; for instance, **S5** has a very simple and elegant formulation in terms of non-contingency.” (MONTGOMERY and ROUTLEY in [79])

The authors were right while arguing that such logics of contingency are very rich from the philosophical perspective, and the motivation used to introduce such logics is used as a motivation to introduce its combinations with epistemic logics. What we investigate here is a philosophical problem which has not been remarked on by either author (An exception is the future contingents): The problem of interactions between contingency and knowledge.

Given the importance of such logics for the development of this book, and given that such logics are not so famous as epistemic logics and modal metaphysical logics for \diamond and \Box , it is explained in detail inside this chapter the basic principles of contingency logics. Contingency logics are investigated in an isolated mode and later combined with epistemic logics. Consider a modal language L composed by the following symbols:

$$L_{\nabla} = \langle \neg, \rightarrow, \wedge, \vee, \nabla \rangle$$

Note that ∇ (contingency) is primitive. One could also take as primitive its dual Δ :

$$L_{\Delta} = \langle \neg, \rightarrow, \wedge, \vee, \Delta \rangle$$

In the first language, the authors proposed the logics of contingency while in the second language they proposed logics with a non-contingency structure. Consider then the first language. The logics of contingency are generated based on it, using the following definition and rule:

1. $\Box\phi =_{def} \neg(\phi \rightarrow \nabla\phi)$

2. $\vdash \phi$ then $\vdash \neg\nabla\phi$

The basic logic with a contingency structure is composed also by the following two axioms:

1. $\nabla\phi \leftrightarrow \nabla\neg\phi$
2. $\phi \rightarrow (\neg\nabla(\phi \rightarrow \psi) \rightarrow (\nabla\psi \rightarrow \nabla\phi))$

The second axiom does not have an intuitive interpretation. And it is very difficult to find one. Such a system composed by two axioms, a definition and one rule generates the modal logic of contingency which is called here \mathbf{T}_∇ . In [79], MONTGOMERY and ROUTLEY present different deductively equivalent logics of contingency and non-contingency which corresponds to the modal logic \mathbf{T} .

The usual operator ∇ applied to propositional variables $\nabla\phi$ means that “ ϕ is contingent”. The dual operator of the contingency modality is the so-called non-contingency operator which is defined $\Delta\phi = \neg\nabla\phi$. As showed by the authors, given a normal modal logic, one can take contingency or non-contingency as primitive operator and define possibility and necessity in the system. Usual modal operators \diamond and \square are interdefinable. Also, ∇ can be defined using \diamond or \square . There are many axiomatizations of the (non-)contingency logics, and there are translations from (non-)contingency logics to normal modal logics. In order to understand this point, consider two languages and two logics such that there are some translations between both sets. In [111] it is showed how to convert contingency and non-contingency logics in logics for necessity and possibility. The argument runs as follows. Take a set of formulas $PROP_\Delta$ and a set of formulas $PROP_\square$. Take a translation t from $PROP_\Delta$ to $PROP_\square$. The classical clauses are translated in the usual way, while the non-contingency has the following translation:

$$t(\Delta\phi) = \square t(\phi) \vee \square\neg t(\phi)$$

Take now a set of formulas $PROP_\nabla$ and a set of formulas $PROP_\diamond$. Take a translation t from $PROP_\nabla$ to $PROP_\diamond$. The classical clauses are translated in the usual way, while the contingency has the following translation:

$$t(\nabla\phi) = \diamond t(\phi) \wedge \diamond\neg t(\phi)$$

Such translations show that we can extract logics for \square and \diamond from logics for Δ and ∇ respectively. Such translations are conservative translations

because the logics generated by such translations prove exactly the same propositions (one can call it a simulation). However, ∇ can be taken as primitive and other modal operators can be defined from it. Obviously, contingency is, therefore, defined as $\nabla\phi = \diamond\phi \wedge \diamond\neg\phi$ and necessity is defined as $\Box\phi = \Delta\phi \wedge \phi$. In the article [79], the authors analyze contingency and non-contingency versions of the modal logics **T**, **S4** and **S5**. For the system **T**, they showed three systems having a non-contingency bases and two systems containing a contingency bases. They also show versions of **S4** and **S5**. Afterwards, they prove that such contingency and non-contingency logics are deductively equivalent to each of the logics **T**, **S4** and **S5**. A very useful and interesting version of such logics which is used here is the simulation of **T**, which can be defined as follows. Take the language L_Δ represented by the structure $\langle \wedge, \vee, \rightarrow, \neg, \Delta \rangle$. Now, take the following axiomatic system generated on the language containing Δ . Then the logics of non-contingency are composed by:

1. $\Delta\phi \leftrightarrow \Delta\neg\phi$
2. $\phi \rightarrow (\Delta(\phi \rightarrow \psi) \rightarrow (\Delta\phi \rightarrow \Delta\psi))$

And the definition and rule below:

1. $\Box\phi =_{def} \phi \wedge \Delta\phi$
2. $\vdash \phi$ then $\vdash \Delta\phi$

As proved in [79], such axiomatic system is deductively equivalent to the logic **T**. One could obtain a version of **S4** by adding for example $\Delta \rightarrow \Delta\Delta\phi$ and could obtain a version of **S5** by adding for example $\Delta\Delta\phi$. For the purposes here just consider the non-contingency version of **T**. The models are the same Kripke models for the logic **T**, which are reflexive.

Some results were obtained by CRESSWELL in [37], HUMBERSTONE in [57], KUHN in [62] and E. ZOLIN in [111]. We discuss briefly each proposal. In the article [37], CRESSWELL studied sufficient conditions to define \Box using Δ . As pointed out by CRESSWELL in the same article, SEGERBERG showed that the definition of $\Box\phi = \Delta\phi \wedge \phi$ just holds in reflexive modal logics which extend **T**. CRESSWELL presents an argument (theorem 9 of [37]) showing that indeed it is possible to define necessity using non-contingency in a modal logic which does not contain reflexivity. His definition is the following: $\Box\phi \leftrightarrow (\Delta\phi \vee (\phi \leftrightarrow \Delta\Delta\phi))$. Although CRESSWELL's argument, the systems studied in this section are reflexive in the sense that

they validate $\Box\phi \rightarrow \phi$. In [57], HUMBERSTONE presented a generalization of the Hilbert-style axiomatic system proposed by MONTGOMERY and ROUTLEY for non-contingency logics. However, HUMBERSTONE's system does not have restrictions in the accessibility relation. He proves in detail soundness and completeness for his general system using the famous technique of canonical models. He also presents some extensions of this system. In the article [62], KUHN analyzed HUMBERSTONE's paper and proposed finite axiomatizations arguing for a minimal non-contingency logic. In a more actual approach in [111], ZOLIN proved soundness and completeness for a large class of non-contingency logics containing versions of axioms such as **D, 4, 5**.

All authors mentioned above do not investigate interactions of (non-)contingency logics with epistemic logic. As already argued for, such interactions are very fruitful from the philosophical viewpoint given that it allows us to study connections between metaphysical and epistemological problems such as those mentioned in the beginning of this chapter. Our approach proposes neither a new axiomatization nor new results on logics of (non-)contingency. What is proposed here is a combination of two logics in the sense of [43](i.e. a fusion). In order to combine (non-)contingency logics with epistemic logics, let's establish the appropriate framework.

2.2.3 Mixing knowledge and contingency

Take a set *PROP* of propositional variables $\{\phi, \psi, \dots\}$ and a simple modal language $L_{\nabla} = \langle \neg, \wedge, \vee, \rightarrow, \nabla \rangle$. In such a language we can define \diamond 's and \Box 's. Possibility is defined as usual: $\diamond\phi = \neg\Box\neg\phi$. And $\Box\phi$ is defined as: $\Box\phi = \neg(\phi \rightarrow \nabla\phi)$. Then a proposition ϕ is said to be contingent if and only if it is possible ϕ and it is possible that the negation of ϕ be true: $\nabla\phi = \diamond\phi \wedge \diamond\neg\phi$. Indeed, propositions about the world are contingent propositions of the form $\nabla\phi$ because given that the world changes its configuration, it follows that some propositions which were true become false and vice-versa. In order to discuss knowledge of contingent propositions, let's add to the simple modal language the knowledge operator K . Take a simple epistemic language L_K such that $L_K = \langle \neg, \wedge, \vee, \rightarrow, K \rangle$. Then, the fusion \oplus of L_{∇} and L_K is defined as the union of all operators in the fused language $L_{\nabla} \oplus L_K = \langle \neg, \wedge, \vee, \rightarrow, \nabla, K \rangle$. In this setting, what can be said about an agent's knowledge of contingent propositions? Is it possible for an agent to know a given proposition ϕ , if it is a contingent one: $(\nabla\phi \rightarrow \diamond K\phi)$? Does an agent know a given proposition ϕ , if it is a contingent one $(\nabla\phi \rightarrow K\phi)$? From epistemic logic, if there is a theorem which proves the proposition,

then it is *known*. What about contingent propositions? Are they known? If the answer is negative, then it validates a particular form of skepticism, especially that which states that “if a proposition is contingent, then it cannot be known” and its stronger version “if a proposition is contingent then it is not known”. A negative answer to both questions would show one of the limits of knowledge: one cannot know contingent propositions and, in fact, one does not know contingent propositions. Indeed, if a proposition is knowable (or is known), then it is not contingent. Let call this form of skeptical attitude “logical skepticism”. In order to develop a logical environment able to formalize and analyze logical skepticism one need a very special formal language which contains at the same time two different modal operators: knowledge and contingency. Indeed, the language one needs here is the one required to formalize FITCH’s paradox: a fusion of two modal languages. Or, differently, a language taking ∇ or Δ as primitives and its fusion with an epistemic language. In the fused language, many different forms of skepticism and interactions can be generated.

Let call skeptical thesis (ST) the following statement:

(ST) If a proposition is contingent, then it is not known.

Such a statement is formalized by:

$$(\nabla\phi \rightarrow \neg K\phi)$$

Clearly, by contraposition, the skeptical thesis can be stated as:

(ST’) If a proposition is known, then it is not a contingent proposition.

The formalization is:

$$(K\phi \rightarrow \neg\nabla\phi)$$

In order to analyze a weaker version of logical skepticism, the possibility operator should be introduced to formulate the skeptical principle in the following way:

(WST) If a proposition is contingent, then it cannot be known.

The logical form of (WST) is:

$$(\nabla\phi \rightarrow \neg\Diamond K\phi)$$

Again, by contraposition, the skeptical thesis can be stated as:

(WST') If a proposition can be known, then it is not a contingent proposition.

Which logical form is:

$$(\diamond K\phi \rightarrow \neg\nabla\phi).$$

By contraposition, and replacing contingency by non-contingency, the skeptical thesis and its weaker forms are formulated in the following way: $K\phi \rightarrow \Delta\phi$ and $\diamond K\phi \rightarrow \Delta\phi$.

Consider now the full skeptical thesis:

(FST) If a proposition ϕ is contingent, then ϕ is not known, and $\neg\phi$ is not known.

This more complicated form of skepticism can be formalized as:

$$\nabla\phi \rightarrow (\neg K\phi \wedge \neg K\neg\phi)$$

Another form to state the full skeptical thesis is:

(FST') If a proposition ϕ is contingent, then it is not possible to know ϕ , and it is not possible to know $\neg\phi$.

Which formalization is:

$$\nabla\phi \rightarrow (\neg\diamond K\phi \wedge \neg\diamond K\neg\phi)$$

Other interactions can be considered assuming a structure which contains \square or \diamond added to K , especially those which manifests connections between the concepts of *knowledge* and *necessity*. For many authors (as for instance [99]), knowledge and necessity are the same concept. Is there, indeed, a distinction between both? Below the reader find interactions of modalities:

(NK) If a proposition is necessary, then it is known.

(KN) If a proposition is known, then it is necessary.

These statements are also strong forms of the followings:

(PK) If a proposition is possible, then it is known.

(KP) If a proposition is known, then it is possible.

But there are also some false interactions (involving non-interdefinable concepts):

(PC) If a proposition is possible, then it is contingent.

(CP) If a proposition is contingent, then it is possible.

Using the fused language containing ∇ and K all those principles relating knowledge and contingency can be analyzed. Considering the fusion $\mathbf{T}^* \oplus \mathbf{T}_\nabla$, and adding extra interactions, some immediate answers can be done to simple non-interactive statements, for instance: (PC) is not the case, using classical propositional logic. (CP) is a tautology, as the reader can easily verify using classical propositional logic. Both are simple because contingency and possibility are definable operators (no interaction, therefore). In the case of interactions of knowledge and possibility, the situation is more complicated. We will leave without examination the following interactions (ϕ did not appear, just the pure interacted form): $\diamond \rightarrow K$, $K \rightarrow \diamond$, $K \rightarrow \square$, $\square \rightarrow K$. To give an answer one should select a definition of the accessibility relation and plausibility relation.

The interaction (ST) is justified by the fact that there is just knowledge of mathematical truths. Somebody could argue that we, in fact, *know* many facts of everyday life. Indeed, what would be happening here is an informal use of the concept of knowledge. It is confused with the concept of belief. One believes that many facts of everyday life will be such and such forever, but there is no guarantee that these facts will not change. Perhaps one day one will wake up and the sun will not be there. This is nothing, but the well-known distinction between knowledge and belief. While knowledge implies truth, the same is not the case for the concept of belief (a detailed treatment to this problem is examined in the next chapter).

Before examining logical skepticism in detail from the logical viewpoint, one interesting point examined by VON WRIGHT in [109] is a relation between knowledge and contingency, considering a language where there are both operators. In the book [109] he argued that if a proposition is contingent and known, then the knowledge one has about it is also contingent:

$$(K\phi \wedge \nabla\phi) \rightarrow \nabla K\phi$$

The argument is the following:

1. $K\phi \wedge \nabla\phi$ (Hypotheses)
2. $K\phi \rightarrow \phi$ (Knowledge's axiom)

3. $\Box(K\phi \rightarrow \phi)$ (Necessitation in 2.)
4. $(\Box K\phi \rightarrow \Box\phi)$ (Kripke's axiom in 3.)
5. $(\neg\Box\phi \rightarrow \neg\Box K\phi)$ (Contraposition in 4.)
6. $(\Diamond\neg\phi \rightarrow \Diamond\neg K\phi)$ (Definition of \Diamond in 5.)
7. $(\Diamond\neg K\phi)$ (From 1 and MP with 6.)
8. $(K\phi \rightarrow \Diamond K\phi)$ (Axiom of **T**.)
9. $(\Diamond K\phi) \wedge (\Diamond\neg K\phi)$ (Contingent knowledge, from HYP and 7.)

VON WRIGHT's argument shows that whatever is known is contingent knowledge: "...If the object of knowledge is contingent, then knowledge of it is contingent too." (VON WRIGHT in [109]) This is a stable example where both operators ∇ and K appear together in the same formalism. The logic above used by VON WRIGHT is the fusion $\mathbf{T}^* \oplus \mathbf{T}_\nabla$. Nevertheless, the author did not consider a logic containing ∇ as a primitive concept, as we did in reproducing his above argument. Clearly, he did not consider a fusion, although his argument holds in a fused logic. One has obviously to take contingency as primitive and define necessity and possibility.

Consider again the following fused modal languages, one for knowledge and another for "non-contingency":

$$L_K \oplus L_\Delta = \langle \neg, \wedge, \vee, \rightarrow, \Delta, K \rangle$$

Take now the following fusion of two axiomatic systems, one for the epistemic operator of knowledge K and other for the non-contingency operator Δ :

1. All tautologies of classical propositional logic CPL;
2. $(K(\phi \rightarrow \psi) \wedge K\phi) \rightarrow K\psi$;
3. $K\phi \rightarrow \phi$;
4. $\Delta\phi \leftrightarrow \Delta\neg\phi$
5. $\phi \rightarrow (\Delta(\phi \rightarrow \psi) \rightarrow (\Delta\phi \rightarrow \Delta\psi))$
6. $\vdash \phi$ implies $\vdash K\phi$

7. $\vdash \phi$ implies $\vdash \Delta\phi$

Add the definition $\Box\phi =_{def} \phi \wedge \Delta\phi$. The two inference rules came from each axiomatic system. And we also have to add the rule of modus ponens. As the reader note the logic above is the fusion $\mathbf{T}^* \oplus \mathbf{T}_\Delta$. We could also use the following fusion:

1. All tautologies of classical propositional logic CPL;
2. $(K(\phi \rightarrow \psi) \wedge K\phi) \rightarrow K\psi$;
3. $K\phi \rightarrow \phi$;
4. $\nabla\phi \leftrightarrow \nabla\neg\phi$
5. $\phi \rightarrow (\neg\nabla(\phi \rightarrow \psi) \rightarrow (\nabla\psi \rightarrow \nabla\phi))$
6. $\vdash \phi$ implies $\vdash K\phi$
7. $\vdash \phi$ then $\vdash \neg\nabla\phi$

Plus the definition $\Box\phi =_{def} \neg(\phi \rightarrow \nabla\phi)$. Such logic is the fusion $\mathbf{T}^* \oplus \mathbf{T}_\nabla$. A very important argument concerning such fusions of languages and logics is the following: The rule $\vdash \phi$ implies $\vdash K\phi$ shows that theorems, mathematical truths, are objects of knowledge. One knows logical theorems. At the same time the rule $\vdash \phi$ then $\vdash \neg\nabla\phi$ shows that mathematical truths, theorems, are not contingent. Such interpretations are compatible with the idea according to which contingent propositions are not known, but just necessary propositions are known.

Up to now the reader found particular examples of fusions of languages and also fusions of axiomatic systems: the syntactical approach. Now let us examine how things are from the semantical viewpoint. Thus, assume two Kripke structures F_T^* and F_{T_Δ} used to interpret the original axiomatic systems \mathbf{T}^* and \mathbf{T}_Δ . Take the fusion of Kripke structures for knowledge and contingency. Such fusion of both structures is the structure $F_T^* \oplus F_{T_\Delta} = \langle W, R, S \rangle$ such that:

1. W is a non-empty set of possible worlds;
2. R is an accessibility relation for contingency: $R \subseteq W \times W$;
3. S is an accessibility relation for knowledge: $S \subseteq W \times W$.

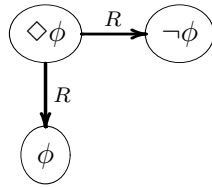
Clearly, given that the original accessibility relations of both logics are reflexive, it follows that the accessibility relations in the fused frames are also reflexive. Assume standard valuations for classical propositional environment and for the knowledge operator. Add the following additional truth-valuation for contingency:

$w \Vdash \nabla\phi$ if and only if there is a world w' such that wRw' , $w' \Vdash \phi$ and there is a world w'' such that wRw'' , $w'' \Vdash \neg\phi$ (w' different from w'')

And the following formal semantics for non-contingency:

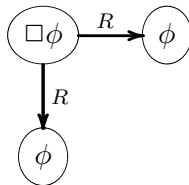
$w \Vdash \Delta\phi$ if and only if for all worlds w' such that wRw' , $w' \Vdash \phi$ or for all worlds w'' such that wRw'' , $w'' \Vdash \neg\phi$

Considering the semantics for the contingency operator, it cannot happen that $w' = w''$ because this would entail a contradiction in each one of the situations. One very interesting fact on possible worlds semantics is that we can make use of very simple diagrams. Needless to say, the difference between possibility, necessity and contingency appears also at the pictorial level:

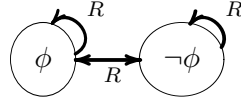


The above figure shows that it suffices that ϕ holds in at least one world to have that ϕ is possible.

Necessity is truth in all possible worlds:



The above semantical clause for contingency is represented by the following diagram (note that the diagram below is the minimal way to represent contingency, which requires at least two worlds, if the accessibility relation is reflexive):

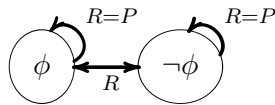


However, without the reflexive condition, we should need at least three worlds to represent contingency. An statement ϕ would be contingent in a world w_1 if and only if there exists a world w_2 accessible from w_1 where ϕ is true, but also another world w_3 accessible from w_1 where ϕ is not true, and therefore its negation is true. The diagram above shows that contingency implies possibility but the converse does not hold. In this setting (i.e. considering fusions of (non-)contingency languages, axiomatic systems and semantics), what could be said about the following principles:

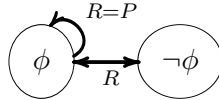
1. (ST) $K\phi \rightarrow \neg\nabla\phi$;
2. (ST') $\nabla\phi \rightarrow \neg K\phi$?

If we want to establish a meaningful relation between knowledge and contingency, we have to add some restrictions in the frames. Given the fusion of frames $F_T^* \oplus F_{T\nabla} = \langle W, R, P \rangle$ we have to put that $P \subseteq R$. In this sense, the accessibility relation for knowledge is a subset of the accessibility relation for contingency. Both (ST) and (ST') have different interpretations. Should we consider *all* contingent propositions or just *some* contingent propositions? The skeptic considered here defends that there is no known contingent proposition. Let us examine such a problem considering that the accessibility relations respect $P \subseteq R$.

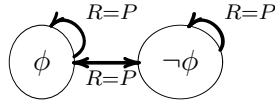
Assume that “All contingent propositions are known”. The logical form of such proposition is $\nabla\phi \rightarrow K\phi$, for all ϕ . This formula is not valid, given that there is a countermodel which falsifies it:



Therefore, one could conclude that there is at least one proposition which is contingent and not known. Additionally, “No contingent proposition is known” would have a countermodel, if one reflexive accessibility relation for knowledge is erased:



Thus one could conclude that there is at least one contingent proposition that is known, against the skeptic. However, the accessibility relation for knowledge cannot be erased, because the frames considered are entirely reflexive for both operators. Therefore, the above countermodel does not hold. Now let us check what would happen if the relations were identical $P = R$. There is no known contingent proposition. If a proposition is contingent, then it is not known, because the minimal model which validates contingency does not validate its knowledge:



Combined Kripke models for knowledge and contingency shows that there are contingent propositions which are not known. Nevertheless, we do not know, from the logical viewpoint, whether all contingent propositions are not known. Consider again the main framework to reason about interactions of knowledge and (non-)contingency. Such a framework is composed by the fusion $\mathbf{T}^* \oplus \mathbf{T}_{\nabla}$ or by the fusion $\mathbf{T}^* \oplus \mathbf{T}_{\Delta}$. Both fused logics are complete, because the operation of fusion preserves completeness when both logics used in the fusion are complete. Thus, given that both \mathbf{T}^* and \mathbf{T}_{Δ} (or \mathbf{T}_{∇}) are complete, it follows that their fusion is also complete. Such fact as mentioned before, has been proved by many authors (KRACHT, WOLTER, SCHURZ, FINE, GABBAY). Considering pure fusions, there are no criteria to relate both operators, because there is no direct interaction axiom. However given the combination of axiomatic systems, there are some curiosities while combining both logics in the three levels: languages, axioms and mod-

els. The first one is a very interesting property which appears in the fusion $\mathbf{T}^* \oplus \mathbf{T}_\Delta$:

$$(K\phi \rightarrow (\Delta(\phi \rightarrow \psi) \rightarrow (\Delta\phi \rightarrow \Delta\psi))).$$

The second surprising property occurs in the fusion $\mathbf{T}^* \oplus \mathbf{T}_\nabla$:

$$(K\phi \rightarrow (\neg\nabla(\phi \rightarrow \psi) \rightarrow (\nabla\psi \rightarrow \nabla\phi))).$$

This shows that fusions of contingency or non-contingency versions of the modal logic \mathbf{T} with their epistemic versions generate automatically interactions of modalities. The first one is generated because we have $\phi \rightarrow (\Delta(\phi \rightarrow \psi) \rightarrow (\Delta\phi \rightarrow \Delta\psi))$ but given that $K\phi \rightarrow \phi$, it follows the automatic interaction. The second one is generated because we have $\phi \rightarrow (\neg\nabla(\phi \rightarrow \psi) \rightarrow (\nabla\psi \rightarrow \nabla\phi))$ and also $K\phi \rightarrow \phi$. Using the transitivity of implication, both interaction axioms are automatically generated. This is surprising, because logicians studying fusions consider that they do not involve any interaction:

“The formation of fusions , or independent joins , is the simplest and perhaps most frequently used way of combining logics. Let \mathbf{L}_1 and \mathbf{L}_2 be two multimodal logics formulated in languages L_1 and L_2 , both containing the language L of classical propositional logic, but having disjoint sets of modal operators. Denote by $L_1 \oplus L_2$ the union of L_1 and L_2 . Then the fusion $\mathbf{L}_1 \oplus \mathbf{L}_2$ of \mathbf{L}_1 and \mathbf{L}_2 is the smallest multimodal logic \mathbf{L} in the language $L_1 \oplus L_2$ containing $L_1 \cup L_2$. In particular, if \mathbf{L}_1 is axiomatized by a set of axioms AX1 and \mathbf{L}_2 is axiomatized by AX2 , then $\mathbf{L}_1 \oplus \mathbf{L}_2$ is axiomatized by the union $AX1 \cup AX2$. This means that no axiom containing modal operators from both languages L_1 and L_2 is required to axiomatize the fusion of \mathbf{L}_1 and \mathbf{L}_2 . The modal operators in \mathbf{L}_1 and \mathbf{L}_2 remain *independent*, they do not interact...” (GABBAY ET AL in [44])

In the introduction some paradoxes related to the combination of logics have been mentioned. One of these paradoxes is the so called *copulation paradox*.⁹ It is intriguing that such fusion generates, up to a certain level, the phenomenon mentioned by BEZIAU. The axiom of knowledge and the distributivity of (non-)contingency together entail one interaction axiom between knowledge and (non-)contingency, producing therefore a new property. This is a case of the copulation paradox, but not based in the

⁹This name has been suggested by BEZIAU. Copulation because the axiomatic systems are together producing a new property.

combination of classical propositional connectives. This last case has been pointed out by BEZIAU in order to show that the combination of two logics in a fusion is not “the smallest logical system for the combined language which is a conservative extension of both” (GABBAY in [43]). What is interesting in the above axiomatic systems is that they manifest an interaction between the notions of knowledge and (non-)contingency. Interactions between modalities are not in general obtained automatically in a fusion. Indeed, interaction axioms are added in a fusion. For this reason, it is rather surprising that the fusion of the above systems generates an interaction. Both interactions of axioms do not show in what sense the accessibility relations are interacting. K. FINE and G. SCHURZ state the following, concerning their results on preservation properties:

“Our transfer results make it unnecessary to establish completeness and other properties separately for stratified multimodal logics, as long as these properties are known to hold for their monomodal components. Thus the results have applications in all areas in which several modal operators which do not interact logically are used.” (K. FINE and G. SCHURZ in [41])

So, what could be said about (ST) and (ST’) in such fusions? If we do not discover how accessibility relations are interacting, then we do not have any correct ground to state whether logical skepticism is sound or not. What would happen if the fusions were expanded in the following way:

1. $T^* \oplus T_{\nabla} \oplus (\nabla\phi \rightarrow \neg K\phi)$
2. $T^* \oplus T_{\Delta} \oplus (K\phi \rightarrow \Delta\phi)$

More interesting cases are those where we extend the fusion by adding two different forms of full skeptical thesis:

1. $T^* \oplus T_{\nabla} \oplus (\nabla\phi \rightarrow (\neg K\phi \wedge \neg K\neg\phi))$
2. $T^* \oplus T_{\nabla} \oplus (\nabla\phi \rightarrow (\neg\Diamond K\phi \wedge \neg\Diamond K\neg\phi))$

Completeness is preserved up to the level where no interaction occurs, but what could be said in the case where we add to the fusions interaction axioms? We come back to this point later. Now, another intriguing fact is that there is a paradox of contingency. Consider (WST) $\Diamond K\phi \rightarrow \neg\nabla\phi$ in the presence of the verification principle $\phi \rightarrow \Diamond K\phi$. If there is a formalism where both hold, then we have the following amazing fact (Assume the fusion proposed in chapter 1 for possibility and knowledge, and take contingency as definable notion):

(Truth implies non-contingency) If a proposition is true, then it is not contingent: $(\phi \rightarrow \neg\nabla\phi)$.

