Supersymmetric matrix models

in

M-theory

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Contents

Notations

In this thesis, the indefinite metric on spacetime is always chosen so that time-like directions correspond to negative eigenvalues and space-like directions to positive eigenvalues. In particular, the minkowskian metric is always: $\eta_{\mu\nu} = diag(-1, +1, \ldots, +1)$. The Dirac matrices are then taken to satisfy the Clifford algebra:

$$\{\gamma^\mu, \gamma^\mu\} = 2\eta^{\mu\nu}\,\mathbb{I}\ . \tag{1}$$

Except stated otherwise, all formulae are expressed in a Majorana representation of the Clifford algebra of $SO(10,1)$ where Γ^{10} is block-diagonal, so that it can also be seen as the chirality matrix of a Majorana-Weyl representation of the Clifford algebra for $SO(9,1)$. In that case, the Dirac matrices can be chosen to be real as in the following Majorana representation:

$$C = \Gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_{16} \\ -\mathbb{I}_{16} & 0 \end{pmatrix}\ , \quad \Gamma^i = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix}\ , \forall\, i = 1, \ldots, 9 \quad \Gamma_* = \Gamma^{10} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}\ , \tag{2}$$

where $\{\gamma^i, \gamma^j\} = 2\delta^{ij}\,\mathbb{I}_{16}$ form a Majorana representation of $SO(9)$. We give an example of such a representation in appendix A. However, most formulae are valid in any representation. From the chirality matrix Γ_*, we define the left-handed and right-handed chiral projection operators by:

$$\mathcal{P}_L = \frac{1}{2}(\mathbb{I} + \Gamma_*) \ \text{ and } \ \mathcal{P}_R = \frac{1}{2}(\mathbb{I} - \Gamma_*)\ . \tag{3}$$

We use the following convention for the anti-symmetrization bracket [] on indices:

$$F_{[i_1 \ldots i_k]} = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} (-1)^{|\sigma|} F_{i_{\sigma(1)} \ldots i_{\sigma(k)}}\ , \tag{4}$$

where $|\sigma| = 0$ or 1 is the signature of the permutation $\sigma \in \mathcal{S}_k$. Although this notation is sometimes ambiguous, in case it acts only on part of the indices enclosed, or when several of these brackets are entangled, its meaning can be inferred by looking at what indices are anti-symmetrized on the other side of the equality. For example, in an equation like:

$$\Gamma^{jk}\Gamma^{i_1 \ldots i_4} = \Gamma^{jki_1 \ldots i_4} - 8\eta^{[j[i_1}\Gamma^{k]i_2 \ldots i_4]} - 12\eta^{[j[i_1}\eta^{k]i_2}\Gamma^{i_2 i_3 i_4]}\ , \tag{5}$$

all anti-symmetrizations on the right-hand side are on the groups of indices $\{j, k\}$ and $\{i_1, \ldots, i_4\}$ separately, as the comparison with the left-hand side suggests.

Furthermore, we choose hermitian generators t^a for the $U(N)$ gauge groups, so that a $U(N)$ matrix is given by:

$$U = \exp(i\theta^a t^a)\ , \tag{6}$$

for N^2 real parameters θ^a. We normalize the t^a so that:

$$[t^a, t^b] = if^{abc}t^c\ , \ \text{and } Tr(t^a \cdot t^b) = \delta^{ab}\ , \tag{7}$$

for a completely anti-symmetric matrix f^{abc}. Matrices in the adjoint representation of $U(N)$ are simply decomposed as: $A = A^a t^a$ on the basis of generators, so that the A^a are real in case of a hermitian operator

Chapter 1

Introduction

I will start with a historical introduction, in order to explain how my work fits in the long line of research towards a quantum theory of gravity and a unified theory of all interactions. I will not attempt to give all relevant references to the subject, since the list would become either endless or incomplete.

After the early successes of quantum mechanics in the description of atomic orbitals, it became apparent that we should try to find a quantum version of Albert Einstein's general theory of relativity. However, a quantum description of classical field theory was first needed. A first example of a quantum field theory with local gauge invariance was found in quantum electrodynamics. The essential new ingredient needed to extract sensible results from quantum electrodynamics is renormalization theory. However, the issue of renormalization is precisely the element that makes the quantization of gravity such a difficult problem.

1.1 Renormalization

Indeed, when we compute quantum corrections to the classical result given by some field theory, many quantities turn out to be ultraviolet divergent. More precisely, such calculation always involve the computation of momentum integrals on the momenta of all virtual particles (those that do not appear in the initial or final states). The momenta p_μ of these particles need not be related to their mass by the relativistic physical on-shell condition $p_\mu p^\mu = -m^2$, so that they can be arbitrarily large, usually leading to divergent integrals. Since these divergences are due to very high momenta, which are related by Fourier transformation to small distances, they can be seen as coming from the shrinking of the virtual particles' loop in the Feynman diagrams to a single point, which means that they are related to the local nature of interactions in quantum field theory.

We can use different regularization methods to exhibit the structure of the divergent contributions to the results (most notably the Pauli-Villars regularization that introduces additional particles of very large masses M_i to make the integrals less divergent, the use of an ultraviolet cutoff Λ to limit the integration range to finite values or the dimensional regularization, that replaces the usual 4-dimensional space-time momentum integrations by their $4+\epsilon$-dimensional counterparts), but of course the results will blow up when we remove the cut-off (i.e. send M_i or $\Lambda \to \infty$ or $\epsilon \to 0$). The way out is to suppose that the bare quantities present in the original action are themselves divergent and rescale the fields, coupling constants and masses of the theory by infinite constants (with an appropriate cutoff-dependence), so that the physical results are finite when expressed in terms of these renormalized quantities. Then, we can rewrite the bare action as a finite physical action plus a number of so-called infinite counterterms. The counterterms can thus be seen as compensating the divergent quantities in the original action to replace them by the finite physical ones. Fortunately, this rather unnatural method allows us to extract the finite physical quantum corrections out of the divergent momentum integrations. It was proven to be applicable to all orders of perturbation theory in quantum electrodynamics and various other quantum field theories.

Despite being both gauge field theories, general relativity is quite different from electrodynamics in various ways. Firstly, being a gauge theory of the Lorentz algebra, the gauge symmetry of the general theory of relativity is non-abelian. An answer to that difficulty was found only much later, when the work of Feynman, Faddeev, Popov, De Witt and Berezin about path integration in quantum theories indicated how to gauge away the unphysical degrees of freedom in a systematic way for a general choice of gauge, allowing us to quantize non-abelian gauge field theories in a covariant way. In fact, this procedure leaves a shadow of the original gauge invariance in the form of the so-called BRST symmetry from the names of Becchi, Rouet, Stora and Tyutin.

In 1973, Veltman and t'Hooft were able to show that the coupled theory of a non-abelian $SU(3)$

gauge field with $N_f < 16$ triplets of fermionic matter fields in the fundamental representation of the gauge group was renormalizable and asymptotically free, two important criteria for a theory to be well-defined. Actually, in that sense, quantum electrodynamics is ill-defined since its coupling constant blows up at very high energies. On the other hand, we don't expect it to make sense at such high energies anyway, since it should be replaced by a more fundamental theory much before (note that it merges with weak interactions at the weak scale and shouldn't be considered separately above that limit).

Unfortunately, general relativity differs from usual non-abelian gauge theories in an essential way. Besides the fact that the local gauge symmetry group is acting directly on the space-time and not in some inner space, a more subtle difference resides in the fact that the gravitational coupling constant is dimensionful, making hopelessly non-renormalizable any direct quantization of general relativity, since the divergences grow worse with any additional graviton loop in the Feynman expansion.

1.2 Quantum gravity

This fact suggests that either standard perturbation theory is ill-defined and the theory only makes sense when treated exactly, which would imply the existence of a non-trivial ultraviolet fixed point, or quantum gravity should not be directly described by the quantization of general relativity, but through a different theory that has a better high-energy behaviour but the same low-energy limit.

The first point of view is usually advocated by the community of researcher studying loop quantum gravity or dynamical triangulations. In loop gravity, one tries to deal directly with the gauge-invariant observable, like Wilson loops, instead of dealing with the graviton field. This theory has some nice features, like giving a random lattice picture (spin network) of space-time and smearing-out the point-like interactions of gravitons to joining and parting interactions of loops (quite similarly to string theory, in fact).

However, it seems extremely difficult to show that it indeed reduces to general relativity in the low-energy limit, since a smooth continuous space-time has to be represented by a network containing an infinite number of links. In other words, the low-energy case corresponds to the situation where the theory is extremely complicated.

The dynamical triangulations approach suffers from the same problem since it attempts to directly discretize space-time with a random triangulation. However, since the model is much simpler, the large N limit of the number of vertices might be computed exactly in lower dimensions and estimated numerically in higher dimensions. However, it doesn't seem to be clear whether it leads to sensible result beyond the lower-dimensional cases where gravity is trivial or topological. In 4-dimensional minkowskian spacetime, it remains to be seen whether the theory has an ultraviolet-stable fixed point where the continuum large N limit can be taken in a sensible way. An example of a dynamical triangulation theory in two dimensions is presented in chapter 2.

There are other theories of that first type like Regge calculus, but I won't describe them all here.

There have also been countless attempts of the second type to describe quantum gravity by theories whose starting point is not pure general relativity. After the discovery of supersymmetry on the string world-sheet, the idea of a boson-fermion symmetry was extended to field theories and it has soon been shown to lead to interesting cancellations between Feynman diagrams, so that there was some hope that supersymmetrizing general relativity could be enough to keep the ultraviolet divergences under control. It turned out that it was not to be, even in the case of the $\mathcal{N} = 1$ supergravity theory in eleven dimensions, where some hope remained until a few years ago. In any case, the most successful of this second type of theories is without any doubt string theory, which we will introduce in the following section.

1.3 String theory

String theory was first proposed as a theory of the strong interaction. Indeed, strong interaction experiments indicated that the spectrum of resonances followed a linear trajectory, usually called Regge trajectory, that links mass and spin through the Regge slope α' as: $m^2 = (J - \alpha(0))/\alpha'$ for some constant $\alpha(0)$. People noticed correctly that a relativistic open string also exhibited such a mass spectrum. Thanks to the infinite tower of higher mass and higher spin states this equivalence predicts, the physical amplitudes exhibit an interesting duality symmetry between s-channel and t-channel results. For that reason, these early string models were called dual resonance models. However, it turned out that the theory is unitary only when the space-time in which the string lives is 26-dimensional and the constant $\alpha(0) = 1$. In particular, there is a scalar state with $J = n = 0$ that has a tachyonic mass given by $m^2 = -1/\alpha'$, that makes the theory unstable.

This theoretically deceptive result, as well as some contradictory later experimental results led to the abandon of this line of work, while quantum chromodynamics became generally accepted as the correct theory describing the strong interactions.

On the other hand, although the open string theory above only has a state of spin 1 in its massless spectrum, the theory of closed strings contains a massless symmetric tensor of spin 2 that we can identify with the graviton, since its low-energy effective theory indeed corresponds to general relativity. It was later realized that the nice high-energy behaviour of the theory (that can be traced back to the smearing-out of the quantum field theory point-like particle interactions) might indeed make it an interesting candidate for a definition of quantum gravity. Furthermore, its supersymmetric extension "only" requires ten dimensions, while the GSO (Gliozzi-Sherk-Olive) projection allows us to remove the tachyonic ground state from the spectrum of the Ramond-Neveu-Schwarz string and obtain spacetime supersymmetric theories. Of course, this conclusion became obvious from the point of view of the Green-Schwarz superstring, which is manifestly supersymmetric in space-time. In fact, certain superstring theories were shown to be free of anomalies and ultraviolet divergences. This revived interest in string theories culminated with the proof of anomaly-cancellation in heterotic theories that paved the way to the study of the possible low-energy particle phenomenology that string theory allows. This period around 1984 became known as the first superstring revolution.

Despite these encouraging early successes, string theory suffers from another kind of problem. It is a first-quantized theory, not a quantum field theory, so that the perturbative expansion in Riemann surfaces of higher genera is not the perturbative expansion of some non-perturbatively defined theory, although it is certainly needed for the unitarity of the theory. This means that we can only derive some low-energy approximation of the equations of motion for the massless fields, so that it is difficult to understand the vacuum structure of the theory in the high-energy régime. An obvious answer to this problem is to try to develop a string field theory. Despites some early successes in the bosonic open string case and some recent results about superstring fields, it seems to be extremely diffcult to obtain a field theory of closed superstrings. Furthermore, it might well be that the non-perturbative sector of superstring theory is not described by superstrings living in a ten-dimensional spacetime, but rather by supermembranes living in an eleven-dimensional spacetime, since everything looks like an additional dimension is unfolding when we raise the value of the coupling constant in type IIA string theory.

This unknown mysterious eleven-dimensional theory has been called M-theory, since we expect it to be the "mother" of all superstring theories, but we are far from having a concrete non-perturbative and background-independent definition of it. We only know that it should have various perturbative régimes corresponding to the known superstring theories in ten dimensions and reduce to eleven-dimensional supergravity in the low-energy limit.

1.4 Branes and dualities

The most significant discovery in the last ten years of research on string theory has been linked to dualities and branes, the so-called second superstring revolution. We understood so far that there were five apparently different consistent superstring theories in flat ten-dimensional minkowskian spacetime, the type IIA and IIB closed superstring theories with $\mathcal{N} = 2$ spacetime supersymmetry on one hand, the type I theory that contains both open strings and closed unoriented strings and the heterotic closed string theories with gauge group $E_8 \times E_8$ and $SO(32)$, all three with $\mathcal{N} = 1$ spacetime supersymmetry, on the other hand. However, it became clear that these theories are all related by various kinds of duality transformations that maps one theory to another, so that the idea emerged that they should all be different perturbative régimes of a unique theory, M-theory. In a fancier language, they should all correspond to various corners of the moduli space of vacua of M-theory.

An essential ingredient in the discovery of this web of dualities was the understanding that string theories should contain D-brane states in their non-perturbative spectrum. Indeed, if one quantizes open string with Dirichlet boundary conditions in certain directions instead of the standard purely Neumann boundary conditions, one can see the hypersurface on which the open-strings'ends are attached as a dynamical object on its own. Such types of hypersurfaces were called D-branes as a reminder of their interpretation as Dirichlet boundary conditions. More interestingly, if one performs a T-duality, i.e. an inversion of the radius of compactification of the spacetime in some direction of the form $\tilde{R} = \alpha'/R$, one can exchange Dirichlet with Neumann boundary conditions, so that the dimension of the boundary state changes in the process. At the same time, the original string theory is usually transformed into some other kind of string theory.

A classic example illustrating this process is the T-duality between type IIA and IIB superstring theories. While the type IIA string theory contains non-perturbative Dp-brane states for all even values of p, type IIB string theory contains them for all odd values of p (this can be seen easily from the central charges appearing in the type IIA and IIB supersymmetry algebras). In consequence, if we toroidally compactify only one dimension, none of them can be self-dual under the inversion of the compactification radius, but they turn out to be dual to each other.

Another type of dualities involves the inversion of the string coupling $\tilde{g}_s = 1/g_s$. It is called S-duality. The canonical example is the self-duality of the type IIB string theory under such a transformation. Another example is the S-duality between type I superstring theory and heterotic string theory with the same gauge group $SO(32)$. There, one has to pay attention that some perturbative states on one side are mapped to D-brane states on the other side, making D-branes an essential ingredient of the duality. Finally, when both kinds of inversions are involved, we speak about U-duality.

Since we lack a complete understanding of the non-perturbative régime of superstring theory, it is difficult to prove that these dualities are indeed correct, since there are few states where one can follow the evolution of one state all the way from the perturbative régime of one theory to the perturbative régime of the other, where things are under control again. Fortunately, some brane states have masses that are protected from quantum corrections by supersymmetry. This happens when they are annihilated by some fraction of the supercharges. Then, they satisfy a so-called BPS (Bogomolnyi-Prasad-Sommerfeld) bound:

$$m^2 = Z^2 \,, \tag{1.1}$$

that relates their mass to their charge. The study of such states has been very useful in testing duality conjectures.

There are also ways to construct non-BPS D-branes, such that they are the lowest-mass states with a certain set of quantum numbers, so that we can expect them to remain stable when we raise the value of the coupling constant. Such states are usually constructed out of a Dp-brane and the corresponding anti-Dp-brane, which are usually unstable since they exchange tachyon states. However, there is a nice way to remove the instability called tachyon condensation, which actually prompted a revival of interest in string field theory, since it is a proper framework to discuss questions of vacuum decay.

Furthermore, the study of the physics of D-branes together with the finding that there should be an eleven-dimensional theory that is the large coupling limit of type IIA string theory also revived the interest in matrix models, making it plausible that M-theory could be (at least, partly) described by such kind of models. I will describe these ideas in the next section.

1.5 Matrix models for M-theory?

Here, I want to develop the idea that the second superstring revolution drastically changed our understanding of the meaning of matrix models, nearly as much as it changed our understanding of string theory. Simple matrix models of the kind presented in chapter 2 were already considered in the late 80's in the context of string theories, but they could only describe systems living in dimensions smaller or equal to 2, making them interesting for the study of conformal field theories coupled to gravity, but not really well-suited to give a non-perturbative description of the ten-dimensional superstring theories. On the other hand, models of the kind presented in chapter 3 and 4 of this thesis were already considered before, although they were not aimed at describing superstring theories.

More specifically, the same Hamiltonian as in the BFSS matrix model was introduced as a matrix regularization of the dynamics of a supermembrane, a connection we will describe in details in section 3.3. However, at that time, it wasn't clear that supermembrane theories could be relevant to the study of non-perturbative string theories and it wasn't clear neither that the same physical theory also described the low-energy dynamics of D0-branes in type IIA string theory. After it was understood that the theory had (unlike the Green-Schwarz superstring) a continuous spectrum of states, it was essentially forgotten for 7 years until the time came when all these connections could be made. More precisely, the second superstring revolution led to the idea that we could decouple all degrees of freedom of type IIA superstring theory except the D0-branes by going to the infinite momentum frame, so that

M-theory in this limit could be described by the simple theory governing the low-energy dynamics of D0-branes. Although it is unessential in this argument, the fact that this theory could also be seen as describing M-theory supermembranes in this particular frame gave a further motivation for studying this model and check whether it could reproduce eleven-dimensional supergravity results.

On the other hand, matrix models of the type discussed in chapter 4 of this thesis were also considered earlier in the rather different context of Yang-Mills theories. Indeed, the study of the large N limit of Yang-Mills integrals showed that many results could be obtained by compactifying all spacetime dimensions to a single point. Such 0-dimensional matrix theories were called totally reduced models. Again, when it became clear from the relation between dimensionally reduced Super Yang-Mills theories and D-brane low-energy dynamics that such an action could also describe type IIB D-instantons, it was understood that it could be related to type IIB superstrings. More precisely, it was shown that the totally reduced super Yang-Mills action can be obtained as the matrix regularization of a type IIB superstring in the Schild gauge. I will also explain this connection thoroughly in chapter 4.

1.6 Matrix models based on $\mathfrak{osp}(1|32,\mathbb{R})$

Since matrix models where the matrix eigenvalues generate the spacetime manifold give an algebraic rather than geometric picture of spacetime, another possible idea is to try to construct a matrix model directly from algebraic considerations and to formulate it in a background-independent way so that it can be considered in situations with various target-spacetime dimensions. If such a model is to be related to M-theory, we should endow it with an algebraic structure inspired from that of eleven-dimensional supergravity. Furthermore, to avoid the puzzle linked to the absence of transverse M5-branes in supersymmetric matrix quantum mechanics, we want to formulate it in such a way that membranes and M5-branes are treated as equally fundamental objects. A possible choice fulfilling these wishes is a model based on the the $\mathfrak{osp}(1|32,\mathbb{R})$ supersymmetry algebra, the minimal supersymmetric extension of the eleven-dimensional Poincaré superalgebra. Though various supersymmetric action can be made out of $\mathfrak{osp}(1|32,\mathbb{R})$ supermatrices, we figured out that a mass term together with a cubic interaction seems to lead to the most interesting results.

Together with Luca Carlevaro and Adel Bilal, we studied such a model in twelve-dimensional and eleven-dimensional contexts. In the latter case, we showed how to perform the infinite momentum frame limit and used the matrix version of T-duality to obtain a supersymmetric matrix quantum mechanics. The constrained fields can be eliminated to obtain a quartic potential for the D0-branes out of the initial cubic potential. However, since we cannot find a closed form for the solutions of the constraints, we use a perturbative technique that leads to an infinite tower of higher-order interactions amongst the physical fields, an apparently very complicated theory. On the other hand, the lowest-order terms reproduce those of the BFSS theory with an additional mass term together with interaction terms involving the 5-brane degrees of freedom. This research was published in [19].

In another research paper detailed in chapter 6, the product of a research collaboration with Takehiro Azuma of Kyôto University, we study the same model, this time expressed as a totally reduced ten-dimensional matrix model. In this context, in absence of a Hamiltonian formulation, there is no direct way to discriminate physical from unphysical fields. However, the constrained fields of the preceding chapter appear as tachyonic fields in this approach. This evidently leads to the conclusion that the trivial vacuum is unstable for these fields. Searching for non-trivial solutions of their equations of motion, we find that they can describe vacua of the fuzzy sphere type. We study two examples in details, the triple fuzzy 2-sphere and the maximally symmetric fuzzy 8-sphere, which are two ways of filling the nine space-like directions. We further discuss their respective stability properties. This research was published in [16].

1.7 Outlook

This thesis is organized as follows. The first part is constituted by the three review chapters on matrix theories, which contain the necessary background to understand which ideas led us to the research projects presented in the second part of this thesis in chapter 5 and 6.

More precisely, chapter 2 aims at giving a first simple example of a duality between a matrix model and a string theory. In particular, we compute a critical exponent, the string susceptibility, on both sides of the correspondence to figure out which specific conformal field theories of matter coupled to low-dimensional quantum gravity can be described by which critical points of the matrix partition function.

Chapter 3 is dedicated to the study of supersymmetric matrix quantum mechanics. That part of the text aims at showing how M(atrix) theory can be considered either as a low-energy effective action for D0-branes or as a matrix regularization of a supermembrane theory in eleven dimensions in the light-cone gauge. The BFSS conjecture on M-theory is also explained there, as well as the puzzle regarding the absence of transverse M5-branes in this theory.

Chapter 4 is an introduction to the totally reduced IIB matrix model. In parallel to the discussion in chapter 3, it shows why the IIB matrix model can be considered either as a low-energy effective action for D-instantons or as the regularization of the Green-Schwarz type IIB superstring action in the Schild gauge. The on-shell closure of the supersymmetry algebra and the D-brane spectrum are also derived there. Finally, it is shown how to deform it slightly to obtain non-commutative curved-space classical solutions of the fuzzy sphere type.

Chapter 5 contains a study of a matrix model with $\mathfrak{osp}(1|32,\mathbb{R})$ supersymmetry. In particular, a supersymmetric matrix quantum mechanics is obtained after the elimination of certain unphysical fields. It is compared to the BFSS matrix model and conjectured to be related to the D0-brane dynamics in a curved background of anti-de Sitter type with non-trivial M5-branes fluxes.

Chapter 6 studies the same model in a different context, where it has similarities with the massive IIB matrix model. In particular, non-commutative classical solutions of the fuzzy sphere type are obtained and their stability is analyzed.

Finally, a few conclusive remarks are given in chapter 7, before going to the technical appendices, where some of the notation used are also introduced (in appendices A and B).

Chapter 2

Matrix models for quantum gravity

Although the results and ideas contained in this chapter are not really needed to understand the following ones, we have included it since it provides an interesting first example of a duality between a matrix theory and a string theory. Furthermore, it illustrates quite well how a matrix theory can be seen as a discretization of the string world-sheet, while the large N limit will be the continuum limit of the discretization. We will use this idea twice more later when we will show in chapter 3 how the BFSS matrix theory can be seen as a matrix regularization of the world-volume theory of a supermembrane and in chapter 4 when we will show how the IKKT matrix model can be derived from a regularization of the world-sheet of the Green-schwarz superstring.

The goal of this chapter is essentially to compute a critical exponent, the string susceptibility, both on the matrix theory and string theory side to show how different kinds of critical points of the matrix model partition function describe various conformal field theories on the string world-sheet, representing low-dimensional quantum gravity coupled to some matter fields.

With this goal in mind, we first explain the discretization procedure and why we expect the sum over all random triangulations of the string world-sheet to reproduce in the large N limit the genus expansion of the string theory partition function. Then, we introduce a computational method, the so-called orthogonal polynomial method, that allows us to compute matrix integrals in a convenient way. Then, we first take the simplest large N limit and show how it predicts the critical behaviour of the partition function in the neighbourhood of the critical coupling. The result we obtain for the string susceptibility corresponds to the genus 0 contribution on the string theory side. Then, we take a more subtle large N double scaling limit that keeps contributions of Riemann surfaces of all genera to the partition function.

In the second part of this chapter, we explain the quantization of a bosonic string in a spacetime of non-critical dimension and how the requirement of conformal symmetry leads to the Liouville action. Finally, we use a scaling argument to compute the string susceptibility and discuss the significance of its possible values and how they relate to specific critical points of the matrix theory partition function.

2.1 Old matrix models & dynamical triangulations

Matrix models were first used in the context of quantum gravity around 1990, when people thought of describing low-dimensional quantum gravity through matrix integrals [73, 49, 102, 101]. The idea behind such models is to discretize the string world-sheet, before taking a continuum limit that would include naturally the contributions of surfaces of all possible genera, thus predicting the genus expansion of perturbative string theory from a non-perturbative theory. The simplest of such models is defined through the discretization of 0-dimensional string theory, whose action is:

$$Z = \sum_{g_S=0}^{\infty} \int \mathcal{D}h e^{-\beta A + \gamma\chi} , \tag{2.1}$$

where h_{ab} is the (euclidean) metric tensor of determinant h on the Riemann surface parametrized by the two coordinates σ^1 and σ^2, $A = \int \sqrt{h} d^2\sigma$ its area, while its genus g_S is related to the Euler characteristic χ through the Gauss-Bonnet theorem:

$$\chi = \frac{1}{4\pi} \int R\sqrt{h}\, d^2\sigma = 2 - 2g_S \tag{2.2}$$

and R is the Ricci curvature of the surface. The metric integration is defined as $\mathcal{D}h = \mathcal{D}h_{11}\mathcal{D}h_{12}\mathcal{D}h_{22}$. On the other hand, β and γ are real parameters ("coupling constants"). We then wish to replace both the metric path integration and the infinite sum over surface topologies by a single sum over all possible discrete geometries of the string world-sheet. It turns out that we can obtain interesting critical behaviour, even if we restrict the possible discretization to equilateral triangulations only. Since a flat hexagon can be covered by 6 equilateral triangles, we can represent positive (resp. negative curvature) at each vertex by the coincidence of less than (resp. more than) 6 triangles. We denote here by N_i the number of triangle touching the vertex i

In appropriate units, the area is simply given by the total number of triangles, i.e. $A = 1/3 \sum_i N_i$, since each triangle has three vertices. The discrete definition of χ is known to be $\chi = V - E + F$, where V, E and F denote the number of vertices, edges and faces, respectively. In a triangular discretization, each face is surrounded by 3 edges, while each edge separates two faces. In consequence, there is a relation $3F = 2E$ that gives the Euler characteristic as: $\chi = V - 1/2F$. Furthermore, V is simply given by $\sum_i 1$, while F is given by $F = A = 1/3\sum_i N_i$, since there is exactly one face per triangle. We thus obtain: $\chi = \sum_i (1 - N_i/6)$ On the other hand, $\chi = \frac{1}{4\pi} \int R\sqrt{h}\, d^2\sigma \to \frac{1}{4\pi}\sum_i (N_i/3)R_i$, so that the discretized Ricci curvature at vertex i is:

$$R_i = \frac{(1 - N_i/6)}{\frac{1}{4\pi} N_i/3} = 2\pi(6 - N_i)/N_i \, . \tag{2.3}$$

2.1.1 Matrix model approximation

Our next task is to construct a matrix integral that produces a diagrammatic expansion that generates all such random triangulations [48]. Of course, triangular vertices are produced by trilinear couplings, while edges (propagators) are produced by bilinear terms. We thus need to consider a matrix integral of the form:

$$\int dM e^{-Tr(\frac{1}{2}M^2 + \frac{g}{\sqrt{N}}M^3)} , \tag{2.4}$$

where M is some $N \times N$ hermitian matrix, g is the coupling constant,

$$dM = \prod_{i=1}^{N} dM_i^i \prod_{i,j=1,i<j}^{N} dReM_j^i \, dImM_j^i \ , \tag{2.5}$$

and we normalize the result by:

$$\int dM e^{-Tr(\frac{1}{2}M^2)} = 1 \ . \tag{2.6}$$

Of course, this integral can be expressed as:

$$\sum_{k=0}^{\infty} (\frac{g}{\sqrt{N}})^k \int dM \ (Tr(M^3))^k \ e^{-Tr(\frac{1}{2}M^2)} \ , \tag{2.7}$$

which is a sum over diagrams containing k vertices. The contribution of each of these diagrams can be computed thanks to:

$$\begin{aligned}\int dM \ M_{j_1}^{i_1} M_{j_2}^{i_2} \cdots M_{j_n}^{i_n} \ e^{-Tr(\frac{1}{2}M^2)} &= \frac{\partial}{\partial J_{i_1}^{j_1}} \cdots \frac{\partial}{\partial J_{i_n}^{j_n}} \int dM \ e^{-Tr(\frac{1}{2}M^2 + JM)}|_{J=0} = \\ &= \frac{\partial}{\partial J_{i_1}^{j_1}} \cdots \frac{\partial}{\partial J_{i_n}^{j_n}} \ e^{Tr(\frac{1}{2}J^2)}|_{J=0} \ ,\end{aligned} \tag{2.8}$$

through the use of the usual generating functional method. This diagrammatic expansion gives a surface with trilinear vertices and diverse polygonal faces, all closed thanks to the matrix traces. By putting a vertex on each face of this diagram and by connecting together with edges all those pair of new vertices that correspond to faces that share a boundary, we obtain the so-called dual diagram, that has triangular faces, but diverse types of vertices. This is of course, the desired random triangulation of a Riemann surface. Since the dualization exchanges the rôles of faces and vertices, the diagram with k vertices of order g^k in the matrix integral corresponds to a random triangulation with k faces in the discretized string theory picture, i.e. to a Riemann surface of area k, which allows us to make the formal identification $g = e^{-\beta}$. Furthermore, because of the multiple matrix traces, the matrix integral generates connected as well as disconnected diagrams. The free energy from the matrix point of view is thus the partition function from the string theory point of view. The precise correspondence is then:

$$e^Z = \int dM e^{-Tr(\frac{1}{2}M^2 + \frac{g}{\sqrt{N}}M^3)} \ , \tag{2.9}$$

Note that a rescaling $M \to \sqrt{N}M$ gives the action as: $NTr(1/2M^2 - gM^3)$ with an overall factor of N. In this formulation, we see that each propagator (edge) is accompanied by a factor of $1/N$, while each vertex of a diagram brings a factor of N. On the other hand, each closed loop (building a face in the diagram) is described by a trace, which brings another factor of N. in consequence, each diagram carries an overall factor of:

$$N^{V-E+F} = N^{\chi} = N^{2-2g_S} \ , \tag{2.10}$$

As χ remains unchanged under the dualization of the diagram, which exchanges V and F, while keeping E, this remains true for the discretization of the world-sheet. In consequence, we have the correspondence: $N = e^{\gamma}$. Furthermore, we can arrange the result:

$$Z(g) = \sum_h N^{2-2h} Z_h(g) \tag{2.11}$$

as a sum over topologies.

2.1.2 The orthogonal polynomial method

There is an efficient method to solve simple matrix models called the orthogonal polynomial method [39]. The idea behind it is to diagonalize the matrix M and integrate over its eigenvalues and the diagonalization matrix instead of the elements of M. Since M is Hermitian, it can be diagonalized by a unitary transformation described by some $U(N)$ matrix U through: $M = U^\dagger D U$, where D is some real diagonal matrix. Since the Jacobian of the variable transformation only depends on infinitesimal variations of D, it can be calculated using the linear approximation: $U = \mathbb{1} + iT + \ldots$, where T is a Hermitian matrix, so that $U^\dagger = \mathbb{1} - iT + \ldots$. In particular, $M = D + i[D,T] + \ldots$. In terms of elementary matrices $E_i{}^j = \vec{e}_i \otimes \vec{e}^{*j}$ (that are defined to have only one non-zero element equal to one at line i and column j), T can be described as: $T = \sum_{i,j} \epsilon^i{}_j E_i{}^j$ with $\epsilon^i{}_j \in \mathbb{C} \forall i,j = 1,\ldots,N, \epsilon_i{}^j = (\epsilon^i{}_j)^*$, while D can be simply described as: $D = \sum_i \lambda_i E_i{}^i$. To derive the Jacobian of the transformation, noting that the (m,n)-th element of $E_i{}^j$ is given by: $(E_i{}^j)^m{}_n = \delta_i^m \delta_j^n$, we should first compute:

$$(E_i{}^j E_k{}^l)^m{}_n = \sum_o \delta_i^m \delta_o^j \delta_k^o \delta_n^l = (E_i{}^l)^m{}_n \delta_k^j \,, \tag{2.12}$$

so that:

$$[E_i{}^j, E_k{}^l] = E_i{}^l \delta_k^j - E_k{}^j \delta_l^i \,. \tag{2.13}$$

In particular, this implies that:

$$[D,T] = \sum_{i,j} \lambda_i (\epsilon^i{}_j E_i{}^j - \epsilon^j{}_i E_j{}^i) \,, \tag{2.14}$$

so that:

$$([D,T])^i{}_j = (\lambda_i - \lambda_j)\epsilon^i{}_j \,. \tag{2.15}$$

This means that M depends only on $\epsilon^i{}_j$ for $i \neq j$, which gives the correct number of independant variables, with the N^2 real independant components of the Hermitian matrix M replaced by the N real eigenvalues λ_i's and the $N(N-1)/2$ complex parameters $\epsilon^i{}_j$ for $i < j$. Thus, while M depends on λ as:

$$\begin{aligned} \frac{\partial M^i{}_i}{\partial \lambda_k} &= \delta_i^k \text{ and} \\ \frac{\partial M^i{}_j}{\partial \lambda_k} &= i\epsilon^i{}_j(\delta_i^k - \delta_j^k) \text{ for } i \neq j \,, \end{aligned} \tag{2.16}$$

M depends on ϵ as:

$$\begin{aligned} \frac{\partial M^i_i}{\partial \epsilon^j_k} &= 0 \text{ and} \\ \frac{\partial M^i_j}{\partial \epsilon^k_l} &= i(\lambda_i - \lambda_j)\delta^i_k \delta^l_j \text{ for } i \neq j \,. \end{aligned} \tag{2.17}$$

Thanks to $\partial M^i_{\ i}/\partial\epsilon^j_k = 0$, $\partial M^i_{\ j}/\partial\lambda_k$ doesn't contribute to the determinant, while $\partial M^i_{\ i}/\partial\lambda_k$ contribute as 1, thus reducing it to:

$$J = \sum_{\pi\in\mathcal{S}_{N^2-N}} (-1)^{|\pi|} \prod_{i,j=1,i\neq j}^{N} \frac{\partial M^i_{\ j}}{\partial\epsilon^{\pi(i)}_{\ \ \pi(j)}} = \prod_{i,j=1,i<j}^{N} (\lambda_i-\lambda_j)^2 \ . \tag{2.18}$$

Furthermore, the action doesn't depend on the ϵ_{ij}'s, which reduces the matrix integral computation to:

$$\int \prod_i d\lambda_i \prod_{i,j=1,i<j}^{N} (\lambda_i-\lambda_j)^2 e^{-V(\lambda)} \ , \tag{2.19}$$

In fact this product of eigenvalue differences can be written as a determinant, called the Vandermonde determinant in the literature. Taking $N=3$ for illustration, the usual properties of the determinant allows to rewrite the jacobian as $\det(\lambda_i^{j-1})$ since:

$$\det(\lambda_i^{j-1}) = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{vmatrix} = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 0 & \lambda_2-\lambda_1 & \lambda_2^2-\lambda_1^2 \\ 0 & \lambda_3-\lambda_1 & \lambda_3^2-\lambda_1^2 \end{vmatrix} = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 0 & \lambda_2-\lambda_1 & \lambda_2^2-\lambda_1^2 \\ 0 & 0 & (\lambda_3-\lambda_1)(\lambda_3-\lambda_2) \end{vmatrix} ,$$

so that for general N:

$$\Delta(\lambda) = \prod_{i,j=1,i<j}^{N} (\lambda_i-\lambda_j) = \det(\lambda_i^{j-1}) \tag{2.20}$$

To compute the integral, we now have to introduce an infinite set of polynomials $\{P_n(\lambda)\}_{n=1}^{\infty}$ orthogonal with respect to the measure:

$$\int d\lambda e^{-V(\lambda)} P_m(\lambda)P_n(\lambda) = h_n\delta_{mn} \ , \tag{2.21}$$

normalized so that $P_n(\lambda) = \lambda^n + \ldots$, hence the constant h_n. We can then replace $\det(\lambda_i^{j-1})$ by $\det(P_{j-1}(\lambda_i))$, since adding polynomial of lower orders in each column doesn't change the determinant. Again, we illustrate this idea for $N=3$:

$$\det(\lambda_i^{j-1}) = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{vmatrix} = \begin{vmatrix} 1 & \lambda_1+c & \lambda_1^2-k\lambda_1+d \\ 1 & \lambda_2+c & \lambda_2^2-k\lambda_2+d \\ 1 & \lambda_3+c & \lambda_3^2-k\lambda_3+d \end{vmatrix} = \det(P_{j-1}(\lambda_i))$$

for any kind of polynomials normalized as $P_n(\lambda) = \lambda^n + \ldots$. Given that property, the task of computing the partition function reduces to the computation of:

$$\int \prod_i d\lambda_i (\det(P_{j-1}(\lambda_i)))^2 e^{-V(\lambda)} = \int \prod_i d\lambda_i \left(\sum_{\pi\in\mathcal{S}_N} \prod_{i=1}^{N} (P_{i-1}(\lambda_{\pi(i)})) \right)^2 e^{-V(\lambda)} \ . \tag{2.22}$$

In each term of the sum, the integrals on the λ_i factorize, giving non-zero results only for terms containing $\prod_{i=1}^N P_{i-1}^2(\lambda_{\pi(i)})$, thanks to the orthogonality property. In other words, cross-terms do not contribute to the result. Furthermore, each term contributes the same $\prod_{i=1}^{N} h_{i-1}$ and there are $N!$ of them. We finally obtain:

$$e^Z = N! \prod_{i=0}^{N-1} h_i = N! h_0^N \prod_{k=1}^{N-1} f_k^{N-k} , \tag{2.23}$$

where $f_k = h_k/h_{k-1}$. Of course, the values of the f_k depend on the measure, or in other words, on the specific matrix model potential.

In the naive (planar) large N limit, we set: $\chi = \lim_{N\to\infty} k/N$ and $f(\chi) = \lim_{N\to\infty} f(k/N) \equiv f_k$. In this limit, we obtain:

$$\frac{1}{N^2} Z = \frac{1}{N^2}\left(\ln(N!) + N\ln(h_0) + \sum_{k=1}^{N-1}(N-k)\ln(f_k)\right) \stackrel{N\to\infty}{\longrightarrow} C + \int_0^1 (1-\chi)\ln(f(\chi))d\chi . \tag{2.24}$$

To determine $f(\chi)$ more precisely, note first that: $\lambda P_n(\lambda) = \sum_{i=0}^{n+1} a_i P_i(\lambda)$ for:

$$a_i = h_i^{-1} \int d\lambda e^{-V(\lambda)} \lambda P_n(\lambda) P_i(\lambda) . \tag{2.25}$$

For even potentials, $\int d\lambda e^{-V(\lambda)}\lambda P_n(\lambda)P_n(\lambda) = 0$, so that $a_n = 0$. Since also $\int d\lambda e^{-V(\lambda)}\lambda P_i(\lambda)P_n(\lambda) = 0$ for all $i < n-1$, λP_n is simply:

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + r_n P_{n-1}(\lambda) \tag{2.26}$$

for some scalar coefficient r_n.

Furthermore:

$$h_n = \int d\lambda e^{-V(\lambda)}\lambda P_{n-1}(\lambda)P_n(\lambda) = \int d\lambda e^{-V(\lambda)}\lambda P_n(\lambda)P_{n-1}(\lambda) = r_n h_{n-1} , \tag{2.27}$$

so that $r_n = f_n$. Similarly, differentiating (2.26), we obtain:

$$\lambda P_n'(\lambda) = P_{n+1}'(\lambda) - P_n(\lambda) + r_n P_{n-1}'(\lambda) , \tag{2.28}$$

so that (using $P_{n+1}'(\lambda) = (n+1)P_n(\lambda) + \sum_{i=0}^{n+1} a_i P_i(\lambda)$):

$$nh_n = \int d\lambda e^{-V(\lambda)}\lambda P_n'(\lambda)P_n(\lambda) = \int d\lambda e^{-V(\lambda)}P_n'(\lambda) r_n P_{n-1}(\lambda) = r_n \int d\lambda e^{-V(\lambda)} V'(\lambda)P_n(\lambda)P_{n-1}(\lambda) , \tag{2.29}$$

after integration by parts.

2.1.3 The genus zero partition function

We now want to use (2.29) to find an expression for $f_n = r_n$ and consequently, for the partition function. Take for example the potential:

$$V(\lambda) = \frac{1}{2g}\left(\lambda^2 + \frac{\lambda^4}{N} + b\frac{\lambda^6}{N^2}\right) , \tag{2.30}$$

with derivative:

$$gV'(\lambda) = \lambda + 2\frac{\lambda^3}{N} + 3b\frac{\lambda^5}{N^2} \ . \tag{2.31}$$

To compute (2.29), we need to calculate integrals of the form: $\int d\lambda e^{-V}\lambda^{2p-1}P_nP_{n-1}$. Using repeatedly (2.26), we can rewrite:

$$\lambda^{2p-1}P_n = \sum_{i=n-2p+1}^{n+2p-1} a_i P_i \ , \tag{2.32}$$

where a_i is a homogeneous polynomial of order $(p+(n-1-i)/2)$ in the variables $r_{n-2p},\dots,r_{n+2p}$ containing $\binom{2p-1}{i}$ positive terms for odd i's and $a_i = 0$ for even i's. Together with the orthogonality relation, this reduces (2.29) to:

$$\begin{aligned} ng = r_n\Big(1 + &\frac{2}{N}(r_{n+1}+r_n+r_{n-1}) + \\ +\frac{3b}{N^2}\{r_{n+1}(r_{n+2}+r_{n+1}+r_n) + &r_n(r_{n+1}+r_n+r_{n-1}) + r_{n-1}(r_{n+1}+r_n+r_{n-1}+r_{n-2})\}\Big) \ . \end{aligned} \tag{2.33}$$

In the large N limit, with $r_{n\pm1}/N \to r(\chi\pm\epsilon)$ for $\epsilon = 1/N$, this goes to:

$$g\chi = r(\chi)(1+6r(\chi)+30br^2(\chi)) + \mathcal{O}(1/N) \text{ corrections } = W(r(\chi)). \tag{2.34}$$

Expanding $W(r)$ near a critical point r_c as: $W(r) = g_c + 1/2W''(r_c)(r-r_c)+\dots$, we can guess the critical behaviour as[1]:

$$r(\chi) - r_c \propto |g_c - g\chi|^{1/2} \ . \tag{2.36}$$

Since (2.27) implies that $r(\chi) = f(\chi)$, we obtain from (2.24) the large N behaviour of the partition function near the critical coupling g_c.

For general even polynomial $V(\lambda) = 1/(2g)\sum_{p=1}^m b_p\lambda^{2p}$, the remark below (2.32) tells us that a_{p-1} will always be a p-th order homogeneous polynomial containing $\binom{2p-1}{p}$ positive terms, so that:

$$g\chi = \sum_{p=1}^{m}\frac{(2p-1)!}{((p-1)!)^2}b_p r^p(\chi) + \mathcal{O}(1/N) \text{ corrections } \ . \tag{2.37}$$

Through fine-tuning enough of the parameters b_p, we can in principle reach an m-th order critical point where $g\chi = W(r)$ is such that all derivatives $W^{(i)}(r=r_c)$ vanish $\forall i = 1,\dots,m-1$ [114]. This more general critical behaviour is then:

$$r(\chi) - r_c \propto |g_c - g\chi|^{1/m} \ . \tag{2.38}$$

This relation[2] defines the critical exponent $\gamma = -1/m$, usually called string susceptibility.

[1]If $b = 0$, we can solve exactly for $r(\chi)$ and we obtain:

$$r - r_c = (1/\sqrt{6})\,(g\chi - g_c)^{1/2} \ . \tag{2.35}$$

for $g_c = -1/24$ and $r_c = -1/12$.

[2]In our example, if we set b at the critical value $b_c = 6/15$, we have $r_c = -1/6$, $g_c = -1/18$ and the tricritical behaviour is then:

$$r(\chi) - r_c = (1/\sqrt[3]{12})\,(g\chi - g_c)^{1/3} \ . \tag{2.39}$$

In such one-matrix models where $f(\chi) = r(\chi)$, the leading singular behaviour of f for large N is then:

$$f(\chi) - f_c \propto |g_c - g\chi|^{-\gamma} \, . \tag{2.40}$$

For $g \sim g_c$, the g-dependent part of the partition function is determined by the region where $\chi \sim 1$, where the above estimate is valid. Putting (2.40) in (2.24) and using $\ln(f) = \ln(f_c) + \ln(1 - a|g_c - g\chi|^{-\gamma}) \propto |g_c - g\chi|^{-\gamma}$ this determines the behaviour of the partition function in the neighbourhood of the critical coupling as:

$$\begin{aligned}\frac{1}{N^2} Z &\sim \int_0^1 (1-\chi)|g_c - g\chi|^{-\gamma} d\chi \sim (1-\chi)|g_c - g\chi|^{-\gamma+1}\Big|_0^1 + \int_0^1 (1-\chi)|g_c - g\chi|^{-\gamma+1} d\chi \quad \sim \\ &\sim |g_c - g\chi|^{-\gamma+2} = g_c^{2-\gamma} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\Gamma(3-\gamma)}{\Gamma(3-\gamma-n)} (g/g_c)^n \end{aligned} \tag{2.41}$$

For large n, the Stirling's formula predicts an asymptotic of the type: $n^{\gamma-3}$ for the coefficients, which determines the contributions of surfaces of large area, i.e. big n, to the partition function, a result that we will test again a string calculation in the end of this chapter to understand which critical points correspond to which kind of non-critical string theories in the continuum.

Of course, there is no full proof that the large N limit of matrix integrals is able to reproduce the properties of the continuum theory, but there is a variety of tests supporting this conjecture (see [115] for a review).

2.1.4 The all genus partition function

To discuss the contribution of higher genus surfaces, we should keep higher terms in the $1/N$ expansion. Coming back to the example treated above, this means that we shall consider the complete expression:

$$\begin{aligned} g\chi &= r(\chi)\Big(1 + 2(r(\chi+\epsilon) + r(\chi) + r(\chi-\epsilon)) + 3b\Big\{r(\chi+\epsilon)(r(\chi+2\epsilon) + r(\chi+\epsilon) + r(\chi)) + \\ &\quad + r(\chi)(r(\chi+\epsilon) + r(\chi) + r(\chi-\epsilon)) + r(\chi-\epsilon)(r(\chi+\epsilon) + r(\chi) + r(\chi-\epsilon) + r(\chi-2\epsilon))\Big\}\Big) = \\ &= r(\chi)(1 + 6r(\chi) + 30b(r(\chi))^2) + \epsilon^2 r(\chi)\Big(2r''(\chi) + 15b\Big(2r(\chi)r''(\chi) + (r'(\chi))^2\Big)\Big) + \\ &\quad + \frac{\epsilon^4}{12} r(\chi)\Big(2r^{(4)}(\chi) + 33b\Big(2r(\chi)r^{(4)}(\chi) + 4r'(\chi)r^{(3)}(\chi) + 3(r''(\chi))^2\Big)\Big) + \mathcal{O}(\epsilon^6) \, . \end{aligned} \tag{2.42}$$

To obtain an interesting large N limit, we want to transform the large N expansion (2.11) into an expansion of the type:

$$Z(g) = \sum_{h=0}^{\infty} \kappa^{2h-2} Z_h(g) \, , \tag{2.43}$$

where κ is kept fixed in the limiting procedure. This is possible if the partition function scales in an appropriate way near critical coupling. We already know that:

$$Z(g) = N^2(|g - g_c|^{5/2} c + \ldots) + \sum_{h=1}^{\infty} N^{2-2h} Z_h(g) \, , \tag{2.44}$$

which suggests to take $\kappa^{-2} = N^2|g - g_c|^{5/2}$. In other words, we want to approach critical radius as:

$$g - g_c = \kappa^{-4/5} N^{-4/5} \tag{2.45}$$

while keeping κ fixed at some finite value. To obtain a simple-looking result, we introduce a new variable x related to χ through: $g\chi - g_c = g^{-4/3}(1/2W''(r_c))^{-1/5}(-6r_c(1+15br_c))^{2/5}N^{-4/5}x$ and the rescaled functional:

$$l(x) = g^{2/3}(1/2W''(r_c))^{3/5}(-6r_c(1+15br_c))^{-1/5}N^{2/5}(r(\chi) - r_c) \tag{2.46}$$

so that:

$$\epsilon \frac{\partial}{\partial \chi} \propto N^{-1/5} \frac{\partial}{\partial x} \ . \tag{2.47}$$

With these rescaled variables the expansion (2.42) turns into:

$$x = l^2(x) - 1/3l''(x) + O(N^{-2/5}) \ . \tag{2.48}$$

In this improved large N limit, the behaviour of the partition function containing the contribution of surfaces with all genera is thus determined by the solutions of the Painlevé I differential equation: $x = l^2(x) - 1/3l''(x)$. Its perturbative solution with leading term $x^{1/2} \propto \kappa^{-2/5}$ is of the form:

$$l(x) = x^{1/2}(1 - \sum_{k=1}^{\infty} u_k x^{-5k/2}) \ , \tag{2.49}$$

where the u_k are all positive, so that the leading term in $Z(\kappa)$ will be of order $x^{5/2} \propto \kappa^{-2}$ as anticipated in (2.43). From (2.46), $l(x) \propto N^{2/5}(r(\chi) - r_c)$ and we can obtain the critical behaviour from (2.24). For large k, the u_k grow as $(2k)!$, so that the perturbative solution is not Borel summable. This might lead to wonder whether perturbative agreement between matrix models and gravity extends to a full non-perturbative equivalence or not.

For the higher order critical point reached by tuning b to critical value $b_c = 6/15$, since $Z_0(g) = aN^2|g - g_c|^{7/3} + \ldots$, we set $\kappa^{-2} = N^2|g - g_c|^{7/3}$. Again, to make the result look simpler, we introduce a new variable through: $g\chi - g_c = 18^{-1/7}(g/N)^{6/7}x$ and rescale:

$$\tilde{l}(x) = (864)^{1/7}(N/g)^{2/7}(r(\chi) - r_c) \ , \tag{2.50}$$

which leads to another differential equation:

$$x = \tilde{l}^3(x) - \tilde{l}(x)\tilde{l}''(x) - 1/2(\tilde{l}'(x))^2 + \alpha \tilde{l}^{(4)}(x) \ . \tag{2.51}$$

where $\alpha = 1/10$ with our choice of potential. We can also solve this differential equation through a perturbative expansion, which will be of the form:

$$\tilde{l}(x) = x^{1/3}(1 - \sum_{k=1}^{\infty} v_k x^{-7k/3}) \ , \tag{2.52}$$

with $x^{-7k/3} \propto \kappa^{2k}$ as it should be. The signs of the v_k depend on the particular value of α. They are all positive for $\alpha < 1/12$, in which case such matrix models describe the unitary theory of quantum gravity in $2D$ coupled to the Ising model. On the contrary, our specific model is not unitary. It rather describes the Yang-Lee edge singularity coupled to gravity.

For a general even potential, we can reach a critical point of order m, in other words with critical exponent $\gamma = -1/m$, where we will have $Z_0(g) = aN^2|g-g_c|^{2-\gamma} + \ldots$. In that case, we will take: $\kappa^{-2} = N^2|g-g_c|^{2-\gamma}$, $g\chi - g_c \propto N^{(\gamma-2)/2}x$ and

$$l(x) \propto N^{2\gamma/(\gamma-2)}(r(\chi) - r_c) \ , \tag{2.53}$$

which will lead to the m^{th}-order hamiltonian density for the Korteweg-de Vries hierarchy of differential equations. We will not explain this connection here, but rather turn our attention to the corresponding string calculations, to establish a relation between m and the number of fields that we couple to two-dimensional gravity in the Liouville approach.

2.2 Non-critical string theory and the Liouville action

Let us here sketch how to treat non-critical bosonic string theory, when $D \neq 26$ [140, 58, 45, 46, 88, 87, 86, 85]. In that case, since there is no conformal symmetry to gauge all three independent components of the world-sheet metric, we should treat the conformal factor as an additional dynamical field, described by a Liouville-type action, as we will explain below, following the conformal gauge approach of [70]. Starting from the (euclidean) string theory partition function:

$$Z = \int \frac{\mathcal{D}h\mathcal{D}X}{Vol(Diff)} e^{-S(X;h) - \frac{\mu}{2\pi}\int \sqrt{h}\, d^2\sigma} \ , \tag{2.54}$$

where $Vol(Diff)$ is the volume of the diffeomorphism group of the string world-sheet, h_{ab} is the world-sheet metric, $h = \det(h_{ab})$ its determinant, $\sigma^{1,2}$ are coordinates on the world-sheet and $S(X;h)$ is the bosonic string action given by:

$$S(X;h) = \frac{1}{8\pi}\int h^{ab}\partial_a X^\mu \partial_b X_\mu \sqrt{h}\, d^2\sigma \tag{2.55}$$

for fields X^μ describing the string propagation in a flat D-dimensional space-time. To define the integration, we need to normalize the measures $\mathcal{D}h$ and $\mathcal{D}X$ in an appropriate way. For example, we can require that:

$$\int \mathcal{D}_h(\delta X) e^{-\int \delta X^\mu \delta X_\mu \sqrt{h}\, d^2\sigma} = 1 \ , \tag{2.56}$$

and similarly for $\mathcal{D}h$ (see [77] for more details). More interesting for us here is the fact that the measures, though diffeomorphism-invariant, are not necessarily invariant under conformal transformations $h_{ab} \to e^\omega h_{ab}$. Since the normalization above depends on h,it turns out that:

$$\mathcal{D}_{e^\omega h} X = e^{\frac{D}{48\pi}S_L(\omega)}\mathcal{D}_h X \ , \tag{2.57}$$

where S_L is called the Liouville action, given by:

$$S_L(X;h) = \int (\frac{1}{2}h^{ab}\partial_a\omega\partial_b\omega + R\omega + \mu e^\omega)\sqrt{h}\, d^2\sigma \ . \tag{2.58}$$

This result can be derived diagrammatically, via the Fujikawa method, or via an index theorem [7]. The metric has also a similar anomalous variation under conformal transformation. Without going into too much details, it involves replacing the metric integration $\int \mathcal{D}h_{11}\mathcal{D}h_{12}\mathcal{D}h_{22}/Vol(Diff)$ by an integration over a conformal factor ϕ and ghosts b and c. Indeed, if we choose a representative metric $\hat{h}_{ab}(\tau)$ for each point of the moduli space of metrics relevant to the considered Riemann surface, any metric can be reached through a diffeomorphism f and a conformal transformation parametrized by ϕ through:

$$f^*h = e^{\phi}\hat{h}(\tau)\ . \tag{2.59}$$

Gauge-fixing the diffeomorphism group to $\hat{h}(\tau)$, we obtain:

$$Z = \int d\tau \mathcal{D}_h\phi \mathcal{D}_h X \mathcal{D}_h(gh) e^{-S(X;h)-S_{gh}(b,c;h)-\frac{\mu}{2\pi}\int\sqrt{h}\,d^2\sigma}\ , \tag{2.60}$$

where $\int d\tau$ is the finite dimensional integration on the moduli space of metrics, $\mathcal{D}_h(gh) = \mathcal{D}_h b_{zz}$ $\mathcal{D}_h c^z \mathcal{D}_h b_{\bar{z}\bar{z}} \mathcal{D}_h c^{\bar{z}}$ is the ghost integration and $S_{gh}(b,c;h)$ is the ghost action from the Faddeev-Popov determinant:

$$S_{gh}(b,c;h) = \int (b_{zz}\nabla_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\nabla_z c^{\bar{z}})\sqrt{h}\,d^2\sigma\ . \tag{2.61}$$

The ghost measure also has a conformal anomaly proportional to the Liouville action when we rescale $h \to e^{\omega}h$:

$$\mathcal{D}_{e^{\omega}h}(gh) = e^{-\frac{26}{48\pi}S_L(\omega)}\mathcal{D}_h(gh)\ . \tag{2.62}$$

In standard bosonic string theory, we would set $D = 26$ to obtain a conformally-invariant quantum theory. Here, we are rather interested in obtaining results that can be compared to matrix theory results, in other words, we want to study low-dimensional non-critical string theories. Since it is difficult to compute the Weyl anomaly in $\mathcal{D}_h\phi$, instead of trying to find the correct action for the Liouville mode ϕ that gives rise to a Weyl-invariant theory, we will assume [126, 65] that it takes a Liouville-type form and impose conformal invariance on the overall path integration:

$$Z = \int d\tau \mathcal{D}_{\hat{h}}\phi \mathcal{D}_{\hat{h}} X \mathcal{D}_{\hat{h}}(gh) e^{-S(X;\hat{h})-S_{gh}(b,c;\hat{h})-\int(k_1\frac{1}{2}\hat{h}^{ab}\partial_a\phi\partial_b\phi+k_2\hat{R}\phi+\mu e^{\lambda\phi})\sqrt{\hat{h}}\,d^2\sigma}\ ,$$

to determine the correct constants k_1, k_2 and λ. This is possible since, contrarily to $\mathcal{D}_h\phi$, the integration measure $\mathcal{D}_{\hat{h}}\phi$ does not depend on ϕ. In other words, it's the measure of integration for a free field. Using (2.57) and (2.62), we find:

$$k_1 = \frac{25-D}{96\pi}\ ,\quad k_2 = \frac{25-D}{48\pi}\ . \tag{2.63}$$

In particular, if we rescale $\phi \to \sqrt{12/(25-D)}\phi$ so that its kinetic term is normalized like that of the X's, the two first terms in the action for ϕ are written as:

$$\frac{1}{8\pi}\int(\hat{h}^{ab}\partial_a\phi\partial_b\phi + Q\hat{R}\phi)\sqrt{\hat{h}}\,d^2\sigma \tag{2.64}$$

with:

$$Q = \sqrt{\frac{25-D}{3}}\ , \tag{2.65}$$

so that the leading short-distance behaviour of the operator product expansion of the energy-momentum tensor $T = -1/2\partial\phi\partial\phi + Q/2\partial^2\phi$ is given by:

$$T(z)T(w) \approx 1/2\frac{c_L}{(z-w)^4} + \dots \tag{2.66}$$

with the central charge $c_L = 1 + 3Q^2$, so that the total conformal anomaly indeed vanishes. To determine λ, or rather some related α for the rescaled ϕ field (chosen so that $\mu e^{\alpha\phi}$ appears in the Liouville part of the action), we require that the actual physical metric is indeed $h_{ab} = \hat{h}_{ab}e^{\alpha\phi}$, so that $\int\sqrt{\hat{h_{ab}}}e^{\alpha\phi}d^2\sigma$ is the actual world-sheet surface. For this sake, we impose that this combination is conformally invariant, or in other words, that $e^{\alpha\phi}$ has conformal weight $(1,1)$. This sets:

$$\alpha = \frac{1}{\sqrt{12}}(\sqrt{25-D} - \sqrt{1-D}) . \tag{2.67}$$

In consequence, for $D \leq 1$, Q and α are real and the Liouville theory is well-defined. For $D \geq 25$, both α and Q are imaginary and we should Wick rotate $\phi \to -i\phi$ to get a real physical metric. This gives a time-like kinetic term to ϕ. Precisely for $D = 25$, ϕ can be seen as a free time coordinate in target-space and we recover the critical bosonic string theory. However, in the most interesting régime $1 < D < 25$, α is complex and Q is imaginary and we do not really know how to make sense of the Liouville approach.

