THE GEOMETRY OF LOGICAL OPPOSITION

PhD Thesis in logic, presented to the Faculty of Humanities
Institute of Philosophy
University of Neuchâtel, Switzerland

For obtaining the title of Philosophy Doctor

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Defended the 13th March 2009

University of Neuchâtel
2009

Université de Neuchâtel unine

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Key words:

geometry of oppositions, n-opposition, square of opposition, logical square, logical hexagon, logical bi-simplexes, logical poly-simplexes, contradiction, modal logic, concept, spatial logics, n-dimensional geometry, central symmetry, simplexes, Aristotelian p^q-semantics, Aristotelian p^q-lattice, conceptual spaces.

Summary:

The present work is devoted to the exploration of some formal possibilities suggesting, since some years, the possibility to elaborate a new, whole *geometry*, relative to the concept of "opposition". The latter concept is very important and vast (as for its possible applications), both for philosophy and science and it admits since more than two thousand years a standard logical theory, Aristotle's "opposition theory", whose culminating formal point is the so called "square of opposition". In some sense, the whole present enterprise consists in discovering and ordering geometrically an infinite amount of "avatars" of this traditional square structure (also called "logical square" or "Aristotle's square"). The results obtained here go even beyond the most optimistic previous expectations, for it turns out that such a geometry exists indeed and offers to science many new conceptual insights and formal tools. Its main algorithms are the notion of "logical bi-simplex of dimension m" (which allows "opposition" to become "n-opposition") and, beyond it, the notions of "Aristotelian p^q-semantics" and "Aristotelian p^q-lattice" (which allow opposition to become p-valued and, more generally, much more fine-grained): the former is a game-theoretical device for generating "opposition kinds", the latter gives the structure of the "opposition frameworks" containing and ordering the opposition kinds. With these formal means, the notion of opposition reaches a conceptual clarity never possible before. The naturalness of the theory seems to be maximal with respect to the object it deals with, making this geometry the new standard for dealing scientifically with opposition phenomena. One question, however, philosophical and epistemological, may seem embarrassing with it: this new, successful theory exhibits *fundamental* logical structures which are shown to be *intrinsically* geometrical: the theory, in fact, relies on notions like those of "simplex", of "n-dimensional central symmetry" and the like. Now, despite some appearances (that is, the existence, from time to time, of logics using some *minor* spatial or geometrical features), this fact is rather

revolutionary. It joins an ancient and still unresolved debate over the essence of mathematics and rationality, opposing, for instance, Plato's foundation of philosophy and science through Euclidean geometry and Aristotle's alternative foundation of philosophy and science through logic. The geometry of opposition shows, shockingly, that the logical square, the heart of Aristotle's transcendental, anti-Platonic strategy is in fact a Platonic formal *jungle*, containing geometrical-logical hyper-polyhedra going into infinite. Moreover, this fact of discovering a lot of geometry inside the very heart of logic, is also linked to a contemporary, raging, important debate between the partisans of "logic-inspired philosophy" (for short, the analytic philosophers and the cognitive scientists) and those, mathematics-inspired, who begin to claim more and more that logic is intrinsically unable to formalise, alone, the concept of "concept" (the key ingredient of philosophy), which in fact requires rather *geometry*, for displaying its natural "conceptual spaces" (Gärdenfors). So, we put forward some philosophical reflections over the aforementioned debate and its deep relations with questions about the nature of concepts. As a general epistemological result, we claim that the geometrical theory of oppositions reveals, by contrast, the danger implicit in equating "formal structures" to "symbolic calculi" (i.e. non-geometrical logic), as does the paradigm of analytic philosophy. We propose instead to take newly in consideration, inspired by the geometry of logic, the alternative paradigm of "structuralism", for in it the notion of "structure" is much more general (being not reduced to logic alone) and leaves room to formalisations systematically missed by the "pure partisans" of "pure logic".

Mots clés:

géométrie des oppositions, n-opposition, carré des oppositions, carré logique, hexagone logique, bi-simplexes logiques, poly-simplexes logiques, contradiction, logique modale, concept, logiques spatiales, géométrie n-dimensionnelle, symétrie centrale, simplexes, p^q-sémantique aristotélicienne, p^q-treillis aristotélicien, espaces conceptuels.

Résumé:

Notre étude est consacrée à l'exploration d'éléments formels suggérant, depuis quelques années, la possibilité d'élaborer une véritable *géométrie* propre au concept d' « opposition ». Ce dernier est très important et omniprésent (quant à ses applications), aussi bien en philosophie qu'en sciences et il admet, depuis plus de 2000 ans, une paradigme logique : la théorie de l'opposition d'Aristote, dont le point culminant au niveau formel est le « carré des oppositions ». En un sens, toute notre démarche consiste dans le fait de découvrir et d'ordonner selon une géométrie un ensemble infini d'« avatars » de cette structure carrée traditionnelle (le « carré logique » ou « carré d'Aristote »). Les résultats obtenus ici vont bien au-delà des attentes les plus optimistes, car il s'avère qu'une telle géométrie existe véritablement et offre à la science grand nombre d'intuitions conceptuelles et d'instruments formels. Ses principaux algorithmes sont les « bi-simplexes logiques de dimension m » (qui ouvrent à la n-opposition) et, au-delà, les notions de « p^q-sémantique » et de « p^q-treillis » aristotéliciens (qui ouvrent à l'opposition p-valuée et, plus généralement, à des oppositions plus fines) : la première est un moyen jeux-théorétique d'engendrer des « catégories d'oppositions », le deuxième donne la structure des « cadres oppositionnels » qui contiennent et ordonnent les catégories d'opposition. Avec ces moyens formels, cette notion atteint à un niveau de clarté conceptuelle jamais connu auparavant. La naturalité de cette théorie est maximale par rapport à son objet d'étude, cette géométrie semble donc pouvoir prétendre au rang de nouveau standard scientifique pour traiter de phénomènes d'opposition. Toutefois, une question philosophique et épistémologique peut troubler : cette théorie, conquérante, montre que des propriétés *fondamentales* de la logique sont *intrinsèquement* géométriques : par les notions de « simplexe », de « symétrie centrale n-dimensionnelle », et par d'autres du même genre. Or, malgré les apparences (qui laisseraient croire cela déjà vu), ce fait est révolutionnaire. Il réveille un débat ancestral, toujours actuel, sur l'essence des

mathématiques et de la rationalité, opposant par exemple le géométrisme (Euclidien) philosophique et scientifique de Platon à sa critique transcendantaliste par la logique d'Aristote. La géométrie des oppositions montre de manière choquante que le carré logique, le cœur stratégique du dispositif transcendantal anti-platonicien d'Aristote, est en fait une *jungle* formelle platonicienne, qui contient élégamment d'infinis hyper-polyèdres. Qui plus est, cette découverte d'une géométrie infinie au cœur même de la logique est également liée à un terrible débat de fond entre les partisans d'une « philosophie guidée par la logique » (les philosophes analytiques et les cognitivistes) et ceux qui, inspirés plutôt par les mathématiques, commencent de plus en plus à avancer que la logique est intrinsèquement incapable de formaliser, en elle-même, le concept de « concept » (l'ingrédient principal de la philosophie), qui requière plutôt l'aide de la *géométrie* afin de déployer naturellement les « espaces conceptuels » (Gärdenfors). Nous proposons donc quelques réflexions au sujet de ce débat et de ses liens profonds avec la nature des concepts. En guise de résultat épistémologique, nous avançons que la théorie géométrique des oppositions révèle, par contraste, le danger inhérent à la réduction des « structures formelles » aux « langages symboliques » (i.e. à la logique non géométrique), ainsi que le fait pourtant, paradigmatiquement, la philosophie analytique. Nous proposons en lieu de cela de réanimer, suite au brillant exemple de la géométrie des oppositions, le paradigme alternatif du « structuralisme », puisque la notion de « structure » est bien plus riche (ne se réduisant pas à la seule logique) et ouvre à des formalisations systématiquement perdues par le fétichisme de la « logique pure ».

à ma mère

al caro ricordo di mio padre

denen ich – wenn nicht alles – unglaublich viel verdanke

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SYNOPSIS

General overview. The object of this study is an attempt of ours to build a general, formal theory of opposition. We rely on a preliminary philosophical analysis determining, among others, three main ways of conceiving "opposition": a static, a dynamic and an intensive one. Our formal approach begins by focussing on the static side of oppositions. We start by showing the existence of a previously unnoticed geometrical-logical parallelism (one generalising Aristotle's "logical square" and Sesmat-Blanché's "logical hexagon of opposition" into the notion of "logical bi-simplex of dimension *n*"), and by developing systematic, formal translation rules allowing to usefully switch from one domain to the other. The one domain is, broadly (but not exclusively) speaking, modal logic, abstract (standard, hybrid, multi-modal, etc.) as well as applied (i.e. alethic, deontic, temporal, epistemic, etc.). The other domain is a new, rich field, made of *n*-dimensional complex, but very regular solids, yielding a new discipline which we, joined by some others, call "*n*-opposition theory" (NOT). Among its current main domains, NOT includes three very important nested branches: "*n*-opposition theory" (the study of the bi-simplexes and their use), "poly-simplicial opposition theory" (the study of the extension of the bi-simplexes) and "symplectic-lattice theory" (the study of the extension of the poly-simplexes). From the first to the third, we go from particular to general. We argue that this constitutes a deepened understanding of the concept of static opposition, which we prove to be, contrary to what was expected, infinitely rich (apparently a new branch of science between logics and mathematics). Additionally, NOT introduces the workable notion of "dynamic opposition" (a theory of the discrete transition between different opposition states) and, apparently, the possible beginnings of a treatment of intensive oppositions (the third qualitative kind of oppositions) by means of the conjectural notion of "opposition field". The applications of this new understanding of opposition phenomena seem quite numerous already, but there will probably be more than the ones already known (due to the extreme generality and exportability of this concept). Remark that, truly speaking, the scope of the present geometrical treatment of oppositions is much wider than the sole realm of modal logic, which it nevertheless maps out. For oppositions also link together *concepts*, and there are strong arguments in favour of thinking that concepts, structurally speaking, have inner geometrical features irreducible to formal logic, especially when the latter is conceived solely as a symbolic calculus. The geometry of oppositions (as explored by NOT) seems to embody another dimension of the logical. With this respect, a reflection is proposed on the difference between a conception of the "formal" reducing it to the logical or to the algebraic, and one open to geometrical (many-dimensional) features *per se*. This concerns, formally as philosophically, a possible future revival of the research paradigm called "structuralism".

In what follows, we give a synthesis of the different sections of the PhD (with less details about the more technical parts II and III).

Philosophical introduction. In three chapters we deal with methodological questions preliminary to our inquiry. First we try to show how very important for thought in general the concept of "opposition" is; related, among others, to the concepts of "negation", "definition", "existence" and "struggle", and to schools of thinking such as, for instance, "gestalt theory" and "structuralism". Second, we face the problem of handling in the best way possible the concept of "concept". Following P. Gärdenfors' warning (in his impressive theory of the "conceptual spaces", 2000), we examine logic and geometry as being two currently alternative paradigms, of which the former is easy to use but rather fruitless, whereas the latter is "close to the facts" but highly complex (geometry is in some sense preferable to logic, but geometry needs reduction). Thirdly we motivate the observable *datum* (which will be confirmed by the rest of the study) that, concerning opposition phenomena, the two rival approaches (logic and geometry) happen to combine coherently (beyond expectations). So, besides allowing us to study opposition with double (powerful), non-conflictive means, this rather exceptional fact, relative to a notion (opposition) common to logic as well as geometry, seems to us to possibly disclose interesting new views on the deep nature of thinking, deductively as well as conceptually (for we keep their distinction). To this issue we will try to come back at the end.

Part I. The classical theory of opposition and its sudden metamorphosis. Here we trace back, along 7 chapters, the path that leads from Aristotle's "opposition theory" (still considered the standard nowadays) to the premises of the "*n*-opposition theory" proposed by us. First, we recall the context (the Greek thought) in which Aristotle's doctrine took place, and the precise way it had to display its elements into the "logical square"(of opposition). Second, we recall the classical formal applications of this logical square, in contemporary mathematics as well as in contemporary logic, where it still holds a crucial position. Third, we recall the main known extra-logical applications of the square of opposition until today. This comprises a series of fields very different in nature (psychology, semiotics, psychoanalysis, anthropology, linguistics, philosophy …). Fourth, we recall one of the most interesting attempts to think opposition beyond Aristotle, that is N.A. Vasil'ev's "imaginary logic" (1910-1913), grounded on some pioneering reflections of his about a *triangular* (the "triangle of contrariety"), rather than the Aristotelian *square* model of logical opposition. Fifth, we show how Vasil'ev's very original and powerful (if not recognised) attempt misses a target that is reached by A. Sesmat's (1951) and R. Blanché's (1953) independent discovery of a "logical hexagon" (of opposition), of which Vasil'ev's triangle of contrariety is one of the ingredients (but not the whole). This hexagonal structure is very powerful and contains the logical square as a particular case or fragment. All the logical (and mathematical) power of the logical square is kept and in fact augmented in the logical hexagon. Sixth, we recall, in addition, the principal known applications of the hexagon, which again goes through a very large scope of varied applications, sometimes quite spectacular and powerful. Seventh, we recall how J.-Y. Béziau (2003), defending the idea of logical paraconsistency against H. Slater's charges (1995), discovered two new "decorations" (one paraconsistent and one paracomplete) of the hexagon (in addition to the classical one), and thus reasons for suspecting the existence of a logical-geometrical structure more complex than the hexagon (possibly three-dimensional), but containing the three known versions of it. Some further discoveries of H. Smessaert and ourselves allowed Béziau's intuition to become quite real, under the species of a solid of opposition called "logical cuboctahedron".

Part II. The new theory of opposition: many opponents. In this part we detail, in 7 more chapters, "*n*-opposition theory" (NOT), the formal framework which emerged as adequate and necessary for housing the geometrical-logical objects seen so far (and *all* the following ones). Using the standard notion of "geometrical simplex of dimension *m*", this gives the new notion of "logical bi-simplex of dimension *m*", of which the logical square, hexagon and cube (a new oppositional structure discovered by us) are the first three elements (logical bi-segment, bi-triangle and bi-tetrahedron). We go from opposition to *n*-opposition in so far as our logical bi-simplexes do allow to think opposition not only for 2 terms (the logical square) or 3 terms (the logical hexagon) but for any number *n* of terms. Since bi-simplexes as such are empty oppositional structures, they need a modal decoration. This is done with the help of a powerful decorating technique, developed (among other important discoveries) by R. Pellissier (2005). Thanks to this, we prove the existence of a series of higher-order "gatherings" (i.e. sets of geometrically interlaced bi-simplexes), the hyper-tetraicosahedra (Pellissier discovered the "logical tetraicosahedron"). Hence we try to construct, relying on the series of the hyper-tetraicosahedra and on Pellissier's powerful method, which we extend as for its scope of applications, a general translation rule from any finite "modal graph" (the graph displaying, for any modal system, its basic modalities and their mutual relations) to its oppositional solid geometry. Remark that if finding a general such algorithm remains hard (each complex case is treatable, but with *ad hoc* applications of Pellissier's method), the theory in itself is very powerful, for it owns a mathematical proof of the fact that future, still unknown oppositional solids will have to be bi-simplexes or gatherings of bi-simplexes and therefore will have to respect constraints already known by the theory. We end this part by a large panel of known applications of it, logical as well as not.

Part III. From static to dynamic opposition. In this part, containing 7 chapters, we show that *n*-opposition theory can be radicalised once more. For the notion of logical bi-simplex can in fact be usefully generalised to that of "logical *p*-simplex". We do this by introducing the game-theoretical notion of "Aristotelian p^q-semantics", a generalisation of the simple combinatorial algorithm (implicitly) used by Aristotle to generate the semantics of the logical square (a quaternary semantics). We show that if a so-called 2^2-semantics (the one of the square) generates logical bi-simplexes, a p^2-semantics generates logical *p*-simplexes and square lattices of length *p* (later we will similarly have 2^q-semantics and then general p^q-semantics). This allows to have many more kinds of opposition, whereas any bi-simplex keeps Aristotle's constraint of admitting only 4 kinds of opposition. This is a big qualitative gain, for it makes opposition qualitatively much more fine-grained. It turns out that *p*-simplexes are perfectly fit for handling many-valued (in fact *p*-valued) oppositions, which is something totally new. But as many-valued logics in general have been heavily charged with the theoretical suspicion of being a formal deceit (Suszko), in order to defend ourselves against a possible similar accusation (being ourselves committed by now to the *p*-valuedness of the concept of static opposition), we recall the very recent first satisfactory theoretical answer given to that accusation (Malinowski, Shramko, Wansing). We suggest (as a conjecture for the future) that the strange but powerful formalisms produced for that occasion (the trans-Suszkian many-valued logics) could be the logical-axiomatical counterpart of the logical *p*-simplexes (or, the other way round, the logical *p*-simplexes could be the geometrical "Aristotelian" counterpart of the trans-Suszkian logics). So to say "coming down to earth", relying on such logical *p*-simplexes, we show some ways of conceiving a formal (geometrical-logical) treatment of dynamical oppositions (i.e. the second qualitative kind of opposition, according to our philosophical categorisation of the beginning), along with a few different complementary strategies raised as answers to five possible challenges addressed by ourselves to NOT. One deals with transitions between differently valued states. However, a further radicalisation of NOT is possible. For, as we will show, all the *p*-simplexes are characterised by so-called “Aristotelian square lattices”. Considerations on the combinatorial possibilities of the Aristotelian general p^q-semantics show that we can have correspondingly, instead of the square ones, a series of “simplectic lattices” (lattices of shapes corresponding to the simplectic series of segment, square, cube, hyper-cube, etc.). This issue, not yet fully explored, could strengthen our means of elaborating a complex theory of “opposition dynamics”, conceived mainly as a study of the possible transitions between fine-grained opposition situations. As for the third kind of qualitative oppositions, the intensive ones, and taking “opposition” as a general term for both repulsive and attractive phenomena, we suggest – because loving does not imply being loved – a strategy aiming at being able to formalise general “asymmetric oppositions” (despite the fact this may seem a *contradictio in adiecto*): if this turns out possible (as we conjecture and explore hypothetically), once mastered it could yield a way of extending NOT by expressing subject-centred “opposition fields” (whereas static oppositions are symmetric and objects-linking), allowing in turn to modelise emotional states and intersubjectivity situations as sedimentations of opposition fields relative to different subjects.

Conclusion. In the last two chapters we discuss what has been seen previously. First comes a recapitulation of the “forgotten things” that opposition phenomena urge to handle theoretically, and second a recapitulation of the kind of conceptual performances that our new theory of opposition allows or does not allow, with particular mention of its opening to *dynamical* and *intensive* aspects, whereas traditionally, opposition is thought of as a *per se* static phenomenon. Consequently, we reconsider Gärdenfors’ challenge to concept theory, by comparing our very young results on opposition dynamics with his own on conceptual spaces. Then, both from the point of view of pure logic and from that of conceptual (non strictly logical) dynamics, we try to position our discoveries with respect to the new science named “universal logic” (Béziau) – a systematic exploration of the infinite multiplicity of logics, with its fractal invariants –, especially from the point of view of the thinkable status of negation inside it. This throws some new light on the essence of negation (classical, paraconsistent, paracomplete, ...) and thus over logic itself, by which move we somehow return to the starting point of the whole adventure (hopefully not without some nice gains).

Introductive, Philosophical Part

THE NEED TO UNDERSTAND WHAT AND HOW OPPOSITION IS

01.
THE IMPORTANCE OF THE CONCEPT OF “OPPOSITION”

In this opening chapter, the first of the three preliminary ones of our study, we want to discuss the concept of “opposition”, in order to show, after recalling it intuitively, that it bears a particular importance among all possible concepts. Our aim here will be solely to present its starting intuitive richness and problematicity, and show that this situation deserves conceptual clarification (which will be the object of our study as a whole). Progressively, after recalling the main intuitive varieties of opposition (with their main motivations and originating grounds), as well as their respective pros and cons, we will justify the evocation of a standard theory of opposition – which is Aristotle’s one, with its formal central piece, the “logical square” and its four “flavours” of opposition – as well as the theoretical reactions and issues it has given rise to (this will be developed more fully in part I of this study). Then we will discuss the possible ways in which this standard theory can be seen and understood, in particular in relation with philosophy, logic and geometry. Starting from this basis, later, we will also prospect some possible developments of this standard theory of opposition, developments which will be the general object of this study. Some of them are recent (part II), and some others are totally new and so far unknown (part III of this study).

01.01. The starting intuition on what is an opposition.

What is “opposition”? Everyone of us knows many intuitive answers (or examples) to this question. For instance, these examples are classical: “black” and “white”, “being” and “not being”, or “true” and “false” can be seen as being opposed, as can be “friend” and “enemy”, or (in some respects) “female” and “male”, and so on.

Nevertheless, a more precise answer to this simple question (“what is opposition?”) is not so easy to give as it could seem. Is, for instance, “friend” really opposed to “enemy”, or must the term opposed to “enemy” be “enemy” itself? (by some kind of autonymy) If so, how to understand the difference between these two oppositions? (actually both are) To take another example, is the opposed term to “male” “female”, or is it more precisely (and vaguely) “not male”? How to relate these two possible answers? (“‘male’ is both opposed to ‘not male’ and to ‘female’) A more important complexifying remark could specify that there seem to be, at a first glance, at least three qualitative kinds of opposition: static, dynamic and intensional: the ones we spoke about (black VS white, etc.) belong to the first kind, the second one being that of the "active oppositions" (competitions, fights, wars, uni- or bi-lateral aggressions, etc.) whereas an instance of the third kind of opposition could be the primitive but strong feeling of a baby disliking something (to which he/she is clearly "opposed"). But in order to postpone clearer technical definitions and theories of opposition, let us limit ourselves to a very clear but necessary remark: there are good reasons to see opposition as being one of the major concepts of human thought. Let us see why.

01.02. Why is this concept so important?

One of the main reasons for it to be so important is that this concept is *crucial*. That is, it lays at the intersection (the "cross") of very different and fundamental fields. First of all, opposition is a concept ruling deeply very fundamental fields, namely the *ontological*, the *practical* and the *theoretical* fields (one could probably add the *ethical* field, if separated from the practical).

If we take "ontology" to mean the general fundamental science of what there is and of what happens, opposition seems to be fundamental for ontology in at least two senses. For if we take ontology to mean, classically enough, the "science" (or the theory) of "being as such" (or: "pure being", "pure existence", etc.), it seems that (1) "being" (or "existing") means "being different from some ontological (or existential) background or *substratum*" (i.e. it means an opposition). (2) Correlatively, if we take "ontology" to mean the science of changing (i.e. becoming, coming to being), "change" (or "becoming") seems to be fundamentally related to the notion of opposition: for real change can occur only between different (and thus opposed) states (the initial and the final ones).

Opposition is also fundamental for the practical sphere (action) in at least three senses. (a) It is necessary in order to have the possibility of "making something change" (for, again, the change can only occur between two somehow opposed things or states). (b) Opposition is necessary for "choosing": the choice, to be such, must oscillate between two opposed alternatives (otherwise, if the two candidates are the same one, it is no real choice). (c) It is necessary in order to have competition (or fight): for competing (or, *a fortiori*, fighting) means being somehow opposed.

Lastly, opposition is fundamental for the theoretical sphere in at least two senses. (i) It is necessary in order to "think", for thinking, as an action, consists, at least partly, in distinguishing things, by opposing them mutually in various degrees (not to mention the fact that thinking is a flux, and thus, a change again, and thus, again, a series of oppositions). (ii) From a more “architectural” point of view, opposition is also necessary in order to make a theoretical foundation: for, as we will recall later, giving a theoretical (at least philosophical) foundation to something consists most of the time in showing that the negation of the starting thing is impossible (*reductio ad absurdum*): which consists in “being opposed to an absurdity”.

These three still unformalised considerations show clearly, beyond rhetoric, the *extreme* importance of the intuitive (and not yet technically specialised) concept of opposition.

01.03. Predominance of its “static” over its “dynamic” aspect

As already mentioned, opposition can be viewed as admitting at least three qualitative kinds: static, as in the case of the classical dichotomies (“black or white”, “true or false”, etc.); dynamic, as in the case of competitions (“I play against you”); or even as intensional (i.e. emotional), as in the case where some “negative” feeling is felt by someone towards something or someone else (“antipathy”, “hate”, “disgust”, “scandal”, etc.). Now, one remarkable point is that the static aspect seems to overcome theoretically the two other aspects. This is probably due to the fact that, from the point of view of pure theory, the static aspect is more abstract (and thus more general, and theoretically prior): as if the dynamic and the intensional oppositions could be constructed on the base of the static one (we will return to this important point later). Remark that from the psychological (genealogical, historical) point of view this would certainly be false: psychology, psychoanalysis and evolutionary cognitive science all teach us that the abstract stage of thought (language, conscious thinking, theory) emerges from a more sensitive basis (like the child’s painful and conceptually vague sensations of opposition). Not so in an abstract, axiomatic view, where it is important to build complex formalisms starting from the simplest ones. The etymology of the word “opposition” seems to confirm this (conceptually useful) predominance of the static aspect: the Latin word “oppositio” comes from “ob-positio” (akin to “ob-jectum”, object), the “position (*positio*) of something in front (*ob*) of something else”. The same can be found in ancient Greek: “αντικειμενον”, analysable as “αντι–κειμενον” means more or less the same thing: the “laying (κεισθαι) one in front (αντι) of the other”. This means that, traditionally (at least in the West), opposition is thought (theoretically) *in the space*, as a play of relative *positions*. And from this spatial point of view, one can remark that there are at least four ways of being spatially opposite, for there can be: a "specular" opposition (like in mirrors), a negative opposition (like between a photograph and its negative image), a differential opposition (the simple fact of occupying a different place) and an ordered opposition (the fact of belonging to an ordered couple – "I'm ahead, you are behind me").

01.04. Two main historical (geographical) paradigms of opposition

Relatively to the previous distinctions, the history of thought has more or less displayed into the geographical space two major traditions of thinking relatively to the concept of opposition. One is the Western (centred on Greek thought), the other is the Eastern (centred on the Chinese thought). The former is more concentrated in the static aspects, with a predilection for sharp distinctions (strong oppositions), the latter is more open to dynamic aspects, and privileges nuances and subtle harmonies (subtle oppositions).