The above sentence can be seen as a refutation of logical skepticism or a new refutation of verificationism, because it implies that all truths are not contingent. However, there are many truths which are contingent. For example “snow is white” is usually true and contingent. Worse, accepting the above principle, contingent propositions are false. This just could be accepted as a very strong notion of truth which would just allows truths as those propositions which are necessary, in such a way that there would not be a way to possible truths and would imply that the world is not contingent. Thus (WST) and (VP) are not compatible. One could think that (WST) is not a plausible option because if it is true, then truth implies non-contingency. Let us check whether contingent propositions can be known $(\nabla\phi \rightarrow \diamond K\phi)$. Assuming that possible knowledge implies truth $(\diamond K\phi \rightarrow \phi)$, it follows that contingent propositions imply truth $(\nabla\phi \rightarrow \phi)$, which is also a strange fact, because some contingent propositions could also be false.

It is interesting to note that taking K and ∇ as primitives, and using the fusion $\mathbf{T}^* \oplus \mathbf{T}_\nabla$ it is problematic for an agent to state that a proposition is contingent and known (or necessary). The reasoning is the following argument (consider that the axiom “knowledge implies necessity” is added to the fusion): From the hypothesis $\nabla\phi \wedge K\phi$, and the interaction axiom “knowledge implies necessity” one gets $\Box\phi$. But from the definition of contingency and some classical propositional logic it follows $\neg\Box\phi$, which is a contradiction. Therefore, by *reductio ad absurdum*, $\neg(\nabla\phi \wedge K\phi)$, assuming that knowledge implies necessity. This shows that in a logic where knowledge implies contingency, it is impossible to know something that is contingent. Hence, an agent cannot state that a given proposition is contingent and known. Indeed, in the logic $\mathbf{T}^* \oplus \mathbf{T}_\Delta$ one could find a fact showing that if a proposition is known, then it is non-contingent. Or, if a proposition is contingent then it is not known $(\vdash K\phi \rightarrow \Delta\phi)$. It is important to see that if a given proposition ϕ is contingent, then it is necessarily contingent, at least if the notion of possibility satisfies **S5**¹⁰. Assuming that ϕ is contingent, and possibility respecting $\diamond\phi \rightarrow \Box\diamond\phi$, then we get that $\Box\nabla\phi$. Thus, if a proposition is contingent then it is necessarily contingent. Considering that if a proposition is necessarily contingent then it is also contingent, both notions are equivalent. Statements of the form “ ϕ is contingent” are necessary truths. Although an agent does know that ϕ is contingent, because contingency of ϕ is necessary, one does not know ϕ .

¹⁰I owe this remark to Arnold Koslow in personal communication.

Given that the world is contingent, and such contingency is reflected in our propositions, it follows that the world is not known. What about the weak version of the skeptical thesis? We do not know, up to now, if it holds or not. To sum up: the world is not an object of knowledge, because: 1) it is contingent and contingent propositions are not known; 2) Knowledge is truth in all possible worlds, and therefore just mathematical truths are known;

2.3 Logical skepticism revisited

A formal analysis of the interactions between the concepts of truth, knowledge and contingency allows a study of logical skepticism. Combinations of knowledge with contingency generate automatic interaction axioms. Systems containing interactions can be used to analyze epistemological questions such as the limits of what can be known: if one knows (or can know) contingent propositions. Indeed, being the mirror of the world, a result obtained in a linguistic construction can be exported to reality. In this sense, from one point of view, given that contingent propositions are not known (or cannot be known), it follows that the world itself is not known (cannot be known). Somebody could argue that a formal approach to the limits of knowledge, despite of interesting, it is extremely artificial because it does not show exactly how people use such concepts. In order to avoid such objections, the following subsections show in what sense a logical approach to the limits of knowledge and philosophical concepts is rather useful.

2.3.1 Knowledge and reality

Reality is a complex collection of state of affairs. Day-by-day there are many events and facts that are present in our lives. We see the news in the newspaper, on the TV, and we have solid reasons to state that we *know* a lot of things. Somebody could argue that the approach defended here is not sufficiently close to reality and normal life because there is *evidence* showing that we indeed *know* some propositions, although we do not have mathematical proofs that these propositions are true. As a reply, the logical skeptic would answer that there are different conceptions and levels of knowledge, and that he/she is supposed to consider a strong conception of knowledge: if an agent *knows* a given proposition, then it cannot be false. The “it cannot be false” has a metaphysical interpretation in relation to a group of possible situations. Despite of the fact that one could argue about the existence of different levels of knowledge, all such levels should confirm

that knowledge entails truth. A conception of knowledge which do not accept the principle “knowledge implies truth” is certainly more related to belief than knowledge.

The approach considered here is based on logical considerations on the structure of knowledge, and a criticism based on a different criteria would be the same as playing a game with different rules and therefore would be the same as playing a different game. If we assume a formal, mathematical and particular notions of contingency and knowledge then we can find arguments showing that skepticism is sound. Evidently there is no doubt that somebody could for example elaborate a logic showing that skepticism is not right. This could be done, even perhaps in an intuitive way. But up to here, what we have is that there is no knowledge of the world. And such a conclusion can be obtained by particular combined logics, especially combining epistemic logics with contingency logics. The fact that there is a model where at least one proposition is contingent but known is an especially artificial argument because it considers the identity between both accessibility relations, and it does erase one of the reflexive accessibility relations.

2.3.2 Two basic criteria

One could think that a formal approach to skepticism (or philosophy) is not sufficiently informative, because one would be able to formalize skepticism (a philosophical theory) but one would not be able to clearly state that skepticism (such a theory) is a plausible philosophical position (or theory) and therefore a formal approach would be, in reality, an empty approach.

If there is no interesting criterion to determine what is plausible from the logical viewpoint, then the above argument would be meaningful and correct. However, given that indeed there are two basic criteria, it follows that the argument is erroneous and it ignores the power of logic in the analysis of philosophical concepts. Let me present two basic criteria and give two examples of how to use the criteria to determine whether a given philosophical theory (or position) is plausible or not.

The first criterion that one should use is *completeness* and the second one is *finite axiomatizability*. Let’s check why these criteria are important. Consider the first criterion. As is well-known, a logical system \mathbf{L} is complete when for all true propositions there is a proof that the propositions are true. Take a logic \mathbf{L} such that \mathbf{L} is not a complete logic in the following sense: there is a true statement without proof. If a logic is incomplete, one can find a proposition which is semantically true although without a proof. Otherwise, if a logic is complete and if a given proposition is true

(valid in the models), then it is also a theorem and therefore, there is a proof. It seems clear that incomplete logics are not so strong in order to help while judging a philosophical position because it is possible to accept the truth of a given statement without being able to prove that it is true. The soundness criterion is the converse of the completeness criterion. If a given logic is not sound, then there are some theorems which are not valid in the models. A philosophical theory which has as its underlying logic an unsound and incomplete formal system does not have, therefore, an interesting value because there are no safety grounds to distinguish between truth and falsity. An example: Suppose that we are trying to determine whether the philosophical position “knowledge implies truth” is a robust philosophical position. In order to do this, we have to find a minimal sound and complete logic where the thesis is a theorem. Take therefore the basic reflexive epistemic logic or one of the fusions considered in this chapter. These logics are sound and complete and validate the knowledge axiom: $K\phi \rightarrow \phi$. Using the completeness criterion, the thesis is a solid philosophical position. But what would happen if somebody finds another incomplete and unsound logic which validates the same axiom? This would lead to the search of more sophisticated criteria. In this sense, we should use general and abstract criteria from universal logic, the general theory of logical systems, and then some properties of the consequence relation should be investigated to determine what is the best logic to analyze a given philosophical thesis.

The second criterion is that of finite axiomatizability. A well-known fact is that a logic L is said to be *finite axiomatizable* if there is a finite set Γ of formulas such that all theorems of the logic are consequences of application of inference rules to its axioms. In a finite axiomatizable logic it is possible to deduce all truths of a given system just by application of its deduction rules. This criterion is important because if it does not hold, then we could suppose that a given philosophical position is solid, but perhaps one new axiom would appear and show that our philosophical notion entails a contradiction (that is a case of monotonicity). One could think that the existence of contradictions in a given theory is not a dangerous thing. This is right, but accepting the environment of classical logic, contradictions can destroy the system by putting all formulas as theorems in the sense that a given philosophical theory is provable but so too its negation. Discussions of paraconsistent logics and other logics able to avoid trivialization are explored in chapter 4 ¹¹. Some other computational aspects of modal logics such as

¹¹According to a famous slogan of N. DA COSTA and other paraconsistent logicians, the problem is not a contradictory theory, but a trivial theory: a theory which proves all

complexity are out of the scope of the present document, but they also constitute important criteria to judge theories, especially when the criteria above are not sufficient.

Clearly, we defend both criteria up to a certain level, but they could be avoided when the theory deserves to be studied by other properties (for instance a very powerful application), as the case of PEANO Arithmetic, ZFC set theory and so on.

Applying the above criteria, what can be said about the philosophical value of logical skepticism? Should we accept or reject skepticism as defined here? Let come back to the basic fusions considered here:

1. $T^* \oplus T_{\nabla}$
2. $T^* \oplus T_{\Delta}$

Such fusions are not able to analyze the skeptical thesis because although they generate one formula where there is interaction of knowledge and contingency, such an interaction is not sufficiently informative to tell us how things are from the semantical viewpoint. In this sense, we should add (ST), (ST'), (FST), (FST') and other variations to both fusions in order to determine the full formal aspect of skepticism:

1. $T^* \oplus T_{\nabla} \oplus (\nabla\phi \rightarrow \neg K\phi)$
2. $T^* \oplus T_{\Delta} \oplus (K\phi \rightarrow \Delta\phi)$
3. $T^* \oplus T_{\nabla} \oplus (\nabla\phi \rightarrow (\neg K\phi \wedge \neg K\neg\phi))$
4. $T^* \oplus T_{\nabla} \oplus (\nabla\phi \rightarrow (\neg\Diamond K\phi \wedge \neg\Diamond K\neg\phi))$

The systems above are viewed as *logics of skepticism*, because they contain both versions of (ST) and versions of (FST). Needless to say that both systems (1) and (2) are deductively and semantically equivalent, but are they complete modal systems? As explained, fusions preserve completeness, and therefore both $T^* \oplus T_{\nabla}$ and $T^* \oplus T_{\Delta}$ are complete. Nevertheless, what would happen adding (ST), (ST'), (FST), (FST') to such fusions respectively? Are they still complete? Our conjecture is *yes*, although we are not able to give a proof of such fact. Completeness preservation of both logics is guaranteed up to the level where no interaction axiom is added. A perfect, full, consistent defence of skepticism should reach a formal proof that the

formulas as theorems.

expanded fusions are complete. To prove completeness, we should find the exact relation between accessibility relations for knowledge and contingency, and determine the exact conditions under the accessibility relations. A natural conjecture is that the canonical models technique can be used here to prove completeness. The skeptical interested in defending logical skepticism from the logical viewpoint is invited to prove completeness of such systems in order to use them against dogmatics who defend that there is knowledge about the world.

One could wonder what can be known at all, if the world cannot be known, according the skeptical thesis and intuitive claims about the relations between knowledge and belief. The answer is not entirely satisfactory, but it leaves at least some hope: we can know, in the strongest conception of the word *knowledge*, just logical and mathematical truths, propositions which could not be false and which are not contingent. One could argue that this is an artificial way to proceed because it is just a manifestation of the epistemic version of necessitation rule: $\vdash \phi \Rightarrow \vdash K\phi$. This rule states that whatever is proved as a theorem is known. In this sense, to know something means to know things that could not be false.

In order to conclude this chapter, let me mention a very classical example given by ARISTOTLE in his book ON INTERPRETATION (Check [3]). This problem states that contingent propositions about the future are neither true nor false. The famous problem called “future contingents” is the following as stated by ARISTOTLE:

“Let me illustrate. A sea-fight must either take place tomorrow or not, but it is not necessary that it should take place tomorrow, neither is it necessary that it should not take place, yet it is necessary that it either should or should not take place tomorrow. Since propositions correspond with facts, it is evident that when in future events there is a real alternative, and a potentiality in contrary directions, the corresponding affirmation and denial have the same character. [...] One of the two propositions in such instances must be true and the other false, but we cannot say determinately that this or that is false, but must leave the alternative undecided. One may indeed be more likely to be true than the other, but it cannot be either actually true or actually false...” (ARISTOTLE in [3])

First, it is interesting to note that the correspondence theory of truth according to which propositions corresponds to facts was already present in the above passage. Second, the logical principle according to which a proposition is either true or false is there too. This problem has been used sometimes to attack the principle of bivalence according to which given a

proposition, it is true or false. Philosophers against bivalence argue, following ARISTOTLE, that contingent propositions about the future are neither true nor false. This looks like just a weak excuse, because although propositions about the future are not *now* neither true nor false, they can be true or false, and therefore bivalence still holds. Thus, there is no problem such as future contingents. And arguing that this is a motivation for introducing a third value is not acceptable. Nevertheless, the fact that *now* a proposition about the future is neither true nor false does not give a lot of information for the conclusion that some propositions about the future have a different truth value. The principle of bivalence and its enemies are not considered here, but the problem is examined from another point of view: are contingent propositions about the future known? If ARISTOTLE were alive, he would probably state the above remark in the scope of epistemic logic replacing all occurrences of the word “necessary” by “known”. Now we have a new version of an old problem: Are future contingents known? The answer follows the same tendency defended above and it is proposed in a form of dialogue with VON WRIGHT. He is one of the most stimulating philosophers who studied informally interactions of modalities (check for instance [108]). Take for example the following problem examined by VON WRIGHT: “Is it logically possible to know the future, what is going to be?” (VON WRIGHT in [108])

But before check one definition of contingency proposed by VON WRIGHT. He announces that:

“A state of affairs or an event is contingent if it is of a generic character such that states (events) of this character sometimes obtain (happen), sometimes not.”.

He argues that there is some knowledge of some future events. Thus, knowledge of contingencies. This is not compatible with logical skepticism, and all propositions stated by VON WRIGHT would be adequate for the logical skeptic if *belief* replaces the place of *knowledge*. He defends that contingent facts of the future are objects of knowledge. The main criticisms to VON WRIGHT is that he does not consider that facts are contingent, and assume that some fact occurs necessarily. Thus, he accepts that some knowledge about the future is possible. One noble example to explain clearly the difference between the kind of interpretation of epistemic logics introduced here and those discussed by VON WRIGHT is the analysis of five propositions discussed by the author. The question “Is it possible to know the future, what is going to be?” is answered comparing the answers of the logical skeptic to VON WRIGHT’s answers based on the analysis of the

same set of propositions. In order to compare both, take a look at some propositions analyzed by VON WRIGHT in [108]:

[1] It is night now, but I know that in a couple of hours it will be daybreak;

[2] Astronomers know that there will be an eclipse of the moon at t . It can be predicted with great exactitude. (A classical example).

[3] I knew that the text which I was jotting down on a sheet of paper when writing this was going to remain legible for a considerable time; that it would not fade away as soon as I had finished writing.

[4] I know that if I jump out of the window of a tall building I shall fall to the ground and hurt myself badly, perhaps be killed. I know that I put my hand in the fire, it will hurt.

[5] I know that I shall die some day in a not very distant future.”

His analysis goes as follows: Items (1) and (2) are object of knowledge because there are some regularities in the nature. We learned about such processes and we do know that they “will continue undisturbed” and “stably”. According to VON WRIGHT “It is sufficient that there is a familiarity with ongoing processes in nature, and no reason for us to assume that they will change or stop.” Case (3) has the same nature, but it is based on human constructions. We know that the process will happen because we have familiarity with the objects involved in the action. For case (4), he argued that we have the best example of knowledge of the future. However, he is very cautious.

Now, take a look at how the skeptic considered here would analyze the same propositions: All items do not express knowledge of a given agent in a given proposition, but belief. Even propositions (4) and (5), which looks like correct from the point of view of a non-logician, are exaggerated. We do not know that the fire will burn, and we also do not know whether we will die. These are beliefs that we accept in order to continue our lives. But there is no knowledge on these facts, simply because one can figure out a situation where the world would be different and, therefore, fire would not burn and we would not die. The fact that we can figure out a possible world with different laws eliminates knowledge. Naturally, VON WRIGHT was an advanced logician and he certainly knew about these other possibilities. Thus, we argued that he is cautious, although he still decided to use the name “knowledge” for those beliefs:

“When, in a concrete situation, I am absolutely confident that if I then put my hand in the fire it is going to hurt, this means that

I have *no* reason for thinking that this situation is exceptional, not of the normal kind. This can be called knowledge of a disjunctive state prevailing for the future, i.e. for the future I am now facing - not perhaps for “all future” and many conceivable situation.” (VON WRIGHT in [109])

The fact that “all future” and “many conceivable situation” are not considered can be used as an argument showing that there is knowledge if the universe of the model is finite and closed. At least from the standard viewpoint of epistemic logics, knowledge in a proposition ϕ means that ϕ is true in all plausible states which the agent considers possible. But for us, knowledge in a proposition ϕ means that ϕ is true in all plausible states which are indeed possible. Suppose, for instance, that the proposition “The fire burns” is true in the actual world and there are just four imaginable worlds from the actual world, and in all these four worlds the proposition is true. In this context, an agent living in the actual world is authorized to state that indeed “someone in the actual world *knows* that fire burns”. However, while analyzing the concept of knowledge, it is not interesting considering this kind of superficial knowledge, because we do not know whether our universe of possible worlds is finite and closed in real life. Indeed, there are no grounds to state that it is finite and closed. Everything indicates that it is not, therefore the field of possibilities does not have a concrete border. In this sense, when analyzing whether a given agent knows a given proposition, the universe, the model where the situation is judged should, at least from the philosophical point of view, continue open. This is important to understand the skeptic thesis defended at the beginning of this chapter. The thesis according to which the world is contingent and the problem of induction are just interesting while considering infinite (or very big) and open universes. There are, of course, some artificial strategies that can be developed to avoid contingency and the problem of induction. Such strategies are basically founded in closed and finite universes. Let’s consider an example related with induction: assume that there is a closed and finite box where an epistemic agent lives, suppose also that all there is exists in that box, and nothing could happen out of the box. Suppose also that there are a finite set of possible states which are conceivable for the agent. Suppose that there are three spiders which live in the box with the epistemic agent and all spiders are white. Accept that it is impossible for something to be a spider and not be white. In this environment, one can state that “All spiders are white” is necessarily true and could not be different. Therefore, the agent would be able to state a necessity about the world. The box is not contingent. In this sense, all skeptical principles would be meaningless.

However, this is a very artificial way to avoid some old problems, which still remain unsolvable while treating open and infinite universes, where usual epistemic agents live. The notion of knowledge considered here is stronger than the concept of knowledge in standard epistemic logics. From the standard viewpoint, an agent knows a given proposition if the proposition is true in all possible worlds which the agent *considers* possible. Our interpretation of knowledge does not accept the subjective content of knowledge. For us, an agent knows a given proposition if the proposition is true in all plausible worlds which are *in fact* possible.

2.4 Fusions, skepticism and the border of concepts

Simple fusions of epistemic and (non-)contingency logics allow the study of some philosophical conceptions such as a particular kind of skepticism. One can apply the mechanism of fusion to combine epistemic and (non-)contingency logic, but if one wants more complex cases, one has to add axioms to the resulting fusion, although the fact that interesting interactions appear in an automatic way. This chapter studied different ways of adding interaction axioms to the fusion as well many manifestations of the skeptical thesis. Although soundness and completeness are preserved for the simple logic, when one adds interaction axioms to the fusion, it remains open to decide whether the resulting logics are sound and complete. This chapter showed that the so-called skeptical thesis, and its variations, is correct in its strong version because in the fused language it is not a theorem and from the semantical level there is no countermodel which falsifies the implication.

The interactions between the epistemic modality of knowledge and the metaphysical modality of contingency introduce a problem in the intersection of epistemology and metaphysics: Is it possible to know the world? A not entirely satisfactory answer has been proposed to this problem based especially in the use of tools coming from combining logics, and intuitive arguments founded in the problem of induction and the contingency of the world. Although reasons to believe that some properties of the world continue being the same and everything working as always, the strong formal and abstract conception of knowledge still remains pure in the sense that there is just knowledge of mathematical truths. Despite the existence of solid intuitive facts pro skepticism, from the logical viewpoint it seems that there is still missing a proof that the logics of skepticism are sound and complete. This is stimulating because while one cannot know the content of the world, one can know its unflexible, universal, abstract and mathematical structure.

Although the logical gap, it seems that the world, as we conceive it, is not known and cannot be known in all its plenitude, considering the problem of induction, the contingency of the world and the notion of knowledge used here.

Chapter 3

Connections: belief and contingency

“Dans *Alice* comme dans *De l'autre côté du miroir*, il s'agit d'une catégorie des choses très spéciales: les événements, les événements purs. Quand je dis “Alice grandit”, je veux dire qu'elle devient plus grande qu'elle n'était. Mais par là-même aussi, elle devient plus petite qu'elle n'est maintenant. Bien sûr, ce n'est pas en même temps qu'elle est plus grande et plus petite. Mais c'est en même temps qu'elle le devient. Elle est plus grande maintenant, elle était plus petite auparavant[...] Telle est la simultanéité d'un devenir dont le propre est d'esquiver le présent...” DELEUZE ET GUATTARI in [38]

The above remark shows a description of the flowing and, therefore, *contingency*. The flowing pointed out by the authors is the one responsible to generate contingencies in the world. In order to understand a general view about the concept of contingency, this work does not consider just a logical and analytical approach to the flowing of reality. From one side it is certainly the case that formal philosophers have developed a very attractive account to contingency. From the other side, contingency has been a main subject in philosophy since pre-socratic fragments of HERACLITUS. Last century, the approach developed by G.DELEUZE AND F.GUATTARI on the notion of “flowing” has been important to the development of what both called *Logique du Sens* or a series of texts on the flowing (Check for instance [38]). It is true that such authors do not explore epistemological concepts which arise in such investigation, but rather they are concentrated in ontological aspects of reality and flowing, and not how an agent knows (or can know) such a reality. The above citation at the beginning of this chapter reflects a paradox. Such paradox is not a logical paradox but an intuitive

paradox. As formulated above, it seems clear that despite of the fact that contingency exists in the world, such a process cannot be contradictory: “Ce n’est pas en même temps qu’elle est plus grande et plus petite.”

Despite some evidences mentioned in previous chapter (i.e. there is no knowledge about the world), one could not avoid the fact that there are *beliefs* about the world. This attitude, belief, which allows some partial understanding of reality is investigated here based on its interactions with the metaphysical modality of contingency. We hope to provide arguments to suggest the reader an interesting conclusion. Another interaction studied is that between belief and knowledge. This chapter is concerned with the logical aspects of belief and its interactions with knowledge and contingency. The idea is to investigate a combination of logics of belief and logics of contingency and analyze some philosophical consequences of such combination. An interaction of knowledge and belief is expressed by a property used called by some authors as *entailment* property (Check HALPERN in [49]). Other interaction investigated is that between contingency logics and logics of belief. Fusions of modal logics of knowledge and belief (and its behavior while adding some interaction axioms such as the thesis according to which “knowledge implies belief”) are very interesting to understand a theory of knowledge, belief and contingency. Some traditional approaches to the knowledge-belief reduction are analyzed. We also investigate the principle according to which there are just beliefs about the world, but not knowledge.

If one decides to combine modalities, the first obvious thing to do is to select the modalities to be combined. Take the modality ∇ and also the modality B . There are two possible combinations using only one time each modality in a single formula:

1. $\nabla B\phi$ (contingent belief)
2. $B\nabla\phi$ (belief about contingencies)

However, such modalities can also appear interacted in the form of implications. Such kind of interaction is, indeed, what is called originally interaction of modalities by researchers studying the theory of combination of logics:

1. $\nabla\phi \rightarrow B\phi$ (contingency implies belief)
2. $B\phi \rightarrow \nabla\phi$ (belief implies contingency)
3. $K\phi \rightarrow B\phi$ (knowledge implies belief)

Such combinations of modalities allows non interactive expressions such as “The proposition ϕ is an object of belief”, “It is believed that ϕ ”, “The proposition ϕ is contingent”, and interactive expressions such as “If a proposition is contingent, then it is an object of belief” and many others.

In a nutshell: this chapter investigates interactions of concepts of knowledge and belief, and its philosophical consequences, as well interactions of belief and contingency and its philosophical relevance. The methodology used is fusions of modal logics (expanded with interaction axioms), in the same fashion of chapter 2. Such approach allows, once again, a study of the bridges between epistemology and metaphysics.