2.2.1 String susceptibility in the Liouville approach

We saw in the part about matrix models that an m-th order critical point determined a scaling behaviour of string susceptibility $\gamma = -1/m$. In this subsection, we want to show that this corresponds to a Liouville string theory embedded in a target-space of dimension $D \leq 1$, so that both approaches can be compared in a meaningful way. We're interested here in the partition function for a world-sheet of fixed area, more precisely:

$$Z[A] = \int d\tau \mathcal{D}_{\hat{h}}\phi \mathcal{D}_{\hat{h}}X \mathcal{D}_{\hat{h}}(gh) e^{-S(X,\phi,b,c,\hat{h})}\, \delta(\int \sqrt{\hat{h}}e^{\alpha\phi}d^2\sigma - A)$$

and we define the string susceptibility γ as the critical exponent in:

$$Z[A] \propto A^{\frac{(\gamma-2)}{2}\chi - 1} ,$$

where χ is here the Euler characteristic of the world-sheet we consider. To determine γ, we note that the integration measure does not change under the shift $\phi \to \phi + \rho/\alpha$ for a constant ρ. On the other hand the action changes as:

$$\frac{Q}{8\pi}\int \hat{R}\phi\sqrt{\hat{h}}\, d^2\sigma \to \frac{Q}{8\pi}\int \hat{R}\phi\sqrt{\hat{h}}\, d^2\sigma + \frac{Q\rho}{8\pi\alpha}\int \hat{R}\sqrt{\hat{h}}\, d^2\sigma . \tag{2.68}$$

so that:

$$Z[A] = \int d\tau \mathcal{D}_{\hat{h}}\phi \mathcal{D}_{\hat{h}}X \mathcal{D}_{\hat{h}}(gh) e^{-S(X,\phi,b,c,\hat{h}) - \frac{Q\rho}{2\alpha}\chi}\, \delta(e^{\rho}\int \sqrt{\hat{h}}e^{\alpha\phi}d^2\sigma - A) = e^{-\frac{Q\rho}{2\alpha}\chi - \rho} Z[e^{-\rho}A]$$

since $\delta(e^\rho f(\phi) - A) = e^{-\rho}\delta(f(\phi) - e^{-\rho}A)$. In particular, we can take $e^\rho = A$ and we obtain the scaling behaviour:

$$Z[A] = A^{-\frac{Q}{2\alpha}\chi - 1} Z[1] \ ,$$

so that:

$$\gamma = 2 - \frac{Q}{\alpha} = \frac{1}{12}(D - 1 - \sqrt{(D-25)(D-1)}) \ . \tag{2.69}$$

When we compare this definition with the matrix model definition:

$$\gamma = -\frac{1}{m} \ , \tag{2.70}$$

we see that m and D should be related as:

$$D = 1 - \frac{6}{m(m+1)} \ , \tag{2.71}$$

so that the large N limit of a one-matrix model with a critical point of order m should describe a theory of two-dimensional gravity coupled with a conformal field theory of central charge $D = 1 - \frac{6}{m(m+1)}$. For example, $m = 2$ corresponds to the pure gravity case $D = 0$, while $m = 3$ corresponds to gravity coupled to a 1/2-boson, i.e. a fermion. This is the conformal field theory of the critical Ising model. Of course, this relation loses meaning for $D > 1$, showing as usually that $D = 1$ is a barrier for the central charge of a conformal field theory.

This chapter only covers very early results about two-dimensional gravity and matrix integrals, but there is a vast literature on the subject that we won't attempt to cover here. Instead, let us point to some good review articles like [40, 90, 66, 89, 8, 9, 47, 50, 100] and the references they contain.

Although the results exposed in this chapter are very interesting and offer a first example of how a matrix model can be used to study the non-perturbative régime of a string theory, it is clear that they are not really relevant to the problem of finding a non-perturbative definition of the known ten-dimensional critical superstring theories, since their scope seems to be limited to the study of low-dimensional systems. On the other hand, following these earlier developpments, other approaches involving more elaborate matrix models have been suggested to provide a non-perturbative definition of type II superstring theories. That is what we will study in the next chapters.

Chapter 3

D0-branes, membranes and the BFSS matrix model

This chapter is certainly the core chapter of this review on matrix models. Most motivations for the research project described in chapter 5 find their source here. Indeed, there have been numerous calculations in BFSS theory that show a remarkable agreement with supergravity results, although more work has to be done in that direction. It is probably the most attractive proposal so far in the search for a non-perturbative definition of M-theory.

We have four main goals in this chapter. First, we want to explain the connection between the low-energy effective action for a Dp-brane in string theory and the $\mathcal{N} = 1$ super Yang-Mills theory dimensionally reduced from 10 to $p + 1$ dimensions. This allows us to see the BFSS matrix model with gauge group $U(N)$ as the low-energy effective action for N D0-branes in type IIA string theory. Second, we want to explain the BFSS conjecture that states that M-theory in the infinite momentum frame can be described by the large N limit of this supersymmetric matrix theory and the related more controversial finite N conjecture of Susskind. Third, we will study the supersymmetry algebra of the model in details to shed light on the brane spectrum of the theory. That will lead us to the conclusion that the transverse M5-branes we would expect from the M-theory point of view seem to be absent of the BFSS theory, perhaps because of the infinite momentum frame limit. Finally, we will discuss a classical theory of supermembranes and we will show how its light-cone frame expression can be regularized through the replacement of Poisson brackets by commutators, leading again to the same supersymmetric matrix quantum mechanics.

To put a long story short, this chapter shows a kind of duality between the world-volume theory of an M2-brane and the low-energy effective action of many D0-branes in type IIA string theory. Note that the we will obtain a parallel result in the following chapter where we will identify the world-sheet theory of a fundamental F1-string with the low-energy effective action of many D(-1)-branes, more properly called D-instantons.

3.1 $\mathcal{N}=1$ super Yang-Mills Theory in 10 dimensions

Since we can see various matrix models [20, 107, 67] as dimensional reductions of $\mathcal{N}$=1 super Yang-Mills Theory (SYM) in 10 dimensions [91, 92, 52], we will first briefly study this case. This theory contains only massless fields, a non-abelian vector field A_μ and an adjoint Majorana-Weyl spinor Ψ of 16 real components, that we can choose to take as left-handed (see the Appendix for conventions on spinors and Dirac matrices). All fields are themselves matrices in the adjoint representation of some Lie group and the theory has a local gauge invariance under its adjoint action. A_μ is the connection for that gauge symmetry and can be developped as $A_\mu = A_\mu^a T^a$ on a basis of the gauge group's Lie algebra given by Hermitian generators satisfying $[T^a, T^b] = if^{abc}T^c$ for some totally anti-symmetric tensor f^{abc} (there exists a basis of the Lie algebra in which f^{abc} is anti-symmetric whenever the Lie algebra is the direct sum of commuting simple and $U(1)$ Lie subalgebras [84, 161]) and $Tr(T^a \cdot T^b) = \delta^{ab}$. Of course, Ψ can also be decomposed as $\Psi = \Psi^a T^a$. We define the gauge covariant derivative as: $[\mathcal{D}_\mu, \Psi] = \partial_\mu \Psi + i[A_\mu, \Psi]$ and the non-abelian field strength tensor as: $F_{\mu\nu} = \frac{1}{i}[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$, so that: $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc} A^b_\mu A^c_\nu$. These fields have a gauge-invariant action in 10 dimensions given by:

$$S_{SYM} = \frac{1}{g_{10}^2} \int Tr\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\Psi} \Gamma^\mu [\mathcal{D}_\mu, \Psi]\right) d^{10}x \ . \tag{3.1}$$

This leads to the following equations of motion:

$$[\mathcal{D}_\mu, F^{\mu\nu}] = \frac{i}{2}\{\bar{\Psi}, \Gamma^\nu \Psi\} \ , \qquad \Gamma^\mu [\mathcal{D}_\mu, \Psi] = 0 \ , \tag{3.2}$$

and it is supersymmetric under the field transformations:

$$\delta_\epsilon A_\mu = -\frac{i}{2} \bar{\epsilon} \Gamma_\mu \Psi \ , \qquad \delta_\epsilon \Psi = -\frac{1}{4} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon \ . \tag{3.3}$$

Indeed, the variation of the action $\delta_\epsilon S_{SYM}$ is

$$\begin{aligned}
&\frac{1}{4g_{10}^2} \int Tr\Big(\frac{i}{2}(\bar{\epsilon}\Gamma^{\mu\nu\rho} F_{\mu\nu} [\mathcal{D}_\rho, \Psi] - \bar{\Psi}\Gamma^{\mu\nu\rho}\epsilon [\mathcal{D}_\rho, F_{\mu\nu}]) - i\bar{\epsilon} F_{\mu\nu} \Gamma^\mu [\mathcal{D}^\nu, \Psi] + i\bar{\Psi}\Gamma^\mu [\mathcal{D}^\nu, F_{\mu\nu}]\epsilon\Big) d^{10}x - \\
&\qquad - \frac{i}{4g_{10}^2} f^{abc} \int (\bar{\epsilon}\Gamma_\mu \Psi^a)(\bar{\Psi}^b \Gamma^\mu \Psi^c) d^{10}x = \\
&= \frac{1}{4g_{10}^2} \int Tr\Big(-\frac{i}{2}(\bar{\epsilon}\Gamma^{\mu\nu\rho}\Psi [\mathcal{D}_\rho, F_{\mu\nu}] + \bar{\Psi}\Gamma^{\mu\nu\rho}\epsilon [\mathcal{D}_\rho, F_{\mu\nu}]) + i\bar{\epsilon}\Gamma^\mu \Psi [\mathcal{D}^\nu, F_{\mu\nu}] + i\bar{\Psi}\Gamma^\mu \epsilon [\mathcal{D}^\nu, F_{\mu\nu}]\Big) d^{10}x - \\
&\qquad - \frac{i}{4g_{10}^2} f^{abc} \int (\bar{\epsilon}\Gamma_\mu \Psi^a)(\bar{\Psi}^b \Gamma^\mu \Psi^c) d^{10}x = \\
&= \frac{-i}{4g_{10}^2} \int Tr\Big(\bar{\epsilon}\Gamma^{\mu\nu\rho}\Psi [\mathcal{D}_\rho, F_{\mu\nu}] + f^{abc} (\bar{\epsilon}\Gamma_\mu \Psi^a)(\bar{\Psi}^b \Gamma^\mu \Psi^c)\Big) d^{10}x \ ,
\end{aligned} \tag{3.4}$$

after using partial integration in the first step and Majorana fermions properties in the second. The first term disappears in all dimensions due to the Bianchi identity $\epsilon^{\mu\nu\rho}[\mathcal{D}_\mu, F_{\nu\rho}] = 0$. The second term vanishes thanks to a Fierz transformation valid in certain special dimensions only, specifically in 3,4,6 and 10 dimensions. The proof of this fact (for D=10) can be found in the Appendix B.4.

3.1.1 Dimensional reduction, D-branes and SYM theory

In fact, besides being a supersymmetric theory of 10-dimensional gauge fields, $\mathcal{N}=1$ super Yang-Mills theory in 10 dimensions with $U(N)$ gauge group can also be seen as some simplified low-energy field theory describing a bunch of N space-filling D9-branes in superstring theory. To be more precise, the bosonic part of the action for a single Dp-brane is given by the Born-Infeld theory (generalized to non-trivial backgrounds) described by [124]:

$$S_{BI} = -T_p \int e^{-\phi} \sqrt{-\det(G_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta})}\, d^{p+1}\xi \;, \tag{3.5}$$

where α, β are world-volume indices taking $p+1$ values from 0 to p. The fields $G_{\alpha\beta}$, $B_{\alpha\beta}$ and ϕ are the pullbacks to the Dp-brane world-volume of the bosonic fields of the massless $\mathcal{N}=1$ supergraviton multiplet in ten dimensions, the metric, the Kalb-Ramond anti-symmetric tensor and the dilaton, respectively. They appear in the massless spectrum of all consistent supersymmetric string theories in 10 dimensions. On the other hand, $F_{\alpha\beta}$ is the field strength tensor of a $U(1)$ gauge field A_α living on the brane's world-volume, giving Chan-Paton charges on the ends of open strings attached to it from a string theory point of view. Finally, for $p=1$, the string tension would be given by $T_s = \frac{1}{2\pi\alpha'}$, which defines the meaning of α', the squared string legth.

Although this action can be shown to reproduce the open bosonic string computations correctly, it is a very difficult problem to generalize it. The supersymmetrization, as well as the inclusion of several interacting D-branes (leading to a non-abelian gauge field) are not completely understood, especially for non-trivial backgrounds (for some recent accounts about it, see [156, 38, 36, 37, 35, 43, 76]). In the single bosonic brane case, the validity of the Born-Infeld action above has been tested through string perturbative calculations by [18, 30, 29, 6]. Such calculations also set the branes' tensions to be:

$$T_p = \frac{1}{\sqrt{\alpha'}} \frac{1}{(2\pi\sqrt{\alpha'})^p} \;. \tag{3.6}$$

For a detailed review of Born-Infeld theory for branes, see for example [139, 138, 137]. However, since Born-Infeld theory is highly non-linear and hard to handle, we will be interested in some simplified versions of it in the present work. For that sake, we start by enumerating a few simplifying assumptions that allow to reach a useful polynomial approximation of the Born-Infeld action. We suppose the following facts:

- The 10-dimensional background space-time metric is flat Minkowskian.
- The brane is sufficiently flat so that we can identify the first $p+1$ space-time coordinates with the world-volume coordinates A_α (an assumption customarily called static gauge).
- There is no background B-field .
- The dilaton field is constant along the brane.
- Both $\partial_\alpha X^a$ (for X^a describing transverse directions with $a = p+1, \ldots, 9$) and $2\pi\alpha' F_{\alpha\beta}$ are not too big on the brane's world-volume and of similar order of magnitude.

Under such conditions, the determinant can be expanded as follows:

$$\det(G_{\alpha\beta}+2\pi\alpha' F_{\alpha\beta}) \approx \det(\eta_{\alpha\beta}+\partial_\alpha X^a\partial_\beta X^a+2\pi\alpha' F_{\alpha\beta}) \approx$$
$$\approx \det(\eta_{\alpha\beta}) - \partial_\alpha X^a \partial^\alpha X^a - \frac{1}{2}(2\pi\alpha')^2 F_{\alpha\beta}F^{\alpha\beta} + \mathcal{O}((\partial X)^4, F^4, (\partial X)^2\cdot F^2)\ . \tag{3.7}$$

The second equality is exact for $p=0,1$, while for example:

$$\det(\eta_{\alpha\beta}+\partial_\alpha X^a\partial_\beta X^a+2\pi\alpha' F_{\alpha\beta}) = \det(\eta_{\alpha\beta}) - \partial_\alpha X^a\partial^\alpha X^a - \frac{1}{2}(2\pi\alpha')^2F_{\alpha\beta}F^{\alpha\beta} -$$
$$-(2\pi\alpha')^2(F^\alpha{}_\gamma F^{\gamma\beta}+\frac{1}{2}F_{\gamma\delta}F^{\gamma\delta}\eta^{\alpha\beta})\partial_\alpha X^a\partial_\beta X^a \tag{3.8}$$

for p=2. In any case, leaving out the fourth-order terms, the action can be approximated through:

$$S_p = -\frac{T_pV_p}{g_s} - \frac{T_p}{4g_s}\int\Big((2\pi\alpha')^2F_{\alpha\beta}F^{\alpha\beta}+2\partial_\alpha X^a\partial^\alpha X^a\Big)\,d^{p+1}x\ , \tag{3.9}$$

where V_p is the brane's volume $V_p=\int\sqrt{\det(\eta_{\alpha\beta})}d^{p+1}x$ and g_s is the string coupling $g_s=\exp(<\phi>)$, determined dynamically by the dilaton expectation value. Taking $p=9$ as a first example, we see that this action corresponds to 10-dimensional YM theory if we take the ten-dimensional Yang-Mills coupling to be:

$$g_{10}^2 = \frac{g_s}{(2\pi\alpha')^2T_9} = (2\pi)^7\alpha'^3 g_s \tag{3.10}$$

In this case, however, the generalization to fermions and several D-branes is rather straightforward, since $\mathcal{N}=1$ SYM theory in $10D$ is essentially unique. However, this correspondence is not limited to D9-branes, we can dimensionally reduce 10-dimensional $\mathcal{N}=1$ super Yang-Mills theory to $p+1$ dimensions and obtain a theory of Dp-branes. To do so, we suppose that the fields do not depend on the toroidally compactified dimensions x^a anymore, so that the derivatives and integrations in these transverse directions are trivial and the covariant derivatives become transverse scalars. Specifically, this reduction involves the replacement:

$$i\mathcal{D}_a = i\partial_a - A_a \longrightarrow (2\pi\alpha')^{-1}X_a\,\forall\, a=p+1,\dots,9 \tag{3.11}$$
$$\Longrightarrow \begin{cases} F_{ab} &= \frac{1}{i}[\mathcal{D}_a,\mathcal{D}_b] \longrightarrow (2\pi\alpha')^{-2}i[X_a,X_b]\,\forall\, a,b=p+1,\dots,9 \\ F_{\alpha b} &= \frac{1}{i}[\mathcal{D}_\alpha,\mathcal{D}_b] \longrightarrow -(2\pi\alpha')^{-1}[\mathcal{D}_\alpha,X_b]\,\forall\, b=p+1,\dots,9,\ \alpha=0,\dots,p\end{cases}\ . \tag{3.12}$$

Starting from (3.1), we obtain the following Dp-brane action:

$$S_p = \frac{(2\pi)^{9-p}R_{p+1}\cdot\ldots\cdot R_9}{4g_{10}^2}\int Tr\Big(-F_{\alpha\beta}F^{\alpha\beta}-2(2\pi\alpha')^{-2}[\mathcal{D}_\alpha,X_a][\mathcal{D}^\alpha,X^a]+$$
$$+(2\pi\alpha')^{-4}[X_a,X_b][X^a,X^b]+2i\bar\Psi\Gamma^\alpha[\mathcal{D}_\alpha,\Psi]+2(2\pi\alpha')^{-1}\bar\Psi\Gamma^a[X_a,\Psi]\Big)d^{p+1}x\ . \tag{3.13}$$

Of course, one should rescale the action so that it remains finite in the limit where the compactification radii become null. A way to do this is to match the normalization of (3.13) with that of the Born-Infeld action (3.5). This gives us:

$$S_p = \frac{1}{g_{p+1}^2} \int Tr\Big(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}(2\pi\alpha')^{-2}[\mathcal{D}_\alpha, X_a][\mathcal{D}^\alpha, X^a] + \frac{1}{4}(2\pi\alpha')^{-4}[X_a, X_b][X^a, X^b] + \\ + \frac{i}{2}\bar{\Psi}\Gamma^\alpha[\mathcal{D}_\alpha, \Psi] + \frac{1}{2}(2\pi\alpha')^{-1}\bar{\Psi}\Gamma^a[X_a, \Psi]\Big) d^{p+1}x \ , \tag{3.14}$$

with a $p+1$-dimensional Yang-Mills coupling given by:

$$g_{p+1}^2 = \frac{g_s}{(2\pi\alpha')^2 T_p} = (2\pi)^{p-2}\alpha'^{\frac{p-3}{2}} g_s \tag{3.15}$$

Note that it is a non-abelian generalization of (3.9), accounting for N Dp-branes if the gauge group is $U(N)$. Indeed, as can be understood easily from open string theory, the $U(1)^N$ gauge symmetry of N D-branes is enhanced whenever two branes are coincident, reaching the maximally symmetric case $U(N)$ when they all lie on top of each other. In the string theory language, if we start with an open string stretched between two D-branes and push them together, the string energy will progressively drop to zero, allowing new massless vectors to appear in the string spectrum.

3.1.2 T-duality

We now want to show how the above dimensional reduction process can be reversed through the matrix analogue of string theory's T-duality. To illustrate the ideas involved, let us first study T-duality in the case of one compact direction, for the space $\mathbb{R}^9 \times S^1$. In string theory, T-duality corresponds to the inversion of the circle radius R_9 to a dual radius $\hat{R}_9 = \alpha'/R_9$. Besides exchanging type IIA with type IIB string theory, such an operation changes the boundary conditions for open strings, exchanging Neumann with Dirichlet boundary conditions in the T-dualized direction (here, X^9). Since the presence of Dirichlet boundary conditions are usually interpreted through the presence of D-branes on which open strings'ends are attached, a T-duality also transforms Dp-branes into D$(p \pm 1)$-branes. More precisely, if a Dp-brane is wrapped onto the compact direction in the original theory, the strings'ends are free to be at any value of X_9 and the boundary conditions are Neumann in that direction. After T-duality, the boundary conditions change to Dirichlet, which means that the strings'ends are kept fixed at some precise value of X_9, thus, the D-brane's position is now fixed in the 9-th direction, meaning that it was unwrapped and has now become a D$(p-1)$-brane. Reciprocally, a Dp-brane that was originally unwrapped in the 9-th direction will become a wrapped D$(p+1)$-brane after such a T-duality.

To describe mathematically D-branes on a circle, it is useful to think of S^1 as being the orbifold $\mathbb{R}/\mathbb{Z}$ [72]. Instead of describing N D-branes moving on S^1, we consider N families of D-branes moving on $\mathbb{R}$, each labelled by some other index $n \in \mathbb{Z}$. Thus, we treat $U(\infty)$ doubly-indexed matrices $X^a_{mi,nj}$ instead of the original $U(N)$ matrices, introducing periodicity constraints to keep the correct number of degrees of freedom. In string theory language, $X_{mi,nj}$ corresponds to a string stretching between the m-th copy of the i-th brane and the n-th copy of the j-th brane. To make the theory invariant under the $\mathbb{Z}$ action, these matrices should be invariant under a simultaneous translation of $2\pi R_9$ in the 9-th direction and a relabelling of the indices $n \to n+1$. More precisely, this means that:

$$\begin{aligned} X^a_{mi,nj} &= X^a_{(m-1)i,(n-1)j} \,, \text{ if } a < 9 \\ X^9_{mi,nj} &= X^9_{(m-1)i,(n-1)j} \,, \text{ if } m \neq n \\ X^9_{ni,nj} &= 2\pi R_9 \delta_{i,j} + X^9_{(n-1)i,(n-1)j} \,. \end{aligned} \tag{3.16}$$

Dropping the i, j indices and writing X^9 as an infinite matrix with $N \times N$ matrix as entries, the constraints give the following aspect to X^9 around the diagonal:

$$X^9 = \begin{pmatrix} \ddots & X_{-1} & X_{-2} & X_{-3} & \ddots \\ X_1 & X_0 - 2\pi R_9 \mathbb{1} & X_{-1} & X_{-2} & X_{-3} \\ X_2 & X_1 & X_0 & X_{-1} & X_{-2} \\ X_3 & X_2 & X_1 & X_0 + 2\pi R_9 \mathbb{1} & X_{-1} \\ \ddots & X_3 & X_2 & X_1 & \ddots \end{pmatrix} , \tag{3.17}$$

where we have set $X_k = X^9_{k0}$. Such a matrix can be interpreted as a matrix representation of the covariant derivative operator:

$$2\pi i\alpha'(\frac{\partial}{\partial \hat{x}_9} + iA_9(\hat{x}_9)) \tag{3.18}$$

acting on a Fourier decomposition of functions of the type:

$$f(\hat{x}) = \sum_n \hat{\phi}_n e^{in\hat{x}_9/\hat{R}_9} \,, \tag{3.19}$$

which are periodic on a circle of radius $\hat{R}_9 = \alpha'/R_9$. The partial derivative part $2\pi i\alpha'\hat{\partial}_9$ acts on it by multiplication of the n-th term in the sum by $-2\pi n\alpha'/\hat{R}_9 = -2n\pi R_9$. Writing the Fourier components as a column vector as:

$$f(\hat{x}) = \begin{pmatrix} \vdots \\ \hat{\phi}_2 \\ \hat{\phi}_1 \\ \hat{\phi}_0 \\ \hat{\phi}_{-1} \\ \hat{\phi}_{-2} \\ \vdots \end{pmatrix} , \tag{3.20}$$

it acts on it like the diagonal matrix: $diag(\ldots, -4\pi R_9, -2\pi R_9, 0, 2\pi R_9, 4\pi R_9, \ldots)$. Furthermore, the connection part:

$$-2\pi\alpha' A(\hat{x}) = -2\pi\alpha' \sum_n \hat{A}_n e^{in\hat{x}_9/\hat{R}_9} \,, \tag{3.21}$$

acts on $\hat{\phi}$ exactly as the remaining part of (3.17) if we identify $X_n \sim -2\pi\alpha'\hat{A}_n$, so that we can indeed identify the action of X^9 on a circle of radius R_9 transverse to the Dp-brane world-volume with the action of the covariant derivative $2\pi i\alpha'(\frac{\partial}{\partial \hat{x}_9} + iA_9(\hat{x}_9))$ on the dual circle of radius α'/R_9 on which the dual D$(p+1)$-brane is now wrapped. In particular, if the circle radius R_9 tends to zero, a flat infinite direction appears in the dual theory, effectively reversing the dimensional reduction process described above.

3.2 The BFSS matrix model

The interesting hypothesis made by Banks, Fishler, Shenker and Susskind [20] in 1996 is that M-theory may be described completely through the dynamics of an infinite number of D0-branes (or D-particles), when expressed in a particular Lorentz frame called the infinite momentum frame (abbreviated as IMF in the following). Although this hypothesis seems too simple to be true at first sight, it passes a few important tests, like the comparison with the $11D$ supergravity result for the two-graviton scattering in the low-energy limit and the correct spectrum of an arbitrary number of 11-dimensional supergraviton multiplets of 256 states.

Such a theory of a big number of N D0-branes can be cast mathematically as a quantum mechanical theory of $N \times N$ matrices with the following action[20, 61]:

$$S = \frac{1}{g^2} \int Tr \left(\frac{1}{2}([\mathcal{D}_t, X_i])^2 - \frac{i}{2}\psi^\dagger[\mathcal{D}_t, \psi] + \frac{1}{4}([X_i, X_j])^2 + \frac{1}{2}\bar{\psi}\Gamma^i[X_i, \psi] \right) dt \,. \tag{3.22}$$

As should be clear from the previous subsection, the BFSS action is the dimensional reduction from 9+1 to 0+1 dimensions of 10-dimensional $\mathcal{N} = 1$ super Yang-Mills theory, in which we replaced: $i\mathcal{D}_i = i\partial_i - A_i$ through X_i (rescaling X_i by a factor of $2\pi\alpha'$ in comparison to (3.14) and writing $g_1 \equiv g$ for simplicity).

Varying this action with respect to the various fields (taking fermionic derivatives to be left-derivatives), we get:

$$\begin{aligned}
\frac{\delta S}{\delta A} &= \frac{i}{g^2}[X^i, [\mathcal{D}_t, X_i]] - \frac{1}{2g^2}\{\Psi^\dagger, \Psi\} \,, & p_A &= \frac{\delta S}{\delta(\partial_t A)} = 0 \,, \\
\frac{\delta S}{\delta X_i} &= -\frac{i}{g^2}[A, [\mathcal{D}_t, X^i]] + \frac{1}{g^2}[X_j, [X^i, X^j]] - \frac{1}{2g^2}\{\bar{\Psi}, \Gamma^i\Psi\} \,, & p_i &= \frac{\delta S}{\delta(\partial_t X^i)} = \frac{1}{g^2}[\mathcal{D}_t, X_i] \,, \\
\frac{\delta S}{\delta \Psi} &= -\frac{1}{g^2}[\Psi^\dagger, A] - \frac{1}{g^2}[\bar{\Psi}, X_i]\Gamma^i - \frac{i}{2g^2}(\partial_t \Psi^\dagger) \,, & p_\Psi &= \frac{\delta S}{\delta(\partial_t \Psi)} = \frac{i}{2g^2}\Psi^\dagger \,.
\end{aligned} \tag{3.23}$$

That gives us the following equations of motion:

$$\begin{aligned}
[\mathcal{D}_t, [\mathcal{D}_t, X^i]] &= [X_j, [X^i, X^j]] - \frac{1}{2}\{\bar{\Psi}, \Gamma^i\Psi\} \,, \\
[\mathcal{D}_t, \Psi] &= -i\Gamma^0\Gamma^i[X_i, \Psi] \,,
\end{aligned} \tag{3.24}$$

and the constraint:

$$\mathcal{C} = [X^i, [\mathcal{D}_t, X_i]] + \frac{i}{2}\{\Psi^\dagger, \Psi\} = 0 \,. \tag{3.25}$$

This also allows us to compute the Hamiltonian:

$$H = Tr(p_i\partial_t X^i - p_\Psi \partial_t \Psi) - L = Tr \left(\frac{g^2}{2}p_i^2 - \frac{1}{4g^2}([X_i, X_j])^2 + ip_\Psi\Gamma^0\Gamma^i[X_i, \psi] + iA\tilde{\mathcal{C}} \right) \,, \tag{3.26}$$

where $\tilde{\mathcal{C}}$ is essentially the constraint (3.25):

$$\tilde{\mathcal{C}} = [p_i, X^i] - \{p_\Psi, \Psi\} = 0 \,. \tag{3.27}$$

This also allows to write the Hamiltonian equations of motion:

$$\partial_t X^i = \frac{\delta H}{\delta p_i} = g^2 p^i - i[A, X^i]\,, \qquad \partial_t p^i = -\frac{\delta H}{\delta X_i} = \frac{1}{g^2}[X_j,[X^i,X^j]] + i\{p_\Psi, \Gamma^0\Gamma^i\Psi\} - i[A,p^i]\,,$$
$$\partial_t \Psi = -\frac{\delta H}{\delta p_\Psi} = -i\Gamma^0\Gamma^i[X_i,\Psi] - i[A,\Psi]\,, \qquad \partial_t p_\Psi = -\frac{\delta H}{\delta \Psi} = i[p_\Psi, X_i]\Gamma^0\Gamma^i + i[p_\Psi, A]\,,$$

which are indeed equivalent to the Lagrangian ones.

3.2.1 The infinite momentum frame

Let us describe the IMF more precisely in this subsection. It was introduced in quantum field theory by Weinberg [160] to simplify perturbation theory, since the vacuum is trivial in the IMF. Moreover, it gives a non-relativistic appearance to a relativistic system, in a sense that we will make more precise below. Starting from a reference frame where particles have an approximately uniform and isotropic spectrum of velocities, we perform a high-velocity relativistic boost in some direction, so that all particles acquire a big component of the velocity in that direction. If the sum of all momenta in this new frame is now $\vec{P}$, we can write the velocity of the k-th particle as:

$$\vec{p}_k = \eta_k \vec{P} + \vec{p}_k^{\perp} \tag{3.28}$$

where $\vec{p}_k^{\perp} \cdot \vec{P} = 0$, $\sum_k \vec{p}_k^{\perp} = 0$ and $\sum_k \eta_k = 1$. It is clear that we can find a big enough boost so that η_k will be positive for all massive particles. For massless particles, however, we cannot change the sign of the momentum of a particle with a boost of subluminal speed in a direction opposite to the original momentum. We therefore suppose in the following that we avoid taking the boost direction (opposite to $\vec{P}$) anti-parallel to any massless particles' momentum. This will be possible except in the very degenerate case where all particles are massless with parallel momenta. Let's now assume that we have reached a frame where all η_k are positive and the boost is big enough so that $\eta_k\|\vec{P}\| \gg \|\vec{p}_k^{\perp}\|$ for all k. Noting $\|\vec{P}\| = P$, the energy of each particle will be given by:

$$E_k = \sqrt{\vec{p}_k^2 + m_k^2} = \eta_k P + \frac{(\vec{p}_k^{\perp})^2 + m_k^2}{2\eta_k P} + \mathcal{O}(P^{-2})\,. \tag{3.29}$$

We can interpret this formula as the non-relativistic energy of a particle of mass $\eta_k P$ shifted by the constant $\eta_k P + m_k^2/(2\eta_k P)$. Note that this is related to the light-cone frame, where one spatial direction is singled out as longitudinal. To use a similar notation, we can denote the longitudinal particle momenta as fractions of the total longitudinal momenta P as: $p_{kL} = \eta_k P$. Then we create light-cone momenta from the temporal and longitudinal momenta as: $p_{k\pm} = E_k \pm p_{kL} = E_k \pm \eta_k P$. In this frame, the mass shell condition reads: $p_{k-}p_{k+} - (\vec{p}_k^{\perp})^2 = m_k^2$, which is equivalent to:

$$E_k - \eta_k P = \frac{(\vec{p}_k^{\perp})^2 + m_k^2}{p_{k+}}\,. \tag{3.30}$$

If the total longitudinal momentum P is very large, all individual p_{kL} will be large, so that $E_k \approx p_{kL}$ and $p_{k+} \approx 2\eta_k \vec{P}$. Thus, the infinite momentum frame can also be seen as a light-cone frame boosted in the longitudinal direction.

3.2.2 The BFSS matrix model as M-theory in the IMF

Now that we have defined the Infinite Momentum Frame, we want to make more precise the BFSS conjecture by explaining the relationship between M-theory in the infinite momentum frame and D0-branes matrix quantum mechanics in the large N limit. To understand why D0-branes might be sufficient to describe M-theory in the IMF, the following arguments are necessary [20]:

- Only states with a positive longitudinal momentum survive as independent dynamical degrees of freedom in the large N limit. The other states get infinite energies and can be integrated out. However, the process of integrating out such modes can determine the dynamics of the remaining degrees of freedom.
- From a type IIA string theory point of view, momenta in the eleventh direction appear as charges under the Ramond-Ramond 1-form.
- Since fundamental strings do not carry any RR charges, they have zero longitudinal momentum and should not appear as physical degrees of freedom in M-theory boosted to the IMF.
- The only objects which are charged under the RR potential in type IIA string theory are the D0-branes. Each D-particle carries a single unit of charge. There are also anti-D-particles with a negative unit of charge, but we only expect positively charged D0-branes to contribute in the IMF limit.
- We also have to consider states carrying N units of charge, which we interpret as bound states of N D0-branes.

Technically, the procedure goes as follows:

- Consider M-theory compactified in the eleventh dimension on a circle of radius R_{10} with states of momentum $P_{10} = N/R_{10}$. It is equivalent to a type IIA string theory with bound states of D0-branes carrying N units of charge under the RR one-form potential.
- Carry out an infine boost in the longitudinal direction. It will send $P_{10} = N/R_{10} \to \infty$ and should decouple the string states given the arguments above, leaving only D0-branes as dynamical degrees of freedom.
- If we want this limit to agree with eleven-dimensional $\mathcal{N} = 1$ supergravity in a minkowskian spacetime in the low-energy limit, we have to take the IMF boost in a way that decompactifies the eleventh dimension by sending $R_{10} \to \infty$.
- This implies of course that N should go to ∞ at the same time.

In the end, one can conjecture that M-theory in the infinite momentum frame could be described by the supersymmetric matrix quantum mechanics for D0-branes in the large N limit and compare its predicitions with supergravity in the low-energy limit. Noticing that tests of this duality were valid at finite N already, Susskind [150] suggested to consider the sector of M-theory compactified on a light-like circle of radius R that contains states of momentum $P^- = N/R$. The discrete light-cone quantization (DLCQ) of this system should be equivalent to the $U(N)$ supersymmetric matrix quantum mechanics for D0-branes. Following ideas from Seiberg [144] and Sen [145], we want to show here how these two proposals relate and explain why type IIA D0-branes in the IMF can indeed be described by the BFSS matrix model Hamiltonian, which is a reasonable conjecture if we can show that:

- type IIA perturbative string theory is valid in this limit, in other words, string theory is weakly-coupled.
- the string scale is big, i.e. the string length is small, so that the string massive modes do not influence the D0-branes' dynamics in this limit. From the D-brane point of view, it means that the terms of higher order in the expansions made in equations (3.7) and (3.9) will be negligible.

To show this, we will consider almost light-like compactifications. Thus, we take a space-time compactified on a one-parameter family of space-like circles of lengths $\sqrt{l^2+\tilde{l}^2}$, parametrized by $\tilde{l}$, through the identification:

$$\begin{pmatrix} x \\ t \end{pmatrix} \sim \begin{pmatrix} x - \sqrt{l^2/2+\tilde{l}^2} \\ t + l/\sqrt{2} \end{pmatrix} . \tag{3.31}$$

Note that the circle becomes time-like in the limit $\tilde{l} \to 0$. If we perform a Lorentz boost:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & \frac{\beta}{\sqrt{1-\beta^2}} \\ \frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} . \tag{3.32}$$

of parameter:

$$\beta = \frac{1}{\sqrt{1+2\tilde{l}^2/l^2}} , \tag{3.33}$$

we reach new coordinates in which the space-like identification reads:

$$\begin{pmatrix} x' \\ t' \end{pmatrix} \sim \begin{pmatrix} x' - \tilde{l} \\ t' \end{pmatrix} . \tag{3.34}$$

In other words, there is a boost relating any space-like circle (even almost time-like) to a purely longitudinal circle, in which case we know the precise relationship between M-theory and type IIA string theory. Indeed, we know that M-theory compactified on a circle of length $\tilde{l} = 2\pi R_{10}$ in the eleventh dimension is equivalent to type IIA superstring theory at string coupling and string length:

$$g_s = \left(\frac{R_{10}}{l_P}\right)^{3/2} , \qquad \alpha' = \frac{{l_P}^3}{R_{10}} . \tag{3.35}$$

where l_P is the eleven-dimensional Planck length and R_{10} is measured in the eleven-dimensional Einstein metric). Note that these two relations combine to $R_{10} = \sqrt{\alpha'} g_s$, if we want to measure R_{10} in string units. In any case, the light-like limit $R_{10} \to 0$ corresponds to small string coupling and big string length, i.e. small string tension, a limit in which string perturbation theory is valid, but SYM theory is not a good approximation of the full Dirac-Born-Infeld theory since α' is large.

Despite this deceptive conclusion, let us nevertheless study the behaviour of D-particles in this limit. If they have a longitudinal momentum in the compact direction $P'^{10} = N/R_{10}$ and a total energy

$$E' = N/R_{10} + \Delta E \tag{3.36}$$

in type IIA string theory, these will transform to:

$$\begin{pmatrix} P^{10} \\ E \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & -\frac{\beta}{\sqrt{1-\beta^2}} \\ -\frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix} \begin{pmatrix} P'^{10} \\ E' \end{pmatrix} \tag{3.37}$$

in the original frame where the compactification was almost light-like. In this frame, the light-front energy is thus:

$$P^- = \frac{1}{\sqrt{2}}(E - P^{10}) = \frac{1}{\sqrt{2}} \frac{1+\beta}{\sqrt{1-\beta^2}} \Delta E = \frac{l}{2\tilde{l}} \left(1 + \sqrt{1 + 2(\tilde{l}/l)^2}\right) \approx \frac{l}{\tilde{l}} \Delta E \ , \tag{3.38}$$

so that we expect type IIA configurations to have energies of the order:

$$\Delta E \approx \frac{\tilde{l}}{l} P^- \ . \tag{3.39}$$

This energy compares to the string scale as:

$$\frac{\Delta E}{1/\sqrt{\alpha'}} \approx \frac{P^-}{l} \tilde{l} \sqrt{\alpha'} = \frac{2\pi P^-}{l} (R_{10} {l_P}^3)^{1/2} \ , \tag{3.40}$$

which means that the energy of interest is smaller than the string scale in the light-like limit, although the latter goes to zero, too. In other words, we are working in units where the D0-branes' light-cone Hamiltonian is zero and the string length, too, although it vanishes slower in the limit we consider. This suggests that it might be possible to find units in which the light-cone energy of D-particles stays finite while the string length diverges when we take the limit, which would fulfill our initial goal. To achieve this, let us introduce a new eleven-dimensional length scale l_{11} such that the energy of interest stays constant in the $R_{10} \to 0$ limit. Since P^- goes to zero like $R_{10}/{l_P}^2$ when $R_{10} \to 0$ (where we have introduced powers of l_P on dimensional ground), we expect it to scale like R_{10}/l_{11}^2 in the new description. To achieve finite light-cone energy for the D-particles, we need to keep the ratio R_{10}/l_{11}^2 fixed in the light-like limit. To be specific, we can fix it to be:

$$\frac{2\pi R_{10}}{l_{11}^2} = \frac{l}{{l_P}^2} \ , \tag{3.41}$$

since the R.H.S stays constant in the limiting procedure, so that we must have $l_{11} \to 0$ when $R_{10} \to 0$. In these new units, the light-cone energy will be:

$$\Delta \hat{E} = \frac{{l_P}^2}{l_{11}^2} \Delta E \approx \frac{2\pi R_{10}}{l_{11}^2} \frac{{l_P}^2}{l} P^- = P^- \ , \tag{3.42}$$

which does not scale in the limit. Furthermore, this theory corresponds to a string theory with new parameters:

$$\hat{g}_s = \left(\frac{R_{10}}{l_{11}}\right)^{3/2} \propto (l_{11})^{3/2} \quad \text{and} \quad \hat{\alpha}' = \frac{{l_{11}}^3}{R_{10}} \propto l_{11} \ , \tag{3.43}$$

so that both the string coupling and the string length vanish in the light-like limit, giving rise to a theory which we can describe through a simple BFSS matrix model.

To summarize what we have done in this subsection, we have shown that M-theory compactified on an almost light-like circle boosted to the infinite momentum frame can be described through the large N limit of a supersymmetric matrix model that gives the dynamics of an infinite number of D0-branes living in a weakly-coupled type IIA string theory with string coupling $\hat{g}_s = (R_{10} l_{11})^{3/2}$ and string length $\sqrt{\hat{\alpha}'} = ({l_{11}}^3/R_{10})^{1/2}$. Both are vanishing in the limit where the circle becomes light-like, which corresponds to the discrete light-cone quantization of M-theory. Of course, one can worry about that limit (or equivalently about infinite boosts). For a discussion about the subtleties involved in this limit and direct comparisons between type IIA superstring theory compactified on light-like and almost light-like circles, one can consult [42, 41], where both perturbative and non-perturbative evidences show that the limit seems to be well-defined. This lets us think that the same might be true of M-theory.

Another point worth further discussion is whether the large N limit of matrix theory really exists, although all calculations performed so far exhibited well-defined large N behaviour. Discussions about this point and the related question about the renormalizability of supermembrane theory can be found in [129].

Finally, there are many works comparing matrix model predictions with $11D$ supergravity results. The reader can consult for example [71] for a one-loop computation of the interaction of a pair of D0-branes, [25, 22, 24, 23] for two-loops results, [111, 154] for leading-order computations of the interaction between pairs of arbitrary background configurations, [135, 136, 118, 119] for the discussion of supersymmetric non-renormalization theorems, [132, 116, 117] for discussion of the one-loop effective action and [153, 69, 134, 133, 68, 104] for 3-body interactions. Such results are also reviewed in [152, 103], where the likelihood of different conjectures about matrix theory is also discussed.

3.2.3 Supersymmetry

Next, we want to turn our attention to the supersymmetry algebra to understand how a theory having only D0-branes as fundamental degrees of freedom can describe other kind of branes as well. Of course, the BFSS matrix model inherits its supersymmetry from $\mathcal{N}$=1 SYM theory in 10 dimensions. However, it is a bit more delicate to prove explicitly. We can infer the transformation rules from (3.3) and (3.11) to be:

$$\delta_\epsilon A_i = -\frac{i}{2}\bar{\epsilon}\Gamma_i\Psi \to \delta_\epsilon X_i = \frac{i}{2}\bar{\epsilon}\Gamma_i\Psi \ , \tag{3.44}$$

$$\delta_\epsilon A_0 = -\frac{i}{2}\bar{\epsilon}\Gamma_0\Psi \to \delta_\epsilon A = -\frac{i}{2}\epsilon^\dagger\Psi \ , \tag{3.45}$$

$$\delta_\epsilon \Psi = -\frac{1}{4}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon \to \delta_\epsilon\Psi = \frac{1}{2}\left([\mathcal{D}_t, X_i]\Gamma^0\Gamma^i - \frac{i}{2}[X_i, X_j]\Gamma^{ij}\right)\epsilon \ . \tag{3.46}$$

This is usually called dynamical or homogeneous supersymmetry in the matrix model literature. There is an additional inhomogeneous supersymmetry, called kinematical supersymmetry, that is simply a translation by a constant fermionic parameter:

$$\delta_{\epsilon'} A_i = 0 \ , \quad \delta_{\epsilon'} A_0 = 0 \ , \quad \delta_{\epsilon'}\Psi = \epsilon'\mathbb{1} \ . \tag{3.47}$$

A detailed proof of the supersymmetry can be found in the Appendix C on the supersymmetry of the BFSS model. These transformations induce corresponding kinematical and dynamical supercharges and we want to compute their anti-commutation relations to see how various brane central charges are obtained as particular configurations of D0-branes. The following results were first obtained by Banks, Seiberg and Shenker in [21], but we will use here a formalism closer to that used by Adler in [2] (though our conventions are different), based on what he calls generalized quantum mechanics or trace quantum mechanics [1, 4, 3].

To study the supersymmetry algebra, we first want to find an expression for the supercharges. For that sake, first note that the Lagrangian is not supersymmetric itself, but transforms into a total derivative under supersymmetry (Cf. Appendix C):

$$\delta L = \frac{i}{4g^2}\partial_t Tr(\bar{\epsilon}\Gamma^i\Psi[\mathcal{D}_t, X_i]) + \frac{1}{8g^2}\partial_t Tr(\epsilon^\dagger\Gamma^{ij}\Psi[X_i, X_j]) \ . \tag{3.48}$$

Secondly, remark that $\delta S = 0$ is valid when both ϵ and ϵ' are constant Grassmann parameters. Let us now suppose they are not, the variation of the action can be read off in the appendix on the supersymmetry of the BFSS model:

$$\begin{aligned}\delta S = &\frac{1}{g^2}\int Tr\Big(-\frac{i}{2}[\mathcal{D}_t, X_i]\bar{\Psi}\Gamma^i(\partial_t\epsilon) - \frac{1}{4}\Psi^\dagger[X_i, X_j]\Gamma^{ij}(\partial_t\epsilon) - i\Psi^\dagger(\partial_t\epsilon')\Big)dt = \\ = &\, i\int \Big(Q^\alpha(\partial_t\epsilon)_\alpha + \tilde{Q}^\alpha(\partial_t\epsilon')_\alpha\Big)\, dt \ .\end{aligned} \tag{3.49}$$

From that variation, we can read off the dynamical supercharge using the supersymmetric version of the Noether theorem:

$$Q^\alpha = -\frac{1}{2g^2}Tr(\bar{\Psi}[\mathcal{D}_t, X_i]\Gamma^i - \frac{i}{2}\Psi^\dagger[X_i, X_j]\Gamma^{ij})^\alpha \ , \tag{3.50}$$

as well as the kinematical supercharge:

$$\tilde{Q}^\alpha = -\frac{1}{g^2}Tr(\Psi^\dagger)^\alpha \ . \tag{3.51}$$

Let's now work in a gauge where $A = 0$ for simplicity. To compute the algebra, we need to introduce the Poisson bracket of two operators corresponding to the Hamiltonian structure of the model:

$$\{A, B\}_{PB} = \frac{\delta A}{\delta(X^i)^b_a}\frac{\delta B}{\delta(p_i)^a_b} - \frac{\delta A}{\delta(p_i)^b_a}\frac{\delta B}{\delta(X^i)^a_b} + \frac{\delta A}{\delta(\Psi_\alpha)^b_a}\frac{\delta B}{\delta(p^\alpha_\Psi)^a_b} + \frac{\delta A}{\delta(p^\alpha_\Psi)^b_a}\frac{\delta B}{\delta(\Psi_\alpha)^a_b} \tag{3.52}$$

In particular, we of course have:

$$\{(X^i)^b_a, (p_j)^d_c\}_{PB} = -\{(p_j)^d_c, (X^i)^b_a\}_{PB} = \delta^i_j\delta^d_a\delta^c_b \ , \quad \{(\Psi_\alpha)^b_a, (p^\beta_\Psi)^d_c\}_{PB} = \{(p^\beta_\Psi)^d_c, (\Psi_\alpha)^b_a\}_{PB} = (\mathcal{P}_L)^\beta_\alpha\delta^d_a\delta^c_b \ .$$

To use those supercharges in the Poisson bracket formula without having to worry about losing track of traces of commutators, we need to symmetrize their expressions with respect to Ψ and p_Ψ in this way:

$$\begin{aligned}Q^\alpha = &Tr(q^\alpha) = \frac{1}{2}Tr\left((-\frac{1}{2g^2}\Psi^\dagger + ip_\Psi)^\beta(g^2 p_i\Gamma^0\Gamma^i - \frac{i}{2}[X_i, X_j]\Gamma^{ij})_\beta{}^\alpha\right) \ , \\ \tilde{Q}^\alpha = &\, Tr(\tilde{q}^\alpha) = Tr\left((-\frac{1}{2g^2}\Psi^\dagger + ip_\Psi)^\alpha\right) \ .\end{aligned} \tag{3.53}$$

Indeed, brane charges are only non-trivial in the large N limit, so that they will appear under the form of traces of commutators. The symmetrization in $\Psi \leftrightarrow p_\Psi$ is a technical trick introduced in [2] that allows us to work directly with the supercharges instead of the supercharge densities, thus taking advantage of the properties of the trace. With this definition, the supercharges can be shown to generate supersymmetry transformations, since:

$$\begin{aligned} \delta_\epsilon X^i =& i\{\epsilon_\beta Q^\beta, X^i\}_{PB} = \frac{i}{2}\bar{\epsilon}\Gamma_i\Psi \ , \\ \delta_\epsilon \Psi_\alpha =& i\{\epsilon_\beta Q^\beta, \Psi_\alpha\}_{PB} = \frac{1}{2}\left([\mathcal{D}_t, X_i]\Gamma^0\Gamma^i - \frac{i}{2}[X_i,X_j]\Gamma^{ij}\right)\epsilon \ , \\ \delta_{\epsilon'} X^i =& i\{\epsilon'_\beta \tilde{Q}^\beta, X^i\}_{PB} = 0 \ , \quad \delta_{\epsilon'}\Psi = i\{\epsilon'_\beta\tilde{Q}^\beta, \Psi_\alpha\}_{PB} = \epsilon'\mathbb{1} \ . \end{aligned} \tag{3.54}$$

Now, we can compute the supersymmetry algebra from the supercharges' Poisson brackets. From now on, we restrict ourselves to the Majorana-Weyl representation of the Clifford algebra of $SO(9,1)$ described in appendix A to simplify the calculations. Then:

$$\begin{aligned} Q^\dagger_\alpha = Q^\top_\alpha =& Tr(q^\top_\alpha) = \frac{1}{2}Tr\left((g^2 p_i\Gamma^0\Gamma^i + \frac{i}{2}[X_i,X_j]\Gamma^{ij})_\alpha{}^\beta(\tilde{q}^\top)_\beta\right) \ , \\ \tilde{Q}^\dagger_\alpha = \tilde{Q}^\top_\alpha =& Tr(\tilde{q}^\top_\alpha) = Tr\left((-\frac{1}{2g^2}\Psi + ip^\top_\Psi)_\alpha\right) \ . \end{aligned} \tag{3.55}$$

First note that:

$$\{(\tilde{q}^\top)_{\alpha a}{}^b, (\tilde{q})^{\beta d}_c\}_{PB} = -\frac{i}{g^2}(\mathcal{P}_L)^\beta_\alpha\delta^d_a\delta^b_c \ , \tag{3.56}$$

so that:

$$\{\tilde{Q}^\top_\alpha, \tilde{Q}^\beta\}_{PB} = -\frac{i}{g^2}(\mathcal{P}_L)^\beta_\alpha Tr(\mathbb{1}) \ , \text{ and} \tag{3.57}$$

$$\{\tilde{Q}^\top_\alpha, Q^\beta\}_{PB} = -\frac{i}{2g^2}Tr\left((g^2 p_i\mathcal{P}_L\Gamma^0\Gamma^i - \frac{i}{2}[X_i,X_j]\mathcal{P}_L\Gamma^{ij})_\alpha{}^\beta\right) \ . \tag{3.58}$$

(3.56) also allows us to compute the bosonic part of $\{Q^\top_\alpha, Q^\beta\}_{PB}$:

$$\begin{aligned} \{Q^\top_\alpha, Q^\beta\}^{bos.}_{PB} =& \frac{1}{4}(g^2 p_i\Gamma^0\Gamma^i + \frac{i}{2}[X_i,X_j]\Gamma^{ij})_{\alpha a}{}^{\gamma b}\{(\tilde{q}^\top)_{\gamma b}{}^a, (\tilde{q})^{\delta d}_c\}_{PB}(g^2 p_k\Gamma^0\Gamma^k - \frac{i}{2}[X_k,X_l]\Gamma^{kl})^{\beta c}_{\delta d} = \\ =& -\frac{i}{4g^2}Tr\left((g^2 p_i\Gamma^0\Gamma^i + \frac{i}{2}[X_i,X_j]\Gamma^{ij})\mathcal{P}_L(g^2 p_k\Gamma^0\Gamma^k - \frac{i}{2}[X_k,X_l]\Gamma^{kl})_\alpha{}^\beta\right) \ . \end{aligned} \tag{3.59}$$

To simplify this expression, we use: $\Gamma^0\Gamma^i\Gamma^0\Gamma^k = \Gamma^{ik} + \delta^{ik}$, $[\Gamma^0\Gamma^k, \Gamma^{ij}] = 2\Gamma^0(\delta^{ik}\Gamma^j - \delta^{jk}\Gamma^i)$ and $1/2\{\Gamma^{ij}, \Gamma^{kl}\} = \Gamma^{ijkl} + \delta^{il}\delta^{jk} - \delta^{ik}\delta^{jl}$. Thus, we obtain:

$$\{Q^\top_\alpha, Q^\beta\}^{bos.}_{PB} = -\frac{i}{2}Tr\left((\frac{g^2}{2}p_ip^i - \frac{1}{4g^2}([X_i,X_j])^2)\mathcal{P}_L - ip_i[X^i,X_j]\mathcal{P}_L\Gamma^0\Gamma^j + \frac{1}{8g^2}\mathcal{P}_L\Gamma^{ijkl}[X_i,X_j][X_k,X_l]\right)_\alpha^\beta \ ,$$

where we remark the 4-form central charge $X_{[i}X_jX_kX_{l]}$, usually identified as the charge of a longitudinal M5-brane wrapped in the longitudinal direction. The other terms involve bosonic Poisson brackets of the type $\{p_i, X^j\}_{PB}$ and contains fermions. To compute them, it is useful to calculate first:

$$\{(p_i)_b{}^a, ([X^j,X^k])_d{}^c\}_{PB} = -\delta^j_i(X^k)_b{}^c\delta^a_d - \delta^k_i(X^j)_d{}^a\delta^b_c + (j \leftrightarrow k) \ . \tag{3.60}$$

The fermionic part of the Poisson bracket is thus:

$$\begin{aligned}\{Q_\alpha^\top, Q^\beta\}_{PB}^{ferm.} &= -\frac{ig^2}{8}(\tilde{q}^\top)_{\gamma a}{}^b \Big[(\Gamma^0\Gamma^i)_\alpha{}^\gamma \{(p_i)_b{}^a, ([X^j,X^k])_d{}^c\}_{PB}(\Gamma_{jk})_\delta{}^\beta + \\ &\qquad + (\Gamma_{jk})_\alpha{}^\gamma\{(p_i)_d{}^c, ([X^j,X^k])_b{}^a\}_{PB}(\Gamma^0\Gamma^i)_\delta{}^\beta\Big](\tilde{q})_c^{\delta d} = \\ &= -\frac{ig^2}{4}Tr(X_j\{\tilde{q}^\gamma,\tilde{q}^\delta\})\Big[(\Gamma^0\Gamma^i\mathcal{P}_L)_{\alpha\gamma}(\mathcal{P}_L\Gamma_i{}^j)_{\delta\epsilon} - (\Gamma_i{}^j\mathcal{P}_L)_{\alpha\gamma}(\mathcal{P}_L\Gamma^0\Gamma^i)_{\delta\epsilon}\Big]\delta^{\epsilon\beta} \,. \end{aligned} \tag{3.61}$$

Using the symmetry of $\Gamma^0\Gamma^i$ and the anti-symmetry of Γ^{ij} and symmetrizing it explicitly under $(\gamma \leftrightarrow \delta)$, the expression in the brackets can be reduced using the following Fierz identity of the Appendix on Majorana fermions of $SO(9,1)$:

$$(\mathcal{P}_L\Gamma^0\Gamma^i)_{\alpha\gamma}(\mathcal{P}_L\Gamma_i{}^j)_{\epsilon\delta} + (\mathcal{P}_L\Gamma_i{}^j)_{\alpha\gamma}(\mathcal{P}_L\Gamma^0\Gamma^i)_{\epsilon\delta} + (\gamma\leftrightarrow\delta) = 2((\mathcal{P}_L\Gamma^0\Gamma^j)_{\gamma\delta}(\mathcal{P}_L)_{\alpha\epsilon} - (\mathcal{P}_L\Gamma^0\Gamma^j)_{\alpha\epsilon}(\mathcal{P}_L)_{\gamma\delta})\,.$$

Thus, we obtain:

$$\{Q_\alpha^\top, Q^\beta\}_{PB}^{ferm.} = -\frac{1}{2}Tr\left(X_j\{p_\Psi,\Psi\}((\mathcal{P}_L\Gamma^0\Gamma^j)_\alpha{}^\beta - X_j\{p_\Psi\Gamma^0,\Gamma^i\Psi\}(\mathcal{P}_L)_\alpha{}^\beta\right)\,. \tag{3.62}$$

Adding the two parts of the calculation and summarizing, we get the supersymmetry algebra of the BFSS matrix model:

$$\begin{aligned}\{\tilde{Q}_\alpha^\top, \tilde{Q}^\beta\}_{PB} &= -\frac{i}{g^2}(\mathcal{P}_L)_\alpha^\beta Tr(\mathbb{1})\,, \\ \{\tilde{Q}_\alpha^\top, Q^\beta\}_{PB} &= -\frac{i}{2}(\mathcal{P}_L\Gamma^0\Gamma^i)_\alpha{}^\beta P_i - \frac{1}{4g^2}(\mathcal{P}_L\Gamma^{ij})_\alpha{}^\beta Tr([X_i,X_j])\,, \\ \{Q_\alpha^\top, Q^\beta\}_{PB} &= -\frac{i}{2}(\mathcal{P}_L)_\alpha{}^\beta H + \frac{1}{2}(\mathcal{P}_L\Gamma^0\Gamma^j)_\alpha{}^\beta Tr\left(p_i[X^i,X_j] + p_\Psi[X_j,\Psi]\right) - \\ &\quad - \frac{i}{16g^2}(\mathcal{P}_L\Gamma^{ijkl})_\alpha{}^\beta Tr\left([X_i,X_j][X_k,X_l]\right)\,.\end{aligned} \tag{3.63}$$

It is worth noting that the M-theory radius is related to the Yang-Mills coupling as $R_{10} = \sqrt{\alpha'}g_s = g^2$, so that $1/g^2 Tr(\mathbb{1}) = N/R_{10} = P_+ = Tr(p_+)$, the longitudinal light-cone momentum, where N is counting the number of D0-branes in the system. On the other hand, the Hamiltonian is $H = P_-$.

We would expect that the supersymmetry of BFSS matrix theory can be obtained from some IMF limit applied to the eleven-dimensional supersymmetry algebra [155, 146]:

$$\{Q_\alpha, Q^\beta\} = -\frac{i}{2}(\tilde{C}\tilde{\Gamma}^M)_\alpha^\beta P_M + \frac{i}{2}(\tilde{C}\tilde{\Gamma}^{MN})_\alpha^\beta Z_{MN} - \frac{i}{2}(\tilde{C}\tilde{\Gamma}^{M_1\ldots M_5})_\alpha^\beta Z_{M_1\ldots M_5} \tag{3.64}$$

Note that the charges Z_{MN} and $Z_{M_1\ldots M_5}$ are given by space integrals of the time components of the associated conserved currents j_{MN0} and $j_{M_1\ldots M_5 0}$, so that anti-symmetry forces the time-components of the charges Z_{M0} and $Z_{M_1\ldots M_4 0}$ to vanish. In the light-cone frame, the eleven coordinates x^M go into the nine space-like x^i and the light-cone coordinates x^+ and x^-, $\tilde{C} = \Gamma^0$ and $\Gamma^{10} \to \mathcal{P}_L$, so that we expect something of the form:

$$\{\tilde{Q}_\alpha, \tilde{Q}^\beta\} = -\frac{i}{2}(\mathcal{P}_L)_\alpha^\beta P_+\,, \tag{3.65}$$

$$\{Q_\alpha, \tilde{Q}^\beta\} = -\frac{i}{2}(\mathcal{P}_L\Gamma^0\Gamma^i)_\alpha^\beta P_i + \frac{i}{2}(\mathcal{P}_L\Gamma^{ij})_\alpha^\beta Z_{ij} - \frac{i}{2}(\mathcal{P}_L\Gamma^0\Gamma^{i_1\ldots i_5})_\alpha^\beta Z_{i_1\ldots i_5} \tag{3.66}$$

$$\{Q_\alpha, Q^\beta\} = -\frac{i}{2}(\mathcal{P}_L)_\alpha^\beta H + \frac{i}{2}(\mathcal{P}_L\Gamma^0\Gamma^i)_\alpha^\beta Z_i - \frac{i}{2}(\mathcal{P}_L\Gamma^{i_1\ldots i_4})_\alpha^\beta Z_{i_1\ldots i_4}\,. \tag{3.67}$$

To summarize our results, we obtained expressions for the charges of a transverse membrane:

$$Z_{ij} = \frac{i}{2R} Tr([X_i, X_j]) \; , \tag{3.68}$$

as well as for a membrane wrapped around the eleventh dimension:

$$Z_i = -iTr(p_i[X^i, X_j] + p_\Psi[X_j, \Psi]) = -iTr(X_j \tilde{C}) \tag{3.69}$$

and an M5-brane wrapped around the eleventh dimension:

$$Z_{ijkl} = \frac{1}{2R} Tr(X_{[i} X_j X_k X_{l]}) \; . \tag{3.70}$$

Note that the central charges indeed vanish at finite N since they are either given by the trace of a commutator or proportional to the constraint $\tilde{C} = 0$.

On the other hand, we have no term proportional to $\Gamma^{i_1 \dots i_5}$ that we could identify as the charge of a transverse M5-brane. This is the heart of the puzzle of the transverse M5-branes in BFSS theory. Its absence is probably related to the IMF limit that breaks the covariance of the model, since purely transverse D-branes cannot be constructed in light-cone perturbative string theory neither, because the Virasoro condition $\partial_s X^- = \partial_s X^a \partial_t X^a$ shows that both Dirichlet and Neumann boundary conditions in transverse directions imply Neumann boundary conditions in the longitudinal direction [21]. From a matrix model point of view, transverse D-branes are not translation invariant in the longitudinal direction, so that they do not remain static objects when we boost the system to the IMF. Note that, however, large transverse strings are allowed in the light-cone formalism, so that the apparent dissymmetry between transverse membranes and transverse M5-branes in the BFSS matrix model might be due to the fact that the membranes are fundamental objects in matrix theory, in a way that we will make more precise in the following sections, while the M5-branes are not. This remark led us to the idea that a fully covariant description of M-theory might involve membranes and M5-branes on an equal footing, ultimately leading to a matrix model of the type presented in [19].