The Western tradition has mainly thought of opposition in three stages. First, Parmenides has sharply distinguished "being" from "not being": doing this he laid the first element of a coherent theory of (static) opposition. His opposed counterpart, Heraclitus, held a non-static view on opposition (which is traditionally called "enantiodromy", the fact that the opposite "run" the one toward the other). Indeed Parmenides thus originated the philosophical-logical very important notion of "absurdity" (ατοπος, the "placeless"), i.e. what cannot (shall not) be conceived, defined nowadays as "contradiction" ("A and not A", "$A \wedge \neg A$"), defined by him as the monstrous "conjunction of being and not being". He thus opened the door to philosophical-logical-scientific thinking, separating it from mythological thinking[1]. Because Parmenides' view (contradiction as a strict opposition between something and its negation) generated some big apories ("motion does not exist", etc.), Plato has deepened this theory by introducing a distinction between "not something" (negation of something) and "other than something" (alterity of something): doing this he introduced one of the first precise theories of the distinctions between different kinds of (static) opposition[2]. Aristotle brought the third element of this story, by conceiving a beautiful theory where there are not two but four kinds of oppositions, all related in such a way that a unique mathematical object expresses them all at once: the "logical square" (or "square of opposition" – of this we will speak in detail in chapter 4). This theory has become standard and is still at the heart of contemporary science (as we will show in ch.05 *infra*).

[1] In what remains to us of his poem "On the nature" ("Περι φυσεως").
[2] In his dialogue *The Sophist*.

On the other side, among the many components of the Eastern tradition let it suffice to mention here quickly, for China, the Daoist (or Taoist) approach (Lao-Zi, Chuang-Zi, Lie-Zi, …), whose fundamental intuitions about the "Dao" (the Way) concern the way all things are dynamically related by the complex opposition of the "Yin" and the "Yang" (we will spend some more words on this in chapter 9). What is generally said about traditional Chinese thought is that it is effective without any reference to some kind of ontology: "process" is preferred to the Western "composition of pre-existing elements"[3]. In India, where ontology is generally present, at least two schools must be remembered for their complex relation to opposition: the "Advaita Vedanta" (i.e. the "non-dualist end of the Veda") of Shankara, which, as the name "Advaita" tells (a-dvaita), is an investigation (rather powerful) over the conceptual intricacies of a "non-dualist" non-monist resolution of metaphysical oppositions[4]; and the Nyaya school of philosophy (founded by Gotama), which developed, with a huge chronological advance with respect to Western thought, a logical calculus with five truth-values: which allows to express finer oppostions (the Buddhist Nagarjuna had already developed a 4-valued logic)[5]. From this point of view there is some kind of true myth (or cliché), possibly justified, of an Eastern more fluid approach to opposition.

01.05. Philosophical problems of the two distinct approaches

The two approaches just seen (Western and Eastern) are interesting in so far they seem to be complementary (if not yet compatible): it seems nice to have both looks on opposition, rigid and flexible. But before asking ourselves if these two can be somehow unified (or integrated), we must review the main problems, rather severe, inherent to both of them (separately).

The Western approach to opposition must face some major problems, related mainly to the fact that if the "rigid opposition" has nice scientific properties (for it opens to logical, mathematical and physical rigour, cf. ch.5 *infra*), it seems to lead to at least two quite difficult philosophical positions. (1) Historically speaking, it has regularly led, as a consequence of its rigid axioms, to hold so-called "necessitarian" positions. Necessitarianism is, broadly speaking, the doctrine according to which "all is necessary (in various ways): there is no place

[3] Elements of this can be found in the reference works of F. Jullien. On him, cf. J. Alluch et alii (eds.), *Oser construire. Pour François Jullien*, Les Empêcheurs de penser en rond, Paris, 2007.
[4] Cf. E. Deutsch, *Advaita Vedanta: A Philosophical Reconstruction*, University of Hawaii Press, Honolulu, 1973.
[5] Cf. J. Filliozat, *Les philosophies de l'Inde*, PUF, Paris, 1987 (1970), p. 81-91.

for real (ontological or psychological) freedom"[6]. Four thinkers at least deserve to be remembered here in this respect. The Megarian Diodorus Cronus (4th century BC) showed that logical bivalence (i.e. our logic, the one with only two truth values, the "true" and the "false"), if assumed to rule ontology, implies that all the future (all its succession) is already determined (we will return to this in chapter 4). In a somehow similar way, Spinoza (in his *Ethics*, around 1676) showed that the traditional metaphysical term "substance", if internalised to ontology through a clear set of definitions (Spinoza uses systematically strict opposition – instead of vague concepts – through an axiomatically grounded series of *reductiones ad absurdum*), implies a metaphysical view where there is one substance and one only, one which is fully determined and leaves no real place (other than illusion) to freedom (and free-will). More recently (around 1941), one of the biggest logicians ever, Kurt Gödel has tried to demonstrate, in a believer's vein, the existence of God (at least of a very abstract metaphysical basis of the traditional Judeo-Christian concept of God) relying on some very fundamental (and complex) notions of mathematical logic: as a result, similarly to Spinoza, he got *volens nolens* to a proof that if contemporary modal logic (an expression of the strict oppositions – in fact an expression of the logical square)[7] is internalised to ontology, the consequence indeed seems to be the demonstration of the necessary existence of a unique being ("God"), but one which has strongly necessitarian properties ("if this God exists, we – and Him/She/It – are not free")[8]. Lastly, in times close to us, the Italian philosopher Emanuele Severino (already mentioned by us) has developed an impressive philosophical system where the Parmenidean notion of "strict opposition" is always kept (against "Platonic betrayals"), so that at the moment it can be said that Severino is the greatest living philosopher of non-contradiction (maybe even the greatest so far *stricto sensu*)[9]. The metaphysical result of his ontological assumption ("never the slightest conceptual contradiction") is a very strange, counterintuitive but still convincing (and troublesome) thesis of the "eternity of all beings",[10] which, again, implies the strict absence of real ontological freedom[11]. In all these four examples, as in some others, it appears clearly that the consequence of axiomatising (and internalising) strict opposition tends to be, regularly, the uneasy (i.e. unpopular) assumption of the strict inexistence of freedom. (2) Additionally, the Western approach to opposition has

[6] Necessitarianism is a more general version of "determinism", possibly immune to the criticisms raised against determinism by the philosophy of quantum mechanics.
[7] Gödel did not stress this geometrical feature, we stress it here in order to keep the relation to opposition clear.
[8] Cf. Wang, H., *Kurt Gödel*, Paris, Armand Colin, 1987, p. 194-195.
[9] Cf. E. Severino, *La struttura originaria*, Adelphi, 1981 (1957).
[10] Actually, very similar to some of the principal "modal realist" views of D. Lewis.

also general difficulties to grasp "change". For, paradoxically enough, if we have seen previously that opposition (of the initial and the final state) is necessary in order to think real change, it is nevertheless true that the very transition itself remains mysterious and requires, at its infinitesimal unescapable levels of transit, an intermediate continuity which can be easily turned into serious paradoxes (of self-contradiction). Thinking through rigid oppositions does not think "change" in itself, it just splits it into arbitrarily small constitutive oppositions. As classical examples of this, one can recall Zeno's demonstration of the inexistence of "motion"; less spectacularly, but still dramatically, Aristotle is obliged, in his general metaphysical system constructed on top of the principle of non-contradiction, to let some degree of contradiction exist at the level of pure change (the articulation "power/act"), in order to make this last possible, as remarked in a very critical vein by Severino.

On the other hand, the Eastern approach to opposition, in turn, has, in some sense, the opposite problem with respect to the Western approach: if it is compatible with common-sense intuition, both for freedom and for change, it is not very science-friendly, it does not leave much theoretical room to precise logical, mathematical or physical development, which requires rigid assumptions over axioms, definitions and rules. As for the 5-valued logics of the Nyaya or the 4-valued logic of Nagarjuna, which resemble the Western Way (logical approaches of opposition) except that they pluralise the notion of truth, it undergoes a different kind of problems that we will recall, as general criticisms addressed to many-valued logic, later (cf. ch.23 *infra*).

So the question may now be raised of having some kind of intermediate solution or mix.

01.06. Internal challenges to the standard (Western) approach

The need to look for some new solution concerning opposition (i.e. other than the bare Greek solution, that is – we will see –, Aristotle's square), "Easternalised" or not, is stressed, inside our own Western tradition, by the constant existence, through history, of explicit strong criticisms against the standard rigid approach, despite its formal logical-mathematical power. There are at least four kinds (or families) of criticisms.

(a) The first one is embodied by "relativism", a recurrent trend of human thought, whose first shocking (but also stimulating) apparition was the so-called Greek Sophistic (5th

[11] Cf. E. Severino, *Essenza del nichilismo*, Adelphi, Milano, 1972; cf. E. Severino, *Destino della necessità*, Adelphi, Milano, 1979.

century BC)[12]. In a same movement, the Sophists criticised in many ways the sacral character of the social order by showing that one of its main hieratic vectors and pillars, the "sacred" and prophetic "poetic language", was totally unjustified to fulfil its alleged normative sacral role. They stressed the many malfunctionings of human language, by discovering many troubling "sophisms" (i.e. strange convincing yet false reasonings), making evident its contingent and *ad hoc* grammatical and semantic conventional essence. They pushed so far this innovative line of thought, that they endorsed the general shocking thesis on the "anti-logic character" of human language: "the structure of the language and that of the world are such that on any given subject two contradictory (i.e. strongly opposed) opinions can always be developed with full coherence" (so Protagoras, paraphrased)[13]. Two things must be remarked here. First, the theory of opposition (at least in its Platonic and Aristotelic parts, if not in the Parmenidean one) has been developed for quite a large extent *precisely* in order to extirpate the "Sophistic danger" (their general relativism, reaching moral values, was seen by Plato and Aristotle as socially seditious)[14]. Second, this kind of intellectual relativism is universal (it is structural): it appears from time to time in very distant and unrelated places of the world (it appeared in old China, for instance)[15], and it newly exists nowadays, especially under the famous name of "Post-Modernism" (with thinkers like Lyotard, Derrida, Rorty, Vattimo, etc.). Relativists typically think that non-contradiction (strict opposition) is far too restrictive[16]. They sometimes propose (as does Vattimo) to develop instead a "weak thought"[17].

(b) The second historical big challenge to standard opposition theory was sent by Hegel, with his concept of "dialectic" (originally inspired by Plotinus and Fichte). Hegel thought that the standard (Aristotelian) opposition was incomplete, for it lost "life" and

[12] The reference work on them is: M. Untersteiner, *I sofisti*, Mondadori, Milano, 1996 (1949). Cf. also G. Romeyer-Dherbey, *Les Sophistes*, PUF, Paris, 1989 (1985).

[13] Diogenes Laertius says literally: "He was the first to maintain that on any thing there are two [solid] discourses contradicting themselves mutually" (quoted by Romeyer-Dherbey, *Op. Cit.*, p.11)

[14] This strong link between theoretical choices in logic and political concerns (in Aristotle) has been put forward by Łukasiewicz in his two writings on Aristotle's "indirect demonstration" of the Principle of Non-Contradiction: J. Łukasiewicz, "Über den Satz des Widerspruchs bei Aristoteles", Bulletin international de l'Académie des sciences de Cracovie, 1910 (French translation: "Sur le principe de contradiction chez Aristote", *Rue Descartes*, April 1991, 1.2); and J. Łukasiewicz, *O zasadzie sprzeczności u Arystotelesa*, (1910) (French translation: *Du principe de contradiction chez Aristote*, L'éclat, Paris, 2000).

[15] This is the Ming Kia, the "School of names" (4th and 3th centuries BC)), cf. M. Kaltenmark, *La philosophie chinoise*, PUF, Paris, 1987 (1972), pp.53-54.

[16] This Post-Modernist position is clearly endorsed by Vattimo, against Severino's strong defence of ontological non-contradiction, cf. Severino E. and Vattimo G., "In cammino verso il nulla. Emanuele Severino", in: G. Vattimo (ed.), *Filosofia al presente*, Garzanti, Milano, 1990 (translated into French by us: "Sur la voie du néant. Emanuele Severino", *Les Cahiers de l'ATP*, (2003), http://alemore.club.fr/CahiersATP.htm).

"process", and had to be better understood theoretically in order to have a real impact over reality. As is well-known, his main idea was that the concept of "concept" gives rise to two possible apprehensions, one by "intellect" and one by "reason" (the distinction of these two "faculties" comes from Kant): the difference of the two, namely the fact the reason is total, but practically finitely unreachable, whereas intellect is accessible but always deadly partial (finite), generates, according to Hegel, an inescapable but fertile complex semantic process of ternary transitions, mental as well as natural, that constitutes an infinite "dialectical process". This is supposed to be generated by opposition ("negation") itself, for, truly understood (i.e. dialectically), this structure has the power of generating, from any possible starting point, and by means of problematic but lively contradictions, a cascade of progressive transformations, explaining all spheres of reality.

(c) One third historical big challenge to opposition theory was advanced by Freud (1915): for his convincing concept of "psychic unconscious structure", which helps much in explaining psychological (important) phenomena previously left as totally impossible to understand, needs in turn the assumption of at least five strange (but empirically convincing) "meta-psychological axioms", among which one postulating the irrelevance of the principle of non-contradiction in the unconscious fields (mainly in the structure of psychopathology and in that of dreams). In other words, Freud put forward the existence of an important sphere of knowledge (and of existence) where "human irrationality" (to be kept distinct from a pure "informal chaos") takes place instead of non-contradictory rationality.

(d) A last noticeable big challenge to standard opposition theory was brought into light (around 1910) by the fundamental part of physical theory named "quantum mechanics" (the physics of the smallest known elements of nature). For this new science of the physical nature, although fully mathematised in theory and in practice, showed unexpectedly formal properties violating some basic assumption of the traditional view on rationality, as for instance the principle of identity, or the principle of definiteness. As is well known, quantum "waves-particles" on the one hand bear properties of two kinds (wave properties and particle properties) which seem heterogeneous and conflictive (strongly opposed, if not strictly); on the other hand, it shows that fundamental quantum entities can (and must) be "undetermined": the opposition between something and its negation becomes here thus fuzzy. As a

[17] This "weak thought" (whose manifesto is in: P.A. Rovatti and G. Vattimo (eds.), *Il pensiero debole*, Feltrinelli, Milano, 1983) is a clear (reversed) echo to the Sophistic "kreitton logos" – κρειττον λογος –, the "stronger argument" of Protagoras.

symptomatic anecdote, Niels Bohr, one of the main theoreticians of quantum mechanics, chose as family blazon the symbol of Daoism, the Yin-Yang[18].

Of these four families of strong opposition to opposition theory at least three (perhaps not the dialectic one) are still very lively and embarrassing for the standard theory of opposition.

01.07. Attempts to somehow mix usefully the two paradigms

Naturally enough, there have been theoretical attempts to combine somehow the two "poles" of the understanding of opposition (the Western and the Eastern). Among the major attempts one can recall at least the six following.

(1) Hegel's dialectic (1800), already mentioned, proposed itself as an ambitious candidate. As is well known, it impressed many minds over quite a long period, also by the successive intervention of Marx's theory (dialectic materialism), which opened the hope of a powerful general application of dialectic "science" to real life (beginning by modifying the social structure). But dialectic seems to many scientists and theoreticians to lack mathematical support and consistency[19].

(2) The already mentioned many-valued logics (the logics admitting other truth values beyond "true" and "false": for instance "undetermined", "almost false", "almost true", etc.), of which we saw the invention was made by early Indian philosophers, re-invented independently in Poland by Jan Łukasiewicz (around 1920), were proposed as a means, inside pure mathematical logic, to be able to express more complex degrees of opposition than those of logical bivalence. Incidentally, Łukasiewicz built his 3-valued logic (and later n-valued logic, $n \in \mathrm{N}$, $n \geq 2$) in order to fight philosophically against the already mentioned necessitarian (determinist) consequences that Diodorus showed to be inherent to logical bivalence as such: Łukasiewicz wanted to separate mathematical logic from determinism[20].

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[18] I.e. the symbol of something more complex than the simple opposition of the black and the white, cf. F. Capra, *Le tao de la physique*, Sand, Paris, 1985 (1975), p. 146 (where the Bohr anecdote is presented and documented with pictures).

[19] The Swiss psychologist J. Piaget has tried to study this notion in a scientific non-Hegelian way, cf. J. Piaget (ed.), *Les formes élémentaires de la dialectique*, Gallimard, Paris, 1980. An anthology of essays trying to formalise dialectic is: D. Marconi (ed.), *La formalizzazione della dialettica. Hegel, Marx e la logica contemporanea*, Rosenberg & Sellier, Torino, 1979. The French philosopher Alain Badiou conceives dialectics as pertaining to a tension between algebra and topology, conceived respectively as a theory of the operations and a theory of the localisations, cf. Badiou, A., *Second manifeste pour la philosophie*, Paris, Fayard, 2009, p. 49-50.

[20] Cf. S. Gottwald, "Many-Valued Logic", *Stanford Encyclopedia of Philosophy*, (internet) (2004); cf. also G. Priest, *An Introduction to Non-Classical Logic*, CUP, Cambridge, 2001; and G. Palau, *Introducción filosófica a las lógicas no clásicas*, Gedisa, Barcelona, 2002.

(3) An answer to the severe criticism made to standard opposition by quantum theory was proposed by quantum physicists themselves (around 1937) with the elaboration of the so-called "quantum logic" (Février-Destouches). These logical systems, relying on special mathematical projective properties carried *ad hoc* in the universe of the axiomatisation of logic, where they had never been seen before, allowed to simulate formally some strange opposition behaviours[21].

(4) In the middle of the twentieth century, a big *virtual* enrichment was brought to the study of opposition (and to mathematics in general) by the gradual development of "category theory", a discipline which turned out to take the place of set theory as the foundation of mathematics. This theory is able to express at best all the "mathematically (and *a fortiori* the logically) possible". In particular, category theory makes it possible to conceive non-temporal formal "variations" (as for instance in the "variant sets"), some kind of "time outside time". Some argue that inside category theory, by the notion of "adjoint", it should be possible to express Hegel's intuition of dialectic (whereas Hegel believed that dialectic exceeded, thanks to the formal irreducibility of time, the essential aporetic static character of mathematics). If not yet clear, the possible contribution of category theory to the understanding of opposition seems promising[22].

(5) One of the biggest changes ever in these matters of strict opposition seemingly occurred around the middle of the twentieth century, when a new kind of answer was given to the problem of the logical "unthinkability" of contradiction. Two logicians, the Polish Jaskowski (in 1948) and the Brazilian da Costa (in 1953) developed independently the idea of having rigorous mathematical systems able to admit (eventually) formal contradictions (or "inconsistencies", like "a∧¬a") *without going "trivial"*, i.e. without admitting that for any formula "b" ("b" may be any formula), "b is a theorem" (trivial systems are *dead* systems). In normal logic, basic standard laws like "(a∧¬a)→b" (the so-called *ex contraditione quodlibet* or *ex falso quodlibet*), together with the *Modus Ponens* (i.e. "if A, and if A→B, then B"), guarantee "logical explosivity": if one contradiction (i.e. a "a∧¬a") appears *anywhere* in the system, the whole system becomes contradictory *everywhere* (every "b" is true, everything is true, therefore "truth" becomes trivial – the predicate "truth" cannot distinguish between good

[21] Cf. A. Wilce, "Quantum Logic and Probability Theory", *Stanford Encyclopedia of Philosophy*, (internet) 2006 (2002); cf. also some pages in: S. Haack, *Deviant Logic, Fuzzy Logic. Beyond the Formalism*, The University of Chicago Press, Chicago and London, 1996 (1974); cf. Y. Gauthier, "La logique quantique", in: *La logique interne des théories physiques*, Bellarmin / Vrin, Montréal et Paris, 1992.

[22] Cf. R. Lavendhomme, *Lieux du sujet. Psychanalyse et mathématique*, Seuil, Paris, 2001, pp.311-316; cf. A. Badiou, *Court traité d'ontologie transitoire*, Paris, Seuil, ch. 13. In ch.17 below we will see some applications of category theory to opposition theory by D. Luzeaux, J. Sallantin and C. Dartnell.

and bad formulas, it applies to all). These axiomatic systems able (by complex mathematical machineries) to inhibit logical explosivity, i.e. these systems able to be at the same time "inconsistent" and "not trivial", are said to be "paraconsistent" (an important well-known subgroup of "paraconsistent logics" is the so-called "relevant logics")[23]. The logician Richard Routley (alias Richard Sylvan) was seemingly the main philosopher of this growing group of logicians. His pupil and colleague Graham Priest has contributed to make this view philosophically popular, defending his own version of paraconsistency, which he calls "dialetheia"[24]. The logician Jean-Yves Béziau (pupil and colleague of da Costa) develops "universal logic", a generalisation of all these logics (cf. ch.10 *infra*).

(6) One last noticeable proposal made in order to cope rigorously but constructively with "irrationalities" (specifically those advocated by Freud in his metapsychology) is Matte Blanco's "bi-logic" (1975). By developing (*in nuce*) a complex interplay of *n*-dimensional geometry and "symmetry-logic" (one where a special operator can sometimes introduce symmetrisations of asymmetric relations) he gives a rather convincing model of unconscious (in the sense of Freud) mental processes (to which we will come back in ch.2)[25].

With the exception of the first proposal (Hegelian dialectics), and leaving for later the examination of Matte Blanco's one (cf. ch.02 *infra*), the other four innovative attempts mentioned here are mathematically sound (viable). Eventually, the problem with them concerns the *philosophical* interpretation of the proposed logical formalism, this one being, in each of these four mentioned cases, not yet fully understood.

01.08. The form of their "transcendental refutations"

One last important point deserves to be mentioned here, for the reflection over the attempts to overcome (or to ameliorate) the classical Western paradigm of opposition has given rise, by reaction, since the very beginning (i.e. since Aristotle himself) to a very interesting structure of thinking, still used nowadays. This structure remains at the heart of many contemporary fundamental discussions in logic or philosophy and is one blocking the possible explorations of "opposition". It is therefore important to know what we are speaking

[23] For paraconsistent logics in general, cf. A. Bobenrieth, *Inconsistencias ¿por qué no? Un estudio filosófico sobre la lógica paraconsistente*, Colcultura, Bogotá, 1996; for relevant logic, cf. F. Rivenc, *Introduction à la logique pertinente*, PUF, Paris, 2005.

[24] Cf. G. Priest, *In Contradiction. A Study of the Transconsistent*, Clarendon Press, Oxford, 2006 (1987).

[25] Cf. A. Moretti, "Trois approches de l'irrationnel : Davidson, Matte Blanco et da Costa", *Noésis*, 2003; "Deux spatialisations convergentes : I. Matte Blanco et P. Gärdenfors", in : M. Sobieszczanski and C. Lacroix (eds.), *Spatialisations en art et sciences humaines*, Peeters, 2004; "Trois approches complémentaires et non

of. One can mention at least 8 important moments of this recurrent philosophical phenomenon.

(1) As already said, Aristotle felt compelled to fight against “antilogism”, the ultra-relativistic doctrine of the Sophists. With this aim he constructed a philosophical system based on evident axioms and on deductions (logic founding metaphysics, metaphysics founding physics, physics founding biology, biology founding psychology, psychology founding ethics and politics, etc.). The main tool for such deductive foundation was “logic” (invented as such by Aristotle), whose pivotal tool, in turn, was the so-called “principle of non-contradiction”[26]. In order to give to his whole system the deepest deductive foundation, he encountered however a problem with logic: how to grant logic’s first axiom itself? (i.e. the principle of non-contradiciton) For, without such a justification, this would seem arbitrary (and it seemed, indeed, to the Sophists’ eyes)[27]; but justifications in Aristotelian style amount to deducing (logically) the (ontologically) posterior from the (ontologically) anterior and, by definition, the principle of non-contradiction is the most anterior of all possible principles. So no further “more fundamental” (more anterior) principle is available: does this mean that there can be no demonstration of it? (and therefore that the whole Aristotelian system lacks of a serious foundation?) Again, this would have been highly problematic, for the Sophists could, for instance, have advocated, against the arbitrariness of the principle of non-contradiction, the authority of a philosopher like Heraclitus (for whom “all flows”, “nothing is fix”, “the opposite do coincide”, etc.). Aristotle’s stroke of genius was to propose an “indirect demonstration” of the principle of non-contradiction, the so-called “elenkhos” (ελεγχος). He argued like this: “Ok, I cannot demonstrate the PNC, since each (direct) demonstration requires a use of the PNC: so any (direct) demonstration of it would be circular. But I can demonstrate it *indirectly*: for, any person wishing to demonstrate (against me) that the PNC is false, will try to claim that the PNC is false and (therefore) that its negation is true; but such a distinction of the false and the true is (at a meta-level) already an (implicit) expression of the PNC! So, it is impossible to negate the PNC, so the PNC is implicitely true. QED”. Remark

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incantatoires de la création artistique : Matte Blanco, Luhmann et Badiou”, in: P.-V. Duquaire, A. Moretti and B. Van Belleghem (eds.), *Electrobolochoc Workshop 2005-2006*, Zuma, Etroussat, 2006.

[26] “It is impossible that the same attribute belongs and does not belong at the same time to the same subject and under the same respect” (“το γαρ αυτο αμα υπαρχειν τε και μη υπαρχειν αδυνατον τω αυτο και κατα το αυτο”, *Metaphysics*, G, 3, 1005b, 19-20).

[27] “[…] the most certain of all principles is the one about which it is impossible to be mistaken. It is, indeed, necessary for such a principle to be the best known (for, we always do mistakes about the things we do not know) and that nothing in it be hypothetical, for, a principle whose possession is necessary in order to understand any other being is no hypothesis, and that which must necessarily be known in order to known anything whatsoever, one must necessarily already possess it before anything else” (*Metaphysics*, G, 3, 1005b, 11-17).

that this argument was extended by Aristotle to any act of meaning (i.e. any "deixis", any act of showing, any act of meaning by saying "that thing", implies the PNC). The elenkhos became a model for all the Western philosophical tradition, as we are going to recall quickly now.

(2) In his most famous and most important contribution to philosophy, despite his anti-Aristotelianism, Descartes can (and must) be seen as having developed a new instance of elenkhos. As a matter of fact, the argument "I think, therefore I am" ("cogito, ergo sum", "je pense, donc je suis") is an indirect demonstration (an elenkhos) of subjective existence. He says: "Ok, I cannot demonstrate directly that I'm existing (life can be highly deceptive, like in dreams). But in case I would not exist (let's suppose), I would not be thinking; but I'm thinking, therefore it can't be that I do not exist (right now), therefore I am (right now)". This argument is supposed (if not to give an "eternal foundation" to the Cartesian philosophy) to show that a radical scepticism over "existence" is not tenable.

(3) Kant not only made the same gesture (the elenkhos) under a new appearance (he founded indirectly – but definitively, according to him – a theory of the human thinking "faculties", a theory of human knowledge, the indirect proof – the elenkhos – taking here the name "transcendental deduction"), but he also gave it a new general name: he called it "transcendental structure" of thought ("X being the transcendental condition of X" meaning by Kant that X is a necessary condition of the bare possibility of Y). This philosophical move turned up to be of the highest importance for the philosophical world, which – fervently or polemically – adopted Kant's main categories (with the partial but meaningful exception of contemporary analytical philosophy)[28].