3.1 Shaking knowledge and belief

Logics of knowledge and belief have a large history and recently there is a booming literature on the subject. At the very beginning, just philosophers were paying attention to epistemic logics, but nowadays a lot of researchers coming from even computer science are interested in the subject given that there are some applications in artificial intelligence, cognitive science and related fields. The applications of epistemic logic in philosophy are also important, although the epistemic logician is rather concentrated in formal and structural properties of belief. There are a lot of articles and books in the literature explaining and showing the relations between epistemic logic and epistemology. In the last century, since the works of G.VON WRIGHT [108] and HINTIKKA [54], the formal theory of knowledge and belief has received great attention. Recently, a new approach based on the notion of change of states and dynamics of belief recreated the discussion between formal theories and epistemology in the field usually called *belief revision theory*.

There are many debates related to the concept of *belief* and the concept of *knowledge*, especially in philosophy of mind. Discussions concerning the content of a given belief and the terminology *propositional attitudes* are avoided in this work because they are mainly used in cognitive science and studies in philosophy of mind with the aim of showing the social and morphological dimension of beliefs. Such investigations intend to answer questions related to the content of a given belief and how the mind captures such a content. Both aspects of beliefs are not target of this investigation, because we are interested in general and abstract forms of beliefs (in the same sense of LENZEN in [69]). This study aims to provide a formal and abstract account to the concept of belief in relation to other concepts such as

contingency. This approach is rather logical and pure, instead of empirical. It means that the main point is to examine universal and general properties of belief and its interactions with other concepts, instead of investigating the way one believe in something and the properties of such belief.

In the previous chapter, we studied the relations between knowledge, contingency and truth. In this chapter the relations between belief, contingency and knowledge are analyzed. Such relations are expressed in the diagram below:



Metatheoretical properties of classical systems of knowledge and belief are well understood, but systems containing interactions between both concepts have been just a little bit explored.¹ What are the real interactions of knowledge and belief? Are they the same concept? One could think that if knowledge is just knowledge of mathematical truths, as defended by the skeptic, why not eliminating such a concept from an epistemology which intends to discover how the relations between agents and the world work? Perhaps knowledge and belief should be rather regarded as equivalent concepts, and perhaps one should accept that even knowledge could imply false propositions. Anyway, as the reader checked in previous chapter, it is plausible to accept that knowledge implies truth, but one cannot forget that such a conception constitutes to some extent a kind of logical dogma, assuming that probably a logician could create a sound, complete and finite axiomatizable logical system where such principle does not hold.

Recently, the theory of belief revision has been extremely investigated. Many philosophers and logicians are trying to understand the flowing of information and its effects in our conception about the world. One interesting approach is that of G. PRIEST in [87] showing how to treat contradictions in the revision of beliefs. Also, up to this point, some ontological and epistemic investigations concerning the interplays of belief and contingency have not been target of investigation. In this sense, there is no logical answer to

¹As far as we know, combinations of doxastic logics with contingency logics were not explored in the literature, but combinations of doxastic logics with epistemic logics for K were treated by some authors, as it is explained later.

some problems stated by skeptics such as the limits of knowledge and what can be known. Indeed, studying the formal and mathematical relations between belief and contingency allows a logical research in order to show that contingent propositions about the world are not known, but they are object of belief.

While combining logics, one can proceed in two different ways to develop a complex logic. One is putting logics into a machine which will generate the combined logic. Other is combining logics by hand, and drawing a logic without any special methodology. Tools for combining logics are recent, and authors who combined logics of knowledge and belief in just one complex system did not use any special methodology such as fibring, fusion or products. It is surprising that one example is HALPERN who wrote the article [49] without using an already defined methodology of combination, although his results are entirely right and elegant. His combination has been manufactured. Another example is that N. DA COSTA in his book [32]. He even proposes a logic for knowledge which is composed by three logics. He introduces three operators: truth, belief and justification. Since knowledge is viewed as “justified true belief”, a natural conjecture would be that knowledge implies justification. But also knowledge implies truth and knowledge implies belief because they are all ingredients of knowledge. Then, he is able to analyze the logic of knowledge using in some sense a procedure of combination. However, he proceeded in the same direction of HALPERN, without a special mechanism for combining his ingredient logics. There are many examples. GABBAY even wrote a chapter of his book [43] showing how different systems from the literature are cases of fibring.

Although PLATO’s definition of knowledge induces one to think that knowledge implies belief, this implication is controversial. It is not easy to design a logic able to express a particular interaction between knowledge and belief. Especially, the formula so called by some authors (check LENZEN in [69] and HALPERN in [49]) as *entailment property* which states that “knowledge implies belief” is particularly problematic. The reason is an argument proposed by LENZEN in [69] according to which any logic accepting the entailment property would not be able to distinguish between both concepts because there is a collapse of knowledge and belief². It is not however clear to determine whether logical systems containing the entailment property are systems which belief and knowledge collapse. Some authors as

²HALPERN in [49] states “In this context, it becomes of interest to reexamine a particular attack against the **S5** interpretation of knowledge that seems to be due originally to LENZEN [1979](...)”. But as we will see later it is not however clear what is the fountain of such deduction.

for example W. VAN DER HOEK in [55] even considered the paradox as a consequence of adding other principles such as $B\phi \rightarrow BK\phi$. He states that KRAUS and LEHMAN discovered such a problem. It is not clear what is the real fountain able to generate the paradox and who discovered such a problem.

Some approaches trying to solve this problem have been proposed in the literature. Let's call this problem *the knowledge-belief reduction*. We do not propose a full solution to the paradox, but an analysis of it from the viewpoint of combining logics, explaining what is the right fusion of languages, axiomatic systems and Kripke models to state and formulate knowledge-belief reduction. In this kind of approach, it should be clear to determine what have to be done to avoid the problem and what is the exact fountain of the problem. The strategy here is not the same of the usual strategies which can be found in the literature.

As we pointed out, one of the main motivations of the combination is to explore the thesis according to which the world is just object of belief, but not object of knowledge. This theory is a complement of the theory presented in previous chapter showing that there is no knowledge about the world. There we showed that to determine whether the skeptical thesis is right or not, one should first determine a very good notion of accessibility relation. And we saw that it depends of the perspective which is accepted on accessibility relations to get a good answer to such a problem. However, we did not find a complete logic for skepticism, but we showed how it should be. If both accessibility relations are equivalent and therefore the same, then the skeptical thesis is correct, but if there is a distinction between both relations then the skeptical thesis could not be correct. In this chapter, we proceed in the same direction but now investigating such problem using the concept of belief instead of the concept of knowledge. We think that the world cannot be known (and it is not known), but there is still some way to open the doors of reality.

3.2 Different conceptions of belief

There are different definitions and notions of belief. One can believe in God, one can believe that it is raining outside. There is however a distinction, in relation to the truth-value, between the belief that someone has in God, and the belief that someone has that it is raining outside. D. HUME already knew that there is an hierarchy of beliefs:

“There is certainly a probability, which arises from a superiority of

chances on any side; and according as this superiority encreases, and surpasses the opposite chances, the probability receives a proportionable encrease, and begets still a higher degree of belief or assent to that side, in which we discover the superiority. If a die were marked with one figure or number of spots on four sides, and with another figure or number of spots on the two remaining sides, it would be more probable, that the former would turn up than the latter; though, if it had a thousand sides marked in the same manner, and only one side different, the probability would be much higher, and our belief or expectation of the event more steady and secure.” (D. HUME in [58])

There are, in fact, many *degrees* of belief because some beliefs are close to the truth than others ³. The case where somebody believes in God is more complicated than the case where someone believes that it is raining outside. In the first case, belief cannot be verified, but in the second case, it is possible to verify whether the proposition object of belief is true or not. If there are different degrees of belief one could wonder how it is possible to create a logic for belief, given that there are so many levels of belief. It is not simple to design a logic of belief given that there are many different conceptions and degrees of belief. In this sense, a process of abstraction can play a particular role. While we are not able to map and understand all manifestations of particular beliefs, we can understand formal, general, abstract and unifying properties of belief. How to formalize therefore all this multiplicity without losing the different meanings of what a belief is?

Logical formalizations try to abstract the content of a given concept by examining its essential properties which cannot change, which cannot be revised, which are not dynamic. Abstraction allows a separation of reality. Given that reality is contingent and therefore the flowing is everywhere it is intriguing how abstracting the form of a given concept instead of investigating its content. As a matter of fact, such a process allows and induces the understanding of very complicated structures. For example, there are many facts constituting the world. Facts describing particular fragments of reality, facts which belongs to particular sciences, but there is something general, not belonging to any particular area, but to the general philosophy. It is exactly here that one can understand one of the tasks of philosophy. Instead of studying the content of a particular fact, the philosopher studies

³It is important to make a distinction between degrees of truth and degrees of belief: degrees of truth are related to an interval of truth-values and in general related to fuzzy logics, where propositions are not just true or false, but can have many degrees of truth between both. Degrees of belief refer to the content of beliefs, and up to a certain level, they depend of degrees of truth.

the structure of facts, and the notion of *fact* in general. This idea can be generalized to all concepts. Take now one particular example. In the case of belief, it seems that the basic essential property of the concept of belief is the idea that a belief could be false. In this sense, a logic of belief should capture the most important, general and abstract features of a belief. Therefore, a logical analysis of the concept of belief is a *formal* study independent of discrete manifestations of some instances of belief. The important properties of belief are those which constitute the most general structure of the concept. Despite of some theories trying to study different degrees of belief, one logical approach is rather pure. This is a particular property of logic. In the same way, we know that there are a lot of different propositions, each one expressing some characteristic of a given reality. When a logician formalizes a proposition he/she is not worried to capture its different contents but rather its form. The same happens in the case of the concept of belief. One question concerning the logical approach to belief is to determine what one can believe. What is the object of a given belief? It is a doxastic state which is manifested in the content of a given proposition. One dogma generally accepted by researchers on belief revision theory is to announce that there is a dynamics in doxastic states, but epistemic states never change. This dynamics is rather explained by those people working on belief revision. The static approach is explained by those people mainly in epistemic logic. That is: belief in flowing is especially object of study of people working on changes of beliefs, revisions etc. The formal logician should be rather interested in formal and pure aspects of the concept.

The discussion of this paragraph is based on the book [46]. GÄRDENFORS pointed out that different from epistemic states, which are static, there is a dynamics in doxastic states and therefore belief changes its content in many different ways. In this sense, GÄRDENFORS argued in [46] that a given belief state can change in many directions: expansion, revision and contraction. Indeed, this kind of change is studied in the well established discipline nowadays: belief revision. According to GÄRDENFORS in [46] - the most famous researcher working in the subject - there are basically two forms of belief revision theory: the foundations theory and the coherence theory. According to the author the first one is engaged in the study of justifications of beliefs. Those who defend foundations theory argue that in order to be accepted a given belief has to be justified, and when it loses its justification, it should be rejected. Those who defend coherence theory argue that in order to be accepted a belief should be consistent with other beliefs. Both are therefore studying the content of beliefs. There are, in fact, many models of belief revision. According to GÄRDENFORS, one of them modelling

changes of belief in the foundation theory and it is based on the idea of nodes and semantic network. This is the so-called *Truth Maintenance System* and proposed by DOYLE ⁴. The other famous approach representing coherence theory is the so-called *AGM model* (ALCHOURRON, GÄRDENFORS and MAKINSON) generated by some postulates trying to model different flows of belief. One interesting property according to the founders of such a model, is that the AGM model is founded on the notion of *belief set* which is a set of propositions respecting the principle that belief sets are closed under logical consequence(For an interesting introduction to this debate and an approach trying to unify both theories check [46]). There are three ways to update a given belief set (we reproduce the content here especially because in chapter 4 we discuss G. PRIEST account to belief revision based on paraconsistent logics). Such ways of updating belief sets are the following (Check [46] for details):

1. (EXPANSION): Given a belief set, a sentence and all its logical consequences are added to the set;
2. (REVISION) Given a belief set, a sentence which is inconsistent with the whole of the belief set is added and to get consistency some old propositions are deleted;
3. (CONTRACTION) Given a belief set, no new sentence is added, but some old proposition is eliminated from the set.

What is curious is that people studying flowing of belief are not so interested in epistemic logic. Probably because they assume that epistemic logic is merely a game using the notion of belief which does not explain, or explain in a very non satisfactory way, what is a belief and its behavior. Clearly, such remark does not apply unrestrictedly, but one can verify some articles on belief revision and note that articles on epistemic logics are rarely mentioned. The vice-versa is also the case. Nevertheless, both aspects of the concept of belief are important to understand what is exactly a belief.

Considering standard criteria, it follows that belief is a weaker notion than knowledge. Knowledge is a sophisticated belief, according to PLATO's definition, but not a simple belief, a belief which is also justified and true. In this sense, if an agent believes in a given proposition it does not follow that the proposition is true, and that it is known. Knowledge is stronger than belief, and knowledge implies, in certain sense, belief. The idea that the

⁴As mentioned by GÄRDENFORS in [46].

world is contingent and therefore cannot be known (and it is not known) is too strong. Nevertheless, sometimes it is reasonable to state that we believe in many facts in such a way that we are close to knowledge.

Another approach to the concept of belief can be done by probability theory. Sometimes beliefs in a given proposition have a very high probability to be true. Some mathematicians developed theories to model this kind of situation, where a given event occurs a very big number of times and therefore one is authorized to say that the belief, although its content could be false, has a very high probability to be true. This analysis of belief take into consideration the content of a given belief. All beliefs describe in a certain way what a given agent accepts about a little piece of reality. The formal analysis of the concept of belief assumes that if there is at least one possible state of affairs where a given proposition is false it is sufficient to state that it is not knowledge. Both concepts are judged according to a given closed universe ⁵. Even knowledge, if the universe was not restricted, it could be false, as in the case where some revisions of statements are done. It would be even possible to state in some sense that there is knowledge about the world, considering closed universes.

The problem of analyzing a given content of a belief or its probability to occur or not in reality is not considered here, because we are just interested in the pure formal aspects of beliefs. Empirical problems related to beliefs are also avoided. For example, if somebody believes that water is indeed H_2O , we are not interested in discovering whether, in fact, there is something which is called water and the chemical composition of such substance is equivalent to H_2O in such a way that the content of the belief is true or false.

Given that the concept of belief does not entail truth, it follows that an agent can believe that something is the case, while it is not. Also, given different degrees of a belief, it follows that if an agent believes in given proposition, then his/her belief can vary the truth value under an interval of truth-values, and then a change in the degrees of truth would imply a change in the degrees of belief. This would suggest an approach to the belief operator using many-valued logics. Although this approach seems to be interesting, it is not able to capture general properties of belief. Such approach to belief using many-valued logics has been studied by DRIANKOV in [39]. What is especially curious here is a result proved by R. SUSZKO according to which all n -valued logics can be reduced to a 2-valued logic

⁵In the previous chapter we provided the reader with an example of how knowledge can even be applied to reality since it is a closed reality.

(See [104]). SUSZKO's argument shows that the bipartition of the universe of truth-values in two classes, the designated and non-designated values, recreate bivalence inside of n -valued logics. Considering SUSZKO's argument as sound, how to interpret the concept of belief? What is the sense of stating that a belief has designated value? What are the degrees of belief? If SUSZKO is right, there are not a lot of degrees of belief, but just two levels. Should the approach developed by DRIANKOV, analyzed in the light of SUSZKO's argument, be revised? If there are just two logical values, how to interpret belief in a many-valued framework?⁶

But how to determine when a belief is true or false? Indeed, what does it mean to state that an agent has a true (or false) belief? Take for instance the proposition "The sun is a star". If, in the world, the sun is really a star, then one can argue that the proposition is true, and therefore if an agent believes in the propositions, then (s)he believes in a true proposition. Otherwise, if in the reality the sun is not a star, then the belief is false. Such a proposition states a contingency about the world: that there is something called sun and such a thing is a star. If it is really the case in reality that the sun is a star then the belief is true (in the present). But given that the proposition states a given configuration of reality, such configuration can be different and the belief can become false.

3.3 Interactions of knowledge and belief

KANT pointed out in [60], the following definitions of opinion, knowledge and belief:

"The holding of a thing to be true, or the subjective validity of the judgment, in its relation to conviction (which is at the same time objectively valid), has the following three degrees: opining, believing, and knowing. Opining is such holding of a judgment as is consciously insufficient, not only objectively, but also subjectively. If our holding of the judgment be only subjectively sufficient, and is at the same time taken as being objectively insufficient, we have what is termed believing. Lastly, when the holding of a thing to be true is sufficient both subjectively and objectively, it is knowledge." (KANT in [60])

⁶It is caricatural that SUSZKO is rarely mentioned by people working on many-valued logics. His argument is, by some miracle, not take into consideration in almost all important handbooks of many-valued logic. Such a fact generates two ideas: or SUSZKO's argument is too bad and wrong and then professional many-valued logicians ignore it or they just do not know so much on Polish logic and therefore they did not know SUSZKO's argument.

Such definitions of belief and knowledge are not based on standard criteria used to define what knowledge and belief are. They are rather vague and make use of two complicated and not defined notions: objectivity and subjectivity. In this sense, KANT's definition is not so interesting, because it uses unclear concepts to define the states of believing and knowing.

Many philosophers would accept that knowledge is a sufficient condition for belief and belief is a necessary condition for knowledge because if we have knowledge then we have also belief, and if we do not have belief, then we do not have knowledge. That knowledge entails belief seems to be a common place. But is it so simple to accept that knowledge entails belief and that belief is part of knowledge? At first glance, it seems natural to accept that knowledge implies belief. However, this principle involves a combination of two modalities in an implication containing what is called an "interaction axiom". It is intriguing that FITCH's paradox nowadays is extremely studied but the problem of interactions between knowledge and belief do not receive much attention. Some approaches to this problem are just philosophically justified and a few exceptions are authors such as HALPERN who developed logical arguments to approach this problem. Another interesting study is that of W. VAN DER HOEK, but both authors do not agree in relation to the real cause able to generate such paradox in modal logics containing at the same time the operators B and K . So, finding a modal formalism containing at the same time interactions between knowledge and belief is difficult, given that there are some complications while adding interaction axioms. One of these problems is that called here knowledge-belief reduction. This paradox shows that knowledge and belief respect the same properties and they are, therefore, the same concept ⁷. Other problem concerns interactions of belief and contingency, and simple fusion of logics of belief and logic for contingency. We do not know what happen if some interaction axiom is added to such a fusion, whether completeness is preserved or not.

One traditional distinction made between knowledge and belief is founded in a common argument within the context of epistemic logics: Given that the basic difference between knowledge and belief is that while knowledge implies truth, belief can implies falsity, it follows that logics of belief do not respect, therefore, the knowledge's axiom. Instead, while creating a modal logic of belief in general the axiom according to which knowledge implies truth is avoided and replaced by another axiom. If an agent is supposed to believe just in true propositions, then knowledge and belief respect the

⁷Such an inference can be done considering LEBNIZ law of indiscernibility of identicals.

same axioms.

Consider now the following language $L_B = \langle \wedge, \vee, \rightarrow, \neg, B \rangle$. In such a language, expressions containing the epistemic operator of belief can be generated and inferences for this connective can be constructed. The epistemic system **KD45** is considered in the literature as the main system to treat the notion of belief. Such a logic is generated based on the above language and it is composed by the following axioms and inference rules:

1. Tautologies of classical propositional logic;
2. $B(\phi \rightarrow \psi) \rightarrow (B\phi \rightarrow B\psi)$;
3. $\neg B \perp$;
4. $B\phi \rightarrow BB\phi$;
5. $\neg B\phi \rightarrow B\neg B\phi$;

And two inference rules:

6. $\vdash \phi$ then $\vdash B\phi$
7. $\phi, \phi \rightarrow \psi$ then ψ

Consider now the following language $L_K = \langle \wedge, \vee, \rightarrow, \neg, K \rangle$. In such a language, expressions containing the epistemic operator of knowledge can be generated and inferences for this connective can be taken into consideration. The epistemic system **S5*** to treat the notion of knowledge is composed by the following axioms and inference rules:

1. Tautologies of classical propositional logic;
2. $K(\phi \rightarrow \psi) \rightarrow (K\phi \rightarrow K\psi)$;
3. $K\phi \rightarrow \phi$;
4. $K\phi \rightarrow KK\phi$;
5. $\neg K\phi \rightarrow K\neg K\phi$;

And two inference rules:

6. $\vdash \phi$ then $\vdash K\phi$
7. $\phi, \phi \rightarrow \psi$ then ψ

The first axiomatic system is one classical presentation of a logic able to capture the content and formalize what is a belief. The above axiomatic system for B is the most traditional doxastic logic that one can find in the literature (see [50] and [51] for explanations on epistemic logics for knowledge and belief.). The second axiomatic system is usually considered a plausible logic to treat the notion of knowledge.

Let's examine truth-conditions for belief and knowledge operators. Considering a Kripke epistemic model, knowledge is defined as in the previous chapter. Belief has a different conception. One could think that belief would be possibility while knowledge would be necessity. But this is not true. Indeed, given a Kripke doxastic model, the truth-condition for the belief operator is:

$$s \models B\phi \text{ if and only if for all } s' \text{ such that } sTs' \text{ and } s' \models \phi$$

Let's reproduce knowledge's condition here:

$$s \models K\phi \text{ if and only if for all } s' \text{ such that } sPs' \text{ and } s' \models \phi$$

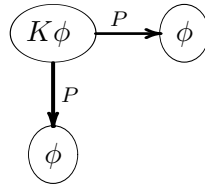
One could think therefore that knowledge and belief are the same concept, assuming just the semantical counterpart of the above axiomatic systems. But the properties on the plausibility relations are different. The relation for belief is not reflexive. This shows that such a conception is not, nevertheless, true because it is the case that the accessibility relation for B is indeed a subset of the accessibility for knowledge K : $T \subseteq P$.⁸ If both accessibility relations were identical, and knowledge equivalent to belief, then knowledge and belief would collapse. Given the above inclusion, it follows that knowledge is stronger than belief. Knowledge is a kind of belief but something more.

Also, given the axiomatic systems **KD45** and **S5***, and given the truth-condition of knowledge and belief, in order to obtain soundness and completeness, properties of the accessibility relations have to be considered. According to general results in the literature, see for instance [51], the system **KD45** is sound and complete with respect to all euclidian, serial and transitive doxastic Kripke models. The system **S5*** is sound and complete with respect to all epistemic models where the accessibility relation is an equiva-

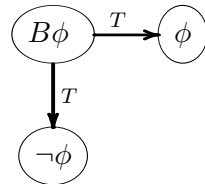
⁸Check MEYER in [78] to verify how to determine such interaction between both operators. But this author studies a combined system of knowledge and belief without using a methodology of combination.

lence relation (reflexive, symmetric and transitive). But what could be said about the fusion $KD45 \oplus S5^*$? We come back to this point later. From the viewpoint of epistemic logics, it is clear what is knowledge and what are its components. Nevertheless, many ingredients of knowledge could be considered. The most traditional are justification and truth. But we could add some other properties to knowledge such as for example evidence, soundness etc ⁹.

Considering the environment of epistemic logics, both knowledge and belief looks like the same, but in a picture everything changes:



The picture shows that if an agent knows a given proposition in a given world, then the proposition should be true in all accessible worlds.



The picture shows that if an agent believes in a given proposition in a given world, then the proposition can be false in some accessible world. Belief can be false, while knowledge cannot be false. If both clauses appear in just one diagram, or in just one Kripke model, then one could say that the accessibility relation for belief is a subset of the accessibility of knowledge, because there are some propositions which are object of knowledge but not only object of belief.

Consider the fusion $KD45 \oplus S5^*$. Such modal system is complete, assuming the above properties of knowledge and belief. This logic has in its

⁹What is well-known is the fact that the classical definition of knowledge suffered many counterexamples, the most famous is that of E.GETTIER in [47].

structure at the same time notions of knowledge and belief, but no interaction between both can be done. Suppose now that we expand the fusion with the entailment property. Then the new logic is $\mathbf{KD45} \oplus \mathbf{S5}^* \oplus (\mathbf{K}\phi \rightarrow \mathbf{B}\phi)$ Such logic is also complete, as proved by HALPERN, W. VAN DER HOEK and many others. It suffices to consider the accessibility relation for belief as a subset of the accessibility relation for knowledge. The problem is in some sense related to collapse of concepts. Indeed, assuming a very strict conception of belief, and some assumptions on its interactions of knowledge, this leads to the following problematic principle:

$$B\phi \rightarrow K\phi$$

This is, obviously, the collapse between knowledge and belief, because the converse holds in the logic $\mathbf{KD45} \oplus \mathbf{S5}^* \oplus (\mathbf{K}\phi \rightarrow \mathbf{B}\phi)$.

The problem is that if someone believes in something he/she should also know the same thing. This cannot be accepted. The argument is explained in detail later. Some weaker version of the paradox states that an agent cannot believe in a given proposition and states that the proposition is false. Such a paradox is called here the knowledge-belief reduction.

3.3.1 The knowledge-belief reduction

In order to understand the knowledge-belief reduction (or LENZEN paradox), let me generate its adequate language and logic. Consider a set *PROP* of propositions and a language $L_B = \langle \neg, \wedge, \vee, \rightarrow, B \rangle$ for belief and $L_K = \langle \neg, \wedge, \vee, \rightarrow, K \rangle$ for knowledge. We need a fused language containing at the same time knowledge and belief.

The paradox occurs inside the fusion of epistemic version $\mathbf{S5}^*$ and the doxastic logic $\mathbf{KD45}$ plus some additional axioms. Then, consider now the following language $L_{KB} = \langle \wedge, \vee, \rightarrow, \neg, B, K \rangle$. In such a fused language, there are two non-interdefinable modal operators and then expressions containing interactions of belief and knowledge can be obtained. The following modal premises are required to obtain the knowledge-belief reduction, obviously in the environment of classical propositional logic (Check HALPERN for complete references in [49] and for explanations on this problem):

- Axiom of Knowledge: $K\phi \rightarrow \phi$
- Axiom of positive introspection for K: $K\phi \rightarrow KK\phi$
- Axiom of negative introspection for K: $\neg K\phi \rightarrow K\neg K\phi$

- Belief's consequence: $(B(\phi \rightarrow \psi) \wedge B\phi) \rightarrow B\psi$
- Bottom particle ¹⁰ is not an object of belief: $\neg B \perp$
- Axiom of positive introspection for B: $B\phi \rightarrow BB\phi$
- Axiom of negative introspection for B: $\neg B\phi \rightarrow B\neg B\phi$

Modus ponens and the inference rules which came from each system.
And two interaction axioms:

- Entailment property: $K\phi \rightarrow B\phi$
- Belief entails BK: $B\phi \rightarrow BK\phi$

HALPERN in [49] states that the entailment property is responsible for generating the paradox, while W. VAN DER HOEK in [55] considers the second interaction as the cause of the problem. The axiomatic system above can be understood in the scope of combining logics. What we have is, indeed, a fusion plus two interaction axioms. The fusion

$$\mathbf{S5^*} \oplus \mathbf{KD45} \oplus (\mathbf{K}\phi \rightarrow \mathbf{B}\phi) \oplus (\mathbf{B}\phi \rightarrow \mathbf{BK}\phi)$$

is a complete logic, as proved in [55]. It is interesting to note that completeness of above logic is not proved using standard techniques of completeness preservation by fusions, but a typical proof by the canonical models method ¹¹.