3.3 The theory of membranes

In this section, we want to explain in more details why the matrix model is related to the propagation of supermembranes in a 10+1-dimenional space-time, a property we would expect from any theory claiming to be a candidate for M-theory. Let us first describe a bosonic membrane moving in a space-time of arbitrary dimension. The most obvious choice would be a Nambu-Goto type action ([127, 95]) proportional to the membrane's area:

$$S_{NG} = -T_M \int \sqrt{-\det(h_{\alpha\beta})} d^3\sigma \; , \tag{3.71}$$

where the σ_α's are 3 coordinates on the membrane's world-volume for $\alpha = 0, 1, 2$, $h_{\alpha\beta}(\sigma)$ is the metric on this world-volume and T_M denotes the membrane tension. In particular, if the membrane lives in a flat D-dimensional space-time with coordinates x^μ, for $\mu = 0, \dots, D-1$, $h_{\alpha\beta}(\sigma)$ is the pullback of the D-dimensional metric and it can be expressed as:

$$h_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x_\mu \; . \tag{3.72}$$

Mimicking the procedure leading to the Polyakov action in string theory ([51, 64]), we introduce an auxiliary world-sheet metric $\gamma_{\alpha\beta}(\sigma)$ and write a free action for the x's, seeing them as bosonic field living on the membrane's world-volume:

$$S_M = -\frac{T_M}{2}\int (\gamma^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x_\mu - 1)\sqrt{-\det(\gamma_{\alpha\beta})}d^3\sigma \ . \tag{3.73}$$

Note that we need to introduce a cosmological term since the theory is not conformally invariant, unlike Polyakov's string. Computing the equations of motion from the variation of $\gamma_{\alpha\beta}$, we obtain:

$$-\gamma^{\alpha\gamma}\gamma^{\beta\delta}h_{\gamma\delta} + \frac{1}{2}\gamma^{\alpha\beta}\gamma^{\gamma\delta}h_{\gamma\delta} - \frac{1}{2}\gamma^{\alpha\beta} = 0 \tag{3.74}$$

Multiplying that equation with $\gamma_{\alpha\epsilon}$ and $\gamma_{\beta\phi}$, we get:

$$h_{\alpha\beta} = \frac{1}{2}\gamma_{\alpha\beta}(\gamma^{\gamma\delta}h_{\gamma\delta} - 1) \ . \tag{3.75}$$

We can finally multiply by $\gamma^{\alpha\beta}$ to obtain:

$$\gamma^{\alpha\beta}h_{\alpha\beta} = \frac{3}{2}(\gamma^{\alpha\beta}h_{\alpha\beta} - 1) \ , \tag{3.76}$$

so that:

$$\gamma^{\alpha\beta}h_{\alpha\beta} = 3 \text{ and } \gamma_{\alpha\beta} = h_{\alpha\beta} \ . \tag{3.77}$$

This shows the classical equivalence of the two formulations. On the other hand, from varying x^μ, we obtain the expected Klein-Gordon equation:

$$\partial_\alpha(\sqrt{-\det(\gamma)}\gamma^{\alpha\beta}\partial_\beta x^\mu) = 0$$

As is noted in [99], there are 6 independent components of the metric γ, but only three reparametrization symmetries of the σ's to fix them. We can however use those to set:

$$\gamma_{00} = -\frac{4}{N^2}\det(h_{ab}) \doteqdot -h \ , \qquad \gamma_{0a} = 0 \ , \tag{3.78}$$

where $a, b, \ldots$ denote space-like world-volume indices. Note that this gauge makes sense in a world-volume of the type $\mathbb{R}\times\Sigma$ where Σ is a Riemann surface of fixed topology. The gauge-fixed action becomes:

$$S_g = -\frac{T_M}{2}\int \left(-\frac{1}{\sqrt{h}}\dot{x}^\mu\dot{x}_\mu + \sqrt{h}(\gamma^{ab}\partial_a x^\mu\partial_b x_\mu - 1)\right)\sqrt{\det(\gamma_{ab})}d^3\sigma \ . \tag{3.79}$$

Using (3.77) to set the remaining components of γ_{ab} to h_{ab} so that $\gamma^{ab}h_{ab} = 2$, we obtain another gauge-fixed action:

$$S_g = \frac{NT_M}{4}\int (\dot{x}^\mu\dot{x}_\mu - h)\, d^3\sigma \ . \tag{3.80}$$

The determinant can be conveniently written in the form of a Poisson bracket defined by $\{f,g\} = \epsilon^{ab}\partial_a f\partial_a g$ with ϵ^{ab} anti-symmetric and $\epsilon^{12} = 1$, since:

$$\frac{N^2}{4}h = \partial_1 x^\mu\partial_1 x_\mu\partial_2 x^\nu\partial_2 x_\nu - \partial_1 x^\mu\partial_2 x_\mu\partial_1 x^\nu\partial_2 x_\nu = \frac{1}{2}(\partial_1 x^\mu\partial_2 x_\nu - \partial_2 x^\mu\partial_1 x_\nu)^2 = \frac{1}{2}\{x^\mu, x^\nu\}\{x_\mu, x_\nu\} \ .$$

With that notation, the action becomes:

$$S_g = \frac{NT_M}{4} \int \left(\dot{x}^\mu \dot{x}_\mu - \frac{2}{N^2} \{x^\mu, x^\nu\}\{x_\mu, x_\nu\} \right) d^3\sigma \ . \tag{3.81}$$

The equations of motion obtained by varying x^μ are:

$$\begin{aligned} \frac{N^2}{4} \ddot{x}^\mu &= \partial_1(\partial_1 x^\mu \partial_2 x^\nu \partial_2 x_\nu - \partial_2 x^\mu \partial_1 x^\nu \partial_2 x_\nu) + \partial_2(\partial_2 x^\mu \partial_1 x^\nu \partial_1 x_\nu - \partial_1 x^\mu \partial_1 x^\nu \partial_2 x_\nu) = \\ &= \partial_1(\{x^\mu, x^\nu\}\partial_2 x_\nu) + \partial_2(\{x^\nu, x^\mu\}\partial_1 x_\nu) = \{\{x^\nu, x^\mu\}, x_\nu\} \ . \end{aligned}$$

However, this system is still constrained by the remaining "shadows" of (3.77). Indeed,

$$\gamma_{\alpha\beta} = h_{\alpha\beta} \to \begin{cases} h_{00} = \dot{x}^\mu \dot{x}_\mu = \gamma_{00} = -h \ , \\ h_{0a} = \dot{x}^\mu \partial_a x_\mu = 0 \ . \end{cases} \tag{3.82}$$

Note that the second line implies in particular:

$$\{\dot{x}^\mu, x_\mu\} = \partial_1(\dot{x}^\mu \partial_2 x_\mu) - \partial_2(\dot{x}^\mu \partial_1 x_\mu) = 0 \ . \tag{3.83}$$

We have thus expressed the closed bosonic membrane theory as a constrained system where the degrees of freedom are D scalars x^μ living on the membrane world-volume of topology $\mathbb{R} \times \Sigma$, where Σ is a Riemann surface. Although it is a much nicer formulation than the original Nambu-Goto theory, it remains very difficult to quantize because of the constraints and the non-linearity of the equations of motion.

As in string theory, the choice of light-cone coordinates for the embedding space-time is a useful step towards quantization. Specifically, we set:

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{D-1}) \tag{3.84}$$

With such coordinates, the flat Minkowski metric is described by: $\eta_{++} = \eta_{--} = 0$, $\eta_{+-} = \eta_{-+} = -1$, $\eta_{i\pm} = \eta_{\pm i} = 0$, while the transverse part is $\eta_{ij} = \delta_{ij}$ as before, so that the scalar product of two vectors becomes:

$$V^\mu W_\mu = V^i W^i - V^- W^+ - V^+ W^- \ . \tag{3.85}$$

Light-cone gauge is defined by the relation:

$$x^+ = \sigma^0 \doteqdot \tau \ . \tag{3.86}$$

This reduces the constraints (3.82) to:

$$\begin{cases} \partial_\tau x^- = \frac{1}{2}(\partial_\tau x^i \partial_\tau x^i + h) = \frac{1}{2}(\partial_\tau x^i \partial_\tau x^i + \frac{2}{N^2}\{x^i, x^j\}\{x_i, x_j\}) \ , \\ \partial_a x^- = \partial_\tau x^i \partial_a x_i \ , \end{cases} \tag{3.87}$$

while the action reads:

$$S_{LC} = \frac{NT_M}{4} \int \left(-2\partial_\tau x^- + \partial_\tau x^i \partial_\tau x_i - \frac{2}{N^2}\{x^i, x^j\}\{x_i, x_j\} \right) d^3\sigma \ . \tag{3.88}$$

The conjugate momenta of x^- and x^i are thus:

$$p_- = -p^+ = \frac{\delta S_{LC}}{\delta(\partial_\tau x^-)} = -\frac{NT_M}{2} , \quad p_i = \frac{\delta S_{LC}}{\delta(\partial_\tau x^i)} = \frac{NT_M}{2}\partial_\tau x^i , \tag{3.89}$$

so that the light-cone Hamiltonian reads:

$$H_{LC} = \int (p_i \partial_\tau x^i + p_- \partial_\tau x^-) d^2\sigma - L = \frac{NT_M}{4} \int \left(\partial_\tau x^i \partial_\tau x_i + \frac{2}{N^2}\{x^i, x^j\}\{x_i, x_j\} \right) d^2\sigma . \tag{3.90}$$

The only remaining constraint comes from (3.83), forcing the transverse degrees of freedom to satisfy:

$$\{\dot{x}^i, x_i\} = 0 . \tag{3.91}$$

Although we simplified the constraints, this theory is still unfortunately rather difficult to quantize, since the equations of motion are still non-linear. This Hamiltonian has a residual invariance under time-independant area-preserving diffeomorphisms that leave the symplectic form invariant.

3.3.1 Matrix regularization of the membrane action

We now want to show how it is possible to regularize this membrane theory and turn it into a matrix model. Following Goldstone and Hoppe ([106]), we will treat the case where the membrane has the topology of a sphere. Functions on such a membrane can be described by three cartesian coordinates χ_1, χ_2 and χ_3 satisfying:

$$(\chi^1)^2 + (\chi^2)^2 + (\chi^3)^2 = 1 . \tag{3.92}$$

In the usual spherical coordinates:

$$\chi_1 = \sin(\theta)\cos(\phi) , \quad \chi_2 = \sin(\theta)\sin(\phi) , \quad \chi_3 = \cos(\theta) , \tag{3.93}$$

we can define a Poisson bracket on the sphere by:

$$\{f, g\} = \frac{1}{\sin(\theta)}\left(\frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \phi} - \frac{\partial g}{\partial \theta}\frac{\partial f}{\partial \phi}\right) . \tag{3.94}$$

With this definition, the χ's satisfy the Poisson algebra:

$$\{\chi^a, \chi^b\} = \epsilon^{abc}\chi^c . \tag{3.95}$$

A representation of this Poisson algebra can be given in terms of any N-dimensional representation of $\mathfrak{su}(2)$. Let us identify:

$$\chi^a \leftrightarrow \frac{2}{N} J^a , \tag{3.96}$$

where the J^a's generate the N-dimensional matrix representation of $SU(2)$ and:

$$[J^a, J^b] = i\epsilon^{abc} J^c . \tag{3.97}$$

This is a useful idea, since we can find a matrix approximation of any function on the sphere in the following way:

- Give a representation of each spherical harmonics $y^{lm}(\vec{\chi}) = \sum_{a_1 \ldots a_l} C^{lm}_{a_1 \ldots a_l} \chi^{a_1} \cdot \ldots \cdot \chi^{a_l}$ with $l < N$ in terms of matrices as

$$Y^{lm} = \sum_{a_1 \ldots a_l} C^{lm}_{a_1 \ldots a_l} J^{a_1} \cdot \ldots \cdot J^{a_l} \, . \tag{3.98}$$

- For each function on the sphere, $f(\vec{\chi}) = \sum_{l,m} k_{lm} y^{lm}(\vec{\chi})$, define its matrix approximation as

$$F = \sum_{l<N,m} k_{lm} Y^{lm} \, , \tag{3.99}$$

  where the sum stops at $l = N-1$, since we cannot represent higher spherical harmonics with an $N \times N$ matrix, because the dimensions match as: $\sum_{l=0}^{N-1} (2l+1) = N^2$.

- Comparing (3.95) with (3.97), we can see that the correspondence (3.96) implies the prescription:

$$\{f, g\} \leftrightarrow \frac{-iN}{2} [F, G] \, . \tag{3.100}$$

- Finally, we have to discretize integrals in this way:

$$\frac{1}{4\pi} \int f d^2\sigma \leftrightarrow \frac{1}{N} Tr(F) \, , \tag{3.101}$$

  where the coefficients are chosen so that both sides agree exactly in the large N limit.

Applying these ideas to discretize our light-cone membrane Hamiltonian H_{LC}, we get the following matrix approximation:

$$H_{LC} = \pi T_M Tr \left(\partial_\tau X^i \partial_\tau X_i - \frac{1}{2} [X^i, X^j][X_i, X_j] \right) \, . \tag{3.102}$$

Recalling from (3.6) that the D2-brane tension was $T_2 = \frac{1}{4\pi^2 \alpha'^{3/2}}$ in string units, the membrane tension should be $T_M = \frac{1}{4\pi^2 l_M^3}$ for some appropriate length scale l_M for the membrane theory. Since $P_i = 2\pi T_M \partial_\tau X^i = \frac{1}{2\pi l_M^3} \partial_\tau X^i$, we obtain:

$$H_{LC} = \frac{1}{2} Tr \left(2\pi l_M^3 P_i P^i - \frac{1}{2} \frac{1}{2\pi l_M^3} [X^i, X^j][X_i, X_j] \right) \, . \tag{3.103}$$

In the case of a membrane living in eleven-dimensional space-time, this agrees exactly with the bosonic part of (3.14) rescaled by a global factor of $2\pi\alpha'$ if we choose l_M so that $2\pi l_M^3 = (2\pi\alpha')^3 g_1^2$. Since $g_1^2 = \frac{g_s}{4\pi^2 \alpha'^{3/2}}$, we should take:

$$l_M = \sqrt{\alpha'} g_s^{\frac{1}{3}} \, . \tag{3.104}$$

This fits very nicely in the big picture, since this is precisely the eleven-dimensional Planck length expressed in string units, as given by (3.35).

3.3.2 Supersymmetric extension of the membrane action

It is indeed possible to extend the membrane action (3.81) to a supermembrane theory. However, as is the case for the Green-Schwarz formulation of the superstring [96, 97, 98], such a supersymmetric extension is only possible in a limited number of dimensions, namely 4, 5, 7 and 11. Note that these are all one dimension higher than in the Green-Schwarz superstring (and $\mathcal{N} = 1$ super Yang-Mills theory) case. Although all these theories are equally acceptable classically, it is believed that only the eleven-dimensional one is free from anomalies in the Lorentz algebra, making eleven-dimensional space-time the natural setting for propagating membranes, just as ten-dimensional space-time is natural for superstrings. We won't go into details about the developments that lead to such theories, here, but we will quote the main results. Details can be found in the original paper of Bergshoeff, Sezgin and Townsend [33, 32] or in the review [74]. Using the convenient supersymmetry-invariant combination:

$$\pi^\mu_\alpha = \partial_\alpha x^\mu + \bar{\psi}\Gamma^\mu\partial_\alpha\psi \ , \tag{3.105}$$

where ψ is a Majorana spinor of $SO(10,1)$. the membrane action in a flat background can be written as:

$$\begin{aligned} S = -\frac{T_M}{2}\int \Bigg(& \sqrt{-\det(\gamma_{\alpha\beta})}(\gamma^{\alpha\beta}\pi^\mu_\alpha\pi^\mu_\beta - 1) - \epsilon^{\alpha\beta\gamma}\Big[\frac{1}{2}\partial_\alpha x^\mu(\partial_\beta x^\nu + \bar{\psi}\Gamma^\nu\partial_\beta\psi) + \\ & + \frac{1}{6}(\bar{\psi}\Gamma^\mu\partial_\alpha\psi)(\bar{\psi}\Gamma^\nu\partial_\beta\psi)\Big]\bar{\psi}\Gamma_{\mu\nu}\partial_\gamma\psi\Bigg) d^3\sigma \ . \end{aligned} \tag{3.106}$$

Besides the reparametrization symmetry that we already had in the bosonic case, this action has another world-volume symmetry, called κ-symmetry. As for the Green-Schwarz superstring, this symmetry is necessary to have a matching of the number of bosonic and fermionic physical degrees of freedom, since it allows to project out half of the fermions. The Wess-Zumino terms proportional to $\epsilon^{\alpha\beta\gamma}$ are indeed chosen so that this symmetry holds. On the other hand, this action exhibits the desired space-time supersymmetry:

$$\delta x^\mu = -\bar{\epsilon}\Gamma^\mu\psi \ , \qquad \delta\psi = \epsilon \ , \tag{3.107}$$

that indeed leaves π^μ_α invariant. The Fierz identity:

$$(\bar{\psi}_{[1}\Gamma^\mu\psi_2)(\bar{\psi}_3\Gamma_{\mu\nu}\psi_{4]}) = 0 \ . \tag{3.108}$$

is needed in the proofs of both supersymmetry and κ-symmetry, which explains why only space-time dimensions 4, 5, 7 and 11 are allowed. Another parallel with the Green-Schwarz superstring lies in the fact that gauge-fixing of the κ-symmetry usually breaks the Lorentz covariance of the action, so that it is difficult to find a way to quantize the theory in a covariant way. It is thus customary to study supermembranes in the light-cone gauge. Without going into details, let us just quote the resulting light-cone hamiltonian:

$$H_{LC} = \frac{NT_M}{4}\int \left(\partial_\tau x^i\partial_\tau x^i + \frac{2}{N^2}\{x^i,x^j\}\{x_i,x_j\} - \frac{2i}{N}\bar{\psi}\Gamma^i\{x_i,\psi\}\right) d^2\sigma \ . \tag{3.109}$$

where κ-symmetry has been used to project out half of the 32 components of ψ, that can now be seen as a Majorana-Weyl spinor of $SO(9,1)$. Applying the matrix regularization procedure to that functional, we obtain as expected the BFSS Hamiltonian (3.26) in the gauge $A = 0$:

$$H_{LC} = 2\pi T_M Tr\left(\frac{1}{2}\partial_\tau X^i \partial_\tau X^i - \frac{1}{4}[X^i, X^j][X^i, X^j] - \frac{1}{2}\bar{\Psi}\Gamma^i[X^i, \Psi]\right) , \tag{3.110}$$

where we should set $2\pi T_M = \frac{1}{2\pi l_P^3}$ as before.

There have been some attempts to treat supermembranes theories in a more covariant way. One can look at [79, 80] for more details. There are also supermembrane actions valid in general supergravity backgrounds [33, 32] expressed in a superspace formulation. Although the precise identification between superfields and component fields is not fully understood yet [63], it is interesting to note that κ-symmetry only holds when the background fields satisfy the equations of motion of eleven-dimensional N=1 supergravity [33], so that supergravity seems to emerge out of supermembranes already at the classical level. Rather than explaining these facts in details here, we will discuss the puzzle related to the continuous spectrum of the membrane theories in the following subsection.

3.3.3 Membranes as a second-quantized theory

De Wit, Lüscher and Nicolai [62] showed that the membrane theory has no mass gap separating the massless degrees of freedom from the rest of the spectrum, as opposite to superstring theory and its equally spaced mass levels. The geometrical intuitive reason for this instability is that the membrane can grow very long spikes at a very small cost of energy. Indeed, a roughly cylindrical spike of length l and radius r has a classical energy $2\pi T_M rl$, proportional to its area. If the spike's radius is very small, so that $r \ll 1/(T_M l)$, the energy cost is very small although the spike can be quite long. A quantum membrane is therefore subject to many fluctuations of this type, making it hard to interpret as a localized object. Note that this phenomenon doesn't occur in string theory since a long string has energy proportional to its length, without any other parameter to make it small. From a theoretical point of view, that instability is clear in the matrix-regularized form of the theory. Indeed, there are flat directions in the potential, since any set of mutually commuting matrices minimize the $Tr(([X^i, X^j])^2)$ potential term. For example, if we take 9 diagonal matrices, they will minimize the potential for any diagonal components. In fact, the off-diagonal components builds an effective confining potential for the diagonal ones in the matrix-regularized bosonic membrane theory,which leads to a discrete spectrum of energy levels for $N \times N$ matrices of fixed sizes. However, supersymmetric compensations in the supermembrane case precisely cancel this effective potential, giving rise to a continuous spectrum of states. This result was first proven in [62] for the $U(2)$ maximally supersymmetric matrix model and subsequently refined in [78]. This apparent disadvantage of matrix theory was later turned into a virtue by the new interpretation that Banks, Fischler, Schenker and Susskind brought up in [20].

From the BFSS point of view, we indeed expect a continuous spectrum of states to appear for any matrix size bigger or equal to 2. Suppose there are two single massless graviton states in M-theory. If they are non-interacting (infinitely far apart), we can represent them as two diagonal components in a $U(2)$ matrix theory. Considering these states together, we can obtain a bound state of arbitrary small energy by taking them to have a large separation and a small relative velocity. This discussion of course extends to higher values of N, where we can change the particle number by creating or destroying bound states of two or many D-particles, which reflects itself in the matrix model formulation by a matrix having a bigger or smaller number of non-interacting diagonal blocks. In this sense, matrix theory should be seen as a second-quantized theory where the particle number is not conserved.

From the membrane point of view, as the building of long spikes has very low energy cost, we can imagine a big membrane decaying in smaller-size membranes joined by very thin tubes. In the limit where the latter become very small, their effect on the classical dynamics becomes negligible and we have an effective system of multiple membranes, even though the topology didn't really change. When quantum effects are taken into account, we can add or remove handles and the topology can also change, so that a consistent quantum supermembrane theory should allow for an arbitrary topology and be a "second quantized" theory from the target-space point of view.

Chapter 4

The IIB matrix model

The aim of this chapter is to introduce the basics of the IIB matrix model that are relevant to the research project of the chapter 6, as well as to make clear how the IIB matrix model positions itself in comparison to the BFSS model in the big string theory picture. We will also alternatively call it IKKT model from the initials of its inventors.

After the definition of the model and a discussion of its classical solutions, we will show how it can be derived from regularizing the Green-Schwarz superstring action for type IIB theory (i.e. with two supercharges of the same chirality) by going into the Schild gauge. Although it has first been obtained from the Nambu-Goto formulation of the superstring action, we propose here an alternative derivation (to my knowledge, new in the literature) starting from the Polyakov-type action, in order to enhance the parallelism with the membrane case of the preceding chapter. This also requires a new choice for the parameters of the κ-symmetry in order to induce the $\mathcal{N}=2$ supersymmetry of the IKKT matrix theory from a combination of the $\mathcal{N}=2$ supersymmetry and the 2 κ-symmetries of the Green-schwarz superstring that preserve the choice of the Schild gauge.

We further study the supersymmetry algebra of the IIB matrix model and show how a convenient combination of the dynamical and kinematical supercharges rotates its strange-looking algebra into the usual $\mathcal{N}=2$ supersymmetry in 10 dimensions. Finally, we compute the anti-commutators of the supercharges in a calculation parallel to that performed in the BFSS case. It allows us to obtain expressions for the D1-brane and D3-brane charges, as one would expect from a theory related to type IIB string theory. This part reviews and details known results, correcting some earlier mistakes in the literature.

Then, we briefly overview related matrix models that have the interesting feature that they possess classical solutions where the spacetime-generating bosonic matrices build various non-commutative spaces. In particular, we show how to obtain fuzzy-sphere solutions of various dimensions in such kind of models. This line of work directly leads to the ideas presented in the research project of chapter 6.

4.1 The IIB matrix model

Another proposal for a non-perturbative definition of superstring theory that takes the form of a matrix theory is the so-called IIB matrix model that was suggested by Ishibashi, Kawai, Kitazawa and Tsuchiya [107] and further refined with Fukuma [81] in 1997, also called IKKT model in the literature. Similarly to the BFSS model, it is also a compactified version of $\mathcal{N}=1$ D=9+1 super Yang-Mills theory with gauge group $U(N)$, with the difference that it is compactified to 0+0 dimension instead of 0+1. Such models having a point-like target space are called totally reduced models [75, 94, 93]. As such, they are not really dynamical, the information is contained in the partition function. We can define the model in two different ways, either through the large N limit of the microcanonical partition function:

$$Z = \int dX d\Psi e^{+S_E^{(1)}} \ ,$$

where $S_E^{(1)} = -H^{(1)}$ is defined in the 10-dimensional Euclidean space by Wick-rotating X_0 and Γ_0 in the manifestly covariant action:

$$S^{(1)}_{IKKT} = \frac{1}{g_0^2} Tr\left(\frac{1}{4}[X_\mu, X_\nu][X^\mu, X^\nu] + \frac{1}{2}\bar{\Psi}\Gamma^\mu[X_\mu, \Psi]\right) \tag{4.1}$$

expressed in terms of hermitian $\sqcap(N)$ matrices X_μ and ψ of large but fixed sizes. In target-space, ψ is a 10-dimensional Majorana-Weyl spinor field with 16 real components that we choose as left-handed. This action is indeed the direct time-like dimensional reduction of the BFSS matrix model of the preceding chapter. Alternatively, the model can be defined through the grand-canonical partition function:

$$Z[\beta] = \sum_{N=1}^{\infty} \int dX d\Psi e^{+S_E^{(2)}[\beta]}$$

where β is the chemical potential dual to the matrix size N and $S_E^{(2)}[\beta]$ is the Wick-rotated version of:

$$S^{(2)}_{IKKT}[\beta] = \frac{1}{2\alpha'^2\beta} Tr\left(\frac{1}{4}[X_\mu, X_\nu][X^\mu, X^\nu] + \frac{1}{2}\bar{\Psi}\Gamma^\mu[X_\mu, \Psi]\right) + \beta N \tag{4.2}$$

It is generally expected that the grand-canonical partition function has a critical behaviour in the limit $\beta \to \beta_c$ (we will give the classical value of β_c later) that is identical to the microcanonical partition function with a large but fixed value of N. In fact, the action (4.2) can also be regarded as an effective action for (4.1) (see section 4.2 of [107] for details).

Issues of convergence are treated in details in Austing's thesis [12] and the previous papers with Wheater [13, 14].

In either case, the classical equations of motion are easily derived as:

$$\begin{aligned} 2[X_\nu, [X^\mu, X^\nu]] - \{\bar{\Psi}, \Gamma^\mu \Psi\} &= 0 \qquad (4.3)\\ \Gamma^\mu[X_\mu, \Psi] &= 0 \ . \qquad (4.4) \end{aligned}$$

We immediately see that any classical configuration with diagonal matrices will solve the equations of motion and minimize the energy functional. In a type IIB language, this corresponds to a bunch of N non-interacting D-instantons. Following [107], we can also construct D-string configurations, by choosing configurations where $[X^\mu, X^\nu]$ is a diagonal matrix for all values of μ and ν.

In the continuum, a string stretching in the direction of x^1 can be realized by:

$$x^0 = \tau \; , \quad x^1 = c^1 + \frac{L}{2\pi}\sigma \; , \quad x^\mu = c^\mu \; \forall \, \mu = 2, \ldots, 9 \; , \tag{4.5}$$

for 9 constants c^μ describing the position of the string of length L in the target-space and $0 \leq \sigma \leq 2\pi$. This system of course satisfies:

$$\{x^0, x^1\} = \frac{L}{2\pi} \; , \tag{4.6}$$

with all other Poisson brackets vanishing. In particular, it satisfies the classical equations of motion:

$$\{x_\mu, \{x^\mu, x^\nu\}\} = 0 \; , \tag{4.7}$$

Seeing large N matrices as operators acting on an N-dimensional space, we can realize the matrix approximation of this system through

$$X^0 = \sqrt{\frac{2}{N}} q \; , \quad X^1 = c^1 \mathbb{1} + \sqrt{\frac{2}{N}} \frac{L}{2\pi} p \; , \quad X^\mu = c^\mu \mathbb{1} \; \forall \, \mu = 2, \ldots, 9 \; , \tag{4.8}$$

if the operators satisfy canonical commutation relations of the kind $[q, p] = i$. Then, it follows that:

$$[X^0, X^1] = \frac{L}{\pi N} \;\text{ and }\; [X_\nu, [X^\mu, X^\nu]] = 0 \; . \tag{4.9}$$

However, there is no finite-dimensional representation of the algebra $[q, p] = i$, so that these states only exists in the $N \to \infty$ limit. This is consistent with the fact that the charges of D-branes configurations are given by traces of commutators that can be non-zero only in the $N \to \infty$ limit.

Later in this chapter, we will study alternative models where a phenomenological term has been added to the original action. Such kind of models have more interesting classical vacua.

4.1.1 The IKKT model as a regularization of the IIB superstring

One of the main motivation leading towards the IKKT action is that it is related to the Green-Schwarz [96, 97] superstring action in the so-called Schild gauge [143]. Let us try to explain this connection here. The IIB superstring is defined in terms of the world-sheet scalar fields x^μ, θ and ψ, which are respectively a target-space vector (describing the embedding of the string in the 10-dimensional target-spacetime) and two Majorana-Weyl target-space spinors of the same chirality (we take them as left-handed here). We first define a convenient combination of these fields:

$$\pi^\mu_\alpha = \partial_\alpha x^\mu - i\bar{\theta}\Gamma^\mu \partial_\alpha \theta + i\bar{\psi}\Gamma^\mu \partial_\alpha \psi \; , \tag{4.10}$$

which is invariant under the supersymmetry transformations:

$$\begin{aligned} \delta x^\mu &= i\bar{\epsilon}\Gamma^\mu \theta - i\bar{\xi}\Gamma^\mu \psi \\ \delta\theta &= \epsilon \; , \quad \delta\psi = \xi \; , \end{aligned} \tag{4.11}$$

Besides the obviously supersymmetric kinetic term in π^2, we also need a Wess-Zumino term to get a κ-symmetric action. Indeed, on a world-sheet with coordinates σ^α, $\alpha = 0,1$ and metric $h_{\alpha\beta}$, the Green-Schwarz action reads:

$$S_{GS} = \frac{1}{4\pi\alpha'} \int \Big[(h^{\alpha\beta}\pi^\mu_\alpha \pi_{\beta\mu})\sqrt{-h} + 2i\epsilon^{\alpha\beta}\partial_\alpha x^\mu(\bar\theta\Gamma_\mu\partial_\beta\theta + \bar\psi\Gamma_\mu\partial_\beta\psi) + 2\epsilon^{\alpha\beta}(\bar\theta\Gamma^\mu\partial_\alpha\theta)(\bar\psi\Gamma_\mu\partial_\beta\psi)\Big] d^2\sigma \ ,$$

where we have written $h = \det h_{\alpha\beta}$. If we define the following projection operators:

$$P^{\alpha\beta}_\pm = \frac{1}{2}(h^{\alpha\beta} \pm \frac{1}{\sqrt{-h}}\,\epsilon^{\alpha\beta}) \ .$$

and a pair of Grassmann parameters κ^1 and κ^2 that are world-sheet vectors and target-space spinors satisfying the self-duality conditions:

$$P^{\alpha\beta}_-\kappa^1_\beta = \kappa^{1\alpha} \ , \quad P^{\alpha\beta}_+\kappa^2_\beta = \kappa^{2\alpha} \ ,$$

we can show that, besides the above supersymmetry, this action is also invariant under the local κ-symmetry defined by:

$$\begin{aligned} \delta_\kappa x^\mu &= i\bar\theta\Gamma^\mu\delta\theta - i\bar\psi\Gamma^\mu\delta\psi \\ \delta_\kappa\theta = 2\Gamma^\mu\pi_{\alpha\mu}\kappa^{1\alpha} \ &, \quad \delta\psi = 2\Gamma^\mu\pi_{\alpha\mu}\kappa^{2\alpha} \\ \delta_\kappa(\sqrt{-h}h^{\alpha\beta}) &= -16i\sqrt{-h}\Big(\partial_\gamma\bar\theta\kappa^{1\beta}P^{\alpha\gamma}_- - \partial_\gamma\bar\psi\kappa^{2\beta}P^{\alpha\gamma}_+\Big) \ . \end{aligned} \tag{4.12}$$

In contrast to usual supersymmetries, the transformation law of x^μ contains both bosonic and fermionic fields in the case of κ-symmetry.

Details of the proof of κ-symmetry may be found in the appendix D on the Green-Schwarz superstring. It turns out that the same Fierz transformation that allows to prove supersymmetry of super Yang-Mills theory in D=10 and of the BFSS matrix model plays a central rôle in the proof of all interesting symmetries of the Green-Schwarz superstring. The following properties of the projection operators are also necessary in the proof and will be needed further in that chapter to understand how the supersymmetry of the IIB matrix model emerges out of the supersymmetry and κ-symmetry of the Green-Schwarz superstring. Firstly, P_- and P_+ are orthogonal projectors:

$$\begin{aligned} P^{\alpha\beta}_\pm h_{\beta\gamma}P^{\gamma\delta}_\pm &= P^{\alpha\delta}_\pm \\ P^{\alpha\beta}_+ h_{\beta\gamma}P^{\gamma\delta}_- &= P^{\alpha\beta}_- h_{\beta\gamma}P^{\gamma\delta}_+ = 0 \ . \end{aligned} \tag{4.13}$$

Secondly, they have the following symmetry property:

$$P^{\alpha\beta}_\pm P^{\gamma\delta}_\pm = P^{\gamma\beta}_\pm P^{\alpha\delta}_\pm \tag{4.14}$$

Now, we want to show how the Green-Schwarz action can be gauged to obtain the Schild action. The Schild gauge corresponds to the choice $\theta = \psi$ where we identify both Majorana-Weyl spinors, which is allowed in type IIB string theory, where both supercharges (and thus, θ and ψ, too) have the same chirality. In such a case; $\pi^\mu_\alpha = \partial_\alpha x^\mu$ and the last term in the action cancels due to anti-symmetry in $\alpha \leftrightarrow \beta$, so that the action reduces to:

$$S_{GS} = \frac{1}{4\pi\alpha'} \int \Big[h^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu\sqrt{-h} + 4i\epsilon^{\alpha\beta}\partial_\alpha x^\mu\bar\psi\Gamma_\mu\partial_\beta\psi\Big] d^2\sigma \ .$$

Varying S_{GS} with respect to $h_{\alpha\beta}$ we obtain the relation:

$$T^{\alpha\beta} = h^{\alpha\gamma}h^{\beta\delta}\partial_\gamma x^\mu \partial_\delta x_\mu - \frac{1}{2}h^{\alpha\beta}h^{\gamma\delta}\partial_\gamma x^\mu \partial_\delta x_\mu = 0 \; ,$$

which is solved by:

$$h_{\alpha\beta} = k\partial_\alpha x^\mu \partial_\beta x_\mu \tag{4.15}$$

for any constant k[1]. The standard choice $k = 1$ leads us to the equivalent Nambu-Goto [127, 95] formulation of the Green-Schwarz superstring:

$$S_{NG} = \frac{1}{2\pi\alpha'}\int \left[\sqrt{-\det(h^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x_\mu)} + 2i\epsilon^{\alpha\beta}\partial_\alpha x^\mu \bar{\psi}\Gamma_\mu \partial_\beta \psi\right] d^2\sigma \; .$$

For later convenience, we rather take $k = 2/\sqrt{N}$ in the following. Furthermore, since we want to have a polynomial action, we will use (4.15) to write $h^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x_\mu \sqrt{-h} = 2\sqrt{-h}$ as:

$$2\frac{-h}{-h}\sqrt{-h} = -2\frac{k^2}{-h}\det(\partial_\alpha x^\mu \partial_\beta x_\mu)\sqrt{-h} \; .$$

To obtain a Schild-type action, we still need to rewrite the determinant in a convenient way as:

$$\begin{aligned}\det(\partial_\alpha x^\mu \partial_\beta x_\mu) &= \partial_0 x^\mu \partial_0 x_\mu \partial_1 x^\nu \partial_1 x_\nu - \partial_0 x^\mu \partial_1 x_\mu \partial_1 x^\nu \partial_0 x_\nu = \\ &= \frac{1}{2}(\partial_0 x_\mu \partial_1 x_\nu - \partial_1 x_\mu \partial_0 x_\nu)(\partial_0 x^\mu \partial_1 x^\nu - \partial_1 x^\mu \partial_0 x^\nu) = \frac{1}{2}(\epsilon^{\alpha\beta}\partial_\alpha x_\mu \partial_\beta x_\nu)(\epsilon^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu) \; .\end{aligned} \tag{4.16}$$

We can now introduce a Poisson bracket similar to the one we used in the membrane theories through:

$$\{f, g\} = \frac{1}{\sqrt{-h}}\epsilon^{\alpha\beta}\partial_\alpha f \partial_\beta g \; , \tag{4.17}$$

so that the action reads:

$$S_{Schild} = \frac{2}{\pi\alpha'}\int \left[-\frac{1}{2N}\{x_\mu, x_\nu\}\{x^\mu, x^\nu\} + \frac{i}{2}\bar{\psi}\Gamma^\mu\{x_\mu, \psi\}\right]\sqrt{-h}\, d^2\sigma \; . \tag{4.18}$$

If we directly apply the regularization procedure that we described in the membrane case (paying attention to the fact that we haven't gauged the metric components here, so that we have to include the density $\sqrt{-h}$ in the relation so that it has an invariant meaning), we can do the replacements:

$$\{f, g\} \to -i\frac{N}{2}[F, G] \; , \quad \frac{1}{4\pi}\int \sqrt{-h}\, d^2\sigma \to \frac{1}{N}Tr(F) \tag{4.19}$$

and we obtain the totally reduced action (4.1):

$$S^{(1)}_{IKKT} = \frac{4}{\alpha'}Tr\left(\frac{1}{4}[X_\mu, X_\nu][X^\mu, X^\nu] + \frac{1}{2}\bar{\Psi}\Gamma^\mu[X_\mu, \Psi]\right) \tag{4.20}$$

for $g_0^2 = \alpha'/4$.

[1] k corresponds to the ratio of the length scales between world-sheet coordinates and target-space coordinates. For example, for a string extended in the direction x^i in a flat spacetime with metric $\eta_{\mu\nu}$, it corresponds to the choice of world-sheet coordinates $\sigma^0 = \sqrt{k}x^0$ and $\sigma^1 = \sqrt{k}x^i$ (indeed, this choice leads to a Minkowskian world-sheet metric $h_{\alpha\beta} = \eta_{\alpha\beta}$)

4.1.2 The grand-canonical formulation from the Schild superstring

Here, we want to show how to obtain a chemical potential for the grand-canonical formulation directly from the Schild formulation. This subsection is not really needed in the following and has been added here for completeness to show why the microcanonical and grand-canonical formulation should be related. As a byproduct, we will obtain the classical value of the critical chemical potential $\beta_c = 1/(8\alpha')$. Of course, we expect the quantum value to be different.

In order to obtain a chemical potential term in the matrix-regularized IIB action, we introduce a cosmological constant term in the Schild action. For that purpose, let us introduce two constants β and γ and write a new Schild action $\tilde{S}_{Schild}$ in this way:

$$\tilde{S}_{Schild}[\beta,\gamma] = \int \left[\gamma\left(-\frac{1}{2N}\{x_\mu, x_\nu\}\{x^\mu, x^\nu\} + \frac{i}{2}\bar{\psi}\Gamma^\mu\{x_\mu,\psi\}\right) + \frac{N}{4\pi}\beta\right]\sqrt{-h}\, d^2\sigma\ . \tag{4.21}$$

As was noted earlier in [107], this action is classically equivalent to the other. Indeed, if we vary $\tilde{S}_{Schild}$ with respect to $\sqrt{-h}$, we obtain:

$$\frac{\gamma}{2N(\sqrt{-h})^2}(\epsilon^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu)(\epsilon^{\gamma\delta}\partial_\gamma x_\mu \partial_\delta x_\nu) + \frac{N}{4\pi}\beta = 0 \tag{4.22}$$

which we can solve for $\sqrt{-h}$ as:

$$\sqrt{-h} = \frac{\sqrt{2\pi}}{N}\sqrt{\frac{\gamma}{\beta}}\sqrt{-(\epsilon^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu)(\epsilon^{\gamma\delta}\partial_\gamma x_\mu \partial_\delta x_\nu)}\ . \tag{4.23}$$

Substituting this constraint into (4.21), we obtain exactly the Nambu-Goto action (4.1.1):

$$\tilde{S}_{NG} = \int \left[\sqrt{\frac{\beta\gamma}{\pi}}\sqrt{-\det(h^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x_\mu)} + \frac{i\gamma}{2}\epsilon^{\alpha\beta}\partial_\alpha x^\mu \bar{\psi}\Gamma_\mu \partial_\beta \psi\right] d^2\sigma\ , \tag{4.24}$$

when $\gamma = 2/(\pi\alpha')$ and $\beta = 1/(8\alpha')$. In order to keep the chemical potential β open, we only want to fix the relative product of the two parameters to:

$$\gamma\beta = \frac{1}{4\pi\alpha'^2}\ , \tag{4.25}$$

while the factor of γ in the fermionic term of (4.24) can be absorbed through an appropriate rescaling of the fermionic variables. This choice leads us to the following form of the Schild superstring action:

$$\tilde{S}_{Schild}[\beta] = \int \left[\frac{1}{4\pi\alpha'^2\beta}\left(-\frac{1}{2N}\{x_\mu, x_\nu\}\{x^\mu, x^\nu\} + \frac{i}{2}\bar{\psi}\Gamma^\mu\{x_\mu,\psi\}\right) - \frac{\alpha' N}{8}\beta\right]\sqrt{-h}\, d^2\sigma\ . \tag{4.26}$$

We can now apply the matrix regularization to obtain the totally reduced action (4.2):

$$S^{(2)}_{IKKT}[\beta] = \frac{1}{2\alpha'^2\beta} Tr\left(\frac{1}{4}[X_\mu, X_\nu][X^\mu, X^\nu] + \frac{1}{2}\bar{\Psi}\Gamma^\mu[X_\mu,\Psi]\right) + \beta N\ . \tag{4.27}$$

As usually in dimensional reductions of $\mathcal{N} = 1$ super Yang-Mills theory, the first prefactor can be absorbed in a redefinition of the fields X^μ and Ψ.

4.2 Supersymmetry of the Schild superstring action

In this subsection, we first want to show how the supersymmetry of the IIB matrix model emerges from the supersymmetry and the κ-symmetry of the type IIB superstring in the Schild gauge. Recalling the supersymmetry transformations:

$$\begin{aligned} \delta_\chi x^\mu = i\bar{\chi}\Gamma^\mu\theta \ , \quad \delta_\xi x^\mu = -i\bar{\xi}\Gamma^\mu\psi \ , \\ \delta_\chi\theta = \chi \ , \quad \delta_\xi\psi = \xi \ , \end{aligned} \tag{4.28}$$

and the κ-symmetry transformations of the IIB superstring action:

$$\begin{aligned} \delta_{\kappa^1} x^\mu = i\bar{\theta}\Gamma^\mu\delta_{\kappa^1}\theta \ , \quad \delta_{\kappa^2} x^\mu = -i\bar{\psi}\Gamma^\mu\delta_{\kappa^2}\psi \ , \\ \delta_{\kappa^1}\theta = 2\Gamma^\mu\pi_{\alpha\mu}\kappa^{1\alpha} \ , \quad \delta\psi_{\kappa^2} = 2\Gamma^\mu\pi_{\alpha\mu}\kappa^{2\alpha} \ , \end{aligned} \tag{4.29}$$

we will identify two combinations of the 4 parameters that lead to new supersymmetry transformations that preserve the Schild gauge. We will then identify the matrix regularization of these transformations with the $\mathcal{N}=2$ supersymmetry of the IIB matrix model. With this goal in mind, we relate κ^1 and κ^2 to χ and ξ in the following way:

$$\kappa^{1\alpha} = \frac{k}{2}\Gamma^\mu\left(\frac{\xi-\chi}{2}\right)\pi_{\beta\mu}P_-^{\alpha\beta} \ , \quad \kappa^{2\alpha} = \frac{k}{2}\Gamma^\mu\left(\frac{\chi-\xi}{2}\right)\pi_{\beta\mu}P_+^{\alpha\beta} \ . \tag{4.30}$$

Although far from obvious, this choice leads to the following fermionic transformations:

$$\begin{aligned} \delta_{\kappa^1}\theta = k\Gamma^\mu\Gamma^\nu\pi_{\alpha\mu}\pi_{\beta\nu}\left(\frac{\xi-\chi}{2}\right)P_-^{\alpha\beta} = \frac{k}{2}h^{\alpha\beta}\pi_{\alpha\mu}\pi_\beta^\mu\left(\frac{\xi-\chi}{2}\right) - \frac{k}{2}\Gamma^{\mu\nu}\frac{\epsilon^{\alpha\beta}}{\sqrt{-h}}\pi_{\alpha\mu}\pi_{\beta\nu}\left(\frac{\xi-\chi}{2}\right) \ , \\ \delta_{\kappa^2}\psi = k\Gamma^\mu\Gamma^\nu\pi_{\alpha\mu}\pi_{\beta\nu}\left(\frac{\chi-\xi}{2}\right)P_+^{\alpha\beta} = \frac{k}{2}h^{\alpha\beta}\pi_{\alpha\mu}\pi_\beta^\mu\left(\frac{\chi-\xi}{2}\right) + \frac{k}{2}\Gamma^{\mu\nu}\frac{\epsilon^{\alpha\beta}}{\sqrt{-h}}\pi_{\alpha\mu}\pi_{\beta\nu}\left(\frac{\xi-\chi}{2}\right) \ . \end{aligned} \tag{4.31}$$

When we fix the gauge $\theta=\psi$, $\pi_{\alpha\mu}\to\partial_\alpha x^\mu$ and $(1/\sqrt{-h})\,\epsilon^{\alpha\beta}\pi_{\alpha\mu}\pi_{\beta\nu}\to\{x_\mu,x_\nu\}$, so that these transformations become:

$$\delta_{\kappa^1}\theta = \frac{k}{2}\left(h^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu - \Gamma^{\mu\nu}\{x_\mu,x_\nu\}\right)\left(\frac{\xi-\chi}{2}\right) \ , \tag{4.32}$$

$$\delta_{\kappa^2}\psi = \frac{k}{2}\left(h^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu + \Gamma^{\mu\nu}\{x_\mu,x_\nu\}\right)\left(\frac{\xi-\chi}{2}\right) \ . \tag{4.33}$$

Then, we eliminate $h^{\alpha\beta}$ from the action by setting $h_{\alpha\beta}=k\partial_\alpha x^\mu\partial_\beta x_\mu$, so that $h^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu = 2/k$ with $k=4/N$. In the Schild action (4.18), the transformations above take the simple form:

$$\delta_{\kappa^1}\theta = \left(\mathbb{1} - \frac{2}{N}\Gamma^{\mu\nu}\{x_\mu,x_\nu\}\right)\left(\frac{\xi-\chi}{2}\right) \ , \tag{4.34}$$

$$\delta_{\kappa^2}\psi = \left(\mathbb{1} + \frac{2}{N}\Gamma^{\mu\nu}\{x_\mu,x_\nu\}\right)\left(\frac{\chi-\xi}{2}\right) \ , \tag{4.35}$$

so that:

$$\delta_\kappa x^\mu = i\bar{\psi}\Gamma^\mu(\delta_{\kappa^1}\theta - \delta_{\kappa^2}\psi) = i\bar{\psi}\Gamma^\mu(\chi-\xi) \ . \tag{4.36}$$

We can now define the following useful combinations of supersymmetry parameters:

$$\epsilon = 2(\chi - \xi) \ , \quad \epsilon' = \frac{\chi + \xi}{2} \ , \tag{4.37}$$

in terms of which the Schild superstring action has the following $\mathcal{N} = 2$ supersymmetry:

$$\begin{aligned}(\delta_\kappa + \delta_\chi + \delta_\xi)x^\mu &= \frac{i}{2}(\bar{\epsilon}\Gamma^\mu \psi) \ , \\ (\delta_{\kappa^1} + \delta_\chi)\theta = (\delta_{\kappa^2} + \delta_\xi)\psi &= \frac{1}{2N}\{x_\mu, x_\nu\}\Gamma^{\mu\nu}\epsilon + \epsilon' \mathbb{1} \ .\end{aligned} \tag{4.38}$$

4.2.1 Supersymmetry of the IIB matrix model

The $\mathcal{N} = 2$ supersymmetry of the IIB matrix model follows immediately from the matrix regularization of the above transformations, giving the homogeneous supersymmetry (usually called dynamical supersymmetry):

$$\begin{aligned}\delta_\epsilon X_\mu &= \frac{i}{2}\bar{\epsilon}\Gamma_\mu \Psi \\ \delta_\epsilon \Psi &= -\frac{i}{4}\Gamma^{\mu\nu}[X_\mu, X_\nu]\epsilon \ ,\end{aligned} \tag{4.39}$$

that is the dimensional reduction of the usual supersymmetry of $\mathcal{N}$=1 super Yang-Mills theory in D=10 and one inhomogeneous supersymmetry, which is a simple fermionic translational invariance:

$$\delta_{\epsilon'} X_\mu = 0 \ , \qquad \delta_{\epsilon'}\Psi = \epsilon' \mathbb{1} \ , \tag{4.40}$$

also called kinematical supersymmetry.

We can show explicitly the invariance of the action under these transformations as:

$$\begin{aligned}\delta_\epsilon S_{IKKT} = \frac{i}{2g^2} Tr\Big(&(\bar{\epsilon_1}\Gamma^\mu[\Psi), X^\nu][X_\mu, X_\nu] + \frac{1}{4}(\bar{\epsilon_1}\Gamma^{\mu\nu}\Gamma^\rho[\Psi), X_\rho][X_\mu, X_\nu] + \\ &+ \frac{1}{4}(\bar{\Psi}\Gamma^\rho\Gamma^{\mu\nu}\epsilon_1)[X_\rho, [X_\mu, X_\nu]]\Big) = 0 \ ,\end{aligned} \tag{4.41}$$

thanks to the Jacobi identity $\Gamma^{\mu\nu\rho}[X_\rho, [X_\mu, X_\nu]] = 0$ and the Majorana property $\bar{\epsilon}\Gamma^\mu\Psi = -\bar{\Psi}\Gamma^\mu\epsilon$, while $\delta_{\epsilon'} S_{IKKT} = 0$ follows trivially from $Tr([\,,]) = 0$.

4.2.2 On-shell closure of the $\mathcal{N} = 2$ supersymmetry algebra

The particular structure of the $\mathcal{N} = 2$ supersymmetry of the IIB matrix model allows us to interpret the eigenvalues of the large N matrices describing the bosonic fields as space-time coordinates [10, 108]. To see why, we want to investigate the structure of the supersymmetry algebra. Computing the commutation relation of two successive supersymmetry transformations, we get:

$$\begin{aligned}[\epsilon_2 Q, \epsilon_1 Q]X_\mu &= \frac{1}{2}\bar{\epsilon}_2\Gamma^\nu\epsilon_1[X_\nu, X_\mu] \\ [\epsilon_2 Q, \epsilon_1 Q]\Psi &= \frac{1}{2}\bar{\epsilon}_2\Gamma^\nu\epsilon_1[X_\nu, \Psi] - \left(\frac{7}{32}\bar{\epsilon}_2\Gamma^\mu\epsilon_1\Gamma_\mu - \frac{1}{64\cdot 5!}\bar{\epsilon}_2\Gamma^{\mu_1\ldots\mu_5}\epsilon_1\Gamma_{\mu_1\ldots\mu_5}\right)\Gamma^\nu[X_\nu, \Psi] \ ,\end{aligned} \tag{4.42}$$

thanks to the Fierz identity of the appendix on Majorana fermions in D=9+1. The two last term are proportional to the equation of motion (4.3). Modulo equations of motion, we thus have:

$$\begin{aligned} [\epsilon_2 Q, \epsilon_1 Q] X_\mu &= [\Lambda, X_\mu] \\ [\epsilon_2 Q, \epsilon_1 Q] \Psi &= [\Lambda, \Psi] \end{aligned} \tag{4.43}$$

with $\Lambda = 1/2 \bar{\epsilon}_2 \Gamma^\nu \epsilon_1 X_\nu$. In a 0-dimensional gauge theory, this is of course simply a $U(N)$ gauge transformation of parameter Λ. Of course, $\{\tilde{Q}, \tilde{Q}\} = 0$. On the other hand:

$$\begin{aligned} [\epsilon_2 \tilde{Q}, \epsilon_1 Q] X_\mu &= \frac{i}{2} \bar{\epsilon}_1 \Gamma_\mu \epsilon_2 \\ [\epsilon_2 \tilde{Q}, \epsilon_1 Q] \Psi &= 0 \; . \end{aligned} \tag{4.44}$$

Introducing the operator P_μ which generates the translational symmetry of the model through:

$$X'_\mu = \exp(i c_\nu P^\nu) X_\mu = X_\mu + c_\nu \mathbb{1} + \mathcal{O}(c^2) \tag{4.45}$$

and the redefined supercharges:

$$\begin{aligned} Q^{(1)} &= Q + \tilde{Q} \\ Q^{(2)} &= i(Q - \tilde{Q}) \; , \end{aligned} \tag{4.46}$$

we obtain the standard $\mathcal{N} = 2$ chiral superalgebra:

$$\{Q^{(i)}_\alpha, Q^{(j)}_\beta\} = i(\mathcal{P}_L C \Gamma^\mu)_{\alpha\beta} P_\mu \delta^{ij} \tag{4.47}$$

up to gauge transformations and equations of motion. This superalgebra, characteristic of $\mathcal{N} = 2$ supersymmetry in 10 dimensions supports the interpretation of X_μ as space-time coordinates. The presence of $\mathcal{N} = 2$ space-time supersymmetry in 10 dimensions and the existence of massless excitations (due to the flat directions in the $Tr([X^\mu, X^\nu])^2$ potential) are essential features for a model claiming to be a non-perturbative definition of type IIB string theory, since they indicate that the matrix model might indeed have the correct low-energy limit, namely type IIB supergravity in 10 dimensions.

4.2.3 Central charges and D-brane spectrum

In the following, we want to perform a similar calculation, but in a way that allows us to keep track of traces of commutators, in order to understand the structure of the possible central charges and the branes' spectrum of the IIB matrix model. This calculation has been first performed in [55]. Although the calculation procedure is in principle the same, we obtain a different result, correcting a mistake in the literature. We use the same Poisson bracket notation as in the BFSS case, even though there is no Hamiltonian formalism for a totally reduced matrix model. Here, we really want to consider $P_\mu = Tr(p_\mu)$ and $i\tilde{Q} = Tr(p_\Psi)$ as translation operators in target-superspace. We recall our definition of the Poisson bracket for the reader's convenience:

$$\{A, B\}_{PB} = \frac{\delta A}{\delta (X^i)^b_a} \frac{\delta B}{\delta (p_i)^a_b} - \frac{\delta A}{\delta (p_i)^b_a} \frac{\delta B}{\delta (X^i)^a_b} + \frac{\delta A}{\delta (\Psi_\alpha)^b_a} \frac{\delta B}{\delta (p^\alpha_\Psi)^a_b} + \frac{\delta A}{\delta (p^\alpha_\Psi)^b_a} \frac{\delta B}{\delta (\Psi_\alpha)^a_b} \tag{4.48}$$

Then, we can define the dynamical and kinematical supercharges Q^α and $\tilde{Q}^\alpha$ so that they generate the supersymmetry transformations of the model as in:

$$
\begin{aligned}
\delta_\epsilon X^\mu =& i\{\epsilon_\beta Q^\beta, X^\mu\}_{PB} \equiv \frac{i}{2}\bar{\epsilon}\Gamma^\mu\Psi \ , \\
\delta_\epsilon \Psi_\alpha =& i\{\epsilon_\beta Q^\beta, \Psi_\alpha\}_{PB} \equiv -\frac{i}{4}[X_\mu, X_\nu]\Gamma^{\mu\nu}\epsilon \ , \\
\delta_{\epsilon'} X^\mu =& i\{\epsilon'_\beta \tilde{Q}^\beta, X^\mu\}_{PB} \equiv 0 \ , \quad \delta_{\epsilon'}\Psi = i\{\epsilon'_\beta \tilde{Q}^\beta, \Psi_\alpha\}_{PB} \equiv \epsilon' \mathbb{I} \ .
\end{aligned}
\tag{4.49}
$$

This corresponds to the choice of left-handed supercharges and leads to the following definitions:

$$
\begin{aligned}
Q^\alpha =& Tr(q^\alpha) = -\frac{1}{2}Tr\left((\bar{\Psi}\Gamma^\mu)^\alpha p_\mu + \frac{1}{2}(p_\Psi)^\beta (\Gamma_{\mu\nu})_\beta{}^\alpha [X^\mu, X^\nu]\right) \ , \\
\tilde{Q}^\alpha =& Tr(\tilde{q}^\alpha) = -iTr\left(p_\Psi^\alpha\right) \ .
\end{aligned}
\tag{4.50}
$$

Now, we can compute the supersymmetry algebra from the supercharges' Poisson brackets. From now on, we restrict ourselves to the Majorana-Weyl representation of the Clifford algebra of $SO(9,1)$ described in the Appendices to simplify the calculations. Then:

$$
\begin{aligned}
Q_\alpha^\dagger = Q_\alpha^\top =& Tr(q_\alpha^\top) = -\frac{1}{2}Tr\left((\Gamma^0\Gamma^\mu\Psi)_\alpha p_\mu - \frac{1}{2}[X^\mu, X^\nu](\Gamma_{\mu\nu})_\alpha{}^\beta (p_\Psi^\top)_\beta\right) \ , \\
\tilde{Q}_\alpha^\dagger = \tilde{Q}_\alpha^\top =& Tr(\tilde{q}_\alpha^\top) = Tr\left((-ip_\Psi^\top)_\alpha\right) \ .
\end{aligned}
\tag{4.51}
$$

Obviously,

$$
\{\tilde{Q}_\alpha^\top, \tilde{Q}^\beta\}_{PB} = 0, \text{ and} \tag{4.52}
$$

$$
\{\tilde{Q}_\alpha^\top, Q^\beta\}_{PB} = \frac{i}{2}Tr\left((\mathcal{P}_L\Gamma^0\Gamma^\mu)_\alpha{}^\beta p_\mu\right) \ . \tag{4.53}
$$

Let us then compute the bosonic part of $\{Q_\alpha^\top, Q^\beta\}_{PB}$:

$$
\begin{aligned}
\{Q_\alpha^\top, Q^\beta\}_{PB}^{bos.} =& \frac{1}{8}(p_\mu\Gamma^0\Gamma^\mu)_{\alpha a}^{\gamma b}\{(\Psi)_{\gamma b}{}^a, (p_\Psi)_c^{\delta d}\}_{PB}([X_\nu, X_\rho]\Gamma^{\nu\rho})_{\delta d}^{\beta c} - \\
&-([X_\nu, X_\rho]\Gamma^{\nu\rho})_{\alpha a}^{\gamma b}\{(p_\Psi^\top)_{\gamma b}{}^a, (\Psi^\top)_c^{\delta d}\}_{PB}(p_\mu\Gamma^0\Gamma^\mu)_{\delta d}^{\beta c} = \\
=& \frac{1}{8}Tr\left(p_\mu[X_\nu, X_\rho](\Gamma^0\Gamma^\mu\mathcal{P}_L\Gamma^{\nu\rho})_\alpha{}^\beta - [X_\nu, X_\rho]p_\mu(\Gamma^{\nu\rho}\mathcal{P}_L\Gamma^0\Gamma^\mu)_\alpha{}^\beta\right) \ .
\end{aligned}
\tag{4.54}
$$

Note that we restrained ourselves from using the cyclicity of the trace in order to keep track of the traces of commutators. However, we can use: $\Gamma^\mu\Gamma^{\nu\rho} = \Gamma^{\mu\nu\rho} + \eta^{\mu\nu}\Gamma^\rho - \eta^{\mu\rho}\Gamma^\nu$ to obtain:

$$
\{Q_\alpha^\top, Q^\beta\}_{PB}^{bos.} = \frac{1}{8}Tr\left([p_\mu, [X_\nu, X_\rho]](\mathcal{P}_L\Gamma^0\Gamma^{\mu\nu\rho})_\alpha{}^\beta + 2\{p_\mu, [X^\mu, X_\nu]\}(\mathcal{P}_L\Gamma^0\Gamma^\nu)_\alpha{}^\beta\right) \ ,
$$

where we remark in particular the 3-form central charge $[p_\mu, [X_\nu, X_\rho]]$. The other terms contained in the dynamical supercharge commutator involve bosonic Poisson brackets of the type $\{p_\mu, X^\nu\}_{PB}$ and contains fermions. We first recall that:

$$
\{(p_\mu)_b{}^a, ([X^\nu, X^\rho])_d{}^c\}_{PB} = -\delta_\mu^\nu (X^\rho)_b{}^c \delta_d^a - \delta_\mu^\rho (X^\nu)_d{}^a \delta_b^c + (\nu \leftrightarrow \rho) \ . \tag{4.55}
$$

so that the fermionic part of the Poisson bracket is:

$$\begin{aligned}\{Q^\top_\alpha, Q^\beta\}^{ferm.}_{PB} = \frac{1}{8}\Big[&(\Gamma^0\Gamma^\mu\Psi)_{\alpha a}{}^b\{(p_\mu)_b{}^a, ([X^\nu, X^\rho])_d{}^c\}_{PB}(p_\Psi\Gamma_{\nu\rho})^{\beta d}_c - \\ &-((\Gamma_{\nu\rho}p^\top_\Psi)_{\alpha a}{}^b\ \{([X^\nu, X^\rho])_b{}^a, (p_\mu)_d{}^c\}_{PB}(\bar\Psi\Gamma^\mu)^{\beta d}_c\Big] = \\ -\frac{1}{4}&(\mathcal{P}_L\Gamma^0\Gamma^\mu)_\alpha{}^\delta(\mathcal{P}_L\Gamma_{\mu\nu})_\gamma{}^\beta Tr\left(\Psi_\delta[X^\nu, p^\gamma_\Psi]\right) - \frac{1}{4}(\mathcal{P}_L\Gamma^0\Gamma^\mu)_\delta{}^\beta(\mathcal{P}_L\Gamma_{\mu\nu})_\alpha{}^\gamma Tr\left((p^\top_\Psi)_\gamma[X^\nu, \Psi^\top_\delta]\right)\ .\end{aligned} \tag{4.56}$$

As we did in the previous chapter, we can use the Fierz identity of Appendix B, which here takes the following form:

$$\begin{aligned}(\mathcal{P}_L\Gamma^0\Gamma^\mu)_\alpha{}^\delta(\mathcal{P}_L\Gamma_{\mu\nu})_\gamma{}^\beta &= -(\mathcal{P}_L\Gamma^0\Gamma^\nu)_\alpha{}^\beta(\mathcal{P}_L)_\gamma{}^\delta + \frac{7}{16}(\mathcal{P}_L\Gamma^0\Gamma^\mu)_\alpha{}^\beta(\mathcal{P}_L\Gamma_\mu\Gamma_\nu)_\gamma{}^\delta + \\ + \frac{5}{96}(\mathcal{P}_L\Gamma^0\Gamma^{\kappa\lambda\mu})_\alpha{}^\beta(\mathcal{P}_L\Gamma_{\kappa\lambda\mu}\Gamma_\nu)_\gamma{}^\delta &- \frac{1}{48}(\mathcal{P}_L\Gamma^0\Gamma^{\kappa\lambda\mu})_\alpha{}^\beta(\mathcal{P}_L\Gamma_{\kappa\lambda\mu\nu})_\gamma{}^\delta - \frac{1}{32\cdot 5!}(\mathcal{P}_L\Gamma^0\Gamma^{\mu_1\ldots\mu_5})_\alpha{}^\beta(\mathcal{P}_L\Gamma_{\mu_1\ldots\mu_5}\Gamma^\nu)_\gamma{}^\delta\ .\end{aligned}$$

Besides several terms proportional to the equations of motion, we get:

$$\frac{1}{4}(\mathcal{P}_L\Gamma^0\Gamma^\nu)_\alpha{}^\beta Tr\left(\Psi^\top[X^\nu, p^\top_\Psi] - p_\Psi[X^\nu, \Psi]\right) + \frac{1}{192}(\mathcal{P}_L\Gamma^0\Gamma^{\kappa\lambda\mu})_\alpha{}^\beta Tr\left(\Psi^\top\Gamma_{\kappa\lambda\mu\nu}[X^\nu, p^\top_\Psi] - p_\Psi\Gamma_{\kappa\lambda\mu\nu}[X^\nu, \Psi]\right)\ ,$$

Putting this result together with the purely bosonic part, we obtain:

$$\begin{aligned}\{\tilde Q^\top_\alpha, \tilde Q^\beta\}_{PB} =& 0\ , \\ \{\tilde Q^\top_\alpha, Q^\beta\}_{PB} =& \frac{i}{2}(\mathcal{P}_L\Gamma^0\Gamma^\mu)_\alpha{}^\beta P_\mu \\ \{Q^\top_\alpha, Q^\beta\}_{PB} =& \frac{1}{4}(\mathcal{P}_L\Gamma^0\Gamma^\nu)_\alpha{}^\beta Tr\left(\{p_\mu, [X^\mu, X_\nu]\} + \Psi^\top[X_\nu, p^\top_\Psi] - p_\Psi[X_\nu, \Psi]\right)\ , \\ &+\frac{1}{8}(\mathcal{P}_L\Gamma^0\Gamma^{\mu\nu\rho})_\alpha{}^\beta Tr\left([p_\mu, [X_\nu, X_\rho]] + \frac{1}{24}\Psi^\top\Gamma_{\mu\nu\rho\sigma}[X^\sigma, p^\top_\Psi] - \frac{1}{24}p_\Psi\Gamma_{\mu\nu\rho\sigma}[X^\sigma, \Psi]\right)\ ,\end{aligned} \tag{4.57}$$

where one can read off the D1-brane and D3-brane central charges in the last expression. These are indeed the branes we expect to see appearing in type IIB string theory.