(4) Another important instance of "elenkhos" (or of "transcendental"), under a new form, is given by Popper's famous argument (mainly used against dialectics and psychoanalysis, for showing that they are not scientific)[29] according to which the transcendental structure of science is "refutability": the "engine" of the positive evolution of science, according to Popper, is not the "dialectic progression of the ideal through the concept" (as think the Hegelians), but the simple and cruel fact that some judgements over the real are open to possible refutations of them (only falsified theories can be said, retrospectively, to have been scientific in their time). Now, Popper maintains that if there is

[28] As is known, some consider it as one of the main historical criteria for distinguishing analytical from continental contemporary philosophy: continental philosophers, in some sense, do worship Kant's criticist philosophy (eventually polemically), whereas analytical philosophers take it just as one philosophical theory among others (and prefer Hume's legacy of an empiricist openness).

[29] Also claiming that the root of all conceptual and political evil comes from Plato's mystical philosophy!

no direct proof of the fact that refutability is the trademark of the scientific enterprise (and for him there is no other), it can be proven indirectly: for, suppose a theoretical corpus claims to be scientific but refuses to give, to the public, ways of testing its adequacy with reality: then there would be no more progress of science. But science is progress (perfectibility), so it can't be an unfalsifiable theory. Once again, we must remark that this elenkhos is impressive (very convincing, if not as surely true as it claims). Elenkhoi are rather convincing structures.

(5) Among many other instances of elenkhos (we could have spoken about Fichte, Husserl and many others) two ones touching mathematical logic deserve to be mentioned here (they will turn up very relevant for our present investigation on "opposition"). The first is proposed by the Polish logician Roman Suszko, with his very strong argument against the possibility of many-valued logics (the famous "Suszko' Thesis"): in an indirect way (as usual, it is an elenkhos) he claimed in 1975 to prove that the so-called "many-valued logics" (i.e. the mathematical algebraic systems allowing to have logical objects mapped into a set of truth-values larger than the standard {0,1} one) are fake: from a really logical point of view (i.e. inside the realm of the "Tarskian consequence operator") they remain two-valued (their extra truth-values – that is, those other than "0" and "1" – are algebraic and not logical). This indirect refutation is very impressive and cast a troubling doubt about the serious character of many-valued logic (Suszko, additionally, gave a formal method for reducing – under certain circumstances … – to 2-valued logics the systems originally many-valued). Remark that this issue is truly crucial to contemporary logic: if confirmed (it still is debated and object of fights and conjectures) it would mean that the foundations of logic are less open than is generally thought ("no multi-polarism, just a duo-pole"). This is tantamount to affirming the existence of strong transcendental structures governing (in a severe, non-liberal way) the logically possible (on this very important issue, very much related to the meaning of opposition, we will come back on ch.23 *infra*).

(6) The second instance of elenkhos touching nowadays contemporary mathematical logic is one due to the Australian philosopher Slater, who in 1995 claimed (we already mentioned it) against paraconsistent logics that these formal systems are in fact *impossible* (this is a very strong claim!). Again, this is an elenkhos for, being himself unable to prove something positive about "transcendental logic" (there is no such logic in mathematics), Slater pretends to prove that whoever tries to hold (as a logician) a paraconsistent position (i.e. whoever pretends to be building a paraconsistent – i.e. inconsistent but non-trivial – logic) indirectly assumes some (implicit) presuppositions (in fact, geometrical ones) that truly speaking embody the negation of the paraconsistent idea (this has to do with the "square of

opposition"). This argument is very important in general (it still is very much discussed nowadays by logicians) and it is very important in the particular case of our present enquiry (we will come back to it in ch.10 below).

(7) Another spectacular example of elenkhos (although a not widely known one) is due to the already mentioned Italian philosopher Emanuele Severino. His argument, directed against the very notion of contradiction, is a philosophical "tour de force": no philosopher (except Parmenides) in the Western tradition has dared assert the *total* absence of contradiction in the world (and from that he derives very strange theses, comparable in strangeness to D. Lewis' "modal realism")[30]. The kernel of Severino's demonstration is developed in his theory of the "originary structure". Remark that he has convincing arguments for showing that Aristotle's elenkhos is a bad reasoning (Severino meditates explicitly the notion of "elenkhos") and that he takes into account, in his "very continental" thinking, formal structures and arguments coming from set theory. From a philosophical point of view, it is a line of thought of the highest importance – both for the content and for the strength of thought – for people concerned, like us, with the concept of "opposition" (he endorses an extreme position, interesting to be explored even if false – more or less as with Wittgenstein's *Tractatus*, which is interesting even now that we know it false, for it is conceptually radical in exploring – and thus elucidating – the classical "picture theory"). Severino's claim, if it were true, would probably be the most severe known transcendental structure up to the present.

(8) One last instance of elenkhos we would like to mention is the one due to the German philosopher K.-O. Apel. His elenkhos is used by him in order to give, against relativism (especially against Postmodernism: Derrida, Rorty, …), a ultimate, inescapable and non-objectionable foundation to "rationality"; and thereafter, to a universal ethic grounded on such a definitely founded rationality[31]. His position is highly interesting for several reasons, among which the facts that (i) he is radical (he pretends to give a foundation to rationality, *mutatis mutandis* like Kant!), (ii) he embodies a renewed (radical) Kantian position for the present days (internalising the "linguistic turn" of contemporary philosophy), (iii) he is one of the main inspirers of Habermas' philosophy (the famous and rather consensual – if not paradox-free – "ethic of discourse") and (iv) he embodies a mixture of continental and Anglo-American deep and serious influences, being thus really a representative philosopher of our

[30] In fact, it can be shown that there is an amazing generally unknown convergence of the two.
[31] Cf. K.-O. Apel, *Transformation der Philosophie. Band 1: Sprachanalytik, Semiotik, Hermeneutik*, Suhrkamp, Frankfurt am Main, 1994 (1973); *Transformation der Philosophie. Band 2: Das Apriori der Kommunikationsgemeinschaft*, Suhrkamp, Frankfurt am Main, 1993 (1973). For an introduction to Apel, cf. W. Reese-Schäfer, *Karl-Otto Apel zur Einführung*, Junius, Hamburg, 1990.

times (Apel relies very seriously, among others, on Peirce, Wittgenstein and Austin). But his main interest for our present purpose is that he gives a new explicit theory of what it means to do an elenkhos in philosophy. According to this theory, an elenkhos is a strategy in order to "grasp" (or seize) transcendental structures (in fact, allegedly, it is the only possible such strategy), and it relies on a grounding principle that is, at the same time, logical, pragmatic and semiotic: the "Principle of the Avoidance of all Performative Self-Contradiction" ("Prinzip des zu vermeidenden performativen Selbstwiderspruchs"). So, this renewed concept of elenkhos is philosophically very important because it is much clearer than its glorious conceptual ancestors (by Aristotle, Descartes and Kant): it specifies its logical, pragmatic and semiotic components (according to contemporary accepted requirements in each of these disciplines).

Now, it must be noted that all these elenkhoi are very powerful theoretically speaking (whence the lasting philosophical glory of their fathers: Aristotle, Descartes, Kant, …). But they are not "scientifically" cogent: they remain *philosophical* arguments (i.e. arguments "going a bit too fast to be true"). These last two assertions of ours may advocate the rigorous testimonial of thought-events like: (a) Łukasiewicz *logical* demonstration of the fact that *logically speaking* Aristotle's indirect demonstration of the principle of non-contradiction (i.e. the elenkhos) is not logically cogent (from which point of view the later emergence of paraconsistent systems is not surprising); (b) the very development of mathematical paraconsistent systems; (c) the development (by Malinowski, Shramko and Wansing) of "non-Suszkian many-valued systems" (i.e. many-valued systems not reducible to 2-valuated ones); and so on. In other words: transcendental arguments are deep, troublesome and inspiring (they often touch the deepest issues at stake), but must be regarded without fear and without submission. So must we do in our present enquiry relatively to the threat of the existence of possible transcendental constraints on opposition.

01.09. The strong need of a deeper understanding of this concept

The conclusion of this chapter is that the field of opposition, remaining positively (i.e. inspiringly) problematical (it deals in a very pure way with the static/dynamic debate), is still open to thought and bears puzzling elements of evidence that, despite its starting *logical* flavour, something of it must be accounted for *geometrically*. This part is important. For, put in other words: there seems to be some not fully justified prejudice inclining scholars to think that, because of its logical nature (opposition is the heart of logic), the concept of opposition is fixedly taken into transcendental nets (the concept of “transcendental structure” – *necessary condition* of *possibility* – in some sense being clearly one grounding in logic). We must be careful with the philosophical prejudices committed by the use of logic.

02.
PITFALLS OF THE CONCEPT OF "CONCEPT"

In this chapter we want to pursue the examination of the concept of "opposition". But here we will try to show that, before going on investigating opposition *per se*, what seen previously about it and its possible "transcendental constraints" (ch.01) obliges us to make some new adapted methodological remarks on the relations between the concepts of "logic" and of "concept". For, logic, although powerful, is not the only possible formal paradigm for thinking, and concepts are not exclusively ordered to the function of logical deduction (we will highlight three different distinct functions of logic for philosophy). On the contrary, we will recall how science seems to show that the realm of concepts is a rather *geometrical* one.

02.01. New starting point: a "transcendental" problem with logic

As we have just seen (ch.01), the concept of opposition, which is a really, crucially, important one, leaves strongly unsatisfied because of some of its corollary bearings (for short, it is too rigid, both as a tool – it is static – and as for its hidden imports – it tends to necessitarianism). This seems to mean that, instead of thinking its knowledge achieved once and for all, we need to investigate it more deeply, looking for possible advances inside the still vague field of "opposition theory". As we have also seen, there are good reasons to take as paradigm of this concept its static side (whence the other, less static sides should be – hopefully – obtainable later), that is, formal logic. For, formal logic, which stems from the Greek theory of opposition (seemingly brought to perfection by Aristotle), is an elegant image of basic opposition (logic is a "black-white" world). This logic has a static, geometric "metaphysical" basis: in a strong sense, "opposition", as we saw, is *ob-positio* (this is the geometrical heart, "position"). But this "minimal geometry" of opposition (in fact reduced, as we will see, to the so-called logical *square*) seems to be "transcendental": it helps explain (logic and oppositions) but remains itself unquestionable (it seems, for instance, to negate the nevertheless promising paraconsistent research issue)[32]. In this respect, the problem is, concretely speaking, that the bare logical study of the concept of opposition (its rigid or static fertile paradigm) seems to stumble over the dispute taking place, at a philosophical meta-level, between, at one side, the openness of logic to infinity (as seem to show paraconsistent

[32] This is "Slater's argument" against paraconsistent logics, invoking the transcendental logic inscribed in the geometry of the square of opposition, cf. ch.10 *infra*.

and many-valued logics) and, at the other side, the finiteness of the self-reflexive, transcendental and foundational argument (as do claim “indirect” demonstrations of transcendental behaviour, for instance in Aristotle, Apel, Suszko, Slater). In other words, if we remain at this stage, there seem to be no novelties at the horizon – the two conflictive positions (the will to know more and the responsibility of being “the transcendental heart of logic”) become somehow paralysed and seemingly fruitless.

Which methodology are we to follow in order to overcome this situation? For, we do want to overcome it. We clearly need to find new bases of reflection.

02.02. We must look for new conceptual hints

As often, the necessary hints do hide before everybody’s eyes. In this sense, in this precise situation just described, there seem to be at least two useful hints before our eyes, if we just open them enough.

(1) One first such needed hint could be the astonishing concord evocation, by Slater and Béziau, during their reciprocal conceptual philosophical fight over logic (cf. ch.10 *infra*), of the importance to come back, while discussing over the possibility of paraconsistent negations, to the *geometrical* underlying basement of negation (i.e. the logical square). As we will see, Slater invokes the square (against Priest’s paraconsistent system LP) and Béziau answers to him (defending the very idea of paraconsistent negation) by recalling the existence of a conservative extension of the square, the “logical hexagon” (of which we will speak in detail in ch. 8 and 9). Now, how comes that, discussing a branch of mathematics highly developed (i.e. formal logic), which seems to have *almost no commerce whatsoever with geometry*,[33] both invoke, at the very heart (i.e. about the operator of “negation”) of this discipline some aspects (the square, the hexagon) which are typically *geometrical*? This is very strange and deserves to be understood.

(2) A second interesting remark, a second possible hint, disguised as an uninteresting detail, is the fact that our legitimate discussion over opposition is in fact one touching “concepts” (the concept of “negation” and that of “logical square”, for instance). And concepts are immediately associated with logic. But do we really know what a concept is? What is, in other words, the concept of “concept”? Is it simply sufficient to put the hands into

[33] This (true) affirmation must be taken *cum grano salis*. There are, of course, studies trying – for instance – to join topology and logic (cf. Tarski, of course, but also – more recently – D. Sarenac, J. Van Benthem and more generally the “Amsterdam school” and its studies over complexity issues in modal logic), not to mention category theory and its reduction of modal logic to the topology of the open.

formal (or philosophical) logic in order to make conceptual clarification? More true-to-the-facts, it seems that we must recognise that, at least at this stage, the concept of “concept” is somehow mysterious.

02.03. Looking for the general structure of concepts

At the present stage of research, one could expect cognitive science (the new face of psychology) to be the specific discipline apt to give an answer to the question “what is a concept?” (for: concepts are the “bricks” of thought). But, historically speaking, *pace* the cognitive sciences, this role (of investigating concepts) has until recent date been played (and is still being played) by philosophy: this is why, before turning as we must to cognitive science, we will briefly sketch, in three movements (tradition, continental, analytical), a review of some of the main philosophical views on the theory of concepts. Only after that we will turn to the “cognitive science view” on concepts and see if it is a convincing starting point for us.

02.03.01. The concept of “concept” inside the philosophical tradition

Inside the philosophical tradition the concept of “concept” has indeed played a major role (philosophy is primarily a play with concepts). Of its rich history here we will try to give only a few major trends. The main theme seems to have been that of the relations between “concept” and “judgement” and this seems to have put into relation concepts with logic (judgements being related to logic). Plato was probably one of the first to propose this notion of concept, under the name “idea” (ιδεα, “what is seen”, i.e. by the mind). The basic intuition is that of a mental image of the real things. As known, Plato defended the paradoxical hypothesis that ideas do exist by themselves (we only perceive them) and give form to reality (the real is a copy of the ideal), of which they are formal models. For reasons not too clear (we know very little of the real Plato) Aristotle strongly refused Plato’s historically prophetic (i.e. confirmed by the history of Western science since Galileo) attachment to mathematics (cf. ch.3) and in particular fought this alleged ontological primacy of ideas (and numbers) over things, proposing instead to understand ideas as mental abstractions of the real things. This apparently commonsensical defence of material reality over ideal reality (which is applauded by many philosophers still today) hid the reduction of the idea of idea (the concept of concept) to an entity formally poorer than in Plato’s case. In other words, and this point seems to be very important, from formally *mathematical* (even if "vague" and mystic in Plato), ideas (or concepts) became formally *logical*, and this in Aristotle's sense: they were conceived as simple piles of "genders" and "specific differences" (a mono-dimensional hierarchy of nested "sub-concepts"[34] – whereas the general mathematical power is *multidimensional*). This link of the concepts with logic was a major change, destined to last (despite appearances) to the present day. After a long period of explicit reigning Aristotelianism (by our present simplification we omit, of course, many other theories – how else?), Descartes developed his anti-scholastic (anti-Aristotelian) philosophy. In this respect, he who was personally a great mathematician (Descartes made several technical discoveries and invented some revolutionary views, especially in geometry), expressed ostensibly his disliking for "the logic of the School" (i.e. the syllogistics used by the Scholastics). But his own idea of "idea" then was more related to a method (the "Cartesian method" of enquiry)[35] than to a formal structure (a formal model of the "concept"), unless for relating "conceptual clarity" (which was sought by the Method) to the criterion of a conscious thinking (by opposition to an unconscious one – it took Leibniz and, later, Schopenhauer and Freud to rediscover the importance of "unconscious thinking"). Remark however that his method, discovered by him while making mathematical discoveries (and thereafter destined to be exported to philosophy), is one rooted in the exploration of space (Descartes is the discoverer of the "analytical geometry"). After him, Spinoza will radicalise this Cartesian method by making it Euclidean: his own system of thought (where concepts are ideas) is expressed *more geometrico* (i.e. like in a demonstration of Euclidean geometry). The psychological stress present in the Cartesian method (as a preference accorded to the conscious against the unconscious) is also present (changed) by Hume who holds a mentalist-associationist position. For him the mental sphere is epistemologically autonomous (mental events must be explained in terms of mental events). Knowledge begins with impressions, and ideas are copies of impressions, whereas general representations (i.e. concepts) are just singular ideas playing a particular role in thinking: "a singular idea serves as an abstract idea when it is associated with a word that triggers the mind to summon forth ideas that resemble this idea")[36]. Hence Hume defines "belief" (i.e. judgement), traditionally conceived as the uniting of ideas, as being nothing but an idea with a particularly high degree of force and vivacity.

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[34] This is traditionally known as "Porphyrius' tree".

[35] As known, theorised primarily in: R. Descartes, *Discours de la méthode*, (1637), but whose ancestor is to be found in his private *Regulae ad directionem ingenii*, ("Rules for directing intelligence"), composed around 1628, but published posthumously.

Against Hume, Kant will hold that, far from being reducible to concepts (ideas), beliefs (judgements) are so important that no concept is possible if it does not involve a judgement. This is codified by the notion of “scheme”. As already hinted at (cf. ch.1 *supra*), starting from Kant, Hegel will develop another view about concepts, where he refuses the Kantian theory of the existence of an exterior bound to the conceptual sphere (the Kantian criticism as a whole is the theory of such a bound)[37]. This is the principle of the reduction of all reality to the ideal through the inner dialectic of the concept. As is well known, Kant and Hegel will have a huge influence over philosophy (especially the continental one) up to the present day. A big change (deprived however of the immediate sensible impact it would have deserved) is brought about by C.S. Peirce, with his three great innovations touching directly the notion of concept: the paradoxical theory of the “realism of the vague”, the theory of the “generalised semiosis” (with its ternary semiotic relation, which crosses indifferently the theoretical and the practical fields – whose slogan is: “everything is sign”) and the “Pragmatist Maxim”, according to which the meaning (the concept) of any idea is nothing more and nothing less than the sum of all the possible changes it produces to the bearer of the idea[38]. This is revolutionary: a concept is a (infinite) sum of possible acts. All these three Peircian elements give a coherent new image of human thinking (and of concepts) and of ontology, where the stress is put on the lack of “atomic constituents” (and philosophical Foundations) and on the necessarily dynamical architecture of meaning (and ontology). Remark however that some of Peirce’s own expressions, stressing the predominance of the concept of “sign”, give rise to the idea that thought is a “calculus over signs”, which can be taken (lazily) as a further confirmation of the standard “logical model of thought” (and concepts)[39]. A very important last benchmark of traditional philosophy to be mentioned here on concept theory is, of course, Frege’s theory. The notion of concept is crucial for him, as testify the titles of several of his main works: *Begriffsschrift*, “Funktion und Begriff”, etc. (“Begriff” being the German for “concept”). Frege, with his “concept-script” for clarifying mathematical thought and with his rigorous theory of meaning (this last in some sense a re-edition of the poorly known ternary one of B. Bolzano), in so far he thinks mathematics as ontologically second with respect to logic,

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[36] Cf. D. Landy, “A (Sellarsian) Kantian Critique of Hume’s Theory of Concepts”, *Pacific Philosophical Quarterly*, Vol.88, Number 4 (December 2007).

[37] This is discussed, as being an issue still pertaining to contemporary philosophy (through the mediation of Wittgenstein, Quine and Davidson) by J. McDowell in *Mind and World*, Harvard UP, Cambridge MA, 1996 (1994).

[38] Peirce says: “Consider what effects, that might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object”.

[39] But such an interpretation is a very false one: mathematical logic is not Peircean, it relies on atoms, it is not semiotic.

frankly pulls the concept of “concept” to logic, radicalising thus, in some sense, the Aristotelian anti-Platonic move[40]. The important criteria for understanding and handling a concept become now those of extension (of a set), of argument (of a function), of intensional definition (of a set’s extension) and so on, as well as the triad of referent (the real counterpart), sense (the way of imagining) and denotation (the ideal object, or truth value). Frege’s conception, perfectly suitable for mathematical logic (which he bears to light fully), embeds, again, the concept of “concept” into the *logic* of judgement (through the development of a general new theory of the logical). If we omit to speak about other important philosophical movements of traditional philosophy like Marxism, hermeneutics, phenomenology, post-modernism – where the influence of Hegel and Kant (or that of their Husserlian and Heideggerian avatars) about concepts is huge and persisting – the general lesson so far, beside the considerable divergences, seems to be that “concepts” are some kind of images of the world, related to words, sharing much with judgements. Because judgements are mainly seen as a matter of logic, concepts are most of the time conceived as deeply related to logic, the question being then that of knowing *which logic* we are speaking of: a transcendental one (for continental philosophy) or the new mathematical one (for analytical philosophy).

02.03.02. The concept of “concept” inside continental philosophy

Inside the contemporary so-called continental tradition, the concept of “concept” still reigns (philosophy is still a play with concepts), but the “hate for logic”, shyly assumed as “prevention against the logical fetishism” (of the analytical “cousins”) is quite a common place (even shared by such mathematics-friendly continental thinkers as A. Badiou, A. Boutot and J.-M. Salanskis)[41]. Quite often, not much is done in order to clarify this ideological refusal of logic, just adopted as such. And this implies adopting a view on concepts which is tributary to the Kantian-Hegelian theory, according to which concepts and judgements are related to a transcendental (ante-mathematical) kind of logic (the really interesting, philosophical logic). A remarkable exception, even if not exempt of (big) polemical aspects,

[40] Despite the apparent “Platonic idealism” (or realism) of his (Bolzanian) theory of the “ideal meanings” (Gedanke).

[41] Cf. A. Badiou, *L’Être et l’événement*, Seuil, Paris, 1988; *Le Nombre et les nombres*, Seuil, 1990; and *Logiques des mondes*, Seuil, Paris, 2006; cf. A. Boutot, *L’invention des formes*, Odile Jacob, Paris, 1993; cf. J.-M. Salanskis, *L’herméneutique formelle*, Éditions du CNRS, Paris, 1991; *Le temps du sens*, HYX, Orléans, 1997; and *Le constructivisme non-standard*, Presses Universitaires du Septentrion, Villeneuve d’Ascq, 1999. Badiou is probably the greatest French continental; Boutot is an Heideggerian pupil of R. Thom; Salanskis is an hermeneutic pupil of J. Petitot.

is the case of one of the greatest continentals, Gilles Deleuze. Independent from Marxism, phenomenology, hermeneutic and post-modernism (and of course from analytic philosophy) he developed in great isolation a very original and powerful philosophical system. Inside of it, he (with Felix Guattari) developed a theory of the "making philosophy" where an argumented distinction is made between "concepts", "percepts" and "affects" ("philosophy is not 'using logic' but 'building concepts'")[42]. According to this astonishing deep theory logic is the proper of the scientific concepts (functional concepts), whereas the philosophical ones are more related to geometry. This position is singular, but has to be remembered.

02.03.03. The concept of "concept" inside (pragma-)analytical philosophy

Let us now turn to the analytical contemporary tradition. Having already mentioned Frege (and Peirce for pragmatism – or pragmaticism, as he particularised it with respect to W. James' own version), we must recall how (the second, non-Tractarian) Wittgenstein brought to philosophy a very important contribution, relatively to the concept of "concept", in so far he very convincingly criticised the "Platonic view" on ideas (according to him implicitly shared by the majority of the philosophers), which conceives them as simple: Wittgenstein showed that the human words, which always play a role of starting point for ideas or concepts (as the everyday word/concepts "beauty", "play", "human", …), in fact bear a complex structure of *bundle*, which, most of the time (if not always), makes it impossible to give, in a Socratic-Platonic style, a unique intensive definition (for an extensive set). Concepts are not essences, but bundles (of locally, non transitively linked properties) In other words, there are "family-relations" giving to a whole bundle an appearance of (conceptual) unity whereas it is (and remains) impossible to give a unique defining property ruling each of the elements of the bundle. Remark that this structure has some geometrical flavour (like in Deleuze's case). Now, contemporary analytical tradition, almost since its origins, seems to be divided into (at least) two poles: a consensus over logic on one side (considering language to be an application field of logic), and the pragmatist-Wittgensteinian view according to which rough practice (i.e. infinite openness to rule invention) is the source of both language and meaning[43]. The former has largely dominated, with its double hope of transforming philosophy into science and of giving birth, via a computational approach (based on mathematical logic) to a scientific theory of mind (of which the project of an Artificial Intelligence based on the model

[42] Cf. G. Deleuze and F. Guattari, *Qu'est-ce que la philosophie?*, Les Éditions de Minuit, Paris, 1991.

of the Turing machine is a corollary). The latter pole tends to remain, formally speaking, in the framework of logic, proposing no radically alternative formalism (as, say, geometry)[44]. A somehow singular position to be mentioned here is the one of G. Priest, who developed, jointly with his dialethic logical systems a philosophical view in which concepts are conceived, similarly to Hegel, as entities essentially inconsistent[45]. If we try now to ask more precise elements, a recent reference study on the notion of "concept" mentions 3 conflictive views on the question of the ontology of concepts and 5 conflictive views on the issue of the structure of concepts[46]. The 3 mentioned ontological positions are: concepts as mental representation, concepts as abilities and concepts as Fregean senses. These are clearly avatars of what we have just discussed so far (no big novelty). The 5 mentioned theories of conceptual structure are the "classical theory", the prototype theory (one abruptly based on probabilities), the theory theory (one drawing an analogy between concepts and whole scientific theories), the conceptual atomism and a vague mixture of the preceding four (explicitly based on the evidence that none of the previous is convincing alone). Remark that this "mixture solution" resembles (at a higher metalevel) to Wittgenstein's "bundle structure" of concepts. We still see no precise formal proposal here other than that of logic (cooked some way or other).

02.03.04. The concept of "concept" inside standard cognitive science

Now, having recalled very briefly the philosophical approaches to concept theory we can see that from the point of view of the new discipline named "cognitive science", despite an apparent large disagreement about the strategies to follow or the fundamental models to adopt (e.g. innatism against empiricism, holism against compositionality, and so on), a deep consensus seems to have emerged around the (tacit) idea that something logical is inside the human mind. This turns to ideology (in the political, bad sense), if one considers that such huge corpuses of empirical evidence and theoretical serious proposition as psychoanalysis tend to be more and more ostracised on the mere ground that "there is no anchorage of that in

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[43] Provided one keeps in mind the lively theoretical link relating, say, Dummett to Frege, McDowell to (the second) Wittgenstein and Brandom to Peirce.