So, in order to obtain the knowledge-belief reduction, assume a proposition $\neg p$ is true but an agent believes that p is true: Bp . The paradox is the following (see [49]):

1. $\neg p \wedge Bp$ (Assumption)
2. $\neg p \rightarrow \neg Kp$ (Contraposition in the axiom of knowledge)
3. $\neg Kp$ (Modus ponens)

¹⁰A classical logician would enjoy a reduction of bottom particles to inconsistencies and contradictions, but such a reduction is not the case, given that in many paraconsistent logics bottom particles are not contradictions.

¹¹Assuming that *knowledge* and *belief* are interdefinable notions, it follows that there is no need of combining logics to model the paradox. However, both notions are not interdefinable, and thus we should combine two different formalisms to express interactions of knowledge and belief.

4. $\neg Kp \rightarrow K\neg Kp$ (Axiom of negative introspection)
5. $K\neg Kp$ (Modus ponens)
6. $K\neg Kp \rightarrow B\neg Kp$ (Instance of entailment property)
7. $B\neg Kp$ (Modus ponens)
8. $Bp \rightarrow BKp$ (Certainty property ¹²)
9. BKp (Modus ponens)
10. $(BKp) \wedge (B\neg Kp)$ (Contradiction)
11. $Bp \rightarrow p$ (Classical operations)

The reader could think that the above paradox is the same of MOORE's paradox. This is not correct. While the knowledge-belief reduction shows that an agent cannot have false beliefs, MOORE's paradox shows that an agent cannot states that p is true and he/she does not believe that p is true. The knowledge-belief reduction requires at least fusions generating a bi-modal logic containing B and K , MOORE's paradox occurs in a monomodal logic containing just B . The knowledge-belief reduction shows that given the entailment property (or another interaction axiom), one get that belief does not imply false propositions. From the philosophical viewpoint, the paradox shows that the concept of knowledge and the concept of belief have the same consequences, but they do not respect the same properties. It shows that belief implies truth and therefore the paradox is problematic, because intuitively belief looks like as a trivial consequence of knowledge, and belief could imply false propositions, belief traditionally does not imply truth necessarily. What the paradox shows is that if an agent knows something, then the agent also believes in the same thing. Knowledge and belief are, according to the paradox, the same concept ¹³. Still one cannot entirely defend that knowledge collapses with belief. We say that a concept ϕ *collapses* with a concept ψ in a given system if and only if there is a formula $\phi \leftrightarrow \psi$ which is true in the system ¹⁴. We do not have such an equivalence between knowledge and belief, although belief satisfies some properties which knowledge also satisfies.

¹²As HALPERN call it in [49].

¹³This paradox is based in a narrow conception of belief which satisfies many properties.

¹⁴Considering such a conception of collapse, is it possible to have two equivalent concepts which do not respect the same properties?

The paradox is complicated because if one accepts that knowledge implies belief then one can prove, under some assumptions, that belief also implies knowledge. What can be done is to examine the axiom and the logic used in the argument, since the axiom according to which knowledge implies beliefs looks like sound. How to avoid such a problem? Here at least we formulated the exact set of axioms, and the respective fusion of models and logics able to generate the paradox has been proposed. The knowledge-belief reduction, and the attempts to solve, is another solid example of the role combining logics play in philosophical issues.

3.4 Combining belief and contingency

In previous section the relations between knowledge and belief were studied considering fusions. Now we go on in the same direction, but combining doxastic logics with contingency logics.

While the world is contingent and cannot be known, because propositions about the world could be false, if they are true and vice-versa, the same is not the case while treating the concept of belief. Indeed, one cannot *know* propositions about the world, but one *believes* in propositions about the world (contingent propositions). In this sense, we should present a logic adding that “contingency implies belief” and other forms of logical skepticism in order to reflect the philosophical position of skepticism. Here we analyze in what sense a hybrid formalism containing at the same time contingency and belief should be formulated using (or not) a principle such:

$$\nabla\phi \rightarrow B\phi$$

And other powerful forms:

1. $\nabla\phi \rightarrow \diamond B\phi$
2. $\nabla\phi \rightarrow (B\phi \vee B\neg\phi)$
3. $\nabla\phi \rightarrow (\diamond B\phi \vee \diamond B\neg\phi)$

Combinations between logics of belief and logics of contingency are done by taking as primitives operators of contingency and belief. In this sense, given two languages L_∇ and L_B , each one containing ∇ and B as primitives their fusion is a language which contains both: $L_\nabla \oplus L_B = \langle \neg, \wedge, \rightarrow, \vee, \nabla, B \rangle$. The fusion of axiomatic systems generated in this language

is composed by the set of axioms of each logic. The axiomatic system considered here is that of **KD45** for belief and **T ∇** for contingency. The fusion is **KD45** \oplus **T ∇** :

1. All tautologies of classical propositional logic;
2. $B(\phi \rightarrow \psi) \rightarrow (B\phi \rightarrow B\psi)$
3. $\neg B \perp$
4. $B\phi \rightarrow BB\phi$
5. $\neg B\phi \rightarrow B\neg B\phi$
6. $\nabla\phi \leftrightarrow \nabla\neg\phi$
7. $\phi \rightarrow (\neg\nabla(\phi \rightarrow \psi) \rightarrow (\nabla\psi \rightarrow \nabla\phi))$

Adding to the above axiomatic system the following deduction rules which are imported by each axiom system:

8. $\vdash \phi$ then $\vdash B\phi$
9. $\vdash \phi$ then $\vdash \neg\nabla\phi$

And the definition $\Box\phi =^{def} \neg(\phi \rightarrow \nabla\phi)$

Such definition shows that truth does not imply contingency. Previous chapter, when we fused an epistemic logic of knowledge with a contingency logic, one version of the copulation paradox mentioned by BEZIAU occurred. In this combination, the fusion is free of the copulation paradox, because no new property is produced while combining both logics. In last section we checked that fusions of contingency logics with epistemic logics entails immediately some interaction axiom between knowledge and contingency. The same is not true here. Constructing a fusion of contingency logics with doxastic logics does not generate interaction axioms automatically. In this sense, if we want some interaction we should add axioms and put restrictions in the accessibility relations. Semantically, given two Kripke structures, each one for a given logic and each one containing its particular accessibility relations, their fusion is defined in a standard way.

What interesting propositions could be proven inside such a logic? It is interesting to note that the logic **KD45** \oplus **T ∇** is sound and complete, because the logic of contingency is deductively equivalent to its **T** version, which is a complete logic and fusions preserve metaproperties. Completeness can be proven in standard way. Now, there is a perfect framework in order to

study relations between both concepts. What interesting interaction axiom should be added in the above axiomatic system in order to be able to analyze the problem of belief in contingent propositions? The reader should note that in the fusion considered in chapter 2, we used the epistemic version of the necessitation rule, and also its version for contingency, to argue that the world cannot be known, because knowledge just applies to theorems, and given that propositions about the world are not theorems, then there is no knowledge about the world. Here there is a new version of the necessitation rule. Using the same argument above, one could conclude that the world is not an object of belief, because just theorems can be an object of belief. This should be right, if the doxastic version of the necessitation rule is considered alone. However, one main criticism is that such a doxastic version of the necessitation rule is not compatible with the intuitions about belief, because belief cannot imply falsity, and just theorems are object of belief, which is not correct. Thus, the intuitive argument behind such a rule should be: Not just theorems are object of belief, but other kinds of propositions too. Belief is, in fact, a subset of knowledge.

Combining logics of contingency and logics of belief (and knowledge) should be expanded by the following interactions according to the intuitions developed in the previous chapter and now:

1. $\nabla\phi \rightarrow B\phi$
2. $\nabla\phi \rightarrow \diamond B\phi$
3. $\nabla\phi \rightarrow \neg K\phi$
4. $\nabla\phi \rightarrow \neg\diamond K\phi$
5. $\nabla\phi \rightarrow (\neg K\phi \wedge \neg K\neg\phi)$
6. $\nabla\phi \rightarrow (\neg\diamond K\phi \wedge \neg\diamond K\neg\phi)$
7. $\nabla\phi \rightarrow (B\phi \vee B\neg\phi)$
8. $\nabla\phi \rightarrow (\diamond B\phi \vee \diamond B\neg\phi)$

There are other possible interactions to reflect logical skepticism, but above we have the main principles. The first clause states that if ϕ is a contingent proposition, then an agent believes in ϕ . Is there therefore a collapse between contingency and belief? The second clause states that if ϕ is contingent, then an agent can believe in ϕ . The third clause states that contingent propositions are not known and the fourth that contingent

propositions are unknowable. After, there are more complicated interactions of concepts. The above clauses together announces once again the skeptical thesis, but now in its full power. There is no knowledge on the world, but just beliefs. These beliefs vary their contents in many possible directions. What is really difficult to discover is whether a logic of skepticism is sound and complete, or if it is incomplete. As stated in the previous chapter these would be a nice criteria to determine whether skepticism is or not a valid position from the logical viewpoint. Case (3) and (4) were examined in previous chapter. What would happen adding the above interactions to the fusion $\mathbf{KD45} \oplus \mathbf{T}_{\nabla}$? If $\nabla\phi \rightarrow B\phi$ is added to the fusion and if a complete logic is obtained as consequence, then one believes in contingent propositions. A weak skepticism would have as logic $\mathbf{KD45} \oplus \mathbf{T}_{\nabla} \oplus (\nabla\phi \rightarrow B\phi)$. If one decides to examine some properties of the above logic, natural questions to be asked are: Is it complete? Sound?

What would happen if a very big logic containing at the same knowledge, belief and contingency is proposed? What rules would be valid? What is such a logic? What could we state about logical skepticism in this logic? Consider the following fusions:

1. $\mathbf{T}^* \oplus \mathbf{T}_{\nabla} \oplus (\nabla\phi \rightarrow \neg K\phi)$
2. $\mathbf{KD45} \oplus \mathbf{T}_{\nabla} \oplus (\nabla\phi \rightarrow B\phi)$

Obviously, if we fuse both logics in just one system, the resulting system is the following:

$$\mathbf{T}^* \oplus \mathbf{T}_{\nabla} \oplus \mathbf{KD45} \oplus (\nabla\phi \rightarrow \neg K\phi) \oplus (\nabla\phi \rightarrow B\phi)$$

This is a triple fusion. The language of such a fusion is composed by three modal operators $L_{KB\nabla} = \langle \wedge, \vee, \rightarrow, \neg, K, B, \nabla \rangle$ and it is obtained by fusion of three languages. The axiomatic system is composed by those axioms of logics \mathbf{T}^* , \mathbf{T}_{∇} and $\mathbf{KD45}$ plus two interaction axioms. Kripke frames for such a system are structures $F = \langle W, R, P, T \rangle$ such that:

1. W is a non-empty set of worlds;
2. R is a relation for contingency;
3. P is a relation for knowledge;
4. T is a relation for belief.

Each relation respects the original properties of the logics used in the fusion. What are the relations between such accessibility relations? Besides any technical question, the most important should be determine how it would be the *full combined logic of skepticism*? We should add to the fusion $T^* \oplus T_{\nabla} \oplus KD45$ ways expressing the full skeptical thesis. Then we should add to the fusion:

$$(\nabla\phi \rightarrow (\neg K\phi \wedge \neg K\neg\phi)) \oplus (\nabla\phi \rightarrow (\diamond B\phi \vee \diamond B\neg\phi))$$

Another option would be:

$$(\nabla\phi \rightarrow (\neg\diamond K\phi \wedge \neg\diamond K\neg\phi)) \oplus (\nabla\phi \rightarrow (\diamond B\phi \vee \diamond B\neg\phi))$$

Unfortunately, we are not able to study in detail all these logics, and the possible variations of logical skepticism. But certainly if the above logics are sound and complete, and we conjecture that most of them are, then there are good logical grounds to state that contingent propositions are not known, but one just believes in contingent propositions. The logical skeptic is again invited to prove basic properties of his/her favorite combined logic of skepticism.

In order to conclude this chapter, let me mention WITTGENSTEIN who has interesting statements about belief. In his *Tractatus*, the philosopher states the following:

“We cannot infer the events of the future from those of the present.
Superstition is nothing but belief in the causal nexus” WITTGEN-
STEIN (5.1361)

The above citation is in the context of the knowledge of future events. The first WITTGENSTEIN can be considered as a good example of logical skeptic. This idea is old, and came from the works of D. HUME. In the chapter 2 we already have shown the influence of HUME’s proposition in WITTGENSTEIN. We argued that the world is contingent and therefore we have the problem of induction. Thus, we cannot know the world, but just believe that it will be in such and such way. Many philosophers already knew about this fact: there is no logical necessary connection between facts, and for this reason WITTGENSTEIN stated that belief in the causal nexus is superstition.

This work used methods for combining logics to examine the consequences of such a conception. Combining logics of contingency with doxastic logics from one side, and combining logics of contingency with epistemic logic from another side, give a powerful method to investigate the relations between metaphysical concepts (possibility and contingency) and epistemic concepts (knowledge and belief).

3.5 Fusions, skepticism and the border of concepts II

Previous chapter gave a very pessimistic conclusion: the world cannot be known. This conclusion has been restored now in the new configuration using the belief operator. The hierarchy of degrees of belief and the probability of each proposition to be true are not explored here because they are not dependent of logical connections between propositions. But the good news of this chapter is that sometimes our beliefs in a given proposition or set of propositions have a high probability of be true and just a small (sometimes really small) of be false. In this sense, although we cannot know the world in its plenitude, we have good reasons to think that the world will continue its regularity in such a way that we can believe that the fire will burn our hands next time we touch it. The following argument from HUME is illustrative:

“What, then, is the conclusion of the whole matter? A simple one; though, it must be confessed, pretty remote from the common theories of philosophy. All belief of matter of fact or real existence is derived merely from some object, present to the memory or senses, and a customary conjunction between that and some other object. Or in other words; having found, in many instances, that any two kinds of objects- flame and heat, snow and cold- have always been conjoined together; if flame or snow be presented anew to the senses, the mind is carried by custom to expect heat or cold, and to believe that such a quality does exist, and will discover itself upon a nearer approach. This belief is the necessary result of placing the mind in such circumstances. It is an operation of the soul, when we are so situated, as unavoidable as to feel the passion of love, when we receive benefits; or hatred, when we meet with injuries. All these operations are a species of natural instincts, which no reasoning or process of the thought and understanding is able either to produce or to prevent.”
(HUME in [58])

It is a commonplace, and has been argued by many philosophers (for instance HUME in [58] and POPPER in [85]), that this belief is fundamental to human beings and without this kind of belief probably human beings would not be on Earth. We agree that the induction problem force us to deny any knowledge about the world but we have to believe that the behavior of the world will not change dramatically and we also have to believe that some regularities, although apparent because indeed they can change, will continue in its development without great changes. Indeed, this assump-

tion is central for the development of empirical sciences which are basically founded in hypothesis and predictions. Beliefs are dynamics because they are related to the content of a given proposition. We can therefore believe in a given content which describes the world. However, it is possible to study belief without considering their content. As illustrated, the induction problem has been stated by HUME and WITTGENSTEIN. K. POPPER also has interesting insights on the relations between the knowledge of the world, its contingency, and beliefs:

“Our propensity to look out for regularities, and to impose laws upon nature, leads to the psychological phenomenon of dogmatic thinking or, more generally, dogmatic behavior: we expect regularities everywhere and attempt to find them even where there are none; events which do not yield to these attempts we are inclined to treat as a kind of 'background noise'... It is clear that this dogmatic attitude, which makes us stick to our first impressions, is indicative of a strong belief.”
(POPPER in [85])

In this chapter, we examined the problem mentioned above in the context of contingency logics combined with epistemic and doxastic logics. We hope have shown that the world can be an object of belief. In this sense, belief is everything we have to understand reality and its details.

In the next chapter we will deal with some examples of antinomies, and how to control such contradictions in the scope of combined concepts.

Chapter 4

Combinations: contradictions, contingency, knowledge and belief

“Le Tout n’a pas d’être. Ou: le concept d’univers est inconsistant”
A. BADIOU in [4].

In this chapter the interactions of metaphysical and epistemic concepts are studied in the context of contradictory environments. Such contexts can appear in sciences, philosophy, mathematics, computer science and even in our decisions in life. Therefore, they deserve attention. A sketch of a methodology able to convert any logic into a paraconsistent logic, and called *paraconsistentization of logics*, is developed.

The existence of logics able to tolerate contradictory reasoning make some philosophers think that there are indeed contradictions which are true. G.PRIEST has investigated in [86] whether the real world can tolerate the existence of real and true contradictions. His argument is interesting (check [88], also [89] and [86]): Contradictions cause explosion because they are always false and for this reason the principle of explosion holds for classical propositional logic. If we accept that there are true contradictions, the antecedent of the principle of explosion is true, and therefore it is not difficult to falsify it finding a false proposition in the conclusion. Indeed, if one is able to find a deduction where the antecedent is true and the conclusion false, and if such antecedent is a contradiction, then the logic is probably a paraconsistent logic. The existence of non-explosive logics provides reason to believe that there are true contradictions. In a paraconsistent logic contradictions do not entail all formulas, and this is the reason why one can

think that there are true contradictions. G. PRIEST's idea deserves a little bit of attention: his argument is not wrong, but it is not necessarily sound, because other ways to eliminate *ex falso sequitur quodlibet* are possible, not just when we have true contradictions. One of these ways is changing the force of conjunction, or even defining a new kind of consequence relation. Therefore, it does not follow from the existence of paraconsistent logics that some contradictions are true. This is the reason why G. PRIEST's argument is not so interesting. If there was just one way to falsify the principle of explosion, clearly G. PRIEST would be right. But there are many different ways from the case where we have true contradictions in the antecedent.

Moreover, about the existence of true contradictions, one can assume that the world is contingent, but there is no foundation to argue that the world is contradictory and that there are true contradictions because generally from a contingent formula, one cannot deduce a contradictory formula, considering contingency logics:

$$(\Diamond\phi \wedge \Diamond\neg\phi) \not\vdash (\phi \wedge \neg\phi)$$

Given the contingency of the world, it is completely irrelevant whether one could or could not find contradictions in the world because even if there are now some true and real contradictions, the flowing of reality can eliminate it. The reader should be ready to accept that the world has sometimes strange rules, and that the way we use to describe such a reality is just one possible partial description which is not the only correct one. Some recent developments in physics should be taken into consideration, as for example some lessons of quantum physics. The existence of contradictory facts, or some apparent facts, can be, indeed, the case. But one needs much more than logic to discover about the nature of such contradictions.

The fact that contradictions exist, be in the language (or even in the world), is already a good reason to investigate the sort of attitude one can have in relation to these strange objects. Strange because a contradiction is something which we cannot imagine (the reader could do a test and try to figure out how a contradictory fact would be. One cannot imagine a contradictory configuration of reality). Human beings are not able to figure out contradictory objects or facts (in chapter one we analyzed briefly the question of imagination, but not exactly its relation with contradiction). We can imagine many things which do not exist in reality, for instance a monster with 15 heads, a human being with wings, but we cannot imagine something contradictory (exception is the case of a liar, she/he cannot imagine, but she/he could state and argue that he/she can imagine a contradiction).

An interesting practical example concerns 3-D virtual communities which simulate the real world: there one can fly, teletransport for different places, but one cannot do a contradictory action, or one cannot have a contradictory fact. Contradictions are not accepted in 3-D virtual environments, which are indeed a simulation of real world. Given that we cannot have in our minds a contradictory image, it follows that a contradiction is not imaginable. Even a logic of imagination should not allow contradictions as imaginable entities. Considering the imagination operator I of chapter 1, a logic of imagination should not be able to validate the following formula:

$$(I\phi \wedge I\neg\phi) \rightarrow I(\phi \wedge \neg\phi)$$

Such a formula means that one could imagine a fact and also could imagine its negation, but one could not imagine both at the same time. It is not related to cases as the one mentioned by NIINILUOTO. He states the following on “contradiction” and imagination:

“But is it possible to imagine a logical contradiction? If I imagine that Ingmar is both dancing and not dancing at the same time, then there must be *two* Ingmars in my field of imagination. This is a violation of the laws of nature (the same physical individual appears in two locations at the same time), but it is not a logical contradiction.”
I. NIINILUOTO in [81].

The above example concerns the distinction studied below: weak (illu-sory) contradictions and strong (real) contradictions. Although the author relates the notions of contradiction and imagination, he does not deal exactly with such interaction, because he is considering a weak contradiction.

Despite we cannot imagine, we can find contradictions in our theories, and every time a contradiction appears, the first reaction is panic and sadness. The fact that there are contradictions (be it apparent or real) can be used as a motivation for studying the ways of blocking the bad character of contradictions, which is characterized especially by its explosion. Contradictions have been, since 2000 years before, a negative thing in such a way that if a theory is contradictory then it should be rejected. From the viewpoint of classical logic, contradictions are, up to now, considered enemies of reason. If one finds a contradiction, the usual feeling is that one found something wrong. In the history of logic and philosophy, G. FREGE’s project of foundation of mathematics finished when B.RUSSELL sent to him a letter announcing the discovery of a contradiction in his main theory. For G.FREGE, at that time, such a fact was a terrible thing. He

gave up his project usually called nowadays *logicism*. If G.FREGE were alive, and if he knew on the existence of paraconsistent logics, the news sent from B.RUSSELL to him would not be so tragic. It should suffice to change the underlying logic for a paraconsistent logic.

Here we are not interested in bottom particles in general, but particular kinds of bottom particles in classical propositional logic: contradictions. Clearly, not all logics have contradictions as bottom particles. Essentially, in paraconsistent logics contradictions are not bottoms, and this is another reason one can erroneously think that there are true contradictions. We try to check what sort of modality can contain contradictions as scope, and how to convert some logics of previous chapters into logics able to treat contradictions without explosion. How to generate paraconsistent logics from logics which are not paraconsistent? Some forms of paraconsistent negations can be formulated using some modalities, but in what sense a contradiction can be an object of belief and knowledge, and in what sense a multi-modal logic can be a paraconsistent logic still has to be investigated.

In order to avoid confusion, it is important to clarify what a contradiction is. Classical propositional logic is not able sometimes to formalize the exact meaning of a given proposition. Suppose for instance that we use the atomic proposition p to represent the statement “It is raining in Neuchâtel”. The negation of such proposition is “It is not raining in Neuchâtel” and should be represented therefore by the negation \neg of p . Apparently stating that “It is raining in Neuchâtel and it is not raining in Neuchâtel” is a contradiction. A computer, for example, would not be able to recognize this as a non-contradictory formula because it would be formalized as $p \wedge \neg p$. But a detailed analysis would show that the above sentence is not necessarily contradictory, in reality. It is perfectly possible that it is raining in the north of Neuchâtel (in the mountains) but not in the south (in the lake). Therefore, classical propositional logic, in some situations, fail while formalizing natural languages. However, for the sake of it, let’s assume that when such a problem appears we have to accept that $\neg\phi$ is the negation of ϕ in exactly all aspects. If one aspect fails, it is not a contradiction, but a illusory contradiction ¹. In this sense, the proposition “It is raining in Neuchâtel” should be completed with the exact latitude and longitude, and its negation should act exactly in the same time and place. If this is not explicit, the computer or a human being should consider these implicit conditions.

¹Sometimes people working on paraconsistent logics think that there are weak contradictions. This fact is wrong, there is not a weak contradiction, because a *weak* contradiction is not a contradiction.

There are a lot of examples. What people call weak contradictions are not contradictions, and paraconsistent logics dealing with weak contradictions seems to be mistake.

The difference is captured by the two diagrams below:



Above, one find a situation where there is a contradictory fact or state. That is our way to represent a real contradiction, because they are in the same world (it means that one is the negation of the other in all aspects).



Which is a weak (or illusory) contradiction, because the worlds are not the same, and therefore the *all* aspects fail. Another interesting example of a illusory (weak) contradiction is the following:

(C) “An object A is at rest and it is not at rest”.

As stated above, an ingenious mind could immediately think that there is a contradiction. And, in fact, just as stated above, it seems a contradiction. But the problem is that there is hidden information. One could say that an object A is at rest *in relation to itself* and an object A is not at rest *in relation to the moon*. If the hidden information appears, then it is easy to see that there is no contradiction. A contradiction is:

(C+) “An object A is at rest in relation to the moon and it is not at rest in relation to the moon at the same time.”

Considering the diagrams above, (C) should be represented, at least while we do not have all required information, using the second diagram, while (C+) using the first diagram. The example mentioned by NIINLU-OTO, and quoted here, seems to fit in the first category: weak contradictions. Such kinds of situation appear everyday. Here, while formalizing contradictions, we assume that one is the real negation of the other, accepting everything we should accept (under all aspects) to respect such a negation .