4.3 Alternative totally reduced models

In the last part of this chapter, we will review recent developments that eventually led us to the research presented in chapter 6. The work we present here is mostly due to Kimura and collaborators. They proposed different variations of the IIB matrix model [109, 120, 121, 123]. Some of these modifications can be performed for a target-space of any dimension, while others are limited to a specific number of dimensions (at least if we want to maintain the covariance). An example of the first kind is described by the following action:

$$S = \frac{1}{g^2}Tr\left(\frac{1}{4}[X_\mu, X_\nu][X_\mu, X_\nu] + \lambda^2 X_\mu X_\mu\right), \tag{4.58}$$

where a mass term has been added to the bosonic part of the original IKKT action (4.1). We will call it the massive IIB matrix model. Such mass terms can appear when we consider a matrix model living in a non-trivial gravitational background. For example, they appear in matrix models living on a pp-wave background [27] and in attempts to describe quantum gravity in de Sitter spaces [83, 125, 54].

The equations of motion given by the massive IIB matrix model are the following:

$$[X_\nu, [X_\mu, X_\nu]] + 2\lambda^2 X_\mu = 0 \tag{4.59}$$

If the mass term has the normal sign ($\lambda^2 < 0$), its addition lifts the flat directions of the potential $Tr([A_\mu, A_\nu])^2$ and confines all the "D-instantons" to the point $X^\mu = 0$. In other words, while any diagonal matrices with arbitrary eigenvalues solved the IKKT equations of motion, all eigenvalues are brought back to zero by the mass term. The situation becomes more interesting if $\lambda^2 > 0$ when the mass term is tachyonic. In such a case, $X^\mu = 0$ is of course an unstable extrema and the system decays to lower-energy states. Let us consider for illustration a simplified version in three Euclidean dimensions. In such a case, we can make the Ansatz that A^i for $i = 1, 2, 3$ are given by an N-dimensional irreducible representation of the $\mathfrak{su}(2)$ Lie algebra. Specifically, we set:

$$X^i = rJ^i_N \tag{4.60}$$

for some constant r, where the J^i_N's are $N \times N$ matrices satisfying:

$$[J^i_N, J^j_N] = 2i\epsilon^{ijk} J^k_N \ , \tag{4.61}$$

which we can define by the following relationship:

$$J^i_N = [\sigma^i \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \mathbb{1} \otimes \sigma^i \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \ldots + \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \sigma^i]_{\text{sym}} \ , \tag{4.62}$$

where the σ^i's are the standard Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{4.63}$$

and $\mathbb{1}$ is the 2×2 identity matrix. The sum contains $N-1$ terms with $N-2$ tensor products and the subscript "sym" means that we keep only the totally symmetric part of the tensor product, corresponding to the irreducible subrepresentation of dimension N. With that definition and the properties $(\sigma^i)^2 = \mathbb{1}$ and $\sum_i (\sigma^i \otimes \sigma^i)_{\text{sym}} = \mathbb{1}$, the X^i's build a fuzzy 2-sphere of radius $R = r\sqrt{N^2-1}$ since they also satisfy:

$$\sum_{i=1}^{3} (X^i)^2 = r^2(N^2-1)\mathbb{1}_{N\times N} \ . \tag{4.64}$$

Putting this fuzzy sphere Ansatz in the equation (4.59), we obtain the simple relation:

$$r^2 = 4\lambda^2 \ . \tag{4.65}$$

Indeed, the radius is real for a positive λ^2 and it would become imaginary for $\lambda^2 < 0$, which is another reflection of the fact that the trivial solution is more stable in that case. In addition, the classical energy $E = -S_E$ of the fuzzy sphere solution is:

$$E = -\frac{\lambda^4}{2g^2} Tr((J_i)^2) = -\frac{\lambda^4}{8g^2}(N^3 - N) = -\frac{\lambda^2}{2g^2} NR^2 \ , \tag{4.66}$$

of course smaller than that of the trivial solution. Worth noting is the fact that it grows with the value of N, i.e. with the radius of the fuzzy sphere, albeit through quantum jumps in energy. We can thus expect the fuzzy spheres to expand until they reach their maximal size where the representation of $\mathfrak{su}(2)$ fills up the whole $N \times N$ hermitian $\mathfrak{u}(N)$ matrices. In that sense, the fuzzy sphere replaces the 3 unstable direction of the trivial solution by 2 stable orbital direction and one radial direction, which might be classically stable, but tends to decay quantum-mechanically into meta-stable spheres of bigger and bigger radii. This process stops when the representation of the rotation algebra becomes maximal. Of course, one should be careful in taking the large N limit in such a case. Besides the classical energy consideration above, there have been recent attempts at showing the quantum stability of such solutions [163]. In [120], where this action is studied in a 4-dimensional context, expansions around the two-dimensional fuzzy sphere and the two-dimensional fuzzy torus have been studied. This background field method naturally leads to non-commutative gauge theories living on such non-commutative backgrounds.

Although the mathematics become more involved, we can extend this framework to higher-dimensional target-spaces and find that such matrix models can describe higher-dimensional fuzzy spheres [122], at least for even fuzzy $2k$-spheres [53, 141, 105, 142], when $D = 2k+1$ is odd. More specifically, we can define the following $2k+1$ matrices constructed as the symmetrized tensor products of $2^k \times 2^k$ Dirac matrices in dimension $2k+1$:

$$J^k_\mu = [\Gamma^{(k)}_\mu \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \ldots + \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \Gamma^{(k)}_\mu]_{\text{sym}} \; , \tag{4.67}$$

where the $\Gamma^{(k)}_\mu$'s build a representation of the Clifford algebra of $SO(2k+1)$ i.e. $\{\Gamma^{(k)}_\mu, \Gamma^{(k)}_\nu\} = 2\delta_{\mu\nu}\mathbb{1}_{2^k \times 2^k}$. Since the cross-products cancel in the commutators (only terms like $\Gamma^{(k)}_{\mu\nu} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$ survive, while terms of the form $\Gamma^{(k)}_\mu \otimes \Gamma^{(k)}_\nu \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$ disappear), their commutators $J^k_{\mu\nu} = 1/2[J^k_\mu, J^k_\nu]$ furnish a representation of the $\mathfrak{so}(2k+1)$ Lie algebra of respective dimensions N_k given by [141, 105]

$$\begin{aligned} N_1 &= (n+1) \; , \quad N_2 = \frac{(n+1)(n+2)(n+3)}{6} \; , \quad N_3 = \frac{(n+1)(n+2)(n+3)^2(n+4)(n+5)}{360} \; , \\ N_4 &= \frac{(n+1)(n+2)(n+3)^2(n+4)^2(n+5)^2(n+6)(n+7)}{302400} \; , \end{aligned} \tag{4.68}$$

if there are n factors in the tensor product. These matrices can be shown to satisfy the following algebraic relations:

$$\begin{aligned} J^k_\mu J^k_\mu &= n(n+2k)\mathbb{1}_{N_k \times N_k} \; , \\ J^k_{\mu\nu} J^k_{\mu\nu} &= -2kn(n+2k)\mathbb{1}_{N_k \times N_k} \; , \\ [J^k_{\mu\nu}, J^k_\rho] &= -2\delta_{\mu\rho} J^k_\nu + 2\delta_{\nu\rho} J^k_\mu \\ [J^k_{\mu\nu}, J^k_{\rho\sigma}] &= 2\left(-\delta_{\mu\rho} J^k_{\nu\sigma} + \delta_{\nu\rho} J^k_{\mu\sigma} + \delta_{\mu\sigma} J^k_{\nu\rho} - \delta_{\nu\sigma} J^k_{\mu\rho}\right) \end{aligned} \tag{4.69}$$

where the commutation relations are simply inherited from those of the Dirac matrices. Then, we can follow the same procedure as before and make the Ansatz

$$X_\mu = r J^k_\mu \; , \forall\, \mu = 1, \ldots, 2k+1 \; , \tag{4.70}$$

for some constant r to be fixed by the equations of motion. Indeed, putting (4.70) into (4.59), we obtain the relation:

$$\lambda^2 = 4kr^2 \tag{4.71}$$

so that the fuzzy sphere solves the equations of motion for the sphere radius $R = \lambda/2\ \sqrt{kn(n+2k)}$. We can also compute its classical energy to be:

$$E = -\frac{\lambda^4}{(2kg)^2}Tr(\frac{1}{4}(J_{\mu\nu})^2 + k(J_\mu)^2) = -\frac{\lambda^4}{8kg^2}n(n+2k)N_k = -\frac{\lambda^2}{2(kg)^2}N_k R^2 \ . \tag{4.72}$$

Of course, all these results reduce to the fuzzy 2-sphere case when we choose $k=1$.

Other kinds of models which admit non-commutative classical non-commutative solutions have been proposed. For example, in [109], a Chern-Simons term has been added to the IIB matrix action. It is also called Myers term, since it may be seen as describing the presence of a background flux [5]. In a 3-dimensional setting, such a matrix model can be defined as:

$$S_{CS} = \frac{1}{g_0^2}Tr\left(\frac{1}{4}[X_\mu, X_\nu][X^\mu, X^\nu] + \frac{1}{2}\bar{\Psi}\sigma^\mu[X_\mu, \Psi] + \frac{2i}{3}\alpha\epsilon^{\mu\nu\rho}X_\mu X_\nu X_\rho - \alpha\bar{\Psi}\Psi\right) \ , \tag{4.73}$$

for some constant α. Contrarily to the massive IIB matrix model, it is translation-invariant in the 3-dimensional space-time generated by the X^μ and it has the $\mathcal{N}=1$ supersymmetry:

$$\begin{aligned} \delta_\epsilon X_\mu &= \frac{i}{2}\bar{\epsilon}\sigma_\mu\Psi \\ \delta_\epsilon\Psi &= -\frac{i}{4}\sigma^{\mu\nu}[X_\mu, X_\nu]\epsilon \ , \end{aligned} \tag{4.74}$$

For $\Psi = 0$, the equations of motion for X^μ are given by:

$$[X_\nu, [X_\mu, X^\nu]] + i\alpha\epsilon^{\mu\nu\rho}[X_\nu, X_\rho] = 0 \tag{4.75}$$

They can of course be solved by the fuzzy 2-sphere Ansatz (4.60) for:

$$r = \frac{\alpha}{2} \ . \tag{4.76}$$

Models that contain the two type of terms (mass and Chern-Simons) have also been studied by Valtancoli in [159, 158], where the stability of the fuzzy sphere solution is discussed, depending from the values of λ and α. Of course, such kind of totally anti-symmetric terms can be added in any dimension, but they lead to very non-linear theories. However, higher-dimensional fuzzy spheres can also appear as solutions in these models, thanks to the duality relation:

$$\epsilon^{\mu_1\cdots\mu_{2k+1}}J^k_{\mu_1}J^k_{\mu_2}\cdots J^k_{\mu_{2k}} = m_k J^k_{\mu_{2k+1}} \ , \tag{4.77}$$

where

$$m_1 = 2i, \quad m_2 = 8(n+2), \quad m_3 = -48i(n+2)(n+4), \quad m_4 = -384(n+2)(n+4)(n+6) \ .$$

The computation of the various m_k's is given in appendix E.

4.4 Other studies of the IIB matrix model

There are other important lines of work on the IIB matrix model that I won't describe in details here, since they are not directly related to my personal work. A particularly important problem to solve is to show that there is a dynamical breaking of the $SO(10)$ covariance in the IIB matrix model, if its solutions are to describe our 4-dimensional spacetime. There are various papers on this issue that point towards such a symmetry breaking to $SO(4)$. The main idea is to integrate out the off-diagonal components of the matrices, as well as the diagonal components of the fermions to obtain an effective action for the eigenvalues of the bosonic matrices that are supposed to generate the spacetime. By doing so, we obtain an effective theory similar to a branched polymer theory. A numerical study of this model shows that different directions seem to have different length scale, which could show how some directions become compact and small [10, 11, 113, 112]. Another interesting approach uses a variational method to determine the most likely resulting configurations [131, 130]. They also conclude from numerical studies that the 4-dimensional configuration might dominate. In any case, it seems that a purely bosonic model doesn't show any symmetry-breaking. It should thus be mediated by the fermionic determinant.

The reader might also be interested in the study of loop equations [81, 10], gauge-invariance and diffeomorphism-invariance in the IIB matrix model [10]. Since the matrix eigenvalues form a kind of random lattice space-time with N points, gauge-invariance appears in a similar fashion to lattice gauge theories, with the difference that the lattice is not fixed. On the other hand, diffeomorphism invariance expresses itself as a permutation symmetry of the lattice points.

Chapter 5

A matrix quantum mechanics with $\mathfrak{osp}(1|32,\mathbb{R})$ supersymmetry

After having introduced various matrix models and explained their various relationships to string theory and brane physics in the review part of this thesis, I will go on with the description of my first research project, realized in collaboration with Luca Carlevaro and Prof. Adel Bilal. As we have discussed in chapter 3, the BFSS conjecture has been shown to reproduce well the eleven-dimensional supergravity results. However, a disagreement seems to exist between the corrections to supergravity predicted by supersymmetric matrix quantum mechanics from three-graviton scattering computations and the R^4-corrections we would expect from type IIA string theory [104]. It is thus possible that matrix theory does not completely fulfill its goal of describing the non-perturbative sector of type IIA string theory. Furthermore, we have seen that transverse M5-branes were absent in the BFSS theory. Although we advocated the idea that it might be an artifact of the infinite momentum frame at the end of subsection 3.2.3, it would be better to have a theory that can take such backgrounds into account. In any case, we also would like to have an eleven-dimensional covariant formulation of the theory, as well as a way to describe gravitational backgrounds other than flat minkowskian space-time and gravitational pp-wave background.

The model presented in this chapter is a tentative proposal to overcome these difficulties by the introduction of a matrix model defined in a background-independant way by its supersymmetry and its action, with no reference to a particular target spacetime. In particular, there is no a priori constraint on topology and dimensionality. However, it is difficult to test its eventual physical relevance in this original abstract formulation, so we look at two possible expressions of its dynamics, in a twelve-dimensional and eleven-dimensional contexts. We are more particularly interested in the latter case, where we want to perform a boost to the infinite momentum frame, to see how it relates to better-known M-theoretical physics.

This way, we obtain a matrix quantum mechanics similar to, but different from the BFSS matrix model. This is not too surprising, since the bosonic part of the symmetry of our model is the conformal group rather than the Lorentz group in ten dimensions, suggesting that our model might rather be related to M-theoretical physics in an AdS_{11}-spacetime background. Unfortunately, very few is known about such physics, so that it is difficult to test this conjecture.

5.1 Introduction

In the absence of a microscopic description of M-theory, some of its expected features can be obtained by looking at the eleven-dimensional superalgebra [155], whose central charges correspond to the extended objects, i.e. membranes and five-branes present in M-theory. Relations with the hidden symmetries of eleven-dimensional supergravity [57] and its compactifications and associated BPS configurations (see e.g. [60, 162] and references therein) underlined further the importance of the algebraic aspects. It has been conjectured [34] that the large superalgebra $\mathfrak{osp}(1|32)$ may play an important and maybe unifying rôle in M and F theory [157].

In this chapter, we will explore further this possible unifying rôle and study its implications for matrix models. One of the main motivations is to investigate the dynamics of extended objects such as membranes and five-branes, when they are treated on the same footing as the "elementary" degrees of freedom. In order to see eleven and twelve-dimensional structures emerge, we have to embed the $SO(10,2)$ Lorentz algebra and the $SO(10,1)$ Poincaré algebra into the large $\mathfrak{osp}(1|32)$ superalgebra. This will yield certain deformations and extensions of these algebras which nicely include new symmetry generators related to the charges of the extended objects appearing in the eleven and twelve-dimensional theories. The supersymmetry transformations of the associated fields also appear naturally.

Besides these algebraic aspects, we are interested in the dynamics arising from matrix models derived from such algebras. Following ideas initially advocated by Smolin [148], we start with matrices $M \in \mathfrak{osp}(1|32)$ as basic dynamical objects, write down a very simple action for them and then decompose the result according to the different representations of the eleven and twelve-dimensional algebras. In the eleven-dimensional case, we expect this action to contain the scalars X_i of the BFSS matrix model and the associated fermions together with five-branes. In ten dimensions, cubic supermatrix models have already been studied by Azuma, Iso, Kawai and Ohwashi [17] (more details can be found in Azuma's master thesis [15]) in an attempt to compare it with the IIB matrix model of Ishibashi, Kawai, Kitazawa and Tsuchiya [107].

To test the relevance of our model, we try to exhibit its relations with the BFSS matrix model. For this purpose, we perform a boost to the infinite momentum frame (IMF), thus reducing the explicit symmetry of the action to $SO(9)$. Then, we integrate out conjugate momenta and auxiliary fields and calculate an effective action for the scalars X_i, the associated fermions, and higher form fields. What we obtain in the end is the BFSS matrix model with additional terms. In particular, our effective action explicitly contains couplings to 5-brane degrees of freedom, which are thus naturally incorporated in our model as fully dynamical entities. Moreover, we also get additional interactions and masslike terms. This should not be too surprising since we started with a larger theory. The interaction terms we obtain are somewhat similar to the higher-dimensional operators one expects when integrating out (massive) fields in quantum field theory. This can be viewed as an extension of the BFSS theory describing M-theoretical physics in certain non-Minkowskian backgrounds.

The outline of this paper is the following: in the next section we begin by recalling the form of the $\mathfrak{osp}(1|32)$ algebra and the decomposition of its matrices. In section 3 and 4, we study the embedding of the twelve-, resp. eleven-dimensional superalgebras into $\mathfrak{osp}(1|32)$, and obtain the corresponding algebraic structure including the extended objects described by a six- resp. five-form. We establish the supersymmetry transformations of the fields, and write down a cubic matrix model which yields an action for the various twelve- resp. eleven-dimensional fields. Finally, in section 5, we study further the eleven-dimensional matrix model, compute an effective action and do the comparison with the BFSS model.

5.2 The $\mathfrak{osp}(1|32,\mathbb{R})$ superalgebra

We first recall some definitions and properties of the unifying superalgebra $\mathfrak{osp}(1|32,\mathbb{R})$ which will be useful in the following chapters. The superalgebra is defined by the following three equations:

$$
\begin{aligned}
[Z_{AB}, Z_{CD}] &= \Omega_{AD} Z_{CB} + \Omega_{AC} Z_{DB} + \Omega_{BD} Z_{CA} + \Omega_{BC} Z_{DA} \;, \\
[Z_{AB}, Q_C] &= \Omega_{AC} Q_B + \Omega_{BC} Q_A \;, \\
\{Q_A, Q_B\} &= Z_{AB} \;,
\end{aligned}
\tag{5.1}
$$

where Ω_{AB} is the antisymmetric matrix defining the $\mathfrak{sp}(32,\mathbb{R})$ symplectic Lie algebra. Let us now give an equivalent description of elements of $\mathfrak{osp}(1|32,\mathbb{R})$. Following Cornwell [56], we call $\mathbb{R}B_L$ the real Grassmann algebra with L generators, and $\mathbb{R}B_{L0}$ and $\mathbb{R}B_{L1}$ its even and odd subspace respectively. Similarly, we define a $(p|q)$ supermatrix to be even (degree 0) if it can be written as:

$$
M = \begin{pmatrix} A & B \\ F & D \end{pmatrix} \,.
$$

where A and D are $p\times p$, resp. $q\times q$ matrices with entries in $\mathbb{R}B_{L0}$, while B and F are $p\times q$ (resp. $q\times p$) matrices, with entries in $\mathbb{R}B_{L1}$. On the other hand, odd supermatrices (degree 1) are characterized by 4 blocks with the opposite parities.

We define the supertranspose of a supermatrix M as[1]:

$$
M^{ST} = \begin{pmatrix} A^\top & (-1)^{deg(M)} F^\top \\ -(-1)^{deg(M)} B^\top & D^\top \end{pmatrix} \,.
$$

If one chooses the orthosymplectic metric to be the following 33×33 matrix:

$$
G = \begin{pmatrix} 0 & -\mathbb{1}_{16} & 0 \\ \mathbb{1}_{16} & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \,,
$$

(where the i is chosen for later convenience to yield a hermitian action), we can define the $\mathfrak{osp}(1|32,\mathbb{R})$ superalgebra as the algebra of $(32|1)$ supermatrices M satisfying the equation:

$$
M^{ST}\cdot G + (-1)^{deg(M)} G\cdot M = 0 \,.
$$

From this defining relation, it is easy to see that an even orthosymplectic matrix should be of the form:

$$
M = \begin{pmatrix} A & B & \Phi_1 \\ F & -A^\top & \Phi_2 \\ -i\Phi_2^\top & i\Phi_1^\top & 0 \end{pmatrix} = \begin{pmatrix} m & \Psi \\ -i\Psi^\top C & 0 \end{pmatrix} \,,
\tag{5.2}
$$

[1] We warn the reader that this is not the same convention as in [15].

where A,B and F are 16×16 matrices with entries in $\mathbb{R}B_{L0}$ and $\Psi = (\Phi_1, \Phi_2)^\top$ is a 32-components Majorana spinors with entries in $\mathbb{R}B_{L1}$. Furthermore, $B = B^\top$, $F = F^\top$ so that $m \in \mathfrak{sp}(32, \mathbb{R})$ and C is the following 32×32 matrix:

$$C = \begin{pmatrix} 0 & -\mathbb{1}_{16} \\ \mathbb{1}_{16} & 0 \end{pmatrix} , \tag{5.3}$$

and will turn out to act as the charge conjugation matrix later on.

Such a matrix in the Lie superalgebra $\mathfrak{osp}(1|32, \mathbb{R})$ can also be regarded as a linear combination of the generators thereof, which we decompose in a bosonic and a fermionic part as:

$$H = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \chi \\ -i\chi^\top C & 0 \end{pmatrix} = h^{AB} Z_{AB} + \chi^A Q_A \tag{5.4}$$

where Z_{AB} and Q_A are the same as in (5.1). An orthosymplectic transformation will then act as:

$$\delta_H^{(1)} = [H, \bullet] = h^{AB}[Z_{AB}, \bullet] + \chi^A [Q_A, \bullet] = \delta_h^{(1)} + \delta_\chi^{(1)} . \tag{5.5}$$

This notation allows us to compute the commutation relations of two orthosymplectic transformations characterized by $H = (h, \chi)$ and $E = (e, \epsilon)$. Recalling that for Majorana fermions $\chi^\top C \epsilon = \epsilon^\top C \chi$, we can extract from $[\delta_H^{(1)}, \delta_E^{(1)}]$ the commutation relation of two symplectic transformations:

$$[\delta_h^{(1)}, \delta_e^{(1)}]_A{}^B = \begin{pmatrix} [h, e]_A{}^B & 0 \\ 0 & 0 \end{pmatrix} , \tag{5.6}$$

the commutation relation between a symplectic transformation and a supersymmetry:

$$[\delta_h^{(1)}, \delta_\chi^{(1)}]_A{}^B = \begin{pmatrix} 0 & h_A{}^D \chi_D \\ i(\chi^\top C)^D h_D{}^B & 0 \end{pmatrix} , \tag{5.7}$$

and the commutator of two supersymmetries:

$$[\delta_\epsilon^{(1)}, \delta_\chi^{(1)}]_A{}^B = \begin{pmatrix} -i(\chi_A (\epsilon^\top C)^B - \epsilon_A (\chi^\top C)^B) & 0 \\ 0 & 0 \end{pmatrix} . \tag{5.8}$$

5.3 The twelve-dimensional case

In order to be embedded into $\mathfrak{osp}(1|32, \mathbb{R})$, a Lorentz algebra must have a fermionic representation of 32 real components at most. The biggest number of dimensions in which this is the case is 12, where Dirac matrices are 64×64. As this dimension is even, there exists a Weyl representation of 32 complex components. We need furthermore a Majorana condition to make them real. This depends of course on the signature of space-time and is possible only for signatures $(10, 2)$, $(6, 6)$ and $(2, 10)$, when (s, t) are such that $s - t = 0 \mod 8$. Let us concentrate in this paper on the most physical case (possibly relevant for F-theory) where the number of timelike dimensions is 2. However, since we choose to concentrate on the next section's M-theoretical case, we will not push this analysis too far and will thus restrict ourselves to the computation of the algebra and the cubic action.

To express the $\mathfrak{osp}(1|32,\mathbb{R})$ superalgebra in terms of 12-dimensional objects, we have to embed the $SO(10,2)$ Dirac matrices into $\mathfrak{sp}(32,\mathbb{R})$ and replace the fundamental representation of $\mathfrak{sp}(32,\mathbb{R})$ by $SO(10,2)$ Majorana-Weyl spinors. A convenient choice of 64×64 Gamma matrices is the following:

$$\Gamma^0 = \begin{pmatrix} 0 & -\mathbb{I}_{32} \\ \mathbb{I}_{32} & 0 \end{pmatrix} , \Gamma^{11} = \begin{pmatrix} 0 & \tilde{\Gamma}^0 \\ \tilde{\Gamma}^0 & 0 \end{pmatrix} , \Gamma^i = \begin{pmatrix} 0 & \tilde{\Gamma}^i \\ \tilde{\Gamma}^i & 0 \end{pmatrix} \quad \forall i = 1,\dots,10, \tag{5.9}$$

where $\tilde{\Gamma}^0$ is the 32×32 symplectic form:

$$\tilde{\Gamma}^0 = \begin{pmatrix} 0 & -\mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{pmatrix}$$

which, with the $\tilde{\Gamma}^i$'s, builds a Majorana representation of the 10 + 1-dimensional Clifford algebra $\{\tilde{\Gamma}^\mu,\tilde{\Gamma}^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_{32}$ for the mostly + metric . Of course, $\tilde{\Gamma}^{10} = \tilde{\Gamma}^0\tilde{\Gamma}^1\dots\tilde{\Gamma}^9$. This choice has $(\Gamma^0)^2 = (\Gamma^{11})^2 = -\mathbb{I}_{64}$, while $(\Gamma^i)^2 = \mathbb{I}_{64}$, $\forall i = 1\dots 10$, and gives a representation of $\{\Gamma^M,\Gamma^N\} = 2\eta^{MN}\mathbb{I}_{64}$ for a metric of the type $(-,+,\dots,+,-)$. As we have chosen all Γ's to be real, this allows to take $B=\mathbb{I}$ in $\Psi^* = B\Psi$, which implies that the charge conjugation matrix $C = \Gamma^0\Gamma^{11}$, i.e.

$$C = \begin{pmatrix} -\tilde{\Gamma}^0 & 0 \\ 0 & \tilde{\Gamma}^0 \end{pmatrix} .$$

This will then automatically satisfy:

$$C\Gamma^M C^{-1} = (\Gamma^M)^\top , \; C\Gamma^{MN}C^{-1} = -(\Gamma^{MN})^\top \tag{5.10}$$

and more generally:

$$C\Gamma^{M_1\dots M_n}C^{-1} = (-1)^{n(n-1)/2}(\Gamma^{M_1\dots M_n})^\top . \tag{5.11}$$

The chirality matrix for this choice will be:

$$\Gamma_* = \Gamma^0\dots\Gamma^{11} = \begin{pmatrix} -\mathbb{I}_{32} & 0 \\ 0 & \mathbb{I}_{32} \end{pmatrix} .$$

We will identify the fundamental representation of $\mathfrak{sp}(32,\mathbb{R})$ with positive chirality Majorana-Weyl spinors of $SO(10,2)$, i.e. those satisfying: $\mathcal{P}_+\Psi = \Psi$, for:

$$\mathcal{P}_+ = \frac{1}{2}(1+\Gamma_*) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_{32} \end{pmatrix} .$$

Decomposing the 64 real components of the positive chirality spinor Ψ into 32+32 or 16+16+16+16, we can write: $\Psi^\top = (0,\Phi^\top) = (0,0,\Phi_1^\top,\Phi_2^\top)$. Because $\overline{\Psi} = \Psi^\dagger\Gamma^0\Gamma^{11} = \Psi^\top C$, this choice for the charge conjugation matrix C is convenient since it will act as C in equation (5.3) (though with a slight abuse of notation), and thus:

$$(0,0,-i\Phi_2^\top, i\Phi_1^\top) = (0,-i\Phi^\top\tilde{\Gamma}^0) = -i\Psi^\top C = -i\overline{\Psi}.$$

5.3.1 Embedding of $SO(10,2)$ in $OSp(1|32,\mathbb{R})$

We would now like to study how the Lie superalgebra of $OSp(1|32,\mathbb{R})$ can be expressed in terms of generators of the super-Lorentz algebra in 10+2 dimensions with additional symmetry generators. In other words, if we separate the $\mathfrak{sp}(32,\mathbb{R})$ transformations h into a part sitting in the Lorentz algebra and a residual $\mathfrak{sp}(32,\mathbb{R})$ part, we can give an explicit description of the orthosymplectic algebra (5.1) in the form of an enhanced super-Lorentz algebra, where the central charges of the super-Lorentz algebra appear as new generators of the enhanced superalgebra.

To do so, we need to expand a symplectic matrix in irreducible tensors of $SO(10,2)$. This can be done as follows:

$$h_A{}^B = \frac{1}{2!}(\mathcal{P}_+\Gamma^{MN})_A{}^B h_{MN} + \frac{1}{6!}(\mathcal{P}_+\Gamma^{M_1\dots M_6})_A{}^B h^+_{M_1\dots M_6} \tag{5.12}$$

where the $+$ on $h_{M_1\dots M_6}$ recalls its self-duality, and the components of h in the decomposition in irreducible tensors of $SO(10,2)$ are given by $h_{MN} = -\frac{1}{32}Tr_{\mathfrak{sp}(32,\mathbb{R})}(h\Gamma_{MN})$ and $h^+_{M_1\dots M_6} = = -\frac{1}{32}Tr_{\mathfrak{sp}(32,\mathbb{R})}(h\Gamma_{M_1\dots M_6})$. Indeed, a real symplectic 32×32 matrix satisfies $m\tilde{\Gamma}^0 = -\tilde{\Gamma}^0 m^\top$, and C acts like $\tilde{\Gamma}^0$ on $\mathcal{P}_+\Gamma^{M_1\dots M_n}$. Furthermore, (5.11) indicates that:

$$C(1+\Gamma_*)\Gamma^{M_1\dots M_n} = (-1)^{n(n-1)/2}((1+(-1)^n\Gamma_*)\Gamma^{M_1\dots M_n})^T C\,. \tag{5.13}$$

Thus, $\mathcal{P}_+\Gamma^{M_1\dots M_n}$ is symplectic iff n is even and $(-1)^{n(n-1)/2} = -1$. For $0\le n\le 6$, this is only the case if $n=2$ or 6. As a matter of fact, the numbers of independent components match since: $12\cdot 11/2 + 1/2\cdot 12!/(6!)^2 = 528 = 16\cdot 33$.

The symplectic transformation δ_h may then be decomposed into irreducible 12-dimensional tensors of symmetry generators, namely the $\mathfrak{so}(10,2)$ Lorentz algebra generator J^{MN} and a new 6-form symmetry generator $J^{M_1\dots M_6}$. To calculate the commutation relations of this enhanced Lorentz algebra, we will choose the following representation of the symmetry generators:

$$J^{MN} = \frac{1}{2!}\mathcal{P}_+\Gamma^{MN}\,, \qquad J^{M_1\dots M_6} = \frac{1}{6!}\mathcal{P}_+\Gamma^{M_1\dots M_6}\,.$$

so that a symplectic transformation will be given in this base by:

$$h = h_{MN}\,J^{MN} + h_{M_1\dots M_6}\,J^{M_1\dots M_6}\,.$$

We will now turn to computing the superalgebra induced by the above bosonic generators and the supercharges for $D=10+2$. The bosonic commutators may readily be computed using:

$$[\Gamma_{M_1\dots M_k},\Gamma_{N_1\dots N_l}] = \begin{cases} \displaystyle\sum_{j=0}^{\lfloor(min(k,l)-1)/2\rfloor}(-1)^{k-j-1}\,2\cdot(2j+1)!\binom{k}{2j+1}\binom{l}{2j+1}\times \\ \qquad\times\eta_{[M_1[N_1}\cdots\eta_{M_{2j+1}N_{2j+1}}\Gamma_{M_{2j+2}\dots M_k]N_{2j+2}\dots N_l]} \quad \text{if } k\cdot l \text{ is even and,} \\ \displaystyle\sum_{j=0}^{(min(k,l)-1)/2}(-1)^j\,2\cdot(2j)!\binom{k}{2j}\binom{l}{2j}\times \\ \qquad\times\eta_{[M_1[N_1}\cdots\eta_{M_{2j}N_{2j}}\Gamma_{M_{2j+1}\dots M_k]N_{2j+1}\dots N_l]} \qquad \text{if } k\cdot l \text{ is odd.} \end{cases} \tag{5.14}$$

On the other hand, for the commutation relations involving fermionic generators, we proceed as follows. We expand equation (5.7) of the preceding chapter in irreducible tensors of $SO(10,2)$:

$$[\delta_\chi,\delta_h] \;=\; -\,\frac{1}{2!}\,\chi^A\,h_{MN}(\mathcal{P}_+\,\Gamma^{MN})^B{}_A Q_B - \frac{1}{6!}\,\chi^A\,h_{M_1\dots M_6}(\mathcal{P}_+\,\Gamma^{M_1\dots M_6})^B{}_A Q_B\;,$$

which is also given by:

$$[\delta_\chi,\delta_h] \;=\; \chi^A\,h_{MN}[Q_A,J^{MN}] + \chi^A\,h_{M_1\dots M_6}[Q_A,J^{M_1\dots M_6}]\;. \tag{5.15}$$

Comparing terms pairwise, we see that the supercharges transform as:

$$[J^{MN},Q_A] \;=\; \frac{1}{2!}\,(\mathcal{P}_+\,\Gamma^{MN})^B{}_A Q_B\;,\qquad [J^{M_1\dots M_6},Q_A] \;=\; \frac{1}{6!}\,(\mathcal{P}_+\,J^{M_1\dots M_6})^B{}_A Q_B\;.$$

Finally, in order to obtain the anti-commutator of two supercharges, we expand the RHS of (5.8) in the bosonic generators J^{MN} and $J^{M_1\dots M_6}$:

$$-\chi^A\epsilon_B\{Q_A,Q^B\} \;\equiv\; [\delta_\chi,\delta_\epsilon] \;=\; \frac{i}{16}(\chi^\top C\Gamma_{MN}\epsilon)J^{MN} + \frac{i}{16}(\chi^\top C\Gamma_{M_1\dots M_6}\epsilon)J^{M_1\dots M_6}\;, \tag{5.16}$$

and match the first and the last term of the equation.

Summarizing the results of this section, we get the following 12-dimensional realization of the superalgebra $\mathfrak{osp}(1|32,\mathbb{R})^2$:

$$\begin{aligned}
[J^{MN},J^{OP}] \;&=\; -4\eta^{[M[O}J^{N]P]}\\
[J^{MN},J^{M_1\dots M_6}] \;&=\; -12\,\eta^{[M[M_1}\,J^{N]M_2\dots M_6]}\\
[J^{N_1\dots N_6},J^{M_1\dots M_6}] \;&=\; -4!\,6!\,\eta^{[N_1[M_1}\,\eta^{N_2\,M_2}\,\eta^{N_3\,M_3}\,\eta^{N_4\,M_4}\,\eta^{N_5\,M_5}\,J^{N_6]M_6]}\\
&\quad+\;2\cdot 6^2\,\eta^{[N_1[M_1}\,\varepsilon^{N_2\dots N_6]M_2\dots M_6]}{}_{AB}\,J^{AB}\\
&\quad+\;4\left(\frac{6!}{4!}\right)^3\,\eta^{[N_1[M_1}\,\eta^{N_2\,M_2}\,\eta^{N_3\,M_3}\,J^{N_4\dots N_6]M_4\dots M_6]}
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
[J^{MN},Q_A] \;&=\; \frac{1}{2}\,(\mathcal{P}_+\,\Gamma^{MN})^B{}_A Q_B\\
[J^{M_1\dots M_6},Q_A] \;&=\; \frac{1}{6!}\,(\mathcal{P}_+\,\Gamma^{M_1\dots M_6})^B{}_A Q_B\\
\{Q_A,Q^B\} \;&=\; -\frac{i}{16}(C\Gamma_{MN})_A{}^B J^{MN} - \frac{i}{16}(C\Gamma_{M_1\dots M_6})_A{}^B J^{M_1\dots M_6}\;,
\end{aligned}$$

where antisymmetrization brackets on the RHS are meant to match the anti-symmetry of indices on the LHS.

[2]Notice that the second term appearing on the right-hand side of the third commutator is in fact proportional to $\Gamma^{M_1\dots M_{10}}$, which, in turn, can be reexpressed as $\Gamma^{M_1\dots M_{10}} = -(1/2)\varepsilon^{AB\,M_1\dots M_{10}}\,\Gamma_{AB}\Gamma_*$. Indeed, in $10+2$ dimensions, we always have:

$$\Gamma^{M_1\dots M_k} = \frac{(-1)^{\frac{(k-1)k}{2}}}{(12-k)!}\varepsilon^{M_1\dots M_k\,M_{k+1}\dots M_{12}}\,\Gamma_{M_{k+1}\dots M_{12}}\Gamma_*$$

5.3.2 Supersymmetry transformations of $12D$ matrix fields

In the following, we will construct a dynamical matrix model based on the symmetry group $\mathfrak{osp}(1|32,\mathbb{R})$ using elements in the adjoint representation of this superalgebra, i.e. matrices in this superalgebra. We can write such a matrix as:

$$M = \begin{pmatrix} m & \Psi \\ -i\Psi^\top C & 0 \end{pmatrix} , \tag{5.18}$$

where m is in the adjoint representation of $\mathfrak{sp}(32,\mathbb{R})$ and Ψ is in the fundamental one. Since M belongs to the adjoint representation, a SUSY will act on it in the following way:

$$\delta^{(1)}_\chi M_A{}^B = \chi^D [Q_D, M]_A{}^B = \begin{pmatrix} -i(\chi_A(\Psi^\top C)^B - \Psi_A(\chi^\top C)^B) & -m_A{}^D\chi_D \\ -i(\chi^\top C)^D m_D{}^B & 0 \end{pmatrix} \tag{5.19}$$

In our particular 12D setting, m gives rise to a 2-form field C (with $SO(10,2)$ indices, not to be confused with the charge conjugation matrix with $\mathfrak{sp}(32,\mathbb{R})$ indices) and a self-dual 6-form field Z^+, as follows:

$$m_A{}^B = \frac{1}{2!}(\mathcal{P}_+\Gamma^{MN})_A{}^B C_{MN} + \frac{1}{6!}(\mathcal{P}_+\Gamma^{M_1\dots M_6})_A{}^B Z^+_{M_1\dots M_6} \,. \tag{5.20}$$

We can extract the supersymmetry transformations of C, Z^+ and Ψ from (5.19) and we obtain:

$$\begin{aligned} \delta^{(1)}_\chi C_{MN} &= \frac{i}{16}\overline{\chi}\,\Gamma_{MN}\Psi \,, \\ \delta^{(1)}_\chi Z^+_{M_1\dots M_6} &= \frac{i}{16}\overline{\chi}\,\Gamma_{M_1\dots M_6}\Psi \,, \\ \delta^{(1)}_\chi \Psi &= -\frac{1}{2}\Gamma^{MN}\chi\, C_{MN} - \frac{1}{6!}\Gamma^{M_1\dots M_6}\chi\, Z^+_{M_1\dots M_6} \,. \end{aligned} \tag{5.21}$$

These formulæ allow us to compute the effect of two successive supersymmetry transformations using (5.11) and (5.14):

$$\begin{aligned} [\delta^{(1)}_\chi, \delta^{(1)}_\epsilon]\Psi &= \frac{i}{16}\Big\{(\overline{\epsilon}\Psi)\,\chi - (\overline{\chi}\,\Psi)\,\epsilon\Big\} \,, \\ [\delta^{(1)}_\chi, \delta^{(1)}_\epsilon] C_{MN} &= \frac{i}{4}\overline{\chi}\left\{\Gamma_{[M}{}^P C_{N]P} + \frac{1}{5!}\Gamma_{[M}{}^{M_1\dots M_5} Z^+_{N]M_1\dots M_5}\right\}\mathcal{P}_+\,\epsilon \,, \\ [\delta^{(1)}_\chi, \delta^{(1)}_\epsilon] Z^+_{M_1\dots M_6} &= \overline{\chi}\left\{\frac{3i}{4}\Gamma_{[M_1\dots M_5}{}^N C_{M_6]N} + \frac{3i}{2}\Gamma_{[M_1}{}^N Z^+_{M_2\dots M_6]N} - \right. \\ &\quad \left. - \frac{5i}{12}\Gamma_{[M_1M_2M_3}{}^{N_1N_2N_3} Z^+_{M_4M_5M_6]N_1N_2N_3}\right\}\mathcal{P}_+\,\epsilon \,, \end{aligned} \tag{5.22}$$

where we used the self-duality[3] of Z^+. At this stage, we can mention that the above results are in perfect agreement with the adjoint representation of $[\delta^{(1)}_\chi, \delta^{(1)}_\epsilon]$ (viz. (5.8)) on the matrix fields.

[3] Z^+ satisfies $Z^+_{M_1\dots M_6} = \frac{1}{6!}\varepsilon_{M_1\dots M_6}{}^{N_1\dots N_6} Z^+_{N_1\dots N_6}$

5.3.3 $\mathfrak{sp}(32,\mathbb{R})$ transformations of the fields and their commutation relation with supersymmetries

To see under which transformations an $\mathfrak{osp}(1|32,\mathbb{R})$-based matrix model should be invariant, one should look at the full transformation properties including the bosonic $\mathfrak{sp}(32,\mathbb{R})$ transformations. In close analogy with equation (5.19), we have the following full transformation law of M:

$$\delta_H^{(1)} M_A{}^B = \left[\begin{pmatrix} h & \chi \\ -i\overline{\chi} & 0 \end{pmatrix}, \begin{pmatrix} m & \Psi \\ -i\overline{\Psi} & 0 \end{pmatrix}\right]_A{}^B, \tag{5.23}$$

implying the following transformation rules:

$$\delta_H^{(1)} m_A{}^B = [h,m]_A{}^B - i(\chi_A \overline{\Psi}^B - \Psi_A \overline{\chi}^B)\,, \tag{5.24}$$
$$\delta_H^{(1)} \Psi_A = h_A{}^C \Psi_C - m_A{}^C \chi_C\,. \tag{5.25}$$

We then want to extract from the first of the above equations the full transformation properties of C_{MN} and $Z^+_{M_1\ldots M_6}$. From (5.17) and (5.22) or directly using (5.14) and the cyclicity of the trace, the bosonic transformations are:

$$\begin{aligned} \delta_h^{(1)} C_{MN} =& \; 4h^P{}_{[N} C_{M]P} + \frac{4}{5!} h^{N_1\ldots N_5}{}_{[N} Z^+_{M]N_1\ldots N_5}\,, \\ \delta_h^{(1)} Z^+_{M_1\ldots M_6} =& \; 12 h_{[M_1\ldots M_5}{}^P C_{M_6]P} - 24\, h^N{}_{[M_1} Z^+_{M_2\ldots M_6]N} - \\ &+ \frac{20}{3} h^{N_1 N_2 N_3}{}_{[M_1 M_2 M_3} Z^+_{M_4 M_5 M_6]N_1 N_2 N_3}\,, \end{aligned} \tag{5.26}$$

while the fermionic part is as in (5.21). If one uses (5.26) to compute the commutator of a supersymmetry and an $\mathfrak{sp}(32,\mathbb{R})$ transformation, the results will look very complicated. On the other hand, the commutator of two symmetry transformations may be cast in a compact form using the graded Jacobi identity of the $\mathfrak{osp}(1|32,\mathbb{R})$ superalgebra, which comes into the game since matrix fields are in the adjoint representations of $\mathfrak{osp}(1|32,\mathbb{R})$.

Such a commutator acting on the fermionic field Ψ yields:

$$\begin{aligned} [\delta_\chi^{(1)}, \delta_h^{(1)}]\,\Psi =& \; -hm\chi + [h,m]\chi \; = \; -mh\chi \; = \\ =& \; -\frac{1}{2!}(\mathcal{P}_+\Gamma^{MN} h\chi) C_{MN} - \frac{1}{6!}(\mathcal{P}_+\Gamma^{M_1\ldots M_6} h\chi) Z^+_{M_1\ldots M_6}\,. \end{aligned} \tag{5.27}$$

The same transformation on m leads to:

$$[\delta_\chi^{(1)}, \delta_h^{(1)}]\, m_A{}^B = i\left(\Psi_A (\chi^\top h^\top C)^B - (h\chi)_A (\Psi^\top C)^B\right)\,, \tag{5.28}$$

which in components reads:

$$[\delta_\chi^{(1)}, \delta_h^{(1)}]\, C_{MN} = \frac{i}{16}\, \chi^\top C h \Gamma_{MN} \Psi\,, \tag{5.29}$$
$$[\delta_\chi^{(1)}, \delta_h^{(1)}]\, Z^+_{M_1\ldots M_6} = \frac{i}{16}\, \chi^\top C h \Gamma_{M_1\ldots M_6} \Psi\,. \tag{5.30}$$

In eqns. (5.27), (5.29) and (5.30), one could write h in components as in (5.12) and use:

$$\Gamma_{M_1\ldots M_k}\Gamma_{N_1\ldots N_l} = \sum_{j=0}^{min(k,l)} (-1)^{j/2(2k-j-1)}\, j! \binom{k}{j}\binom{l}{j} \eta_{[M_1[N_1}\ldots\eta_{M_jN_j}\Gamma_{M_{j+1}\ldots M_k]N_{j+1}\ldots N_l]} \tag{5.31}$$

to develop the products of Gamma matrices in irreducible tensors of $SO(10,2)$ and obtain a more explicit result. The final expression for (5.27) and (5.30) will contain Gamma matrices with an even number of indices ranging from 0 to 12, while in (5.29) the number of indices will stop at 8. Since we won't use this result as such in the following, we won't give it here explicitly.

5.3.4 A note on translational invariance and kinematical supersymmetries

At this point, we want to make a comment on the so-called kinematical supersymmetries that have been discussed in the literature on matrix models ([107], [17]). Indeed, in the IIB matrix model, commutation relations of dynamical supersymmetries do not close to give space-time translations, i.e. they do not shift the target space-time fields X^M by a constant vector.

However, as was pointed out in [107] and [17], if one introduces so-called kinematical supersymmetry transformations, their commutator with dynamical supersymmetries yields the expected translations by a constant vector, as we explained in subsection 4.2.2. By kinematical supersymmetries, one simply means translations of fermions by a constant Grassmannian odd parameter. In our case, this assumes the form:

$$\begin{aligned} \delta_\xi^{(2)} C_{MN} \;=\; \delta_\xi^{(2)} Z^+_{M_1\ldots M_6} \;&=\; 0\;, \qquad\qquad \delta_\xi^{(2)}\Psi \;=\; \xi\;, \\ \Longrightarrow [\delta_\xi^{(2)},\delta_\zeta^{(2)}]M \;&=\; 0 \end{aligned} \tag{5.32}$$

Since there is no vector field to be interpreted as space-time coordinates in this 12-dimensional setting, it is interesting to look at the interplay between dynamical and kinematical supersymmetries (which we denote respectively by $\delta^{(1)}$ and $\delta^{(2)}$) when acting on higher-rank tensors. In our case:

$$[\delta_\chi^{(1)},\delta_\xi^{(2)}]C_{MN} \;=\; -\frac{i}{16}(\chi^\top C\Gamma_{MN}\xi)\;, \qquad [\delta_\chi^{(1)},\delta_\xi^{(2)}]Z^+_{M_1\ldots M_6} = -\frac{i}{16}(\chi^\top C\Gamma_{M_1\ldots M_6}\xi)\;. \tag{5.33}$$

Thus, $[\delta_\chi^{(1)},\delta_\xi^{(2)}]$ applied to p–forms closes to translations by a constant p-form, generalizing the vector case mentioned above.

For fermions, we have as expected:

$$[\delta_\chi^{(1)},\delta_\xi^{(2)}]\Psi = 0\;. \tag{5.34}$$

It is however more natural to consider dynamical and kinematical symmetries to be independent. We would thus expect them to commute. With this in mind, we suggest a generalized version of the translational symmetries introduced in (5.32):

$$\delta_K^{(2)}\Psi = \xi, \quad \delta_K^{(2)} C_{MN} = k_{MN}, \quad \delta_K^{(2)} Z^+_{M_1\ldots M_6} = k^+_{M_1\ldots M_6}\;. \tag{5.35}$$

It is then natural that the matrix

$$K = \begin{pmatrix} k & \xi \\ -i\xi^\top C & 0 \end{pmatrix} \tag{5.36}$$

should transform in the adjoint of $\mathfrak{osp}(1|32,\mathbb{R})$, which means that:

$$\delta_H^{(1)} k_A{}^B = [h,k]_A{}^B - i(\chi_A(\xi^\top C)^B - \xi_A(\chi^\top C)^B) \tag{5.37}$$

$$\delta_H^{(1)} \xi_A = h_A{}^C \xi_C - k_A{}^C \chi_C. \tag{5.38}$$

We can now compute the general commutation relations between translational symmetries $M \to M + K$ and $\mathfrak{osp}(1|32,\mathbb{R})$ transformations and conclude that these operations actually commute:

$$[\delta_H^{(1)}, \delta_K^{(2)}]M = 0\ . \tag{5.39}$$

5.3.5 Twelve-dimensional action for supersymmetric cubic matrix model

We will now build the simplest gauge- and translational-invariant $\mathfrak{osp}(1|32,\mathbb{R})$ supermatrix model with $U(N)$ gauge group. For this purpose, we promote each entry of the matrix M to a hermitian matrix in the Lie algebra of $\mathfrak{u}(N)$ for some value of N. We choose the generators $\{t^a\}_{a=1,\dots,N^2}$ of $\mathfrak{u}(N)$ so that: $[t^a,t^b] = if^{abc}t^c$ and $Tr_{\mathfrak{u}(N)}(t^a\cdot t^b) = \delta^{ab}$.

In order to preserve both orthosymplectic and gauge invariance of the model, it suffices to write its action as a supertrace over $\mathfrak{osp}(1|32,\mathbb{R})$ and a trace over $\mathfrak{u}(N)$ of a polynomial of $\mathfrak{osp}(1|32,\mathbb{R}) \otimes \mathfrak{u}(N)$ matrices. Following [148], we consider the simplest model containing interactions, namely: $STr_{\mathfrak{osp}(1|32,\mathbb{R})}Tr_{\mathfrak{u}(N)}(M[M,M]_{\mathfrak{u}(N)})$. For hermiticity's sake one has to multiply such an action by a factor of i. We also introduce a coupling constant g^2. This cubic action takes the following form:

$$\begin{aligned} I = \frac{i}{g^2} STr_{\mathfrak{osp}(1|32,\mathbb{R})}Tr_{\mathfrak{u}(N)}(M[M,M]_{\mathfrak{u}(N)}) \ &= \ -\frac{1}{g^2} f^{abc} STr_{\mathfrak{osp}(1|32,\mathbb{R})}(M^aM^bM^c) \ = \\ = -\frac{1}{g^2} f^{abc}\left(Tr_{\mathfrak{sp}(32,\mathbb{R})}(m^am^bm^c) + 3i\Psi^{a\top}Cm^b\Psi^c\right)& \end{aligned} \tag{5.40}$$

which we can now express in terms of 12-dimensional representations, where the symplectic matrix m is given by (5.20).

Let us give a short overview of the steps involved in the computation of the trace in (5.40). It amounts to performing traces of triple products of m^a's over $\mathfrak{sp}(32,\mathbb{R})$, i.e. traces of products of Dirac matrices. We proceed by decomposing such products into their irreducible representations using (5.31). The only contributions surviving the trace are those proportional to the unit matrix. Thus, the only terms left in (5.40) will be those containing traces over triple products of 2-forms, over products of a 2-form and two 6-forms, and over triple products of 6-forms, while terms proportional to products of two 2-forms and a 6-form will yield zero contributions.

The two terms involving Z^+'s (to wit CZ^+Z^+ and $Z^+Z^+Z^+$) require some care, since $\Gamma^{A_1\dots A_{12}}$ is proportional to $\Gamma_\star$ in $12D$, and hence $Tr(\mathcal{P}^+\Gamma^{A_1\dots A_{12}}) \propto Tr(\Gamma_\star^2) \neq 0$. Since double products of six-indices Dirac matrices decompose into $\mathbb{I}$ and Dirac matrices with 2, 4 up to 12 indices, their trace with Γ^{MN} will keep terms with 2, 10 or 12 indices (the last two containing Levi-Civita tensors) while their trace with $\Gamma^{M_1\dots M_6}$ will only keep those terms with 6, 8, 10 and 12 indices.

Finally, putting all contributions together, exploiting the self-duality of Z^+ and rewriting cubic products of fields contracted by f^{abc} as a trace over $\mathfrak{u}(N)$, we get:

$$I = \frac{32i}{g^2} Tr_{\mathfrak{u}(N)} \Bigg(C_M{}^N \, [C_N{}^O, C_O{}^M]_{\mathfrak{u}(N)} \; - \; \frac{1}{20} \, C_A{}^B \, [Z_B^{+M_1 \ldots M_5}, Z^{+}_{M_1 \ldots M_5}{}^{A}]_{\mathfrak{u}(N)} + \\ + \; \frac{61}{2(3!)^3} \, Z^{+}_{ABC}{}^{DEF} \, [Z^{+}_{DEF}{}^{GHI}, Z^{+}_{GHI}{}^{ABC}]_{\mathfrak{u}(N)} + \\ + \; \frac{3i}{64} \Psi^\top C \mathcal{P}_+ \Gamma^{MN} [C_{MN}, \Psi]_{\mathfrak{u}(N)} \; + \; \frac{3i}{32 \cdot 6!} \Psi^\top C \mathcal{P}_+ \Gamma^{M_1 \ldots M_6} [Z^{+}_{M_1 \ldots M_6}, \Psi]_{\mathfrak{u}(N)} \Bigg)$$

where we have chosen: $\varepsilon^{0\ldots 11} = \varepsilon_{0\ldots 11} = +1$, since the metric contains two time-like indices. Similarly, one can decompose invariant terms such as $STr_{\mathfrak{osp}(1|32,\mathbb{R})} Tr_{\mathfrak{u}(N)}(M^2)$ and $STr_{\mathfrak{osp}(1|32,\mathbb{R})} Tr_{\mathfrak{u}(N)}([M,M]_{\mathfrak{u}(N)}[M,M]_{\mathfrak{u}(N)})$, etc. While it might be interesting to investigate further the $12D$ physics obtained from such models and compare it to F-theory dynamics, we will not do so here. We will instead move to a detailed study of the better known $11D$ case, possibly relevant for M-theory.

5.4 Study of the 11D M-theory case

We now want to study the $11D$ matrix model more thoroughly. Similarly to the 12 dimensional case, we embed the $SO(10,1)$ Clifford algebra into $\mathfrak{sp}(32,\mathbb{R})$ and replace the fundamental representation of $\mathfrak{sp}(32,\mathbb{R})$ by $SO(10,1)$ Majorana spinors. A convenient choice of 32×32 Gamma matrices are the $\tilde{\Gamma}$'s we used in the $12D$ case. We choose them as follows:

$$\tilde{\Gamma}^0 = \begin{pmatrix} 0 & -\mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{pmatrix} , \; \tilde{\Gamma}^{10} = \begin{pmatrix} 0 & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{pmatrix} , \; \tilde{\Gamma}^i = \begin{pmatrix} \gamma^i & 0 \\ 0 & -\gamma^i \end{pmatrix} \quad \forall i = 1, \ldots, 9, \tag{5.41}$$

where the γ^i's build a Majorana representation of the Clifford algebra of $SO(9)$, $\{\gamma^i, \gamma^j\} = 2\delta^{ij}\mathbb{I}_{16}$. As before, we have $\tilde{\Gamma}^{10} = \tilde{\Gamma}^0\tilde{\Gamma}^1 \cdots \tilde{\Gamma}^9$ provided $\gamma^1 \cdots \gamma^9 = \mathbb{I}_{16}$, since we can define γ^9 to be $\gamma^9 = \gamma^1 \cdots \gamma^8$. This choice has $(\tilde{\Gamma}^0)^2 = -\mathbb{I}_{32}$, while $(\tilde{\Gamma}^M)^2 = \mathbb{I}_{32}$, $\forall M = 1 \ldots 10$ and gives a representation of $\{\tilde{\Gamma}^M, \tilde{\Gamma}^N\} = 2\eta^{MN}\mathbb{I}_{32}$ for the choice $(-,+,\ldots,+)$ of the metric. As we have again chosen all $\tilde{\Gamma}$'s to be real, this allows to take $\tilde{B} = \mathbb{I}$ in $\Psi^* = \tilde{B}\Psi$, which implies that the charge conjugation matrix is $\tilde{C} = \tilde{\Gamma}^0$. Moreover, we have the following transposition rules for the $\tilde{\Gamma}$ matrices:

$$\tilde{C}\tilde{\Gamma}^{M_1 \ldots M_n}\tilde{C}^{-1} = (-1)^{n(n+1)/2}(\tilde{\Gamma}^{M_1 \ldots M_n})^\top \tag{5.42}$$

We will identify the fundamental representation of $\mathfrak{sp}(32,\mathbb{R})$ with a 32-component Majorana spinor of $SO(10,1)$. Splitting the 32 real components of the Ψ into $16+16$ as in: $\Psi^\top = (\Phi_1^\top, \Phi_2^\top)$, we can use the following identity:

$$(-i\Phi_2^\top, i\Phi_1^\top) = -i\Psi^\top\tilde{\Gamma}^0 = -i\Psi^\top\tilde{C} = -i\overline{\Psi}$$

to write orthosymplectic matrices again as in (5.2).