[44] There is, of course, a virtual (but very important) geometrical element in Wittgenstein's notion of bundle. Similarly, one could imagine something analogous in Peirce's notion of "pragmatist meaning" (Peirce touched somehow geometry by his famous "existential graphs", which remained however – like logical diagrams in general – a tool subordinate to logic). But both hints remain formally poor. This seems to be the reason why many Anglo-American people come back to logic again and again: "if not, what else?".

[45] Cf. G. Priest, *In Contradiction. A Study of the Transconsistent*, Clarendon Press, Oxford, 2006 (1987).

[46] Cf. S. Laurence and E. Margolis, "Concepts", *Stanford Encyclopedia of Philosophy*, 2006 (internet).

mathematical logic"[47]. This consensus on the use of logic as main formal tool for building models is paradigmatically instantiated by Fodor's postulation of the existence of a (mysterious!) "language of thought" (or "mentalese"), if one remembers (as one must) that in the Anglo-American (or analytical) context the equation of "language" with "logic" (natural language as just one particular – a bit tricky and ugly – formal language) is a rather unquestioned starting point[48].

From this point of view, our starting question in this chapter ("what is a concept?") can be claimed to be related to one of the major and most dramatic philosophical facts of the twentieth century: the intuition or hope (or dream?), typical of Anglo-American "analytical philosophy", that logic *could be* for philosophy (and for humanities in general) what mathematics *is* (since Galileo) for physics (and thereafter science in general): that is, the very key of an amazingly bright and endless cognitive-epistemic success (making slowly but surely of philosophy a *science*). But is this, the belief that the structure of "concepts" is essentially *logical*, the only conceivable way? Again: what about *geometry*?

02.04. Gärdenfors: "Conceptual spaces" VS "logic of concepts"

Does cognitive science *really* confirm the primacy of the logical for understanding what a concept is? Many are of course the answers, for cognitive science, let us say it again, is not unified. I believe however that one approach is until now much more convincing, despite its unpopular (in terms of "trends") claims[49]. The Swede Peter Gärdenfors was famous for his contribution to "belief revision theory" ("G" stands for "Gärdenfors" in the famous and successful "AGM" model for belief-revision). But in 2000 he proposed a radically new

[47] Among the valuable exceptions one must mention, for analytical philosophy, D. Davidson and, for cognitive science (or more precisely neuroscience), A. Damasio: both authors claim that psychoanalysis must be recovered one day or other if a serious enquiry is to be at stake on irrationality (akrasia phenomena, etc.) and/or emotions. Cf. D. Davidson, "Paradoxes of irrationality", in: Wollheim R. and Hopkins J. (eds.), *Freud, A collection of Critical Essays*, Cambridge, CUP, 1982; cf. A. Damasio, *Descartes' Error. Emotion, Reason, and the Human Brain*, Putnam Books, 1994.

[48] The already quoted paper by Laurence and Margolis underlines that Fodor's theory of "mentalese" (and more deeply: the Representationalist Theory of Mind, RTM) is a reference (at least an accepted starting point for discussion) to all (the "mental representation view on concepts" is said to be dominant in cognitive science). As for the "symptoms" of this unquestioned linguistic logicism, we think, say, to R. Montague's approach to linguistic semantics as well as to computational linguistics in general. These are fine and valuable researches, but there are nevertheless serious grounds for doubting that they are the last word on the structure of real language (they are nice and useful fictions, but fictions nevertheless)

[49] It is remarkable (or symptomatic of a bad functioning of the intellectual community) that the approach I'm now going to recall is not mentioned, not even in bibliography, in the already quoted reference study over "Concepts" (by Laurence and Margolis). Remarkable for three reasons at least: (1) the thesis proposed is very strong and new (and interesting); (2) the author of it is an authority in the field; (3) the publisher of the book is also an authoritative one (the MIT Press).

understanding of concepts[50]. As I already offered a quite extended review of his thought in a paper of mine of 2004,[51] and as it is not the primary object of investigation here, I will be very brief in recalling his line of thought.

Gärdenfors starts by remarking that "concepts" are a very important ingredient of cognition. He then recalls that there are mainly two approaches to mental representation nowadays: the symbolic approach, based on the Turing machine and conceiving "thinking" as a symbolic calculation; and the connexionist approach, based on artificial neural nets (ANN) and conceiving thinking as a neural infinite-dimensional space (the second approach being nowadays more "trendy" than the first). He points out that in neither approach the very first notion to be modelled for understanding "concepts", i.e. "similarity", is modelled. This despite the fact that, from the point of view of real-life psychology, "similarity" is the most important capability we have (with respect to the handling of concepts). And this simple capability, in fact, turns logic mad. He then explains why it is so: the symbolic approach is nice (we all like logical calculations, they are easy to understand) but stupid (i.e. conceptually impotent), whereas the connexionist one is smart and grounded on reality (the real structure of the brain) but ugly (that is, too complex to be seriously used to speak about precise given concepts). This leads him to realise that there seemingly exists, between the ontologically real level of connexionism (the neurones) and the idealistic utopian level of the symbolic calculus (the paper-level), an intermediate level where concepts "do live". This level is emergent with respect to the connexionist neural rough basement (*via* a reduction of the complexity of the infinite-dimensional neural space) and is more real than the foolish linear (mono-dimensional) realm of the Turing-machine calculations. So, it is an emergent level where concepts are sub-spaces of the global infinite-dimensional space of the neural network[52]. And this conjecture seems more than realistic after reflection: Gärdenfors shows that there is a way, coherent with this, to conceive "concepts" (and properties, and the like…) in terms of structures of bundles of perception-dimensions (a conceptual space is then a space furnished with certain perceptual co-ordinates; an object, abstract or concrete, relatively to a given such space, is a point in this space, i.e. a vector; and "similarity" between two such objects in a given such conceptual space is a function of their distance – the Euclidean distance – in that space). Such a view

[50] P. Gärdenfors, *Conceptual Spaces. The Geometry of Thought*, The MIT Press, Cambridge MA, 2000.
[51] Cf. A. Moretti, "Deux spatialisations convergentes. I. Matte Blanco et P. Gärdenfors", in: M. Sobieszczanski and C. Lacroix (ed.), *Spatialisation en art et sciences humaines*, Peeters, 2004.
[52] One could say: a level of one infinite-dimensional space (the neural level), a level of many n-valued spaces (the conceptual level), and a level of mono-dimensional spaces (the symbolic level, symbolic being taken in the sense of the formal treatment of symbolic recursive calculation). Hence, in a sense, the lack of interest of logic for space.

(detailed by Gärdenfors along the many points of the possible consequences and the known challenges of it for the study of cognition) shows evident superiority of such an approach over the "logical" ones (no matter how complex the logic used, Gärdenfors stresses and argues this point extensively and convincingly), in particular for grasping the static as well as the dynamic (for instance non-monotonic) properties of the realm of concepts[53]. The mind, from the point of view of its being able to cope easily with concepts (primarily *via* its ability to compare them and to grasp easily similarities), seems thus to be partly made of many conceptual n-dimensional (and topologically variable) spaces (again, interpreted as emergent constant localities over the whole space of the possible neural association patterns)[54].

As long as we are concerned here with this question (our own question being that of the structure of "opposition"), the relevant point to be retained from Gärdenfors' new theory of "concepts" is important: there is a clear sign coming from high-level cognitive science (if not from trendy but conceptually lazy cognitive science) that the concept of "concept" is (alas!) primarily not logical, but geometrical instead. This new (but convincing) fact must confirm our doubts about the existence of transcendental, allegedly already known ("since the Greeks", "since Frege", …) "boundaries of thinkability" of the concept of "opposition". On the contrary, it seems, more than ever, that the strange fact that the standard model of opposition (the logical *square*) is a partly geometrical model (as is its recent conservative extension, the logical *hexagon*) is not "just a strange meaningless detail" with respect to "the huge importance it has, as a transcendental unchanging structure, for logic in general". We can also remark that this argued discovery (by Gärdenfors) that "logic is not the right tool for 'concepts'" (the right tool for "concepts" being some kind of geometry) bears deep similarities with Deleuze and Guattari's analysis. In what follows we will draw attention on another meaningful such unexpected similarity (one with psychoanalysis).

02.05. Psychoanalysis with cognitive science

Arguments similar (to those just seen), although very different, could be found in the psychanalytical theory of Ignacio Matte Blanco. We have seen (ch. 1) that psychoanalysis,

[53] Gärdenfors shows, for instance, that the well-known non-monotonic properties of belief-revision (with respect to the categorisation of concepts: as when I know that Tweety is a bird, and then later understand that she also is a penguin) are very natural (to understand and to formalise) from the geometrical point of view, whereas the same phenomenon, handled logically (i.e. with non-monotonic or default logics, or the like) gives rise to all kinds of problems and *ad hoc* formal solutions which always remain very far from any possible reasonable realistic understanding both of the studied phenomenon and of its bio-physical neural basement.

when taken seriously (as it seemingly should) offers a big challenge to logic (i.e. to the world of “black and white” opposition). This theoretician (a psychiatrist at the origin) has suggested very convincingly that Freud’s five “metapsychological axioms” (1915), pointing the most strange features of the unconscious, may be explained formally by using a strange but easy to obtain (mentally) non-standard logic, “bi-logic”. The same phenomena, according to him, can also be formalised by focussing on strange behaviours of the mental representations in “mental spaces” of varying dimensionality (admitting geometrical dimension of them very high). I claim that there is, when examined closely, a deep and astonishing convergence between Matte Blanco’s and Gärdenfors’ theories (they concern two correlated and yet distinct faces of the “mind”, the unconscious and the conscious ones). This also suggests that logic itself cannot be taken as a proof of the fact that psychoanalysis (being « illogical ») does not exist[55].

But what about logic itself, then ?

02.06. Three philosophical functions of logic

What is the role of logic within philosophy or thought in general, then? Far from thinking that logic is useless (except for giving work to computer scientists and AI engineers), it seems to me that, so far, at least three functions must be mutually isolated, for they correspond to three different realities.

(1) One first role of logic is, naturally enough, to reason, in the sense of making deductions (think of syllogistics by Aristotle). As such, logic may as well (a) study the existing human reasonings, in order to classify them, the valid ones as well as the fallacious ones (this is what does contemporary philosophical logic); (b) study the abstract “logical possibilities”, free from any anchorage to psychological materials (this is what pure mathematical logic does).

(2) A second role of logic, quite distinct from the first, consists in allowing definitions. For a definition, precise as it will, is not a deduction, but the creation of some kind of “concept” (or formal object). This part of logic is the one closer to the analysis of concepts.

(3) A third role of logic, clearly perceivable from the point of view of pure philosophy, is one of “founding” (i.e. giving a foundation). Many among the biggest philosophical

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[54] The form and the mutual disposition of the different dimensions in a conceptual space may vary: there are conceptual spaces with radial dimensions, other with circular ones, and of course some have different kinds at the same time.

[55] Again, we study in some detail the elements of such an uxpected theoretical convergence in our paper “Deux spatialisations convergentes: I. Matte Blanco et P. Gärdenfors”, *op. cit.*.

systems, at least among those of continental flavour (Aristotle, Descartes, Kant, Fichte, Husserl, Apel, Severino, …), rely on some kind of “transcendental reasoning” (whatever the given name) which equals a foundation. Generally it consists in assessing something “architectonically first” (from the point of view of the elaboration of a monument of thought) by way of a more or less explicit (or more or less disguised) form of *reductio ad absurdum* (e.g. Aristotle’s “ελεγχος”, Descartes’ “*cogito*” or Apel’s “*Prinzip des zu vermeidenden Selbstwiderspruch*”).

It seems plausible to state, as we do, that among these three kinds of “logical action”, the first is certainly legitimate (if not exhaustive). The second one seems to be only useful in quite restricted cases (scientific formal definitions – mainly mathematics and logics); in the general case of the need to understand (or define) a concept, Gärdenfors’ “conceptual spaces”, or something similar, will be much more adequate and powerful. The third action of logic is rather slippery: it is formally rigorous, *provided that* the criterion for judging if something is “absurd” is granted; outside the zone of validity of this criterion, the foundation fails to work.

02.07. Beware of logic, trust geometry. By the way, which one?

In what follows we must try to get emancipated from pure logic, avoiding the pitfall of the « all logic ». We must cope with the « dangerous bet » which says that logic is the ultimate tool of philosophy. However, the problem with Gärdenfors’ analysis is that it is frustrating: it does not give us a magic stick in change of logic (our previous magic stick). Even if, along with Gärdenfors’ analysis (2000) we do not know yet of something geometrical comparable to logic, we must not despair before having better explored geometry. But which kind of geometry, then? For there are many…

03.
THINKING NEWLY THROUGH GEOMETRY

According to what seen previously, we think that we must *think* geometry, and re-think logic through it. In other words, in this chapter (1) at (very) large scale we face the problem of the sense of a philosophical "geometric turn" of future philosophy; (2) more simply, we try to collect some possible useful ideas over geometry and philosophy, some conceptual benchmarks allowing us to locate the very geometrical-logical enquiry (on "opposition") which will take place in the rest of this work. By analysing the philosophical path which led many thinkers to literally "run away" from geometry, we show that, in many respects, this "geometric turn" is (or will be) in fact a long expected come-back.

03.01. The strange starting point: geometry disappeared

We have seen that the heart of logical negation has to do with simple geometry (squares and – we will see in ch. 8 – hexagons). We have also seen, thanks to Gärdenfors, that logic is misleading with respect to "concepts" (for them, logic is not the adequate tool): concepts deserve geometry. But then two questions arise at once: (a) how to understand *concretely* Gärdenfors' analysis? (how to develop concretely a science of the "conceptual spaces"? – we will not handle this question now) And, as a corollar, why then, historically speaking, such an exaggeration concerning the importance of logic for philosophy? (damaging geometry) Is there a problem with the philosophical understanding of logic? As we will see, nowadays, in the so-called "Priest/Slater/Béziau" debate (cf. ch. 10 *infra*), the very boundary of logic is more and more indexed over (polygonal) geometry. How come that this "geometrical side" of logic, seemingly very important, has not been seriously questioned before? We see two possibilities: (1) it could be because "geometrical logic" leads to nothing (no results to be found). But this is demonstrably false, for, as we will show (Parts II-III) many new *geometrical* results, inside logic, can be found. (2) Then it could be because of something else. This could be a contingent matter, or one admitting complex but understandable reasons, useful to be remembered and meditated. Again, as we will be able to show that there is a whole geometrical world behind (and inside) logic (Parts II-III), the "contingent oblivion" explanation seems to run short. And as a matter of fact, relying mainly on the incredible recently discovered whole story of Euclidean and non-Euclidean geometry (cf. § 03.02 *infra*), we will adopt the last answering position, the one according to which the vicissitudes of the role of geometry in philosophy have an interesting "political" (i.e. ideological, opaquely motivated) story, a story still heavy for our own choices of theoretical strategies. So, in this chapter, we want to have a preliminary look to some of the philosophical arguments, explicit or implicit, conscious or unconscious, that made active what turned out to be a true conceptual blockade to geometry.

03.02. An ancient scandalous discovery, and a sharp controversy

The standard, academic way of understanding Plato today, from a philological-philosophical point of view, is determined by Schleiermacher (the founder of hermeneutics, in Hegel's time)[56]. This is so in the sense that a majority of scholars tend to study his puzzling thought by reading his *Dialogues*, and refusing other textual resources, such as doxography (i.e. what people said of Plato, his teaching and his writings in his own time, or later, but inside the school – the "Academy" – which he founded (this research institution, one of the very first universities in the world, lasted for many centuries).

03.02.01. Problems with reading Plato

But this way of studying and understanding him bears a problem: we know that we lost most of the texts of the ancient (Greek) world. For instance, we know by some authors that in their time the "real great thing" in philosophy was not Aristotle (or Plato), but the Stoics. As is known, these last were supposed to have developed a very coherent system of thought (we know, we logicians, that their logic contained important elements – propositional calculus – absent in Aristotle famous logic). And we have none of their writings, seemingly lost for ever, burnt in the many fires or censored by the "rodent critic of rats" (Marx). One could answer to this: "Yes, but in Plato's case, we have all his writings!" (his *Dialogues*). However, the problem with Plato remains, for several reasons. (1) His dialogues, as a whole, make no coherent system, which is rather puzzling provided everyone recognises in Plato's reasoning the thinking style of a first rate genius; where is the system gone? Many answer that Plato's genius is one of incompleteness: he had the extreme wisdom not to close his thought into some theory, he remained "systematically alive", conceptually open forever. (2) But unmistakable elements contradict strongly such a poetic reading: how come that when

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[56] This point is studied and explained with much detail and theoretical strength by the Italian philosopher and philologist Giovanni Reale in his *Per una nuova interpretazione di Platone. Rilettura della metafisica dei grandi dialoghi alla luce delle "Dottrine non scritte"*, Milano, Vita e Pensiero, 1997 (1984), p. 3-134. Reale is one of the leaders of the so-called "Tübingen-Milano school", of which we are going to speak.

Aristotle makes his strongest critic against Plato, in the books M and N of his *Metaphysics*, he mentions very carefully and richly, with many conceptual details, theories – very theoretical – of Plato of which no trace other than very vague hints (vaguer than what Aristotle quotes extensively) remain in Plato's written dialogues? For instance, whereas the *Dialogues* give the impression that Plato's main contribution to philosophy is the "theory of Ideas", Aristotle (his pupil for over 20 years in the Academy) discusses extensively and in first place (as being the main philosophical issue, as being Plato's main and most important theory) Plato's "theory of the Principles" (the Εν and the αοριστος Δυας, the "One" and the "indefinite Two"), a theory absent from the *Dialogues*, where such two principles appear as secondary and curious things. (3) Plato himself wrote in many places (as in the 7th of his *Letters* – the one recognised by all as not spurious) that of the philosophically most important points he will not *write* (and this is coherent with theories of him present in the *Dialogues* where he claims that the true, deep and serious philosophy must remain *oral*). (4) We know by serious doxography (even by people of the very time of Plato, including Aristotle who, again, was his direct pupil) that Plato founded and directed a "school" (named "Academy", from the name of the hill where it laid), a research centre where young skilful people were selected, nourished and encouraged to make mathematical and philosophical research). Are we consequently authorised, under the excuse that no "dialogues" of these people arrived to us to deduce that there was no thought (no interesting philosophy) going on there? This is nonsense.

03.02.02. Plato's unwritten doctrines (αγραφα δογματα)

Such elements, with several other similar ones were gathered by some German reputed scholars (firstly by H. Krämer and K. Gaiser, the pupils of the famous German philologist W. Schadewaldt) in the fifties and used to propose a new standard for reading and understanding Plato (the one of the "School of Tübingen", later also called the "Tübingen-Milano School"). Its main methodological point is to "read" Plato through lateral doxography (reading Plato by taking seriously into account all that the tradition starting from him said of him, of his school and of his thought – his system!). In some sense, it is like paleontology: dinosaurs are reconstructed (hypothetically, of course) starting from the small fragments of fossilised bones available to us. And this method gives us a new vision of ancient philosophy, a much more coherent one. As for Plato, it seems that there was a Platonic *System* of philosophy, based indeed not on the "theory of Ideas", but on the "theory of the Principles" (whence Ideas and Ideal Numbers where derivable). Plato's Academy was really a research centre with a religious-political dimension: it was a kind of "intelligent sect" or "secret society", comparable to two of its ancestors, Pythagora's school and Parmenides school. In this research centre, the main direction was mathematical rigour, the model of such rigour being geometry: at the entrance stand, it is a well known traditional anecdote, the words:

ΜΗΔΕΙΣ ΑΓΕΩΜΕΤΡΗΤΟΣ ΕΙΣΙΤΩ,

(medeis ageometretos eisito)

"No one shall enter here if he is not a geometrician"

It is in this centre that some very important Greek discoveries in mathematics were made, among which:

- the theory of proportions: a/b = c/d (hence probably a stress put on analogic thought, including even the use of myths);
- the idea of axiomatic deduction system (later popularised and refined logically by Aristotle – in the Academic view the formal studies where not logical, logic as we know it seems to have been the invention of Aristotle, the Academic axiomatic was a purely mathematical one, embodied in geometry);
- axiomatic geometry (which later will be called "Euclid's geometry", Euclid being a later compiler, the research results having taken place previously);
- the "Platonic solids", that is the five only possible regular polyhedra (tetrahedron, cube, octahedron, dodecahedron, icosahedron)

As for Plato's system, its ontological architecture seems to have been a proportioned one, taking for model and guide the Pythagorean "tetraktys" (the triangle made of ten points and embodying the remarkable sum "1 + 2 + 3 + 4 = 10"), made of four levels of being (each corresponding to one of the four previous numbers): the Principles, Ideas, the mathematical entities and the sensible things (at each level new subdivisions tend to repeat the basic ontological scheme). Remark that it seems that Plato's system, seen as a whole, was a rather tragic one (at Plato's own eyes), because of the predominant place in it of the "indefinite Two" (which was for him a principle of badness – the Good being the One).

03.02.03. A shock inside the Academy: geometry is not one

Now, the important point for our enquiry is that, according to the Tübingen school, there seems to be evidence that a big (in fact a huge) shock occurred about geometry inside Plato's Academy. We must remember that geometry was being studied in order to oppose it to the threatening Sophists: Plato's school was ideologically engaged against increasing relativism. Geometry, as an ideological weapon, was meant to represent science (επιστημη, "episteme", what stands without falling) against relativism (επιστημη against δοξα, "doxa", the opinion). And geometry was being elaborated, for the first time in the history of humanity (the Egyptian geometry, very developed, was mostly empirical) as a mathematical deductive system. But while investigating geometry axiomatically, the researchers of the Academy discovered that one very intuitive property (i.e. the fact that given a right line and a point exterior to it, one line and one only, among all the infinite number of those which pass by this point, is parallel to the starting one) cannot be deduced by means of the axioms of geometry: despite its intuitive character it is a postulate. And when they tried to show, by a *reductio ad absurdum* (by considering the hypothetical case where such very natural property would be false), the necessity of having such a property (because absurd consequences would have been derivable by its supposed negation), they discovered, instead of the expected result, that nothing prevents from constructing new geometries where such a property indeed fails. But these are horrible geometries, contradicting intuition. They do not legitimate reason, they do destroy its image. This result was a catastrophe, it seemed to go in the direction of the Sophists. Apparently, there was a vast debate inside the Academy (a debate including the young Aristotle), but it was chosen not to let the (bad) news go outside the walls of the Academy.

Some remarks are capital here. (i) This kind of case is not new: something very similar happened with the Pythagorean school (this is an established and accepted historical fact), which, investigating arithmetic (and giving to the laws of arithmetic a deep philosophical and religious meaning) discovered against its will the existence of the so-called "irrational numbers" (more precisely, they discovered the square root of two, "$\sqrt{2}$"), the numbers which are not reducible to finite fractions (or: the decimal numbers with a non-periodical infinity of number after the comma). The tradition says that in the Pythagorean school it was decided to keep this discovery hidden (so big was the philosophical shock with it – it could diminish the credibility of the Pythagorean numerologist philosophy); tradition also says that when one of the members of the sect revealed the secret to the world, the sect decided to kill him (and did it). (ii) Whereas one can think it unbelievable that the ancient Greek discovered a mathematical astonishing result that we discovered only in the 19^{th} century, the problem may be reversed: it would be highly unplausible that those who discovered (and founded) Euclidean geometry were not able to see that one of its postulates, if changed, can give rise to different geometries. (iii) Such a (seemingly) strange idea ("the Greeks having discovered 4 centuries BC non-Euclidean geometry, which we discovered in the 19th century") was held independently by several scholars before the Tübingen school did it, among which the renown historian of ancient science C. Mugler and … the logician and philosopher C.S. Peirce![57] (iv) To those objecting (rightly) that it is strange (apparently) not to see any sign of such a past "epistemic bomb" it must be replied that, (a) the secret was kept relatively well (and that there was no scientific need to develop the discovery, whereas the discovery of the irrationals was very useful for making calculations) and (b) indeed, there are testimonials of it, in particular by Aristotle (!), who sometimes, mainly in his *Nicomachean Ethics*, speaks strangely (not as if he were dealing with something absurd) of having triangles whose sum of the inner angles is different from two right angles[58].

03.02.04. The decline of the Academy and the disappearance of geometry

So what happened? Clearly we do not know. But it seems, with all the uncertainty due to the already mentioned extreme poverty of what was conserved for us of the ancient philosophy, that the Platonic Academy declined slowly, also helped in this by the political fall of Athens (first under rival Greek cities like Sparta and Thebes, then under the Macedonian empire and finally under Rome). But even independently from this, the Academy was challenged by rival schools: the one of the Stoics (of which we don't know much), the one of Epicurus (whose writings have as well almost entirely been lost) and the one of Aristotle's himself (the Lyceus), who could have become the Academy's new leading figure after Plato and who chose instead (at least we presume) to make secession and open his own school. In all this seemingly the role of geometry became poorer and poorer.

03.03. A very heavy Aristotelian choice

Aristotle clearly could not have succeeded to Plato (unless he had made an intra-Academical huge revolution): for he held that mathematics are dangerous for philosophy. The reasons of such an original anti-Platonic position are multiple, but among them something seems to be going on with geometry. Aristotle's alternative proposal for understanding the

[57] Cf. Mugler, Ch., *Platon et la Recherche Mathématique de son Epoque*, Strasbourg-Zürich, A. W. van Bekhoven, 1948, p.141; Peirce, C. S., *The New Elements of Mathematics*, (edited by C. Eisele), vol. III/1, The Hague-Paris, Walter de Gruyter, 1976, p. 704.

[58] Cf. the reference work: I. Toth, *Aristotele e i fondamenti assiomatici della geometria. Prolegomeni alla comprensione dei frammenti non-euclidei nel "Corpus Aristotelicum"*, Vita e Pensiero, Milano, 1997. In French one can read the excellent compendium of M.-D. Richard, *L'enseignement oral de Platon*, Paris, Cerf, 1986.

intimate structure of the real will be destined to a large (if not continuous) success, partly lethal to the philosophical “geometrical thinking”.

03.03.01. Aristotle’s background: the departure from Platonism

So, why did he escape Platonism, he who could have been the “new Plato” inside the Academy? From the point of view of the arguments (we cannot explore the point of view of psychology) it seems clear that the turning point is mathematics. Aristotle finds mathematics dangerous (for philosophy) for several reasons. One could be (but this is conjectural) that they lead to necessitarianism: if there are traces of such a position by Socrates and Plato, Aristotle clearly fought this doctrine heavily, as appears in his dispute with the Megarians (cf. ch. 04 *infra*). Another reason of hostility may have been some kind of allergy to the “mathematical fetichism” of the Platonists. He says: “But Mathematics have become, for the Modern, the whole philosophy, despite the fact that they claim that one should practice it only in order to better approach the rest” (Aristotle, *Metaphysics* A, $992a^{32}$-b^{1}). In other words, Plato’s fetichism of the geometrical objects (the five regular solids, meant to be the “atoms” of the natural elements), is potentially ludicrous (one could think of a play of Aristophanes thereupon, as the comedy he wrote against Socrates)[59]. A third reason, possibly related to the previous one, is relative to the reconstruction offered by the Tübingen-Milano School: if geometry failed, inside the Academy, to achieve the philosophical task it had (i.e. giving a decisive weapon against the Sophist’s relativism) then mathematics could fall in disgrace. And against the Sophist’s sophisms (which, once more, could have deep political effects on the Athenian society), Aristotle may have preferred to develop a defensive (“transcendental”) strategy consisting in elaborating a canon of verbal reasoning: logic and syllogistics. This development of logic instead of geometry (as a general method of thinking) was parallel to (and complementary with) the development of a science, biologically flavoured, of the process of “becoming”.