4.1 Can contradictions be object of knowledge and belief?

In previous chapters the limits of notions such as knowledge and belief were investigated considering its interactions especially with the concept of contingency. One of the problems which remains unsolved is to determine what happen in combined modal contexts when we are reasoning in the presence of contradictions and inconsistencies. Additionally, interactions of modalities should also be investigated in its relations with contradictory environments. This problem is related to I. KANT's antinomies of pure reason. I. KANT showed that for some special questions such as whether time and space have or not a start and an end, whether the soul is immortal or not, whether there is or not a God, the reason can both prove the thesis and the antithesis. One could find plausible arguments showing that ϕ and the negation of ϕ are true. KANT called this kind of problems as antinomies and argued that we can neither know nor believe in such things, given the contradictory character of the statements. The following passage shows KANT's attitude towards some antinomies:

“When principles which are regulative are regarded as constitutive, and employed as objective principles, contradictions must arise; but if they are considered as mere maxims, there is no room for contradictions of any kind, as they then merely indicate the different interests of reason, which occasion differences in the mode of thought. In effect, Reason has only one single interest, and the seeming contradiction existing between her maxims merely indicates a difference in, and a reciprocal limitation of, the methods by which this interest is satisfied.”
(I. KANT in [60])

Are the antinomies treated by KANT real contradictions? Or are the antinomies just illusory (weak) contradictions? This is difficult to say. Logics able to tolerate contradictions were proposed in the last century and sometimes they are called paraconsistent logics. Recently, N. DA COSTA and S. FRENCH in [33] proposed a logic of belief able to support contradictory beliefs. The idea of generating a contradiction-tolerating epistemic logic can be developed by replacing the propositional classical environment for a paraconsistent propositional logic. Such a work can be used to clarify what happens when an agent believes in a contradiction or when an agent believes in some given proposition and the same agent also believes in the negation of such a proposition. They said:

“To conclude, then, we assert that there do exist situations, both in everyday life and in science, which involved the holding of contradictory beliefs. Any attempt to formalize such belief systems must therefore invoke some kind of non-classical logic.” (N. DA COSTA and S. FRENCH in [33])

N. DA COSTA and his collaborators even said in many articles (see for example [35]) that all logics and mathematics can be adapted in order to become paraconsistent. The idea consists in replacing the underlying logic by a non-explosive logic able to tolerate contradictions. This strategy is artificial, but it works in many cases. It makes us remember VAN BENTHEM’s metaphor on the radio mentioned in chapter one.

Combining fused modal logics with some paraconsistent logics can be used in order to even analyze mathematically KANT’s position. Indeed, we are forced to develop a method to generate paraconsistent logics from logics which are not paraconsistent. This chapter deals with such a methodology and we show how this method can be applied to paraconsistentize some logics proposed in previous chapters in such a way that it will be able to tolerate contradictions and inconsistencies.

While analyzing the skeptical thesis sometimes we could face very complicated situations where both a proposition and its negation hold. Sometimes one can find contradictions in the context of interactions of two different families of concepts. It is exactly the existence of such situations which is one of our motivations in developing a method able to generate paraconsistent logics from logics which are not paraconsistent. But as the reader will note, such methodology does not propose a magic algorithm, neither just one way to generate a paraconsistent logic from a logic which is not paraconsistent, especially because the basic definition of what a paraconsistent logic is still problematic.

How to treat contradictions in the context of metaphysical and epistemic concepts? How in general to convert a logic into a paraconsistent logic? One recent approach to the problem of contradictions in the context of modal concepts is that of J. WILLIAMS in [113]. The author presented a distinction between inconsistency and contradiction using the notion of belief.² In the third chapter we checked the belief revision theory proposed by P. GÄRDENFORS and his collaborators, they had a rule for belief revision stating that the belief set should be re-organized when contradictory beliefs

²Another approach proposing a distinction between inconsistency and contradiction is that of W.CARNIELLI, M.E.CONIGLIO and J.MARCOS in [29] and this last one in [29.1], but using other arguments to show that both concepts are not equivalent. Both WILLIAMS and CARNIELLI ET AL have the same conclusion, but a different argument.

were present. G. PRIEST constructed a paraconsistent belief revision theory in [87]. A kind of belief revision theory which allows the presence of contradictions without explosion. This is a particular case of our method of paraconsistentization applied to belief revision. Both WILLIAMS and PRIEST approach are particular cases of a general process called paraconsistentization of logics. This process has as motivation to treat inconsistencies and contradictions in the scope of modal operators. The idea of paraconsistentization is not that of proposing a new logic or families of logics, but proposing a methodology to unify the different ways one can use to find a paraconsistent logic.

In general, a *contradiction* is a proposition which has the following logical form: $\phi \wedge \neg\phi$. It is possible to combine the concept of contradiction in such a way that there are two basic forms involving the logical form of contradiction, knowledge and belief:

1. $B\phi \wedge \neg B\phi$
2. $K\phi \wedge \neg K\phi$

Propositions above are just instantiations of a contradictory logical form. The logical form of a contradiction is replaced by its modal version. But what could be said of:

1. $B\phi \wedge B\neg\phi$
2. $K\phi \wedge K\neg\phi$

$B\phi \wedge B\neg\phi$ states that agents can have contradictory beliefs, because an agent can believe in ϕ and also in its negation. However, the second formula states that an agent can have contradictory knowledge. Contradictory belief seems to be plausible but contradictory knowledge looks insane, at least at first glance. What about the forms below:

1. $B(\phi \wedge \neg\phi)$
2. $K(\phi \wedge \neg\phi)$

Neither belief in a contradiction nor knowledge of a contradiction seems to be plausible. Although it is possible for an agent $B\phi \wedge B\neg\phi$, it is a strange fact to deduce $B(\phi \wedge \neg\phi)$. In the logic \mathbf{T}^* or its extensions, $K(\phi \wedge \neg\phi)$ should not be accepted, given that it would imply $(\phi \wedge \neg\phi)$. But are there

formalism where such a deduction holds? What would happen in the case where contradictions appear in the context of combined modal operators ³?

ARISTOTLE once said (and proposed the most famous formulation of the principle of non-contradiction):

“There is a principle in things, about which we cannot be deceived, but must always, on the contrary, recognize the truth: the same thing cannot at the one and the same time be and not be, or admit any other similar pair of opposites.” (ARISTOTLE in [2])

While dealing with the limits of knowledge, one can wonder what can be known. One of the biggest open challenges of epistemology, and what should be an answer to KANT’s antinomies of Pure Reason, is to determine what can be known and how reason treat interactive modalities in the context of contradictory environments. T. WILLIAMSON, at the end of this book [115] even states that:

“What we have not seen is a route to knowing that when the pair consists of a proposition and its negation. Yet we may plausibly conjecture that, in some sense of “impossible”, we can know of some propositions both that they are true or false and that it is impossible to know them to be true and impossible to know them to be false. We are only beginning to understand the deeper limits of our knowledge.[WILLIAMSON in [115]]”

This chapter can be also seen as an answer to WILLIAMSON’s questions about the limits of knoweldge. WILLIAMSON’s book is one of the most important books which has been written in the context of formal epistemology, but as the above remark shows, the problem of knowledge in the context of contradictions has not been examined. Some exceptions are those approaches of N. DA COSTA and S. FRENCH. This problem deserves a detailed examination here. To do it, the method of paraconsistentization has been introduced. In order to study contradictions in the context of combined modalities, and to realize such research we propose the method of paraconsistentization. By the end of the chapter, the methodology is applied to paraconsistentize some fusions of modal concepts, and to deal with problems related to logical skepticism.

³The relations between modal logics and paraconsistent logics are studied in the literature by many authors. Even some modal paraconsistent systems have been proposed, examples which interest for us here are those of [33] and [35].

4.2 Paraconsistentization of logics

Given a logic, how can one obtain its paraconsistent counterpart? Given a paraconsistent logic, how to obtain the non-paraconsistent counterpart of this logic? Many systems of paraconsistent logics were already proposed using different methods and techniques. Despite this fact, there is not a singular way to generate paraconsistent logics. This section examines the problem of a unification of all these ways to create paraconsistent logics from logics which are non-paraconsistent by means of what is called *paraconsistentization of logics*. The plurality of ways which can be used to create a paraconsistent logic are examined: defining paraconsistent negations in the language (BEZIAU's argument in [14] showing how to generate paraconsistent logics from modal logics and vice-versa), by restrictions in the axioms (the example of some modal paraconsistent logics), fibring a given logic with a paraconsistent one and changing valuations (many-valued logics). Afterwards, a classification and definitions of different paraconsistentizations are determined. This work is included in a more general project, that of universal logic. A very interesting remark of BLACKBURN ET AL in [17], in the spirit of universal logic, because they avoid the study of particular logics and they are concentrated in general properties of logical systems reflects also the intentions of this chapter:

“The reader looking for a catalogue of facts about his or her favorite modal system probably will not find it here. But such a reader may well find the technique needed to algebraize it, to analyze its expressive power, to prove a completeness result, or to establish its decidability or undecidability...” (BLACKBURN ET AL in [17])

We do not propose techniques for mathematical projects on paraconsistent logics, but we propose a way to unify paraconsistent logics instead of concentrating on particular systems. Paraconsistentization is the general theory of paraconsistent logics. Therefore, it is for paraconsistent logics what universal logic is for logics in general. Such kind of approach is still missing in the context of paraconsistent logic. There are many very interesting systems of paraconsistent logic, but there is no such general technique to treat paraconsistent logics from an abstract viewpoint. In this sense, modal logic is much more developed than paraconsistent logic, which seems a non-uniform area of research where many developments are missing, beginning by the very definition of what is a paraconsistent logic. One of the problems of paraconsistent logics is that people studying it are more interested in particular systems and families of logics than in a general approach to paraconsistency.

MARCUS KRACHT in [64] also has a interesting remark in the same direction of BLACKBURN ET AL:

“The book treats modal logic as a theory, with several subtheories, such as completeness theory, correspondence theory, duality theory and transfer theory...Moreover, we do not proceed by discussing one logic after the other; rather, we shall be interested in general properties of logics and calculi and how they interact” (KRACHT in [64])

Both citations above reflect a new tendency in logical approaches: the search for general tools and techniques instead of an inquiry of particular systems of logic. Such systems are important especially while trying to apply a given formal tool in a particular area such, for example, linguistics, computer science or philosophy. A universal approach to logic provides an interesting unification of the subject. But note that there are a lot of complications in such approach. It is very difficult to speak in a general way while the main purpose is application-oriented. If one intends to apply a given logical system to some particular problem, there is no reason to stay at the abstract level.

Paraconsistent logics, as is well-known, were developed some years ago by S. JAŚKOWSKI, who introduced a calculus able to tolerate contradictions in reasoning (see [59]). A few years later, N. DA COSTA proposed an hierarchy of paraconsistent logics which would be known as the C-systems (see [34]). All these logics are not trivial in the presence of contradictory propositions, because it is not possible to deduce everything (i.e. all formulas) from a contradictory set of propositions. The paraconsistency journey has attracted many researchers around the world, which were responsible for contributing to the development of the basic concepts and techniques for paraconsistent logics. So, what happened with paraconsistent logics was the same of what happened with logics in general: a proliferation of systems. In this sense, it is possible to find a great variety of paraconsistent logics in the literature: the logic **LP** proposed in [89], the recent **LFIs** approach proposed in [29] and many others. There are many paraconsistent logics but also many definitions of paraconsistent logic (see [13] and [29] for discussions on possible definitions of paraconsistent logics): paraconsistent logics are contradictory but non-trivial logics or, what is the same, paraconsistent logics are non-explosive logics (see [59] and [34]), paraconsistent logics are logics with a paraconsistent negation (see [13]), etc. Despite of this multiplicity, let's assume that all those definitions are equivalent in the sense that they are all based in the simple fact that contradictions do not lead to trivialization. This idea looks like interesting, but as BEZIAU pointed out in [13], the

existence of paraconsistent logics is not guaranteed by such a definition, because many logics can be seen as paraconsistent based in this negative criteria. If no positive criteria are found, some discussions about the history of paraconsistent logics are completely without meaning and just can be used to polemical proposals, given that perhaps there is no paraconsistent logic. It would be the same of a research trying to discover who created the 8-D virtual dimensional world, without knowing whether such a world exists. Before discussing who created paraconsistent logics we have to determine whether such kind of logics exists. In this sense, some more basic questions can be stated:

1. What is a paraconsistent logic?
2. Given a logic, how can one obtain its paraconsistent counterpart?
3. Given a paraconsistent logic, how to obtain the non-paraconsistent counterpart of this logic?

Explaining and analyzing such questions and the different possible partial solutions to the problems is not easy. It is difficult to create a general theory to develop paraconsistent logics from certain logics which are not paraconsistent. Such difficulty exists especially because there is no clear definition of paraconsistent logics and logical systems are presented in multiple ways. Some examples of how turning some combined modal logics into paraconsistent logics are important to deal with contradictions when these are object of knowledge and belief (and its interactions).

As explained, N. DA COSTA has stated some years ago in many different articles that all logics can be adapted in order to become paraconsistent, but he never analyzed or showed how to realize methodically *paraconsistentization*. With the aim of expressing the idea of introducing a paraconsistent dimension into a given logic let me propose, first, the verb *to paraconsistentize* which means *the process of turning a given logic into a paraconsistent logic* and, afterwards, the substantive *paraconsistentization* which means not just one, but a plurality of methods and strategies to paraconsistentize a given logic. In the same sense that it is possible to add a temporal dimension in a given logic (see [45]) in order to make it a temporal logic, and in the same sense that it is possible to add a fuzzy dimension in a given logic (see [43]) in order to make it a fuzzy logic, it is also possible to introduce paraconsistency in a given logic in order to make it a paraconsistent logic ⁴.

⁴Indeed, the idea of paraconsistentization has been inspired by the idea of M. FINGER and D. GABBAY which appears in a method called temporalization. Check the article

The motivation of paraconsistentization of logics is the same motivation of paraconsistent logics. So, the question *Why paraconsistentization of logics?* can be reduced to the question *Why paraconsistent logics?*, but noting that for us here paraconsistentization is important because it is a tool for converting some modal logics into paraconsistent logics while some application should be implemented. It is important to pay attention to the fact that just creating a paraconsistent logic is not the same as creating a paraconsistent logic from a given logic which is not paraconsistent. In this sense, the first question above is not the same of the question: *How can one create a paraconsistent logic?*

In what follows, it is examined different kinds of paraconsistentization that can be found in the literature and in particular, two ways that can be used to paraconsistentize a given logic. Besides, it is proposed a general way to paraconsistentize classical propositional logic and some of its fused modal extensions.

4.2.1 The case of modal logics

Case I

One example of paraconsistentization of logics has been developed by BEZIAU in [14]. Let me re-construct his argument using a new environment. A signature S_i is a collection of operators or connectives with different arities i . Take two signatures S_n and S'_n such that S_n is composed by S_0 = propositional variables; S_1 = $\{\sim\}$; S_2 = $\{\wedge, \vee, \rightarrow\}$; And S'_n is composed by: S'_0 = propositional variables; S'_1 = $\{\neg, \diamond\}$ S'_2 = $\{\wedge, \vee, \rightarrow\}$.

Assume that the signatures S_n and S'_n generate the collections $PROP$ and $PROP'$ of propositions, respectively, where $PROP$ is a paraconsistent signature and $PROP'$ is a modal signature. Define a function h from $PROP$ to $PROP'$ such that:

1. $h(p)=p$, if p is an atomic proposition;
2. $h(\phi * \psi) = h(\phi) * h(\psi)$, $*$ \in $\{\wedge, \vee, \rightarrow\}$;
3. $h(\sim \phi) = \diamond \neg h(\phi)$

Consider now two logics L and L' defined using the general abstract approach of universal logic. It means that L = $\langle PROP, \vdash_L \rangle$ (a paraconsistent logic based on a paraconsistent signature) and L' = $\langle PROP', \vdash_{L'} \rangle$

[45], where the authors show a methodology to introduce time dimension in some given logics.

(a modal logic based in a modal signature) where \vdash_L and $\vdash_{L'}$ are tarskian consequence relations. So, in order to generate a paraconsistent logic from a modal logic and vice-versa it is sufficient to have:

$$\Gamma \vdash_L \phi \text{ iff } h(\Gamma) \vdash_{L'} h(\phi)$$

This fact guarantees that a modal logic has been generated from a paraconsistent logic. Indeed, BEZIAU shows that it is possible to define in the modal logic **S5** a paraconsistent negation and, therefore, the modal logic **S5** is a paraconsistent logic. His argument attempts to show that it is always possible to find a paraconsistent logic associated to a modal logic and vice-versa. It can be viewed, then, as the first example of paraconsistentization of logics, in particular, modal logics. So, what is required in this kind of paraconsistentization is the definition of paraconsistent negation in a given logic by certain conservative translations. According to D. BATENS, in personal communication, this is not a legitimate paraconsistentization because the logics used as input-output are intact, without any change. But given the definition of paraconsistentization as a process of turning a given logic into a paraconsistent logic, it follows that the above argument can be viewed as a kind of paraconsistentization because it turns a modal logic into a paraconsistent logic using translations.

Case II

The second example of paraconsistentization of logics is structurally different from the first one and consists in turning modal logics into paraconsistent modal logics, but now changing the content of the input logic. This method is particularly interesting while generating paraconsistent versions of some combined modal logics that we studied in chapter one, two and three. The idea is the same, but now combined frames have to be paraconsistentized. Let me take as example the logic **CiT** proposed in [36]. This logic can be viewed as a paraconsistentization of the modal logic **KT**. How to paraconsistentize **KT** and, indeed, all family of normal modal logics? We have to proceed in two different levels.

The first one is at the **deductive level**. Take some standard presentation of classical logic at this level (sequent calculus, natural deduction, axioms etc) and then eliminate from it all forms of explosion by re-constructing the signature. Let us assume the standard presentation of modal logics, i.e., Hilbert-style presentation. In order to paraconsistentize at the deductive level, we have to replace the propositional classical fragment by a para-

consistent logic which means that the signature of the logic is replaced by another one and the set of axioms become weaker.

The second one is at the **semantical level**. Given a Kripke model $M = \langle W, R, V \rangle$, we have to rewrite the clauses for satisfiability respecting the paraconsistentization at the deductive level, i.e, if we paraconsistentize at the deductive level using, for instance, the paraconsistent logic **CiT** we have to rewrite the valuations in the worlds using the valuations of the original logic **CiT**. Formally, if the axiom $(\phi \rightarrow (\neg\phi \rightarrow \psi))$ is replaced by $(\circ\phi \rightarrow (\phi \rightarrow (\neg\phi \rightarrow \psi)))$ to obtain a paraconsistent logic from positive fragment of classical logic (see the case of **LFIs** in [29]), then the semantical counterpart has to behave in the same sense, i.e, replacing the old classical valuations for non-truth functional valuations of some given paraconsistent logic, in this particular case those of **CiT**:

1. $M, w \not\vdash \phi$ implies $M, w \vdash \neg\phi$;
2. $M, w \vdash \neg\neg\phi$ implies $M, w \vdash \phi$;
3. $M, w \vdash \circ\phi$ iff $M, w \not\vdash \phi$ or $M, w \not\vdash \neg\phi$;
4. $M, w \vdash \neg\circ\phi$ iff $M, w \vdash \phi$ and $M, w \vdash \neg\phi$;

The worlds are now paraconsistent worlds, and the valuations are adapted in order to make the system complete. For more details, see the article [36.1]. One important fact is that one can paraconsistentize a given logic and obtain very different logics. This would happen if, instead of **Ci**, one take as base, for instance, the logic **C1**. The valuations, then, should behave like those of the non-truth-functional semantics of **C1**. Another interesting approach which can be viewed as a kind of paraconsistentization is that of C. MCGINNIS [74] where the author proposes a paraconsistent deontic logic whose formulas in the scope of deontic operators are paraconsistent, although they behave classically in other fragments. His logic is a paraconsistent counterpart of a usual deontic logic.

Case III

In the environment of combination of logics, the method developed by GABBAY and known as fibring can be used to solve certain simple paraconsistentizations. One example is that proposed in [22] which consists in fibring a given modal logic with a paraconsistent logic in order to create a modal paraconsistent logic. What is different from the above case? Here everything works in an *industrial* way, because there is a method to create a new logic

from two logics, while before, let me call it a paraconsistentization *by hand*, no general method is used to get the new logic. However, the logic **CiT** could be reached by pure fibring of the positive fragment of the modal logic **KT** with the logic **Ci** given that no interaction axioms are present in **CiT**. Fibring looks like sometimes a very artificial method which does not lead to desired results. And the original idea of D. GABBAY looks like very far away of the kind of fibring developed nowadays. The example studied in [22] shows exactly that fibring the modal logic **KD** with **C1D** is not sufficient to get the original deontic paraconsistent logic of [35], but to get a weaker logic. And this happens in all kinds of logics with axioms interacting operators of different signatures. But the simple fact of adding an interaction axiom to the result logic leads to that deontic paraconsistent logic.

The case of many-valued logics

How to paraconsistentize the many-valued logic \mathbf{L}_3 ? Take a particular presentation of the logic \mathbf{L}_3 . It is possible to paraconsistentize this logic in a plurality of different ways, but such paraconsistentization should be able, at least, to create a non-explosive logic having all desired properties of the input logic. In order to paraconsistentize many-valued logics, one can apply the following ways:

- Paraconsistentization using translations to define a paraconsistent negation inside \mathbf{L}_3 using BEZIAU's strategy;
- Applying SUSZKO's reduction theorem to get a bivalent logic and then eliminating at the deductive level all kinds of explosion and paraconsistentizing at the semantical level adapting the valuations and respecting the deductive paraconsistentization in order to get completeness and other metatheoretical properties preserved;
- Fibring the logic with a paraconsistent logic in the sense of GABBAY;
- By algebraic games changing the designated values or changing the semantics of the operators (computers can easily transform \mathbf{L}_3 into a paraconsistent logic).

Indeed, those strategies can be applied to a large class of many-valued logics, from LUKASIEWICZ's logic to GÖDEL's logics.

4.2.2 The problem of a paraconsistent negation

The reader just saw an example of how to turn a modal logic into a paraconsistent logic and vice-versa, but is the concept of paraconsistent logic clear enough? Are there paraconsistent logics? Let's examine a central question before continuing.

It is important to note that the very definition of paraconsistent logic is problematic, and some logicians even argue that maybe there is no paraconsistent logic (Check BEZIAU in [15] replying SLATER in [103]). H. SLATER also argued against the existence of paraconsistent logic, but SLATER argument is rather different from that of BEZIAU. SLATER argues that when we negate paraconsistently a formula it is not a negation-contradictory relation (in the square of opposition of ARISTOTLE) which is playing a role, but a negation-subcontrary. In this sense, some people can even think that there are true contradictions, because they are considering subcontraries, which can be true at the same time, and not contradictories. The problem is related to positive and negative criteria which are used to define what is a paraconsistent logic and to the basic definition of paraconsistent logic based in the rejection of the principle of explosion. BEZIAU argued in [13] that if we define a paraconsistent logic as a logic which does not respect the principle of explosion we have a negative criterion based on the *rejection* of a given principle. The problem of this rejection is that there are many operators which can be defined in different logical systems and these operators are not explosive. Let's illustrate the problem using a very simple example. Take classical propositional logic, one can define in it four different unary operators (table 4.1):

Table 4.1: Unary operators

ϕ	$\star_1\phi$	$\star_2\phi$	$\star_3\phi$	$\star_4\phi$
T	T	T	F	F
F	F	T	F	T

The reader can easily check that just \star_2 is a paraconsistent negation, because it is not explosive (table 4.2).

Then one can define a formula inside classical propositional logic which is not explosive, considering classical \wedge . For example:

$$\star_2\phi = \phi \wedge \phi$$

This makes us think that classical logic has a paraconsistent negation,

Table 4.2: Unary operators

ϕ	$\star_2\phi$	ψ
T	T	T
T	T	F
F	T	T
F	T	F

and therefore even classical logic is a paraconsistent logic. Indeed, classical logic has infinite many paraconsistent negations:

$$\begin{aligned}\star_n\phi &= ((\phi \wedge \phi) \wedge \phi) \wedge \dots \wedge \phi_n \\ \star_{n+1}\phi &= (((\phi \wedge \phi) \wedge \phi) \wedge \dots) \wedge \phi_n \wedge \phi_{n+1}\end{aligned}$$

The above argument (table 4.1) shows the existence of 4 unary operators. The first one is classical negation, the second is a kind of paraconsistent negation, and the third and fourth are top and bottom particles respectively.

There are indeed many variations of paraconsistent negations that can be found in classical propositional logic. Traditionally, the roots of this kind of argument originated in BEZIAU where he showed how to define a paraconsistent negation using modalities (The argument has been explained in previous section).

If one accepts that a paraconsistent logic is a logic which has a paraconsistent negation, and if one accepts that paraconsistent logics are non-explosive logics because one can find in it contradictions which are not bottom particles, then classical propositional logic is a paraconsistent logic because it has many non-explosive paraconsistent negations. How to deal with this problem? Probably this is the most important problem related to paraconsistent logics. Many papers have been written on the subject but no full definition of paraconsistent logic has been proposed. BEZIAU argued in [13] that a good paraconsistent negation has also positive properties. In this sense, in order to be a paraconsistent negation an operator has to be something more than just a non-explosive operator. One should also add positive properties to paraconsistent negation. BEZIAU also states that a definition is negative if it is based on the rejection of \vdash ($\not\vdash$). Otherwise, the definition is positive. In this sense we should find rules which paraconsistent negations respect, and not just reject. The question consists in discovering what positive properties a negation should respect. Despite the fact of finding positive properties of paraconsistent negations, one can accept in an *ad hoc* manner that the essential property of paraconsistent logic is to be non-explosive.

4.2.3 Desparaconsistentization of logics?

What about the contrary process of paraconsistentization? Let me call it *desparaconsistentization* of logics. Given a paraconsistent logic, how to obtain its non-paraconsistent counterpart? One simple example is the case of logics of formal inconsistency in [29]. These logics are paraconsistent logics, but they become non-paraconsistent if we add to it the consistency of a given schema of formulas as theorem. Indeed, adding consistency as theorem leads the **LFIs** to classical logic. This is one good example of what is called desparaconsistentization. Other example is that proposed by BEZIAU in [14]: each paraconsistent logic has associated to it a modal logic. Nevertheless, it is still missing a general methodology showing how to convert a paraconsistent logic into a logic which is not paraconsistent. Given a paraconsistent logic, one could, at least in principle, desparaconsistentize it.

4.3 Paraconsistentization of classical propositional logic

This section explain in detail what is paraconsistentization - the general theory of paraconsistent logics - and how to paraconsistentize classical propositional logic. Issues such as preservation of some metatheoretical properties such as soundness and completeness are also discussed. Two methods of paraconsistentization are introduced and its basic properties are analyzed from the abstract, syntactical and semantical viewpoint.

Paraconsistentization of logics is the process of turning a given logic into a paraconsistent one. However, a sufficient condition to have a good paraconsistentization is that basic metatheoretical properties like soundness and completeness are preserved. In the following, two methods to paraconsistentize classical propositional logic **CPL** are presented: reduced and extended paraconsistentization. Each kind of paraconsistentization is defined at three different levels of abstraction:

1. Abstract dimension (languages and structures);
2. Syntactical dimension (axiomatic systems, natural deduction etc);
3. Semantical dimension (valuations and models).

By the end, one example of application of paraconsistentization in the problem of fused modal logics and interactions of two different families of

modalities is presented.