5.4.1 Embedding of the 11D super-Poincaré algebra in $\mathfrak{osp}(1|32,\mathbb{R})$

In $11D$, we can also express the $\mathfrak{sp}(32,\mathbb{R})$ transformations in terms of translations, Lorentz transformations and new 5-form symmetries, by defining:

$$h = h_M P^M + h_{MN} J^{MN} + h_{M_1\ldots M_5} J^{M_1\ldots M_5} \,. \tag{5.43}$$

With the help of (5.14), we can compute this enhanced super-Poincaré algebra as in dimension 12, using the following explicit representation of the generators:

$$P^M = \tilde{\Gamma}^M \,, \quad J^{MN} = \frac{1}{2}\tilde{\Gamma}^{MN} \,, \quad J^{M_1\ldots M_5} = \frac{1}{5!}\tilde{\Gamma}^{M_1\ldots M_5} \tag{5.44}$$

In order to express everything in terms of the above generators, we need to dualize forms using the formula: $(-1)^{\frac{k(k-1)}{2}} \varepsilon^{M_1\ldots M_{11}} \tilde{\Gamma}_{M_{k+1}\ldots M_{11}} = -(11-k)!\tilde{\Gamma}^{M_1\ldots M_k}$. This leads to the following superalgebra:

$$\begin{aligned}
[P^M, P^N] &= 4J^{MN} \\
[P^M, J^{OP}] &= 2\eta^{M[O}P^{P]} \\
[J^{MN}, J^{OP}] &= -4\eta^{[M[O}J^{N]P]} \\
[P^M, J^{M_1\ldots M_5}] &= -\frac{2}{5!}\varepsilon^{MM_1\ldots M_5}{}_{N_1\ldots N_5} J^{N_1\ldots N_5} \\
[J^{MN}, J^{M_1\ldots M_5}] &= -10\,\eta^{[M[M_1} J^{N]M_2\ldots M_5]} \\
[J^{M_1\ldots M_5}, J^{N_1\ldots N_5}] &= -\frac{2}{(5!)^2}\varepsilon^{M_1\ldots M_5 N_1\ldots N_5}{}_A P^A + \frac{1}{(3!)^2}\eta^{[M_1[N_1}\eta^{M_2\,N_2}\varepsilon^{M_3\ldots M_5]N_3\ldots N_5]}{}_{O_1\ldots O_5} J^{O_1\ldots O_5} + \\
&\quad + \frac{1}{3!}\eta^{[M_1[N_1}\eta^{M_2\,N_2}\eta^{M_3\,N_3}\eta^{M_4\,N_4} J^{M_5]N_5]} \\
[P^M, Q_A] &= (\tilde{\Gamma}^M)^B{}_A Q_B \\
[J^{MN}, Q_A] &= \frac{1}{2}(\tilde{\Gamma}^{MN})^B{}_A Q_B \\
[J^{M_1\ldots M_5}, Q_A] &= \frac{1}{5!}(\tilde{\Gamma}^{M_1\ldots M_5})^B{}_A Q_B \\
\{Q_A, Q^B\} &= \frac{i}{16}(\tilde{C}\tilde{\Gamma}_M)_A{}^B P^M - \frac{i}{16}(\tilde{C}\tilde{\Gamma}_{MN})_A{}^B J^{MN} + \frac{i}{16}(\tilde{C}\tilde{\Gamma}_{M_1\ldots M_5})_A{}^B J^{M_1\ldots M_5} \,.
\end{aligned} \tag{5.45}$$

Note that this algebra is the dimensional reduction from $12D$ to $11D$ of (5.17). In particular, the first three lines build the $\mathfrak{so}(10,2)$ Lie algebra, but appear in this new 11-dimensional context as the Lie algebra of symmetries of AdS_{11} space (it is of course also the conformal algebra in 9+1 dimensions). We may wonder whether this superalgebra is a minimal supersymmetric extension of the AdS_{11} Lie algebra or not. If we try to construct an algebra without the five-form symmetry generators, the graded Jacobi identity forbids the appearence of a five-form central charge on the RHS of the $\{Q_A, Q^B\}$ anticommutator. The number of independent components in this last line of the superalgebra will thus be bigger on the LHS than on the RHS. This is not strictly forbidden, but it has implications on the representation theory of the superalgebra. The absence of central charges will for example forbid the existence of shortened representations with a non-minimal eigenvalue of the quadratic Casimir operator $C = -1/4 P_M P^M + J_{MN} J^{MN}$ ("spin") of the AdS_{11} symmetry group (see [128]). More generally, in

11D, either all objects in the RHS of the last line are central charges (this case corresponds simply to the $11D$ Super-Poincaré algebra) or they should all be symmetry generators. Thus, although it is not strictly-speaking the minimal supersymmetric extension of the AdS_{11} Lie algebra, it is certainly the most natural one. That's why some authors [34] call $\mathfrak{osp}(1|32,\mathbb{R})$ the super-AdS algebra in $11D$. Here, we will stick to the more neutral $\mathfrak{osp}(1|32,\mathbb{R})$ terminology. Furthermore, $\mathfrak{osp}(1|32,\mathbb{R})$ is also the maximal finite-dimensional (non-central) $\mathcal{N}=1$ extension of the AdS_{11} algebra. In principle, one could consider even bigger superalgebras, but we will not investigate them in this article.

It is also worth remarking that similar algebras have been studied in [31] where they are called topological extensions of the supersymmetry algebras for supermembranes and super-5-branes.

5.4.2 The supersymmetry properties of the 11D matrix fields

Let us now look at the action of supersymmetries on the fields of an $\mathfrak{osp}(1|32,\mathbb{R})$ eleven-dimensional matrix model. We expand once again the bosonic part of our former matrix M on the irreducible representations of $SO(10,1)$ in terms of 32-dimensional Γ matrices:

$$m \,=\, X_M\,\widetilde{\Gamma}^M \,+\, \frac{1}{2!}\,C_{MN}\,\widetilde{\Gamma}^{MN} \,+\, \frac{1}{5!}\,Z_{M_1\dots M_5}\,\widetilde{\Gamma}^{M_1\dots M_5}\;,$$

where the vector, the 2- and 5-form are given by:

$$X_M \,=\, \frac{1}{32}\,Tr_{\mathfrak{sp}(32,\mathbb{R})}(m\,\widetilde{\Gamma}_M)\,,\; C_{MN} \,=\, -\frac{1}{32}\,Tr_{\mathfrak{sp}(32,\mathbb{R})}(m\,\widetilde{\Gamma}_{MN})\,,\; Z_{M_1\dots M_5} \,=\, \frac{1}{32}\,Tr_{\mathfrak{sp}(32,\mathbb{R})}(m\,\widetilde{\Gamma}_{M_1\dots M_5})\,.$$

Let us give the whole $\delta_H^{(1)}$ transformation acting on the fields (using the cyclic property of the trace, for instance: $Tr([h,m]\widetilde{\Gamma}^M)=Tr(h[m,\widetilde{\Gamma}^M])$):

$$\begin{aligned}
\delta_H^{(1)}\,X^M \,&=\, 2\left(h^{MQ}\,X_Q \,+\, h^Q\,C_Q{}^M \,-\, \frac{1}{(5!)^2}\,\varepsilon^{MM_1\dots M_5}{}_{N_1\dots N_5}h^{N_1\dots N_5}\,Z_{M_1\dots M_5}\right) \,-\, \frac{i}{16}\,\chi^\top\widetilde{\Gamma}^0\widetilde{\Gamma}^M\,\Psi\,,\\
\delta_H^{(1)}\,C^{MN} \,&=\, -4\,\left(h^{[M}X^{N]} \,-\, h^{[M}{}_Q C^{N]Q} \,+\, \frac{1}{4!}\,h_{M_1\dots M_4}{}^{[M}\,Z^{N]M_1\dots M_4}\right) \,+\, \frac{i}{16}\,\chi^\top\widetilde{\Gamma}^0\widetilde{\Gamma}^{MN}\Psi\,,\\
\delta_H^{(1)}\,Z^{M_1\dots M_5} \,&=\, 2\left(\frac{1}{5!}\varepsilon^{M_1\dots M_5}{}_{N_1\dots N_5 Q}\,h^{N_1\dots N_5}\,X^Q \,+\, 5\,h_Q{}^{[M_1\dots M_4}\,C^{M_5]Q} \,-\, 5\,h_Q{}^{[M_1}\,Z^{M_2\dots M_5]Q} \,+\right.\\
&\quad\left. +\,\frac{1}{5!}\,\varepsilon^{M_1\dots M_5}{}_{ON_1\dots N_5}\,h^O\,Z^{N_1\dots N_5} \,-\, \frac{1}{3\cdot 4!}h^{O_1\dots O_5}\,\varepsilon_{O_1\dots O_5 N_1 N_2 N_3}{}^{[M_1 M_2 M_3}\,Z^{M_4 M_5]N_1 N_2 N_3}\right) \,-\\
&\quad -\,\frac{i}{16}\,\chi^\top\widetilde{\Gamma}^0\widetilde{\Gamma}^{M_1\dots M_5}\Psi\,,\\
\delta_H^{(1)}\,\Psi \,&=\, \left(h_M\,\widetilde{\Gamma}^M \,+\, h_{MN}\,\widetilde{\Gamma}^{MN} \,+\, h_{M_1\dots M_5}\,\widetilde{\Gamma}^{M_1\dots M_5}\right)\Psi \,-\\
&\quad -\,\widetilde{\Gamma}^M\chi X_M \,-\, \frac{1}{2}\widetilde{\Gamma}^{MN}\chi C_{MN} \,-\, \frac{1}{5!}\widetilde{\Gamma}^{M_1\dots M_5}\chi Z_{M_1\dots M_5}\,,
\end{aligned}$$

where the part between parentheses describes the symplectic transformations, while the remainder represents the supersymmetry variations. Note that we used $(-1)^{\frac{k(k-1)}{2}}\varepsilon^{M_1\dots M_{11}}\widetilde{\Gamma}_{M_{k+1}\dots M_{11}}=-(11-k)!\widetilde{\Gamma}^{M_1\dots M_k}$ in $\delta_H^{(1)}\,Z^{M_1\dots M_5}$ to dualize the Dirac matrices when needed.

5.4.3 Eleven-dimensional action for a supersymmetric matrix model

As in the $12D$ case, we will now consider a specific model, invariant under $U(N)$ gauge and $\mathfrak{osp}(1|32,\mathbb{R})$ transformations. The simplest such model containing interactions and "propagators" is a cubic action along with a quadratic term. Hence, we choose:

$$I = STr_{\mathfrak{osp}(1|32,\mathbb{R})\otimes\mathfrak{u}(N)}\left(-\mu\, M^2 + \frac{i}{g^2}\, M[M,M]_{\mathfrak{u}(N)}\right)\,. \tag{5.46}$$

Contrary to a purely cubic model, one loses invariance under $M \to M + K$ for a constant diagonal matrix K, which contains the space-time translations of the BFSS model. In contrast with the BFSS theory, our model doesn't exhibit the symmetries of flat $11D$ Minkowski space-time, so we don't really expect this sort of invariance. However, the symmetries generated by P^M remain unbroken, as well as all other $\mathfrak{osp}(1|32,\mathbb{R})$ transformations. Indeed, the related bosonic part of the algebra (5.45) contains the symmetries of AdS_{11} as a subalgebra, and as was pointed out in [83] and [54], massive matrix models with a tachyonic mass-term for the coordinate X's fields appear in attempts to describe gravity in de Sitter spaces (an alternative approach can be found in [125]). Note that we take the opposite sign for the quadratic term of (5.46), this choice being motivated by the belief that AdS vacua are more stable than dS ones, so that the potential energy for physical bosonic fields should be positive definite in our setting.

The computation of the 11-dimensional action for this supermatrix model is analogous to the one performed in 12 dimensions. We remind the reader that each entry of the matrix M now becomes a hermitian matrix in the Lie algebra of $\mathfrak{u}(N)$ for some large value of N whose generators are defined as in the $12D$ case.

After performing in (5.46) the traces on products of Gamma matrices, it comes out that the terms of the form XXX, XXZ, XCC, CCZ and XCZ have vanishing trace (since products of Gamma matrices related to these terms have decomposition in irreducible tensors that do not contain a term proportional to $\mathbb{I}_{32}$) so that only terms of the form XXC, XZZ, CZZ, CCC, ZZZ will remain from the cubic bosonic terms. As for terms containing fermions and the mass terms, they are trivial to compute. Using (5.31) and the usual duality relation for Dirac matrices in $11D$, one finally obtains the following result:

$$\begin{aligned} I =\; & -32\mu\, Tr_{\mathfrak{u}(N)}\left\{X_M X^M - \frac{1}{2!}\, C_{MN}C^{MN} + \frac{1}{5!}\, Z_{M_1\ldots M_5}Z^{M_1\ldots M_5} + \frac{i}{16}\overline{\Psi}\Psi\right\} + \\ & + \frac{32i}{g^2}\, Tr_{\mathfrak{u}(N)}\Bigg(3\, C_{NM}\,[X^M,X^N]_{\mathfrak{u}(N)} - \varepsilon^{M_1\ldots M_{11}}\left\{\frac{3}{(5!)^2}\, Z_{M_1\ldots M_5}\,[X_{M_6}, Z_{M_7\ldots M_{11}}]_{\mathfrak{u}(N)} - \right. \\ & \left. - \frac{2^3\, 5^2}{(5!)^3}\, Z_{M_1\, M_2\, M_3}{}^{AB}\,[Z_{AB\, M_4\, M_5\, M_6}, Z_{M_7\ldots M_{11}}]_{\mathfrak{u}(N)}\right\} + \frac{3}{4!}\, C_{MN}\,[Z_{A_1\ldots A_4}{}^N, Z^{A_1\ldots A_4\, M}]_{\mathfrak{u}(N)} + \\ & + C_{MN}\,[C^N{}_O, C^{OM}]_{\mathfrak{u}(N)} + \frac{3i}{32}\left\{\overline{\Psi}\,\widetilde{\Gamma}^M[X_M,\Psi]_{\mathfrak{u}(N)} + \frac{1}{2!}\,\overline{\Psi}\,\widetilde{\Gamma}^{MN}\,[C_{MN},\Psi]_{\mathfrak{u}(N)} + \right. \\ & \left. + \frac{1}{5!}\,\overline{\Psi}\,\widetilde{\Gamma}^{M_1\ldots M_5}\,[Z_{M_1\ldots M_5},\Psi]_{\mathfrak{u}(N)}\right\}\Bigg)\,. \end{aligned} \tag{5.47}$$

5.5 Dynamics of the $11D$ supermatrix model and its relation to BFSS theory

Now, we will try to see to what extent our model may describe at least part of the dynamics of M-theory. Since the physics of the BFSS matrix model and its relationships to 11D supergravity and superstring theory are relatively well understood, if our model is to be relevant to M-theory, we expect it to be related to BFSS theory at least in some régime. To see such a relationship, we should reduce our model to one of its ten-dimensional sectors and turn it into a matrix quantum mechanics.

5.5.1 Compactification and T-duality of the $11D$ supermatrix action

If we want to link (5.47) to BFSS, which is basically a quantum mechanical supersymmetric matrix model, we should reduce the eleven-dimensional target-space spanned by the X^M's to 10 dimensions, and, at the same time, let a "time" parameter t appear. At this stage, the world-volume of the theory is reduced to one point. We start by decompactifying it along two directions, following the standard procedure outlined in [151]. Namely, we compactify the target-space coordinates X_0 and X_{10} on circles of respective radii $R_0 = R$ and $R_{10} = \omega R$. We introduce the rescaled field $X'_{10} \equiv X_{10}/\omega$ which has the same $2\pi R$ periodicity as X_0. We can then perform T-dualities on X_0 and X'_{10} to circles of dual radii $\widehat{R} \equiv l_{11}^2/R$ (parametrized by τ and y), where l_{11} is some scale, typically the 11-dimensional Planck length. The fields of our theory, for simplicity denoted here by Y, now depend on the world-sheet coordinates τ and y as follows:

$$Y(\tau, y) \; = \; \sum_{m,n} Y_{mn} \, e^{i(m\tau + ny)/\widehat{R}} \; . \tag{5.48}$$

As a consequence, we now need to average the action over τ and y with the measure $d\tau dy/(2\pi\widehat{R})^2$. Finally, one should identify under T-duality:

$$X_0 \sim 2\pi l_{11}^2 \Big(i\partial_\tau - A_\tau(\tau, y)\Big) \triangleq i\widehat{\mathcal{D}}_\tau \; , \quad X_{10} \equiv \omega X'_{10} \sim 2\pi\omega l_{11}^2 \Big(i\partial_y - A_y(\tau, y)\Big) \triangleq i\omega\widehat{\mathcal{D}}_y \; , \tag{5.49}$$

where A_τ and A_y are the connections on the $U(N)$ gauge bundle over the world-sheet. For notational convenience, we rewrite $\phi \triangleq C_{0\,10}$, $F_{\tau y} \triangleq -i\,[\widehat{\mathcal{D}}_\tau, \widehat{\mathcal{D}}_y]$ and $\widetilde{\Gamma}_* \triangleq \widetilde{\Gamma}_{10}$ and encode the possible values of the indices in the following notation:

$$\begin{aligned} A,\; B \;&=\; 0, \ldots, 10 \; , \qquad i,\; j,\; k \;=\; 1, \ldots, 9 \; , \\ \alpha \;&=\; 1, \ldots, 10 \; , \qquad \beta \;=\; 0, \ldots, 9 \; . \end{aligned}$$

Then, the compactified version of (5.47) reads:

$$
\begin{aligned}
I_c = \frac{32i}{g^2}\int \frac{d\tau dy}{(2\pi\widehat{R})^2}\,Tr_{\mathfrak{u}(N)}\Bigg(& -6\,C_{i0}\,i[\widehat{\mathcal{D}}_\tau, X_i] \,+\, 6\omega\, C_{i10}\,i[\widehat{\mathcal{D}}_y, X_i] \,+\, \frac{3}{32}\,\overline{\Psi}\,\widetilde{\Gamma}_0\,[\widehat{\mathcal{D}}_\tau,\Psi] - \\
& - \frac{3\omega}{32}\,\overline{\Psi}\,\widetilde{\Gamma}_*\,[\widehat{\mathcal{D}}_y,\Psi] \,-\, \frac{3}{(5!)^2}\,\varepsilon_{\alpha_1\cdots\alpha_{10}0}\,Z_{\alpha_1\cdots\alpha_5}\,i[\widehat{\mathcal{D}}_\tau, Z_{\alpha_6\cdots\alpha_{10}}] \,+\, \frac{3\omega}{(5!)^2}\,\varepsilon^{\beta_1\cdots\beta_{10}10}\,Z_{\beta_1\cdots\beta_5}\,i[\widehat{\mathcal{D}}_y, Z_{\beta_6\cdots\beta_{10}}] + \\
& + 6i\omega\,\phi\,F_{\tau y} \,+\, 3\,C_{ij}\,[X_j, X_i] \,+\, \frac{3}{(5!)^2}\,\varepsilon^{A_1\cdots A_{10}}{}_j\,Z_{A_1\cdots A_5}\,[X_j, Z_{A_6\cdots A_{10}}] - \\
& - \frac{2^3 5^2}{(5!)^3}\,\varepsilon^{A_1\cdots A_{11}}\,Z_{A_1A_2A_3}{}^{B_1B_2}\,[Z_{B_1B_2A_4A_5A_6}, Z_{A_7\cdots A_{11}}] \,+\, \frac{3}{4!}\Big\{C_{ij}\,[Z_{j\,A_1\cdots A_4}, Z_i{}^{A_1\cdots A_4}] - \\
& - 2\,C_{i0}\,[Z_{0\,\alpha_1\cdots\alpha_4}, Z_{i\,\alpha_1\cdots\alpha_4}] \,+\, 2\,C_{i10}\,[Z_{10\,\beta_1\cdots\beta_4}, Z_i{}^{\beta_1\cdots\beta_4}] \,-\, 2\,\phi\,[Z_{10\,i_1\cdots i_4}, Z_{0\,i_1\cdots i_4}]\Big\} + \\
& + C_{ij}\,[C_{jk}, C_{ki}] \,+\, 3\,C_{i0}\,[C_{k0}, C_{ki}] \,-\, 3\,C_{i10}\,[C_{k10}, C_{ki}] \,+\, 6\,\phi\,[C_{k10}, C_{k0}] + \\
& + \frac{3i}{32}\Big\{\overline{\Psi}\,\widetilde{\Gamma}_i\,[X_i,\Psi] \,+\, \frac{1}{2!}\,\overline{\Psi}\,\widetilde{\Gamma}_{ij}[C_{ij},\Psi] \,-\, \overline{\Psi}\,\widetilde{\Gamma}_i\widetilde{\Gamma}_0\,[C_{i0},\Psi] \,+\, \overline{\Psi}\,\widetilde{\Gamma}_i\widetilde{\Gamma}_*\,[C_{i10},\Psi] \,-\, \overline{\Psi}\,\widetilde{\Gamma}_0\widetilde{\Gamma}_*\,[\phi,\Psi] + \\
& + \frac{1}{5!}\,\overline{\Psi}\,\widetilde{\Gamma}^{A_1\cdots A_5}\,[Z_{A_1\cdots A_5},\Psi]\Big\} \,+\, i\mu g^2\Big(\widehat{\mathcal{D}}_\tau\widehat{\mathcal{D}}_\tau \,-\, \omega^2\,\widehat{\mathcal{D}}_y\widehat{\mathcal{D}}_y \,+\, X_iX_i \,+\, \frac{i}{16}\overline{\Psi}\Psi \,+\, \phi^2 - \\
& - \frac{1}{2!}\,C_{ij}C_{ij} \,+\, C_{i0}C_{i0} \,-\, C_{i10}C_{i10} \,+\, \frac{1}{5!}Z_{A_1\cdots A_5}Z^{A_1\cdots A_5}\Big)\Bigg)\,.
\end{aligned}
\tag{5.50}
$$

Repeated indices are contracted, and when they appear alternately up and down, minkowskian signature applies, whereas euclidian signature is in force when both are down.

5.5.2 Ten-dimensional limits and IMF

Since the BFSS matrix model is conjectured to describe M-theory in the infinite momentum frame, we shall investigate our model in this particular limit. For this purpose, let's define the light-cone coordinates $t_+ \equiv (\tau + y)/\sqrt{2}$ and $t_- \equiv (\tau - y)/\sqrt{2}$ and perform a boost in the y direction. In the limit where the boost parameter u is large, the boost acts as $(t_+, t_-) \xrightarrow{\sim} (ut_+, u^{-1}t_-)$, or as $(\tau, y) \xrightarrow{\sim} \sqrt{2}(ut_+, ut_+)$ on the original coordinates. In particular, when $u \to \infty$, the t_- dependence disappears from the action and we can perform the trivial t_- integration. The dynamics is now solely described by the parameter $t \equiv \sqrt{2}ut_+$, which is decompactified through this procedure. In particular, both $\widehat{\mathcal{D}}_\tau$ and $\widehat{\mathcal{D}}_y$ are mapped into $\widehat{\mathcal{D}}_t$.

So far, the ratio of the compactification radii ω is left undetermined and it parametrizes a continuous family of frames. It affects the kinetic terms as:

$$
\begin{aligned}
I_c = \frac{32i}{g^2}\lim_{u\to\infty}\int_{-\pi\widehat{R}u}^{\pi\widehat{R}u}\frac{dt}{2\sqrt{2}\pi\widehat{R}u}\,Tr_{\mathfrak{u}(N)}\Bigg(& -6\,\Big(C_{i0} - \omega\,C_{i10}\Big)\,i[\widehat{\mathcal{D}}_t, X_i] \,+\, \frac{3}{32}\,\overline{\Psi}\,\Big(\widetilde{\Gamma}_0 - \omega\,\widetilde{\Gamma}_*\Big)\,[\widehat{\mathcal{D}}_t,\Psi] - \\
& - \frac{3}{(5!)^2}\,\varepsilon_{\alpha_1\cdots\alpha_{10}0}\,Z_{\alpha_1\cdots\alpha_5}\,i[\widehat{\mathcal{D}}_t, Z_{\alpha_6\cdots\alpha_{10}}] \,+\, \frac{3\omega}{(5!)^2}\,\varepsilon^{\beta_1\cdots\beta_{10}10}\,Z_{\beta_1\cdots\beta_5}\,i[\widehat{\mathcal{D}}_t, Z_{\beta_6\cdots\beta_{10}}] \,+\, \ldots\Bigg)
\end{aligned}
\tag{5.51}
$$

In order to have a non-trivial action, as in the BFSS case, we must take the limit $u \to \infty$ together with $N \to \infty$ in such a way that $N/(\widehat{R}u) \to \infty$. In the following, we will write $\overline{R} \equiv \widehat{R}u$, implicitly take the limit $(\overline{R}, N) \to \infty$ and let t run from $-\infty$ to ∞.

In the usual IMF limit, one starts from an uncompactified X_0. In our notation, this corresponds to $R \to \infty$, i.e. to the particular choice $\omega = R_{10}/R \to 0$. So, in the IMF limit, all terms proportional to ω drop out of (5.51). In the following chapters, we will restrict ourselves to this case, since we are especially interested in the physics of our model in the infinite momentum frame.

5.5.3 Dualization of the mass term

Let us comment on the meaning of the $\widehat{\mathcal{D}}_t^2$ term arising from the T-dualization of the mass term $Tr((X_0)^2)$, which naively breaks gauge invariance. To understand how it works, we should recall that the trace is defined by the following sum:

$$Tr_{\mathfrak{u}(N)}(-\widehat{\mathcal{D}}_t^2) \;=\; -\sum_a \langle u_a(t)|\widehat{\mathcal{D}}_t^2|u_a(t)\rangle \;=\; \sum_a \|i\widehat{\mathcal{D}}_t|u_a(t)\rangle\|^2 \quad . \tag{5.52}$$

for a set of basis elements $\{|u_a(t)\rangle\}_a$ of $\mathfrak{u}(N)$, which might have some t-dependence or not. If the $|u_a(t)\rangle$ are covariantly constant, the expression (5.52) is obviously zero. Choosing the $|u_a(t)\rangle$ to be covariantly constant seems to be the only coherent possibility. Such a covariantly constant basis is:

$$|u_a(t)\rangle \;\triangleq\; e^{-i\int_{t_0}^t A_0(t')\,dt'}|u_a\rangle \quad ,$$

(where the $|u_a\rangle$'s form a constant basis, for instance, the generators of $\mathfrak{u}(N)$ in the adjoint representation). Now, t lives on a circle and the function $\exp(\int_{t_0}^t A_0(t')\,dt')$ is well-defined only if the zero-mode $A_0^{(0)} = 2\pi n$, $n \in \mathbb{Z}$. But we can always set $A_0^{(0)}$ to zero, since it doesn't affect the behaviour of the system, as it amounts to a mere constant shift in "energy". With this choice, we can integrate $\widehat{\mathcal{D}}_t$ by part without worrying about the trace.

5.5.4 Decomposition of the five-forms

In (5.50), the only fields to be dynamical are the X_i, the $Z_{\alpha_1\cdots\alpha_5}$ and the Ψ. The remaining ones are either the conjugate *momentum*-like fields when they multiply derivatives of dynamical fields, or *constraint*-like when they only appear algebraically.

Thus, the C_{i0} and $\overline{\Psi}$ have a straightforward interpretation as *momenta* conjugate respectively to the X_i and to Ψ. For the 5-form fields $Z_{A_1\ldots A_5}$ however, the matter is a bit more subtle, due to the presence of the $11D$ ε tensor in the kinetic term for the 5-form fields. Actually, the real degrees of freedom contained in $Z_{A_1\cdots A_5}$ decompose as follows, when going down from $(10+1)$ to 9 dimensions:

$$\Omega^5(\mathcal{M}_{10,1},\mathbb{R}) \quad\longrightarrow\quad 3\times\Omega^4(\mathcal{M}_9,\mathbb{R})\oplus\Omega^3(\mathcal{M}_9,\mathbb{R}) \quad . \tag{5.53}$$

To be more specific (as in our previous convention, $i_k = 1,\ldots,9$ are purely spacelike indices in $9D$), the 3-form fields on the RHS of (5.53) are $Z_{i_1i_2i_30,10} \triangleq B_{i_1i_2i_3}$, while the 4-form fields are $Z_{i_1i_2i_3i_410} \triangleq Z_{i_1i_2i_3i_4}$, $Z_{i_1i_2i_3i_40} \triangleq H_{i_1i_2i_3i_4}$ and[4] $\Pi^{i_1\cdots i_4} \triangleq 1/5!\,\varepsilon^{j_1\cdots j_5 i_1\cdots i_4 0,10} Z_{j_1\cdots j_5}$; these conventions

[4] Using

$$\varepsilon^{j_1\cdots j_N i_{N+1}\cdots i_9 0,10}\;\varepsilon_{k_1\cdots k_N i_{N+1}\cdots i_9 0,10} \;=\; -(9-N)!\sum_\pi \sigma(\pi)\prod_{n=1}^N \delta^{j_n}_{k_{\pi(n)}} \quad ,$$

where π is any permutation of $(1,2,..,N)$ and $\sigma(\pi)$ is the signature thereof, this relation can be inverted: $Z_{i_1\cdots i_5} = \frac{1}{4!}\,\varepsilon_{i_1\cdots i_5 j_6\cdots j_9}\,\Pi^{j_6\cdots j_9} \quad ,$

allow us to cast the kinetic term for the 5-form fields into the expression $6/4!\,\Pi^{i_1\cdots i_4}\,[\widehat{\mathcal{D}}_t, Z_{i_1\cdots i_4}]$, while B and H turn out to be *constraint*-like fields, the whole topic being summarized in Table 1.

dynamical var.	*number of real comp.*	*conjugate momenta*	*constraint-like*	*number of real comp.*
X_i	9	C_{i0}	C_{ij}	36
			C_{i10}	9
			ϕ	1
$Z_{i_1\cdots i_4}$	126	$\Pi_{i_1\cdots i_4}$	$H_{i_1\cdots i_4}$	126
			$B_{i_1 i_2 i_3}$	84
Ψ	32	$\overline{\Psi}$		

Table 1: *Momentum*-like and *constraint*-like auxiliary fields

We see that longitudinal 5-brane degrees of freedom are described by the 4-form $Z_{i_1\cdots i_4}$, while transverse 5-brane fields $Z_{i_1\cdots i_5}$ appear in the definition of the conjugate momenta. As they are dual to one another, we could also have exchanged their respective rôles. Both choices describe the same physics. We can thus interpret these degrees of freedom as transverse 5-branes, completing the BFSS theory, which already contains longitudinal 5-branes as bound states of D0-branes.

Choosing the $\varepsilon_{i_1\cdots i_9}$ tensor in 9 spatial dimensions to be:

$$\varepsilon_{i_1\cdots i_9} \triangleq \varepsilon_{i_1\cdots i_9}{}^{0,10} = -\varepsilon_{i_1\cdots i_9 0,10} \quad ,$$

we can express the action I_c in terms of the degrees of freedom appearing in Table 1 (note that from now on all indices will be down, the signature for squared expressions is Euclidean and we write $\mathcal{D}_t$ instead of $\widehat{\mathcal{D}}_t$):

$$\begin{aligned}
I_c \;=\; & \frac{8\sqrt{2}i}{\pi g^2 \overline{R}} \int dt\, Tr_{\mathfrak{u}(N)} \Bigg(-6i\, C_{i0}\, [\mathcal{D}_t, X_i] \;-\; \frac{i}{4}\, \Pi_{i_1\cdots i_4}\, [\mathcal{D}_t, Z_{i_1\cdots i_4}] \;+\; \frac{3}{32}\, \overline{\Psi}\, \widetilde{\Gamma}_0\, [\mathcal{D}_t, \Psi] \;+\; 3\, C_{ij}\, [X_j, X_i] - \\
&+ \left(\Pi_{i_1 i_2 i_3 j}\, [X_j, B_{i_1 i_2 i_3}] \;-\; \frac{1}{4\cdot 4!}\, \varepsilon_{i_1\cdots i_8 j} Z_{i_1\cdots i_4}\, [X_j, H_{i_5\cdots i_8}] \right) \;+\; \frac{1}{3!\cdot 4!}\, W(Z,\Pi,H,B) + \\
&+ \frac{1}{2} \Bigg\{ C_{ij}\, K_{ij}(Z,\Pi,H,B) \;-\; 2\, C_{i0} \left(\frac{1}{4\cdot 4!} \varepsilon_{i\, j_1\cdots j_4 k_1\cdots k_4} [H_{j_1\cdots j_4}, \Pi_{k_1\cdots k_4}] \;+\; [Z_{i\, j_1 j_2 j_3}, B_{j_1 j_2 j_3}] \right) + \\
&+\; 2\, C_{i10} \left(\frac{1}{4\cdot 4!} \varepsilon_{i\, j_1\cdots j_4 k_1\cdots k_4} [Z_{j_1\cdots j_4}, \Pi_{k_1\cdots k_4}] \;-\; [H_{i\, j_1 j_2 j_3}, B_{j_1 j_2 j_3}] \right) \;-\; \frac{1}{2}\, \phi\, [Z_{i_1\cdots i_4}, H_{i_1\cdots i_4}] \Bigg\} + \\
&+\; C_{ij}\, [C_{jk}, C_{ki}] \;+\; 3\, C_{i0}\, [C_{k0}, C_{ki}] \;-\; 3\, C_{i10}\, [C_{k10}, C_{ki}] \;+\; 6\, \phi\, [C_{k10}, C_{k0}] + \\
&+\; \frac{3i}{32} \Bigg\{ \overline{\Psi}\, \widetilde{\Gamma}_i\, [X_i, \Psi] \;+\; \frac{1}{2!}\, \overline{\Psi}\, \widetilde{\Gamma}_{ij} [C_{ij}, \Psi] \;-\; \overline{\Psi}\, \widetilde{\Gamma}_i \widetilde{\Gamma}_0\, [C_{i0}, \Psi] \;+\; \overline{\Psi}\, \widetilde{\Gamma}_i \widetilde{\Gamma}_*\, [C_{i10}, \Psi] - \\
&-\; \overline{\Psi}\, \widetilde{\Gamma}_0 \widetilde{\Gamma}_*\, [\phi, \Psi] \;+\; \frac{1}{4!}\, \overline{\Psi}\, \widetilde{\Gamma}_{i_1\cdots i_4} \widetilde{\Gamma}_* [Z_{i_1\cdots i_4}, \Psi] \;+\; \frac{1}{4!}\, \overline{\Psi}\, \widetilde{\Gamma}_{i_1\cdots i_4} \widetilde{\Gamma}_0 \widetilde{\Gamma}_* [\Pi_{i_1\cdots i_4}, \Psi] + \\
&-\; \frac{1}{4!}\, \overline{\Psi}\, \widetilde{\Gamma}_{i_1\cdots i_4} \widetilde{\Gamma}_0 [H_{i_1\cdots i_4}, \Psi] \;-\; \frac{1}{3!}\, \overline{\Psi}\, \widetilde{\Gamma}_{i_1 i_2 i_3} \widetilde{\Gamma}_0 \widetilde{\Gamma}_* [B_{i_1 i_2 i_3}, \Psi] \Bigg\} \;+\; \mu g^2 i \Bigg\{ (X_i)^2 \;+\; \frac{i}{16}\, \overline{\Psi}\Psi \;+\; \phi^2 - \\
&-\, \frac{1}{2!}\, (C_{ij})^2 \;+\; (C_{i0})^2 \;-\; (C_{i10})^2 \;+\; \frac{1}{4!} \left((Z_{i_1\cdots i_4})^2 \;+\; (\Pi_{i_1\cdots i_4})^2 \;-\; (H_{i_1\cdots i_4})^2 \;-\; 4\, (B_{i_1 i_2 i_3})^2 \right) \Bigg\} \Bigg) \;.
\end{aligned} \tag{5.54}$$

We have abbreviated two lengthy expressions in the result above to make it shorter: on one hand, the term coupling the various 5-form components to the C_{ij}:

$$K_{ij}(Z,\Pi,H,B) \triangleq [Z_{j\,k_1k_2k_3}, Z_{i\,k_1k_2k_3}] + [\Pi_{j\,k_1k_2k_3}, \Pi_{i\,k_1k_2k_3}] - 3[B_{j\,k_1k_2}, B_{i\,k_1k_2}] - [H_{j\,k_1k_2k_3}, H_{i\,k_1k_2k_3}]\,,$$

on the other hand, the trilinear couplings amongst the 5-form components:

$$\begin{aligned} W(Z,\Pi,H,B) \;\triangleq\;& \varepsilon_{i_1\cdots i_9}\Big\{ B_{i_1i_2\,j}\,\left(2\,[\Pi_{j\,i_3i_4i_5}, \Pi_{i_6\cdots i_9}] - [Z_{j\,i_3i_4i_5}, Z_{i_6\cdots i_9}] - [H_{j\,i_3i_4i_5}, H_{i_6\cdots i_9}]\right) + \\ & + \frac{2}{3}\,B_{i_1i_2i_3}\,\left([B_{i_4i_5i_6}, B_{i_7i_8i_9}] + [Z_{i_4i_5i_6\,j}, Z_{j\,i_7i_8i_9}] - [H_{i_4i_5i_6\,j}, H_{j\,i_7i_8i_9}]\right)\Big\} \\ & + (3!)^2\,\Pi_{i_1i_2j_1j_2}\,[Z_{j_1j_2k_1k_2}, H_{k_1k_2i_1i_2}] \quad . \end{aligned}$$

5.5.5 Computation of the effective action

We now intend to study the effective dynamics of the X_i and Ψ fields, in order to compare it to the physics of D0-branes as it is described by the BFSS matrix model. For this purpose, we start by integrating out the 2-form momentum-like and constraint-like fields, which will yield an action containing the BFSS matrix model as its leading term with, in addition, an infinite series of couplings between the fields. Similarly, one would like to integrate out the Z-type momenta and constraints Π, H and B, to get an effective action for the 5-brane (described by Z_{ijkl}) coupled to the D0-branes. We will however not do so in the present paper, but leave this for further investigation.

To simplify our expressions, we set:[5]

$$\beta \triangleq \mu g^2 \quad , \qquad\qquad \gamma \triangleq \frac{8\sqrt{2}}{\pi g^2 \overline{R}} \quad ,$$

and write (5.54) as (after taking the trace over $\mathfrak{u}(N)$):

$$I_c = \gamma \int dt\, \left(\beta(\mathbf{C}_i^a)^\intercal(\mathcal{J}_{ij}^{ab} + \Delta_{ij}^{ab})\mathbf{C}_j^b + \mathbf{C}_i^a\cdot\mathbf{F}_i^a + \mathcal{L}_C + \mathcal{L}_\phi + \widehat{\mathcal{L}}\right) \quad . \tag{5.55}$$

For convenience, we have resorted to a very compact notation, where:

$$\mathbf{C}_i^a \triangleq \begin{pmatrix} C_{i0}^a \\ C_{i10}^a \end{pmatrix}, \qquad \mathcal{J}_{ij}^{ab} \triangleq \begin{pmatrix} -\delta^{ab}\delta_{ij} & 0 \\ 0 & \delta^{ab}\delta_{ij} \end{pmatrix}, \qquad \Delta_{ij}^{ab} \triangleq \frac{3f^{abc}}{\beta}\begin{pmatrix} C_{ij}^c & \phi^c\delta_{ij} \\ -\phi^c\delta_{ij} & -C_{ij}^c \end{pmatrix} \quad ,$$

and where the components of the vector $\mathbf{F}_i^a = \begin{pmatrix} F_i^a \\ G_i^a \end{pmatrix}$, are given by the following expressions:

$$\begin{aligned} F_i \;&\triangleq\; 6\,[\mathcal{D}_t, X_i] - \frac{i}{4\cdot 4!}\varepsilon_{i\,j_1\cdots j_4k_1\cdots k_4}\,[H_{j_1\cdots j_4}, \Pi_{k_1\cdots k_4}] - i\,[Z_{i\,j_1j_2j_3}, B_{j_1j_2j_3}] - \frac{3}{32}\{\overline{\Psi}, \widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\} \quad , \\ G_i \;&\triangleq\; \frac{i}{4\cdot 4!}\varepsilon_{i\,j_1\cdots j_4k_1\cdots k_4}[Z_{j_1\cdots j_4}, \Pi_{k_1\cdots k_4}] - i\,[H_{i\,j_1j_2j_3}, B_{j_1j_2j_3}] + \frac{3}{32}\{\overline{\Psi}, \widetilde{\Gamma}_i\widetilde{\Gamma}_*\Psi\} \quad . \end{aligned}$$

[5]If we consider X and hence C, Z and Ψ to have the engineering dimension of a length, then so has β, while γ has dimension (length)$^{-4}$.

Note that we have written $if^{abc}\overline{\Psi}^b\widetilde{\Gamma}_{\dots}\Psi^c$ as $\{\overline{\Psi},\widetilde{\Gamma}_{\dots}\Psi\}^a$ with a slight abuse of notation. The remaining terms in the action (5.55) depending on C_{ij} and ϕ are contained in

$$\begin{aligned} \mathcal{L}_C &\triangleq \frac{\beta}{2}(C^a_{ij})^2 + E^a_{ij}C^a_{ij} - f^{abc}C^a_{ij}C^b_{jk}C^c_{ki} \quad , \\ \mathcal{L}_\phi &\triangleq -\beta(\phi^a)^2 + J^a\phi^a \quad , \end{aligned}$$

with the following definitions

$$\begin{aligned} E_{ij} &\triangleq \frac{i}{2}K_{ij} + 3i\,[X_i,X_j] + \frac{3}{64}\{\overline{\Psi},\widetilde{\Gamma}_{ij}\Psi\} \quad , \\ J &\triangleq \frac{-i}{4}[Z_{i_1\cdots i_4},H_{i_1\cdots i_4}] - \frac{3}{32}\{\overline{\Psi},\widetilde{\Gamma}_0\widetilde{\Gamma}_*\Psi\} \quad , \end{aligned}$$

and finally $\widehat{\mathcal{L}}$ is the part of I_c in (5.54) independent of C_{ij}, C_{i10}, C_{i0} and ϕ. in other words the part containing only dynamical fields (fermions Ψ and coordinates X_i) as well as all fields related to the 5-brane (the dynamical ones: Z and Π, as well as the constrained ones: B and H).

Now, (5.55) is obviously bilinear in the $\mathbf{C}^a_i$ (note that Δ^{ab}_{ij} *is* symmetric, since C_{ij} is actually antisymmetric in i and j). So one may safely integrate them out, after performing a Wick rotation such as

$$t \to \tau = it \quad , \qquad C_{i10} \to \overline{C}_{i10} = \pm iC_{i10} \quad .$$

The indeterminacy in the choice of the direction in which to perform the Wick rotation will turn out to be irrelevant after the integration of C_{i10} (indeed, this $\pm$ sign appears in each factor of ϕ and each factor of G, which always come in pairs).

We then get the Euclidean version of (5.55):

$$I_{\mathrm{E}} = \gamma\int d\tau\,\left(\beta(\overline{\mathbf{C}}^a_i)^\intercal(\mathbb{I}^{ab}_{ij} + \overline{\Delta}^{ab}_{ij})\overline{\mathbf{C}}^b_j + (\overline{\mathbf{C}}^a_i)^\intercal\overline{\mathbf{F}}^a_i - \mathcal{L}_C - \mathcal{L}_\phi - \widehat{\mathcal{L}}\right) \quad ,$$

where the new rotated fields assume the following form:

$$\overline{\mathbf{C}}^a_i \triangleq \begin{pmatrix} C^a_{i0} \\ \overline{C}^a_{i10} \end{pmatrix} , \qquad \overline{\mathbf{F}}^a_i \triangleq \begin{pmatrix} -F^a_i \\ \pm iG^a_i \end{pmatrix} ,$$

$$\mathbb{I}^{ab}_{ij} \triangleq \begin{pmatrix} \delta^{ab}\delta_{ij} & 0 \\ 0 & \delta^{ab}\delta_{ij} \end{pmatrix} , \qquad \overline{\Delta}^{ab}_{ij} \triangleq \frac{3f^{abc}}{\beta}\begin{pmatrix} -C^c_{ij} & \pm i\phi^c\delta_{ij} \\ \mp i\phi^c\delta_{ij} & C^c_{ij} \end{pmatrix} .$$

The gaussian integration is straightforward, and yields, after exponentiation of the non trivial part of the determinant:

$$\begin{aligned} &\int D\overline{C}_{i10}\,DC_{i0}\,\exp\Big\{-I_{\mathrm{E}}\Big\} \\ &\quad\propto \exp\Big\{-\frac{1}{2}Tr\left(\ln(\mathbb{I}^{ab}_{ij} + \overline{\Delta}^{ab}_{ij})\right) - \gamma\int d\tau\,\left(-\frac{1}{4\beta}(\overline{\mathbf{F}}^a_i)^\intercal(\mathbb{I}^{ab}_{ij} + \overline{\Delta}^{ab}_{ij})^{-1}\overline{\mathbf{F}}^b_j - \mathcal{L}_C - \mathcal{L}_\phi - \widehat{\mathcal{L}}\right)\Big\} \quad . \end{aligned}$$

The term quadratic in $\mathbf{F}$ is obviously tree-level, whereas the first one is a 1-loop correction to the effective action. The 1-loop "behaviour" is encoded in the divergence associated with the trace of an operator, since

$$Tr\widehat{O} = \int d\tau\,O^i{}_i(\tau)\langle\tau|\tau\rangle = \Lambda\int d\tau\,O^i{}_i(\tau) \quad , \tag{5.56}$$

where the integration in Fourier space is divergent, and has been replaced by the cutoff Λ. Transforming back to real Minkowskian time t, we obtain the following effective action

$$I_{\text{eff}} = \gamma \int dt \left(\widehat{\mathcal{L}} + \mathcal{L}_C + \mathcal{L}_\phi + \frac{1}{4\beta} (\overline{\mathbf{F}}_i^a)^\intercal (\mathbb{1}_{ij}^{ab} + \overline{\Delta}_{ij}^{ab})^{-1} \overline{\mathbf{F}}_j^b - \frac{\Lambda}{2\gamma} \big(\ln(\mathbb{1} + \overline{\Delta}(t)) \big)_{ii}^{aa} \right) . \tag{5.57}$$

5.5.6 Analysis of the different contributions to the effective action

The natural scale of (5.57) is β, which is proportional to the mass parameter μ. We therefore expand (5.57) in powers of $1/\beta$, which amounts to expanding (5.57) in powers of $\overline{\Delta}$. Now, this procedure must be regarded as a formal expansion, since we don't want to set β to a particular value. However, this formal expansion in $1/\beta$ actually conceals a true expansion in $[X_i, X_j]$, which should be small to minimize the potential energy, as will become clear later on.

First of all, let us consider the expansion of the tree-level term up to $\mathcal{O}(1/\beta^3)$. The first order term is given by:

$$\frac{1}{\beta} \int dt \, (\overline{\mathbf{F}}_i^a)^\intercal \overline{\mathbf{F}}_i^a = \frac{1}{\beta} \int dt \, Tr \Big((F_i)^2 - (G_i)^2 \Big) .$$

Since F_i contains $[\mathcal{D}_t, X_i]$ and $\{\overline{\Psi}, \Psi\}$, while G_i contains only $\{\overline{\Psi}, \Psi\}$ (ignoring Z-type contributions), this term will generate a kinetic term for the X^i's as well as trilinear and quartic interactions.

The second-order term is:

$$\frac{1}{\beta} \int dt \, (\overline{\mathbf{F}}_i^a)^\intercal \overline{\Delta}_{ij}^{ab} \overline{\mathbf{F}}_j^b = \frac{3i}{\beta^2} \int dt \, Tr \bigg(C_{ij} \Big\{ [F_i, F_j] - [G_i, G_j] \Big\} - 2\, \phi \, [F_i, G_i] \bigg).$$

All vertices generated by this term contain either one C, with 2 to 4 X or Ψ, or one ϕ, with 3 or 4 X or Ψ.

Finally, the third-order contibution is as follows:

$$\begin{aligned} \frac{1}{\beta} \int dt \; (\overline{\mathbf{F}}_i^a)^\intercal (\overline{\Delta}^2)_{ij}^{ab} \overline{\mathbf{F}}_j^b = -\frac{3^2}{\beta^3} \int dt \; Tr \bigg(& [F_i, C_{ij}][C_{jk}, F_k] - [G_i, C_{ij}][C_{jk}, G_k] + \\ & + [F_i, \phi][\phi, F_i] - [G_i, \phi][\phi, G_i] + 2\, [G_i, C_{ij}][\phi, F_j] - 2\, [F_i, C_{ij}][\phi, G_j] \bigg) , \end{aligned}$$

producing vertices with 2 ϕ's or 2 C's, together with 2 to 4 X or Ψ, as well as vertices with 1 ϕ or 1 C, with 3 to 4 X or Ψ.

Next we turn to the 1-loop term, where we expand the logarithm up to $\mathcal{O}(1/\beta^3)$. Because of the total antisymmetry of both f^{abc} and C_{ij}, one has $Tr\overline{\Delta} = 0$, so that the first term cancels. Now, keeping in mind that

$$f^{abc} f^{bad} = -C_2(\mathfrak{ad}) \delta^{cd} \qquad \text{and} \qquad f^{amn} f^{bno} f^{com} = \frac{1}{2} C_2(\mathfrak{ad}) f^{abc} \quad ,$$

$C_2(\mathfrak{ad})$ referring to the quadratic Casimir operator in the adjoint representation of the Lie algebra, one readily finds:

(i). $Tr\overline{\Delta}^2 = \left(\frac{3}{\beta}\right)^2 2iC_2(\mathfrak{ad})\Lambda \int dt \, Tr \Big((C_{ij})^2 - 9(\phi)^2 \Big)$, (ii). $Tr\overline{\Delta}^3 \;=\; -\left(\frac{3}{\beta}\right)^3 C_2(\mathfrak{ad})\Lambda \int dt\, Tr\Big(C_{ij}[C_{jk},C_{ki}]\Big)\,.$

In other words, the 1-loop correction (i) renormalizes the mass terms for C_{ij} and ϕ in $\widetilde{I}_c$ as follows:

- Mass renormalization for C_{ij}: $\frac{1}{2}\gamma\beta \longrightarrow \frac{1}{2}\gamma\beta\left(1+\frac{3^2}{\gamma\beta^3}C_2(\mathfrak{ad})\Lambda\right)$
- Mass renormalization for ϕ: $\gamma\beta \longrightarrow \gamma\beta\left(1+\frac{3^4}{2\gamma\beta^3}C_2(\mathfrak{ad})\Lambda\right)$

Whereas the 1-loop correction (ii) renormalizes the trilinear coupling between the C_{ij} in I_c:

- Renormalization of the $C_{ij}[C_{jk},C_{ki}]$ coupling: $\gamma \longrightarrow \gamma\left(1-\frac{3^2}{2\gamma\beta^3}C_2(\mathfrak{ad})\Lambda\right)$

Up to $Tr\overline{\Delta}^3$, the 1-loop corrections actually only renormalize terms already present in I_c from the start. This is not the case for the higher order subsequent 1-loop corrections: there is an infinite number of such corrections, each one diverging like Λ. A full quantization of (5.57) is obviously a formidable task, which we will not attempt in the present paper. A sensible regularization of the divergent contributions should take into account the symmetries of the classical action, which are not explicit anymore after performing T-dualities and the IMF limit. However, since our model is quantum-mechanical, we believe it to be finite even if we haven't come up with a fully quantized formulation.

Summing up the different contributions computed in this section, one gets the following 1-loop effective action up to $\mathcal{O}(1/\beta^3)$:

$$\begin{aligned}\frac{1}{\gamma}\;\; I_{\text{eff}} \;&=\; \int dt\,\left(\mathcal{L}_C+\mathcal{L}_\phi+\widehat{\mathcal{L}}\right)+\frac{\gamma}{4\beta}\int dt\; Tr\left(F_i^2-G_i^2\right)\;-\\ &-\frac{3i\gamma}{4\beta^2}\int dt\,Tr\Big(C_{ij}\Big([F_i,F_j]-[G_i,G_j]\Big)-2\,\phi\,[F_i,G_i]\Big)+\frac{\gamma\lambda}{2\beta^2}\int dt\,Tr\Big(C_{ij}^2-9\phi^2\Big)\;-\\ &-\frac{9\gamma}{4\beta^3}\int dt\,Tr\Big([F_i,C_{ij}][C_{jk},F_k]\;-\;[G_i,C_{ij}][C_{jk},G_k]\;+\;[F_i,\phi][\phi,F_i]\;-\;[G_i,\phi][\phi,G_i]\;+\\ &+2\,[G_i,C_{ij}][\phi,F_j]\;-2\,[F_i,C_{ij}][\phi,G_j]\Big)-\frac{i\lambda\gamma}{2\beta^3}\int dt\,Tr\Big(\,C_{ij}\,[C_{jk},C_{ki}]\Big)\;+\;\mathcal{O}(1/\beta^4)\end{aligned} \tag{5.58}$$

where λ is proportional to the cutoff Λ:

$$\lambda \triangleq \frac{9\,C_2(\mathfrak{ad})\Lambda}{\gamma}\qquad .$$

Note that the $\mathcal{O}(1/\beta^4)$ terms that we haven't written contain at least three powers of C_{ij} or ϕ.

5.5.7 Iterative solution of the constraint equations

The 1-loop corrected action (5.58) still contains the constraint fields C_{ij} and ϕ, which should in principle be integrated out in order to get the final form of the effective action. Since I_{eff} contains arbitrarily high powers of C_{ij} and ϕ, we cannot perform a full path integration. We can however solve the equations for C_{ij} and ϕ perturbatively in $1/\beta$. This allows to replace these fields in (5.58) with the solution to their equations of motion. Thus, in contrast with the preceeding subsection, here we remain at tree-level.

The equation of motion for C_{ij} may be computed from (5.58), and reads:

$$\begin{aligned} C_{ij} &+ \frac{1}{\beta}\Big(E_{ij} + 3i[C_{jk}, C_{ki}]\Big) + \frac{1}{\beta^3}\left(\frac{3}{4}i\Big\{[G_i, G_j] - [F_i, F_j]\Big\} + \lambda C_{ij}\right) + \\ &+ \frac{1}{\beta^4}\frac{9}{2}\bigg(\Big\{[F_{[i}, [C_{j]k}, F_k]] - [G_{[i}, [C_{j]k}, G_k]] + [G_{[i}, [\phi, F_{j]}]] - [F_{[i}, [\phi, G_{j]}]]\Big\} + \\ &- \frac{i\lambda}{3}[C_{jk}, C_{ki}]\bigg) + \mathcal{O}(1/\beta^5) = 0\,, \end{aligned} \tag{5.59}$$

while the equation of motion for ϕ is:

$$\begin{aligned} \phi &- \frac{1}{2\beta}J - \frac{3}{\beta^3}\left(\frac{i}{4}[F_i, G_i] - 3\lambda\phi\right) + \frac{3^2}{4\beta^4}\Big([F_i, [F_i, \phi]] - \\ &- [G_i, [G_i, \phi]] + [F_i, [C_{ij}, G_j]] - [G_i, [C_{ij}, F_j]]\Big) + \mathcal{O}(1/\beta^5) = 0\,. \end{aligned} \tag{5.60}$$

By solving the coupled equations of motion (5.59) and (5.60) recursively, one gets C_{ij} and ϕ up to $\mathcal{O}(1/\beta^5)$. We can safely stop at $\mathcal{O}(1/\beta^5)$, because the terms contributing to that order in (5.59) and (5.60) are, on the one hand, $\beta^{-1}\Lambda(\delta/\delta C_{ij})Tr\overline{\Delta}^4$ and $\beta^{-1}\Lambda(\delta/\delta\phi)Tr\overline{\Delta}^4$, whose lowest order is $\mathcal{O}(1/\beta^8)$, and on the other hand $\beta^{-2}(\delta/\delta C_{ij})\mathbf{F}^\intercal\overline{\Delta}^3\mathbf{F}$ and $\beta^{-2}(\delta/\delta\phi)\mathbf{F}^\intercal\overline{\Delta}^3\mathbf{F}$, whose lowest order is $\mathcal{O}(1/\beta^7)$, so that the eom don't get any corrections from contributions of $\mathcal{O}(1/\beta^4)$ coming from I_{eff}.

Subsequently, the $1/\beta$ expansion for C_{ij} reads

$$\begin{aligned} C_{ij} = &- \frac{1}{\beta}E_{ij} + \frac{3i}{\beta^3}\left([E_{ik}, E_{kj}] + \frac{1}{4}[F_i, F_j] - \frac{1}{4}[G_i, G_j]\right) + \frac{\lambda}{\beta^4}E_{ij} + \\ &+ \frac{9}{\beta^5}\bigg(-2[E_{[ik}, [E_{kl}, E_{lj]}]] + \frac{1}{2}[E_{[ik}, [F_k, F_{j]}]] - \frac{1}{2}[E_{[ik}, [G_k, G_{j]}]] + \frac{1}{2}[[E_{[ik}, F_k], F_{j]}] - \\ &- \frac{1}{2}[[E_{[ik}, G_k], G_{j]}] + \frac{1}{4}[G_{[i}, [F_{j]}, J]] - \frac{1}{4}[F_{[i}, [G_{j]}, J]]\bigg) + \mathcal{O}(1/\beta^6) \quad , \end{aligned} \tag{5.61}$$

and the expansion for ϕ:

$$\begin{aligned} \phi = &\ \frac{1}{2\beta}J + \frac{3i}{4\beta^3}[F_i, G_i] - \left(\frac{3}{2}\right)^2\frac{\lambda}{\beta^4}J - \\ &- \frac{9}{8\beta^5}\Big([F_i, [F_i, J]] - [G_i, [G_i, J]] - 2[[F_i, E_{ij}], G_j] + 2[[G_i, E_{ij}], F_j]\Big) + \mathcal{O}(1/\beta^6)\,. \end{aligned} \tag{5.62}$$

Now, plugging the result for C_{ij} and ϕ into I_{eff}, one arrives at the "perturbative" effective action, which we have written up to and including $\mathcal{O}(1/\beta^5)$, since the highest order ($\mathcal{O}(1/\beta^3)$) we calculated in I_{eff} is quadratic in C and ϕ[6], and since the $\mathcal{O}(1/\beta^4)$-terms in (5.58) only generate $\mathcal{O}(1/\beta^7)$ - terms.

[6] note that their expansion starts at $\mathcal{O}(1/\beta)$

This effective action takes the following form:

$$
\begin{aligned}
\frac{1}{\gamma} I_{\text{eff}} \;=\; & \int dt \Bigg(\widehat{\mathcal{L}} + \frac{1}{4\beta} Tr\left(F_i^2 - G_i^2 + J^2 - 2\,(E_{ij})^2\right) + \\
& + \frac{i}{\beta^3} Tr\bigg(- E_{ij}[E_{jk}, E_{ki}] + \frac{3}{4} E_{ij}\Big\{[F_i, F_j] - [G_i, G_j]\Big\} + \frac{3}{4} J[F_i, G_i]\bigg)\Bigg) + \\
& + \frac{\lambda}{2\beta^4} Tr\Big((E_{ij})^2 - \frac{9}{4} J^2\Big) + \frac{9}{2\beta^5} Tr\Big(([E_{ik}, E_{kj}])^2 + \frac{1}{16}([F_i, F_j] - [G_i, G_j])^2 - \\
& + \frac{1}{2}[E_{ik}, E_{kj}]\big([F_i, F_j] - [G_i, G_j]\big) - \frac{1}{8}([F_i, G_i])^2 - \frac{1}{2}\Big\{([F_i, E_{ij}])^2 - ([G_i, E_{ij}])^2\Big\} + \\
& + \frac{1}{4}\Big\{([F_i, J])^2 - ([G_i, J])^2\Big\} - \frac{1}{2}[G_i, E_{ij}][J, F_j] + \frac{1}{2}[F_i, E_{ij}][J, G_j]\Big)\Bigg) + \mathcal{O}(1/\beta^6)\,.
\end{aligned}
$$

At that point, we can replace the aliases E, F, G and J by their expression in terms of the fundamental fields X, Ψ, Z, Π, B and H. The result of this lengthy computation (already to order $1/\beta$) is presented in the Appendix. Here, we will only display the somewhat simpler result obtained by ignoring all 5-form induced fields. Furthermore, we will remove the parameter β from the action, since it was only useful as a reminder of the order of calculation in the perturbative approach. To do so, we absorb a factor of $1/\beta$ in every field, as well as in $\mathcal{D}_t$ (so that the measure of integration scales with β). Thus, β only appears in the prefactor in front of the action, at the 4^{th} power. This is similar to the case of Yang-Mills theory, where one can choose either to have a factor of the coupling constant in the covariant derivatives or have it as a prefactor in front of the action. To be more precise, we set:

$$
\Theta \;=\; \frac{1}{4\sqrt{6}\beta}\,\Psi\,, \quad \widetilde{X}_i \;=\; \frac{1}{\beta}X_i\,, \quad \widetilde{A}_0 \;=\; \frac{1}{\beta}A_0\,, \quad G \;=\; 9\beta^4\gamma\,, \quad \tilde{t} \;=\; \beta t\,,
$$

and similarily for the Z sector: $(Z, \Pi, H, B) \to (Z/\beta, \Pi/\beta, H/\beta, B/\beta)$.