03.03.02. Aristotle’s philosophical framework: a founded theory

Again, the question of a deep judgement over the genesis of Aristotle’s thinking innovations, these last being so important for human thought, is not easy: we lost most of Aristotle’s writings (he wrote many dialogues, which were reputed even for their writing style

[59] It is the comedy *The Clouds*, where Socrates lives on a suspended basket, in order to be able to contemplate things from above.

– whereas we tend to repeat in a silly way that "Aristotle is a poor writer") and we ignore the real paternity of what we call "his remaining writings", possibly the school annotations taken by some of his students or collaborators. Nevertheless, Aristotle is a grounding figure for the human thought, and this mostly *precisely* because of his complex opposition to his master, Plato (himself a grounding figure). When Aristotle discovers or invents, as he does, many of our most basic concepts ("energy", "act", etc.), he endorses, against Plato, a paradigmatic anti-mathematical position (in this respect he is paradigmatic). Now, Aristotle seems to have been fully aware of this intra-academical controversy about the possible geometries (i.e. about the impossibility to prove that the Euclidean one was the only possible one). So Aristotle probably (or possibly) chose another strategy (in order to pursue the fight against the Sophists), an indirect one, that of a transcendental foundation (the argument of the onto-logical impossibility of getting rid of the principle of non-contradiction). All this converges, in Aristotle, on his theory of opposition, and on the logical square (cf. ch. 4 *infra*).

First, Aristotle establishes biology instead of mathematics (geometry), as a formal paradigm of ontology (the science of pure being) and of the science of all there is (he abandons mathematics, which is a huge change with respect to Platonism)[60]. He abandons the model of (Euclidean) geometry. Second, he chooses foundation (certain and stable foundation) instead of complex infinity (this strive for final foundation is, again, motivated by the fight against the Sophists and their dangerous and frightening relativism). But what can be put forward, as a model of certainty, instead of mathematics? Here comes, thirdly, one of Aristotle's most original inventions (in some sense – i.e. despite its bad effects –, a stroke of genius). He decides to look for something certain inside the art of reasoning itself. In his theory there is a pivotal role of logic, and particularly of the formal law called "principle of non-contradiction".

Differently from geometry, this works in a complex but beautiful way, interlacing the theory of judgement, the theory of thinking, the theory of being and the theory of becoming. Making a clear anachronism, one could be tempted to say, in Hintikkian terms, that if we have with Plato an implicit conception of "logic as calculus", with Aristotle we find, in one of its first occurrences, an explicit conception of "logic as universal language"[61].

[60] Of course, what Aristotle meant by biology is not the same with the current meaning of the word.

[61] Cf. Hintikka, J., "Is Truth Ineffable?", in: (without author), *Les formes actuelles du vrai*, Entretiens de Palerme, Palermo, Enchiridion, 1989, p. 89-120; where Hintikka a conceptual distinction elaborated (if not invented) by Martin Kusch..

03.03.03. Aristotle's philosophical destiny (a curse for geometry)

As is known, in the ancient times Aristotle was, truly speaking, eclipsed philosophically by the Stoics (in Rome's time the two main philosophies fighting for supremacy were stoicism and Epicurism, with a clear predominance of the first). Most of what we have of Aristotle's writings had been brought in Rome at the fall of Athens and codified by Andronicus of Rhodes. Another school, neo-Platonism, offered a synthesis of Plato and Aristotle. With respect to geometry a singular fact is that Proclus, one of the greatest neo-Platonist (Vth century of our era) tried to demonstrate the problematic Euclidean postulate, showing by that – if we are, as we are, to take seriously the reconstruction by the Tübingen-Milano School of the dramatic Academic work on the foundations of geometry – that the philosophical community near to Plato's tradition (as were the neo-Platonist) had totally forgotten the very problem at the origin of the disgrace of the Movement[62]. But after the fall of Rome (476), which destroyed so many of the ancient works which had resisted so far, the slow and inglorious rebirth of philosophy (through the monasteries and through the rude invention of European school by Alcuin for Charlemagne in the VIIIth century) consisted mainly in the elaboration of the theological doctrine of Christendom (after the fathers of the Church)[63]. This was done with very poor means and under constant peril of invasion. One of the main theoretical elaborations was the work of the copists, with what it implied of glosses and doxographies written by the copists inside the borders of the circulating manuscripts. On the basis of what was available, among which elements of Aristotle's theory of logic (the *Organon*), the education of the Western scholars could redevelop some of the ancient *trivium* (the mastery of language was decomposed into grammar, rhetoric and logic): this explains the then growing success of logic: it offered conceptual light in a universe violently deprived of written philosophical culture. So the intellectual strengths inside the protected monasteries (i.e. early Scholastics), concentrated in thinking refinements of logic (and thereby of theological paradoxes). The Arabian, then flourishing civilisation (of El Andalus) offered – between two battles – to the West (then unable to read Greek) the rediscovery of some of the ancient texts lost with the fall of Rome. Finally, Saint Thomas succeeded in imposing his own

[62] The fact that such a huge philosopher as Proclos devotes a commentary to Euclid's *Elements*, in a context where geometry is not anymore a crucial issue, seemingly comes from the fact that he's nevertheless aware of the geometric legacy of Plato's school, of the importance of geometry for it. And this is visible in Plato's writings. But so many centuries have passed that he has lost the thread of what went on dramatically at the origin (and was kept secret rather efficiently, despite Aristotle's several hints to it in his writings, unintelligible for the non-initiated). Proclos case would show that, after centuries, the secret was lost (forgotten) even inside the Academy itself (of which Proclos is one of the heirs).

[63] Cf. K. Flasch, *Introduction à la philosophie médiévale*, Editions universitaires de Fribourg, Fribourg, 1992.

conceptual synthesis of catholic theology and Aristotelian philosophy, which established in a stable way Aristotelianism as the standard of the Western thought starting from the XIIIth century.

03.04. Descartes trying to come back to mathematics

The transition from the domination of Aristotelism to modernity was partly progressive, but partly it was sudden. Inside a vision of the world largely inspired by Christianity (after the fall of Rome) and Aristotelianism (as we saw), one major change toward a revaluation of the role of mathematics for philosophy was brought by Copernicus' discovery of the "false but useful mathematical fiction of heliocentricism". As is known, Galileo made a major contribution – he claimed that heliocentricism was not a "fiction" and he successfully (for science, if not for his personal life) generalised the mathematisation from cosmology to physics establishing the non-Aristolelian, mathematical laws of motion and gravity – which he claimed to be clearly Platonic:

> "Philosophy is written in this huge book constantly open before our eyes (I'm speaking of the universe), but one cannot understand it if one has not previously learned to understand the language, and to know the letters, in which it is written. For it is written in mathematical language, and its letters are triangles, circles and other geometrical figures, and without these means it is impossible for mankind to understand its words; without these it is like turning round hopelessly in some obscure labyrinth" (*Il Saggiatore*, in: *Opere*, VI, pp.232-233).

René Descartes (XVIth century) tried to make a philosophical new synthesis, inside Christianity, of this rediscovery of the importance of mathematics for science. During his own intellectual evolution, the French philosopher and mathematician was impressed by at least 6 points: (1) the uselessness of the traditional knowledge (ancient languages and traditional philosophy – i.e. Scholastic); (2) the study of the body of the human dead (the discovery of a "machinery" inside human anatomy); (3) Galileo's discoveries (gravitation, astronomy – the power of mathematics); (4) his own discoveries in mathematics (like "analytical geometry"); (5) his own discovery of a general method (a method both coming from his mathematical discoveries and leading to them); (6) his own discovery of the (transcendental) argument "I think, therefore I am" ("*cogito ergo sum*"). Hence Descartes formulated the project of building a radically new philosophy, bringing this last back to mathematical rigour. According to Descartes logic is sterile, whereas mathematics are the way (analytical geometry) – to free-will. The world is ruled by a strict mechanism which also rules "life": animals are mindless machines, whereas humans are machines ruled by an exterior soul.

Descartes' highly innovative project turned out to be, in some sense, a catastrophe: for, as his brilliant follower, the Dutch philosopher Baruch Spinoza shows it (around 1675), this project, once purified of its *ad hoc* Christian postulates (through the use of the formal model of Euclidean geometry in Spinoza's *Ethica more geometrico demonstrata*), leads in fact to a strict necessitarist view (where free-will is just an illusion or a wishful thinking, and God a concept representing a mindless and aboulic, all encompassing mechanism).

The attempts in eliminating necessitarianism from the post-Cartesian rationalistic turn of philosophy, as paradigmatically by Leibniz, give remedies against necessitarianism which are largely perceived as sadly (and dramatically) unconvincing. Leibniz' optimistic solution is at least partly inspired by his mathematical discovery of integral calculus (this discovery relying on a certain new use of the notion of infinity). This solution is mainly based on [1] an invocation of the existence of the infinite in nature ("the labyrinth of human free-will"), [2] a monadologic Theodicea (the divine system of pre-established harmony) and [3] in the project of a renewal of logic ("universal logic" as a unique calculus for introducing definite unquestionable human-friendly truth on all topics inside human knowledge).

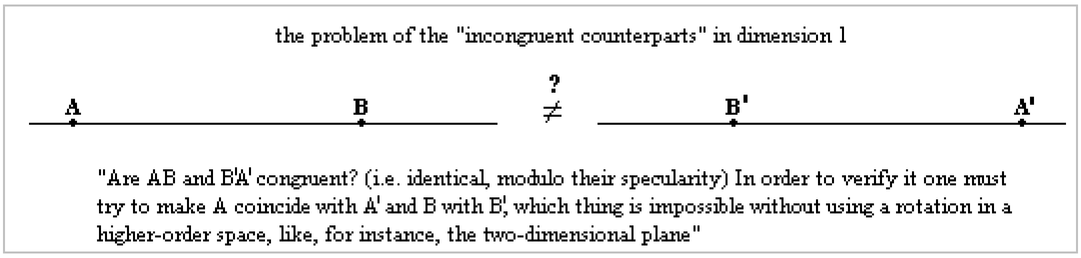
In some sense, the following German-speaking philosophy, largely determined by Kant, is centered in a restless re-examination of Leibniz' optimistic views (popularised and officialised by Christian Wolff), perceived as too optimistic, and in a constant effort to neutralise Spinoza's frightening view of a necessitarian world.

03.05. Kant renewing, for us, heavily the heavy choice

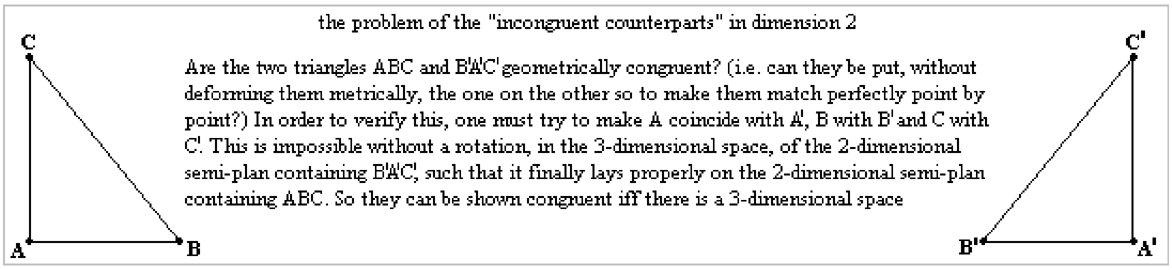
Because of its extreme importance for the tradition, it is necessary for us to understand something of Kant's very original, but also very Aristotelian contribution to the question of the formal means of philosophy, in order to put some kind of diagnosis over the evolution of the Western philosophy, both generally and in the particular case under examination here: the question of the relations of logic and geometry. As it is known, Kant's original philosophy begins with an inner critic of Leibnizian philosophy: there is a clear horror for the necessitarianism implicitely present in it, and also annoying in the natural sciences (with Newton) and in theology (with Luther). If one also recalls Kant's horror for Hume's reasonable theoretical (not practical) skepticism (epistemic strong relativism), it becomes clearer why Kant's main solution, aiming at determining a sharp and definite anti-skeptical fix point (in the *Prolegomena* he speaks of a "all or nothing" solution, meaning that the Critique of pure Reason is done once and for all for the eternity of the future thinking generations, and of aiming at a "critical knowledge entirely achieved": i.e. with no possible return), seems to be the invention of a "transcendental faculty" beyond geometry. Geometry, even if the thing is not so visible, is a major issue with Kant.

Indeed, relying on Aristotle's limited (that is, "anti-infinite") conception of logics while developing his own "criticist philosohy",[64] Kant remarks further, and affirms as a definite mathematical truth, that geometry cannot be more than tri-dimensional (there is no fourth – and a fortiori no fifth and so on – geometrical dimension)[65]. But in so doing he is not simply saying aloud what everybody knows and thinks very naturally at that time ("the world is such that geometrical dimensions stop to three, nothing intelligible or simply conceivable is behind"). He is also, because of the programmatic transcendental character of his theory, laying a very heavy judgement on the possible (un)usefulness of geometry for "pure" thinking in general.

Kant asks how it is possible to compare two linear specular oriented segments (i.e. two one-dimensional objects), for judging if they are the same; he answers: "by using a plane (i.e. the two-dimensional) in order to make the two segments rotate out of their respective linear universe and, if possible, coincide point by point" (in a same line). Here geometry helps the thinking of differences.

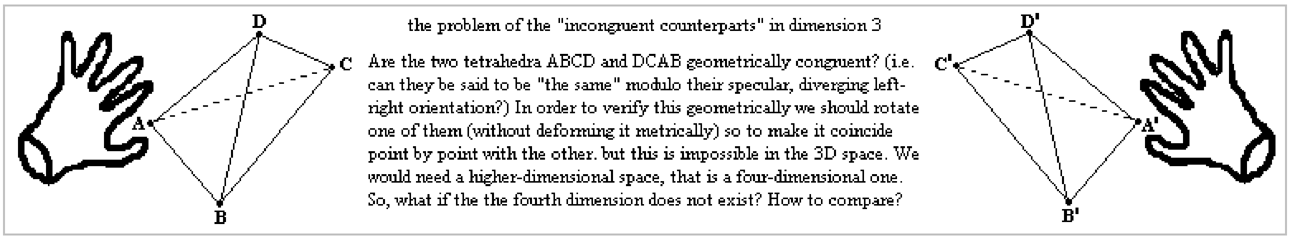


What about two seemingly specular triangles? (i.e. two two-dimensional objects) He answers: we can cope with that by using space (i.e. the three-dimensional) in order to make these two otherwise congruent triangles become point by point comparable in the same plane.



[64] As is known, Kant will say that "After Aristotle logic has made no real further step forward", which means: logic is eternal, indubitable and simple, admitting no future discoveries and surprises within its scope.
[65] It is known that mathematicians like Gauss (not to speak about Bolzano) felt very uneasy with these views of Kant over the formally thinkable.

But then what about two seemingly specular hands, a left and a right one? (i.e. two three-dimensional objects) The strategy used in the two previous cases cannot work here, for now we would need, in order to try to make the two hands (the left and the right) coincide point by point, a higher space of rotation (i.e. the four-dimensional), but this space does not exist (there is no geometrical fourth-dimension), so we cannot perceive the identity of two three-dimensional counterparts, unless we use some mental power other than geometry…".



So, observing that the mind is able, despite the transcendental "tri-dimensional limitation" of geometry due to the impossibility of having a fourth geometrical dimension, to grasp tri-dimensional cases of "incongruent counterparts" (whereas one- or two-dimensional incongruent counterparts geometrically require two- or three-dimensional spaces to be resolved), he "deduces" (by a hidden *reductio ad absurdum*) that the human mind has a superior faculty of intellection which is extra-geometrical (and this faculty takes precisely the place of the mathematically impossible fourth dimension and moreover of mathematical thinking in general)[66].

He consequently explains more generally the human ability to do pure mathematics (that is: to reach scientific absolute certainty) by his "doctrine of faculties" and his "transcendental logic", all this remaining grounded on Aristotle's logic (that is: the logical square, cf. ch.4 *infra*).

This neutralisation of geometry is the somehow hidden move sustaining Kant's well-known invention of "transcendental logic" (which is seemingly in fact a false logic). This move pretends (and believes) to solve both problems: (i) the danger of geometry (cf. Spinoza) and (ii) the danger of relativism (cf. Hume). But this Kantian move ("critic philosophy"), this is our central point, is a very heavy one for philosophy, for at least two reasons (related): (a) it eliminates geometry from the scene of the "theoretically important means" (as we saw with the argument of the incongruent counterparts) and (b) it is explicitly transcendental: again

[66] There are six places in the work of Kant where this argument can be seen repeated. One of the most famous is in the *Prolegomena*, § 12 and 13. On these topics, cf., among many others, Remnant, P., "Incongruent counterparts and absolute space", *Mind*, 72, 1963, p. 393-399; Huenemann, C., "A Note on the Argument for the Non-Spatiotemporality of Things in Themselves", *Kant-Studien*, 83. Jahrg., 1993, p. 381-383; Rusnock P. and George R., "A Last Shot at Kant and Incongruent Counterparts", *Kant-Studien*, 86 Jahrg., 1995, p. 257-277.

(this point is very important and dramatic), Kant's "critics" (his three main treatises) pretend to be demonstrated once and for all, with no permissible theoretical come-back. And he pretends they are rationally unavoidable, that is: in the future of thinking mankind (to make the point clear Kant also refers to possible living rational beings from other planets) what will follow theoretically (i.e. all future philosophy) will have to obey the theoretical "theorems" of Kant's critical thought. In other words, Kant's historically successful philosophy is one claiming the fundamental primacy of time over space[67]. This way, by a move congenital to, but different from Aristotle's "heavy move" (cf. *supra*), Kant renews heavily this heavy Aristotelian anti-mathematical (and anti-geometrical) choice.

The inadequacy of Kant's theory to really explain the essence of mathematics and infinite logic (and of geometry), despite the common success of this theory, is shown by isolated pioneers of thought, the Austrian mathematician and philosopher (and theologian) B. Bolzano and the American logician and philosopher (and semiotician) C.S. Peirce. As a matter of fact, Bolzano (and later Peirce) showed that there are, hidden, huge problems in Kant's explanation of mathematics and logics, for, according to Bolzano, (i) Kant's "synthetic a priori judgements" do not suffice to explain mathematics and (ii) explaining mathematics (as Bolzano does scientifically) does not need synthetic a priori judgements". More generally, Bolzano, against the Aristotelian anti-Platonic tradition, shows the existence in mathematics of the "actual infinite"[68]. As is known, according to Peirce there also are logical problems with Kant's theory of judgements (his "table of categories")[69]. Further, Bolzano shows that there are "mathematical monsters", fractal objects denying the Kantian lazy identification of mathematics with intuition, and showing the treasures which geometry, with its infinity, still may offer to human thinking.

Beyond Bolzano's and Peirce's deep pioneering (and often academically unpopular) remarks, mathematics teach us, against Kant, at least three things (three important revolutions in the science of geometry),articulating the fact that Euclidean geometry can be expanded beyond what was imaginable in at least three ways. [1] First of all, the fourth geometrical dimension, although counter-intuitive, does indeed exist[70]; [2] second, mathematics also

[67] Which explains in part the constant implicit reference made to Kant by the thinkers of the phenomenological tradition, who seem to rely on him as for the matters of logical-transcendental structures and for the main categorical choices of philosophy (they appreciate, even if they modify it, the notion of transcendental subject).
[68] On these matters, we relied mainly on Laz, J., *Bolzano critique de Kant*, Paris, Vrin, 1993; Laz, J., "Un platonicien débridé? Bolzano, critique de l'intuitionnisme kantien", *Philosophie*, N.27, summer 1990, p. 13-29.
[69] Cf. Peirce, C. S., "On a new list of categories", in: *Collected Papers*, Harvard university Press, Vol. 4, § 2.
[70] As is known, the cultural shock of the discovery of higher-space was used by Edwin A. Abbott for writing his delicious fiction of a travel between worlds of different geometrical dimensions. Cf. Abbott, E. A., *Flatland. A Romance of Many Dimensions*, New York, Dover Publications, 1992 (1884).

teaches us that there are geometrical possibilities outside the classical ones established by Euclidean geometry (i.e. with “non-Euclidean geometry”: we discussed in this very chapter the possible Greek first discovery of this; we will see then how a Russian logician will try to use this geometrical impulsion in order to think something logically new cf. ch. 7 *infra*); [3] thirdly, more recently, but coherently with discoveries of Galileo (the paradoxes of the infinite sets of numbers) and Bolzano over the infinite (and positive paradoxes of the infinite), also the so-called fractal geometry (mainly developed conceptually by the French and American mathematician Benoit Mandelbrot), of which we will see in ch. 11 *infra* a logical expression.

03.06. Post-Kantian and Anglo-analytical logical confusions

The destiny of Kant’s legacy (itself rooted in an Aristotelian anti-Platonic revolution) seems to be rather grim for geometry. The direct heirs of Kant, the so-called continental philosophers gave, without beating about the bush, a loud and clear “farewell” not only to geometry, but even to real logic (and even when, like nowadays, it should not be permitted to ignore anymore the extreme richness and relevance of logic for thinking, especially the “non-standard”). Priority was given, inside the fundamental methodology of philosophy, to the concepts of subject, life, existence and action.

But things weren’t much happier for the geometrical approach (to the universe of forms) from the side of the “enemies of Kant”, that is, the British thinkers who (rightly) refused to enter the non-empirical realm of transcendental certainty. This heroic breed, now called (mainly) analytical philosophers gave, at least, highly valuable work as for logic (by a fantastic renewal of logic, and a deep exploration of language). But it also gave, alas, some kind of logical fetishism, by neglecting other approaches to the formal, and a certain amount of ideological extremism, by the dream, still alive, to *extirpate* (almost) all the rival approaches not in accordance with the analytical standard. Which is a major (conceptual) crime: the formal “has no ultimate form”, serving it requires being open to forms (shapes) always different: it requires loving structures for their diversity. On the contrary, in the realm of the (more than) friends of logic, the practice and the pleasure of finding “structures” is lost, all that is looked for is “logical languages”. But this is a loss of mathematical power, for there are things (shapes) which cannot be done (seized) by logic alone, no matter how complex this last. From this side of the “after Kant”, the unique alternative to logic, when its limits are perceived, is said to be “natural language” (by which sometimes analytical and continental do encounter and shake their incongruent hands, like Apel tries to do it). But this, as logic, is, in some sense, much more temporal than geometrical: geometry is clearly and lastingly neglected.

03.07. The logic of the geometrical possibilities

All in all there seem to have been, at least in the Western tradition, two major paradigms of scientificity: Euclidian geometry and Aristotelian logic

We mentioned Kant’s claims over geometry and stressed how the geometrical revolutions (hyper, non-Euclidean and fractal geometry, the list being not even complete) occurred after his work. If consequently Kant is proven to be at best incomplete (but this is a false understanding of it), at worst strongly mistaken (which seems much more trueto the facts), the question arises then as to knowing what can be said, nowadays, from a scientific (that is, formal) point of view, about Aristotle’s logical alternative to geometry, and in particular about the strange, generally unquestioned fact that its heart, the “logical square” (of opposition) is a … square! (that is, a *geometrical* object) We suggest that this question must, accordingly, be handled both ways : logically of course, *but also geometrically* (which is never done).

The object of what follows (i.e. all of the present study) will be to check a conjecture according to which “opposition”, as a subject matter of scientific investigation, besides Aristotle’s classical impressive theory of it, admits an infinite but not obscure complexity: a fractal complexity, a geometrical *n*-dimensional one. An interlace of logic and geometry. This conjecture, if verified, will constitute a philosophical challenge to both Aristotle’s and Kant’s philosophical “heavy positions”.

Part I

THE CLASSICAL THEORY OF OPPOSITION: THE LOGICAL SQUARE AND HEXAGON

04.
ARISTOTLE'S "THEORY OF OPPOSITION": THE "LOGICAL SQUARE"

In this chapter we recall the creation (or discovery) of the "logical square" (or "square of opposition"). This goes mainly through at least two historical moves: the elaboration of its underlying logic, mainly by Aristotle (~384-322 BC), and the proposition of the geometrical-logical object itself, the logical square, by some logicians following the Stagirite, Apuleius (~125-170) and Boethius (480-525), (and perhaps already by Aristotle himself, this point is historically undecided). This logical-geometrical structure is of the highest importance to opposition theory and to logic (and mathematics) in general, as we are going to argue through the rest of this work. But despite its universal success until today (which we will recall in ch. 5 *infra*), it has borne since its Aristotelian beginnings some more or less mysterious and unresolved drawbacks or unclearness.

04.01. Aristotle's philosophical background and own framework

We saw in chapter 3 some of the main elements of Aristotle's Platonic complex (and conflictive) background: nourished at the "school of Ideas" (the prestigious philosophical-mathematical-political research centre called "Academy", in Athens), he finally developed a theory (based on what he baptised "logic") rejecting Ideas (the structures enquired by Plato and his school) to a minor role (for Aristotle, ideas are not entities existing *per se*, they are only ways of speaking inscribed in the "material things", ideas are the result of "abstraction" applied to reality). We also recalled in its main lines his own conceptual framework, an "energetic" ontology of the "becoming" process, based on the hypothesis that things result from the inescapable union – συνολον, *synholon* – of a "matter" and of a "form", and on the articulation of "power" and "act": what has the power to exist may, possibly, commit the act of existing, this eventuality being related with some kind of freedom. This conceptual framework coincided, to a large extent, with a (violent) departure from Platonism (i.e. a divorce with respect to the fundamental idea of choosing mathematics as the guideline of philosophy and ontology – Aristotle preferred, so to say, an approach more based on biological motion and qualitative change). Now, it can be useful, in order to focus on his theory of opposition, which will be the starting point of our present formal work (Parts II and III *infra*), to focus more on two other important conceptual fights of Aristotle, namely the one opposing him to the Megarians (in order to restrict logic from its necessitarian, painful ontological implications, as we already recalled in ch. 1 *supra*) and the one opposing him to the relativism of the Sophists. It is while elaborating his answer to these challenges that Aristotle got to shape more precisely his logic as being, in some sense, a "logic of the square". So let us first say some words about the anti-sophistic root of his logic.