Reduced paraconsistentization of classical propositional logic

A reduced paraconsistentization generates a paraconsistent logic from a logic which is not paraconsistent. The idea consists in a reduction of structures, languages, axioms and semantics.

Abstract level

Let explain how to turn classical propositional logic into a paraconsistent propositional logic from the abstract viewpoint (i.e. considering logics as structures). In order to introduce paraconsistentization of **CPL** at the abstract level, let's start by the well-known concept of a logic due to TARSKI. A *logic* is a structure $\langle F, \Vdash \rangle$ such that:

1. F is a set of propositions;
2. \Vdash is a tarskian consequence relation between sets of propositions and propositions: $\Vdash \subseteq \wp(F) \times F$;

All logics can be represented at this level of abstraction. But this level is not rich enough to allow specific discussions as those of preservation of properties like soundness and completeness, because there is just one kind of logical consequence, and we need a distinction between proof-theoretical and model-theoretical aspects of logical consequence to talk about metaproperties. Also, paraconsistentization cannot be applied to such definition of logic, because all logics are at the same level of abstraction and, therefore, they could be considered the same.

The first step to paraconsistentize **CPL**, from the abstract point of view, is to consider the classical signature S_n such that S_n is composed by S_0 =propositional variables; $S_1 = \{\neg\}$; $S_2 = \{\wedge, \vee, \rightarrow\}$ and then the natural rules for generating **CPL** language.

In [9], the author defines classical propositional logic using the notion of mathematical structure. In his sense, **CPL** can be defined at the abstract level using a classical signature, in the following way:

$$CPL = \langle F, \neg, \wedge, \vee, \rightarrow, \Vdash \rangle$$

Some comments: the above operators are classical connectives, in the sense that they respect the rules of classical propositional logic. The \neg is, therefore, a classical negation. The author uses \vdash instead of \Vdash , but given

that usually \vdash represents deductive consequence relation, then the symbol \Vdash is a more abstract relation which is neither deductive nor semantic, and it is thus the option here.

Given the above abstract conception of **CPL**, the *paraconsistentization of CPL at the abstract level* is a structure $P(CPL) = \langle F, \sim, \wedge, \vee, \rightarrow, \Vdash \rangle$ where \sim is a paraconsistent negation which is not explosive. The classical negation was just replaced by a paraconsistent negation, i.e, a connective which does not respect the principle of explosion. In this sense, we get a paraconsistent structure (i.e. a structure containing a paraconsistent negation).

Syntactical reduced paraconsistentization

Given classical propositional logic in a particular Hilbert-Style presentation, how to obtain paraconsistent counterparts of this presentation? Classical propositional logic can be axiomatized by different sets of axioms. The first step is to assume a particular axiomatization of classical propositional logic. Let me use that proposed by KLEENE (1952) (as cited by E. MENDELSON [76]):

1. $(\phi \rightarrow (\psi \rightarrow \phi))$;
2. $(\phi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \gamma))$;
3. $((\phi \wedge \psi) \rightarrow \phi)$;
4. $((\phi \wedge \psi) \rightarrow \psi)$;
5. $(\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi)))$;
6. $(\phi \rightarrow (\phi \vee \psi))$;
7. $(\psi \rightarrow (\phi \vee \psi))$;
8. $((\phi \rightarrow \gamma) \rightarrow ((\psi \rightarrow \gamma) \rightarrow ((\phi \vee \psi) \rightarrow \gamma)))$;
9. $((\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \neg\psi) \rightarrow \neg\phi))$;
10. $(\neg\neg\phi \rightarrow \phi)$.

Bi-implication is defined as usual and the only rule of inference is *modus ponens*. Thus, the second step is to analyze what is the type of mathematical structure associated to such an axiomatic system (in this case it is classical propositional logic). Thus, we have to replace in the classical signature the

classical negation by a weaker negation which is not explosive in order to obtain a paraconsistent signature, following the same strategy of abstract reduced paraconsistentization. So, we have: $S_1 = \{\sim\}$; The third step is to replace in the axiomatic system all occurrences of \neg by \sim . So, as a consequence:

1. $((\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \sim \psi) \rightarrow \sim \phi))$;
2. $(\sim \sim \phi \rightarrow \phi)$.

This is the partial syntactical reduced paraconsistentization of classical propositional logic. One can wonder whether the axioms (9) and (10) are no more theorems. In order to conclude the paraconsistentization, one has to find new general laws to replace the axioms where the old negation appears. But this can just be realized using the third level below.

Semantical reduced paraconsistentization

Given classical propositional logic presented as a collection of classical valuations, how to obtain paraconsistent counterparts of this presentation? Now we have to paraconsistentize classical propositional logic from the semantical point of view. In order to do this, take the set of classical valuations as usual for every formulas ϕ and ψ :

1. $v(\phi \wedge \psi) = 1$ iff $v(\phi) = 1$ and $v(\psi) = 1$
2. $v(\phi \vee \psi) = 1$ iff $v(\phi) = 1$ or $v(\psi) = 1$
3. $v(\phi \rightarrow \psi) = 1$ iff $v(\phi) = 0$ or $v(\psi) = 1$
4. $v(\phi) = 1$ iff $v(\neg \phi) = 0$

These valuations define classical propositional logic from the semantical viewpoint. As we changed the occurrences of \neg by \sim in the syntactical paraconsistentization, we have now to adapt the valuations where \neg occurs replacing it by:

1. $v(\phi) = 0$ implies $v(\sim \phi) = 1$

Or any other semantical clause which will generate a paraconsistent negation (i.e. a non-explosive operator). A very interesting result that should be

proved, and we are not able to do it here, is to prove that syntactical and semantical reduced paraconsistentization of logics together preserve soundness and completeness.

It is not surprising enough that the axioms (9) and (10) are no more valid. Indeed, in this particular case it works if we begin directly by selecting the biggest positive fragment of classical propositional logic and add to it a paraconsistent negation with the respective valuation. But this would not be the case if the logic in question was not classical propositional logic. After, we have to select other axioms to replace (9) and (10). This can be done in a plurality of ways and each way would define a particular paraconsistent logic. We could, for example, follow the same approach of the C-systems of N. DA COSTA by defining the well-behavior of a formula and adding new versions of the axioms (9) and (10).

We labelled this kind of paraconsistentization as *reduced* because it generates a subsystem of the logic which is paraconsistentized by means of restrictions in the abstract, syntactical and semantical level.

Extended paraconsistentization of classical propositional logic

Now, the procedure goes in other direction. Instead of restricting the syntactical and semantical sides of classical propositional logic, the strategy consists in adding a paraconsistent negation in classical propositional logic. We labelled this kind of paraconsistentization as *extended* because it generates an extension of the logic which is paraconsistentized by means of extensions of the abstract, syntactical and semantical level. An extended paraconsistentization obtains a paraconsistent logic from a logic which is not paraconsistent in different levels. The idea consists in an extension of structures, languages, axioms and semantics.

Abstract level

Extended paraconsistentization, instead of reducing the expressive power of the language in order to paraconsistentize it, one goes other way by treating paraconsistent negation as an extra operator! In this sense, this kind of paraconsistentization generates paraconsistent logics which are *extensions* of classical propositional logic in the same sense that modal logics extend it. The *paraconsistentization of CPL at the abstract level* is a structure $P_E(CPL) = \langle F, \sim, \neg, \wedge, \vee, \rightarrow, \Vdash \rangle$ where \sim is a paraconsistent negation which is not explosive. Layering the cake in order to be able to concentrate in details, consider the definition of logical system almost similar to that of

CALEIRO ET AL in [23] but without the requirement of soundness. A logic system is a triple $L = \langle F, \vdash, \models \rangle$ such that:

1. F is a set of propositions;
2. \vdash is a deductive consequence relation;
3. \models is a semantical consequence relation;

Then, the *abstract extended paraconsistentization* of an explosive logic $L_1 = \langle F_1, \vdash_1, \models_1 \rangle$ is a non-explosive logic $L_2 = \langle F_2, \vdash_2, \models_2 \rangle$ such that:

1. $F_1 \subseteq F_2$;
2. $\vdash_1 \subseteq \vdash_2$;
3. $\models_1 \subseteq \models_2$.

The classical negation is not replaced by a paraconsistent negation, but it is added besides the classical negation. In this sense, we get a paraconsistent structure (i.e. a structure containing a paraconsistent negation) using an extension of it. It would be especially interesting proving that abstract extended paraconsistentization of logics preserves soundness and completeness.

Syntactical extended paraconsistentization

After the abstract level, adding a paraconsistent negation to **CPL** with the following deductive rule also induces a paraconsistent counterpart of CPL:

1. $\vdash \phi \Rightarrow \vdash \neg \sim \phi$

The attentive reader will note that a paraconsistentization by extension consists in generating a modal logic from classical logic by the introduction of a paraconsistent negation viewed as a modality. The idea works especially when the definition of \sim corresponds to $\diamond \neg$.

Paraconsistentization of CPL at the semantical level

In order to get an extended paraconsistentization from the semantical side, one should add a paraconsistent negation to **CPL** with the following valuation:

$v(\sim \phi) = 1$ iff there is a valuation v' such that vRv' and $v'(\phi) = 0$

Indeed, it is easy to verify that the principle of explosion is not valid anymore if we consider the above valuation for paraconsistent negation in **CPL**.

We saw two different kinds of paraconsistentization: reduced and extended. Each one can be divided into abstract, syntactical and semantical paraconsistentization.

Paraconsistentizations: preservation properties

A paraconsistentization is *full* when it is done in the abstract, syntactical and semantical levels. And we say that a paraconsistentization is *plausible* when it preserves basic properties of the paraconsistentized system.

In the same way that in combining logics one can study preservation properties, here it is also the case. The reader saw that we did not discuss the question of *transfer theorems* and *preservation properties* of the two proposed methods of paraconsistentization. The idea consists in studying what kind of properties could be preserved by reduced and extended paraconsistentizations. Take classical propositional logic as an example. It is sound and complete. Does a full paraconsistentization of **CPL** generates a paraconsistent logic also sound and complete? Given that a paraconsistentization generates a paraconsistent logic, it follows that one could represent the relations between paraconsistentization and paraconsistent logics using the Square. These problems certainly deserve a detailed study in future work.

4.3.1 Fusions and paraconsistentization

There are many open problems related to paraconsistentization. The process of turning a given logic into a paraconsistent one can be realized in many different senses, as we saw, and, probably, there is not a magic formula, applicable to all logics, able to give a paraconsistent logic as output of a given logic which is not paraconsistent. As in the combination of logics, here also the important fact is to select the right level of abstraction related to logical systems and, after, define particular paraconsistentizations.

The most important problem related to paraconsistentization of logics is completeness preservation, not studied here. The task is to determine whether completeness is preserved under paraconsistentization. For particular paraconsistentizations is not a difficult question to prove that metathe-

oretical properties such completeness are usually preserved. However, at a more general level it is difficult to prove preservation properties. Not all paraconsistentizations are methods for combining logics, but in all paraconsistentizations the question of preservation of properties appears.

Others problems are those of selecting your favorite non-paraconsistent logic and paraconsistentize it or, otherwise, desparaconsistentize a paraconsistent logic. We conjecture that PRIEST's paraconsistent belief revision can be obtained by an unknown paraconsistentization of classical belief revision, and such methodology should be generated. It would be a kind of paraconsistentization of a logical structure. N. DA COSTA and S. FRENCH system of paraconsistent doxastic logic is also obtained by paraconsistentization of the modal logic **KD45**.

What is the relation between combined modalities and paraconsistency? We have examined such relations while showing some ways to paraconsistentize a logic, especially the case of modal logics. As showed by BEZIAU, paraconsistent logics can be generated from modal logics and vice-versa. The relation between modalities and paraconsistency is not so well understood. There are some researchers which defend that both constitute the same class of logics, if no positive criteria are given to negation.

The logic of belief proposed by N. DA COSTA and S. FRENCH is a kind of paraconsistentization in the same way that the PRIEST paraconsistent belief revision is also a paraconsistentization. The logic proposed by N. DA COSTA and S. FRENCH is a good example of paraconsistentization of doxastic logic. Instead of a logic with all properties of classical propositional logic, the authors change the classical basis by a paraconsistent logic. In this sense, they are paraconsistentizing a logic changing the valuations in the worlds. The idea to paraconsistentize modal logics is basically this one: just change the worlds in order to obtain paraconsistent valuations. In this way, there are paraconsistent modal logics. But as we saw, there are also other ways to generate a paraconsistent logic from logics which are not paraconsistent. Such a methodology allows the construction of paraconsistent logics in almost a mechanical fashion. What is important to do is to find a good definition of paraconsistent logic, just in this sense we will be able to create a good method of paraconsistentization. In this chapter, we considered paraconsistent logics as logics which are not explosive, without any positive property. But we noted and explained the problem of positive properties. The conclusion of this chapter is that paraconsistentization of logics can be an interesting tendency in a time where the universal approach to logic is the rule. However, we are just at the beginning of understanding of how such method works.

The motivation of the methods of paraconsistentization of logics is to propose a way to unify the plurality of paraconsistent logics one can find in the literature. Such methodology is important and can be applied every time somebody find inconsistencies and contradictions in some theories. Paraconsistentization appears as a chapter of the big book of universal logic, and as “a general tool to help paraconsistent logicians lost in the jungle of paraconsistent logics.”⁵ Moreover, let me paraconsistentize the logics for the limits of knowledge proposed in the first chapter of this text following the same kind of paraconsistentization we did for **CPL**. And let’s check in what sense the combined logic of skepticism should be paraconsistentized.

4.4 Paraconsistentizing fusions

Now, we should paraconsistentize two of the combined modal logics in previous chapter in order to understand how to deal with contradictions in the limits of knowledge and belief.

Consider the fusion proposed in the first chapter. The fusion of the axiomatic systems **K** and **T*** is composed by all tautologies of classical propositional logic and by the following set of modal axioms:

1. $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$;
2. $(K(\phi \rightarrow \psi) \wedge K\phi) \rightarrow K\psi$;
3. $K\phi \rightarrow \phi$;

And three inference rules:

4. $\vdash \phi$ then $\vdash \Box\phi$;
5. $\vdash \phi$ then $\vdash K\phi$;
6. *MP*

From the semantical viewpoint, the frames that should be used to model such a logic are obtained by fusions of Kripke frames: $F_1 \oplus F_2 = \langle W, R, P \rangle$ where:

1. W is a set of possible worlds;
2. R is an accessibility relation for \Diamond ;

⁵Paraphrasing BEZIAU’s statement on logics in general in the article [6].

3. P is a reflexive accessibility relation for K .

$F_1 = \langle W, R \rangle$ is a frame for the alethic modal logic \mathbf{K} and $F_2 = \langle W, P \rangle$ is a frame for the epistemic modal logic \mathbf{T}^* .

How to paraconsistentize such a logic? There are different ways to proceed. Let examine some possibilities. The first one is to define a translation in order to define a paraconsistent negation inside the fusion. In this sense, there would be one non-explosive operator. The second one is to paraconsistentize the logic by fibring it with a paraconsistent logic. The third example consists in a reduced paraconsistentization operating at three levels: abstract, syntactical and semantical.

From the abstract viewpoint the above fusion is a structure S such that $S = \langle F, \neg, \wedge, \vee, \rightarrow, \Box, K, \Vdash \rangle$. The reduced abstract paraconsistentization of the structure S generates the structure $P(S)$ such that $S = \langle F, \sim, \wedge, \vee, \rightarrow, \Box, K, \Vdash \rangle$. Given that the structure $P(S)$ contains a paraconsistent negation, we have to erase the classical fragment of the fusion and replace it by some paraconsistent logic, say for instance $\mathbf{C1}$. In this way, at the deductive level the logic obtained is a modal logic which is an extension of a paraconsistent logic $\mathbf{C1}$. From the semantical viewpoint, now we have to re-write all classical clauses using $\mathbf{C1}$ -valuations. In this sense, we obtain a paraconsistent version of the fusion. Such logic is able to tolerate contradictory formulas without leading to explosion, and can even be used to avoid some paradoxes. The same strategy is, in fact, valid for all fusions proposed in previous chapters. Solutions to classical paradoxes using non-classical logics, especially paraconsistent logics, are not rare in the literature. Many authors used paraconsistent logic to avoid explosion and, consequently, trivialization. For instance, in studying the collapse between knowledge and belief, it seems obvious that a paraconsistentization of the fusion involved to derive such a paradox would also eliminate the problem. In this sense, a solution to the collapse knowledge and belief would involve a paraconsistent version of the fusion.

One point concerning logical skepticism: if it entails contradictions, then it suffices to paraconsistentize it in order to get a paraconsistent logical skepticism. Consider one version of logical skepticism which contains the axiom

$$\nabla\phi \rightarrow B\phi$$

An obvious argument: Assuming that a given proposition is contingent, then its negation is also contingent. Therefore, one could show that if a

proposition ϕ is contingent, the one believes in ϕ and one believes in $\neg\phi$. This is a contradictory belief:

$$B\phi \wedge B\neg\phi$$

Here we could just paraconsistentize the combined logic of skepticism, or we could just change its classical fragment by a paraconsistent one, for instance those proposed by N. da COSTA and S. FRENCH. In this sense, the following statement would not be a theorem:

$$(B\phi \wedge B\neg\phi) \rightarrow \psi$$

Otherwise, we could just state that the version of logical skepticism which entails a contradictory belief is not interesting enough, and we would prefer a different one.

Chapter 5

Network: a universal approach to concepts

“Un concept ne va jamais seul. Sans parler du réseau infiniment complexe qui le relie, de proche en proche, l’ensemble des autres concepts et qui fait de cet ensemble, comme de celui des mots qui l’expriment, un système global dont aucun élément ne recoit sa détermination exacte que de son rapport à la totalité, chaque concept se trouve rattaché, par des liens beaucoup plus serrés, à un groupe restreint d’autres concepts, qui forment avec lui une famille. Comme les familles humaines, ces familles de concepts s’ordonnent selon certaines structures de parenté, qu’un schéma intuitif, de nature géométrique ou topologique, vient le plus souvent soutenir: échelles, spectres, figures bipolaires, accolades, arbres, étoiles, rosaces, tableaux à double ou triple entré, etc.” (R. BLANCHÉ in [18])

The titles of the previous chapters showed how each explored interactions, combinations and interconnections involving two families of concepts called modalities work. In those chapters we studied the technique of combining different modalities and explored the philosophical consequences of such combinations. Now we consider the relationship of modalities from a more abstract viewpoint. Instead of studying particular combinations of modalities, we intend to analyze the relationship of modalities from a universal perspective. Previous chapter titles included names indicating a kind of interaction between concepts. Now that we are exploring abstract properties of such concepts, the title is introduced by the abstract notion of *network*. This is intended to express the movement towards abstraction. The above citation from BLANCHE touches on the basic topic of this research: a study of the complex network which is constituted by combined modalities. As

well as being complex, it is also incomplete. In this chapter only a small portion of that network of relationships will be considered. BLANCHE's idea consists in studying such relations using geometrical figures such as the square and the hexagon. This kind of theory is also explored in this chapter especially one generalization of the notion of hexagon proposed by BEZIAU in [12] and MORETTI in [80].

It is not easy to develop an abstract approach to modalities without considering particular modalities. Up to this point, the focus has been on specific concrete logics and instances of concrete combined modalities. Now, an abstract approach to modalities is proposed: one which aims to be general and universal. Such an approach must make use of abstract mathematical tools and concepts. We begin by examining the possible role of category theory as an abstract formal tool and setting for this project. Recently, category theory has a lot of applications in mathematics and logic: the very notion of a category, composed of a collection of objects and morphisms between objects calls to mind possible worlds semantics where possible worlds are objects and morphisms are accessibility relations. Moreover, one can consider categories where modal formulas are objects and proofs between formulas are morphisms.

The first general and formal approach explored in this chapter consists exactly in taking advantage of some basic and simple categorial constructions to propose an abstract approach to metaphysical and epistemic modalities, and its combinations. Although we are not able to give a concrete example of application, we show a very strong argument to accept that categorial constructions can be used for modelling philosophical concepts.

The second approach which is considered here is that of n -dimensional modal logics. What appearance do metaphysical and epistemic modalities assume from the bidimensional or n -dimensional point of view? In the first chapter we studied an interaction, indeed, a mixture, between possibility and knowledge in just one complex modality called knowability constructor (or operator). In what sense might n -dimensional modal logics give a good account of the modalities explored in this work? Unfortunately, limitation of hypothesis precludes the full investigation of this problem here, but at least we aim to provide some clues to the philosophical significance of n -dimensional constructions in the study of metaphysical and epistemic modalities.

The third approach, which emerges as a consequence of the two above, is based on the n -opposition theory developed by BEZIAU in [12] and MORETTI in [80]. The question of what kind of geometrical configuration corresponds to which given specific modal logic and n -dimensional modali-

ties is taken up . The interesting point of such a theory is that it allows a schematic account of modal relationships.

The fourth general approach is that of KOSLOW in [61], based on the notion of an implicational structure. Such an approach defines a modal operator in a very general, abstract and mathematical way, and can be viewed as the most general account of modalities to be found in the literature. KOSLOW did not develop another new logical system, but rather a new approach to logic. His work is an undoubted contribution to the field of universal logic.

So, to summarize : the general, mathematical and abstract approaches to modalities that we study here are the following:

1. Category theory;
2. The n -dimensional modal logic;
3. The n -opposition theory;
4. Structuralist theory of modalities;

It is beyond the scope of this chapter to present detailed research results in each one of these topics. What we intend to do here is to show how such alternative frameworks for the study of the universal aspects of modal notions may offer a better general understanding of the essence and meaning of combined modalities. Some aspects of general perspectives are still missing. The study of the relations between the above approaches is presented here in the context of universal logic. In the four research areas mentioned above, if one examines modal operators and modalities from the viewpoint of each of these approaches, one is not studying particular logics, but rather a general theory of modalities (which belongs to some given logical systems). Therefore, such a project naturally constitutes a part of the subject matter of universal logic: the general theory of logics.

There are different kinds and different families of modal concepts. Given such a plurality, how are we to unify the concept of modality? How to organize the study of modalities which belong to distinct families? Clearly, a more abstract approach to particular kinds of modalities would not do the job. We are not now interested in particular modalities, but in an overall theory of modalities, including the way modalities act on modalities. How are we to study such concepts from an abstract viewpoint? Is there a general theory of modalities? What tools can be used for such a study? The next sections contain some attempts to answer these questions.

5.1 Philosophical categorification

Logic is used by philosophers in order to explain philosophical analysis.¹ Category theory is a branch of mathematics which has attained great technical sophistication, provided deep insights into the unity of basic mathematical constructions, and which finds increasing applications also in the study of philosophical theories. It raises hard questions as to the meaning of its fundamental notions and basic constructions in relation to issues of mathematical ontology, and perhaps for the way in which metaphysical issues in general should be approached. The object of this section is to justify the claim that category theory is relevant for such philosophical tasks, and especially for our philosophical account of modalities.

5.1.1 A categorial approach

There is no unique way to identify what counts as part of analytical philosophy. However we can specify some general features of this kind of philosophical activity. These include:

1. The rigorous definition of concepts;
2. The use of logic and formal methods;
3. The attempts to solve precisely-posed problems through carefully constructed arguments.

There are, of course, other kinds of philosophical activity². It is conceded that one can also use metaphors or different tools such as literature, even including poetry, to undertake some philosophical tasks.

This section proposes an approach drawing on particular formal tools and techniques which have been proposed to aid philosophers. Just to give a brief example : look at the next table where on one side there are mathematical constructions and on the other the philosophical concepts which they can serve to formalize.

This is just a general sketch. These concepts are related and for this reason it is possible to use a plurality of abstract mathematical structures to formalize, at the same time, different philosophical concepts. In previous chapters we presented some applications of logic to model skepticism, problems in the limits of knowledge and contradictory epistemic and doxastic

¹This section, with some changes, has been accepted for publication in Noésis (Nice).

²[38] is a book devoted to the creation of concepts where the authors use others tools to their philosophical purposes.

Table 5.1: Mathematical constructions and philosophical concepts

mathematical tools	philosophical concepts
modal logics	necessity, possibility and contingency
epistemic logics	knowledge, belief
temporal logics	past, present, future
paraconsistent logics	contradiction, consistency
logics in general	predicates, relations, objects, properties
combined logics	interactions of concepts
first-order logic	truth, falsity, structures, language

states. Now we set out another concrete application of logic in philosophy, one of the most original and interesting revolutions which in recent years challenged the way we traditionally regard the relation between necessary truths and *a priori* knowledge.

5.1.2 Logic applied to philosophy

The general assumption of this work is that logic can be applied to philosophy. Many examples can be found in the literature: LEWIS' counterpart theory, KRIPKE's account of necessary *a posteriori* truths, the definition of knowledge and a great variety of direct applications such as in the case of logical constructions with clear ontological meaning: modal logics for metaphysical modalities, deontic logics for ethical concepts, epistemic logics for epistemological concepts such as knowledge and belief, paraconsistent logics for inconsistent non-trivial theories, combined logics for complex systems involving interactions and so on. Let us return to the beginning of our account and review what a proposition is and the logical classification of different kinds of propositions in order to introduce KRIPKE's argument on the existence of necessary *a posteriori* truths proposed in [65].

Propositions: classification

Philosophers usually accept that propositions are linguistic entities which can be true or false³. In fact, in relation to truth-values, statements can be true or false for operative purposes. In addition there are logics able

³It is assumed that the words *propositions*, *statements* and *sentences* all have the same reference. In analytical philosophy, many distinctions were realized between each one. But for theoretical reasons let me assume that they are the same.

to deal with three truth-values and indeed continuously many values between truth and falsity. Classical valuations are functions between a set of propositions and the two-element set of truth-values. But it is effectively possible to extend all these valuations to the closed interval $[0;1]$. So, there are propositions which are neither true nor false, but have an intermediate truth-value. However such a multi-valued semantical environment can be regarded as bivalent, given that it is always possible to reconstruct bivalence via the notion of designated and non-designated values (using SUSZKO's argument). Let us suppose *ad hoc* that our propositions are either true or false. A proposition is true if and only if it is the mirror of the world. According to the theory of truth proposed by TARSKI in [105], a proposition " ϕ " is true if and only if ϕ . This conception of truth played an important role in this book, and it was used in connection to logical atomism.