With this redefinition, it becomes clear that our developpment is really an expansion in higher commutators and not in β. It makes thus sense to limit it to the lowest orders since the commutators should remain small to minimize the potential energy. To get a clearer picture of the final result, we will put all the 5-form-induced fields (Z, Π, H, B) to zero. For convenience we will still write $\widetilde{X}$ as X

and $\tilde{t}$ as t in the final result, which reads:

$$
\begin{aligned}
I(X,\Theta) =& \frac{1}{G}\int dt\, Tr_{\mathfrak{u}(N)}\Bigg(([\mathcal{D}_t,X_i])^2+\frac{1}{2}([X_i,X_j])^2+i\overline{\Theta}\tilde{\Gamma}_0[\mathcal{D}_t,\Theta]-\overline{\Theta}\tilde{\Gamma}_i[X_i,\Theta]- \\
&-\frac{1}{9}(X_i)^2-\frac{2i}{3}\overline{\Theta}\Theta-3[\mathcal{D}_t,X_i]\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\}-\frac{3i}{2}[X_i,X_j]\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}+ \\
&+\frac{9}{4}(\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\})^2-\frac{9}{4}(\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\})^2+\frac{9}{4}(\{\overline{\Theta},\tilde{\Gamma}_0\tilde{\Gamma}_*\Theta\})^2-\frac{9}{8}(\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\})^2+ \\
&+3[X_i,X_j][[X_j,X_k],[X_k,X_i]]-9[X_i,X_j][[\mathcal{D}_t,X_i],[\mathcal{D}_t,X_j]]- \\
&-\frac{3^3 i}{2}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[[X_j,X_k],[X_k,X_i]]+\frac{3^4}{2^2}[X_i,X_j][\{\overline{\Theta},\tilde{\Gamma}_{jk}\Theta\},\{\overline{\Theta},\tilde{\Gamma}_{ki}\Theta\}]- \\
&-\frac{3^4 i}{2^3}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[\{\overline{\Theta},\tilde{\Gamma}_{jk}\Theta\},\{\overline{\Theta},\tilde{\Gamma}_{ki}\Theta\}]+\frac{3^3 i}{2}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[[\mathcal{D}_t,X_i],[\mathcal{D}_t,X_j]]+ \\
&+3^3[X_i,X_j][[\mathcal{D}_t,X_i],\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_0\Theta\}]-\frac{3^4 i}{2^2}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[[\mathcal{D}_t,X_i],\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_0\Theta\}]- \\
&-\frac{3^4}{2^2}[X_i,X_j][\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\},\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_0\Theta\}]+\frac{3^5 i}{2^3}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\},\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_0\Theta\}]+ \\
&+\frac{3^4}{2^2}[X_i,X_j][\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\},\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_*\Theta\}]-\frac{3^5 i}{2^3}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\},\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_*\Theta\}]- \\
&-\frac{3^4 i}{2}\{\overline{\Theta},\tilde{\Gamma}_0\tilde{\Gamma}_*\Theta\}[[\mathcal{D}_t,X_i],\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\}]+,\frac{3^5 i}{2}\{\overline{\Theta},\tilde{\Gamma}_0\tilde{\Gamma}_*\Theta\}[\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\},\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\}]\Bigg)+ \\
&+\text{eighth-order interactions.}
\end{aligned}
$$

We see that the first four terms in this action correspond to the BFSS matrix model, but with a doubled number of fermions. So, in order to maintain half of the original supersymmetries (i.e. $\mathcal{N}=1$ in $10D$), one could project out half of the original fermions with $\mathcal{P}_- \xrightarrow{\text{IMF}} (1+\tilde{\Gamma}_*)/2$. Finally, in addition to the BFSS-like terms, we have mass terms and an infinite tower of interactions possibly containing information about the behaviour of brane dynamics in the non-perturbative sector.

5.6 Discussion

After a general description of $\mathfrak{osp}(1|32)$ and its adjoint representation, we have studied its expression as a symmetry algebra in $12D$. We have described the resulting transformations of matrix fields and their commutation relations. Finally, we have proposed a matrix theory action possessing this symmetry in $12D$. We have then repeated this analysis in the 11-dimensional case, where $\mathfrak{osp}(1|32)$ is a sort of super-AdS algebra. Compactification and T-dualization of two coordinates produced a one-parameter family of singular limiting procedures that shrink the world-sheet along a world-line. We have then identified one of them as the usual IMF limit, which gave rise to a non-compact dynamical evolution parameter that has allowed us to distinguish dynamical from auxiliary fields. Integrating out the latter and solving some constraints recursively, we have obtained a matrix model with a highly non-trivial dynamics, which is similar to the BFSS matrix model when both X^2 and multiple commutators are small. The restriction to a low-energy sector where both X^2 and $[X,X]$ are small seems to correspond to a space-time with weakly interacting (small $[X,X]$) D-particles that are nevertheless not far apart (small X^2). The stable classical solutions correspond to vanishing matrices, i.e. to D-particles stacked at the origin, which displays some common features with matrix models in pp-wave backgrounds (see for instance [27, 59, 44]).

Since the promotion of the membrane charges in the $11D$ super-Poincaré algebra to symmetry generators implied the non-commutativity of the P's, and thus the AdS_{11} symmetry, the membranes are responsible for some background curvature of the space-time. Indeed, since the C_{MN} don't appear as dynamical degrees of freedom, their rôle is to produce the precise tower of higher-order interactions necessary to enforce such a global symmetry on the space-time dynamically generated by the X_i's. The presence of mass terms is thus no surprise since they were also conjectured to appear in matrix models aimed at describing gravity in de Sitter spaces, albeit with a tachyonic sign reflecting the unusual causal structure of de Sitter space ([83, 54]). One might also wonder whether the higher interaction terms we get are somehow related to the high energy corrections to BFSS one would obtain from the non-abelian Dirac-Born-Infeld action. Another question one could address is what kind of corrections a term of the form $STr_{\mathfrak{osp}(1|32)\otimes\mathfrak{u}(n)}([M,M][M,M])$ would induce.

It would also be interesting to investigate the dynamics of the 5-branes degrees of freedom more thoroughly by computing the effective action for Z (from I_{eff} of the Appendix) and give a definite proposal for the physics of 5-branes in M-theory. Note that there is some controversy about the ability of the BFFS model to describe transverse 5-branes (see e.g. [82, 21] and references therein for details). Our model would provide an interesting extension of the BFSS theory by introducing in a very natural way transverse 5-branes (through the fields dual to Z_{ijkl}) in addition to the D0-branes bound states describing longitudinal 5-branes, which are already present in BFSS theory.

5.7 Appendix

We give here the complete effective action at order $1/\beta$.

$$
\begin{aligned}
I_{\rm eff} \;=\; & \frac{1}{G}\int dt\, Tr_{\mathrm{u}(N)}\Bigg(-\beta\Big\{(X_i)^2+\frac{i}{16}\,\overline{\Psi}\Psi+\frac{1}{4!}\Big((Z_{i_1\cdots i_4})^2+(\Pi_{i_1\cdots i_4})^2-(H_{i_1\cdots i_4})^2-4\,(B_{i_1i_2i_3})^2\Big)\Big\}+\\
+\; & \Big\{\frac{1}{4}\Pi_{i_1\cdots i_4}[\mathcal{D}_t,Z_{i_1\cdots i_4}]+\frac{3i}{32}\overline{\Psi}\widetilde{\Gamma}_0[\mathcal{D}_t,\Psi]+i\Pi_{i_1i_2i_3\,j}[X_j,B_{i_1i_2i_3}]-\frac{i}{4\cdot 4!}\varepsilon_{i_1\cdots i_8 j}Z_{i_1\cdots i_4}[X_j,H_{i_5\cdots i_8}]+\\
+\; & \frac{i}{3!\cdot 4!}\varepsilon_{i_1\cdots i_9}\Big(B_{i_1i_2\,j}\Big(2\,[\Pi_{j\,i_3i_4i_5},\Pi_{i_6\cdots i_9}]+[Z_{j\,i_3i_4i_5},Z_{i_6\cdots i_9}]-[H_{j\,i_3i_4i_5},H_{i_6\cdots i_9}]\Big)+\\
+\; & \frac{2}{3}\,B_{i_1i_2i_3}\Big([B_{i_4i_5i_6},B_{i_7i_8i_9}]+[Z_{i_4i_5i_6\,j},Z_{j\,i_7i_8i_9}]-[H_{i_4i_5i_6\,j},H_{j\,i_7i_8i_9}]\Big)\Big)+\\
+\; & \frac{i}{4}\,\Pi_{i_1i_2j_1j_2}\,[Z_{j_1j_2k_1k_2},H_{k_1k_2i_1i_2}]-\frac{3}{32}\Big(\overline{\Psi}\,\widetilde{\Gamma}_i\,[X_i,\Psi]+\frac{1}{4!}\,\overline{\Psi}\,\Big(\widetilde{\Gamma}_{i_1\cdots i_4}\widetilde{\Gamma}_*[Z_{i_1\cdots i_4},\Psi]+\\
+\; & \widetilde{\Gamma}_{i_1\cdots i_4}\widetilde{\Gamma}_0\widetilde{\Gamma}_*[\Pi_{i_1\cdots i_4},\Psi]-\widetilde{\Gamma}_{i_1\cdots i_4}\widetilde{\Gamma}_0[H_{i_1\cdots i_4},\Psi]-4\,\widetilde{\Gamma}_{i_1i_2i_3}\widetilde{\Gamma}_0\widetilde{\Gamma}_*[B_{i_1i_2i_3},\Psi]\Big)\Big)\Big\}+\\
+\;\frac{1}{4\beta}\; & \Big\{36\,([\mathcal{D}_t,X_i])^2-\frac{i}{8}\varepsilon_{ij_1\cdots j_8}[\mathcal{D}_t,X_i][H_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}]-12i\,[\mathcal{D}_t,X_i][Z_{ij_1\cdots j_3},B_{j_1\cdots j_3}]-\\
-\; & \frac{9}{8}[\mathcal{D}_t,X_i]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\}-\frac{1}{16}[H_{i_1\cdots i_4},\Pi_{j_1\cdots j_4}]\Big([H_{i_1\cdots i_4},\Pi_{j_1\cdots j_4}]-16[H_{i_1i_2i_3j_4},\Pi_{j_1j_2j_3i_4}]+\\
+\; & 36[H_{i_1i_2j_3j_4},\Pi_{j_1j_2i_3i_4}]-16[H_{i_1j_2j_3j_4},\Pi_{j_1i_2i_3i_4}]+[H_{j_1j_2j_3j_4},\Pi_{i_1i_2i_3i_4}]\Big)-\\
-\; & \frac{1}{2\cdot 4!}\varepsilon_{ij_1\cdots j_8}[H_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}][Z_{ik_1\cdots k_3},B_{k_1\cdots k_3}]+\frac{i}{2^9}\varepsilon_{ij_1\cdots j_8}[H_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\}-\\
-\; & ([Z_{ij_1\cdots j_3},B_{j_1\cdots j_3}])^2+\frac{3i}{16}[Z_{ij_1\cdots j_3},B_{j_1\cdots j_3}]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\}+\frac{9}{2^{10}}(\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\})^2+\\
+\; & \frac{1}{16}\,[Z_{i_1\cdots i_4},\Pi_{j_1\cdots j_4}]\Big([Z_{i_1\cdots i_4},\Pi_{j_1\cdots j_4}]-16[Z_{i_1i_2i_3j_4},\Pi_{j_1j_2j_3i_4}]+36[Z_{i_1i_2j_3j_4},\Pi_{j_1j_2i_3i_4}]-\\
-\; & 16[Z_{i_1j_2j_3j_4},\Pi_{j_1i_2i_3i_4}]+[Z_{j_1j_2j_3j_4},\Pi_{i_1i_2i_3i_4}]\Big)-\frac{1}{2\cdot 4!}\varepsilon_{ij_1\cdots j_8}[Z_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}][H_{ik_1\cdots k_3},B_{k_1\cdots k_3}]-\\
-\; & \frac{i}{2^9}\varepsilon_{ij_1\cdots j_8}[Z_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_*\Psi\}+([H_{ij_1\cdots j_3},B_{j_1\cdots j_3}])^2+\frac{3i}{16}[H_{ij_1\cdots j_3},B_{j_1\cdots j_3}]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_*\Psi\}-\\
-\; & \frac{9}{2^{10}}(\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_*\Psi\})^2-\frac{1}{16}([Z_{i_1\cdots i_4},H_{i_4\cdots i_4}])^2+\frac{3i}{2^6}[Z_{i_1\cdots i_4},H_{i_4\cdots i_4}]\{\overline{\Psi},\widetilde{\Gamma}_0\widetilde{\Gamma}_*\Psi\}+\frac{9}{2}([B_{ik_1k_2},B_{jk_1k_2}])^2+\\
+\; & \frac{1}{2}([Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}])^2+[Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}][\Pi_{il_1l_2l_3},\Pi_{jl_1l_2l_3}]+\frac{1}{2}([\Pi_{ik_1k_2k_3},\Pi_{jk_1k_2k_3}])^2-\\
-\; & 3[Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}][B_{il_1l_2},B_{jl_1l_2}]-[Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}][H_{il_1l_2l_3},H_{jl_1l_2l_3}]+\frac{9}{2^{10}}(\{\overline{\Psi},\widetilde{\Gamma}_0\widetilde{\Gamma}_*\Psi\})^2-\\
-\; & 3\,[\Pi_{ik_1k_2k_3},\Pi_{jk_1k_2k_3}][B_{il_1l_2},B_{jl_1l_2}]-[\Pi_{ik_1k_2k_3},\Pi_{jk_1k_2k_3}][H_{il_1l_2l_3},H_{jl_1l_2l_3}]+\\
+\; & 3\,[B_{ik_1k_2},B_{jk_1k_2}][H_{il_1l_2l_3},H_{jl_1l_2l_3}]+\frac{1}{2}([H_{ik_1k_2k_3},H_{jk_1k_2k_3}])^2-6\,[Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}][X_i,X_j]+
\end{aligned}
$$

$$
\begin{aligned}
+ \quad & \frac{3i}{32}[Z_{ik_1k_2k_3}, Z_{jk_1k_2k_3}]\{\overline{\Psi}, \widetilde{\Gamma}_{ij}\Psi\} - 6[\Pi_{ik_1k_2k_3}, \Pi_{jk_1k_2k_3}][X_i, X_j] + \frac{3i}{32}[\Pi_{ik_1k_2k_3}, \Pi_{jk_1k_2k_3}]\{\overline{\Psi}, \widetilde{\Gamma}_{ij}\Psi\} + \\
+ \quad & 18\,[B_{ik_1k_2}, B_{jk_1k_2}][X_i, X_j] - \frac{9i}{32}[B_{ik_1k_2}, B_{jk_1k_2}]\{\overline{\Psi}, \widetilde{\Gamma}_{ij}\Psi\} + 6\,[H_{ik_1k_2k_3}, H_{jk_1k_2k_3}][X_i, X_j] - \\
- \quad & \frac{3i}{32}[H_{ik_1k_2k_3}, H_{jk_1k_2k_3}]\{\overline{\Psi}, \widetilde{\Gamma}_{ij}\Psi\} + 18\,([X_i, X_j])^2 - \frac{9i}{16}[X_i, X_j]\{\overline{\Psi}, \widetilde{\Gamma}_{ij}\Psi\} - \frac{9}{2^{11}}(\{\overline{\Psi}, \widetilde{\Gamma}_{ij}\Psi\})^2 \Bigg\} + \\
+ \quad & \mathcal{O}(1/\beta^3)\Bigg) .
\end{aligned}
$$

Chapter 6

Classical curved-space solutions of an $\mathfrak{osp}(1|32,\mathbb{R})$ totally reduced matrix model

We have seen in section 4.3 that various extensions of the IIB matrix model have been proposed to go away from the usual classical commutative vacuum of the theory. In some sense, such models can be seen as phenomenological guesses tailored at obtaining certain kind of possible non-perturbative vacua, in order to study the perturbative non-commutative dynamics of the IIB matrix model in the neighbourhood of such non-commutative spacetimes.

The motivation for this second research project, realized in collaboration with Takehiro Azuma from Kyôto University, is that we noticed a similarity between the massive cubic $\mathfrak{osp}(1|32,\mathbb{R})$ matrix model introduced in the preceding chapter and the IIB massive matrix model studied by Kimura in [120]. This led us to conjecture that the $\mathfrak{osp}(1|32,\mathbb{R})$ matrix model could also have interesting non-commutative static solutions, when expressed as a totally reduced model.

This turned out to be indeed the case and we could show that fuzzy sphere solutions were energetically favoured compared to the trivial solution. However, we can at best find meta-stable vacua of this kind. The instability that generates the non-commutative vacua is also present in the vicinity of the non-trivial vacua (at least in certain directions).

6.1 Introduction

One of the interesting proposal for a constructive definition of superstring theory [20, 107, 67, 110] is a formulation through a large N reduced model. The best-known model of this kind is the so-called IIB matrix model [107, 81] that we have described in chapter 4.

Another intriguing attempt for a constructive definition of superstring theory is the background-independent matrix model based on the Lie superalgebra $\mathfrak{osp}(1|32,\mathbb{R})$ [148, 147, 17, 15, 19] that we studied in the preceding chapter in the context of 12-dimensional and 11-dimensional backgrounds. Considered in a 10-dimensional context, it is a natural generalization of the IIB matrix model, in which both bosons and fermions are unified into a single supermultiplet. $\mathfrak{osp}(1|32,\mathbb{R})$ has been known as the unique maximal simple Lie superalgebra with 32 fermionic generators [28]. In a 10-dimensional representation, the smallest irreducible spinors are the 16-components chiral spinors, so that the 32 fermionic generators can be decomposed in two chiral spinors of equal or opposite chiralities. The former and the latter respectively correspond to the type IIA and IIB superstring theories.

In the papers [17, 15], it has been attempted to clarify the relation between a purely cubic $\mathfrak{osp}(1|32,\mathbb{R})$ supermatrix model and the IIB matrix model, paying particular attention to the structure of the supersymmetry algebra. Here, we will instead study its classical solutions and show that it is difficult to find a truly stable vacuum configuration.

Though the IIB matrix model only possesses flat commutative spacetime as a classical solution, we will see that the cubic matrix model exhibits curved-space solutions. Since large N reduced models are expected to be an eligible framework to describe gravitational interactions, it is essential to have the possibility of describing curved spacetimes manifestly and study perturbations around curved backgrounds.

There were some earlier approaches to construct totally reduced matrix models which have some curved space as a classical solution, so that it becomes possible to perform perturbations around this curved background. Such models have been discussed briefly in chapter 4.3. They typically have classical solutions given by a set of fields satisfying some Lie algebra. Thus, such a massive IIB matrix model can be expanded around various curved spaces. In [120], expansions around the two-dimensional fuzzy sphere and the two-dimensional fuzzy torus have been studied.

In this paper, we take this latter approach in order to describe a curved background spacetime by considering an $\mathfrak{osp}(1|32,\mathbb{R})$ supermatrix model with a mass term. We analyze how the massive supermatrix model incorporates the non-commutative curved-space classical solutions.

This paper is organized as follows: In Section 2, we give a brief review of the $\mathfrak{osp}(1|32,\mathbb{R})$ Lie superalgebra and associated supermatrix models. In Section 3, we suggest an Ansatz that allows to solve the equation of motions of the massive supermatrix model and leads to solution of the fuzzy sphere-type. We describe in detail two of these solutions, one exhibiting $SO(3) \times SO(3) \times SO(3)$ symmetry and the other exhibiting $SO(9)$ symmetry and compare their stability properties. This leads us to a more general discussion of a possible brane nucleation process in such totally reduced matrix models. Then, we make a few remarks on the structure of the supersymmetry transformations in our model. Finally, we summarize the results presented in this work in section 4 and indicate there a few directions for future research on this topic.

6.2 $\mathfrak{osp}(1|32,\mathbb{R})$ supermatrix model with a mass term

L. Smolin proposed a cubic matrix model [148, 147] based on the Lie superalgebra $\mathfrak{osp}(1|32,\mathbb{R})$. The action is constructed from a matrix M belonging to $\mathfrak{osp}(1|32,\mathbb{R})$, whose entries are promoted to large N Hermitian matrices. Note that we changed the notation compared to the published version of the paper, in order to agree with the definitions given in the preceding chapter, so that they differ from those taken in [15]. In particular, this choice amounts to take the matrices in the (even part) of the Lie superalgebra $\mathfrak{osp}(1|32,\mathbb{R})$ to be of the form:

$$M = \begin{pmatrix} m & \psi \\ -i\bar{\psi} & 0 \end{pmatrix}, \tag{6.1}$$

with a minus sign in front of $i\bar{\psi}$ and not a plus sign as in [15, 17]. In this matrix, ψ is a 32 components Majorana spinor and m belongs to the Lie algebra $\mathfrak{sp}(32,\mathbb{R})$.

6.2.1 Action

We consider an $\mathfrak{osp}(1|32,\mathbb{R})$ supermatrix model with a mass term included, expecting similarities with the massive IIB matrix model studied in [120]. We consider the following action, with a mass term added to the pure cubic $\mathfrak{osp}(1|32,\mathbb{R})$ supermatrix model:

$$\begin{aligned} S &= Tr_{\mathfrak{u}(N)}\left[str_{\mathfrak{osp}(1|32)}\left(-3\mu M^2 + iM[M,M]\right)\right] = \\ &\quad Tr_{\mathfrak{u}(N)}\left[3\mu(-m_p{}^q m_q{}^p - 2i\bar{\psi}^p\psi_p) + i\left(m_p{}^q[m_q{}^r, m_r{}^p] + 3i\bar{\psi}^p[m_p{}^q,\psi_q]\right)\right]. \end{aligned} \tag{6.2}$$

where $p,q,r,\cdots = 1,\ldots,32$ and $\mu > 0$. In this model, each element of the $\mathfrak{osp}(1|32,\mathbb{R})$ supermatrices is promoted to an $N\times N$ hermitian matrix. This action is invariant under $U(N)$ gauge transformations and $OSp(1|32,\mathbb{R})$ orthosymplectic transformations, and these two symmetries are decoupled, since they do not act on the same indices. Since we want to consider this model in a 10-dimensional spacetime context, we decompose the bosonic part m as follows:

$$m = W\Gamma^* + A_\mu\Gamma^\mu + B_\mu\Gamma^{\mu *} + \frac{1}{2!}C_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2} + \frac{1}{4!}H_{\mu_1\cdots\mu_4}\Gamma^{\mu_1\cdots\mu_4 *} + \frac{1}{5!}Z_{\mu_1\cdots\mu_5}\Gamma^{\mu_1\cdots\mu_5}, \tag{6.3}$$

where $\mu_i = 0,\ldots,9$ and Γ^* is the chirality operator. Then, the relevant part of the action (6.2) is expressed as (writing simply Tr instead of $Tr_{\mathfrak{u}(N)}$ from now on):

$$\begin{aligned} S &= 96\mu Tr\left(-W^2 - A_\mu A^\mu + B_\mu B^\mu + \frac{1}{2}C_{\mu_1\mu_2}C^{\mu_1\mu_2} - \frac{1}{4!}H_{\mu_1\cdots\mu_4}H^{\mu_1\cdots\mu_4} - \right. \\ &- \left.\frac{1}{5!}Z_{\mu_1\cdots\mu_5}Z^{\mu_1\cdots\mu_5} - \frac{i}{16}\bar{\psi}\psi\right) + 32iTr\left(3C_{\mu_1\mu_2}[B^{\mu_1},B^{\mu_2}] + C_{\mu_1\mu_2}[C^{\mu_2}{}_{\mu_3},C^{\mu_3\mu_1}]\right) + \\ &+ \text{ cubic interactions involving } (W,A,H,Z,\psi), \end{aligned} \tag{6.4}$$

while the full result can be found in the first appendix and the detailed computation in [15]. In the purely cubic supermatrix model (without mass term, which has been studied in [148, 17, 15]), the rank-2 field $C_{\mu_1\mu_2}$ possesses a cubic interaction term but has no quadratic term. This has been a severe obstacle to the appearance of a Yang-Mills-like structure in the supermatrix model, because it has been impossible to identify $C_{\mu_1\mu_2}$ with the commutators of the rank-1 fields $[B_{\mu_1}, B_{\mu_2}]$ (or $[A_{\mu_1}, A_{\mu_2}]$). In the 11-dimensional case, this difficulty has been overcome in [19] through the addition of a mass term, and we thus expect this model to contain the massive IIB matrix model, the bosonic part of which has been studied in [120] to investigate perturbation theory around noncommutative curved-space backgrounds.

6.3 Resolution of the equations of motion

We proceed to search for possible curved-space classical configurations solving the equations of motion that follow from the action 6.4). To get a clearer picture of the problem, we now set the fermions and the positive squared-mass bosonic fields to zero:

$$\psi = W = A_\mu = H_{\mu_1\cdots\mu_4} = Z_{\mu_1\cdots\mu_5} = 0. \tag{6.5}$$

Since their masses are positive (at least in the spatial directions, while the time-like direction of quantum fields is generally unphysical), (6.5) is a stable classical solution for fixed B_μ and $C_{\mu\nu}$. Furthermore, we choose to think about the tachyonic 10-dimensional vector field B_μ, rather than the well-defined A_μ as the space-time generating field, in order to obtain a curved-space non-commutative vacuum. The classical equations of motion for the remaining tachyonic fields B_μ and $C_{\mu\nu}$ following from (6.4) are

$$B_\mu = -i\mu^{-1}[B^\nu, C_{\mu\nu}], \tag{6.6}$$

$$C_{\mu\nu} = -i\mu^{-1}([B_\mu, B_\nu] + [C_\mu{}^\rho, C_{\nu\rho}]). \tag{6.7}$$

Although it is difficult to solve these equations in full generality, the equation of motion for $C_{\mu\nu}$ suggests to take $C_{\mu\nu} \propto [B_\mu, B_\nu]$ for B_μ's satisfying a fairly simple commutator algebra. If we look for objects having a clear geometrical interpretation, it is tempting to look for solutions building fuzzy spheres.

6.3.1 $SO(3) \times SO(3) \times SO(3)$ classical solution

The simplest tentative solution is the product of three fuzzy 2-spheres with the symmetry $SO(3) \times SO(3) \times SO(3)$. Such a system is described by $N \times N$ hermitian matrices building a representation of the $\mathfrak{so}(3)$ Lie algebra in the following way:

$$[B_i, B_j] = i\mu r \epsilon_{ijk} B_k, \qquad B_1^2 + B_2^2 + B_3^2 = \mu^2 r^2 \frac{N^2-1}{4} \mathbf{1}_{N\times N} \text{ for } (i,j,k = 1,2,3) \tag{6.8}$$

with similar relations for $i,j,k = 4,5,6$ and $i,j,k = 7,8,9$, trivial commutators for indices that do not belong to the same group of 3, and $B_0 = 0$ (ϵ_{ijk} is defined as usually). We can in principle choose three representations of different dimensionalities $N^{(1)}$, $N^{(2)}$ and $N^{(3)}$ for the three fuzzy spheres and put them together in a matrix of size $N \geq N^{(1)} + N^{(2)} + N^{(3)}$, More precisely, for $i = 1, 2, 3$:

$$B_i = \mu r \begin{pmatrix} J^i_{N^{(1)}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ for } i = 1,2,3\ , \quad B_i = \mu r \begin{pmatrix} 0 & 0 & 0 \\ 0 & J^i_{N^{(2)}} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ for } i = 4,5,6\ , \tag{6.9}$$

$$B_i = \mu r \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & J^i_{N^{(3)}} \end{pmatrix}, \text{ for } i = 7,8,9\ , \tag{6.10}$$

where $J^i_{N^{(k)}}$ is the i-th generator of the $N^{(k)}$-dimensional irreducible representation of the $\mathfrak{su}(2)$ Lie algebra with $[J^i, J^j] = i\epsilon_{ijk}J^k$. This set of fields (6.8) describes a space formed by the Cartesian product of three fuzzy spheres located in the directions (x_1, x_2, x_3), (x_4, x_5, x_6) and (x_7, x_8, x_9), whose respective radii are $\mu r\sqrt{(N^{(i)})^2 - 1}/2$ ($(N^2-1)/4$ is the quadratic Casimir operator of the $\mathfrak{so}(3)$ Lie algebra). Note that any positive integer values of $N^{(i)}$ are possible here, since $N^{(i)}$ indexes the dimensions of irreducible representations. For $SO(3)$, the irreps have dimensions $N = 2j+1$, for all integer values of the spin j. However, we can also use spinorial representations with half-integer spins in this case. It's better to consider this classical solution instead of the single $SO(3)$ fuzzy sphere

$$[B_i, B_j] = i\mu r\epsilon_{ijk}B_k \text{ (for } i,j,k = 1,2,3), \quad B_\mu = 0 \text{ (for } \mu = 0,4,5,\cdots,9), \tag{6.11}$$

because the solution $B_4 = \cdots = B_9 = 0$ would be unstable in the directions 4 to 9 due to the negative squared mass[1] of the rank-1 fields B_μ. Without restricting the generality, we can focus on the first sphere located in the direction (x_1, x_2, x_3), since the three fuzzy spheres all share the same equations of motion.

In the framework of fuzzy 2-spheres, we can solve the equations of motion (6.6) and (6.7) with the following Ansatz for the rank-2 field C_{ij}:

$$C_{ij} = f(r)\,\epsilon_{ijk}B_k, \forall\, i,j,k \in \{1,2,3\},\ \{4,5,6\} \text{ or } \{7,8,9\}\,. \tag{6.12}$$

where $f(r)$ is a function depending on the radius parameter r, and $C_{\mu\nu} = 0$ if μ and ν do not belong to the same triplet of indices. Indeed, the equation of motion (6.7) reduces then to:

$$\epsilon_{ijk}B_k(-f(r) + r + rf^2(r)) = 0\ , \tag{6.13}$$

for all three fuzzy spheres (6.13) has two solutions: $f_\pm(r) = \frac{1\pm\sqrt{1-4r^2}}{2r}$. When we plug this result in the equation of motion for B (6.6), this leads to

$$B_i(1 - 2rf_\pm(r)) = 0. \tag{6.14}$$

This gives the same condition on the radius parameter r for both $f_+(r)$ and $f_-(r)$, namely:

$$\sqrt{1-4r^2} = 0. \tag{6.15}$$

Therefore, when we assume the Ansatz (6.12), we obtain the classical solution (6.8) with the radius parameter set to $r = \frac{1}{2}$, which is fortunately real. Indeed, $r^2 \leq 0$ would indicate that the fuzzy sphere

[1] The classical solution with $B_0 = 0$ has no problem, because it has a positive mass unlike the other directions of the field B.

solution is unstable. For example, in the IIB massive matrix model described by (4.58), the sign of the squared radius of the fuzzy 2-sphere is linked to the sign of the mass term in the action and it would become negative for a correct-sign mass term, which is to be expected, since in that case, the trivial commutative solution becomes the stable vacuum of the theory.

We next want to discuss the stability of the $SO(3) \times SO(3) \times SO(3)$ classical solution in more qualitative terms [109, 159, 158]. To this end, we compare the energy of the trivial commutative solution $B_\mu = 0$ with that of the fuzzy-sphere solution. The classical energy for $B_\mu = 0$ is obviously $E_{B_\mu=0} = -S_{B_\mu=0} = 0$. [2]

In the $SO(3) \times SO(3) \times SO(3)$ fuzzy-sphere background, the 2-form field C_{ij} is

$$C_{ij} = \epsilon_{ijk} B_k. \tag{6.16}$$

Therefore, the total energy is

$$E_{SO(3)^3} = -S_{SO(3)^3} = -64\mu \sum_{\mu=1}^{9} Tr(B_\mu B^\mu) = -4\mu^3 \sum_{i=1}^{3} N^{(i)}(N^{(i)}-1)(N^{(i)}+1) \,. \tag{6.17}$$

This result shows that the $SO(3) \times SO(3) \times SO(3)$ fuzzy-sphere classical solution has a lower energy compared to the trivial commutative solution and hence a higher probability. However, since the spheres' radii are proportional to N, it also shows that bigger spheres are more stable, so that the radial directions are not really stable, although they can only change by quantum jumps in energy, while maintaining the symmetry. Moreover, the $C_{\mu\nu}$'s for μ and ν belonging to different triplets of indices are still unstable. We should also give them a non-trivial vacuum expectation value. We could do that without disturbing the fuzzy-sphere solution by choosing them so that they commute with the B_μ's and satisfy:

$$C_{\mu\nu} = -\frac{i}{\mu}[C_\mu{}^\rho, C_{\nu\rho}] \,. \tag{6.18}$$

An example of a solution to this coupled set of equations can be found by taking the following triplets of fields to separately form a representation of $\mathfrak{su}(2)$ each: $\{C_{14}, C_{47}, C_{71}\}$, $\{C_{24}, C_{49}, C_{92}\}$, $\{C_{34}, C_{48}, C_{83}\}$, $\{C_{25}, C_{58}, C_{82}\}$, $\{C_{35}, C_{57}, C_{73}\}$, $\{C_{15}, C_{59}, C_{91}\}$, $\{C_{36}, C_{69}, C_{93}\}$, $\{C_{16}, C_{68}, C_{81}\}$, $\{C_{26}, C_{67}, C_{72}\}$. Concretely, one can set:

$$C_{14} = \mu J^1_{N^{(a)}} \,,\; C_{47} = \mu J^2_{N^{(a)}} \,,\; C_{71} = \mu J^3_{N^{(a)}} \,, \tag{6.19}$$

for each triplet of $C_{\mu\nu}$'s (for various choices of representation sizes $N^{(a)}$, $a = 4, \ldots, 12$) and arrange the different representations of $\mathfrak{su}(2)$ in a big $\mathfrak{u}(N)$ matrix in such a way that they mutually commute.

6.3.2 Other curved-space solutions and the fuzzy 8-sphere

So far, we have analyzed the simplest curved-space solution, the $SO(3) \times SO(3) \times SO(3)$ triple fuzzy spheres. Here, we consider other curved-space classical solutions. The fuzzy $2k$-spheres[53, 141, 105,

[2] Recall that in our notation, the action is minus the potential. Since we now consider a classical solution with $B_0 = 0$ (thus no need of Wick rotation), the energy is simply minus the classical action in which we substitute the solution.

142], which exhibit a $SO(2k+1)$ symmetry, are constructed by the following n-fold symmetric tensor product of $(2k+1)$-dimensional gamma matrices:

$$B_p^{SO(2k+1)} = \frac{\mu r}{2}[(\Gamma_p^{(k)} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}) + \cdots + (\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \Gamma_p^{(k)})]_{\rm sym}, \tag{6.20}$$

where p runs over $1, 2, \cdots, 2k+1$. $\Gamma_p^{(2k)}$ are $2^k \times 2^k$ gamma matrices, and build a representation of the $SO(2k+1)$ Clifford algebra; i.e. $\{\Gamma_p^{(k)}, \Gamma_q^{(k)}\} = 2\delta_{pq}\mathbf{1}_{2^k \times 2^k}$. These matrices satisfy the following algebraic relations:

$$B_p^{SO(2k+1)} B_p^{SO(2k+1)} = \frac{\mu^2 r^2}{4} n(n+2k)\mathbf{1}_{N_k \times N_k}, \tag{6.21}$$

$$B_{pq}^{SO(2k+1)} B_{pq}^{SO(2k+1)} = -(\frac{\mu r}{2})^4 8kn(n+2k)\mathbf{1}_{N_k \times N_k}, \tag{6.22}$$

$$[B_{pq}^{SO(2k+1)}, B_s^{SO(2k+1)}] = \mu^2 r^2(-\delta_{ps}B_q^{SO(2k+1)} + \delta_{qs}B_p^{SO(2k+1)}), \tag{6.23}$$

$$[B_{pq}^{SO(2k+1)}, B_{st}^{SO(2k+1)}] = \mu^2 r^2(\delta_{qs}B_{pt}^{SO(2k+1)} + \delta_{pt}B_{qs}^{SO(2k+1)} - \delta_{ps}B_{qt}^{SO(2k+1)} - \delta_{qt}B_{ps}^{SO(2k+1)}), \tag{6.24}$$

where $B_{pq}^{SO(2k+1)} = [B_p^{SO(2k+1)}, B_q^{SO(2k+1)}]$ furnishes (up to a normalization factor) a representation of the $\mathfrak{so}(2k+1)$ Lie algebra and N_k is the dimension of the fully symmetrized irreducible representation for the $SO(2k+1)$ fuzzy sphere. More precisely, defining $J_{pq} \equiv 1/(\mu r)B_{pq}^{SO(2k+1)}\, \forall p, q = 1, \ldots, 2k+1$ and $J_{p0} \equiv 1/(\mu r)B_p^{SO(2k+1)}\, \forall p = 1, \ldots, 2k+1$, the relations (6.23) and (6.24) together with the definition of $B_{pq}^{SO(2k+1)}$ become equivalent to a Lorentz algebra $\mathfrak{so}(2k+1, 1)$ for $\{J_{MN}\}_{M,N=0}^{2k+1}$. Of course, this is because the commutation relations (6.23) and (6.24) are inherited from those of the gamma matrices. Thanks to these relations, we expect that the equation of motion can be solved for all even-dimensional fuzzy spheres in a similar fashion to the fuzzy 2-spheres. In other words, this means that the $SO(2k+1)$ fuzzy spheres will provide us with a whole set of curved classical solutions for some precise values of the parameter r. In addition, the $B_p^{SO(2k+1)}$'s satisfy the following self-duality relation:

$$\epsilon_{p_1 \cdots p_{2k+1}} B_{p_1}^{SO(2k+1)} B_{p_2}^{SO(2k+1)} \cdots B_{p_{2k}}^{SO(2k+1)} = \left(\frac{\mu r}{2}\right)^{2k-1} m_k B_{p_{2k+1}}^{SO(2k+1)}, \tag{6.25}$$

where

$$m_1 = 2i, \quad m_2 = 8(n+2), \quad m_3 = -48i(n+2)(n+4), \quad m_4 = -384(n+2)(n+4)(n+6), \tag{6.26}$$

which is a consequence of the duality relations for odd-dimensional Gamma matrices. We give hints for the computation of these coefficients in Appendix 2 of this chapter and a detailed computation in Appendix E. This whole setup extends the $SO(3)$ case to higher dimensions[3].

[3]It generalizes the fuzzy 2-sphere case with $k=1$ for which the totally symmetric space on which $B_i^{SO(3)}B_i^{SO(3)}$ is proportional to the identity is $(n+1)$-dimensional. For $N = n+1$, the radius of the $SO(3)$ fuzzy sphere is indeed:

$$\frac{\mu^2 r^2}{4} n(n+2) = (\mu r)^2 \frac{N^2-1}{4}.$$

The relation (6.21) actually corresponds to the Casimir of the $\mathfrak{so}(3)$ Lie algebra. And (6.25) is trivially equivalent to the commutation relation $[B_i^{SO(3)}, B_j^{SO(3)}] = i\mu r \epsilon_{ijk} B_k^{SO(3)}$.

Another possible classical solution of our massive supermatrix model is the single $SO(9)$ fuzzy sphere. The analysis is similar to the $SO(3)\times SO(3)\times SO(3)$ fuzzy spheres. Here, the indices $p, q, \cdots$ run over $1, 2, \cdots, 9$. For the $SO(9)$ fuzzy-sphere classical solution, we likewise assume the following Ansatz for the rank-2 fields C_{pq}:

$$C_{pq}^{SO(9)} = -i\mu^{-1} g(r) B_{pq}^{SO(9)}. \tag{6.27}$$

Then, the equation of motion (6.7) implies

$$\frac{-i}{\mu} B_{pq}^{SO(9)} (-g(r) + 1 + 7r^2 g^2(r)) = 0. \tag{6.28}$$

We again have two choices for the function $g(r)$:

$$g_{\pm}(r) = \frac{1 \pm \sqrt{1 - 28r^2}}{14r^2}. \tag{6.29}$$

The equation of motion (6.6) for the rank-1 field B_p gives

$$B_p^{SO(9)} (1 - 8r^2 g_{\pm}(r)) = 0. \tag{6.30}$$

Now, unlike the case of the $SO(3)\times SO(3)\times SO(3)$ fuzzy spheres, $1 - 8r^2 g_-(r) = 0$ does not have any real positive solution for r. However, there is exactly one such solution for $1 - 8r^2 g_+(r) = 0$, which is $r = \frac{1}{8}$.

More generally, for an $SO(2k+1)$ fuzzy sphere, the same Ansatz would give

$$\begin{aligned} g_{\pm}(r) &= \frac{1 \pm \sqrt{1 - 4(2k-1)r^2}}{2(2k-1)r^2}\,, \\ 1 - 2kr^2 g_{\pm}(r) &= 0\,,\ \text{solvable only for } g_+(r) \text{ at } r = \frac{1}{2k}. \end{aligned}$$

We discuss the stability of the $SO(9)$ fuzzy-sphere classical solution by computing its classical energy. At the classical level, we obtain

$$E_{SO(9)} = -\frac{5}{8}\mu^3 n(n+8) N_4. \tag{6.31}$$

N_4 is given in [141, 105] by[4]

$$N_4 = \frac{(n+1)(n+2)(n+3)^2(n+4)^2(n+5)^2(n+6)(n+7)}{302400}. \tag{6.32}$$

In contrast with the $SO(3)\times SO(3)\times SO(3)$ case, N can take here only certain precise values. For example, the smallest non-trivial representation ($n = 1$) has dimension 16, the following one ($n = 2$)

[4] Generally, N_k is known to be of the order $\mathcal{O}(n^{\frac{k(k+1)}{2}})$, and more explicitly,

$$N_1 = (n+1), \quad N_2 = \frac{(n+1)(n+2)(n+3)}{6}, \quad N_3 = \frac{(n+1)(n+2)(n+3)^2(n+4)(n+5)}{360}.$$

126, then 672, etc... The classical energy for the $SO(3) \times SO(3) \times SO(3)$ fuzzy-sphere solution is of the order $\mathcal{O}(-\mu^3 n^3) = \mathcal{O}(-\mu^3 N^3)$ while that of the $SO(9)$ fuzzy-sphere solution is of the order $\mathcal{O}(-\mu^3 n^{12}) = \mathcal{O}(-\mu^3 N^{\frac{6}{5}})$. Therefore, at large N, the $SO(3) \times SO(3) \times SO(3)$ triple fuzzy-sphere solution is energetically favored compared to the $SO(9)$ solution at equal size N of the matrices. The presence of a spherical solution for all N in the $SO(3)\times SO(3)\times SO(3)$ case may indeed be a stabilizing factor. On the other hand, at equal value of n, whose physical meaning is less clear, the fuzzy 8-sphere solution has lower energy.

Note that the single $SO(q)$ fuzzy spheres for $q \leq 8$ are not stable classical solutions of our model. When the $SO(q)$ sphere occupies the direction $x_1, x_2, \cdots, x_q$, the solution $B^{SO(q)}_{q+1} = B^{SO(q)}_{q+2} = \cdots = B^{SO(q)}_9 = 0$ is trivially unstable because of the negative mass squared. On the other hand, the Cartesian product of several fuzzy spheres, such as $SO(3) \times SO(6)$, are interesting candidates for classical solutions.

6.3.3 Nucleation process of spherical branes

Starting from an empty spacetime, it is interesting to try to guess how spherical brane configurations could be successively produced through a sequence of decays into energetically more favorable meta-stable brane systems. The reader may have noticed that we have so far limited ourselves to the study of curved branes building irreducible representations of their symmetry groups. This could seem at first to be an unjustified prejudice, but it turns out that such configurations are energetically favored at equal values of N. For example, for $SO(3)$, an irreducible representation $\mathcal{R}_N$ of dimension N contributes:

$$E_{\mathcal{R}_N} = -4\mu^3(N^3 - N)$$

per fuzzy 2-sphere, while a reducible representation $\mathcal{R}_{N_1}\oplus\ldots\oplus\mathcal{R}_{N_m}$ of equal dimension $N_1+\ldots+N_m = N$ would contribute;

$$E_{\mathcal{R}_{N_1}\oplus\ldots\oplus\mathcal{R}_{N_m}} = -4\mu^3\sum_{i=1}^{m}(N_i^3 - N_i).$$

This is obviously a less negative number, especially for big values of N. A similar conclusion was reached in [109] for the case of a Euclidean 3-dimensional IIB matrix model with a Chern-Simons term and it seems to be a fairly general feature of matrix models admitting non-trivial classical solutions. This property is particularly clear for low-dimensional branes, since the classical energy is of order $\mathcal{O}(-\mu^3 N^3)$ for $SO(3)$, but it remains true for any $SO(2k+1)$ fuzzy-sphere solution, whose energy is of order $\mathcal{O}(-\mu^3 N^{1+4/(k(k+1))})$, which also shows that low-dimensional configurations are favored. As we noticed in the preceding subsection, this latter fact might be a consequence of the fact that there are more irreducible representations available for low-dimensional fuzzy spheres, which makes it easier for them to grow in radius through energetically favorable configurations. A third obvious fact is that configurations described by representations of high dimensionality are preferred.

Put together, these comparisons give us a possible picture for the branes nucleation process in this and similar matrix models. As they appear, configurations of all spacetime dimensions described by small representations will be progressively absorbed by bigger representations to form irreducible ones, that will slowly grow in this way to bigger values of N. Parallel to that, branes of higher dimensionalities will tend to decay into a bunch of branes of smaller dimensionalities, finally leaving only 2-spheres and noncommutative tori of growing radii. If the size of the Hermitian matrices is left open, as is the case in a grand-canonical approach to completely reduced models, where the path integration contains a sum on the matrix size, no configuration will be truly stable, since the size of the irreducible representations will grow continuously.

Of course, this is a relatively qualitative study, which could only be proven correct by a full quantum statistical study of the model. However, it seems to be an interesting proposal for the possible physics of such theories.

6.3.4 Supersymmetry

We next comment on the structure of the supersymmetry. The biggest difference with the purely cubic supermatrix model, due to the addition of the mass term, is that this model is not invariant under the inhomogeneous supersymmetry

$$\delta_\xi m = 0, \quad \delta_\xi \psi = \xi, \tag{6.33}$$

which is a translation of the fermionic field. However, this model has 2 homogeneous supersymmetries in 10 dimensions, which are part of the $\mathfrak{osp}(1|32,\mathbb{R})$ symmetry:

$$\delta_\epsilon M = \left[\begin{pmatrix} 0 & \epsilon \\ -i\bar{\epsilon} & 0 \end{pmatrix}, \begin{pmatrix} m & \psi \\ -i\bar{\psi} & 0 \end{pmatrix} \right] = \begin{pmatrix} -i(\epsilon\bar{\psi} - \psi\bar{\epsilon}) & -m\epsilon \\ -i\bar{\epsilon}m & 0 \end{pmatrix}, \tag{6.34}$$

which transform the bosonic and fermionic fields as

$$\delta_\epsilon m = -i(\epsilon\bar{\psi} - \psi\bar{\epsilon}), \quad \delta_\epsilon \psi = -m\epsilon. \tag{6.35}$$

In the IIB matrix model, the supersymmetry has to balance between a quartic term $Tr([A_\mu, A_\nu])^2$ and a trilinear contribution $Tr\bar{\psi}\Gamma^\mu[A_\mu,\psi]$ in the action (4.1), which implies that the SUSY transformation of the fermionic field has to be bilinear in the bosonic field. On the other hand, the homogeneous supersymmetries are all linear in the fields in the purely cubic supermatrix model [17, 15]. By incorporating the mass term, we are allowed to integrate out the rank-2 field $C_{\mu_1\mu_2}$ by solving the classical equation of motion iteratively as in [19][5].

Thanks to this procedure, the homogeneous SUSY transformation for the fermionic field becomes

$$\delta_\epsilon \psi = \frac{i}{2\mu}[B_{\mu_1}, B_{\mu_2}]\Gamma^{\mu_1\mu_2}\epsilon + \cdots, \tag{6.36}$$

while the transformation of the field B_μ is

$$\delta_\epsilon B_\mu = \frac{1}{32} tr_{32\times 32}(i(\epsilon\bar{\psi} - \psi\bar{\epsilon})\Gamma_{\mu *}) = \frac{i}{16}\bar{\epsilon}\Gamma_{\mu *}\psi. \tag{6.37}$$

[5]The Yang-Mills-like structure of the homogeneous SUSY transformation on the fermion comes from $-\frac{1}{2}C_{\mu\nu}\Gamma^{\mu\nu}\epsilon$, which is a part of $\delta_\epsilon\psi = -m\epsilon$. The explicit form of the iterative solution of the equations of motion (6.6) and (6.7) is

$$\begin{aligned} C_{\mu\nu} &= -i\mu^{-1}[B_\mu, B_\nu] + i\mu^{-3}[[B_\mu, B_\rho], [B_\nu, B^\rho]] \\ &\quad -i\mu^{-5}[[B_\mu, B_\rho], [[B_\nu, B_\chi], [B^\rho, B^\chi]]] + i\mu^{-5}[[B_\nu, B_\rho], [[B_\mu, B_\chi], [B^\rho, B^\chi]]] + \mathcal{O}(\mu^{-7}). \end{aligned}$$

In that sense, the mass term is essential to realize the Yang-Mills-like structure for the homogeneous supersymmetries. On the other hand, if we want to preserve the homogeneous supersymmetries, we cannot just put a mass term for $C_{\mu_1\mu_2}$ by hand, the $\mathfrak{osp}(1|32,\mathbb{R})$ symmetry forces all fields to share the same mass, since they all lie in the same multiplet. In particular, we are forced to introduce a mass term for the fermions as well, which breaks the inhomogeneous supersymmetries. In other words, it seems difficult to have super Yang-Mills-type structure for both homogeneous and inhomogeneous supersymmetries in the context of supermatrix models.

Indeed, in contrast with the purely cubic supermatrix model[17, 15], which has twice as many SUSY parameters, the massive supermatrix model has only $\mathcal{N}=2$ SUSY in 10 dimensions, because it lacks the inhomogeneous supersymmetries. In consequence, we cannot realize the translation of the vector field A_μ as a commutator of two linear combinations of the homogeneous and inhomogeneous supersymmetries (6.35) as in the IIB matrix model, where it leads to the interpretation of the eigenvalues of A_μ as spacetime coordinates. On the contrary,

$$[\delta_\epsilon,\delta_\chi]m = -i[(\epsilon\bar{\chi}-\chi\bar{\epsilon}),m], \quad [\delta_\epsilon,\delta_\chi]\psi = -i(\epsilon\bar{\chi}-\chi\bar{\epsilon})\psi, \tag{6.38}$$

vanishes up to an $\mathfrak{sp}(32,\mathbb{R})$ rotation. This is in clear contrast with the supersymmetry algebra of the IIB matrix model, which contains more than gauge transformations.

6.4 Conclusion and outlook

In this paper, we have investigated a supermatrix model based on $\mathfrak{osp}(1|32,\mathbb{R})$ with a mass term and a cubic interaction. To be able to describe the gravitational interaction in terms of large N reduced models, we must understand how the reduced models can describe physics in curved spacetimes. Although the IIB matrix model only possesses flat spacetime as a classical solution, by adding a tachyonic mass term as in [120], we can obtain new classical solutions building curved space backgrounds. Following this idea, we have expected that massive supermatrix models could also exhibit similar properties leading to non-trivial classical solutions. In particular, we have investigated fuzzy-sphere solutions with symmetries $SO(3)\times SO(3)\times SO(3)$ and $SO(9)$, and calculated the parameter determining the quantization step separating the radii of configurations described by different representations of the Lie algebra. We have then discussed their respective likelihood by comparing their energy at the classical level, which gave us a way to understand a possible dynamical evolution of the solutions through successively more favorable brane configurations.

It is an intriguing issue to search for other stable curved-space classical solutions. For example, an $SO(3)\times SO(6)$ fuzzy sphere could be a promising candidate. Indeed, the expansion around this classical solution may be related in some way to the BMN matrix model [27, 59, 149, 26, 44], which appears as the discrete light-cone quantization of D0-brane in the M-theory pp-wave background [27]. However, to study this case explicitly, we first have to analyze how the equations of motion (6.6) and (6.7) can be treated in the case of odd fuzzy spheres. Indeed, as is outlined in [142], the commutator $[B_i,B_j]$ does not correspond to a representation of $SO(2k)$ for odd fuzzy spheres, which makes the analysis much more involved. However, the construction of an $SO(3)\times SO(6)$ classical solution would show how transverse 5-branes can appear in this model. The $SO(4)\times SO(5)$ case should proceed along similar lines. Another case that can be investigated is a solution of the type $SO(2)\times SO(2)\times SO(5)$, in which the two first circles build a noncommutative torus as in [120].

It is important to notice that such backgrounds can never really be a local minima of our tachyonic potential. Indeed, decomposing:

$$B_\mu = \tilde{B}_\mu + \hat{B}_\mu \ , \quad C_{\mu\nu} = \tilde{C}_\mu + \hat{C}_{\mu\nu} \ , \tag{6.39}$$

in a background part $\tilde{F}$ and a fluctuation part $\hat{F}$, we can write the background-field method effective action for B and C as:

$$\begin{aligned} S &= S_b(\tilde{B},\tilde{C}) + 96\mu Tr\Big(\hat{B}_\mu\hat{B}^\mu + \frac{1}{2}\hat{C}_{\mu\nu}\hat{C}^{\mu\nu}\Big) \\ &+ 32iTr\Bigg(3\tilde{C}_{\mu\nu}[\hat{B}^\mu,\hat{B}^\nu] + 6\hat{C}_{\mu\nu}[\tilde{B}^\mu,\hat{B}^\nu] + 3\hat{C}_{\mu\nu}[\hat{B}^\mu,\hat{B}^\nu] + 3\tilde{C}_{\mu\nu}[\hat{C}^\nu{}_\rho,\hat{C}^{\rho\mu}] + \hat{C}_{\mu\nu}[\hat{C}^\nu{}_\rho,\hat{C}^{\rho\mu}]\Bigg) \ , \end{aligned} \tag{6.40}$$

where S_b is the classical value of the action in the chosen background. Now, if we choose a particular perturbation in direction 1, $\hat{B}_1 \neq 0$ with all other $\hat{B}_\mu$'s and $\hat{C}_{\mu\nu}$'s vanishing, only the tachyonic mass term will contribute to the configuration's energy and we can obtain states of arbitrarily low energy by growing B_1. This conclusion doesn't depend on the specific background chosen, only on the fact that terms linear in the perturbations vanish in a background satisfying the classical equations of motion. In other words, the potential has no local minima, only saddle points. It is thus worth asking whether we can build a matrix model that has fuzzy-sphere solutions that are at least local minima of the potential, or even global minima. This could be a direction for further research.

6.5 Appendix 1: Massive supermatrix model action

$$\begin{aligned} S \ = \ & 96\mu Tr\Bigg(-W^2 - A_\mu A^\mu + B_\mu B^\mu + \frac{1}{2}C_{\mu_1\mu_2}C^{\mu_1\mu_2} - \frac{1}{4!}H_{\mu_1\cdots\mu_4}H^{\mu_1\cdots\mu_4} \\ & -\frac{1}{5!}Z_{\mu_1\cdots\mu_5}Z^{\mu_1\cdots\mu_5} - \frac{i}{16}\bar{\psi}\psi\Bigg) \\ + \ & 32iTr\Bigg(-3C_{\mu_1\mu_2}[A^{\mu_1},A^{\mu_2}] + 3C_{\mu_1\mu_2}[B^{\mu_1},B^{\mu_2}] + 6W[A_\mu,B^\mu] + C_{\mu_1\mu_2}[C^{\mu_2}{}_{\mu_3},C^{\mu_3\mu_1}] \\ & +\frac{1}{4}B_{\mu_1}[H_{\mu_2\cdots\mu_5},Z^{\mu_1\cdots\mu_5}] - \frac{1}{8}C_{\mu_1\mu_2}(4[H^{\mu_1}{}_{\rho_1\rho_2\rho_3},H^{\mu_2\rho_1\rho_2\rho_3}] + [Z^{\mu_1}{}_{\rho_1\cdots\rho_4},Z^{\mu_2\rho_1\cdots\rho_4}]) \\ & +\frac{3}{(5!)^2}\epsilon^{\mu_1\cdots\mu_{10}}\left(-W[Z_{\mu_1\cdots\mu_5},Z_{\mu_6\cdots\mu_{10}}] + 10A_{\mu_1}[H_{\mu_2\cdots\mu_5},Z_{\mu_6\cdots\mu_{10}}]\right) \\ & +\frac{200}{(5!)^3}\epsilon^{\mu_1\cdots\mu_{10}}\left(5H_{\mu_1\cdots\mu_4}[Z_{\mu_5\mu_6\mu_7}{}^{\rho\chi},Z_{\mu_8\mu_9\mu_{10}\rho\chi}] + 10H_{\mu_1\cdots\mu_4}[H_{\mu_5\mu_6\mu_7}{}^{\rho},H_{\mu_8\mu_9\mu_{10}\rho}]\right. \\ & \left. +6H^{\rho\chi}{}_{\mu_1\mu_2}[Z_{\mu_3\mu_4\mu_5\rho\chi},Z_{\mu_6\cdots\mu_{10}}]\right)\Bigg) \\ - \ & 3Tr\Bigg(\bar{\psi}\Gamma^*[W,\psi] + \bar{\psi}\Gamma^\mu[A_\mu,\psi] + \bar{\psi}\Gamma^{\mu*}[B_\mu,\psi] + \frac{1}{2!}\bar{\psi}\Gamma^{\mu_1\mu_2}[C_{\mu_1\mu_2},\psi] \\ & + \frac{1}{4!}\bar{\psi}\Gamma^{\mu_1\cdots\mu_4*}[H_{\mu_1\cdots\mu_4},\psi] + \frac{1}{5!}\bar{\psi}\Gamma^{\mu_1\cdots\mu_5}[Z_{\mu_1\cdots\mu_5},\psi]\Bigg) \end{aligned} \tag{6.41}$$

6.6 Appendix 2: Notations and useful formulae

In this appendix, we give hints for the derivations of the coefficients m_k in the self-duality relation (6.25) for the $SO(2k+1)$ fuzzy sphere. A full proof can be found in Appendix E. It is trivial that $m_1 = 2i$ as explained in the footnote 3, while the computation of the coefficient m_2 can be found in [53]. In this appendix, we give formulae that are useful to derive m_3 and m_4. We set $\frac{\mu r}{2} = 1$ and omit "$_{\text{sym}}$", with the understanding that these formulae are only valid in the fully symmetrized representations.

In general, we have:

$$\sum_{l=1}^{2k+1} (\Gamma_l^{(k)} \otimes \Gamma_l^{(k)}) = (\mathbf{1}_{2^k\times 2^k} \otimes \mathbf{1}_{2^k\times 2^k}),$$
$$\sum_{l_1,l_2=1}^{2k+1} (\Gamma_{l_1 l_2}^{(k)} \otimes \Gamma_{l_1 l_2}^{(k)}) = -2k(\mathbf{1}_{2^k\times 2^k} \otimes \mathbf{1}_{2^k\times 2^k}) \tag{6.42}$$

More specifically, to compute m_8, we also need the following formulae (where summation on all $l_i = 1, \dots, 7$ is implicit):

$$(\Gamma_{l_1 l_2 l_3}^{(3)} \otimes \Gamma_{l_1 l_2 l_3}^{(3)}) = -18(\mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8}), \tag{6.43}$$
$$(\Gamma_{l_1 l_2}^{(3)} \otimes \Gamma_{l_3}^{(3)} \otimes \Gamma_{l_1 l_2 l_3}^{(3)}) = -6(\mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8}), \tag{6.44}$$
$$(\Gamma_{l_1 l_2}^{(3)} \otimes \Gamma_{l_3 l_4}^{(3)} \otimes \Gamma_{l_1\cdots l_4}^{(3)}) = 24(\mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8}), \tag{6.45}$$
$$(\Gamma_{l_1 l_2}^{(3)} \otimes \Gamma_{l_3 l_4}^{(3)} \otimes \Gamma_{l_5 l_6}^{(3)} \otimes \Gamma_{l_1\cdots l_6}^{(3)}) = -48(\mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8}). \tag{6.46}$$

Chapter 7

Conclusions and outlook

In this thesis, we have discussed in general the different proposals that have been made in the literature to obtain a non-perturbative definition of string theory through the use of matrix models. These models show many of the features we would expect of a non-perturbative definition of string theory. In particular, the BFSS matrix model is closely related to type IIA string theory and eleven-dimensional supergravity, while the IIB matrix model is closely related to type IIB string theory. We have shown how they can be obtained as a matrix regularization of a supermembrane living in 11 dimensions and of a superstring theory living in 10 dimensions, respectively. On the other hand, it remains unclear at present if any of them can capture the whole dynamics of M-theory. In particular, we don't know if they can reproduce the complicated (and widely unknown) vacuum structure of M-theory, since we only know how to formulate them around certain simple backgrounds. Further study of dynamical symmetry-breaking in the IIB matrix model is clearly important in that respect, as is the formulation of matrix models in various backgrounds.

The type of matrix models that constituted the main subject of my doctoral research have been first proposed in order to get a manifestly background-independant formulation of M-theory, since their definition doesn't depend on the dimension of the target-spacetime. It can be seen as a step further towards a purely algebraic formulation of quantum gravity, instead of the usual geometrical one. This can be motivated by the common belief that the structure of quantum spacetime at very small scales is fuzzy, exhibiting discreteness and non-commutativity.

Together with my collaborators, we have tried to study it in the more familiar contexts of ten, eleven, and twelve dimensions and compared their properties with better-known models. In all cases, cubic supermatrix models are much more complicated, making them difficult to quantize properly. However, the large number of fields comes together with a large symmetry, making it plausible that they could be equivalent to simpler-looking models in certain limits, once gauge-equivalent configurations are eliminated, depending on what part of the symmetries we take as being local, respectively global.

It is interesting to note that the apparent instability due to the presence of cubic bosonic terms shows itself in different disguises in chapter 5 and 6. In chapter 5, the fields having a tachyonic mass-term turn out (after compactification and T-duality) to be constrained unphysical fields. Integrating them out, we generate an effective potential for the remaining fields whose bosonic part is a polynomial containing even powers of the fields only, positive-definite at least until 4th order. Unfortunately, this expansion leads to an infinite number of higher-order interactions, making it difficult to study this effective theory quantum-mechanically. It would be interesting (though far from obvious) to find a comparatively simple function of the fields that can be developped perturbatively to produce such an expansion, in order to understand its structure better.

In chapter 6, where we haven't performed any T-duality, we cannot discriminate between physical and unphysical fields. Instead, the tachyonic fields trigger successive decays to vacuum configurations of lower energies. Examples of such vacua are given by various types of fuzzy sphere configurations. Although we show that none of them can be truly stable in this particular model, it could be interesting to understand in which kind of matrix models such solutions could become stable.

One can also wonder whether an expansion of the model around such a non-commutative vacuum would actually produce an effective action similar to the one described in chapter 5 or not. This could also be an interesting future direction of research.