04.02. Keeping the Socratic-Platonic fight against the Sophists

The opposition to the sophistic movement seems to have been a Socratic trademark. Of course, it is difficult to know exactly if it is Socrates or Plato who is speaking in Plato's Dialogues, our main philosophical source on the Socratic thought[71]. Seemingly both felt compelled to resist the intellectual relativistic trend burning the Greek minds at that time, by developing rationally acceptable standards of non-deceitful reasoning.

04.02.01. What the Sophistic movement, ambiguously, meant

In order to have a clear grasp of the Aristotelian intervention (at least concerning the object of our enquiry), one must try to remember the exact meaning of the Sophistic movement. Seemingly it was composed of at least two antithetic aspects, constitutive of its high ambiguity. As a matter of fact, on the one hand the Sophistic movement constituted an intellectually very high moment of the history of human thought: desacralising the human language, by the use of strange reasoning rooted in an unlimited use of the grammatical, semantical and logical resources, it liberated from many mental blockades made of religious dogmas or traditional unquestioned ways of thinking or behaving. In this sense, and specifically by the discovery and the exhibition of "paradoxes" (i.e. shocking new truths, delivered by unusual reasoning), the sophistic relativism was a rather healthy pre-condition of the further development of high-level philosophy (what we still call the "Greek miracle"). But on the other hand, this generalised tendency to "break all the limits" (at least provided some kind of "style" made such ruptures acceptable and pleasant) gave rise to innumerable (and often intentional) misuses (the so-called "sophisms"), tendentious, brilliant but unclear reasonings where brilliant people progressively became preferable – for aesthetic reasons – to

[71] As is known, other contemporary reports on Socrates (his life, his deeds), like those of Xenophon or Aristophanes, lack philosophical interest. So the temptation for us is great to follow Plato's descriptions (in his *Dialogues*), Plato being at least clearly apt to seize (and express) Socrates' philosophical dimension. But, then again, nothing guarantees that he has not created more or less severe distorsions of Socrates' real thought.

wise, or reasonable, or just (or decent) people. In this sense, the increasing presence of accepted sophistic reasoning (and of sophistic teaching institutions), for allowing no future limitation (of the excesses) to become visible at the horizon, made the Sophistic movement elicit (potentially huge) fears and resentments, not only among the aristocratic keepers of the traditional ways of thinking and being, but also among those just seeking science and political stability[72]. As we said, Aristotle as well as his philosophical "father" (Plato) and "grandfather" (Socrates) manifestly decided to oppose themselves to the Sophists, and the main way to do so was to fight the "sophisms" at their root.

04.02.02. What "sophisms" are. Their difference from "paradoxes"

Here as well (as with respect to the historical Socrates), it is difficult to judge something (i.e. the doctrines of the Sophists) that seems to have been severely corrupted: the Sophists' real thought is textually lost for us, except for fragments, and these fragments are mainly constituted by the malevolent (or at least polemical) quotations made by their opponents: Plato and Aristotle. So many authors, especially in the 20th century (preceded by the German philologist-philosopher Nietzsche), have developed studies in order to restore the real, glorious thought of many Sophist thinkers[73]. Nevertheless, there seems to be a reasonably shared consensus over the fact that some productions of the Sophists – *pace* other more respectable ones – were pieces of intentionally corrupted reasoning, sometimes made in order to play with words, sometimes made in order to push human reason to its limits, but sometimes made in order to deliberately deceive. These productions are called "sophisms". Roughly speaking, a sophism is a reasoning bearing the appearance of correctness, but not correctness itself. Remark that the criteria of correctness were not clear at that time. Corruptions of reasoning may be of several kinds, generally related to specific aspects of language (grammar) and thought (logic or other). For instance, corruptions can be brought by undue symmetrisations (so to say, "aRb" is skilfully substituted by "bRa"), or by the undue identification of the part with the whole, or *vice versa* (a "mereological" irregularity), and so on[74]. Another important class of sophisms is related to bad uses of the concept of "negation"

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[72] Remark that this point is complex, for: (1) several Sophists were also scholars (for instance serious and brilliant mathematicians), (2) the very idea of science (i.e. our Western one, the so-called επιστημη, *episteme*), in a relevant sense, may be shown to have been originated exactly by opposition to the relativism of the Sophists; and (3) the sophistic movement favoured one kind of viable political life: the so-called Greek democracy.

[73] Such a trend is often nourished of a philosophical suspicion against the "Platonic turn" of the Western world.

[74] The symmetrisations of thought, interpreted as a natural but complex (non-trivial) meaningful feature of the Freudian "unconscious", have been studied by the aforementioned I. Matte Blanco in *The Unconscious as*

(and more generally of "opposition"). It is on this point that we will concentrate now, in order to follow Aristotle. But before that we must remark that sophisms often also bear a shock value, for they often present themselves as paradoxes. And it is because paradoxes can shake the firmest foundations (conceptual as well as moral or political) that thinkers like Socrates, Plato and Aristotle, very aware of foundational problematics, structures and issues (theoretical as well as practical), were very worried about the shocking reasonings produced by the Sophists. Reducing the frightening paradoxes of the latter to ludicrous sophisms was an issue of major importance.

04.03. From the study of "negation" to "opposition theory"

So, studying one special (very important) kind of possible corruption of thought, Aristotle develops a theory of opposition, which turns up to be related to a quaternary combinatory. Aristotle fights the Sophists in many places of his known writings. The discussion on opposition, however, is mainly afforded in the *Περι ερμηνειας* (*On Interpretation*), a study where he investigates language, and more precisely: "what is the noun and what is the verb; then what is the negation, the affirmation, the statement and the discourse"[75]. After having summarised Plato's treatment of the negations of nouns, concerning opposition, the philosophical problem tackled by Aristotle is that of investigating the negations not of nouns but of *modal* contexts, i.e. of sentences where the subject-predicate articulation[76] (like in "Socrates is mortal") is *modulated* (often by an adverb): like in "Socrates is *necessarily* mortal" (or equivalently: "it is *necessary* that Socrates be mortal"), or "Socrates is *probably* wise", etc. As we recalled it (cf. § 01.04 *supra*), for the general notion of "opposition" (or negation, or contrariety, etc.), the two main theories available to Aristotle's time were Parmenides' and Plato's one. Parmenides had warned against committing contradiction (mixing one thing and its negation), while Plato had taught that there are in fact two kinds of negations (of nouns) not to be confused ("otherness" and "negation"). Remark that in fact no really precise, all-encompassing definition of "contradiction", "opposition" or "negation" was therefore available at that time: these notions were synonymous; distinguishing them was just a matter of literary taste. Aristotle will add

Infinite Sets. An Essay in Bi-logic, *op. cit*. Matte Blanco shows that a very big amount of known "thought irregularities" (among which mereological ones) can be traced back "bi-logically" to undue symmetrisations.

[75] Aristotle, *On Interpretation*, § 1.

[76] This fundamental articulation is typical of Aristotle's approach (cf. his logic of "subject and predicate" and his metaphysics of "substance and attribute") and in fact of Indo-European languages.

new (formal, in fact modal-logical) elements to these studies of negation by clarifying the case of sentences (modal or quantified – modality being a special case of quantification). Briefly speaking, Aristotle will focus on the problem (intuitively known by the Greek-speaking people of his times – including Plato – but nevertheless not formalised) of having two possible ways of placing a “negation sign” in a Greek sentence (containing modalisation or quantification), depending on whether the negation bears on the whole sentence (i.e. on the main verb, giving the lexical form ουκ, *ouk*) or only on its main (modal or quantificational) sub-part (the negation taking then the lexical form μη, *me*). Philosophically speaking, beyond grammar, it is clear for Aristotle that “ouk” and “me” are the same thing (i.e. a negation, a conceptual refusal), but he wants to get some more formal knowledge about them and their possible non-deceitful (non-sophistic) interaction and conceptual use. For sentences with many, possibly nested negations become quickly hard to understand clearly. So, because they are philosophically important (many paradoxes use them) and they possibly make use of the two ways of negating (ουκ and μη), Aristotle first considers *modal* sentences like:

1-“Socrates is necessarily mortal” - αναγκαιον Σωκρατην ειναι θνητον[77].

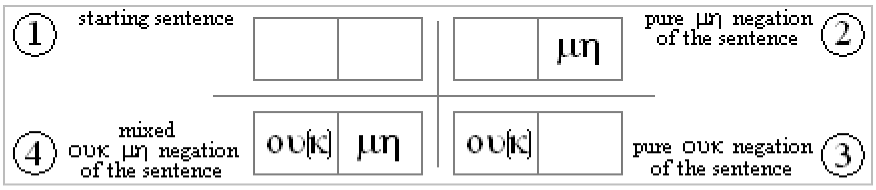
And he asks: how many ways are there of using some kind of “negation” on this sentence? As an answer (dictated by the Greek “grammar”, but pertaining to “logic”, except the latter did not yet exist), Aristotle finds the following three possibilities:

2-“Socrates is necessarily *non*-mortal” - αναγκαιον Σωκρατην μη ειναι θνητον

3-“Socrates is *not* necessarily mortal” - ουκ αναγκαιον Σωκρατην ειναι θνητον

4-“Socrates is *not* necessarily *non*-mortal” – ουκ αναγκαιον Σωκρ. μη ειναι θνητον

In fact, this results from a simple quaternary combinatory of the Greek negations ουκ and μη.



Remark that Aristotle explicitly says that he is investigating a subject where the many cultural and linguistic contingencies, otherwise important, play no part (*On Int.*, § 1). In order to make this point clearer (to himself as well as to the potential readers or conceptual opponents), Aristotle tackles the same “linguistic” (in fact: “logical”) problem from the point of view of “quantity”, that is: from the point of view of mathematics, the most reliable of all sciences. As a matter of fact, there is a family of sentences, “quantified” (quantification being

[77] This precise example is a fictitious one simplifying a reasoning otherwise much more complex in the Aristotelian text. But we remain faithful to his line of reasoning of which we give here the gist on oppositions.

mathematically clearly checkable), totally parallel to the one, "modal", just examined[78]. Take for instance the sentence:

i-"Every Greek is mortal" - Πας Ελλην θνητος

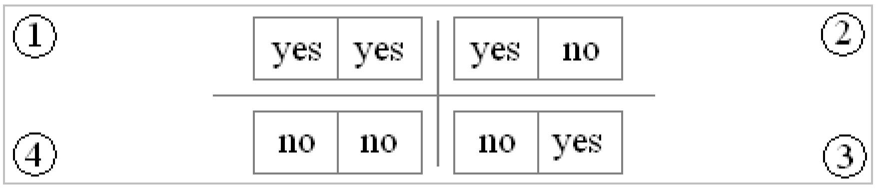
How many ways are there to "make opposition" to this sentence (by some kind of "negation"), independently of the language used to utter it? Aristotle finds the following three answers (and no others), which any thinking human being can find (thanks to the universality of the basic mathematical reasoning over quantities), whatever her/his vernacular language:

ii-"Every Greek is immortal" - Πας Ελλην αθανατος

iii-"Not every Greek is mortal" - Ου πας Ελλην θνητος

iv-"Not every Greek is immortal" - Ου πας Ελλην αθανατος

(here instead of "μη" the Greek language uses a negation directly incorporated in the adjective: as in English, instead of "non-mortal" there is "immortal", αθανατος, *athanatos*) This "quaternarity" has to do with the fact that, again (but this time purely conceptually, with no commitment to the specificity of the Greek grammar), there are all in all two "places" (in the sentence) admitting a possible negation: the quantification degree (universal or particular, "all" or "some") and the assertion polarity (positive or negative, "yes" or "no"): as for instance in "[*Every* (resp. *some*) Greek | *is* (resp. *isn't*) mortal]", [quantity | quality]. This gives, again, this kind of quaternary scheme (but this time a mathematically founded one).



But then one sees, as Aristotle did, that this behaviour is totally isomorphic with the one of the modal sentences: "logic" (and in fact mathematics) commands, beyond the human linguistic singularities, that, so to say,

[1] "[all | yes]" (= [not some | not yes])

may be modified by "opposition" in three and only three ways:

[2] "[all | not yes]" (= "[not some | yes]"),

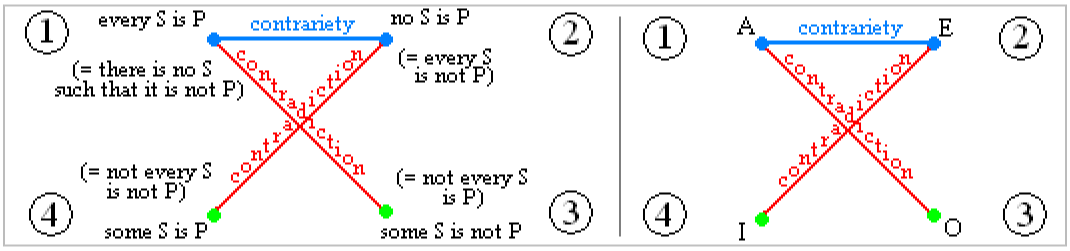
[3] "[not all | yes]" (= "[some | not yes]") and

[4] "[not all | not yes]" (= "[some | yes]").

On this basis, where there are four and only four members of a complete range of oppositions of a starting term (forming possibly some kind of square without sides or

[78] Linguistic "quantification" deals with current expressions like "all", "every", "none", "any", "some", etc.
.

diagonals), Aristotle introduces clear definitions for two such kinds of oppositions (one will embody each of the two diagonals, the other will embody the upper side of the square): first, the one leading from "[all | yes]" to "[not all | yes]" (and the other way round) and the one leading from "[all | not yes]" to "[not all | not yes]" (and the other way round); second, the one leading from "[all | yes]" to the "[all | not yes]" one (and the other way round). He calls the first "contradiction" (αντιφασις, *antiphasis*): this is the pure negation, obtained by putting a negation operator just in front of the starting sentence (modal or quantified). He calls the second "contrariety" (εναντιον, *enantion*): this is a kind of specular, mirror-like negation, where the maximal "intensity" (i.e. the quantificational totality or the modal necessity) is kept unchanged but is given the other direction (positive or negative). If we name, as did the Western, Latin and Medieval tradition, A, E, O and I the four possible Aristotelian positions (turning clockwise and starting from the topmost left), Aristotle's doctrine can be represented by the following quaternary scheme, a square with some but not all its sides (we take it from Terence Parsons, and name it "anti-gamma scheme", for evident formal reasons – it has the shape of an upside-down gamma)[79].



An important further point to be remarked is that the contradictories allow, for each of the four terms (the corners of the anti-gamma square), two possible ways of naming it: for example, the A position can be read as itself (= "Every S is P") or, equivalently, as the contradictory negation of its contradictory term (= There is no S such that it is not P"). The same remark concerns each of the four A, E, I, O terms: each has two (logically) equivalent readings. A problem with this theory is that, despite its being clearly quaternary (from the point of view of its possible geometrical positions, the A, E, I, O), Aristotle only named three of its six possible *relations* (by "relations", we mean here the possible couples of terms: A–E, A–O, A–I, E–O, E–I, I–O). He only named one of the four sides of the square (the A–E) and its two diagonals (A–O and I–E). But truly speaking, his writings tell more, but without giving new names (i.e. other than "contradiction" and "necessity"). They say for instance: (1) that I and O can be true together but cannot be false together; (2) that A implies I, and E implies O. So he has given more than hints concerning the three remaining relations. But no

[79] Cf. Parsons, T., "The traditional square of opposition", *Stanford Encyclopedia of Philosophy*, (2006).

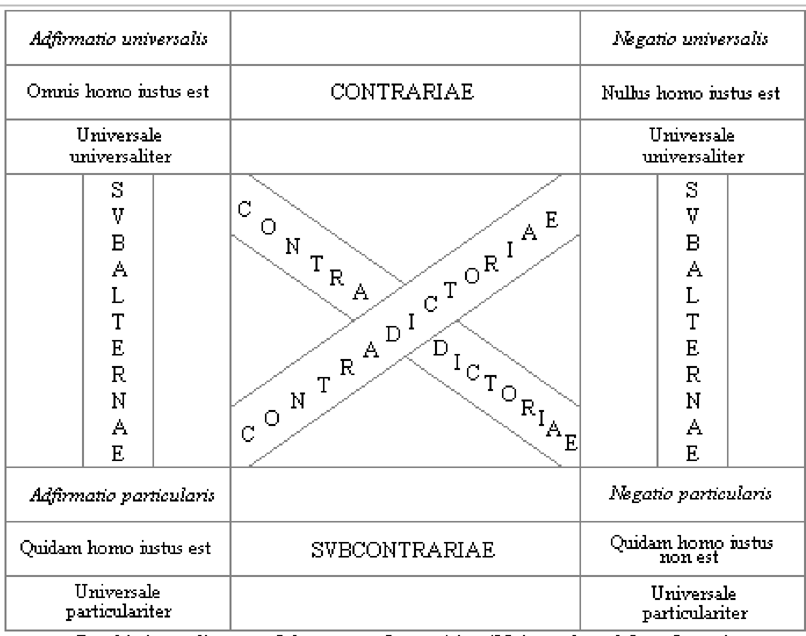
new names. On this problem of the partial absence of names for the new concepts we come back in a while.

04.04. Aristotle’s opposition theory is a “logical square”

As we said, Aristotle’s theory of opposition is fine (combinatorially elegant and logically powerful), but despite the technically precise introduction and definition of the notions of “contradiction” and “contrariety”, it lacks some more names for its possible other “oppositional” concepts (the three other sides of the anti-gamma square). As a matter of fact, the missing relations can be characterised by intrinsic distinctive logical properties, and can therefore be given specific names.

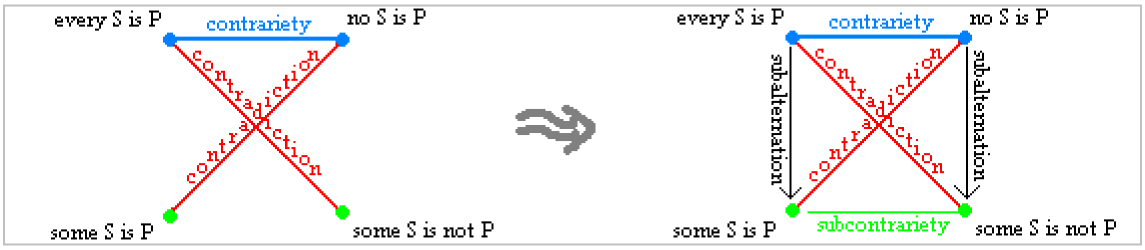
04.04.01. Apuleius’ and Boethius’ “Aristotelian logical square”

Some successors of Aristotle, i.e. Apuleius and Boethius, proposed, centuries later, a small geometrical device in order to express instantly, at a glance, Aristotle’s fundamental logical principles (non-contradiction, excluded middle, etc.): the so-called “square of opposition” (or “logical square”, or “Apuleius’ square”, or “Aristotle’s square”). Here is its version by Boethius.



Boethius' own diagram of the square of opposition (Meiser, adapted from Seuren)

This was achieved by completing what we called Aristotle’s (implicit) anti-gamma scheme (the sidesless square), by filling in the 3 missing relations (i.e. the three missing sides of the square structure). Using the Aristotelian logic, they determined that the property of the mysterious bottom horizontal segment of the square consists in joining two sentences never false together but possibly true together: they called it “subcontrariety”, meaning by that “the relation which, in the square structure, is situated *under the contrariety* bar”. As for the two remaining sides of the square, the two vertical ones, Aristotle’s logic allowed to see that they simply consist of a logical implication: the upper term always implies the lower one (its vertical “projection”, so to say). They called this kind of relation “subalternation”, that is, submission (so to say: the first commands, or dominates the second). So, finally the complete structure turns out to be a square, officially named, again, “square of opposition” or “logical square”.



Regarding whether or not Aristotle himself was aware of this “logical-geometrical” object, the question remains open. Since the first known occurrence of it came centuries later (with Apuleius), one would doubt it. But, truly speaking, in Aristotle’s own writings (cf. *On Interpretation*, § 13), one finds some kind of seminal square representation of the logic of the opposition relations. Aristotle calls it a “drawing”, or a “sketch” (*υπογραφη*, hypographe).

δυνατον ειναι ενδεχομενον ειναι ουκ αδυνατον ειναι ουκ αναγκαιον ειναι	ου δυνατον ειναι ουκ ενδεχομενον ειναι αδυνατον ειναι αναγκαιον μη ειναι
δυνατον μη ειναι ενδεχομενον μη ειναι ουκ αδυνατον μη ειναι ουκ αναγκαιον μη ειναι	ου δυνατον μη ειναι ουκ ενδεχομενον μη ειναι αδυνατον μη ειναι αναγκαιον ειναι

possible that it be contingent that it be not impossible that it be not necessary that it be	not possible that it be not contingent that it be impossible that it be necessary that it be not
possible that it be not contingent that it be not not impossible that it be not not necessary that it be not	not possible that it be not not contingent that it be not impossible that it be not necessary that it be

(this “sketch” is proposed at the beginning of the aforementioned paragraph, in order to study the modal “consecutions”; but quickly Aristotle proves that this sketch is mistaken, and goes on in his reasoning without drawing a further graphical sketch).

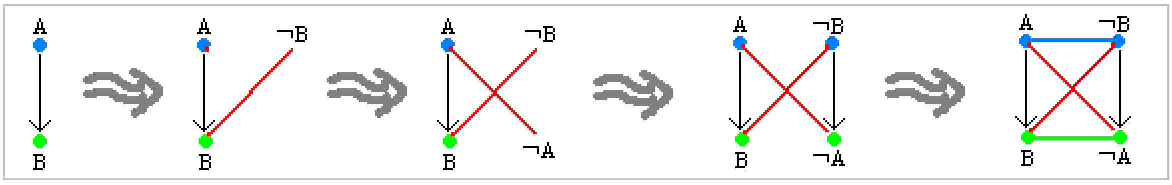
Of course, this is not yet the classical logical square, but it seems to show that Aristotle was not too far from it, something “was in the air”.

Note that this last is what philology would call an απαξ λεγομενον (*hapax legomenon*), i.e. a “one-shot-only”, something occurring just one time (something mysterious)[80]. Notice also that in some sense the square of opposition contradicts Aristotle's anti-mathematical and anti-geometrical (i.e. anti-Platonic) spirit (but this is not astonishing, since Aristotle got to it (or to its "anti-gamma" close ancestor) by reflecting on quantification and therefore – *volens nolens* – on mathematics). And, *a posteriori*, the square is some kind of strong philosophical rival to Euclid's (i.e. Plato's) geometry for embodying the role of transcendental structure of rational thinking rigour (for the square is very abstract and deep). This opposition to the Academic model is very Aristotelian, but the way of doing it (it does so *geometrically*) is not Aristotelian at all, in spirit. It is as if the only way of fighting the "mathematism" of Plato's School consisted in building an anti-Platonic *geometrical*-logical amulet (and fetish): the logical square. Almost an Apelian performative self-contradiction (cf. § 01.08 *supra*). But let us turn now to the square's inner properties.

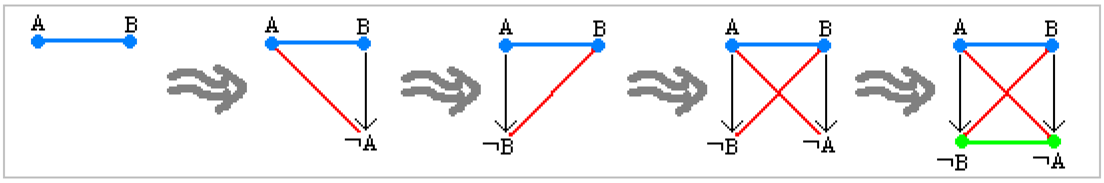
04.04.02. First main properties of the logical square: its universality

From a purely formal point of view, some properties of the logical square deserve to be noticed, such as the following ones (they show that logical squares are present almost everywhere in logic)[81].

(1) Each logical arrow (i.e. each logical implication) generates a logical square (by the law known as "contraposition" – if a term implies another one, the negation of the second implies the negation of the first):



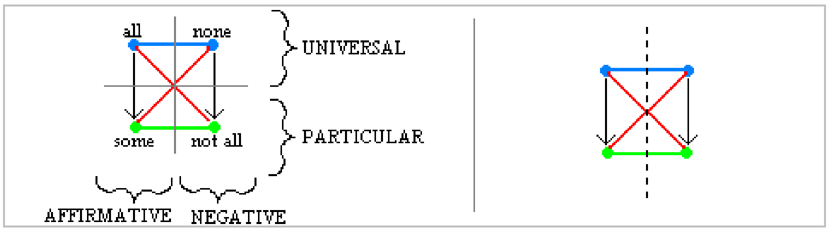
(2) Each contrariety (of two terms) implies that each of its terms implies the negation of the other, and thus implies a logical square:



[80] Hegel would say, as he did when he wanted to criticise, in the philosophy of his fellow-rival Schelling, the absence of a justification for his sudden introduction of the concept of "absolutness": "a gunshot in the dark", i.e. something arriving with no visible reason (a concept without conceptual framework).

[81] In the rest of this study we will adopt the following convention (not of ours, we take it from Béziau, cf. ch. 10 *infra*): in logical squares contrariety will be represented in blue, contradiction in red, subcontrariety in green and subalternation in black (sometimes in grey).

(3) A logical square can be seen as composed of two crossed bi-partitions, a specular (i.e. symmetric) horizontal one (affirmative/negative) and a hierarchic (i.e. asymmetric) vertical one (universal/particular). Aristotle's traditional reading uses "quality" (ποιοτης, *poiotes*) – for the affirmative/negative distinction – and "quantity" (ποσοτης, *posotes*) – for the universal/particular distinction.



Note, again, that the logical square has one (and only one) symmetry axis: the vertical one, dividing as a mirror the left from the right side of the square. Notice also the beauty of the traditional combinatorial definition of the three symmetric Aristotelian relations:

- two things are contradictory if they cannot both be false and cannot both be true;
- two things are contrary if they can both be false but cannot both be true;
- two things are subcontrary if they cannot both be false but can both be true.

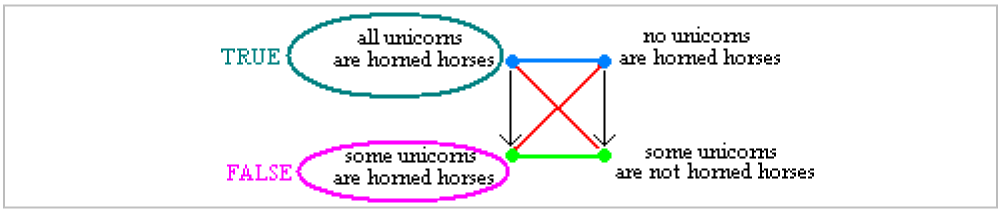
The fourth Aristotelian relation, subalternation, is asymmetrical and hasn't been linked to this kind of combinatorics (on this argument we come back on ch. 18 *infra*). Aristotle calls it "contradictories of contraries". This is tantamount to subalternation (as one can check in the square), but seems to show that Aristotle didn't consider them as real "opposition" relations (on this argument we will come back on ch. 10, 18 and 24 *infra*).