Propositions can also be classified in relation to their metaphysical status. This means that statements can be judged according to the status of the proposition not just in the actual world, but in relation to a multiplicity of worlds. In this sense, propositions can be possible, impossible, contingent or necessary ⁴. These variations have become the playground of those who study modal logics. All these notions can be formalized in modal languages. Using the Kripke model, in basic modal logics, it is possible to formalize the metaphysical status of a proposition. In this sense, a proposition is necessary if and only if it is true in all possible worlds related via an accessibility relation to the world where the proposition is assessed as true. And a proposition is possible if and only if it is true in at least one possible world thus related to the original world.

Concerning their epistemological status, propositions can be conceived as *a priori* or *a posteriori*. Philosophers argue that a proposition is *a priori* if it is possible to discover its truth-value without appeal to empirical data. Otherwise, the proposition is *a posteriori* ⁵. A classical philosopher would argue that the statement "An object a is identical to a " is *a priori* because it is possible to discover its truth independent of the world, given that it is just a logical consequence of the logical principle that "Every object is identical to itself" - Kripke even states that it is bizarre that any philosopher could doubt the truth of this principle (see [65]). The classification of propositions can be better understood checking the next table.

With this picture in mind, it becomes natural to suppose that it is pos-

⁴In last chapters we studied interactions of such modalities with epistemic modalities.

⁵The reader can wonder that such a division is not in relation to known propositions, propositions object of belief etc. We decided to use *epistemological* here to designate how an agent can have access to the content of a given proposition.

Table 5.2: Classification of propositions

truth-value	metaphysical	epistemological
true	necessity	<i>a priori</i>
intermediate values	intermediate modalities	is there sth here?
false	possibility?	<i>a posteriori</i>

sible to combine different kinds of classifications, for instance: a true, necessary and *a priori* proposition. It is well known that traditional philosophers as LEIBNIZ and KANT, and more recently, logical positivists, suggested that if a proposition is true and necessary, then it is also *a priori*. The thesis according to which

A proposition is necessary if and only if it is *a priori*.

is hereafter in this text called the Pure Thesis (PT). (PT) has dominated much of the philosophical tradition and it has widely been considered crazy to doubt its truth.

Kripke's necessary *a posteriori* truths

As we know, the intuitive idea of a modal language is already found in the works of ARISTOTLE in [2] and LEIBNIZ in [68]. However, the birth of formal modal logic took place when logicians attempted to provide mathematical models for modal notions such as necessity. At a particular point in this history, S. KRIPKE proposed the framework of possible world semantics⁶ and proved completeness of some modal logics using this method in [66]. Alongside his work in pure modal logic, KRIPKE has also contributed greatly to philosophy. He has presented an argument against the validity of (PT). He proposed a proposition which is true, necessary, but *a posteriori*. This generates a new bridge between metaphysics and epistemology. In order to explain KRIPKE's strategy it is helpful to introduce the notions of designator, rigid designator and proper name. These are usually defined as follows (these definitions can be found in [65] and [65.1]):

1. A designator is a name which refers to something in the world;
2. A proper name is a designator which refers uniquely to one object in each occurrence;

⁶The reader interested in possible worlds semantics and a complete guide to its history should check the work [31].

3. A designator is rigid if and only if it refers to the same object in all possible worlds.

Given these definitions, it follows that proper names are rigid designators. Definite descriptions, in contrast, are not rigid designators, because their reference can change from world to world. KRIPKE's argument (see [65] and [65.1]) makes direct use of the Leibnizian rule of the indiscernibility of identicals (LLII)

$$\forall x, y((x = y) \rightarrow (F(x) \rightarrow F(y)))$$

It is also important to consider a special modal version of the principle of identity (SVI):

$$\forall x \Box(x = x)$$

Given all these premisses, KRIPKE's reasoning is as follows: Suppose there is an identity statement between proper names of the form $a = b$. Then it can be true or false. But the truth-value of that statement cannot be known as *a priori* given that there is no logical way to infer that, indeed, a and b refer to the same object. Therefore, we have to inspect the world and discover empirically if it is true. So, if $a = b$ is true, then it is *a posteriori*. Then, if it is accepted that $a = b$ is true, then, as an instantiation of (LLII), define F as "something is necessarily identical to a ". So, if a has the property F , then a is necessarily identical to a . The same holds for b . Using the definition of F and an instantiation of (LLII):

$$(a = b) \rightarrow (\Box(a = a) \rightarrow \Box(a = b))$$

But our supposition states that $a = b$. Then, it follows by *modus ponens* that

$$\Box(a = a) \rightarrow \Box(a = b)$$

However, given (SVI), it is possible to conclude, for a particular object a , that

$$\Box(a = a)$$

From this fact, it is possible to deduce that

$$\Box(a = b)$$

Then, identity statements between different proper names, if they are true, they are necessary and, at the same time, we can know the truth of the statement via empirical investigation. Thus there are instances of what KRIPKE called necessary *a posteriori* truths. As we told, this is a different way to propose a connexion between epistemology and metaphysics, and it shows the value of logic in philosophy. An interesting topic is to analyze KRIPKE's argument from the viewpoint of interactions between contingency logics and epistemic logics.

Some other examples

Many other examples of such applications of logic could be mentioned, for instance (some explored in this book):

1. Approaches combining epistemic and doxastic logics with contingency logics, such as those developed in chapters 2 and 3;
2. Combinations of metaphysical and epistemic logics in order to model the limits of knowledge and formalizations of the verification principle;
3. Attempts at modelling contradictory scenarios involving knowledge and belief.

There are also criticisms which have been brought against such uses of logic and mathematical tools in philosophy. For instance, G-C. ROTA in [95] and BEZIAU and SARENAC's paper attacking some uses of possible worlds semantics as fashionable nonsense (see [16]). However, it appears that the use of logic in philosophy in general is well established.

In the next section, the project called philosophical categorification is studied in order to sketch a categorial approach to questions related to the interactions of modalities ⁷.

5.1.3 Category theory and philosophy

There are essentially three celebrated conceptions of the notion of category. One is that found in ARISTOTLE or KANT. This notion is concerned with the classification of the most fundamental features of reality or of thought.

⁷Philosophical categorification is the philosophical counterpart of categorification introduced by [5]. The purpose of relating category theory and philosophy is not entirely new, others philosophers already thought about the subject, and such relations are becoming central theme of research. One recent book which is connected with such idea is that of BADIOU in [4].

There is also the mathematical notion which was defined in the sense of contemporary category theorists.

It is in this mathematical sense that the term is used here. It is suggested that category theory can be used to formalize philosophical theories and concepts in much the same way that logic has been used. It is defended the claim that categories in the mathematical sense have an important connection on ontological inquiry, in that they have a role to play in elucidating the general structure of the world, though a subtly different role from their ancient metaphysical namesake.

5.1.4 Categories and logics

This subsection proposes one argument to show that category theory is relevant in philosophy. One attempted application is also developed.

Logical operators

The first link between categories and the study of logic appears already in the topic of logical operators. The special property of these operators is that they formalize philosophical concepts. Logical operators can be represented in categories using the notion of arrow (or morphism). Conjunction, disjunction, negation and implication have such particular categorial representations. There are also attempts to represent in categorial terms modalities such as necessity. Concepts traditionally conceived as philosophical still operate in categorial structures i.e. in the setting of (mathematical) category theory - e.g. truth, falsity, contradiction, existence, object etc. Recent work in categorial logic has shown how to use categories to understand logical notions. For example, the representation of logical operators inside categories is achieved via particular categories, in which propositions are objects and proofs are arrows (See LAMBEK in [67] and GOLDBLATT in [48]). In such a category, for example, logical operators are identified with various kinds of universal constructions. Conjunction can be viewed as product, while its dual, disjunction, can be viewed as co-product. Bottom and top elements are initial and terminal objects. If one can thus represent the usual propositional or predicative logical operators inside categories, why not also represent modalities and its interactions via arrows? But how to proceed in detail? What kind of category is the correct setting for such representations?

Combination of logics

The second link between logic and categories appears in the methods for combining logical systems. These methods are used to construct powerful logics able to formalize more complex propositions involving multiple philosophical notions. As mentioned in this work, there are many ways in which we can combine logics: fusion, products, synchronization, fibring etc. These methods can be represented by means of universal constructions in categories, given that logics are also understood as objects of categories (for instance, [28] and [24]). So, it is useful to use categories to elucidate how to combine logics and to help in the task of finding powerful logics to formalize philosophical concepts. Those fusions of Kripke models studied in some chapters of this document, and also their products, can be described using arrows. What kind of universal construction corresponds to such fusions and products?

Philosophy of Logic

A very celebrated, known and important category called a topos has as its internal logic a system which is not classical, but intuitionistic. This means that the internal logic of a topos is in general isomorphic to a Heyting algebra rather than a Boolean algebra and hence satisfies intuitionistic logic. In general, the topos-theoretic universe is not a classical one. Indeed, it is possible to obtain different logics inside different categories. The philosophical problems posed by these researches in category theory are of very great importance to the general philosophy of logic. Given that logic has been such an important tool in studying concepts from such areas as epistemology and metaphysics, it is natural to conjecture that *category theory is important in philosophy*.

Basic connections

The main argument for such a conjecture is the following:

1. Logic has been an excellent tool for philosophers trying to understand philosophical concepts and to study the structure and functioning of natural language;
2. Logic can itself be viewed and studied inside category theory. Some connections between logics and category theory were adduced above, such as the categorial representation and study of logical operators and methods for combining logics;

THEREFORE:

3. Category theory should be a good tool for philosophers formalizing natural languages and trying to understand philosophical concepts.

Concrete examples of direct uses of category theory in philosophy are as yet rather sparse. In the next subsection, it is proposed a project which would operate rather in the reverse direction : an investigation of the ontological content and consequences of (mathematical) category theory.

5.1.5 Categories and philosophy

The study of the relations between philosophy and category theory are not still largely explored. In this section, we touch on some of the more evident connections.

Categories and ontology

The variety of ontological questions related to categories is so great that would be possible, as a very far-ranging project for the longer term, to undertake the re-construction of traditional systems of ontology from the categorial point of view. It is also possible to construct a criterion of existence using the concept of a category. For instance, suppose that there is an abstract category C composed by a set of objects and a set of arrows satisfying the relevant properties to be a category. The following criterion of existence could be proposed:

Something exists if and only if it is a member of C .

A particular criterion of existence for mathematical objects would then be:

A mathematical object exists if and only if it is defined in a category.

It is important to note that the mathematical conception of category cannot be applied to unspecified collections of objects. Does this signal some inadequacy in the notion from the standpoint of the aims of general ontology? Or does it rather reveal something about the aims of traditional ontology? Maybe we should enlarge our notion of category in order to formalize more concepts. Using an enlarged conception of category (or perhaps even graph theory), might it be possible to replace QUINE's ontological criterion [91] with a new one:

being is being defined in a category.

Categoricism (or philosophical categorification) is the label chosen here for the philosophical project according to which

1. It is possible to re-construct, formalize and clarify philosophical concepts and theories as well it is possible to solve philosophical problems using categorical tools;
2. Categories have an ontological dimension in the sense that categories reflect general and abstract properties of reality in general, not just mathematical reality;
3. Mathematical truths are to be regarded as resting on truths about categories (this claim can be seen as a categorical version of logicism)

Categoricism (Philosophical categorification) is one further chapter of the attempt to use mathematical tools fruitfully in philosophy. Some philosophical uses of category theory already proposed include: the theory of reference of proper names (see [82] and [72]), some philosophical aspects of categories (see [73]) and see [84]⁸, and a role for categories in the analysis of the concept of identity (see [94]).

Categories once again

We conjecture that it is both possible to apply categories in philosophy via their already well-developed role in representing logical notions and constructions. And that there is also a direct application which involves, amongst other topics, an inquiry into the overall ontological content of category theory. There are many projects which naturally present themselves as aspects of the program of categoricism. A maximally ambitious project would be attempt to re-formulate all possible philosophical concepts (for example: combined notions, being, necessity, knowledge, existence etc) using categorical notions (and why not abstract notions of graph theory?). In this

⁸These two last authors also argue, independent of my approach, and using different arguments, that categories can be applied to philosophy and that itself has a philosophical content. This shows that the idea is not so new. The general question relating category theory and philosophy could be announced in a more abstract way, for any kind of subject: Given a subject A , it can be used to understand a philosophical notion and has itself a philosophical dimension. This means that more important than staying in this abstract level is to be able to provide clear examples of application.

sense, one might construct a categorial theory of properties, a categorial theory of deontic concepts (for instance), and proposing a categorial approach to the whole of classical philosophical notions. If analytical philosophers apply logic to philosophy, the next natural step should be to apply categories. Philosophers who formed logical atomists, notably B. RUSSELL and WITTGENSTEIN held that there is a direct logical correspondence between the structure of propositions and the structure of the world ⁹. One might explore a claim in the same spirit that there is such a correspondence between the structure of categories and the world.

5.1.6 A system of philosophy based on category theory?

Is it possible to develop a system of concepts using a weak category theory (which has a different definition of category) as the basis to model a partial ontology and epistemology? Such idea tries to use the abstractions provided by category theory in order to model the world in its relation with epistemic agents. But it should not intend to be true. It would be interesting to construct a work in the style of the *Tractatus* in which category theory replaced the role played there by logic. A system should be presented just to give a possible example of how to apply category theory in philosophy. The reader has been shown ways in which category theory may play an important role in philosophy, and has also been given an account of the role which categorial logic plays in the program of philosophical categorification. The basic argument is this: given that logic allows a clear understanding of natural languages, and given that category theory can be used to express logical concepts, it follows that category theory allows a clear understanding of natural languages and consequently of philosophical concepts and theories. What is still missing is a concrete example, of a kind akin to KRIPKE's necessary *a posteriori* truths, showing how to apply categorial tools in philosophy. Or a concrete system of concepts generated by categorial tools.

5.2 The universal approach to modalities

Category theory provides a powerful, and mathematically interesting framework for the characterization of modalities, building on the way the “arrows” notation allows the representation of the usual (non-)truth-functional operators. This section studies an alternative approach. The Program of universal logic proposed by BEZIAU is intended to provide both overall orientation

⁹The reader can find a brief explanation of such a conception in chapters 2 and 3.

and effective new tools for the logician “who is lost in the jungle of possible logics” (BEZIAU in [6]). These tools contribute to a more general approach to the study of particular modalities as a novel and very interesting way of studying the interactions of modalities from a general perspective. This is in line with the aim of universal logic to study general properties of logics independent of the formalization of particular logical systems. Besides combining logics, category theory, some other instruments could be considered tools of universal logic.

5.2.1 Structuralist theory of modalities

KOSLOW in his book [61] proposed a very general approach to the study of modalities. The structuralist approach to logical systems which he developed is based on the notion of implicational structure. This provides a framework for studying the most general characteristics of modalities and logical operators, in the spirit of the search for what is universal in logic. KOSLOW’s treatment of modalities illustrates a very general, elegant and subtle approach to logical form based on the notion of implicational structure, KOSLOW argues that a modal operator is always modal with respect to the implication relation. A modal operator should distribute over the implication relation but should not distribute over the converse of the implication relation. From this starting point, he proves a number of properties of modal operators. In this sense modal operators are functions on implication structures. This gives a powerful abstract way of unifying the notion of modality. But what does it bring to the study of the notion of combined modality? In particular, does KOSLOW’s approach permit the combination of different implicational structures? How are we to define the combination of implicational structures? Unfortunately, the scope of this work does not permit a detailed treatment of such problems.

5.2.2 The n -dimensional modal logics

Another abstract approach to modalities is via the study of n -dimensional modal logics generated by the product of modal logics, as mentioned earlier. Introducing n -dimensional modal logics allows us to create modalities of different dimensions. We saw many different modalities defined in the monodimensional environment. Indeed almost all the best known and oldest established systems of modal logic, such as **T**, **S4**, **S5** are monodimensional modal logics because the consequence relation is defined between worlds and formulas. All modalities in such logics are simple modalities

defined in a monodimensional reality. What is interesting is that one can give a two-dimensional and, in general, n -dimensional account of modalities if one define the consequence relation between pairs or n -tuples of worlds and formulas. The knowability modality studied in chapter one is a typical example of a two-dimensional modality. But as the reader saw, we can have even \diamond 's and \square 's defined in a two dimensional environment. In this sense, we have horizontal and vertical \diamond 's and \square 's.

A celebrated property of two-dimensional modal logic is that it can be represented in the plane. It is not surprising that such logics and modalities can be generalized up to n -modalities. One can model horizontal possibility using a two dimensional frame.

$$\diamond\phi \longrightarrow \phi$$

And also vertical possibility:

$$\begin{array}{c} \phi \\ \uparrow \\ \diamond\phi \end{array}$$

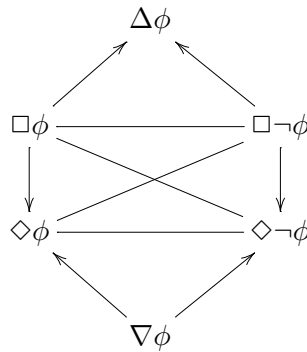
Clearly, both these models suppose that we have an underlying cartesian plane, and each formula holds at a given point (x, y) in the plane. The same applies for \square 's and other modalities. The n -dimensional modal logic is a very interesting tool for modelling combined concepts which involve the interaction of modalities.

5.2.3 The n -opposition theory

The use of geometrical figures to capture basic structures involved in our thought goes back at least to ARISTOTLE with the square of opposition. R. BLANCHÉ in [18] is one of the first who recently again took up the systematic study of such figures in relation to modal concepts. As already remarked, he generalized the square of opposition to a hexagon. Many philosophers have since studied further examples of such configurations and discovered a wide variety of different geometrical means whereby we can represent modalities. Such attempts to represent modalities as BLANCHÉ's hexagon and J.-Y. BEZIAU and A. MORETTI's n -opposition theory - both

of which use a generalization of the Aristotelian square of opposition - also help us to understand the differences between the concepts represented and to avoid confusion between the notions of possibility and contingency. The idea is to find a geometrical representation of the notion of deduction between formulas. If there is a proof in a given logic of the formula ψ from the formula ϕ then there is a vertex of a geometrical figure which corresponds to that deduction. Sometimes such proofs rest on relations between concepts (contraries, sub-alternation, subcontraries etc). Discovering different proofs between formulas allows us to fill in the vertices of a given figure. Sometimes such concepts are contradictories, sometimes not.

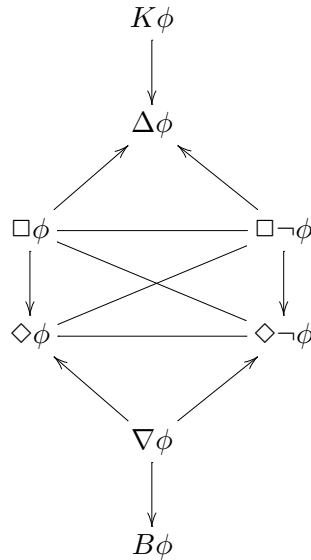
The most elaborate geometrical structures corresponding to the logical relationship of modal notions recently investigated are those which are the subject of BEZIAU's and MORETTI's n -opposition theory (viz the logical tetradehedron) (see BEZIAU in [12] and MORETTI in [80]) . In order to achieve the logical tetradehedron, there are three different hexagons, the first proposed by BLANCHÉ and called the classical hexagon of modalities, and two others proposed by BEZIAU. The hexagon below, which is a generalization of ARISTOTLE's Square of Opposition, is better expressed in the following diagram due to BLANCHÉ:



It is exactly when we put all the hexagons together that we can find the 3-D geometric object. Trying to update the Square of Oppositions, but now for epistemic modalities which interact with metaphysical modalities, one could suppose that there is an epistemic square interacting knowledge and contingency. In the same way that one could draw the epistemic square, one could generate the doxastic square, where the belief modality is interacting with contingency. One could also propose a very big diagram composed

of both the epistemic square and the doxastic square and its interactions. All these diagrams should show clearly the interactions that one can find in studying knowledge, belief and contingency.

It is very difficult to find a place for some other modalities such as ∇ , $\neg\Diamond$ and $\Diamond\neg$ and its interactions with epistemic concepts in a geometrical configuration. Considering just simple modal logics, some interesting variations of such diagrams can be obtained (see BEZIAU in [12]). The difficulty is to find a good diagram in the case where non-interdefinable modalities are investigated. In chapter 2 and 3 we studied interactions between knowledge and belief and (non)-contingency modalities. How can we express such interactions using BLANCHE's hexagon? Given that we do not know precisely all relations between modalities, maybe there is a better place to put K and B in the diagram. In this sense, the following structure is proposed to represent the relations between the logics studied in chapters 2, 3 and 4:



The arrows between epistemic/doxastic modalities and the usual modalities of necessity, possibility, impossibility and non-possibility are missing. How to find a place for them? The above figure is called here *The Drop of Descartes*, as should represent the general combined logic of skepticism proposed by the end of chapter 3.

Probably, a new configuration is required. However, up to now that above is all we have. An interesting question at this point is: Where is the

place of the knowledge modality in the hexagon? The classical hexagon of modalities, as well as MORETTI's and BEZIAU's hexagons, can be reformulated by considering its interactions with the knowledge operator. Indeed, one problem in the approaches cited hitherto is that they do not consider interactions and therefore they use in the geometrical figures just those modalities that are interdefinable. The logic which underlies the above decoration is a contingency version of the logic **T** combined with its epistemic version. And the idea is to find diagrams for those combined logics explored in chapters 2 and 3. The purpose of n -opposition theory is to find geometrical structures associated with some given proof-theoretic systems. The aim should be to provide an example of modal diagrams using some recent developments in n -opposition theory.

A more general question can be posed: What is the place of knowledge in the logical tetracahedron? This inquiry lies outside the scope of this book. The task of finding diagrams for two-dimensional and n -dimensional modalities is still open and is probably the most important problem in n -opposition theory, because a real theory of n -opposition can only be built up if it incorporates the study of n -dimensional modalities.

One of the most important facts related to such geometrical representations is that human thought seems to be structured in certain triads: 1) Never, always, sometimes; 2) Everywhere, nowhere, somewhere 3) Possible, impossible, Necessary; 4) All, some, none; Are there other terms between such concepts? Is there for example something between "never" and "sometimes"? If it is reasonable to represent such concepts schematically, why not explore a more general approach unifying different concepts? The task of finding possible diagrams for the n -opposition theory within the scope of multidimensional and n -dimensional modal logics is an aspect of the program of providing a universal and diagrammatic account of modalities. Some such diagrams were explained and elucidated in the previous chapters. Here the general theory of n -opposition is presented as a way of unifying all squares (and hexagons) in just one big overall figure: it is a tool for universal logic. The relative lack of development that the theory of opposition has experienced since it was founded by ARISTOTLE is underlined by the fact that the first Scientific Congress on the subject of the Square of Opposition took place in the course of the year 2007 in the city of Montreux, Switzerland (CH). Only 25 centuries after its initial intuitions is the study of the square becoming a really fruitful topic of research !

5.3 The duality of reality

One interesting characteristic of reality and of the universe, in general, is its duality. We say that reality or the universe is dual when it can be partitioned into two different aspects which are complementary. The dual aspect of reality manifests itself in our everyday life. It is commonplace to apply dichotomies in our expressions and actions. Dichotomies are present everywhere: good/bad, beautiful/ugly, love/hate and many others. What is really interesting is that such dualities appear at the most basic levels in our description of the most general structure of reality. Even in mathematics and logic, the most pure branches of science, one can find a manifestation of dichotomies. It is especially interesting to note in connection with the topic of this book what place the modalities of knowledge, belief, contingency and the interactions of these occupy in relation to such duality principles. One very important illustration is provided by the basic logical concept of negation, which appears in set theory as the complement operation and in category theory as the duality principle. Such relationships are captured by STONE's representation theorem. One could argue that duality is not a primary property of the universe because sometimes we better represent notions using triadic relationships, as pointed out below. Such triadic relationships are ubiquitous, are everywhere, but in fact it can be shown that these, too, depend on duality principles. The basic argument for the centrality of duality to our understanding is a mathematical one, based on the observation that in mathematics duality is the rule rather than the exception.

The first duality which is of interest appears in logic in the most general form in the so-called *bottom particles* and *top particles*. A bottom particle plays the role of a formula from which in a given system it is possible to prove every formula. It is usual to represent bottom particles in the following way: $\perp \vdash A$, for all A 's. The dual concept to that of bottom particle is that of top particle. This kind of particle can be represented using the rule: $A \vdash \top$, for all A 's. These concepts are duals because if we negate a bottom particle we obtain a top particle and vice-versa. There is another duality in logic, manifested by negations of formulas containing conjunctions or disjunctions. One says that conjunctions and disjunctions are duals, but this kind of duality is not the focus of our interest here.

A second duality is that which appears in set theory. Sets are defined by the axioms stipulated in the classical Zermelo-Fraenkel axiomatic set theory (ZF). In set theory, the set called the *universe* is the largest (and in one sense the most basic) set one can find. Note that the universe is not the same

thing as the set of all sets, which is an impossible notion from the classical point of view. The universe, in set theory, is the field of all possible sets and it is generated by a recursive argument. Usually it is denoted by U . Given an element x and a set A , U can be defined as follows : there exists a U such that for all A , $A \subseteq U$. In set theory, the dual of the universal set is the empty set, which is a set such that no element belongs to it. The universe and the empty set are well known duals, in the sense that the complement of the empty set is the universe and the complement of the universe is the empty set.

The third kind of duality is that which appears in category theory in the notions of terminal and initial objects. An object of a given category is terminal if every arrow of the category arrives in that object, and initial if all arrows depart from that object. Using the duality principle of category theory one can easily represent the flowing of initial objects into terminal objects and vice-versa.

What is particularly noteworthy in studying mathematical and logical duality is that given that both reflects the most general structure of reality, it follows that both show how reality is divided. The notions of bottom particles, empty set and initial objects are all on the same side of the dichotomy, because bottom particles in logic correspond to the empty set in set theory and to initial objects in category theory. Top particles, the universal set and terminal objects are all on the other side by the same argument. Another remarkable dichotomy is that between true and false. Needless to say to which side each belongs. Let us call the first side the positive facet of reality and the other side the negative. What about knowledge and belief, necessity and possibility (and finally contingency)? One of the tasks of a general abstract theory of modalities should be to locate their structure with respect to such dualities/dichotomies as those mentioned above. The argument used here to find a place for necessity and knowledge, from one side, and possibility and belief from the other side, is in certain sense obvious. The fact that knowledge/necessity implies truth in all possible worlds, leads us to think that the place of both these are on the positive side of reality, while belief and possibility are on the negative side, because they do not entail global truth. But in this case it is really difficult to interchange the sides. One could argue against a role for notions of duality here, and defend the view that we simply have a variety of different structures. One could appeal to fuzzy logics and many-valued logics to justify the view that the structure of the universe in relation to semantic notions is not well represented by the duality of : 0 and 1. This view is not wrong, but it is itself a parasitary consequence of the above arguments. Some formal approaches

aimed at restoring the duality are found in the work of SUSZKO.