As we mentioned in the last subsection of chapter 4, the study of this kind of solutions can also be performed in the perhaps physically more realistic case (though algebraically less interesting) of the massive IIB matrix model. In particular, stability issues are more likely to be understood in that case, since we have more control on the quantum theory. In particular, it is important to understand well the features of non-commutative spacetimes in general and fuzzy spheres in particular, since they could provide interesting vacua of M-theory whatever the ultimate formulation of it will be.

To summarize the present status of research on cubic supermatrix models based on the superalgebra $\mathfrak{osp}(1|32,\mathbb{R})$, it's fair to say that we still do not know very much about their actual physical relevance. However, the study of their properties in various contexts has brought some interesting conclusions, that could hopefully be of some help in the broader context of the search for a constructive definition of M-theory, at least if matrix models have some rôle to play in this game.

Appendix A

Majorana fermions in various spacetime dimensions and signatures

This is a résumé of the construction of real Majorana representations of the Clifford algebra in various space-time dimensions and signatures. Let us start with the trivial 2-dimensional case: We use the standard definition for Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.1}$$

For the euclidean metric, to satisfy $\{\gamma^i, \gamma^j\} = \delta^{ij}$, we can take:

$$\gamma^1 = \sigma^1 , \qquad \gamma^2 = \sigma^3 . \tag{A.2}$$

For the Minkowskian metric, to satisfy $\{\gamma^i, \gamma^j\} = \eta^{ij}$ with $\eta^{ij} = diag(-1, +1)$, we can take:

$$\gamma^0 = i\sigma^2 , \qquad \gamma^1 = \sigma^1 . \tag{A.3}$$

The chirality matrix is then $\gamma_2 = \gamma^0\gamma^1 = \sigma^3$ and the representation is obviously both Majorana and Weyl. Of course, in 2+1 dimensions, we can use exactly the same γ^i's to get a Majorana representation. On the other hand, there isn't any such representations in 3 or 4 Euclidean dimensions. On the other hand, in 3+1 dimensions, we can take:

$$\gamma^0 = \sigma^1 \otimes i\sigma^2 , \quad \gamma^1 = \sigma^1 \otimes \sigma^3 , \quad \gamma^2 = \sigma^3 \otimes \mathbb{I} , \quad \gamma^3 = \sigma^1 \otimes \sigma^1 . \tag{A.4}$$

Then, the chirality matrix is $\gamma^4 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma^2 \otimes \mathbb{I}$. Note that it is not real, which reflects the incompatibility of the Majorana and Weyl conditions in $3{+}1D$. Such a Majorana-Weyl representation exists however in 2+2 D, if one takes:

$$\gamma^0 = i\sigma^2 \otimes \sigma^1 , \quad \gamma^1 = \sigma^1 \otimes \mathbb{I} , \quad \gamma^2 = i\sigma^2 \otimes i\sigma^2 , \quad \gamma^3 = i\sigma^2 \otimes \sigma^3 . \tag{A.5}$$

Indeed, the chirality matrix will be: $\gamma^4 = -\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma^3 \otimes \mathbb{I}$, real in this case. This five matrices of course provides us with a Majorana representation in $3{+}2D$ as well. There are no such representations in 5 Euclidean or Minkowskian dimensions.

In 6 dimensions, Majorana representations exist for signatures (6,0), (4,2) and (3,3), the latter being Majorana-Weyl, but not in the Minkowskian case. In 4+2 dimensions, we can choose:

$$\gamma^0 = i\sigma^2\otimes\mathbb{I}\otimes\mathbb{I},\ \gamma^1 = \sigma^1\otimes\sigma^1\otimes\sigma^1,\ \gamma^2 = \sigma^1\otimes\sigma^1\otimes\sigma^3,\ \gamma^3 = \sigma^1\otimes\sigma^3\otimes\mathbb{I},\ \gamma^4 = \sigma^3\otimes\mathbb{I}\otimes\mathbb{I},\ \gamma^5 = \sigma^1\otimes\sigma^1\otimes i\sigma^2.$$

The chirality matrix is then:$\gamma^6 = -i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5 = \sigma^1\otimes\sigma^2\otimes\mathbb{I}$ is imaginary, as should be expected since $4-2 \neq 0 \mod 8$. Multiplying γ^6 by i, we can however create a Majorana representation in 4+3 dimensions. For 3+3 dimensions, we just have to exchange the respective roles of γ^6 and γ^4, to obtain the desired Majorana-Weyl representation. In 6 Euclidean dimensions, however, the Majorana representation is purely imaginary and can be taken as:

$$\gamma^1 = \sigma^2\otimes i\sigma^2\otimes i\sigma^2,\ \gamma^2 = \sigma^2\otimes\sigma^1\otimes\mathbb{I},\ \gamma^3 = \sigma^2\otimes\sigma^3\otimes\mathbb{I},\ \gamma^4 = i\sigma^1\otimes\mathbb{I}\otimes i\sigma^2,\ \gamma^5 = i\sigma^3\otimes\mathbb{I}\otimes i\sigma^2,\ \gamma^6 = \mathbb{I}\otimes i\sigma^2\otimes\sigma^1\ .$$

It leads to a chirality matrix $\gamma^7 = \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6 = \mathbb{I}\otimes\sigma^2\otimes\sigma^3$, which is imaginary, too, giving us a Majorana representation in 7 Euclidean dimension as well.

In 7+1 dimensions, the Majorana representation is again purely imaginary and can be chosen as:

$$\gamma^0 = i\sigma^1\otimes i\sigma^2\otimes\sigma^1\otimes i\sigma^2,\ \gamma^1 = i\mathbb{I}\otimes\sigma^1\otimes i\sigma^2\otimes\sigma^3,\ \gamma^2 = i\mathbb{I}\otimes\sigma^1\otimes\mathbb{I}\otimes i\sigma^2,\ \gamma^3 = i\mathbb{I}\otimes\sigma^3\otimes\sigma^1\otimes i\sigma^2,$$
$$\gamma^4 = \sigma^2\otimes i\sigma^2\otimes\sigma^1\otimes i\sigma^2,\ \gamma^5 = i\sigma^3\otimes\sigma^3\otimes\sigma^3\otimes i\sigma^2,\ \gamma^6 = i\sigma^3\otimes\sigma^3\otimes i\sigma^2\otimes\mathbb{I},\ \gamma^7 = i\sigma^3\otimes i\sigma^2\otimes\mathbb{I}\otimes\mathbb{I}\ .$$

In that case, the chirality matrix is: $\gamma^8 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6\gamma^7 = \mathbb{I}\otimes\sigma^1\otimes\sigma^2\otimes\sigma^1$, imaginary again, so that we have a Majorana representation in 8+1 dimensions, too. In 8 Euclidean dimensions, however, we can find both real and imaginary Majorana representations. The imaginary case is obvious from cancelling γ^0 above, so let's turn to the real case. We can choose:

$$\gamma^1 = \sigma^1\otimes\mathbb{I}\otimes\mathbb{I}\otimes\sigma^1,\ \gamma^2 = \sigma^1\otimes\mathbb{I}\otimes\mathbb{I}\otimes\sigma^3,\ \gamma^3 = \mathbb{I}\otimes\mathbb{I}\otimes i\sigma^2\otimes i\sigma^2,\ \gamma^4 = \mathbb{I}\otimes i\sigma^2\otimes\sigma^3\otimes i\sigma^2,$$
$$\gamma^5 = \sigma^1\otimes i\sigma^2\otimes\sigma^1\otimes i\sigma^2,\ \gamma^6 = \sigma^3\otimes\mathbb{I}\otimes\sigma^1\otimes\mathbb{I},\ \gamma^7 = i\sigma^2\otimes i\sigma^2\otimes\sigma^1\otimes\mathbb{I},\ \gamma^8 = \sigma^3\otimes\sigma^1\otimes\sigma^3\otimes\mathbb{I}\ ,$$

so that the chirality matrix will be: $\gamma^9 = \gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6\gamma^7\gamma^8 = \sigma^3\otimes\sigma^3\otimes\sigma^3\otimes\mathbb{I}$, which also gives the Majorana representation in 9 Euclidean dimensions, which is used extensively throughout this thesis. For completeness, let us mention the existence of a real Majorana representation in 5+3 dimensions:

$$\gamma^1 = \sigma^1\otimes\sigma^1\otimes\sigma^1\otimes\sigma^1,\ \gamma^2 = \sigma^1\otimes\sigma^1\otimes\sigma^1\otimes\sigma^3,\ \gamma^3 = \sigma^1\otimes\sigma^1\otimes\sigma^3\otimes\mathbb{I},\ \gamma^4 = \sigma^1\otimes\sigma^3\otimes\mathbb{I}\otimes\mathbb{I},$$
$$\gamma^5 = \sigma^3\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I},\ \gamma^6 = \sigma^1\otimes\sigma^1\otimes\sigma^1\otimes i\sigma^2,\ \gamma^7 = \sigma^1\otimes\sigma^1\otimes i\sigma^2\otimes\mathbb{I},\ \gamma^8 = \sigma^1\otimes i\sigma^2\otimes\mathbb{I}\otimes\mathbb{I}\ ,$$

with an imaginary chirality matrix: $\gamma^9 = -i\gamma^1\gamma^2\gamma^3\gamma^4\gamma^5\gamma^6\gamma^7\gamma^8 = \sigma^2\otimes\mathbb{I}\otimes\mathbb{I}\otimes\mathbb{I}$, which illustrates the absence of Majorana representations in 6+3 dimensions. However, multiplied by i, it allows us to give a real Majorana representation in 5+4 dimensions, which of course, also exists in 4+4 dimensions by elimination of one of the first five γ's.

Since these reality properties are periodic modulo 8, these basic representations allow to construct all higher-dimensional representations by a simple recursive method. Before illustrating this fact, let us recall the general picture we could exhibit:

If there are s space-like dimensions and t time-like dimensions, there exists a Majorana representation of the Clifford algebra $\{\gamma^\mu, \gamma^\mu\} = 2\eta^{\mu\nu}\mathbb{I}$ of $SO(s,t)$ in which all matrices can be chosen to be:

$$\begin{cases} \text{real} & \text{if } s-t = 0, 1 \text{ or } 2 \mod 8\ , \\ \text{imaginary} & \text{if } s-t = 0, 6, \text{ or } 7 \mod 8\ , \end{cases} \tag{A.6}$$

if the metric $\eta^{\mu\nu}$ is chosen to have s positive and t negative eigenvalues (if one does the opposite choice, the reversed rules will apply).

Moreover, given a Majorana representation for the Clifford algebra of $SO(s,t)$, we can obtain immediately the corresponding representation for $SO(t,s)$ by multiplying by i all γ matrices.

Let us now show how we can obtain a real (resp. imaginary) Majorana representation of the Clifford algebra of $SO(8+s,t)$ from the ones for $SO(s,t)$ and $SO(9)$ (resp. $SO(8,1)$). If the Majorana representation for $SO(s,t)$ is real, denoting $SO(s,t)$ real Dirac matrices by $\{\gamma^\alpha\}_{\alpha=0}^{s+t-1}$ and $SO(9)$ real Dirac matrices by $\{\gamma^i\}_{i=1}^{9}$, the desired representation is obtained as:

$$\Gamma^\alpha = \gamma^\alpha \otimes \gamma^9 \,\forall\, \alpha = 0,\ldots,s+t-1\;, \quad \Gamma^{s+t-1+i} = \mathbb{1} \otimes \gamma^i \,\forall\, i = 1,\ldots,8\;. \tag{A.7}$$

On the other hand, if the Majorana representation for $SO(s,t)$ is imaginary, denoting $SO(s,t)$ imaginary Dirac matrices by $\{\gamma^\alpha\}_{\alpha=0}^{s+t-1}$ and $SO(8,1)$ imaginary Dirac matrices by $\{\gamma^i\}_{i=0}^{8}$, we can choose:

$$\Gamma^\alpha = \gamma^\alpha \otimes i\gamma^0 \,\forall\, \alpha = 0,\ldots,s+t-1\;, \quad \Gamma^{s+t-1+i} = \mathbb{1} \otimes \gamma^i \,\forall\, i = 1,\ldots,8\;. \tag{A.8}$$

Majorana representations of even bigger Clifford algebras can be constructed along the same lines, but we won't detail it here.

Appendix B

Properties of Majorana fermions in $10D$ and various identities

B.1 Majorana-Weyl spinors in 9+1 dimensions

The formulæ below are valid for $\eta_{\mu\nu} = diag(-1,+1,\ldots,+1)$ and anti-commuting (grassmannian) spinors. Complex conjugation is determined by the matrix B, chosen as the product of all imaginary Dirac matrices. Consequently, it satisfies:

$$B^2 = \mathbb{1}\ , \quad B^\top = B\ , \quad B\Gamma^\mu B^{-1} = (\Gamma^\mu)^*\ , \quad B\Gamma^{\mu_1\cdots\mu_n}B^{-1} = (\Gamma^{\mu_1\cdots\mu_n})^* \tag{B.1}$$

That operator allows us to define charge conjugation of a spinor by:

$$\Psi^C = B\Psi^* \tag{B.2}$$

In particular, Majorana spinors are self-conjugate, thus they satisfy:

$$\Psi = B\Psi^* \tag{B.3}$$

Consequently, the charge conjugation matrix $C = B\Gamma^0$ act as:

$$\Psi^C = C\overline{\Psi}^\top\ , \tag{B.4}$$

so that Majorana spinors satisfy:

$$\Psi = C\overline{\Psi}^\top\ , \quad \overline{\Psi} = \Psi^\top C\ . \tag{B.5}$$

C determines the transposition properties of the Dirac matrices as:

$$C^2 = -\mathbb{1}\ , \quad C^\top = -C\ , \quad C\Gamma^\mu C^{-1} = -(\Gamma^\mu)^\top\ , \quad C\Gamma^{\mu_1\cdots\mu_n}C^{-1} = (-1)^{\frac{n(n+1)}{2}}(\Gamma^{\mu_1\cdots\mu_n})^\top\ . \tag{B.6}$$

Finally, Hermitian conjugation is determined by $\Gamma^0 = BC$ as:

$$(\Gamma^0)^2 = -\mathbb{1}\ , \quad \Gamma^0\Gamma^\mu(\Gamma^0)^{-1} = -(\Gamma^\mu)^\dagger\ . \quad \Gamma^0\Gamma^{\mu_1\cdots\mu_n}(\Gamma^0)^{-1} = (-1)^{\frac{n(n+1)}{2}}(\Gamma^{\mu_1\cdots\mu_n})^\dagger\ . \tag{B.7}$$

We can now determine which Majorana fermion bilinears are symmetric or anti-symmetric.

$$\bar{\epsilon}\Gamma^{\mu_1\dots\mu_n}\lambda = \epsilon^\top C\Gamma^{\mu_1\dots\mu_n}C(\Gamma^0)^\top(\lambda^\dagger)^\top = -\epsilon^\top\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}(-1)^{|\sigma|}C\Gamma^{\mu_{\sigma(1)}}C^{-1}\dots C\Gamma^{\mu_{\sigma(n)}}C^{-1}(\Gamma^0)^\top(\lambda^\dagger)^\top =$$
$$= -\epsilon^\top\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}(-1)^{|\sigma|+n}(\Gamma^{\mu_{\sigma(1)}})^\top\dots(\Gamma^{\mu_{\sigma(n)}})^\top(\Gamma^0)^\top(\lambda^\dagger)^\top =(\lambda^\dagger\Gamma^0\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}(-1)^{|\sigma|+n}\Gamma^{\mu_{\sigma(n)}}\dots\Gamma^{\mu_{\sigma(1)}}\epsilon)^\top$$
$$= (-1)^{n+\frac{n(n-1)}{2}}(\bar{\lambda}\Gamma^{\mu_1\dots\mu_n}\epsilon)^\top = (-1)^{\frac{n(n+1)}{2}}\bar{\lambda}\Gamma^{\mu_1\dots\mu_n}\epsilon\ . \tag{B.8}$$

Similarly, one can prove that:

$$\epsilon^\dagger\Gamma^{\mu_1\dots\mu_n}\lambda = \begin{cases} (-1)^{\frac{n(n-1)}{2}}\lambda^\dagger\Gamma^{\mu_1\dots\mu_n}\epsilon & \text{if } \exists i \text{ s.t. } \mu_i = 0 \\ (-1)^{\frac{(n+1)(n+2)}{2}}\lambda^\dagger\Gamma^{\mu_1\dots\mu_n}\epsilon & \text{if} \mu_i \neq 0\,\forall i \end{cases} \tag{B.9}$$

With those relations at hand, we can look at the Hermiticity properties of Majorana fermion bilinears. Indeed,

$$(\bar{\epsilon}\Gamma^{\mu_1\dots\mu_n}\lambda)^\dagger = \lambda^\dagger(\Gamma^{\mu_1\dots\mu_n})^\dagger(\Gamma^0)^\dagger\epsilon = -(-1)^{\frac{n(n+1)}{2}}\lambda^\dagger\Gamma^0\Gamma^{\mu_1\dots\mu_n}(\Gamma^0)^{-1}\Gamma^0\Gamma^0(\Gamma^0)^{-1}\epsilon =$$
$$= -(-1)^{\frac{n(n+1)}{2}}\lambda^\dagger\Gamma^0\Gamma^{\mu_1\dots\mu_n}\epsilon = -\bar{\epsilon}\Gamma^{\mu_1\dots\mu_n}\lambda \tag{B.10}$$

Similarly, one can also prove that:

$$(\epsilon^\dagger\Gamma^{\mu_1\dots\mu_n}\lambda)^\dagger = -\epsilon^\dagger\Gamma^{\mu_1\dots\mu_n}\lambda\ , \tag{B.11}$$

so that all Majorana fermion bilinears are anti-hermitian.

B.2 Weyl spinors

In all even dimensions, we can define a chirality operator as the product of all even-dimensional Dirac matrices. In 10D, we can define:

$$\Gamma_* = \pm\Gamma^0\cdot\ldots\cdot\Gamma^9 \tag{B.12}$$

so that $\{\Gamma_*, \Gamma^\mu\} = 0\,\forall\mu$. which differentiates left-handed fermions: $\Gamma_*\Psi_L = \Psi_L$ from right-handed fermions satisfying $\Gamma_*\Psi_R = -\Psi_R$. It is also customary to define left-handed and right-handed chiral projectors as:

$$P_L = \frac{1+\Gamma_*}{2}\ , \qquad P_R = \frac{1-\Gamma_*}{2}\ , \tag{B.13}$$

which of course satisfy:

$$P_L^2 = P_L\ , \qquad P_R^2 = P_R\ , \qquad P_LP_R = P_RP_L = 0 \tag{B.14}$$
$$\Gamma^\mu P_L = P_R\Gamma^\mu\ , \qquad \Gamma^\mu P_R = P_L\Gamma^\mu \tag{B.15}$$

This implies that certain Weyl fermion bilinears are null. Explicitly:

$$\overline{\epsilon_{L/R}}\Gamma^{\mu_1\dots\mu_k}\lambda_{L/R} = \overline{\epsilon_{L/R}}P_{R/L}\Gamma^{\mu_1\dots\mu_k}P_{L/R}\lambda_{L/R} = \tag{B.16}$$
$$= \overline{\epsilon_{L/R}}\Gamma^{\mu_1\dots\mu_k}\frac{1\pm(-1)^{k+1}\Gamma_*}{2}\frac{1\pm\Gamma_*}{2}\lambda_{L/R} = \begin{cases} 0 & \text{if } k \text{ is even} \\ \overline{\epsilon_{L/R}}\Gamma^{\mu_1\dots\mu_k}\lambda_{L/R} & \text{if } k \text{ is odd} \end{cases} \tag{B.17}$$

Similarly, one easily shows that:

$$\overline{\epsilon_{L/R}}\Gamma^{\mu_1\cdots\mu_k}\lambda_{R/L} = \begin{cases} \overline{\epsilon_{L/R}}\Gamma^{\mu_1\cdots\mu_k}\lambda_{R/L} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases} \tag{B.18}$$

B.3 Dualization of Dirac matrices

A consequence of the definition of the chirality operator as the product of all Dirac matrices is that:

$$\Gamma_*\Gamma^{\mu_1\cdots\mu_k} = \pm\frac{(-1)^{\frac{(k-1)(k-2)}{2}}}{(10-k)!}\epsilon^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_{10-k}}\Gamma^{\nu_1\cdots\nu_{10-k}} \ , \tag{B.19}$$

if we choose $\epsilon^{01\ldots9} = -\epsilon_{01\ldots9} = +1$, the $\pm$ being determined by the choice of sign in the definition of Γ_* above. Of course, since:

$$\epsilon^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_{10-k}}\epsilon_{\mu_1\cdots\mu_k}{}^{\rho_1\cdots\rho_{10-k}} = -k!(10-k)!\delta^{\rho_1\cdots\rho_{10-k}}_{[\nu_1\ldots\nu_{10-k}]} \ , \tag{B.20}$$

double dualization amounts to identity:

$$\Gamma^{\mu_1\cdots\mu_k} = \pm\frac{(-1)^{\frac{(k-1)(k-2)}{2}}}{(10-k)!}\epsilon^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_{10-k}}\cdot(\pm)\Gamma_*\frac{(-1)^{\frac{(9-k)(8-k)}{2}}}{k!}\epsilon^{\nu_1\ldots\nu_{10-k}}{}_{\rho_1\ldots\rho_k}\Gamma_*\Gamma^{\rho_1\cdots\rho_k} = \tag{B.21}$$

$$= -(-1)^{\frac{(k-1)(k-2)+k(k-1)}{2}+k(10-k)}\Gamma^{\mu_1\cdots\mu_k} = \Gamma^{\mu_1\cdots\mu_k} \ . \tag{B.22}$$

As is usual, defining the dual Gamma matrix as:

$$(*\Gamma)^{\mu_1\cdots\mu_{10-k}} = \pm\frac{(-1)^{\frac{k(k-1)}{2}}}{k!}\epsilon^{\mu_1\cdots\mu_{10-k}}{}_{\nu_1\ldots\nu_k}\Gamma_*\Gamma^{\nu_1\cdots\nu_k} \ , \tag{B.23}$$

we can easily see that $(*\Gamma)^{\mu_1\cdots\mu_5} = \pm\frac{1}{5!}\epsilon^{\mu_1\cdots\mu_5}{}_{\nu_1\ldots\nu_5}\Gamma_*\Gamma^{\nu_1\cdots\nu_5}$, which allows to decompose $\Gamma^{\mu_1\cdots\mu_5}$ in a self-dual and an anti-self-dual part as:

$$\Gamma^{\mu_1\cdots\mu_5} = \frac{1}{2}\{(\Gamma+*\Gamma)^{\mu_1\cdots\mu_5} + (\Gamma-*\Gamma)^{\mu_1\cdots\mu_5}\} = \Gamma_+ + \Gamma_- \ . \tag{B.24}$$

Note that the choice of sign in the definition of Γ_* determines what we call self-dual and anti-self-dual later.

B.4 Fierz identities in 9+1 dimensions

Given the orthogonality relation:

$$Tr(\Gamma^{\mu_1\cdots\mu_k}\Gamma^{\nu_1\cdots\nu_l}) = 32k!(-1)^{\frac{k(k-1)}{2}}\delta^{kl}\eta^{\mu_1[\nu_1}\cdots\eta^{\mu_n\nu_n]} \tag{B.25}$$

we get the completeness relation:

$$\delta^\delta_\alpha\delta^\beta_\gamma = \frac{1}{32}\sum_{k=0}^{10}\frac{(-1)^{\frac{k(k-1)}{2}}}{k!}(\Gamma^{\mu_1\cdots\mu_k})^\beta_\alpha(\Gamma_{\mu_1\ldots\mu_k})^\delta_\gamma \tag{B.26}$$

which implies the Fierz transformation rule:

$$\Psi_\alpha \bar{\epsilon}^\beta = -\frac{1}{32}\sum_{k=0}^{10}\frac{(-1)^{\frac{k(k-1)}{2}}}{k!}(\Gamma^{\mu_1\dots\mu_k})^\beta_\alpha(\bar{\epsilon}\Gamma_{\mu_1\dots\mu_k}\Psi) \tag{B.27}$$

Using the dualization formulæ above, the completeness relation can be reduced to:

$$\begin{aligned}\delta^\delta_\alpha\delta^\beta_\gamma = &\frac{1}{32}\sum_{k=0}^{4}\frac{(-1)^{\frac{k(k-1)}{2}}}{k!}\left[(\Gamma^{\mu_1\dots\mu_k})^\beta_\alpha(\Gamma_{\mu_1\dots\mu_k})^\delta_\gamma + (-1)^k(\Gamma_*\Gamma^{\mu_1\dots\mu_k})^\beta_\alpha(\Gamma_*\Gamma_{\mu_1\dots\mu_k})^\delta_\gamma\right] + \\ &+\frac{1}{32\cdot 5!}(\Gamma^{\mu_1\dots\mu_5})^\beta_\alpha(\Gamma_{\mu_1\dots\mu_5})^\delta_\gamma\ ,\end{aligned} \tag{B.28}$$

which is a useful formulation if we deal with Weyl spinors.

B.5 Fierz relation for the supersymmetry algebra of the IIB matrix model

In this section, we want to prove that:

$$(\overline{\epsilon_{[1}}\Gamma_i\Psi)(\Gamma^{ij}\epsilon_{2]})_\alpha = -(\overline{\epsilon_1}\Gamma^j\epsilon_2)\Psi_\alpha + \frac{7}{16}(\overline{\epsilon_1}\Gamma_i\epsilon_2)(\Gamma^i\Gamma^j\Psi)_\alpha - \frac{1}{32\cdot 5!}(\overline{\epsilon_1}\Gamma^{\mu_1\dots\mu_5}\epsilon_2)(\Gamma_{\mu_1\dots\mu_5}\Gamma^j\Psi)_\alpha\ , \tag{B.29}$$

when all three fermions are Majorana-Weyl left-handed fermions. To do so, we need to compute a few products of Gamma matrices. Since ϵ_1 and ϵ_2 are both left-handed, only terms with odd number of indices will contribute to the sum. To compute the term with one index, note that:

$$\Gamma^k\Gamma^{ij} = \Gamma^{ijk} - \eta^{jk}\Gamma^i + \eta^{ik}\Gamma^j \ \longrightarrow\ \Gamma^k\Gamma^{ij}\otimes\Gamma_k\Gamma_i = \Gamma^{ijk}\otimes\Gamma_{ki} + \Gamma_i\otimes\Gamma^i\Gamma^j + 8\Gamma^j\otimes\mathbb{1} \tag{B.30}$$

To compute the term with three indices, we use:

$$\begin{aligned}\Gamma^{klm}\Gamma^{ij} &= \Gamma^{ijklm} + 6\eta^{[i[k}\Gamma^{j]lm]} - 6\eta^{[i[k}\eta^{j]l}\Gamma^{m]} \ \longrightarrow \\ \Gamma^{klm}\Gamma^{ij}\otimes\Gamma_{klm}\Gamma_i &= \Gamma^{ijklm}\otimes\Gamma_{klmi} - 3\Gamma^{klm}\otimes\Gamma^j{}_{klm} + 21\Gamma^{jlm}\otimes\Gamma_{lm} + 48\Gamma_i\otimes\Gamma^i\Gamma^j - 48\Gamma^j\otimes\mathbb{1}\end{aligned} \tag{B.31}$$

Finally, we wish to compute the term with 5 indices:

$$\begin{aligned}\Gamma^{k_1\dots k_5}\Gamma^{ij} =&\Gamma^{ijk_1\dots k_5} + 10\eta^{[i[k_1}\Gamma^{j]k_2\dots k_5]} - 20\eta^{[i[k_1}\eta^{j]k_2}\Gamma^{k_3\dots k_5]} \ \longrightarrow \\ \Gamma^{k_1\dots k_5}\Gamma^{ij}\otimes\Gamma_{k_1\dots k_5}\Gamma_i =& -5!(\Gamma^{klm}\otimes\Gamma^j{}_{klm} - \Gamma_*\Gamma^{klm}\otimes\Gamma_*\Gamma^j{}_{klm} + \\ &+ 25(\Gamma^{jk_1\dots k_4}\otimes\Gamma_{k_1\dots k_4} - \Gamma_*\Gamma^{jk_1\dots k_4}\otimes\Gamma_*\Gamma_{k_1\dots k_4})\end{aligned} \tag{B.32}$$

which allows to write the Fierz formula above as:

$$
\begin{aligned}
(\overline{\epsilon_1}\Gamma_i\Psi)(\Gamma^{ij}\epsilon_2)_\alpha = & -\frac{1}{32}\Big[(\overline{\epsilon_1}\Gamma^k\Gamma^{ij}\epsilon_2)(\Gamma_k\Gamma_i\Psi)_\alpha - (\overline{\epsilon_1}\Gamma_*\Gamma^k\Gamma^{ij}\epsilon_2)(\Gamma_*\Gamma_k\Gamma_i\Psi)_\alpha - \\
& -\frac{1}{6}(\overline{\epsilon_1}\Gamma^{klm}\Gamma^{ij}\epsilon_2)(\Gamma_{klm}\Gamma_i\Psi)_\alpha + \frac{1}{6}(\overline{\epsilon_1}\Gamma_*\Gamma^{klm}\Gamma^{ij}\epsilon_2)(\Gamma_*\Gamma_{klm}\Gamma_i\Psi)_\alpha \\
& +\frac{1}{5!}(\overline{\epsilon_1}\Gamma^{k_1\ldots k_5}\Gamma^{ij}\epsilon_2)\ (\Gamma_{k_1\ldots k_5}\Gamma_i\Psi)_\alpha\Big] = \qquad \text{(B.33)} \\
= & -\frac{1}{16}\left[(\overline{\epsilon_1}\Gamma^k\Gamma^{ij}\epsilon_2)(\Gamma_k\Gamma_i\Psi)_\alpha - \frac{1}{6}(\overline{\epsilon_1}\Gamma^{klm}\Gamma^{ij}\epsilon_2)\ (\Gamma_{klm}\Gamma_i\Psi)_\alpha + \frac{1}{2\cdot 5!}(\overline{\epsilon_1}\Gamma^{k_1\ldots k_5}\Gamma^{ij}\epsilon_2)(\Gamma_{k_1\ldots k_5}\Gamma_i\Psi)_\alpha\right] = \\
= & -\frac{1}{16}\Big[(\overline{\epsilon_1}\Gamma^{ijk}\epsilon_2)(\Gamma_{ki}\Psi)_\alpha + (\overline{\epsilon_1}\Gamma^i\epsilon_2)(\Gamma_i\Gamma^j\Psi)_\alpha + 8(\overline{\epsilon_1}\Gamma^j\epsilon_2)\Psi_\alpha - \frac{1}{6}(\overline{\epsilon_1}\Gamma^{jk_1\ldots k_4}\epsilon_2)(\Gamma_{k_1\ldots k_4}\Psi)_\alpha + \\
& +\frac{1}{2}(\overline{\epsilon_1}\Gamma^{klm}\epsilon_2)({\Gamma^j}_{klm}\Psi)_\alpha - \frac{7}{2}(\overline{\epsilon_1}\Gamma^{ijk}\epsilon_2)(\Gamma_{ki}\Psi)_\alpha - 8(\overline{\epsilon_1}\Gamma^i\epsilon_2)(\Gamma_i\Gamma^j\Psi)_\alpha + 8(\overline{\epsilon_1}\Gamma^j\epsilon_2)\Psi_\alpha - \\
& -(\overline{\epsilon_1}\Gamma^{klm}\epsilon_2)({\Gamma^j}_{klm}\Psi)_\alpha + \frac{5}{4!}(\overline{\epsilon_1}\Gamma^{jk_1\ldots k_4}\epsilon_2)(\Gamma_{k_1\ldots k_4}\Psi)_\alpha\Big] = \\
= & -\frac{1}{16}\Big[-\frac{5}{2}(\overline{\epsilon_1}\Gamma^{ijk}\epsilon_2)(\Gamma_{ki}\Psi)_\alpha - 7(\overline{\epsilon_1}\Gamma^i\epsilon_2)(\Gamma_i\Gamma^j\Psi)_\alpha + 16(\overline{\epsilon_1}\Gamma^j\epsilon_2)\Psi_\alpha - \frac{1}{2}(\overline{\epsilon_1}\Gamma^{klm}\epsilon_2)({\Gamma^j}_{klm}\Psi)_\alpha \\
& +\frac{1}{24}(\overline{\epsilon_1}\Gamma^{jk_1\ldots k_4}\epsilon_2)(\Gamma_{k_1\ldots k_4}\Psi)_\alpha\ \Big]
\end{aligned}
$$

On the other hand:

$$\frac{1}{5!}\Gamma^{k_1\ldots k_5}\otimes\Gamma_{k_1\ldots k_5}\Gamma^j = \frac{1}{4!}(\Gamma^{jk_1\ldots k_4}\otimes\Gamma_{k_1\ldots k_4} - \Gamma_*\Gamma^{jk_1\ldots k_4}\otimes\Gamma_*\Gamma_{k_1\ldots k_4}) \qquad \text{(B.34)}$$

and:

$$\frac{1}{3!}\Gamma^{klm}\otimes\Gamma_{klm}\Gamma^j = \frac{1}{3!}\Gamma^{klm}\otimes{\Gamma_{klm}}^j + \frac{1}{2}\Gamma^{jkl}\otimes\Gamma_{kl} \qquad \text{(B.35)}$$

so that:

$$
\begin{aligned}
(\overline{\epsilon_1}\Gamma_i\Psi)(\Gamma^{ij}\epsilon_2)_\alpha = & -(\overline{\epsilon_1}\Gamma^j\epsilon_2)\Psi_\alpha + \frac{1}{16}\Big[7(\overline{\epsilon_1}\Gamma^i\epsilon_2)(\Gamma_i\Gamma^j\Psi)_\alpha + \frac{5}{3!}(\overline{\epsilon_1}\Gamma^{klm}\epsilon_2)(\Gamma_{klm}\Gamma^j\Psi)_\alpha + \\
& -\frac{1}{3}(\overline{\epsilon_1}\Gamma^{klm}\epsilon_2)({\Gamma_{klm}}^j\Psi)_\alpha - \frac{1}{2\cdot 5!}(\overline{\epsilon_1}\Gamma^{k_1\ldots k_5}\epsilon_2)\ (\Gamma_{k_1\ldots k_5}\Gamma^j\Psi)_\alpha\Big]\ . \qquad \text{(B.36)}
\end{aligned}
$$

Since ϵ_1 and ϵ_2 are Majorana, the terms with a 3-indices Dirac matrix $(\overline{\epsilon_1}\Gamma^{(3)}\epsilon_2)$ vanish when we anti-symmetrize on $(1 \leftrightarrow 2)$, giving the desired expression.

B.6 Fierz relation to prove supersymmetry of the SYM action in 10D

Here, we want to prove that the following relation holds:

$$\epsilon_{abc}(\bar{\Psi}^a\Gamma^\mu\Psi^b)(\Gamma^\mu\Psi^c)_\alpha = 0 \qquad \text{(B.37)}$$

for any triplet of left-handed (9+1)-dimensional Majorana spinors. Using B.28, the $(a \leftrightarrow b)$ anti-symmetry and the Majorana and Weyl properties of fermions, only terms with 1 or 5 indices survive in the completeness relation and we obtain:

$$\epsilon_{abc}(\bar{\Psi^b}\Gamma^\mu\Psi^c)(\Gamma_\mu\Psi^a)_\alpha = \frac{1}{2}\epsilon_{abc}\Big((\bar{\Psi^b}\Gamma^\mu\Psi^a)(\Gamma_\mu\Psi^c)_\alpha - \frac{1}{32\cdot 6}(\bar{\Psi^b}\Gamma^{\mu_1\ldots\mu_4\mu}\Psi^a)(\Gamma_{\mu_1\ldots\mu_4\mu}\Psi^c)_\alpha\Big) \ . \tag{B.38}$$

However, we still haven't used the $(b \leftrightarrow c)$ anti-symmetry. For that sake, we multiply Ψ^c by the chiral projector $\mathcal{P}_L$, write explicitly the 3 different terms in the anti-symmetrization, and "divide" by Ψ^c in every term on the L.H.S., which gives the matrix:

$$-\frac{1}{3}\epsilon_{abc}\Big((\bar{\Psi^b}\Gamma^\mu\mathcal{P}_L)^\beta(\Gamma_\mu\Psi^a)_\alpha + (\bar{\Psi^a}\Gamma^\mu\mathcal{P}_L)^\beta(\Gamma_\mu\Psi^b)_\alpha + (\bar{\Psi^a}\Gamma^\mu\Psi^b)(\Gamma^\mu\mathcal{P}_L)^\beta_\alpha)\Big) \ . \tag{B.39}$$

To extract the 1-index part of the development of this matrix on the complete basis of 10-dimensional Dirac matrices, we multiply it with $(\Gamma^\rho)^\alpha_\beta$ (making use of the orthogonality property B.25) and we get:

$$\begin{aligned}&-\frac{1}{3}\epsilon_{abc}\Big((\bar{\Psi^b}\Gamma^\mu\mathcal{P}_L\Gamma^\rho\Gamma_\mu\Psi^a) + (\bar{\Psi^a}\Gamma^\mu\mathcal{P}_L\Gamma^\rho\Gamma_\mu\Psi^b) + (\bar{\Psi^a}\Gamma^\mu\Psi^b)Tr(\Gamma^\mu\mathcal{P}_L\Gamma^\rho)\Big) = \\ &=\frac{1}{3}\epsilon_{abc}(-8-8+16)(\bar{\Psi^a}\Gamma^\rho\Psi^b) = 0 \ .\end{aligned}$$

since $\Gamma^\mu\Gamma^\rho\Gamma_\mu = -8\Gamma^\rho$. Similarly, using: $\Gamma^\mu\Gamma^{\nu_1\ldots\nu_5}\Gamma_\mu = 0$ and the orthogonality property B.25, it is trivial to show that the 5-indices part vanishes as well, ending the proof.

B.7 Fierz relation to compute the supersymmetry algebra of matrix theory

Starting from the Fierz relation of the preceding paragraph:

$$\epsilon_{abc}(\bar{\Psi^a}\Gamma^\mu\Psi^b)(\Gamma^\mu\Psi^c)_\alpha = 0 \tag{B.40}$$

and cancelling the Ψ's we can write it as:

$$\sum_{\mathcal{S}_3}[(\mathcal{P}_L\Gamma^0\Gamma^\mu)_{\alpha\beta}(\mathcal{P}_L\Gamma^0\Gamma_\mu)_{\gamma\delta}] = 0 \ , \tag{B.41}$$

or equivalently:

$$\sum_{\mathcal{S}_3}[(\mathcal{P}_L\Gamma^0\Gamma^i)_{\alpha\beta}(\mathcal{P}_L\Gamma^0\Gamma_i)_{\gamma\delta} - (\mathcal{P}_L)_{\alpha\beta}(\mathcal{P}_L)_{\gamma\delta}] = 0 \ , \tag{B.42}$$

where the sum is taken on all cyclic permutations of β, γ and δ and roman letters denote space-like indices. If we multiply that equation with $(\Gamma^0\Gamma^j)_{\epsilon\alpha}$, use $\Gamma^\mu\Gamma^\nu = \Gamma^{\mu\nu} + \eta^{\mu\nu}\mathbb{1}$ and relabel the indices, we get:

$$\begin{aligned}&-(\mathcal{P}_L\Gamma^{ij})_{\alpha\beta}(\mathcal{P}_L\Gamma^0\Gamma_i)_{\gamma\delta} - (\mathcal{P}_L\Gamma^{ij})_{\alpha\delta}(\mathcal{P}_L\Gamma^0\Gamma_i)_{\beta\gamma} - (\mathcal{P}_L\Gamma^{ij})_{\alpha\gamma}(\mathcal{P}_L\Gamma^0\Gamma_i)_{\delta\beta} + \\ &+(\mathcal{P}_L)_{\alpha\beta}(\mathcal{P}_L\Gamma^0\Gamma^j)_{\gamma\delta} + (\mathcal{P}_L)_{\alpha\delta}(\mathcal{P}_L\Gamma^0\Gamma^j)_{\beta\gamma} + (\mathcal{P}_L)_{\alpha\gamma}(\mathcal{P}_L\Gamma^0\Gamma^j)_{\delta\beta} - \\ &-(\mathcal{P}_L\Gamma^0\Gamma^j)_{\alpha\beta}(\mathcal{P}_L)_{\gamma\delta} - (\mathcal{P}_L\Gamma^0\Gamma^j)_{\alpha\delta}(\mathcal{P}_L)_{\beta\gamma} - (\mathcal{P}_L\Gamma^0\Gamma^j)_{\alpha\gamma}(\mathcal{P}_L)_{\delta\beta} = 0 \ .\end{aligned} \tag{B.43}$$

On the other hand, we could also multiply it by the symmetric matrix $(\Gamma^0\Gamma^j)_{\gamma\epsilon}$, which gives

$$\begin{aligned}-(\mathcal{P}_L\Gamma^0\Gamma_i)_{\alpha\beta}(\mathcal{P}_L\Gamma^{ij})_{\gamma\delta} + (\mathcal{P}_L\Gamma^0\Gamma_i)_{\alpha\delta}(\mathcal{P}_L\Gamma^{ij})_{\beta\gamma} + (\mathcal{P}_L\Gamma^{ij})_{\alpha\gamma}(\mathcal{P}_L\Gamma^0\Gamma_i)_{\delta\beta} +\\ +(\mathcal{P}_L\Gamma^0\Gamma^j)_{\alpha\beta}(\mathcal{P}_L)_{\gamma\delta} + (\mathcal{P}_L\Gamma^0\Gamma^j)_{\alpha\delta}(\mathcal{P}_L)_{\beta\gamma} + (\mathcal{P}_L)_{\alpha\gamma}(\mathcal{P}_L\Gamma^0\Gamma^j)_{\delta\beta} -\\ -(\mathcal{P}_L)_{\alpha\beta}(\mathcal{P}_L\Gamma^0\Gamma^j)_{\gamma\delta} - (\mathcal{P}_L)_{\alpha\delta}(\mathcal{P}_L\Gamma^0\Gamma^j)_{\beta\gamma} - (\mathcal{P}_L\Gamma^0\Gamma^j)_{\alpha\gamma}(\mathcal{P}_L)_{\delta\beta} = 0\ .\end{aligned} \tag{B.44}$$

Adding these two expressions, we obtain:

$$\begin{aligned}-(\mathcal{P}_L\Gamma^{ij})_{\alpha\beta}(\mathcal{P}_L\Gamma^0\Gamma_i)_{\gamma\delta} - (\mathcal{P}_L\Gamma^0\Gamma_i)_{\alpha\beta}(\mathcal{P}_L\Gamma^{ij})_{\gamma\delta} - (\mathcal{P}_L\Gamma^{ij})_{\alpha\delta}(\mathcal{P}_L\Gamma^0\Gamma_i)_{\beta\gamma} + (\mathcal{P}_L\Gamma^0\Gamma_i)_{\alpha\delta}(\mathcal{P}_L\Gamma^{ij})_{\beta\gamma} +\\ + 2(\mathcal{P}_L)_{\alpha\gamma}(\mathcal{P}_L\Gamma^0\Gamma^j)_{\delta\beta} - 2(\mathcal{P}_L\Gamma^0\Gamma^j)_{\alpha\gamma}(\mathcal{P}_L)_{\delta\beta} = 0\ ,\end{aligned} \tag{B.45}$$

or equivalently:

$$(\mathcal{P}_L\Gamma^{ij})_{\alpha\beta}(\mathcal{P}_L\Gamma^0\Gamma_i)_{\gamma\delta} + (\mathcal{P}_L\Gamma^{ij})_{\gamma\delta}(\mathcal{P}_L\Gamma^0\Gamma_i)_{\alpha\beta} + (\beta \leftrightarrow \delta) = 2\left((\mathcal{P}_L)_{\alpha\gamma}(\mathcal{P}_L\Gamma^0\Gamma^j)_{\delta\beta} - (\mathcal{P}_L)_{\delta\beta}(\mathcal{P}_L\Gamma^0\Gamma^j)_{\alpha\gamma}\right)\ .$$

This is the expression we need for the computation of the supersymmetry algebra.

Appendix C

Proof of the supersymmetry of the BFSS matrix model

In this appendix, we detail the proof of the supersymmetry of the BFSS matrix model, keeping all total derivatives, in such a way that the Noether supercharges can be read off easily.

C.1 Dynamical supersymmetry

Acting on the BFSS action:

$$S = \frac{1}{g^2}\int Tr\left(\frac{1}{2}([\mathcal{D}_t, X_i])^2 - \frac{i}{2}\psi^\dagger[\mathcal{D}_t,\psi] + \frac{1}{4}([X_i,X_j])^2 + \frac{1}{2}\bar{\psi}\Gamma^i[X_i,\psi]\right) dt\ . \tag{C.1}$$

with the supersymmetry transformations:

$$\delta_\epsilon X_i = \frac{i}{2}\bar{\epsilon}\Gamma_i\Psi\ , \qquad \delta_\epsilon A = -\frac{i}{2}\epsilon^\dagger\Psi \tag{C.2}$$

$$\delta_\epsilon\Psi = \frac{1}{2}\left([\mathcal{D}_t, X_i]\Gamma^0\Gamma^i - \frac{i}{2}[X_i,X_j]\Gamma^{ij}\right)\epsilon\ , \qquad \delta_\epsilon\bar{\Psi} = \frac{1}{2}\left(\epsilon^\dagger\Gamma_i[\mathcal{D}_t, X_i] + \frac{i}{2}\bar{\epsilon}\Gamma^{ij}[X_i,X_j]\right)\ , \tag{C.3}$$

we obtain the four following pieces:

$$\begin{aligned}
Tr\left(\delta([\mathcal{D}_t,X_i])^2\right) &= Tr\left(\epsilon^\dagger[\Psi, X^i][\mathcal{D}_t, X_i] + i[\mathcal{D}_t,\bar{\epsilon}\Gamma^i\Psi][\mathcal{D}_t,X_i]\right) = T_1 + T_2\ ,\\
\frac{1}{2}Tr\left(\delta([X_i,X_j])^2\right) &= Tr\left(i\bar{\epsilon}\Gamma_i[\Psi,X_j][X^i,X^j]\right) = T_3\ ,\\
-iTr\left(\delta(\Psi^\dagger[\mathcal{D}_t,\Psi])\right) &= Tr\left(-\frac{i}{2}\bar{\epsilon}\Gamma^i[\mathcal{D}_t,\Psi][\mathcal{D}_t,X_i] + \frac{1}{4}\epsilon^\dagger\Gamma^{ij}[X_i,X_j][\mathcal{D}_t,\Psi] - \frac{i}{2}\Psi^\dagger[(\epsilon^\dagger\Psi),\Psi] - \right.\\
&\left. -\frac{i}{2}\bar{\Psi}\Gamma^i[\mathcal{D}_t,[\epsilon\mathcal{D}_t,X_i]] - \frac{1}{4}\Psi^\dagger\Gamma^{ij}[\mathcal{D}_t,\epsilon[X_i,X_j]]\right) = T_4+T_5+T_6+T_7+T_8\ ,\\
Tr\left(\delta(\bar{\Psi}\Gamma^i[X_i,\Psi])\right) &= Tr\left(-\frac{1}{2}\epsilon^\dagger\Gamma^i\Gamma^j\Psi[X_j,[\mathcal{D}_t,X_i]]) - \frac{i}{4}\bar{\epsilon}\Gamma^{ij}\Gamma^k\Psi[X_k,[X_i,X_j]] + \frac{1}{2}\psi^\dagger\Gamma^i\Gamma^j\epsilon[X_i,[\mathcal{D}_t,X_j]] - \right.\\
&\left. -\frac{i}{4}\bar{\Psi}\Gamma^i\Gamma^{jk}\epsilon[[X_j,X_k],X_i] - \frac{i}{2}(\bar{\Psi}\Gamma^i[\Psi),(\bar{\epsilon}\Gamma_i\Psi)]\right) = T_9+T_{10}+T_{11}+T_{12}+T_{13}\ .
\end{aligned} \tag{C.4}$$

Let's first treat the contributions with two derivatives, T_2, T_4 and T_7. Using properties of Majorana fermions and cyclicity of the trace, these 3 terms add to:

$$T_2 + T_4 + T_7 = \frac{i}{2}\partial_t Tr(\bar{\epsilon}\Gamma^i\Psi[\mathcal{D}_t, X_i]) - iTr(\bar{\Psi}\Gamma^i(\partial_t\epsilon)[\mathcal{D}_t, X_i]) \ . \tag{C.5}$$

The first term is a total derivative and will vanish in δS and the second term is zero for a constant supersymmetry parameter ϵ, showing that the supersymmetry is only global. However, we keep track of such terms to compute the supercharges through the Noether theorem.

Then, let's study terms with one derivative. Besides the tricks already used above, we need to use $\Gamma^i\Gamma^j = \Gamma^{ij} + \delta^{ij}$ and $\Gamma^{ij}[X_i, [\mathcal{D}_t, X_j]] = \frac{1}{2}\Gamma^{ij}[\mathcal{D}_t, [X_i, X_j]]$, in T_9 and T_{11}, which gives us:

$$T_1 + T_5 + T_8 + T_9 + T_{11} = \frac{1}{4}\partial_t Tr(\epsilon^\dagger\Gamma^{ij}\Psi[X_i, X_j]) - \frac{1}{2}Tr(\Psi^\dagger\Gamma^{ij}(\partial_t\epsilon)[X_i, X_j]) \ . \tag{C.6}$$

Then, there are two kinds of terms with zero derivatives. Let us first treat T_3, T_{10} and T_{12}. Using $\Gamma^{ij}\Gamma^k = \Gamma^{ijk} + \Gamma^i\delta^{jk} - \Gamma^j\delta^{ik}$ and $\Gamma^i\Gamma^{jk} = \Gamma^{ijk} + \Gamma^k\delta^{ij} - \Gamma^j\delta^{ik}$, and noting that $\Gamma^{ijk}[X_i, [X_j, X_k]] = 0$ because of the Jacobi identity, these three terms exactly cancel:

$$T_3 + T_{10} + T_{12} = 0 \tag{C.7}$$

Finally:

$$T_6 + T_{13} = \frac{1}{2}f^{abc}[(\bar{\epsilon}\Gamma^i\Psi^a)(\bar{\Psi}^b\Gamma^i\Psi^c) + (\bar{\epsilon}\Gamma^0\Psi^a)(\bar{\Psi}^b\Gamma^0\Psi^c)] = 0 \tag{C.8}$$

for Majorana-Weyl fermions in 9+1 dimensions. It's not surprising to see the appearance of the same Fierz identity needed to prove the supersymmetry of $\mathcal{N} = 1$ super Yang-Mills theory in 10 dimensions, since the BFSS matrix model is just a dimensional reduction of the latter.

C.2 Kinematical supersymmetry

The kinematical supersymmetry is much simpler to prove, since only fermionic terms of the action contribute to $\delta_{\epsilon'}S$. Varying the field as follows:

$$\delta_{\epsilon'}X_i = 0, \quad \delta_{\epsilon'}A = 0, \quad \delta_{\epsilon'}\Psi = \epsilon' \ , \tag{C.9}$$

we obtain the following contributions:

$$\begin{aligned} -iTr\left(\delta(\Psi^\dagger[\mathcal{D}_t, \Psi])\right) &= Tr\left(-i\epsilon'^\dagger[\mathcal{D}_t, \Psi] - i\Psi^\dagger[\mathcal{D}_t, \epsilon']\right) = T_1' + T_2' \ , \\ Tr\left(\delta(\bar{\Psi}\Gamma^i[X_i, \Psi])\right) &= Tr\left(\bar{\epsilon}'\Gamma^i[X_i, \Psi] + (\bar{\Psi}\Gamma^i[X_i, \epsilon']\right) = T_3' + T_4' \ . \end{aligned}$$

Obviously:

$$T_1' + T_2' = -i\partial_t Tr(\epsilon'^\dagger\Psi) - 2iTr(\Psi^\dagger(\partial_t\epsilon')) \ , \tag{C.10}$$

while:

$$T_3' + T_4' = 0 \ . \tag{C.11}$$

Appendix D

Symmetries of the Green-Schwarz superstring

In this appendix, we give further details concerning the diverse symmetries of the Green-Schwarz superstring action, in particular global space-time supersymmetry, local fermionic κ-symmetry and bosonic λ-symmetry.

D.1 $\mathcal{N}=2$ Supersymmetry

We first recall the form of the Green-Schwarz superstring action in our conventions.

$$S_{GS} = \frac{1}{2\pi\alpha'}\int \Big[(h^{\alpha\beta}\pi^\mu_\alpha\pi_{\beta\mu})\sqrt{-h} + 2i\epsilon^{\alpha\beta}\partial_\alpha x^\mu(\bar\theta\Gamma_\mu\partial_\beta\theta + \bar\psi\Gamma_\mu\partial_\beta\psi) + 2\epsilon^{\alpha\beta}(\bar\theta\Gamma^\mu\partial_\alpha\theta)(\bar\psi\Gamma_\mu\partial_\beta\psi)\Big]d^2\sigma \ ,$$

where we have set:

$$\pi^\mu_\alpha = \partial_\alpha x^\mu - i\bar\theta\Gamma^\mu\partial_\alpha\theta + i\bar\psi\Gamma^\mu\partial_\alpha\psi \ . \tag{D.1}$$

The 2 supersymmetry transformations that leave π^μ_α invariant are given by:

$$\begin{aligned} \delta_\epsilon x^\mu = i\bar\epsilon\Gamma^\mu\theta\,, \quad \delta_\xi x^\mu = -i\bar\xi\Gamma^\mu\psi \\ \delta_\epsilon\theta = \epsilon\ , \quad \delta_\xi\psi = \xi\ , \end{aligned} \tag{D.2}$$

for two Grassmann parameters ϵ and ξ that are constant on the world-sheet. Under these transformations, the first term is of course invariant, let us first check how the Chern-Simons term transforms under ϵ:

$$-\pi\alpha'\delta_\epsilon\mathcal{L} = -i\epsilon^{\alpha\beta}\partial_\alpha x^\mu(\bar\epsilon\Gamma_\mu\partial_\beta\theta) + \epsilon^{\alpha\beta}(\bar\epsilon\Gamma^\mu\partial_\alpha\theta)(\bar\theta\Gamma_\mu\partial_\beta\theta) \ .$$

The first term can be written as a total derivative, while we can use integration by part on the second one to transform part of it as in:

$$\epsilon^{\alpha\beta}(\bar\epsilon\Gamma^\mu\partial_\alpha\theta)(\bar\theta\Gamma_\mu\partial_\beta\theta) = \partial_\alpha\big(\epsilon^{\alpha\beta}(\bar\epsilon\Gamma^\mu\theta)(\bar\theta\Gamma_\mu\partial_\beta\theta)\big) - \epsilon^{\alpha\beta}(\bar\epsilon\Gamma^\mu\theta)(\partial_\alpha\bar\theta\Gamma_\mu\partial_\beta\theta)\ , \tag{D.3}$$

so that we can rewrite it (modulo total derivatives) as:

$$\begin{aligned}\epsilon^{\alpha\beta}(\bar\epsilon\Gamma^\mu\partial_\alpha\theta)(\bar\theta\Gamma_\mu\partial_\beta\theta) = \frac{2}{3}\epsilon^{\alpha\beta}(\bar\epsilon\Gamma^\mu\partial_\alpha\theta)(\bar\theta\Gamma_\mu\partial_\beta\theta) - \frac{1}{3}\epsilon^{\alpha\beta}(\bar\epsilon\Gamma^\mu\theta)(\partial_\alpha\bar\theta\Gamma_\mu\partial_\beta\theta) = \\ = \frac{2}{3}\Big((\bar\epsilon\Gamma^\mu\dot\theta)(\bar\theta\Gamma_\mu\theta') + (\bar\epsilon\Gamma^\mu\theta')(\dot{\bar\theta}\Gamma_\mu\theta) + (\bar\epsilon\Gamma^\mu\theta)(\bar\theta'\Gamma_\mu\dot\theta)\Big) = 0 \ ,\end{aligned} \tag{D.4}$$

thanks to the usual Fierz transformation, which also allows to prove supersymmetry in super Yang-Mills and BFSS theory. The proof of the second supersymmetry follows exactly the same pattern, so we will not detail it here.

D.2 κ-symmetry

Then, we turn to the more mysterious local fermionic κ-symmetry of the superstring. Again, we recall the convention we use for the reader's convenience:

$$\begin{aligned}\delta_\kappa x^\mu = i\bar\theta\Gamma^\mu\delta_\kappa\theta - i\bar\psi\Gamma^\mu\delta_\kappa\psi \\ \delta_\kappa\theta = 2\Gamma^\mu\pi_{\alpha\mu}\kappa^{1\alpha} \ , \quad \delta\psi = 2\Gamma^\mu\pi_{\alpha\mu}\kappa^{2\alpha} \\ \delta_\kappa(\sqrt{-h}h^{\alpha\beta}) = -16i\sqrt{-h}(\partial_\gamma\bar\theta\kappa^{1\beta}P_-^{\alpha\gamma} - \partial_\gamma\bar\psi\kappa^{2\beta}P_+^{\alpha\gamma}) \ ,\end{aligned} \tag{D.5}$$

for a pair of Grassmann parameters κ^1 and κ^2 that are world-sheet vectors and target-space spinors. We impose the following "self-duality" conditions on the κ's:

$$P_-^{\alpha\beta}\kappa^1_\beta = \kappa^{1\alpha} \ , \quad P_+^{\alpha\beta}\kappa^2_\beta = \kappa^{2\alpha} \quad .$$

with respect to the following projection operators:

$$P_\pm^{\alpha\beta} = \frac{1}{2}(h^{\alpha\beta} \pm \frac{1}{\sqrt{-h}}\epsilon^{\alpha\beta}) \ .$$

Note that P_- and P_+ are orthogonal projectors since:

$$\begin{aligned}P_\pm^{\alpha\beta}h_{\beta\gamma}P_\pm^{\gamma\delta} = P_\pm^{\alpha\delta} \\ P_+^{\alpha\beta}h_{\beta\gamma}P_-^{\gamma\delta} = P_-^{\alpha\beta}h_{\beta\gamma}P_+^{\gamma\delta} = 0 \ .\end{aligned} \tag{D.6}$$

To prove these relationships, one essentially has to show that:

$$\epsilon^{\alpha\beta}h_{\beta\gamma}\epsilon^{\gamma\delta} = -h\cdot h^{\alpha\beta} \ ,$$

which simply follows from the inversion rule for 2×2 matrices:

$$h^{\alpha\beta} = \frac{1}{h}\begin{pmatrix} h_{11} & -h_{01} \\ -h_{10} & h_{00}\end{pmatrix} \ .$$

Another useful identity is:

$$P_\pm^{\alpha\beta}P_\pm^{\gamma\delta} = P_\pm^{\gamma\beta}P_\pm^{\alpha\delta} \ . \tag{D.7}$$

We of course only need to check that it is true for $\beta \neq \delta$. It turns out that:

$$4P_{\pm}^{00}P_{\pm}^{11} = h^{00}h^{11} \text{ while } 4P_{\pm}^{10}P_{\pm}^{01} = h^{01}h^{10} + \frac{1}{h} \ , \tag{D.8}$$

which indeed agree since $\det(h^{-1}) = 1/h$. In particular, this relation allows us to see that:

$$P_{\pm}^{\alpha\beta}P_{\pm}^{\gamma\delta}\pi_{\alpha}^{\mu}\pi_{\gamma}^{\nu}\Gamma_{\mu}\Gamma_{\nu} = P_{\pm}^{\alpha\beta}P_{\pm}^{\gamma\delta}\pi_{\alpha}^{\mu}\pi_{\gamma\mu} \ , \tag{D.9}$$

without the anti-symmetric combination.