04.04.03. Traditional problems with the logical square

The logical square has been widely used (especially in the Middle Ages, inside the Scholastic) in order to teach logic: it enables to grasp visually (and memorise) some of the most important basic properties of logic. But it also has some drawbacks. One historical problem (possibly the main one) of the logical square is what is called "undue existential import": the square as such (i.e. read without any restriction over its application domain or scope of quantification) suggests that each universal property (i.e. using "all" or "none") implies logically the *existence* (i.e. the use of "some") of at least one individual instantiating that property. But this seems to reduce to a rather bad (and wicked) sophism, as for instance the following:

"all unicorns are horses bearing a horn on their forehead,

hence there exists at least one unicorn, a horse bearing a horn on her/his forehead"!



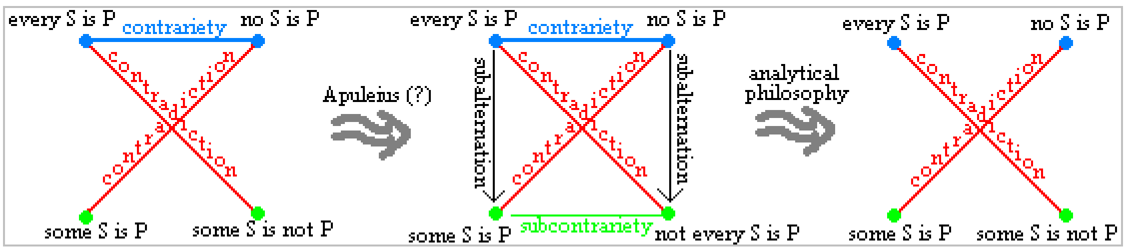
Commonsensically, it seems clear that here the truth is that "unicorns" do not *really* exist, although in what Peirce would have called their paper-existence (which exists indeed), it is true to say that they are horses bearing a horn on the forehead. People like the British philosopher and logician Bertrand Russell (one of the founders of analytical philosophy), by way of a logical analysis of the sentence, would express it in some way like this:

$\forall x(Ux \leftrightarrow Hx \wedge Cx)$

(with: Ux = "x is a unicorn", Hx = "x is a horse", Cx = "x bears a horn")

and, because "$\forall x$" (with $x \in E$) does not imply "$E \neq \varnothing$", this does not allow the logical deduction of anything as dreadful as "$\exists x(Ux \leftrightarrow Hx \wedge Cx)$" (i.e. the real existence of unicorns).

So, the former reasoning (not Russell's one!) is a sophism in so much as it presupposes tacitly (and unduly) that we are absolutely unable to speak (without commitment) about non-existing things (meaning by that: "all what we speak of, therefore immediately exists"!). Now, translated in the language of the logical squares, this simply means that logical squares are "ok" when (and only when …) their quantifying domain is non-empty, which seems to mean that whenever we speak (as we often do, very naturally) about non-existing things, "logical squares" do not apply. And this is a pity, for "logic" traditionally is a means of speaking abstractly and hypothetically about any conceivable thing (including, if not especially, non-existent things). That is why the American philosopher and logician Terence Parsons says that the logical square has left its place, contemporary-logically speaking, to some kind of (what I would call an) "X-scheme" (or "X-square", or Boolean square", cf. ch. 5 *infra*).



In the X-scheme (the "castrated logical square", so to speak!) the vertical arrows have disappeared because empty domains of quantification do not make impossible the simultaneous truth of the sentence "every x does S" and falsity of the sentence "some x does S": so an arrow, there, would express that the true implies the false. The blue contrariety upper horizontal bar has disappeared because the case of the empty domain of quantification allows the upper two positions of the square (its two upper corners) to be true together (the empty set supports any property), which contradicts the definition of contrariety. The green subcontrariety lower horizontal bar has disappeared because in the case of an empty domain, both lower positions of the square (its two lower corners) are false (at the same time), which contradicts the definition of subcontrariety.

But actually, if things are not as simple as the logical square may tell naively, things are neither as simple as the bare negation of the existence (and/or of the conceptual interest) of the logical square (as many – and especially many analytical philosophers – would like to think)[82]: for instance, logical squares apply very, very nicely to entities such as … the mathematical ones! (cf. ch. 5 *infra*).

This issue of the undue existential import has been largely debated over centuries (especially in the Middle Ages). Many solutions have been provided to the disagreement caused by it (for instance forbidding the reading "some": interpreting the O corner as "not all" and not as "some not"). An interesting contemporary treatment of it, very well informed and justifying with contemporary scientific (logical and psycho-linguistic) arguments the interest of returning to the study of the logical square and its inner medieval intricacies, can be found in the recent work of the Dutch logician and psycho-linguist Pieter Seuren (cf. § 06.06.08).

04.05. The fight with Diodorus Cronus: against necessitarianism

Now, a second problem intervenes, after the fight against the Sophists, in the elaboration of Aristotle's reflection on the concept of opposition. This influence will be less visible, but nevertheless very important. This time it is related not to Sophists, but to a major philosophical rival, the Megarian Diodorus Cronus (4^{th} century BC). For short, the problem is that Aristotle's theory (i.e. logics, and mainly his implicit square structure), intended to dissipate a major source of sophisms (those due to improper uses of oppositions), thus developed, leads in fact to a major "paradox" (pointed out by Diodorus), worse than the ones

[82] Not to speak about continental philosophers, of course. Terence Parsons mentions the exception constituted by the American analytic philosopher P. F. Strawson.

of the Sophists, one consisting in strongly negating the existence of *any* real freedom. Why so?

04.05.01. Aristotle's commitment to the "Principle of strict bivalence"

What we highlighted graphically in § 04.04.02 *supra* on the remarkable properties of the square is technically associated with the notion of "logical bivalence". P. Seuren characterises Aristotle's progressive discovery of the main features of his logic as follows:

> "As a student in Plato's Academy, Aristotle discovered the formal-computational aspect of the fact that two sentences (propositions) *P* and *Q* may be CONTRARY (written as P><Q): *P* and *Q* cannot be true, but may be false, simultaneously on account of the meanings of *P* and *Q*. This made him realize the related fact that a sentence (proposition) *P* may ENTAIL a sentence *Q* (written as P ⊢¬Q): whenever *P* is true, *Q* is also true on account of the meanings of *P* and *Q*. These facts are related, because the Aristotelian PRINCIPLE OF STRICT BIVALENCE requires that if P >< Q then P ⊢¬Q and Q ⊢¬P, and vice versa, if P ⊢¬Q or Q ⊢¬P then P >< Q ("¬" stands for the standard bivalent sentential negation, widely believed to correspond to *not* in English). He also discovered that sentential negation ("it is not true that …", written as "¬") is the only reliable creator of the relation of CONTRADICTION (if *P* is true, ¬*P* is false, and vice versa). Under the Principle of Strict Bivalence, double sentential negation cancels out: for any proposition *P*, ¬¬*P* is equivalent with *P* (written as "¬¬P≡P")." (Seuren, P., *The Victorious Square*, draft 2007, forthcoming, p. 7).

We already mentioned several times Aristotle's attachment to the principle of non-contradiction (cf. ch. 1 and 3 *supra*). Two other important principles of his logic, deeply related to that of contradiction (as intuitively suggests the previous quotation from Seuren), are the one called "excluded middle" ("true or false, no other option") and the one called "bivalence". Now, although the question is generally much debated ("The relations between the principles of bivalence, excluded middle and non-contradiction are not necessarily clear")[83], one can quite safely characterise "bivalence" as meaning: "A proposition can be true or false" ("or" being exclusive). But what does this mean philosophically? More or less this: Aristotle, as a result of his fight against the Sophists, delivers a vision of the world where *conceptual stability* is celebrated and codified, among other logical means, in terms of things being necessarily either true or false. But this anti-sophistic remedy opens up a new, unexpected serious problem.

[83] Béziau, J.-Y., "Bivalence, Excluded Middle and Non Contradiction", Behounek L. (ed.), *The Logica Yearbook 2003*, Prague, Academy of Sciences, 2003, p. 73-84; one of Béziau's main claims is that, paradoxically, one cannot say that "Principle of Non-Contradiction + Principle of the Excluded Middle = Principle of Bivalence".

04.05.02. Diodorus' challenge to the possibility of "possibility"

Now, Diodorus developed a famous argument, reputed to be one of the most impressive in the Greek philosophy of all times. Diodorus' own writings have been lost since, but the argument, called the Master argument (κυριευιος λογος, *kurieuios logos*), has been since reconstructed thanks to some indirect sources (mainly Epictetus' *Discourses*). Epictetus speaks of the Master Argument in these terms:

> "Here are, I think, the points starting from which the Master argument is laid: there is, for any of these three propositions, a conflict among any two of them and the third: "Any true proposition concerning the past is necessary. The impossible cannot follow logically from the possible. The possible is what is not actually true and will not be true". Diodorus, having remarked this conflict, used the truthfulness of the first two in order to demonstrate the following: 'No thing is possible if neither it is actually true nor it will be in the future' "[84].

Epictetus follows by explaining how other combinatorial choices over the possible abandons of one of the three "points" generated some of the main positions of ancient philosophy (mainly those of Diodorus, Philon and Cleanthes). The French philosopher and epistemologist Jules Vuillemin, one of the finest interpreters of the question of the Master Argument, goes further. First he shows that many other first-order philosophers of ancient Greece felt compelled to face that challenge (among whom Plato, Aristotle, Epicurus and, much later, the skeptical Carneades). Then he recalls that the Master Argument probably appeared as a reaction of Diodorus to a reaction of Aristotle against Plato's *Timaios*[85] in the *De Coelo*. Diodorus observed that some of the cardinal conceptual strategies developed by Aristotle against Plato, with the adjunction of some further elements, could produce this terrible argument which he chose to produce. And this argument, even beyond what we can understand of it nowadays, was one of the masterpieces of the ancient thought, along with Zeno's paradoxes. In other words, we know that it was a very serious argument, hiding very deep logical intricacies which never disappeared (the argument was never "once and for all" defeated, not even nowadays)[86]. The debate on it implied building different kinds of logics. Like the Stoics a bit later, the Megarians developed a logic alternative to Aristotle's syllogistics (as is known, the former consists more or less in contemporary propositional calculus, whereas the latter is a fragment of contemporary monadic predicate calculus). And the differences in the way chosen in order to "think logically" yielded (and/or were motivated

[84] Quoted from Vuillemin, J., *Nécessité ou contingence. L'aporie de Diodore et les systèmes philosophiques*, Paris, Minuit, 1984, p. 15.
[85] Vuillemin, J., *Nécessité ou contingence, op. cit.*,

by) different ontological options. Like the Stoics, the Megarians tended to believe that there is no human freedom. In particular, Diodorus applied directly logic to time[87]. This reduction of modal logic to “tense logic” (where “necessary” is expressed by “always” and “possible” is expressed by “sometimes”) gives easily and impressively a strong intuition of necessitarism.

Diodorus says, for short: the essence of logic implies the non-existence of freedom; logic being as it is (i.e. rigid in its truth-values, with its yes-no binarism), its rigidity spreads over time, in particular over the future. For, logic showing two-valuedness (assertions on “things” can and must be either true or false), any future event (conceived as an assertion over the reality of its content), as any being, must be in turn (right now) two-valued, that is: true or false. But then the logical value of something at one time, being eternally fixed (at that time), any future event must be already determined (a future event, with respect to a precise instant, no matter which, is already true or false – no matter if *we* still ignore this truth-value). This means that, beyond appearances, the only possible things (or events) are those that in fact are (already) necessary: possibility (conceived as ontological openness) is only an illusion. In other words, according to Diodorus and his school (in virtue of his own solution of the Master Argument – i.e. giving away the third premise), modal logic, the logic of the notions of “possible”, “necessary” and the like admits the fundamental law:

“X is possible → X is necessary”.

Which means: things are either necessary or impossible.

As “historical proof” of the cogentness of Diodorus’ argument beyond his contemporaries (among whom Aristotle), let us recall that more than 2000 years later, the Polish logician and philosopher Jan Łukasiewicz, in order to answer Diodorus’ challenge, created the first many-valued logic, a tri-valued logical system, and moreover *n*-valued logics (the so-called “many-valued logic”, of which we will speak again in ch. 23 *infra*). And some decades later, another giant of logic and (analytic) philosophy, the New-Zealander Arthur

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[86] Vuillemin argues that all philosophy, traditional as contemporary, can be seen as a possible answer to the challenge of the Master argument, for this argument displays a profound abstract typology of the philosophical possibilities in general.

[87] Cf. Belna, J.-P., *Histoire de la logique*, Paris, Ellipses, 2005, p. 32; cf. also Gardies, J.-L., *La logique du temps*, Paris, PUF, 1975, p. 21-30. The links of Aristotle’s modal logic to time seem to be less direct, even if, paradoxically, the very idea of a reduction of the modal to the temporal operators seems to have been an insight of Aristotle. Cf. Hintikka, J., *Time and Necessity: Studies in Aristotle’s Theories of Modality*, Oxford, Clarendon Press, 1973. I owe this remark and its Hintikkian reference to Professor D. Schultess.

Prior, also got inspiration from Diodorus and made some of his greatest discoveries (mostly related to "tense logic") while trying to fight against him[88]. What about Aristotle?

04.05.02. Aristotle's new remedy: the concept of "contingent futures"

The problem for Aristotle is that his beautiful logic commits him to a (for him) terrible, undesired philosophical consequence. As a matter of fact, if he leaves his logical principles unrestricted, the future vanishes in its openness, dominated by a fixed eternity. As Jean-Louis Gardies says, quoting Aristotle's *De interpretatione*:

> "[…] if one admits that the *principle of the excluded middle* – which says that among any proposition and its negation, at least one must be necessarily true –, also holds for any future event, then "nothing is, or becomes, neither under the effect of chance nor in an undetermined way, nothing which could, in the future, indifferently be or not be; on the contrary, everything comes from necessity, without the slightest indeterminacy". Aristotle went on like this: 'According to this reasoning, there would be no more reason to make deliberations, no more reason to care about something, believing that if we accomplish this action, this result will follow, and that if we do not accomplish it, this result will not follow' " (Gardies, J.-L., *La logique du temps*, *op. cit.*, p. 20-21).

Remark that it is not clear which relation such determinist-necessitarian views of the Megarian school do hold with respect to Aristotle's own philosophical close ancestors, for Socrates as well as Plato do show some strange elements of determinist belief. "Socratic determinism" is the view according to which any human being *always* does exactly what he/she *intimately and unconsciously* thinks (or feels) good for him/her[89]. It is known that the Megarians advocated Socrates as one of their sources of philosophical inspiration (Euclid of Megara, the founder of the School, was a direct pupil of Socrates).

However, Aristotle himself clearly did not appreciate Diodorus' "gift" at all. A view in which freedom is only an illusion is unacceptable for him, it is an existentially self-contradictory issue (life would not have any meaning any more).

Aristotle's own solution, submitted to the logic of the Master argument (something in the assumption must be abandoned or the argument must be shown fallacious in some point) consists of two points: (1) one must distinguish between absolute necessity and conditional necessity; (2) one has to limit the principle of bivalence[90]. In other words, Aristotle's answer

[88] The question of the contingent futures has been much debated in all times. Among the many people having discussed it in this century, one can mention N. von Hartmann (who concentrated on Diodorus' ontology), J. Łukasiewicz, A.N. Prior and G.H. von Wright (the last three fought against the Diodorean intuition as logicians). On Hartmann, cf. Morgenstern, M., *Nicolai Hartmann zur Einfürung*, Hamburg, Junius, 1997, p. 66-74.

[89] Blaise Pascal says : "For the one who is going to hang himself, the rope is the best conceivable friend". Of course, against this view Aristotle developed a study of the weakness of the human will, called ακρασια (akrasia). We studied a contemporary discussion of this issue, linked to psychoanalysis and logic, in our "Trois approches de l'irrationnel: Davidson, Matte Blanco and da Costa", *Noésis*, No. 5, Vol. 2, 2003.

[90] Cf. Vuillemin, J., *Nécessité ou contingence*, *op. cit.*, p.150.

will consist in changing the interpretation of logic (changing the notion of logical truth). As is known, Aristotle will say that, due to the distinction between the things existing in act and those existing only potentially, the logical principle in question must not be applied without restrictions to both, but only to the things existing in act. The things belonging now to the future are now only potential, so the principle of the excluded middle applies to them now not in a distributed way on each, but only on their disjunction:

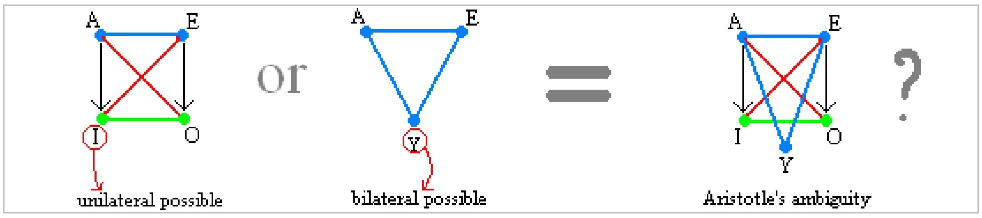
- not: " "tomorrow it is necessary that A" or "tomorrow it is necessary that not A"
  (which would be choosing between two necessities, thus spousing absolute necessity)
- but: " it is necessary that "tomorrow A or not A" "
  (which leaves the inner structure of tomorrow undetermined and untouchable)

So, for Aristotle, there exist contingent (i.e. non-necessary) things in the future, the so-called "contingent futures", and logic must reflect this point of reality: logic must obey to free life.

Note the incredible violence made to logic: moral-existential arguments determine what must be taken for the ontology of "truth" (some could have expected it going the other way round).

04.06. But Aristotle hesitates between a square and a triangle…

The important point in this chapter 4 was to understand both the genesis and the structure of the logical square. The structure is of course the most important thing (and we already mentioned almost all its main features), but its genesis will turn up crucial as well in the rest of our present study, for some strange, unresolved problems seem to be tied to this structure – which should be logico-mathematical and therefore at least ideally unhistorical – because of some of the vicissitudes linked to its creation. And this has to do with the aforementioned fight against Diodorus. As a matter of fact, philosophically speaking, in this context where there is a fight going on in order to "save the possible", by trying to refuse Diodorus' ontological law "X is possible if and only if X is necessary", Aristotle also distinguishes, in fact, between two meanings of "possible": one so-called "unilateral possible" (one incompatible with impossibility but compatible with necessity) and one so-called "bilateral possible" (one incompatible both with impossibility *and* with necessity). And in fact this distinction turns out to be also a hesitation. Aristotle hesitates between a square (of opposition) and a triangle (of contrariety)[91].



There are indeed, from the point of view of the metaphysics that Aristotle develops (partly against Diodorus), both a "unilateral possible" and a "bilateral possible" (this distinction is not totally absurd). The philosophical reason beyond this is a hesitation between two kinds of "possibility", one specific to the "sub-lunar world" (the earthly world of corruption), the other specific to the "supra-lunar world" (the celestial world of eternal and perfect movement). But the logical square only expresses the unilateral possible, whereas the bilateral possible may be expressed by a triangle of contrarieties heterogeneous with respect to the square. This hesitation seems to have lasted until the end of Aristotle's thinking life; Aristotle will keep, philosophically speaking, both solutions, despite the fact that they seem to be mutually conflictive (at least from the point of view of their formal understanding). In other words, this hesitation is a defect, a formal irresolution.

So, in some sense taken between the Sophists' relativism and Diodorus' necessitarism, because he keeps both necessity (Platonic celestial mathematism) and freedom (the refusal of Diodorus' necessitarism), Aristotle seemingly remains with a powerful but ambiguous theory of opposition (on this we will come back on ch. 8 *infra*).

04.07. Aristotle's pivotal role inside pure philosophy

At a rather early stage of Western philosophy, Aristotle embodies the heavy choice, against the openness of geometry to the infinite, of a *founded* "transcendental scheme" (the principle of non-contradiction and, symbolically, the logical square): this instead of an infinite (mathematic-friendly) openness (Euclidean geometry and infinite mathematics in general) compatible with some kind of non-lethal relativism (or scepticism). This choice may have been a heavy one in terms of future consequences.

As a matter of fact, as already said (ch. 1 and 3 *supra*), the list of people who will follow Aristotle's spirit is philosophically impressive, for it is constituted of giants (Descartes, …, Kant, Fichte, …, Husserl, …, Apel, …): it contains some of the greatest thinkers of the Western tradition. On the contrary, the few people who fight him (Bolzano, Peirce, in some respect the "second" Wittgenstein, …) are mainly isolated thinkers who like (and practise) mathematics and feel uneasy with the philosophical theories advocating unchangeable and

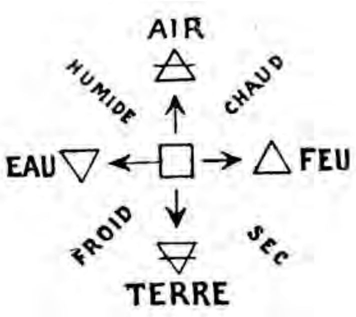
[91] Cf. Gardies, J.-L., *Essai sur la logique des modalités*, Paris, PUF, 1979.

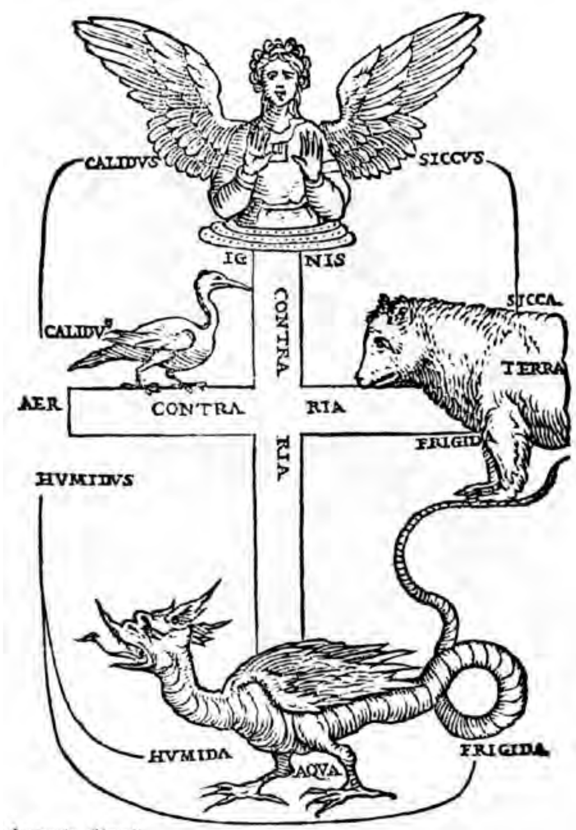
foundational transcendental structures. As for the "square of opposition", in some sense one of Aristotle's philosophic keys[92], it is remarkable to see that it is a piece of his theory which, differently from many others, has resisted rather well through time, as we are going to see in what follows (ch. 5 *infra*).

04.08. Aristotle's "square influence" over the Western culture

Before quitting Aristotle, let us mention one last point. It is well known that his general influence over Western culture (and not only Western, if one thinks of the Arabic culture) has been huge. Here we want to recall briefly how such an influence did not concern only pure logical theory, but also aspects that are very much abandoned nowadays (we are thinking of "crazy" – apologies to the kind reader who believes in them! – but popular things like astrology, esotericism, magic, parapsychology, etc…).

One interesting point to grasp in this respect is that in such fields Aristotle's theory of opposition was (and still remains) very present, even at an iconographic level (cf. the following traditional figures)[93].





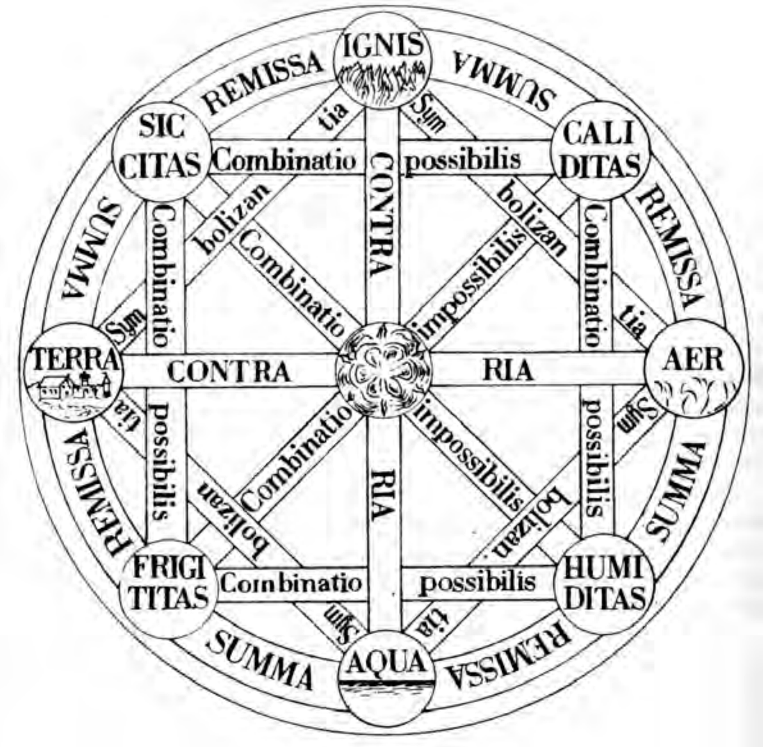
Les quatre éléments

[92] For instance, the key concept (for Aristotle's metaphysics) of "substance" is characterised as being *opposed* to its correlated concept of "attribute" on the grounds that the first does not admit *contraries*, whereas the second does (cf. Aristotle, *Categories*, 2b, 24). As already mentioned (ch. 1 *supra*), the notion of opposition (and hence, implicitly, that of square) is essential in order to think the "first of all principles", the principle of non-*contradiction*. And so on. The square of opposition emits a long shadow in Aristotle's philosophy.

[93] The first two pictures are taken from O. Wirth, *Le symbolisme astrologique*, Paris, Dervy. The third one is taken from Caron M. and Hutin S., *Les alchimistes*, Paris, Seuil, 1970 (1950).

The main mediation between such a monument of human thought (Aristotle) and more popular (and less demanding) “visions of the world” seems to have been Aristotle’s theory of change (mainly in his treatise *De corruptione*) and its link with Aristotle’s version of the probably more ancient “theory of the four elements”: earth, fire, air, water (each element results from the combination of “warm or cold” with “dry or wet”, the “basic natural qualities”). This is understandable, since the theory of change is fundamental to those willing to practise some kind of divination (prediction of the future) or some kind of “magic transformation”. As it is known, these materials have, with others, become the lore of the long-lasting minor (or sub-) culture known as esoteric (including in this wide expression alchemy, astrology, magic, divination, parapsychology, etc.). To us (scholars, philosophers, scientists), at least at a research level, such a vision of the world seems irremediably lost[94]. From this respect it is puzzling or even funny to see that such deep minds as Galileo, Kepler, Leibniz and Newton may have more or less adopted (or at least feigned to adopt sometimes) such a scientifically strange posture towards “the unnatural”.