Where in this general picture is the place of contingency and the other intermediate notions? This question suggests a very general metaphysical principle: Reality has a fundamentally dual nature, and the two opposed aspects are complementary, but there is also typically something further intervening between them. This is the crucial point: what intervenes between both sides is partially captured in such notions as existence, the flow of becoming, contingency, passage, the processual aspect of Being. It is exactly here that one can locate what happens between both sides: the flowing.

How can we best represent such a bipartition in the frame of the universe? We provide here an artificial schema intended to give a general view of this idea. The structure of reality could be something composed by an essential duality: from one side we would have concepts such as 0, BOTTOM, EMPTY, FALSE, INITIAL, BELIEF and NOTHINGNESS. From the other side we would have their duals, respectively: 1, TOP, UNIVERSE, TRUTH, TERMINAL, KNOWLEDGE, BEING. The above mathematical and logical dualities suggest that dichotomies are everywhere and that such a dual structure is, therefore, the main characteristic of the universe.

Conclusion

Although the empirical sciences are very important for human beings and have to be explored and developed, what we have argued for is that science is *a priori* impossible just because its propositions are about the world. The philosophical argument is: Given that the world is contingent, follows that the world is not known (and it cannot be known), assuming that knowledge is just knowledge of formal and necessary truths.

The most general purpose of this work has been to study the interactions of metaphysical and epistemic modalities. Combinations between both families of modalities allow a very interesting study of a great variety of problems in metaphysics and epistemology, and the installation of a bridge between both areas of study. The study of the mixing of modalities provides an understanding of some of the problems connecting the theory of knowledge with the theory of reality: Problems such as the limits of knowledge imposed by FITCH's paradox, the problem of whether the world can be known, or if it is only an object of belief in the way claimed by the logical skeptic, and the existence of a contradictory structure disclosed in the limiting structure of knowledge. These are typical examples of such relations between epistemological and ontological issues. Their study serves to confirm that one of the important interplays between both kinds of inquiry is constituted by the topic of the combination of modalities.

This research started as an attempt to show that skepticism was a sound philosophical position. The reader found in it (I hope), important results which can assist the development of philosophy (it is an important part of my contention here that philosophy indeed develops through time). Some developments were outlined in the last chapters. Let me summarize once again what are the main contributions of this book.

To begin with particular results:

1. The correct language and logic in which to formalize FITCH's paradox is composed by a fusion of modal languages and modal logics.

Moreover, given the correct framework, it allows us to find a counter-model to Fitch's deduction. This result is the first example of how fusions of modal logics play an important role in modelling the limits of knowledge; The proposal of the knowability modality;

2. The presentation of a formalism combining epistemic logics of knowledge and logics of contingency with or without some interactions between both operators. This formalism has been used to argue that contingent propositions cannot be known (and they are not known). This is a conclusion related to the problem of induction and with the contingency of the world. The world cannot be (and it is not) object of knowledge;
3. The presentation of a system combining epistemic logics of belief and logics of contingency with or without interactions between both operators. This formalism was used to show that contingent propositions can be the object of belief. The world, although not the object of knowledge, is an object of belief; Formulations of many kinds of skepticism using fusions extended with interaction axioms;
4. We showed how some combined logics can deal with contradictory theories. When contradictions appear in the limiting relationships of knowledge and belief (and its interactions with contingency), one can tolerate such contradictions by converting some of the underlying logics employed in the representation of belief and knowledge into paraconsistent logics using a special mechanism to realize such a task; The relations between modalities and paraconsistency were clarified by means of a general theory of paraconsistent logics;
5. An outline of some alternative frameworks for studying the relations between different modalities using as abstract tools such mathematical and logical resources as category theory, n -opposition theory, n -dimensional modalities and the structuralist theory of logics: tools for universal logic.

The basic ideas that were used to develop the above approaches and that I claim bring something in relatively new to their study, were as follows:

1. Combining logics for applications to metaphysics and epistemology. This is something largely new, although it is true that some logicians have already had some powerful intuitions about, and even developed

some initial applications; but I believe that in this work we have provided specific and developed examples of how to use techniques for combining logics in philosophy.

2. The paraconsistentization of logics: the general theory of paraconsistent logics. This strategy has been used to “paraconsistentize” the logics combined using the idea above; Such a general method allows us to generate paraconsistent logics from logics which are not paraconsistent;
3. Philosophical categorification: the abstract approach to modalities. The categorial approach to philosophical concepts and theories;

Some minor developments were also proposed here: the existence of some real universal logical principles governing all possible logical systems. Of course it is conceded that the presentation here is necessarily very brief and simplified. I do not set out to provide a full and complete guide to the logical structure of the universe, nor do I think such a thing is possible ¹⁰. But such simplifications are unavoidable in order to model a given fragment of reality. Although simplifications, they point the way to less partial and more comprehensive proposals.

Recent developments in computer science and especially the growth of the internet has altered some aspects of the way that we conceive the world and our relationship to it. I offer some examples to explain my point.

The first one is the development of 3-dimensional worlds where we are able to reproduce in virtual reality some aspects of the world in which we live. The birth of some 3-D communities whose participants/inhabitants can use a special kind of virtual money, one can buy virtual things and live in almost the same way we live in our everyday reality. Probably, these kind of communities will develop beyond limits which we can today scarcely imagine. However, just consider the concrete fact that there already are such 3-D communities. In such communities, one can select a body, and one can consider that this virtual body has known the things that one know, because it is one’s representation. To study how knowledge, contingency and belief (as well its interactions) work in such a world may involve radically novel epistemological and ontological notions. I think that this is an interesting area of investigation which deserves the attentiton of serious researchers in philosophy, and not only science fiction writers and software engineers.

¹⁰Some attempts to describe a complete guide of the law of the universe have been proposed in the literature and are nowadays best-sellers. Check for example PENROSE [83]

Unfortunately, I did not develop my thoughts on this topic here, though I hope to do so elsewhere.

The second example is the recognition that the role of knowledge (in the sense that this involves an agent with some information) in society is not so important in our times as it formerly was. In the past, one could be proud if (s)he knowing how to compute fast, memorize a given body of information (perhaps an epic poem) etc. Recently, with the development of computers, such accomplishments have been made to seem petty. In the future, probably, googling will be developed to the point that we can access all informations in any data base very fast. Indeed, internet chat already allows a very different scenario in interactions between holders of information from anything that was previously familiar. Sometimes, in a given public virtual discussion, one does not have some given information, but if one is in a chatroom, one could easily access google in real time and present the information. Our understanding of aspects of knowledge and belief, as well as of the combinations of epistemic modalities with metaphysical modalities has to take such developments into consideration.

One could here pose a metaphilosophical question : what is the task of philosophy in our times, when the sciences are so highly developed? Long ago, the philosopher was someone who was expected to be able to give a general account of reality, and describe even the basic structure and content of the world and its objects. In Greek philosophy, it is common to find philosophers trying to explain reality and its laws. The most famous example is that of ARISTOTLE. And for centuries, philosophy has always been considered to comprise the attempt to offer a general description of reality. Even today, some attempts to offer a complete guide to the laws of the universe are presented in the literature and indeed are sometimes best-sellers. See for example the recent astonishing publishing success of PENROSE. What then is the object of philosophy? Philosophers turned to language, to study its relations with the world. However, language itself already has a proprietor : logic (in the sense that the study of language is subject to logical norms). A philosopher could also study and use languages in other senses, but here the arts and poetry have already stipulated their claim to the imaginative use of language. So what are philosophers supposed to study? What is philosophy? On one side, the general study of the world appears to belong to the sciences. The study of language, on the other side, increasingly belongs to logic and mathematics. It is not easy to see a distinct subject matter for the philosopher. This book suggested a way to proceed in philosophy which is not dependent on a particular scientific position. The philosopher cannot seriously investigate the overall contents of the world in the fashion of the

various sciences. For example, it would not be sensible for somebody with a disease today to think of going for treatment to someone calling themselves a philosopher (in the way that it might have been in the past). Nor would somebody who desires to create a new software package think of going to a philosopher, but rather to a computer scientist. The task of philosophy is to propose models for the general structure of reality, which are not dependent on the particular sciences. For example, instead of studying a particular process in the world such as how to produce a software package, a philosopher studies the basic definitions of a process. Such activity is important, because it is exactly through such an activity that good general models and guiding ideas can appear. Although philosophy may no longer appear to be helpful in the detailed study of world, it can still be of use in generating fruitful reflections about its overall structure.

This work can be viewed as attempt to model the structure of reality in general and the possibility of knowing such a reality. In Chapter 1, we proposed a model of the limits of knowledge, and a way of dealing with the verification principle using fusions of modal logics and products of modal logics. In Chapters 2 and 3 we discussed what can be known in general principle (KANT famously studied the same question, though in a manner based on another approach). Our approach has been based on combining logics, using modal formalisms where interactions between modalities can be defined and studied. In the fourth chapter we showed how to treat contradictions appearing in the limits of what can be known and in the last chapter we proposed a general theory of modality based on our intuitions concerning combined modal systems and outlined four alternative formal frameworks for their study. The current task of philosophy is to propose models of the general structure of reality. In order to realize such a task, formal tools drawn from mathematics and logic are indispensable. The reflections which were offered concerning the philosophical consequences of category theory also furnish a good example of the kinds of task proper to philosophy. In this connection, it is to be noted that whether a model of reality based on category theory is true or not is not to be regarded as a particularly fruitful, and possibly not even a meaningful question. Rather the problem is to determine whether an approach (e.g. to questions of general ontology) can make use of categorial constructions in an illuminating and fruitful way. Interest and beauty are what are chiefly at issue here, at least in the initial stage of the quest, rather than any premature attempt to characterize the kind of correspondence which might be captured by such methods from the standpoint of traditional ontological inquiry.

I hope to have indicated some new directions and to have presented

some original ideas that deserve to be investigated in further detail. In this sense, I humbly invite the reader to join in the quest, continuing the development of the lines of inquiry begun here, in the hope of achieving a fuller understanding and a broader body of technical results than one could hope to have achieved in this initial work.

History and Interview

After all our journey, the reader finds now the history of this research and an interview that I gave to the book [52], where I presented in an informal way the content of this book. Such history and interview appear in the text with the aims of providing a general view of my philosophical projects.

The history of this research

This research started 1st august 2003 and finished 31th july 2007. During this period, I developed the main ideas and concepts which the reader found in the text. This is the result of a grant of the **Swiss National Science Foundation**. Without this grant, it would be *impossible* to realize this research. I've been invited for Prof. Beziau to join his research team on Universal Logic at the Université de Neuchâtel. It was always important for me that this research should display the result of an original and more or less interesting investigation. The results found here are within the subject of universal logic, the general theory of logical systems.

During the above period, I was lucky to present the main ideas and concepts of this thesis in a few places. The chapters of this book were presented in many seminars, conferences, congresses and workshops. Below the complete list of my talks:

2003

- Methods for combining logics. *1st Seminar for Swiss PhD students in Logic* - University of Neuchâtel - NE - Switzerland
- Possibility and knowledge. *Séminaire de L'Institut de Logique* - University of Neuchâtel - Switzerland

2004

- Methods for combining logics. *La demi-journée scientifique du Centre romand de Logique* - University of Lausanne - Switzerland
- Logiques, catégories et concepts. *Colloque: La philosophie entre l'unité / pluralité logique et l'unité / pluralité mathématique* - University of Nice Sophia Antipolis - France (Thanks to Alessio Moretti for the invitation)
- Combining possibility and knowledge. *Workshop on the combination of logics: theory and applications* - Technical University of Lisbon - Portugal
- General theory on the combination of logics. *2nd Seminar for Swiss PhD students in Logic* - University of Bern - Switzerland
- Categories and philosophy. *5th Swiss Meeting of Graduate Students in Philosophy* - University of Bern - Bern - Switzerland
- Categories and philosophy. *Third International Workshop on the History and Philosophy of Logic, Mathematics and Computation* - University of the Basque Country - Spain
- Paraconsistency and knowability (Joint work with W.A. Carnielli and M.E. Coniglio). *The Logic Tea* - Institute for Logic, Language and Computation - University of Amsterdam - The Netherlands (Thanks to Hartmut Fitz for the invitation)
- Fusion and product of logics (Fusion et produits des logiques). *Séminaire de L'Institut de Logique* - University of Neuchâtel - Switzerland
- Modal paraconsistent logics (joint work with W. A. Carnielli and M. E. Coniglio). *Séminaire de L'Institut de Logique* - University of Neuchâtel - Switzerland

2005

- Logiques, catégories et concepts. *École Doctorale Romande en Philosophie* - University of Fribourg - Switzerland

- Les différents aspects du concept de bivalence. *From Philosophy of Logic to Philosophical Logic: the case of bivalence*. University of Nancy 2 - France (Thanks to Fabien Schang for the invitation)
- Paraconsistentization of logics? *Logic Seminar* - Ghent University - Centre for Logic and Philosophy of Science - Belgium (Thanks to Diderik Batens and Joke Meheus for the invitation)
- Paraconsistentization of logics. *Logic-Philosophical Workshop* - Torun - Poland (Thanks to Tomasz Jarmuzek for the invitation)
- Paraconsistentization of logics: the general theory of paraconsistent logics. *3rd meeting of the Vicious Circle Society* - Haus der Universität Bern - Switzerland
- Proposition, Identité et Théorie des Catégories. *Rencontres du Centre Romand en Logique, Histoire et Philosophie des Sciences* University of Lausanne - Switzerland
- Possibilité, Connaissance et Paradoxe. *Colloque des doctorants de l'Université de Genève* - University of Geneva - Switzerland
- Ferramentas e conceitos para filósofos. *Workshop: Futuro e Limites da Filosofia (The future and limits of philosophy)* Goiânia - Brazil

2006

- Combining possibility and knowledge. *1st Philosophy Graduate Conference at the Central European University* - Budapest - Hungary
- Possibilité, Connaissance et Paradoxe. *Séminaire de L'institut de Philosophie* - Neuchâtel - University of Neuchâtel - Switzerland
- The Church-Fitch paradox and combining logics. *Logic and Computation Seminar* - Technical University of Lisbon - Center for Logic and Computation - Portugal
- The Church-Fitch paradox and combining logics. *Seminários de Lógica Matemática* - University of Lisbon/Technical University of Lisbon - Department of Mathematics 06 JUNE -2006 - Portugal (Thanks to Fernando Ferreira for the invitation)

- The Church-Fitch paradox and combining logics. *Logica 2006* - The Institute of Philosophy, Academy of Sciences of the Czech Republic - Hejnice Monastery - 2006 - Czech Republic
- Combining logics: its philosophical foundations and applications. *SOPHA : Société de Philosophie Analytique*. University of Aix en Provence - France
- Philosophical categorification. *Categorification in algebra and topology*. Uppsala University - Sweden
- Acontecimento e contingência. *Workshop: Futuro e Limites da Filosofia*. Federal University of Goiás - Brazil

2007

- Combining knowledge and contingency. *Workshop: Geneva-Barcelona*, Université de Genève, Switzerland.
- Imagination, possibility and the square of opposition (with Jean-Yves Béziau and Gillman Payette). *The Square of Opposition International Congress Montreux*, Switzerland.
- Combining knowledge and contingency. *Séminaire PHILFORM*, IH-PST, Université Paris 1 France. (Thanks to Gabriel Sandu and Carlo Proietti for the invitation)
- Connaissance, croyance et contingence. *Séminaires des Archives Poincaré*. Université de Nancy, France. (Thanks to Fabien Schang for the invitation)

Some passages of the present book appeared also in the form of publications:

- Towards a general theory of the combination of logics. *Aspects of Universal Logic*. (2004) Eds: J-Y. Béziau, A. Costa-Leite A. Facchini. Centre de Recherches Sémiologiques. Travaux de logique. N.17.
- (Extended abstract) Combining possibility and knowledge *Proceedings of the Workshop on the combination of logics: theory and applications*. 2004) Eds: W. A. Carnielli, F. Dionísio and P. Mateus. IST - Department of Mathematics - Center for Logic and Computation - Lisbon, Portugal.

- Logics, categories and concepts - *Noesis*. Department of Philosophy and Department of Mathematics, University of Nice, France, 2006. (accepted for publication).
- Fusions of modal logics and Fitch's paradox. *Croatian Journal of Philosophy*. (2006). (2(17)), Croatia.
- Combining modal concepts: philosophical applications. (2007) *Logica Yearbook 2006*, Publishing House Filosofia, Prague, Czech Republic.
- Interview with A. Costa-Leite. In *Masses of Formal Philosophy*. (2006) Eds: V. Hendricks and J. Symons. Automatic Press, New York/London.

During the period of research I had the opportunity of organizing one very big event:

- First World Congress and School on Universal Logic, Switzerland, Montreux, 2005.

And I am currently organizing:

- Second World Congress and School on Universal Logic, Xi'an, China, 2007.

I've also organized two workshops in Goiânia, Brazil:

- Workshop: The future and limits of philosophy - Goiânia, Brazil, 2005.
- Workshop: The future and limits of metaphysics - possibility, necessity and contingency - Goiânia, Brazil, 2006

I am forever indebt with all those philosophers and logicians who attended these meetings.

A research is not just writing articles and books, but also some meta-works. I have been editor of the following books:

- Aspects of universal logic (editors: J-Y. Béziau, A. Costa-Leite and A. Facchini). Institute of Logic, University of Neuchâtel - Switzerland - December 2004.
- Handbook of the UNILOG'05. Book of abstracts of UNILOG'05.
- Perspectives of Universal Logic. (2007). Polimetrica, Monza, Italy.

- Handbook of the UNILOG'07. Book of abstracts of UNILOG'07.

I am also indebt with all authors who contributed for the above books. During the research I teached the following courses at the University of Neuchâtel:

- Applications de la logique à la philosophie. Institute of Logic - UniNE - 2004/2005
- Logique Philosophique et la Philosophie de la Logique. Institute of Logic - UniNE - 2005/2006
- Est-il raisonnable de se contredire? (Seminar - UniNE - with Jean-Yves Béziau - Institut de Psychologie - 2006/2007)

I officialy visited the following Institutions while writing the thesis:

1. University of Amsterdam. From February to July. (2005).
2. Technical University of Lisbon. Department of Mathematics. From April to June. (2006).
3. École Normale Supérieure. Group Pensée des Sciences. From April to July. (2007).

I describe now what happened in each one of the cities where I *visited* out of Switzerland.

- (AMSTERDAM - NL) While living in Amsterdam, I was also editing some books and organizing the UNILOG'05. At that time, I met many important logicians who helped me with some suggestions. Maarten Marx who accepted me at the University of Amsterdam was my supervisor. We met sometimes to discuss n -dimensional modalities. I attended freely a course on modal logics given by Yde Venema. I thank Venema for accepting me in his group. I also met Johan van Benthem who suggested interesting ideas on dynamic epistemic logic that unfortunately I did not develop. Amsterdam is a very interesting place for a logician. I found the environment there perfect for a logical life. There, I discovered the method of paraconsistentization of logics and, although I've been there to study modal logics, I've not been able to avoid the birth of the general theory of paraconsistent logics. I wrote in Amsterdam a long part of chapters 4 and 5.

- (LISBON - PT) Lisbon was also interesting for my project, especially because there we can find some of the most important logical combinatorators: A. Sernadas and C. Caleiro. I've been there working especially with Caleiro, who gave me very good directions and wake me up from my dogmatic dream (to copy Kant referring to Hume) on my project concerning the verification principle and Fitch's paradox. I owe to Caleiro the hypothesis that the paradox was happening because there was a confusion between two accessibility relations. In Lisbon, I was able to conclude chapter 1 and set out the main framework of chapters 2 and 3.
- (PARIS - FR) Paris arrived when the thesis was ready, and my most important project was still missing some improvements. There I've been working with Andrei Rodin at the École Normale de Paris. Paris is a city of philosophers. Its streets, environment, everything there invites for a good reflection. Living in Paris allowed me a great revision of the whole of the thesis.

I wrote many passages in beautiful Switzerland, especially in Neuchâtel and Fribourg.

- (NEUCHÂTEL/FRIBOURG - CH)

Being alone in the middle of Swiss mountains allowed me a very productive period of research. In Switzerland, I worked a lot on the idea that combining logics can be a very important philosophical tool while I was studying products and trying to model the modality of knowability. In Switzerland, I created the general theory of combining logics, trying to answer its basic question. I also developed in Switzerland the main arguments to study the combination of contingency logics and epistemic logics, and philosophical categorification.
- (GOIÂNIA - BR) Some small passages of the book have been written in my hometown Goiânia, Brazil, when I've been there for holidays. Alain Badiou [4] has an interesting statement about the place where I came from: "Le plan directeur est le développement d'une simple croix tracée sur un plateau désert de l'État de Goiás. Là est le centre, comme est la définition géométrique d'un point: intersection de deux droites". (ALAIN BADIOU, Logiques des Mondes, 2006).

The research started here is just small step towards a global understanding of how interactions of concepts works. I hope that the readers will continue some lines developed here.

Interview with A. Costa-Leite

This interview is part of the book [52]¹¹.

- Why were you initially drawn to formal methods?

My first contact with formal methods occurred when I was studying Popper's book *Conjectures and Refutations*. I discovered the induction problem and then I decided that I should study Hume's work to understand it. The *Enquiry Concerning Human Understanding* showed me that, indeed, we have good reasons to accept that the world is contingent. My interest in empirical sciences was, therefore, destroyed because I was looking for answers that could not be false. At this time, I came back to the study of mathematics and logic because I supposed that I would discover the most general structure of reality and some necessary truths in these areas. I realized that I should study Wittgenstein's *Tractatus* because he had proposed a kind of philosophy where formal methods were the main tool. I studied the *Tractatus* and then I felt that it was exactly the kind of thing that I was looking for. Afterwards, I started studying contemporary formal philosophers and when I finished my undergraduate studies I wrote a monograph on Kripke's contributions to philosophy. Afterwards, in my Master's thesis, I studied Fitch's paradox and I proposed a solution to it using some non-classical modal logics.

- What example(s) from your work (or the work of others) illustrates the role formal methods can play in philosophy?

I would mention, first, the role of methods for combining logics in philosophy. As Gabbay pointed out in *Fibring Logics*, we know that there are complicated statements in natural languages involving a lot of different concepts which cannot be formalized using a very simple formalism. Let me illustrate the problem. Suppose, for instance, that we are trying to determine if the statement "Contingent propositions can be known" is true or not. This proposition has two different non-interdefinable modalities: contingency and knowledge. Thus, a simple epistemic logic cannot be used. We have to, at least, combine a logic for contingency with an epistemic logic in order to formalize the statement. Recently, there are a lot of methods for combining logics. My main research consists in applications of those methods in epistemology and metaphysics. I show how they can help us in the

¹¹Some passages of the interview were presented and discussed in the main text.

analysis of different philosophical problems, principles and concepts such as skepticism and some paradoxes.

Second, I would mention my project called philosophical categorification. Logic is a powerful tool which philosophers utilise to better understand their problems and concepts. As is well-known, the relationships between logic and category theory (in the sense of Mac Lane) appear in different levels: 1) logical operators can be represented in categories, since objects are propositions and morphisms are proofs (for instance the works of Lambek and Goldblatt); 2) logics can also be assumed as objects of categories where morphisms are translations (for example, some articles of Carnielli and Coniglio); 3) methods for combining logics are universal constructions in some categories where objects are logics (the approach developed by Seradas and Caleiro). Indeed, there are in the literature some other examples of how category-theoretic concepts can replace logical concepts. However, given that logic is a very important tool in order to understand concepts from philosophical areas such as epistemology and metaphysics, a natural conjecture is that category theory can also play an important role in philosophy. Category theory is a tool which can be used in philosophical theories and itself has an ontological status (some attempts of applying categories in philosophy are those of Badiou, Marquis, Rodin and Reyes). Philosophical categorification is the philosophical counterpart of categorification introduced by some mathematicians (Dolan and Baez), but replacing logical concepts for categorial concepts, and also set-theoretic notions by category-theoretic notions in order to investigate philosophical concepts.

- What is the proper role of philosophy in relation to other disciplines?

Philosophy can play a very important role especially in the process of generating new ideas and concepts. Also, philosophy helps in the analysis of main concepts of a given discipline. I should select an example: take the role of philosophy in mathematics, especially because this is a book on how mathematical methods can be applied to philosophical issues. Why not the converse process? Consider a mathematician who studies the concept of *set* and argues that sets are generated by a certain axiom. The mathematician decides to mathematically develop then an axiomatic set theory. When a mathematician examines the nature of a given axiom, its ontological content, why a set is really a set, how we can know the properties of a set and many other foundational questions, then philosophy is playing a role.

- What do you consider the most neglected topics and/or contributions in late 20th century philosophy?

On one hand, the applications of category theory in philosophy and the study of the philosophical content of categories are the most neglected topics in late 20th century philosophy. However, there are other hard problems which have been neglected and that will be probably forever neglected at least in the academic level: the meaning of life, what is death etc. The formal philosopher just ignores such problems. Maybe formal philosophy can even help in the treatment of complicated philosophical questions.

On the other hand, one of the most important contributions on how formal methods can play a role in philosophy is Kripke's argument that there are necessary *a posteriori* truths (necessary truths which can only be known *a posteriori*). This was a real revolution in philosophy, given that necessity was always related to *a priori* knowledge. This kind of philosophical progress was just possible because formal tools such as modal logic were playing a special role. Another contribution which should be mentioned is possible worlds semantics. Nowadays, it is very difficult to find a philosopher who does not use such a tool.

- What are the most important open problems in philosophy and what are the prospects for progress?

This is a very interesting question. I think that the most important open problem in philosophy is to present some concrete examples of philosophical categorification and show that, therefore, category theory is a very general, abstract and important philosophical tool. The problem of generating a system of ontology (answering questions such as what is an object? what is a property? what is an adequate criterion of existence etc) using category theory is one of the most important open problems which can be solved. Some other examples are: how to define metaphysical and epistemic modalities without using possible worlds semantics? How to define the notion of contingency in a categorical approach?

However, there are real open problems, much more important than any other problem of any area of science, that no solution has been announced for centuries and looks like that the situation will not change in the near future: What is the meaning of life? What is death?

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