With these tools in hand, let us first rewrite the action completely in term of π_{α}^{μ} instead of $\partial_{\alpha}x^{\mu}$. Indeed, since terms like $\epsilon^{\alpha\beta}(\bar\theta\Gamma^{\mu}\partial_{\alpha}\theta)(\bar\theta\Gamma_{\mu}\partial_{\beta}\theta)$ vanish:

$$S_{GS} = \frac{1}{2\pi\alpha'}\int\Big[(h^{\alpha\beta}\pi_{\alpha}^{\mu}\pi_{\beta\mu})\sqrt{-h} + 2i\epsilon^{\alpha\beta}\pi_{\alpha}^{\mu}(\bar\theta\Gamma_{\mu}\partial_{\beta}\theta + \bar\psi\Gamma_{\mu}\partial_{\beta}\psi) - 2\epsilon^{\alpha\beta}(\bar\theta\Gamma^{\mu}\partial_{\alpha}\theta)(\bar\psi\Gamma_{\mu}\partial_{\beta}\psi)\Big]d^2\sigma \ , \tag{D.10}$$

First, we note that:

$$\delta_{\kappa}\pi_{\alpha}^{\mu} = 2i\partial_{\alpha}\bar\theta\Gamma^{\mu}\delta_{\kappa}\theta - 2i\partial_{\alpha}\bar\psi\Gamma^{\mu}\delta_{\kappa}\psi = 4i\partial_{\alpha}\bar\theta\Gamma^{\mu}\Gamma^{\nu}\kappa^{1\beta}\pi_{\beta\nu} - 4i\partial_{\alpha}\bar\psi\Gamma^{\mu}\Gamma^{\nu}\kappa^{2\beta}\pi_{\beta\nu} \ . \tag{D.11}$$

Varying π_{α}^{μ} in the first term and θ and ψ in the second gives:

$$\begin{aligned} A = \sqrt{-h}h^{\alpha\beta}\pi_{\alpha}^{\mu}\delta_{\kappa}(\pi_{\beta\mu}) + 2i\epsilon^{\alpha\beta}\pi_{\alpha}^{\mu}\delta_{\kappa}(\bar\theta\Gamma_{\mu}\partial_{\beta}\theta + \bar\psi\Gamma_{\mu}\partial_{\beta}\psi) = 2i\epsilon^{\alpha\beta}\Big(\partial_{\beta}\big(\pi_{\alpha}^{\mu}(\bar\theta\Gamma^{\mu}\delta_{\kappa}\theta + \bar\psi\Gamma^{\mu}\delta_{\kappa}\psi)\big) - \\ -(\partial_{\beta}\pi_{\alpha}^{\mu})(\bar\theta\Gamma^{\mu}\delta_{\kappa}\theta + \bar\psi\Gamma^{\mu}\delta_{\kappa}\psi)\Big) + 16i\sqrt{-h}\pi_{\alpha}^{\mu}\pi_{\gamma}^{\mu}\big(\partial_{\beta}\bar\theta P_{-}^{\alpha\beta}\kappa^{1\gamma} - \partial_{\beta}\bar\psi\Gamma^{\mu}P_{+}^{\alpha\beta}\kappa^{2\gamma}\big) \ , \end{aligned} \tag{D.12}$$

where we integrated by parts terms with $\partial_{\beta}\delta\theta$ or $\partial_{\beta}\delta\psi$ and used (D.11), (D.6) and (D.9). This result justifies our choice in (D.5) for the transformation rule of $\sqrt{-h}h^{\alpha\beta}$, which allows us now to obtain:

$$\delta_{\kappa}(\sqrt{-h}h^{\alpha\beta})\pi_{\alpha}^{\mu}\pi_{\beta\mu} + A = 2i\epsilon^{\alpha\beta}\Big(\partial_{\beta}\big(\pi_{\alpha}^{\mu}(\bar\theta\Gamma^{\mu}\delta_{\kappa}\theta + \bar\psi\Gamma^{\mu}\delta_{\kappa}\psi)\big) - (\partial_{\beta}\pi_{\alpha}^{\mu})(\bar\theta\Gamma^{\mu}\delta_{\kappa}\theta + \bar\psi\Gamma^{\mu}\delta_{\kappa}\psi)\Big) \ , \tag{D.13}$$

where:

$$\partial_{\beta}\pi_{\alpha}^{\mu} = -i\partial_{\beta}\bar\theta\Gamma^{\mu}\partial_{\alpha}\theta + i\partial_{\beta}\bar\psi\Gamma^{\mu}\partial_{\alpha}\psi \ . \tag{D.14}$$

Let us now turn our attention to the terms:

$$B = 2i\epsilon^{\alpha\beta}\delta_{\kappa}(\pi_{\alpha}^{\mu})(\bar\theta\Gamma_{\mu}\partial_{\beta}\theta + \bar\psi\Gamma_{\mu}\partial_{\beta}\psi) = -4\epsilon^{\alpha\beta}(\partial_{\alpha}\bar\theta\Gamma^{\mu}\delta_{\kappa}\theta - \partial_{\alpha}\bar\psi\Gamma^{\mu}\delta_{\kappa}\psi)(\bar\theta\Gamma_{\mu}\partial_{\beta}\theta + \bar\psi\Gamma_{\mu}\partial_{\beta}\psi) \tag{D.15}$$

and

$$\begin{aligned} C = -2\epsilon^{\alpha\beta}\delta_{\kappa}(\bar\theta\Gamma^{\mu}\partial_{\alpha}\theta)(\bar\psi\Gamma_{\mu}\partial_{\beta}\psi) = \\ 2\epsilon^{\alpha\beta}\Big(2(\partial_{\alpha}\bar\theta\Gamma^{\mu}\delta_{\kappa}\theta)(\bar\psi\Gamma_{\mu}\partial_{\beta}\psi) + (\bar\theta\Gamma^{\mu}\delta_{\kappa}\theta)(\partial_{\alpha}\bar\psi\Gamma_{\mu}\partial_{\beta}\psi) + 2(\bar\theta\Gamma^{\mu}\partial_{\alpha}\theta)(\partial_{\beta}\bar\psi\Gamma_{\mu}\delta_{\kappa}\psi) - \\ (\partial_{\alpha}\bar\theta\Gamma^{\mu}\partial_{\beta}\theta)(\bar\psi\Gamma_{\mu}\delta_{\kappa}\psi) - \partial_{\alpha}\Big((\bar\theta\Gamma^{\mu}\delta_{\kappa}\theta)(\bar\psi\Gamma_{\mu}\partial_{\beta}\psi) - (\bar\theta\Gamma^{\mu}\partial_{\beta}\theta)(\bar\psi\Gamma_{\mu}\delta_{\kappa}\psi)\Big)\Big) \ , \end{aligned} \tag{D.16}$$

so that they add to:

$$B + C = 2\epsilon^{\alpha\beta}\Big(-2(\partial_\alpha\bar{\theta}\Gamma^\mu\delta_\kappa\theta)(\bar{\theta}\Gamma_\mu\partial_\beta\theta) + 2(\partial_\alpha\bar{\psi}\Gamma^\mu\delta_\kappa\psi)(\bar{\psi}\Gamma_\mu\partial_\beta\psi) + \tag{D.17}$$
$$+ (\bar{\theta}\Gamma^\mu\delta_\kappa\theta)(\partial_\alpha\bar{\psi}\Gamma_\mu\partial_\beta\psi) - (\partial_\alpha\bar{\theta}\Gamma^\mu\partial_\beta\theta)(\bar{\psi}\Gamma_\mu\delta_\kappa\psi) - \partial_\alpha\Big((\bar{\theta}\Gamma^\mu\delta_\kappa\theta)(\bar{\psi}\Gamma_\mu\partial_\beta\psi) - (\bar{\theta}\Gamma^\mu\partial_\beta\theta)(\bar{\psi}\Gamma_\mu\delta_\kappa\psi)\Big)\Big)$$

This sums up with what we obtained before in a nice way to give (up to total derivatives):

$$2\pi\alpha'\delta_\kappa\mathcal{L} = 2\epsilon^{\alpha\beta}\Big(-2(\partial_\alpha\bar{\theta}\Gamma^\mu\delta_\kappa\theta)(\bar{\theta}\Gamma_\mu\partial_\beta\theta) + (\bar{\theta}\Gamma^\mu\delta_\kappa\theta)(\partial_\alpha\bar{\theta}\Gamma^\mu\partial_\beta\theta) +$$
$$+2(\partial_\alpha\bar{\psi}\Gamma^\mu\delta_\kappa\psi)(\bar{\psi}\Gamma_\mu\partial_\beta\psi) - (\bar{\psi}\Gamma^\mu\delta_\kappa\psi)(\partial_\alpha\bar{\psi}\Gamma^\mu\partial_\beta\psi)\Big) = 0 \tag{D.18}$$

as we already proved in the part concerning supersymmetry in (D.4). Indeed, this is a local symmetry, since we never needed to suppose anything about the (σ^0, σ^1)-dependence of κ.

D.3 λ-symmetry

The Green-Schwarz action exhibits another local symmetry, customarily called λ-symmetry. In contrast to κ-symmetry, λ-symmetry involves a bosonic parameter. It is defined by the following transformations:

$$\delta_\lambda\theta = \sqrt{-h}P_-^{\alpha\beta}\partial_\beta\theta\lambda_\alpha\ , \quad \delta_\lambda\psi = \sqrt{-h}P_+^{\alpha\beta}\partial_\beta\psi\lambda_\alpha\ , \tag{D.19}$$
$$\delta_\lambda x^\mu = i\bar{\theta}\Gamma^\mu\delta\theta - i\bar{\psi}\Gamma^\mu\delta\psi\ .$$

Actually the calculation is very similar to the proof of κ-symmetry, except that we do not vary the metric. Following the same steps as above, we obtain:

$$2\pi\alpha'\delta_\kappa\mathcal{L} = 4i\sqrt{-h}\pi_\alpha^\mu\big(P_-^{\alpha\beta}\partial_\beta\bar{\theta}\Gamma^\mu\delta_\lambda\theta - P_+^{\alpha\beta}\partial_\beta\bar{\psi}\Gamma^\mu\delta_\lambda\psi\big) +$$
$$+ 2\epsilon^{\alpha\beta}\Big(-2(\partial_\alpha\bar{\theta}\Gamma^\mu\delta_\lambda\theta)(\bar{\theta}\Gamma_\mu\partial_\beta\theta) + (\bar{\theta}\Gamma^\mu\delta_\lambda\theta)(\partial_\alpha\bar{\theta}\Gamma^\mu\partial_\beta\theta) +$$
$$+ 2(\partial_\alpha\bar{\psi}\Gamma^\mu\delta_\lambda\psi)(\bar{\psi}\Gamma_\mu\partial_\beta\psi) - (\bar{\psi}\Gamma^\mu\delta_\lambda\psi)(\partial_\alpha\bar{\psi}\Gamma^\mu\partial_\beta\psi)\Big) - \tag{D.20}$$
$$- 2i\epsilon^{\alpha\beta}\Big(\partial_\beta\big(\pi_\alpha^\mu(\bar{\theta}\Gamma^\mu\delta_\lambda\theta + \bar{\psi}\Gamma^\mu\delta_\lambda\psi)\big) - i\partial_\alpha\big((\bar{\theta}\Gamma^\mu\delta_\lambda\theta)(\bar{\psi}\Gamma_\mu\partial_\beta\psi) - (\bar{\theta}\Gamma^\mu\partial_\beta\theta)(\bar{\psi}\Gamma_\mu\delta_\lambda\psi)\big)\Big)$$

Besides the total derivatives of the last line, the two intermediate lines vanish because of the same Fierz identity we already used twice in this appendix, while the first line vanishes because of the particular form of the λ-transformations. Indeed:

$$P_-^{\alpha\beta}\partial_\beta\bar{\theta}\Gamma^\mu\delta_\lambda\theta = \sqrt{-h}P_-^{\alpha\beta}P_-^{\gamma\delta}\partial_\beta\bar{\theta}\Gamma^\mu\partial_\delta\theta\lambda_\gamma\ , \tag{D.21}$$

and such terms vanish because of the symmetry in $\beta \leftrightarrow \delta$ of $P_-^{\alpha\beta}P_-^{\gamma\delta}$.

Appendix E

Proof of the duality relations for fuzzy spheres

In this appendix, we give the derivation of the coefficients m_k in the self-duality relation (6.25) for the $SO(2k+1)$ fuzzy spheres. In this appendix, we define the $2^k \times 2^k$ gamma matrices in the $(2k+1)$-dimensional Euclidean space $\Gamma^{(k)}_p$ by the following recursive relation:

$$\Gamma^{(k+1)}_p = \Gamma^{(k)}_p \otimes \sigma_2 = \begin{pmatrix} 0 & -i\Gamma^{(k)}_p \\ i\Gamma^{(k)}_p & 0 \end{pmatrix}, \quad \Gamma^{(k+1)}_{2k+2} = \mathbf{1}_{2^k\times 2^k} \otimes \sigma_1 = \begin{pmatrix} 0 & \mathbf{1}_{2^k\times 2^k} \\ \mathbf{1}_{2^k\times 2^k} & 0 \end{pmatrix},$$
$$\Gamma^{(k+1)}_{2k+3} = \mathbf{1}_{2^k\times 2^k} \otimes \sigma_3 = \begin{pmatrix} \mathbf{1}_{2^k\times 2^k} & 0 \\ 0 & -\mathbf{1}_{2^k\times 2^k} \end{pmatrix}, \tag{E.1}$$

where the index p runs over $p = 1, 2, \cdots, 2k+1$. The 3-dimensional gamma matrices are identical to the Pauli matrices: $\Gamma^{(1)}_i = \sigma_i$. Under this notation, we obtain

$$\sigma_1\sigma_2 = i\sigma_3, \quad \Gamma^{(2)}_1\Gamma^{(2)}_2\Gamma^{(2)}_3\Gamma^{(2)}_4 = \Gamma^{(2)}_5, \quad \Gamma^{(3)}_1\Gamma^{(3)}_2\cdots\Gamma^{(3)}_6 = -i\Gamma^{(3)}_7, \quad \Gamma^{(4)}_1\Gamma^{(4)}_2\cdots\Gamma^{(4)}_8 = -\Gamma^{(4)}_9. \tag{E.2}$$

It is trivial that $m_1 = 2i$ for the $SO(3)$ fuzzy sphere, as we have explained in the footnote 3. Then, we start with the coefficient m_2. In this appendix, we omit the subscript "$_{\rm sym}$", which indicates that the tensor product is restricted to the fully symmetric subspace.

E.1 Computation of m_2

We first perform the computation of m_2 for the $SO(5)$ fuzzy sphere[53]. We frequently utilize the following identity for the symmetric tensor product:

$$\sum_{i=1}^{3}(\sigma_i \otimes \sigma_i) = (\mathbf{1}_{2\times 2} \otimes \mathbf{1}_{2\times 2}). \tag{E.3}$$

Now, we consider the case in which $n = 2$ for brevity; i.e. the $SO(5)$ fuzzy sphere is described by the 2-fold symmetric tensor products as

$$B^{SO(5)}_p = [(\Gamma^{(2)}_p \otimes \mathbf{1}_{4\times 4}) + (\mathbf{1}_{4\times 4} \otimes \Gamma^{(2)}_p)]. \tag{E.4}$$

Then, the left-hand side of (6.25) is

$$\begin{aligned} & \epsilon_{p_1\cdots p_4 5} B^{SO(5)}_{p_1} B^{SO(5)}_{p_2} B^{SO(5)}_{p_3} B^{SO(5)}_{p_4} \\ = \; & \epsilon_{p_1\cdots p_4 5}[(\Gamma^{(2)}_{p_1\cdots p_4} \otimes \mathbf{1}_{4\times 4}) + (\mathbf{1}_{4\times 4} \otimes \Gamma^{(2)}_{p_1\cdots p_4}) + 2(\Gamma^{(2)}_{p_1 p_2} \otimes \Gamma^{(2)}_{p_3 p_4})]. \end{aligned} \tag{E.5}$$

We do not lose any generality if we set $p_5 = 5$, and the indices $p_1, \cdots, p_4$ run over $1, 2, 3, 4$. The first two terms give $4! = 24$ of $(\Gamma^{(2)}_{1234} \otimes \mathbf{1}_{4\times 4}) + (\mathbf{1}_{4\times 4} \otimes \Gamma^{(2)}_{1234})$, to constitute $24 B^{SO(5)}_5$. On the other hand, the third term is computed as

$$\begin{aligned} & 2\epsilon_{p_1\cdots p_4 5}(\Gamma^{(2)}_{p_1 p_2} \otimes \Gamma^{(2)}_{p_3 p_4}) = 4\epsilon_{ijk}[(\Gamma^{(2)}_{ij} \otimes \Gamma^{(2)}_{k4}) + (\Gamma^{(2)}_{k4} \otimes \Gamma^{(2)}_{ij})] \\ = \; & -8[(\Gamma^{(2)}_{k45} \otimes \Gamma^{(2)}_{k4}) + (\Gamma^{(2)}_{k4} \otimes \Gamma^{(2)}_{k45})] \\ = \; & -8[(\sigma_k \otimes (-i\mathbf{1}_{2\times 2})) \otimes (\sigma_k \otimes (-i\sigma_3)) + (\sigma_k \otimes (-i\sigma_3)) \otimes (\sigma_k \otimes (-i\mathbf{1}_{2\times 2}))] \\ = \; & 8[(\mathbf{1}_{2\times 2} \otimes \sigma_3) \otimes (\mathbf{1}_{2\times 2} \otimes \mathbf{1}_{2\times 2}) + (\mathbf{1}_{2\times 2} \otimes \mathbf{1}_{2\times 2}) \otimes (\mathbf{1}_{2\times 2} \otimes \sigma_3)] = 8 B^{SO(5)}_5. \end{aligned} \tag{E.6}$$

By the same token, this kind of contribution makes $8(n-1)B^{SO(5)}_5$ for any n. Altogether, we have $m_2 = 8(n+2)$.

E.2 Computation of m_3

The computation of m_3 for the $SO(7)$ fuzzy sphere goes in the similar way. In this computation, we use the formulæ

$$\sum_{l=1}^{5}(\Gamma^{(2)}_l \otimes \Gamma^{(2)}_l) = (\mathbf{1}_{4\times 4} \otimes \mathbf{1}_{4\times 4}), \quad \sum_{l_1, l_2 = 1}^{5}(\Gamma^{(2)}_{l_1 l_2} \otimes \Gamma^{(2)}_{l_1 l_2}) = -4(\mathbf{1}_{4\times 4} \otimes \mathbf{1}_{4\times 4}). \tag{E.7}$$

Now, we set $p_7 = 7$ without loss of generality, and consider the 3-fold tensor product. The left-hand side of (6.25) is now

$$\begin{aligned} & \epsilon_{p_1\cdots p_6 7} B^{SO(7)}_{p_1} B^{SO(7)}_{p_2} \cdots B^{SO(7)}_{p_6} \\ = \; & \epsilon_{p_1\cdots p_6 7}[\{(\Gamma^{(3)}_{p_1\cdots p_6} \otimes \mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8}) + (\mathbf{1}_{8\times 8} \otimes \Gamma^{(3)}_{p_1\cdots p_6} \otimes \mathbf{1}_{8\times 8}) + (\mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8} \otimes \Gamma^{(3)}_{p_1\cdots p_6})\} \end{aligned} \tag{E.8}$$

$$+3\{(\Gamma^{(3)}_{p_1\cdots p_4} \otimes \Gamma^{(3)}_{p_5 p_6} \otimes \mathbf{1}_{8\times 8}) + (\text{5 other permutations of this kind})\} \tag{E.9}$$

$$+6(\Gamma^{(3)}_{p_1 p_2} \otimes \Gamma^{(3)}_{p_3 p_4} \otimes \Gamma^{(3)}_{p_5 p_6})]. \tag{E.10}$$

- We first consider the contribution of (E.8). Since there are $6! = 720$ ways to contract the indices $p_1, \cdots, p_6$, this gives

$$-720i[(\Gamma^{(3)}_7 \otimes \mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8}) + (\mathbf{1}_{8\times 8} \otimes \Gamma^{(3)}_7 \otimes \mathbf{1}_{8\times 8}) + (\mathbf{1}_{8\times 8} \otimes \mathbf{1}_{8\times 8} \otimes \Gamma^{(3)}_7)] = -720i B^{SO(7)}_7.$$

- We then go on to the contribution of (E.9):

$$
\begin{aligned}
& \epsilon_{p_1\cdots p_6 7}(\Gamma^{(3)}_{p_1\cdots p_4}\otimes\Gamma^{(3)}_{p_5p_6}\otimes\mathbf{1}_{8\times 8}) \\
= \ & \epsilon_{l_1\cdots l_5 67}[4(\Gamma^{(3)}_{l_1\cdots l_3 6}\otimes\Gamma^{(3)}_{l_4l_5}\otimes\mathbf{1}_{8\times 8})+2(\Gamma^{(3)}_{l_1\cdots l_4}\otimes\Gamma^{(3)}_{l_5 6}\otimes\mathbf{1}_{8\times 8})] \\
= \ & (4!)i[(\Gamma^{(3)}_{l_4l_5 7}\otimes\Gamma^{(3)}_{l_4l_5}\otimes\mathbf{1}_{8\times 8})+2(\Gamma^{(3)}_{l_5 67}\otimes\Gamma^{(3)}_{l_5 6}\otimes\mathbf{1}_{8\times 8})] \\
= \ & (4!)i[((\Gamma^{(2)}_{l_4l_5}\otimes\sigma_3)\otimes(\Gamma^{(2)}_{l_4l_5}\otimes\mathbf{1}_{2\times 2})\otimes\mathbf{1}_{8\times 8})-2((\Gamma^{(2)}_{l_5}\otimes\mathbf{1}_{2\times 2})\otimes(\Gamma^{(2)}_{l_5}\otimes\sigma_3)\otimes\mathbf{1}_{8\times 8})] \\
= \ & -(4!)i[4((\mathbf{1}_{4\times 4}\otimes\sigma_3)\otimes\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8})+2(\mathbf{1}_{8\times 8}\otimes(\mathbf{1}_{4\times 4}\otimes\sigma_3)\otimes\mathbf{1}_{8\times 8})],
\end{aligned}
$$

  where the indices $l_1, l_2, \cdots$ run over $1, 2, \cdots, 5$ and we have utilized the formulæ (E.7). Summing up all 6 permutations, we obtain $-864iB_7^{SO(7)}$. When we extend this argument for the general n-fold tensor product, the result is $-432i(n-1)B_7^{SO(7)}$.

- Lastly, we investigate the term (E.10):

$$
\begin{aligned}
& 6\epsilon_{p_1\cdots p_6 7}(\Gamma^{(3)}_{p_1p_2}\otimes\Gamma^{(3)}_{p_3p_4}\otimes\Gamma^{(3)}_{p_5p_6}) \\
= \ & 12\epsilon_{l_1\cdots l_5 67}[(\Gamma^{(3)}_{l_1l_2}\otimes\Gamma^{(3)}_{l_3l_4}\otimes\Gamma^{(3)}_{l_5 6})+(2\text{ other permutations})] \\
= \ & -12(2!)i[(\Gamma^{(3)}_{l_3l_4l_5 67}\otimes\Gamma^{(3)}_{l_3l_4}\otimes\Gamma^{(3)}_{l_5 6})+(\text{perm.})] \\
= \ & 24i[((\Gamma^{(2)}_{l_3l_4l_5}\otimes\mathbf{1}_{2\times 2})\otimes(\Gamma^{(2)}_{l_3l_4}\otimes\mathbf{1}_{2\times 2})\otimes(\Gamma^{(2)}_{l_5}\otimes\sigma_3))+(\text{perm.})] \\
= \ & 24i[((\Gamma^{(2)}_{l_3l_4}\Gamma^{(2)}_{l_5}\otimes\mathbf{1}_{2\times 2})\otimes(\Gamma^{(2)}_{l_3l_4}\otimes\mathbf{1}_{2\times 2})\otimes(\Gamma^{(2)}_{l_5}\otimes\mathbf{1}_{2\times 2}) \\
& \quad +2(\Gamma^{(2)}_{i_4}\otimes\mathbf{1}_{2\times 2})\otimes((\Gamma^{(2)}_{i_3}\Gamma^{(2)}_{i_4}-\delta_{i_3i_4}\mathbf{1}_{4\times 4})\otimes\mathbf{1}_{2\times 2})\otimes(\Gamma^{(2)}_{i_3}\otimes\sigma_3))+(\text{perm.})] \\
= \ & -96i[(\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8}\otimes(\mathbf{1}_{4\times 4}\otimes\sigma_3))+(\mathbf{1}_{8\times 8}\otimes(\mathbf{1}_{4\times 4}\otimes\sigma_3)\otimes\mathbf{1}_{8\times 8}) \\
& \quad +((\mathbf{1}_{4\times 4}\otimes\sigma_3)\otimes\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8})] = -96iB_7^{SO(7)}.
\end{aligned}
$$

  For the general n-fold symmetric tensor product, we obtain $-48i(n-1)(n-2)B_7^{SO(7)}$.

We sum up all the contributions of (E.8), (E.9) and (E.10) to obtain $m_3 = -48i(n+2)(n+4)$.

E.3 Computation of m_4

We next go on to the coefficient m_4 for the $SO(9)$ fuzzy sphere. We repeat the same procedure, but the computation is rather complicated. We exploit the following formulæ in this computation:

$$(\Gamma^{(3)}_{l}\otimes\Gamma^{(3)}_{l}) = (\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8}), \tag{E.11}$$

$$(\Gamma^{(3)}_{l_1l_2}\otimes\Gamma^{(3)}_{l_1l_2}) = -6(\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8}), \tag{E.12}$$

$$(\Gamma^{(3)}_{l_1l_2l_3}\otimes\Gamma^{(3)}_{l_1l_2l_3}) = -18(\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8}), \tag{E.13}$$

$$(\Gamma^{(3)}_{l_1l_2}\otimes\Gamma^{(3)}_{l_3}\otimes\Gamma^{(3)}_{l_1l_2l_3}) = -6(\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8}), \tag{E.14}$$

$$(\Gamma^{(3)}_{l_1l_2}\otimes\Gamma^{(3)}_{l_3l_4}\otimes\Gamma^{(3)}_{l_1\cdots l_4}) = 24(\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8}), \tag{E.15}$$

$$(\Gamma^{(3)}_{l_1l_2}\otimes\Gamma^{(3)}_{l_3l_4}\otimes\Gamma^{(3)}_{l_5l_6}\otimes\Gamma^{(3)}_{l_1\cdots l_6}) = -48(\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8}\otimes\mathbf{1}_{8\times 8}). \tag{E.16}$$

We set $p_9 = 9$, and the indices $p_1, p_2, \cdots$ and $l_1, l_2, \cdots$ respectively run over $1, 2, \cdots, 8$ and $1, 2, \cdots, 7$. We consider the 4-fold tensor product

$$\begin{aligned} & \epsilon_{p_1 \cdots p_8 9} B^{SO(9)}_{p_1} B^{SO(9)}_{p_2} \cdots B^{SO(9)}_{p_8} \\ = \; & \epsilon_{p_1 \cdots p_8 9} [((\Gamma^{(4)}_{p_1 \cdots p_8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) + \cdots + (\mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16} \otimes \Gamma^{(4)}_{p_1 \cdots p_8})) \end{aligned} \tag{E.17}$$

$$+4((\Gamma^{(4)}_{p_1 \cdots p_6} \otimes \Gamma^{(4)}_{p_7 p_8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) + (11 \text{ other permutations})) \tag{E.18}$$

$$+6((\Gamma^{(4)}_{p_1 \cdots p_4} \otimes \Gamma^{(4)}_{p_5 \cdots p_8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) + (5 \text{ other permuations})) \tag{E.19}$$

$$+12((\Gamma^{(4)}_{p_1 \cdots p_4} \otimes \Gamma^{(4)}_{p_5 p_6} \otimes \Gamma^{(4)}_{p_7 p_8} \otimes \mathbf{1}_{16\times 16}) + (11 \text{ other permutations})) \tag{E.20}$$

$$+24(\Gamma^{(4)}_{p_1 p_2} \otimes \Gamma^{(4)}_{p_3 p_4} \otimes \Gamma^{(4)}_{p_5 p_6} \otimes \Gamma^{(4)}_{p_7 p_8})]. \tag{E.21}$$

- (E.17) trivially gives $-(8!) B^{SO(9)}_9$.

- The contribution of (E.18) is computed as follows:

$$\begin{aligned} & \epsilon_{p_1 \cdots p_8 9} (\Gamma^{(4)}_{p_1 \cdots p_6} \otimes \Gamma^{(4)}_{p_7 p_8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\ = \; & 6(5!)(\Gamma^{(4)}_{l_6 l_7 9} \otimes \Gamma^{(4)}_{l_6 l_7} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) + 2(6!)(\Gamma^{(4)}_{l_7 89} \otimes \Gamma^{(4)}_{l_7 8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\ = \; & 6(5!)((\Gamma^{(3)}_{l_6 l_7} \otimes \sigma_3) \otimes (\Gamma^{(3)}_{l_6 l_7} \otimes \mathbf{1}_{2\times 2}) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\ & -2(6!)((\Gamma^{(3)}_{l_7} \otimes \mathbf{1}_{2\times 2}) \otimes (\Gamma^{(3)}_{l_7} \otimes \sigma_3) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\ = \; & -(6!)[6((\mathbf{1}_{8\times 8} \otimes \sigma_3) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\ & \qquad +2(\mathbf{1}_{16\times 16} \otimes (\mathbf{1}_{8\times 8} \otimes \sigma_3) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16})]. \end{aligned}$$

  We sum up all 12 permutations to obtain $-69120 B^{SO(9)}_9$ ($-23040(n-1) B^{SO(9)}_9$ for the general n-fold tensor product).

- We go on to the contribution of (E.19):

$$\begin{aligned} & \epsilon_{p_1 \cdots p_8 9} (\Gamma^{(4)}_{p_1 \cdots p_4} \otimes \Gamma^{(4)}_{p_5 \cdots p_8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\ = \; & -4(4!)[(\Gamma^{(4)}_{l_5 l_6 l_7 89} \otimes \Gamma^{(4)}_{l_5 l_6 l_7 8} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) + (\Gamma^{(4)}_{l_5 l_6 l_7 8} \otimes \Gamma^{(4)}_{l_5 l_6 l_7 89} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16})] \\ = \; & 4(4!)[((\Gamma^{(3)}_{l_5 l_6 l_7} \otimes \mathbf{1}_{2\times 2}) \otimes (\Gamma^{(3)}_{l_5 l_6 l_7} \otimes \sigma_3) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\ & \qquad +((\Gamma^{(3)}_{l_5 l_6 l_7} \otimes \sigma_3) \otimes (\Gamma^{(3)}_{l_5 l_6 l_7} \otimes \mathbf{1}_{2\times 2}) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16})] \\ = \; & -72(4!)[((\mathbf{1}_{8\times 8} \otimes \sigma_3) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16}) \\ & \qquad +(\mathbf{1}_{16\times 16} \otimes (\mathbf{1}_{8\times 8} \otimes \sigma_3) \otimes \mathbf{1}_{16\times 16} \otimes \mathbf{1}_{16\times 16})]. \end{aligned}$$

  Therefore, when we sum all the permutations, (E.19) gives $-31104 B^{SO(9)}_9$ ($-10368(n-1) B^{SO(9)}_9$ for the general n).

- We next investigate the terms (E.20). Together with all the permutations, this gives

$-41472B_9^{SO(9)}$ $(-6912(n-1)(n-2)B_9^{SO(9)}$ for any n) due to the following considerations:

$$\begin{aligned}
&\epsilon_{p_1\cdots p_8 9}(\Gamma^{(4)}_{p_1\cdots p_4}\otimes\Gamma^{(4)}_{p_5p_6}\otimes\Gamma^{(4)}_{p_7p_8}\otimes\mathbf{1}_{16\times16})\\
=\ &(4!)[-((\Gamma^{(3)}_{l_4\cdots l_7}\otimes\sigma_3)\otimes(\Gamma^{(3)}_{l_4l_5}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_6l_7}\otimes\mathbf{1}_{2\times2})\otimes\mathbf{1}_{16\times16})\\
&\quad+2((\Gamma^{(3)}_{l_5l_6l_7}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_5l_6}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_7}\otimes\sigma_3)\otimes\mathbf{1}_{16\times16})\\
&\quad+2((\Gamma^{(3)}_{l_5l_6l_7}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_7}\otimes\sigma_3)\otimes(\Gamma^{(3)}_{l_5l_6}\otimes\mathbf{1}_{2\times2})\otimes\mathbf{1}_{16\times16})]\\
=\ &-(4!)[24((\mathbf{1}_{8\times8}\otimes\sigma_3)\otimes\mathbf{1}_{16\times16}\otimes\mathbf{1}_{16\times16}\otimes\mathbf{1}_{16\times16})\\
&\quad+12(\mathbf{1}_{16\times16}\otimes((\mathbf{1}_{8\times8}\otimes\sigma_3)\otimes\mathbf{1}_{16\times16}+\mathbf{1}_{16\times16}\otimes(\mathbf{1}_{8\times8}\otimes\sigma_3))\otimes\mathbf{1}_{16\times16})],
\end{aligned}$$

where we have used the formulæ (E.14) and (E.15).

- Lastly, (E.21) gives $-2304B_9^{SO(9)}$ $(-384(n-1)(n-2)(n-3)B_9^{SO(9)}$ for any n):

$$\begin{aligned}
&24\epsilon_{p_1\cdots p_8 9}(\Gamma^{(4)}_{p_1p_2}\otimes\Gamma^{(4)}_{p_3p_4}\otimes\Gamma^{(4)}_{p_5p_6}\otimes\Gamma^{(4)}_{p_7p_8})\\
=\ &48[(\Gamma^{(3)}_{l_1l_2}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_3l_4}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_5l_6}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_1\cdots l_6}\otimes\sigma_3)\\
&+(\Gamma^{(3)}_{l_3l_4}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_5l_6}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_1\cdots l_6}\otimes\sigma_3)\otimes(\Gamma^{(3)}_{l_1l_2}\otimes\mathbf{1}_{2\times2})\\
&+(\Gamma^{(3)}_{l_5l_6}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_1\cdots l_6}\otimes\sigma_3)\otimes(\Gamma^{(3)}_{l_1l_2}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_3l_4}\otimes\mathbf{1}_{2\times2})\\
&+(\Gamma^{(3)}_{l_1\cdots l_6}\otimes\sigma_3)\otimes(\Gamma^{(3)}_{l_1l_2}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_3l_4}\otimes\mathbf{1}_{2\times2})\otimes(\Gamma^{(3)}_{l_5l_6}\otimes\mathbf{1}_{2\times2})]=-2304B_9^{SO(9)}.
\end{aligned}$$

Here, we have exploited the formula (E.16).

When we sum up the contributions of (E.17) $\sim$ (E.21), we obtain $m_4=-384(n+2)(n+4)(n+6)$.

Bibliography

[1] Stephen L. Adler. "Generalized quantum dynamics". *Nucl. Phys.*, B415:195–242, 1994. hep-th/9306009.

[2] Stephen L. Adler. "The matrix model for M theory as an exemplar of trace (or generalized quantum) dynamics". *Phys. Lett.*, B407:229–233, 1997. hep-th/9703053.

[3] Stephen L. Adler. "Poincaré supersymmetry representations over trace class non-commutative graded operator algebras". *Nucl. Phys.*, B499:569–582, 1997. hep-th/9703132.

[4] Stephen L. Adler and Andrew C. Millard. "Generalized quantum dynamics as prequantum mechanics". *Nucl. Phys.*, B473:199–244, 1996. hep-th/9508076.

[5] Anton Yu. Alekseev, Andreas Recknagel, and Volker Schomerus. "Non-commutative world-volume geometries: Branes on SU(2) and fuzzy spheres". *JHEP*, 09:023, 1999. hep-th/9908040.

[6] Enrique Alvarez, Jose Luis F. Barbon, and Javier Borlaf. "T-duality for open strings". *Nucl. Phys.*, B479:218–242, 1996. hep-th/9603089.

[7] Orlando Alvarez. "Differential geometry in string models". Lectures given at Workshop on Unified String Theory, Santa Barbara, CA, Jul 29 - Aug 16, 1985.

[8] Luis Alvarez-Gaumé. "Random surfaces, statistical mechanics and string theory". *Helv. Phys. Acta*, 64:359–526, 1991.

[9] Luis Alvarez-Gaumé and Cesar Gomez. "Topics in Liouville theory". Lectures at the Spring School on String Theory and Quantum Gravity, Trieste, Italy, Apr 15-23, 1991.

[10] Hajime Aoki, Satoshi Iso, Hikaru Kawai, Yoshihisa Kitazawa, and Tsukasa Tada. "Space-time structures from IIB matrix model". *Prog. Theor. Phys.*, 99:713–746, 1998. hep-th/9802085.

[11] Hajime Aoki, Satoshi Iso, Hikaru Kawai, Yoshihisa Kitazawa, Tsukasa Tada, and Asato Tsuchiya. "IIB matrix model". *Prog. Theor. Phys. Suppl.*, 134:47–83, 1999. hep-th/9908038.

[12] Peter Austing. *Yang-Mills matrix theory.* PhD thesis, 2001. hep-th/0108128.

[13] Peter Austing and John F. Wheater. "The convergence of Yang-Mills integrals". *JHEP*, 02:028, 2001. hep-th/0101071.

[14] Peter Austing and John F. Wheater. “Convergent Yang-Mills matrix theories”. *JHEP*, 04:019, 2001. hep-th/0103159.

[15] Takehiro Azuma. “Investigation of Matrix Theory via Super Lie Algebra”. Master's thesis, 2001. hep-th/0103003.

[16] Takehiro Azuma and Maxime Bagnoud. “Curved-space classical solutions of a massive supermatrix model”. *Nucl. Phys.*, B651:71–86, 2003. hep-th/0209057.

[17] Takehiro Azuma, Satoshi Iso, Hikaru Kawai, and Yuhi Ohwashi. “Supermatrix models”. *Nucl Phys. B*, 610:251, 2001. hep-th/0102168.

[18] Constantin Bachas. “D-brane dynamics”. *Phys. Lett.*, B374:37–42, 1996. hep-th/9511043.

[19] Maxime Bagnoud, Luca Carlevaro, and Adel Bilal. “Supermatrix models for M-theory based on $\mathfrak{osp}(1|32,\mathbb{R})$”. *Nucl. Phys.*, B641:61–92, 2002. hep-th/0201183.

[20] Tom Banks, Willy Fischler, Stephen H. Shenker, and Leonard Susskind. “M theory as a matrix model: a conjecture”. *Phys. Rev. D*, 55:5112, 1997. hep-th/9610043.

[21] Tom Banks, Nathan Seiberg, and Stephen H. Shenker. “Branes from Matrices”. *Nucl. Phys. B*, 443:41, 1997. hep-th/9612157.

[22] Katrin Becker. “Testing M(atrix) theory at two loops”. *Nucl. Phys. Proc. Suppl.*, 68:165–171, 1998. hep-th/9709038.

[23] Katrin Becker and Melanie Becker. “Complete solution for M(atrix) theory at two loops”. *JHEP*, 09:019, 1998. hep-th/9807182.

[24] Katrin Becker and Melanie Becker. “On graviton scattering amplitudes in M-theory”. *Phys. Rev.*, D57:6464–6470, 1998. hep-th/9712238.

[25] Katrin Becker, Melanie Becker, Joseph Polchinski, and Arkady A. Tseytlin. “Higher order graviton scattering in M(atrix) theory”. *Phys. Rev.*, D56:3174–3178, 1997. hep-th/9706072.

[26] Stefano Bellucci and Corneliu Sochichiu. “Fermion doubling and BMN correspondence”. 2003. hep-th/0302104.

[27] David Berenstein, Juan M. Maldacena, and Horatiu Nastase. “Strings in flat space and pp waves from N = 4 super Yang Mills”. *JHEP*, 04:013, 2002. hep-th/0202021.

[28] E. Bergshoeff and A. Van Proeyen. “The many faces of $OSp(1|32)$”. *Class. Quant. Grav.*, 17:3277, 2000. hep-th/0003261.

[29] Eric Bergshoeff and Mees De Roo. “D-branes and T-duality”. *Phys. Lett.*, B380:265–272, 1996. hep-th/9603123.

[30] Eric Bergshoeff, Mees de Roo, Michael B. Green, Goerge Papadopoulos, and Paul K. Townsend. “Duality of Type II 7-branes and 8-branes”. *Nucl. Phys.*, B470:113–135, 1996. hep-th/9601150.

[31] Eric Bergshoeff and Ergin Sezgin. "Super p-Branes Theories and New Spacetime Superalgebras". *Phys. Lett. B*, 354:256, 1995. hep-th/9504140.

[32] Eric Bergshoeff, Ergin Sezgin, and Paul K. Townsend. "On 'spinning' membrane models". *Phys. Lett.*, B209:451, 1988.

[33] Eric Bergshoeff, Ergin Sezgin, and Paul K. Townsend. "Properties of the eleven-dimensional super membrane theory". *Ann. Phys.*, 185:330, 1988.

[34] Eric Bergshoeff and Antoine Van Proeyen. "The unifying superalgebra $OSp(1|32)$". In *To be published in the proceedings of International Conference on Quantization, Gauge Theory, and Strings: Conference Dedicated to the Memory of Professor Efim Fradkin, Moscow, Russia, 5-10 Jun 2000*, 2001. hep-th/0010194.

[35] Eric A. Bergshoeff, Adel Bilal, Mees de Roo, and Alexander Sevrin. "Supersymmetric non-abelian Born-Infeld revisited". *JHEP*, 07:029, 2001. hep-th/0105274.

[36] Eric A. Bergshoeff, Mees de Roo, and Alexander Sevrin. "Non-abelian Born-Infeld and κ-symmetry". *J. Math. Phys.*, 42:2872–2888, 2001. hep-th/0011018.

[37] Eric A. Bergshoeff, Mees de Roo, and Alexander Sevrin. "On the supersymmetric non-abelian Born-Infeld action". *Fortsch. Phys.*, 49:433–440, 2001. hep-th/0011264.

[38] Eric A. Bergshoeff, Mees de Roo, and Alexander Sevrin. "Towards a supersymmetric non-abelian Born-Infeld theory". *Int. J. Mod. Phys.*, A16:750–758, 2001. hep-th/0010151.

[39] Daniel Bessis, Claude Itzykson, and Jean-Bernard Zuber. "Quantum field theory techniques in graphical enumeration". *Adv. Appl. Math.*, 1:109–157, 1980.

[40] Adel Bilal. "2-D gravity from matrix models: An Introductory review, and particularities of antisymmetric matrix models". Invited Lecture Given at the 14th Johns Hopkins Workshop on Current Problems in Particle Theory - Nonperturbative Methods in Low Dimensional Quantum Field Theories, Debrecen, Hungary, Aug 27-30, 1990.

[41] Adel Bilal. "A comment on compactification of M-theory on an (almost) light-like circle". *Nucl. Phys.*, B521:202–216, 1998. hep-th/9801047.

[42] Adel Bilal. "DLCQ of M-theory as the light-like limit". *Phys. Lett.*, B435:312–318, 1998. hep-th/9805070.

[43] Adel Bilal. "Higher-derivative corrections to the non-abelian Born- Infeld action". *Nucl. Phys.*, B618:21–49, 2001. hep-th/0106062.

[44] Giulio Bonelli. "Matrix strings in pp-wave backgrounds from deformed super Yang-Mills theory". *JHEP*, 08:022, 2002. hep-th/0205213.

[45] Eric Braaten, Thomas Curtright, and Charles B. Thorn. "Quantum Bäcklund transformation for the Liouville theory". *Phys. Lett.*, B118:115, 1982.

[46] Eric Braaten, Thomas Curtright, and Charles B. Thorn. "An exact operator solution of the quantum Liouville field theory". *Ann. Phys.*, 147:365, 1983.

[47] Edouard Brézin. "Matrix models of two-dimensional quantum gravity". Prepared for Les Houches Summer School on Gravitation and Quantization, Session 57, Les Houches, France, 5 Jul - 1 Aug 1992.

[48] Edouard Brézin, Claude Itzykson, Giorgio Parisi, and Jean-Bernard Zuber. "Planar diagrams". *Commun. Math. Phys.*, 59:35, 1978.

[49] Edouard Brézin and Vladimir A. Kazakov. "Exactly solvable field theories of closed strings". *Phys. Lett.*, B236:144–150, 1990.

[50] Edouard Brézin and Spenta R. Wadia, editors. *The Large N expansion in quantum field theory and statistical physics: From spin systems to two-dimensional gravity.* Singapore, Singapore: World Scientific (1993) 1130 p.

[51] Lars Brink, Paolo Di Vecchia, and Paul S. Howe. "A locally supersymmetric and reparametrization invariant action for the spinning string". *Phys. Lett.*, B65:471–474, 1976.

[52] Lars Brink, John H. Schwarz, and Joël Sherk. "Supersymmetry, Supergravity theories and the dual spinor model". *Nucl. Phys.*, B122:253, 1977.

[53] Judith Castelino, Sang-Min Lee, and Washington Taylor. "Longitudinal 5-branes as 4-spheres in matrix theory". *Nucl. Phys.*, B526:334–350, 1998. hep-th/9712105.

[54] Andrew Chamblin and Neil D. Lambert. "Zero-branes, quantum mechanics and the cosmological constant". *Phys. Rev.*, D65:066002, 2002. hep-th/0107031.

[55] Iouri Chepelev, Yuri M. Makeenko, and Konstantin Zarembo. "Properties of D-Branes in Matrix Model of IIB Superstring". *Phys. Lett. B*, 400:43, 1997. hep-th/9701151.

[56] John F. Cornwell. *Group Theory in Physics*, volume III. Academic Press, 1989.

[57] Eugène Cremmer, Bernard Julia, and Joël Scherk. "Supergravity theory in 11 dimensions". *Phys. Lett.*, B76:409–412, 1978.

[58] Thomas L. Curtright and Charles B. Thorn. "Conformally invariant quantization of the Liouville theory". *Phys. Rev. Lett.*, 48:1309, 1982.

[59] Keshav Dasgupta, Mohammad M. Sheikh-Jabbari, and Mark Van Raamsdonk. "Matrix perturbation theory for M-theory on a PP-wave". *JHEP*, 05:056, 2002. hep-th/0205185.

[60] Bernard de Wit. "M-theory duality and BPS-extended supergravity". *Int. J. Mod. Phys. A*, 16:1002, 2001. hep-th/0010292.

[61] Bernard de Wit, Jens Hoppe, and Hermann Nicolai. "On the quantum mechanics of supermembranes". *Nucl. Phys.*, B305:545, 1988.

[62] Bernard de Wit, Martin Lüscher, and Hermann Nicolai. "The supermembrane is unstable". *Nucl. Phys.*, B320:135, 1989.

[63] Bernard de Wit, Kasper Peeters, and Jan Plefka. "Superspace geometry for supermembrane backgrounds". *Nucl. Phys.*, B532:99–123, 1998. hep-th/9803209.

[64] Stanley Deser and Bruno Zumino. "A complete action for the spinning string". *Phys. Lett.*, B65:369–373, 1976.

[65] Eric D'Hoker and Pawel S. Kurzepa. "2-D quantum gravity and Liouville theory". *Mod. Phys. Lett.*, A5:1411–1422, 1990.

[66] Philippe Di Francesco, Paul Ginsparg, and Jean Zinn-Justin. "2-D Gravity and random matrices". *Phys. Rept.*, 254:1–133, 1995. hep-th/9306153.

[67] Robbert Dijkgraaf, Erik Verlinde, and Herman Verlinde. "Matrix string theory". *Nucl. Phys.*, B500:43–61, 1997. hep-th/9703030.

[68] Michael Dine, Robert Echols, and Joshua P. Gray. "Tree level supergravity and the matrix model". *Nucl. Phys.*, B564:225–240, 2000. hep-th/9810021.

[69] Michael Dine and Arvind Rajaraman. "Multigraviton scattering in the matrix model". *Phys. Lett.*, B425:77–85, 1998. hep-th/9710174.

[70] Jacques Distler and Hikaru Kawai. "Conformal field theory and 2-D quantum gravity or who's afraid of Joseph Liouville?". *Nucl. Phys.*, B321:509, 1989.

[71] Michael R. Douglas, Daniel Kabat, Philippe Pouliot, and Stephen H. Shenker. "D-branes and short distances in string theory". *Nucl. Phys.*, B485:85–127, 1997. hep-th/9608024.

[72] Michael R. Douglas and Gregory W. Moore. "D-branes, Quivers, and ALE Instantons". 1996. hep-th/9603167.

[73] Michael R. Douglas and Stephen H. Shenker. "Strings in less than one-dimension". *Nucl. Phys.*, B335:635, 1990.

[74] Michael J. Duff. "Supermembranes". 1996. Based on lectures given at the T. A. S. I. Summer School, University of Colorado, Boulder, June 1996; the Topical Meeting, Imperial College, London, July 1996 and the 26th British Universities Summer School in Theoretical Elementary Particle Physics, University of Swansea, September 1996. Dedicated to the memory of Abdus Salam. 76 pages, Latex, TASI96. hep-th/9611203.

[75] Tohru Eguchi and Hikaru Kawai. "Reduction of dynamical degrees of freedom in the large N gauge theory". *Phys. Rev. Lett.*, 48:1063, 1982.

[76] Angelos Fotopoulos and Arkady A. Tseytlin. "On gravitational couplings in D-brane action". *JHEP*, 12:001, 2002. hep-th/0211101.

[77] Daniel Friedan. "Introduction to Polyakov's string theory". To appear in Proc. of Summer School of Theoretical Physics: Recent Advances in Field Theory and Statistical Mechanics, Les Houches, France, Aug 2-Sep 10, 1982.

[78] Jürg Fröhlich and Jens Hoppe. "On zero-mass ground states in super-membrane matrix models". *Commun. Math. Phys.*, 191:613, 1998. hep-th/9701119.

[79] Kazuo Fujikawa and Kazumi Okuyama. "On a Lorentz covariant matrix regularization of membrane theories". *Phys. Lett.*, B411:261–267, 1997. hep-th/9706027.

[80] Kazuo Fujikawa and Kazumi Okuyama. "SO(9,1) invariant matrix formulation of a supermembrane". *Nucl. Phys.*, B510:175–198, 1998. hep-th/9709044.

[81] Masafumi Fukuma, Hikaru Kawai, Yoshihisa Kitazawa, and Asato Tsuchiya. "String field theory from IIB matrix model". *Nucl. Phys. Proc. Suppl.*, 68:153–164, 1998.

[82] Ori J. Ganor, Sanjaye Ramgoolam, and Washington Taylor. "Branes, Fluxes and Duality in M(atrix)-Theory". *Nucl Phys. B*, 492:191, 1997. hep-th/9611202.

[83] Yi-hong Gao. "Symmetries, Matrices, and de Sitter Gravity". 2001. hep-th/0107067.

[84] Murray Gell-Mann and Sheldon L. Glashow. "Gauge Theories of Vector Particles". *Ann. Phys. (N.Y.)*, 15:437, 1961.

[85] Jean-Loup Gervais and André Neveu. "The dual string spectrum in Polyakov's quantization. 1". *Nucl. Phys.*, B199:59, 1982.

[86] Jean-Loup Gervais and André Neveu. "Dual string spectrum in Polyakov's quantization. 2. mode separation". *Nucl. Phys.*, B209:125, 1982.

[87] Jean-Loup Gervais and André Neveu. "New quantum treatment of Liouville field theory". *Nucl. Phys.*, B224:329, 1983.

[88] Jean-Loup Gervais and André Neveu. "Green functions and scattering amplitudes in Liouville string field theory, 1". *Nucl. Phys.*, B238:396, 1984.

[89] Paul Ginsparg. "Matrix models of 2-d gravity". 1991. hep-th/9112013.

[90] Paul Ginsparg and Gregory W. Moore. "Lectures on 2-D gravity and 2-D string theory". 1993. hep-th/9304011.

[91] Ferdinando Gliozzi, Joël Sherk, and David Olive. "Supergravity and the spinor dual model". *Phys. Lett.*, B65:282, 1976.

[92] Ferdinando Gliozzi, Joël Sherk, and David Olive. "Supersymmetry, Supergravity theories and the dual spinor model". *Nucl. Phys.*, B122:253, 1977.

[93] Antonio Gonzalez-Arroyo and Chris P. Korthals Altes. "Reduced model for large N continuum field theories". *Phys. Lett.*, B131:396, 1983.

[94] Antonio Gonzalez-Arroyo and Masanori Okawa. "Twisted reduced chiral model". *Nucl. Phys.*, B247:104, 1984.

[95] Tohru Goto. "Relativistic quantum mechanics of one-dimensional mechanical continuum and subsidiary condition of dual resonance model". *Prog. Theor. Phys.*, 46:1560, 1971.

[96] Michael B. Green and John H. Schwarz. "Supersymmetrical dual string theory". *Nucl. Phys.*, B181:502–530, 1981.

[97] Michael B. Green and John H. Schwarz. "Supersymmetrical string theories". *Phys. Lett.*, B109:444–448, 1982.

[98] Michael B. Green and John H. Schwarz. "Covariant description of superstrings". *Phys. Lett.*, B136:367–370, 1984.

[99] Michael B Green, John H. Schwarz, and Edward Witten. *Superstring theory: Introduction*, volume I. Cambridge University Press, 1987.

[100] David J. Gross. "The C = 1 matrix models". In *Jerusalem 1990/1991, Proccedings, Two dimensional quantum gravity and random surfaces* 143-173.

[101] David J. Gross and Alexander A. Migdal. "A nonperturbative treatment of two-dimensional quantum gravity". *Nucl. Phys.*, B340:333–365, 1990.

[102] David J. Gross and Alexander A. Migdal. "Nonperturbative two-dimensional quantum gravity". *Phys. Rev. Lett.*, 64:127, 1990.

[103] Robert Helling. "Scattering in supersymmetric M(atrix) models". *Fortsch. Phys.*, 48:1229–1310, 2000.

[104] Robert Helling, Jan Plefka, Marco Serone, and Andrew Waldron. "Three graviton scattering in M-theory". *Nucl. Phys.*, B559:184–204, 1999. hep-th/9905183.

[105] Pei-Ming Ho and Sanjaye Ramgoolam. "Higher dimensional geometries from matrix brane constructions". *Nucl. Phys.*, B627:266–288, 2002. hep-th/0111278.

[106] Jens Hoppe. "Quantum theory of a relativistic surface". In *Florence 1986, Proceedings, Constraint's theory and relativistic dynamics*, 267-276.

[107] Nobuyuki Ishibashi, Hikaru Kawai, Yoshihisa Kitazawa, and Asato Tsuchiya. "A Large-N Reduced Model as Superstring". *Nucl Phys. B*, 498:467, 1997. hep-th/9612115.

[108] Satoshi Iso and Hikaru Kawai. "Space-time and matter in IIB matrix model: Gauge symmetry and diffeomorphism". *Int. J. Mod. Phys.*, A15:651–666, 2000. hep-th/9903217.

[109] Satoshi Iso, Yusuke Kimura, Kanji Tanaka, and Kazunori Wakatsuki. "Noncommutative gauge theory on fuzzy sphere from matrix model". *Nucl. Phys.*, B604:121–147, 2001. hep-th/0101102.

[110] Hiroshi Itoyama and Akira Tokura. "USp(2k) matrix model: F theory connection". *Prog. Theor. Phys.*, 99:129–138, 1998. hep-th/9708123.

[111] Daniel Kabat and Washington Taylor. "Linearized supergravity from matrix theory". *Phys. Lett.*, B426:297–305, 1998. hep-th/9712185.

[112] Hikaru Kawai, Shoichi Kawamoto, Tsunehide Kuroki, Toshihiro Matsuo, and Shin'ichi Shinohara. "Mean field approximation of IIB matrix model and emergence of four dimensional space-time". *Nucl. Phys.*, B647:153–189, 2002. hep-th/0204240.

[113] Hikaru Kawai, Shoichi Kawamoto, Tsunehide Kuroki, and Shin'ichi Shinohara. "Improved perturbation theory and four-dimensional space- time in IIB matrix model". *Prog. Theor. Phys.*, 109:115–132, 2003. hep-th/0211272.

[114] Vladimir A. Kazakov. "The appearance of matter fields from quantum fluctuations of 2-D gravity". *Mod. Phys. Lett.*, A4:2125, 1989.

[115] Vladimir A. Kazakov and Alexander A. Migdal. "Recent progress in the theory of noncritical strings". *Nucl. Phys.*, B311:171, 1988.

[116] Yoichi Kazama and Tetsuyuki Muramatsu. "Fully Off-shell Effective Action and its Supersymmetry in Matrix Theory". *Class.Quant.Grav.*, 18:2277, 2001. hep-th/0103116.

[117] Yoichi Kazama and Tetsuyuki Muramatsu. "Fully off-shell effective action and its supersymmetry in matrix theory II". *Class. Quant. Grav.*, 18:5545–5560, 2001. hep-th/0106218.

[118] Yoichi Kazama and Tetsuyuki Muramatsu. "A Theorem on the Power of Supersymmetry in Matrix Theory". *Nucl. Phys. B*, 613:17, 2001. hep-th/0107066.

[119] Yoichi Kazama and Tetsuyuki Muramatsu. "Power of supersymmetry in D-particle dynamics". 2002. hep-th/0210133.

[120] Yusuke Kimura. "Noncommutative gauge theories on fuzzy sphere and fuzzy torus from matrix model". *Prog. Theor. Phys.*, 106:445–469, 2001. hep-th/0103192.

[121] Yusuke Kimura. "Noncommutative gauge theory on fuzzy four-sphere and matrix model". *Nucl. Phys.*, B637:177–198, 2002. hep-th/0204256.

[122] Yusuke Kimura. "On higher dimensional fuzzy spherical branes". 2003. hep-th/0301055.

[123] Yoshihisa Kitazawa. "Matrix models in homogeneous spaces". *Nucl. Phys.*, B642:210–226, 2002. hep-th/0207115.

[124] Robert G. Leigh. "Dirac-Born-Infeld action from Dirichlet sigma-model". *Mod. Phys. Lett.*, A4:2767, 1989.

[125] Miao Li. "Matrix model for de Sitter". 2002. hep-th/0106184.

[126] Nikolaos E. Mavromatos and Jose Luis Miramontes. "Regularizing the functional integral in 2-D quantum gravity". *Mod. Phys. Lett.*, A4:1847, 1989.

[127] Yoichiro Nambu. "Duality and hydrodynamics". 1970. Lectures at the Copenhagen symposium.

[128] Hermann Nicolai. “Representations of Supersymmetry in Anti-de Sitter space.”. 1984. Presented at Spring School on Supergravity and Supersymmetry, Trieste, Italy, Apr 4-14, 1984. Published in Trieste School (1984) 368.

[129] Hermann Nicolai and Robert Helling. “Supermembranes and M(atrix) theory”. Lectures given by H. Nicolai at the Trieste Spring School on Non-Perturbative Aspects of String Theory and Supersymmetric Gauge Theories, 23 - 31 March 1998. 2000. hep-th/989103.

[130] Jun Nishimura, Toshiyuki Okubo, and Fumihiko Sugino. “Convergent Gaussian expansion method: Demonstration in reduced Yang-Mills integrals”. *JHEP*, 10:043, 2002. hep-th/0205253.

[131] Jun Nishimura and Fumihiko Sugino. “Dynamical generation of four-dimensional space-time in the IIB matrix model”. *JHEP*, 05:001, 2002. hep-th/0111102.

[132] Yuji Okawa. “Higher-derivative terms in one-loop effective action for general trajectories of D-particles in matrix theory”. *Nucl. Phys.*, B552:447–460, 1999. hep-th/9903025.

[133] Yuji Okawa and Tamiaki Yoneya. “Equations of motion and Galilei invariance in D-particle dynamics”. *Nucl. Phys.*, B541:163–178, 1999. hep-th/9808188.

[134] Yuji Okawa and Tamiaki Yoneya. “Multi-body interactions of D-particles in supergravity and matrix theory”. *Nucl. Phys.*, B538:67–99, 1999. hep-th/9806108.

[135] Sonia Paban, Savdeep Sethi, and Mark Stern. “Constraints from extended supersymmetry in quantum mechanics”. *Nucl. Phys.*, B534:137–154, 1998. hep-th/9805018.

[136] Sonia Paban, Savdeep Sethi, and Mark Stern. “Supersymmetry and higher derivative terms in the effective action of Yang-Mills theories”. *JHEP*, 06:012, 1998. hep-th/9806028.

[137] Joseph Polchinski. “Lectures on D-branes”. 1996. hep-th/9611050.

[138] Joseph Polchinski, Shyamoli Chaudhuri, and Clifford V. Johnson. “Notes on D-Branes”. 1996. hep-th/9602052.

[139] Joseph G. Polchinski. *An introduction to the bosonic string*, volume 1. Cambridge University Press, 1998.

[140] Alexander M. Polyakov. “Quantum geometry of bosonic strings”. *Phys. Lett.*, B103:207–210, 1981.

[141] Sanjaye Ramgoolam. “On spherical harmonics for fuzzy spheres in diverse dimensions”. *Nucl. Phys.*, B610:461–488, 2001. hep-th/0105006.

[142] Sanjaye Ramgoolam. “Higher dimensional geometries related to fuzzy odd- dimensional spheres”. *JHEP*, 10:064, 2002. hep-th/0207111.

[143] Alfred Schild. “Classical null strings”. *Phys. Rev.*, D16:1722, 1977.

[144] Nathan Seiberg. “Why is the matrix model correct?”. *Phys. Rev. Lett.*, 79:3577–3580, 1997. hep-th/9710009.

[145] Ashoke Sen. "D0 branes on T(n) and matrix theory". *Adv. Theor. Math. Phys.*, 2:51–59, 1998. hep-th/9709220.

[146] Ergin Sezgin. "The M-Algebra". *Phys. Lett. B*, 392:323, 1997. hep-th/9609086.

[147] Lee Smolin. "The cubic matrix model and a duality between strings and loops". 2000. hep-th/0006137.

[148] Lee Smolin. "M theory as a matrix extension of Chern-Simons theory". *Nucl. Phys. B*, 591:227, 2000. hep-th/0002009.

[149] Corneliu Sochichiu. "Continuum limit(s) of BMN matrix model: Where is the (nonabelian) gauge group?". 2002. hep-th/0206239.

[150] Leonard Susskind. "Another conjecture about M(atrix) theory". 1997. hep-th/9704080.

[151] Washington Taylor. "Lectures on D-branes, Gauge Theory and M(atrices)". *Phys. Rev. D*, 55:5112, 1997. hep-th/9801182.

[152] Washington Taylor. "The M(atrix) Model of M-theory". volume Iceland 1999, page 5112, 1999. hep-th/0002016.

[153] Washington Taylor and Mark Van Raamsdonk. "Three-graviton scattering in matrix theory revisited". *Phys. Lett.*, B438:248–254, 1998. hep-th/9806066.

[154] Washington Taylor and Mark Van Raamsdonk. "Supergravity currents and linearized interactions for matrix theory configurations with fermionic backgrounds". *JHEP*, 04:013, 1999. hep-th/9812239.

[155] Paul K. Townsend. "M theory from its superalgebra". In *Proceedings of the NATO Advanced Study Institute on Strings, Branes and Dualities, Cargese*, page 141, 1997. hep-th/9712004.

[156] Arkady A. Tseytlin. "Born-Infeld action, supersymmetry and string theory". 1999. hep-th/9908105.

[157] Cumrun Vafa. "Evidence for F-theory". *Nucl. Phys. B*, 469:403, 1996. hep-th/9602022.

[158] Paolo Valtancoli. "Solitons for the fuzzy sphere from matrix model". 2002. hep-th/0209117.

[159] Paolo Valtancoli. "Stability of the fuzzy sphere solution from matrix model". 2002. hep-th/0206075.

[160] Steven Weinberg. "Dynamics at Infinite Momentum". *Phys. Rev.*, 150:1313, 1966.

[161] Steven Weinberg. *The Quantum Theory of Fields*, volume II. Cambridge University Press, 1996.

[162] Peter C. West. "E_{11} and M theory.". *Class. Quant. Grav.*, 18:4443, 2001. hep-th/0104081.

[163] Yu-Jie Zhong and Wung-Hong Huang. "Quantum Energy of Fuzzy Sphere: Gaussian Variational Method". 2003. hep-th/0303196.