Anyway, relative to our purposes here, we can observe that this is often the principal place where Aristotle’s intuitions over the ontology (the geometry) of opposition are expressed. It is here that the word “contrary” plays a visible part. As an example, the following figure comes from Leibniz’s treatise *Dissertatio de arte combinatoria* of 1690 (the *dissertatio* had been defended in 1666 in Leipzig, it was published in 1690 against Leibniz’ will)[95].



Remark that this oppositional figure, fully Aristotelian (philosophically speaking), is not a logical square. What is it then? (we will come back to it on ch. 17 *infra*)

[94] With the notable exception of eminent thinkers like C.-G. Jung, S. Lupasco, G. Durand and some others, overtly committing – not without scandal and serious suspicions of intellectual (self-)deceit – to esotericism.
[95] Cf. Leibniz, G. W., *Scritti di logica - I*, Roma-Bari, Laterza, 1992.
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Having recalled this culturally, if not scientifically, important and still lasting influence (at a popular level), we can now turn to the real scientific (and philosophical) core of the logical square's crucial interest for thought.

05.
FORMAL APPLICATIONS OF THE LOGICAL SQUARE

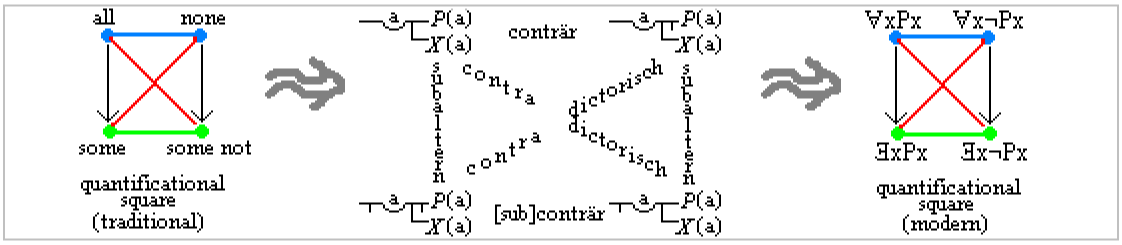
As we already said, Aristotle's logic has been strongly modified or relativised by the "logical revolution" which started with Boole and pursued with (among others) Peirce, Frege and Russell-Whitehead. But one piece of Aristotle's logic didn't quite move: and this was precisely the "logical square". In the previous chapter we hinted at the fact that there are two main instances of contemporary formal applications of the square of opposition: one quantificational (with ∀ and ∃) and one modal logical (with □ and ◊). In this chapter, we are going to show that these two original characterisations of the square (plus some others) are precisely those guaranteeing its actual scientific success. In order to do so, we will recall the two main domains inside formal science where the logical square seems to still play a crucial part: mathematics (*via* quantification theory and mathematical analysis) and logic (*via* propositional and predicate calculus and modal logic).

05.01. Three fundamental logical squares in mathematics

Logical squares rule mathematics in at least three domains: (1) general mathematical reasoning, (2) mathematical analysis and (3) topology.

05.01.01. The square in general mathematical reasoning

The first use of the square of opposition is a mathematical one. The contemporary quantification is more or less the one fixed by Frege (1879) in his invention of "first order logic". Now, from a geometrical point of view, things are almost the same as with the old square. This fact is remarkable: Frege changes the shape of logic drastically, he quits the predication of Aristotelian syllogistics and develops a more general "theory of functions" (taking logical predication as a particular case of mathematical functionality); and he adds propositional logic to this reformed predication calculus. Nevertheless, fundamentally, he keeps the square (cf. figure, in the middle).



As a side note, it must be remarked that in his own graphical representation of 1879 (using his "concept-script", or "Begriffsschrift"), he made a famous typo: on the lower horizontal edge of the square, he wrote "contrary" (*conträr*) instead of "subcontrary" (*subconträr* – in the figure we added "[sub]" to "conträr"). This mistake of his possibly shows that Aristotle's opposition theory had lost its appeal for working logicians by Frege's time: the notion of subcontrariety seems to be irrelevant, meaningless, at least enough so not to be distinguished from contrariety.

Mathematics (general mathematical reasoning) deals fundamentally with quantification, for any theorem just belongs to either the class of universal statements ("all object such that … is …"), or to that of existential ones ("there exists some objects [at least one] … such that …"); in other words, doing mathematics implies fundamentally handling quantifiers: the ∀ one ("for all") and the ∃ one ("there are some"). And the quantifiers still respect the fundamental, formal constraints embodied by the square of opposition. This is a first point showing the extreme importance (and standardness) of the logical square for contemporary mathematics.

05.01.02. The square in mathematical analysis

Now, the logical square turns out to be essential for "mathematical analysis". The latter, the mathematics of the continuous[96], can be seen as dealing mainly with majorations (≥ and >) and minorations (≤ and <), that is: "order relations" (analysis is, in some sense, the consequence of the topology of R, the study of the properties of a totally ordered infinite set). The most precise mathematical results are obtained, in the traditional exposition of analysis, as series of converging series of minorations and/or majorations (as in Leibniz, Cauchy and Bolzano): in this sense, for instance, one of the most important basic notions of mathematical analysis is that of "least upper bound" (and the like).

As a classical (joint) example of the crucial importance of these two basic notions (order and quantification), think of Bolzano's revolutionary (non-Aristotelian) definition of continuity, where he says that an interval is continuous if and only if, for any of its points, one can find points of the same interval as close to it as one wants:

"$\forall\varepsilon$ and $\forall P$, $\exists Q$ such that $|P\text{-}Q| < \varepsilon$"[97],

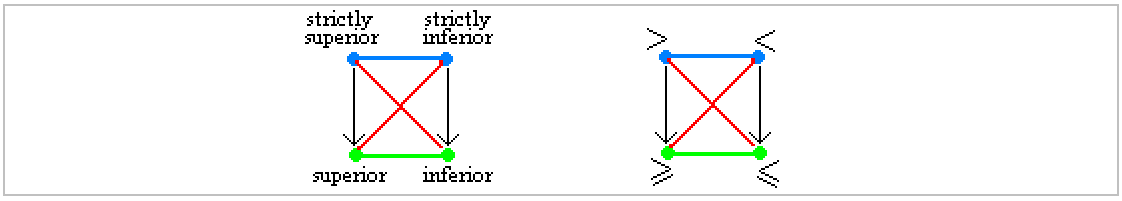
[96] Whereas arithmetic is the mathematics of the discrete, algebra is the mathematics of the abstract, while geometry and topology are the mathematics of the spatial.
[97] Granger, G.G., "Le concept de continu chez Aristote et Bolzano", *Études Philosophiques*, 1969, 4, p. 522.

to which the now standard definition of the continuity of a function makes an echo:

> "Let *f* be a function defined on an interval I, with values on R and let *a* be a point of I; we say that *f* is continuous in *a* if $\forall\alpha>0$ $\exists\beta>0$ such that if $|x-a|\leq\beta$ and if $x\in$I, then $|f(x)-f(a)|\leq\alpha$".

It is clear that these very important definitions (continuity of an interval and of a function), still standard in mathematics nowadays, use both the universal and the existential quantifiers.

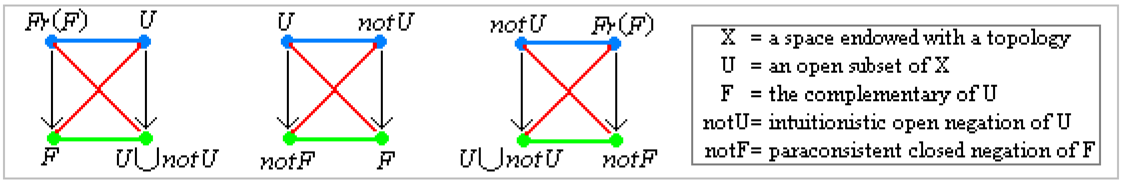
Now, not only the quantifiers $\forall$ and $\exists$, but also both these fundamental order ingredients (inferior, $\leq$, and strictly inferior, $<$, and their symmetric $\geq$ and $>$) of mathematical analysis can be expressed by the square (cf. next figure, for the "square of orders").



This means that it is by having somehow such a "square of orders" in mind that working mathematicians move through their demonstrations in analysis.

05.01.03. The square in the topology of logic

But the logical square is also essential, among others, for topology. The French mathematician and logician Régis Pellissier has recently discovered three instances of logical square related to the topology of logic, that is topology related to logical notions as the intuitionist and the paraconsistent negations (studied as phenomena implying topological notions such as frontier and open sets).



(we are going to speak again of Pellissier in the § 9 and 12.01 *supra*)

So, the square not only perfectly orders geometrically the standard quantifiers of modern (and contemporary) mathematics ($\forall$ and $\exists$), but can also express the order relations ($<$ and $\leq$), as well as some very technical notions of contemporary topology! And the same can be shown for other mathematical notions. All this shows how very important this small geometrical-logical object is (and remains) for contemporary mathematics, of which it seems to be some kind of logical transcendental condition.

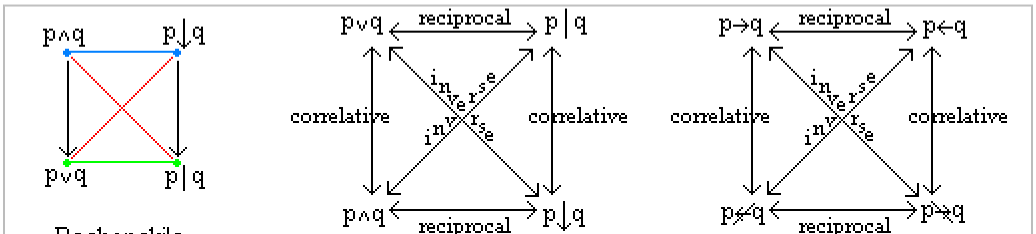
Remark that this astonishing generality seems, after reflection, to be no real mystery: the logical square is a general theory of “negation”. And mathematics cannot avoid using such a notion of negation at fundamental levels.

05.02. The squares of modalities in modal logic

The square of opposition is essential to logic for at least three reasons: (1) it is involved in propositional calculus (*via* the unary and the binary connectives), (2) it is involved in predicate calculus (*via* the quantifiers), and (3) it is involved in modal logic (*via* the modalities).

05.02.01. The squares of the logical binary connectives

Another place in logic where the logical square proves to be very powerful (in fact inescapable) is the space of the 16 binary connectives of propositional logic. These are, of course, absolutely crucial to logic (even if they can be expressed by a combination of a small number of them). Again, they can (and must) be ordered by suitable logical squares, as different authors showed independently in the middle of the 20th century. Bochenski has proposed a “square of junctors”, whereas Piaget has proposed two “quaterns”[98].



Bochenski's "square of junctors"

Piaget's "A quatern" of junctions and "B quatern" of implications

Remark however that Piaget’s notion of quatern is not exactly the same as the logical square (it bears a more psychological and dynamical flavour, cf. ch.6). But one can put instead of them classical logical squares (cf. ch.8). Of this argument we will speak again later (cf. ch.6, 8, 17 and 22 *infra*).

[98] We study in some detail this question in Moretti, A., “The 3D model for logical binary connectives deepening Blanché’s 2-dimensional model”, (to be submitted).

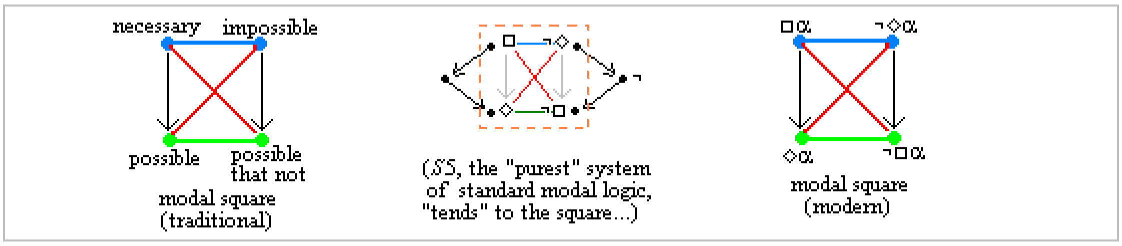
05.02.02. The square and the predicate calculus

As already mentioned, the logical square is essential to the quantification theory. Therefore, predicate logic being mainly submitted to predication, it is easy to prove that the logical square is truly essential to that part of logic.

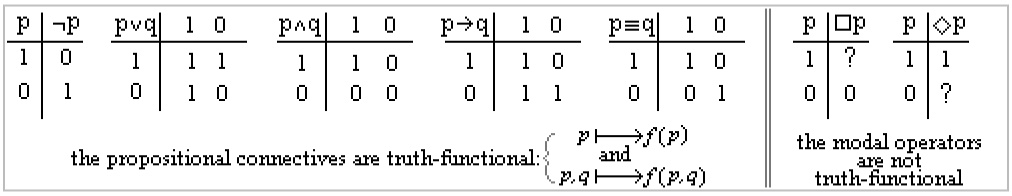
05.02.03. The square and modal logic

The second big scientific use of the logical square, as far as we know, is the one made by formal logic, and more precisely by "modal logic" (the branch of mathematical logic allowing calculations over interpreted notions, such as the notions, or modalities, of "possible", "impossible", "future", "past", "know that", "believes that", "obligatory", "allowed", "forbidden", etc.). This is not surprising, for modal logic can be seen as a systematic study of the possible restrictions made to the quantification of first order logic.

Historically, the re-birth of modal logic after its Aristotelian beginning and its medieval (scholastic) revival, has been carried on by C.I. Lewis. This logician showed that the new Frege-Russell-Whitehead logic could be arranged so as to express a logical axiomatic calculus for the classical modal notions (so far deprived of such an axiomatic calculus). This gave his famous S1 to S5 systems – S5 being standard logic). Now, if one draws the "modal graph" of such systems, i.e. the oriented graph of their "basic modalities" (i.e. the modalities which cannot be reduced logically to any simpler ones), one sees that the logical square reappears.



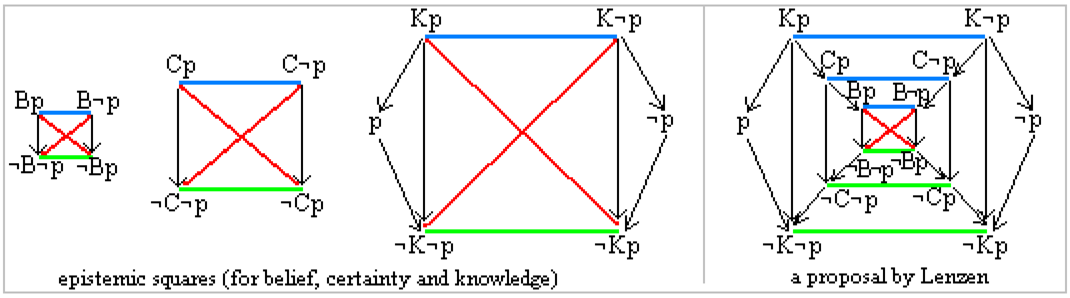
Lewis' modal logic was partly problematic, for it was unable to truth-functionally express some important modal laws (and notably the modal operators).



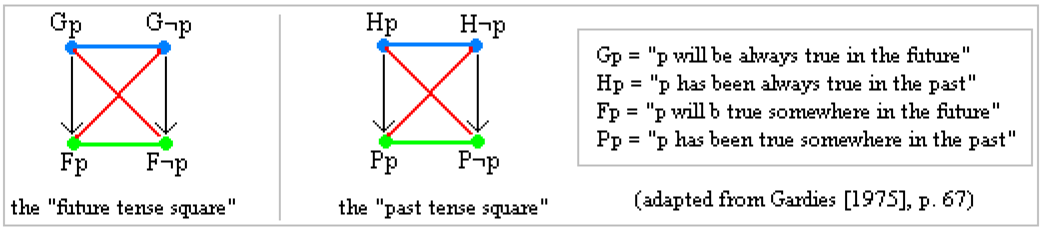
These problems were solved by the development of "possible world semantics" (by Kripke, Kanger, Hintikka, Guillaume and others), where the introduction of the notion of "possible world" offered a new scope for quantification: the modal properties of a logical proposition in a given world are truth-functionally calculated through quantification over the possible worlds (*grosso modo*, possibility in a world is logically equivalent to truth in some world, whereas necessity in a world is logically equivalent to truth in all worlds). So, because we know (after possible world semantics) that modal logic relies on quantification theory (it is a quantification over the set of the possible worlds), it is intuitively not astonishing that the logical square, which expresses fundamental quantificational properties (cf. ch.4) remains valid in contemporary modal logic.

As a matter of fact, in all abstract modal systems the logical square seems to be respected. Again, it seems to be the same kind of transcendental condition of formal thinking. And this still holds for applied modal logic.

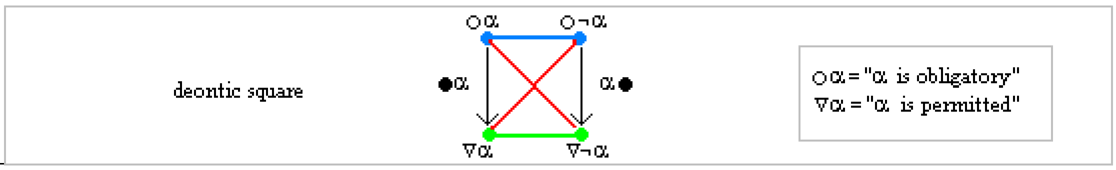
For instance, logical squares are present in so-called epistemic logic (the logic allowing calculations about "knowledge" and "belief" modalities). W. Lenzen proposes a graphical unified model where three epistemic squares (for, respectively, belief, certainty and knowledge) are nested (cf. figure)[99].



Something similar can be said for the so-called "tense logic" (one of the modal logics of time) which admits two "tense squares" (cf. figure)[100].



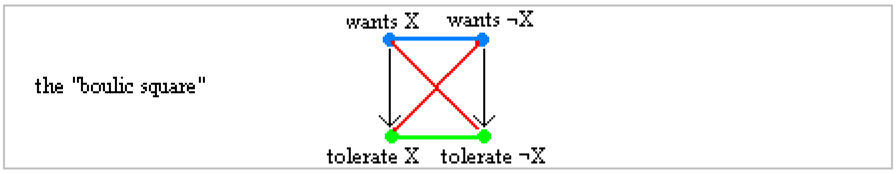
The juridical version of modal logic, the so-called "deontic logic" (the one dealing with the modalities "obligatory", "forbidden", "permitted", etc.) behaves similarly: it admits a "deontic square" (cf. figure).



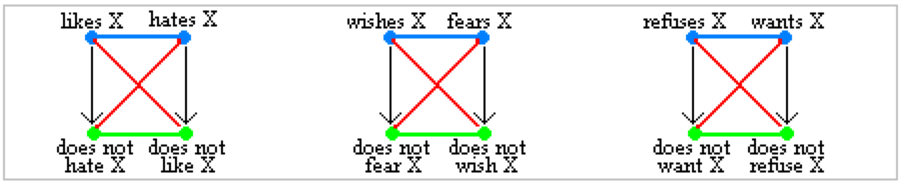
[99] I wish to thank professor Wolfgang Lenzen for this personal communication (June 2007).
[100] Cf. Gardies, *La logique du temps*, Paris, PUF, 1975, p. 67.

It can be noted that in different domains, small changes (small additions to the basic square) can occur: the "K" operator (the "know that" modality) of epistemic logic has four more arrows with respect to the square (Kp→p, p→¬K¬p, K¬p→¬p, ¬p→¬Kp); the deontic square has two more points (α and ¬α) with respect to the square. But the square itself is kept. In this sense, it seems to be "transcendental".

As a last remark here, we can imagine "home-made modal logical squares", for it is in fact quite simple to build logical squares of almost everything. Take for example the notion of "wanting". It can lead to a "boulic square" (βουλομαι, *boulomai*, "I want" in ancient Greek).



In this vein, one can mention Béziau, Costa-Leite and Payette's researches in order to think some kind of "logical square of imagination". Other possible directions are "liking", "wishing", "refusing".



In all cases, the general construction rule seems to be: (1) if two things are opposed (= contrary), their negations are subcontrary and the 4 of them form a logical square (cf. ch.4); (2) if one thing (logically) implies another thing, this first implication together with its contraposed implication form a logical square (cf. ch.4); (3) in language, the linguistic opposition of terms (cf. Saussure) is such that, in building logical squares, one almost always begins by taking into account the two upper (contrary) terms. The two lower terms are derived (by negation) and sometimes not lexicalised.

All these considerations suggest, again, that the logical square could be something unavoidable: some kind of transcendental structure of logic, as we saw it is of mathematics. But the relevance of the logical square for logic can be shown to be even more profound.

05.03. Is there something more abstract behind the square?

Several authors have proposed theories in order to generalise (or deeply understand) the laws ruling (or being expressed by) the logical square. Let us mention three of them[101]. Let me remark preliminarily that what seems common to these otherwise rather unrelated approaches is that they propose and study some kind of generalised quantification theory.

05.03.01 Gottschalk's "quaternality theory" for getting more abstract

In 1953, W. H. Gottschalk pointed out that if it "is well-known that every involution in a logical or mathematical system gives rise to a theory of duality", it is generally ignored "that every involution in a logical or mathematical system gives rise to a theory of *quaternality* and that the *square of quaternality*, of which the classical squares of opposition are special cases, provides a diagrammatic representation for much of the theory of quaternality"[102].

[1] $\varphi^{NN}=\varphi^{CC}=\varphi^{DD}=\varphi$
$\varphi^{CD}=\varphi^{DC}=\varphi^{N}$
$\varphi^{DN}=\varphi^{ND}=\varphi^{C}$
$\varphi^{NC}=\varphi^{CN}=\varphi^{D}$
(the group of quaternality)

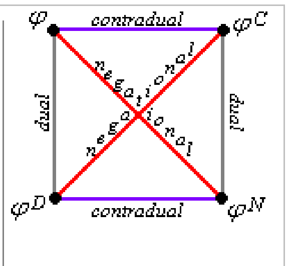
[1]-[5]: the law of quaternality

[2] $\vdash\varphi^{N}\leftrightarrow\neg\varphi$, $\vdash\varphi^{C}\leftrightarrow\neg\varphi^{D}$, $\vdash\varphi^{D}\leftrightarrow\neg\varphi^{C}$

[3] The following statements are pairwise equivalent:
$\vdash\varphi$, $\vdash\neg\varphi^{N}$, $\vdash\varphi^{C}$, $\vdash\neg\varphi^{D}$

[4] The following statements are pairwise equivalent:
$\vdash\varphi\rightarrow\psi$, $\vdash\psi^{N}\rightarrow\varphi^{N}$, $\vdash\varphi^{C}\rightarrow\psi^{C}$, $\vdash\psi^{D}\rightarrow\varphi^{D}$

[5] The following statements are pairwise equivalent:
$\vdash\varphi\leftrightarrow\psi$, $\vdash\varphi^{N}\leftrightarrow\psi^{N}$, $\vdash\varphi^{C}\leftrightarrow\psi^{C}$, $\vdash\varphi^{D}\leftrightarrow\psi^{D}$



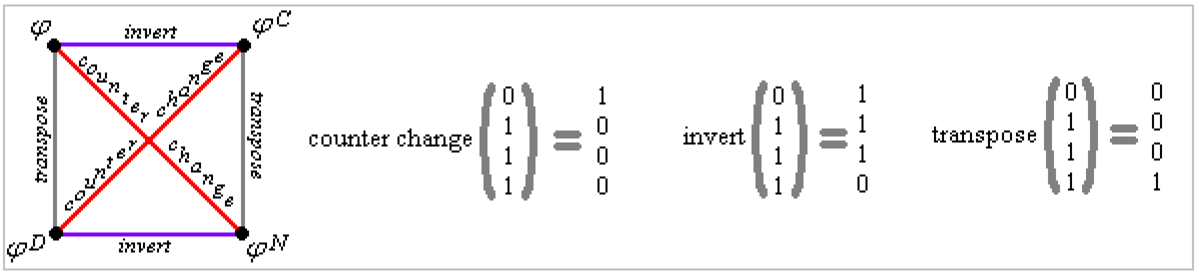
the square of quaternality

Gottschalk's general idea is therefore that the sphere of duality phenomena (very general in mathematics and logics) is in fact *ipso facto* a sphere of quaternality phenomena. In order to show it, he makes the following construction. (1) He assumes axiomatically, beside the propositional calculus, a set of couples of *dual constants*: {T and F}, {∧ and ∨}, {→ and ¬←}, {← and ¬→}, {↔ and ¬↔}, {↑ and ↓}, and {∀ and ∃}; to this he adds one self-dual notion: the constant "¬" (i.e. negation). (2) He then defines three new operations on a propositional formula φ: the *negational* of φ, noted φ^N; the *contradual* of φ, noted φ^C; and the *dual* of φ, noted φ^D; the bare formula φ and its three transformations φ^N, φ^C and φ^D (through the three operations *N*, *C* and *D*) are said to be mutually a *quaternal*[103]. (3) He then exhibits five *laws of quaternality* (cf. figure). (4) These laws are in fact embodied by two mathematical entities: (a) algebrically speaking by a *group of quaternality* and, geometrically

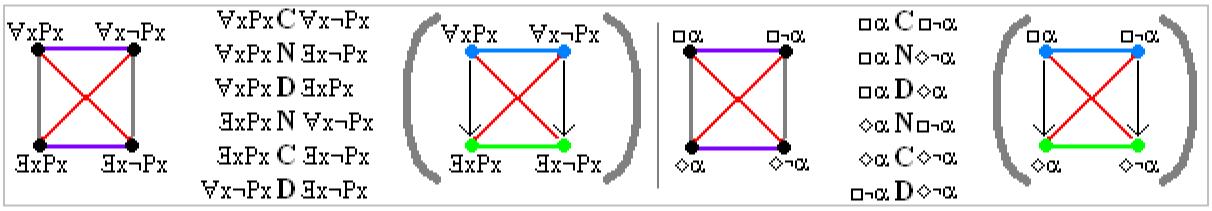
[101] A fourth one, that of J. Piaget, will be mentioned in ch. 4 *infra*.
[102] W. H. Gottschalk, "The Theory of Quaternality", *The Journal of Symbolic Logic*, Vol. 18, No. 3, Sept. 1953.
[103] The *negational* of φ is obtained by interchanging the negated and unnegated variables and by interchanging the dual constants of φ; the *contradual* of φ is obtained by interchanging the negated and unnegated variables of φ; the *dual* of φ is obtained by interchanging the dual constants of φ.

speaking, by a *square of quaternality* (we propose to colour the latter in purlple, grey and red).

(5) Gottschalk then gives an intuitive characterisation of the three operations of point 2 *supra* in terms of operations on truth-tables (here they are named counterchange, invert, transpose).

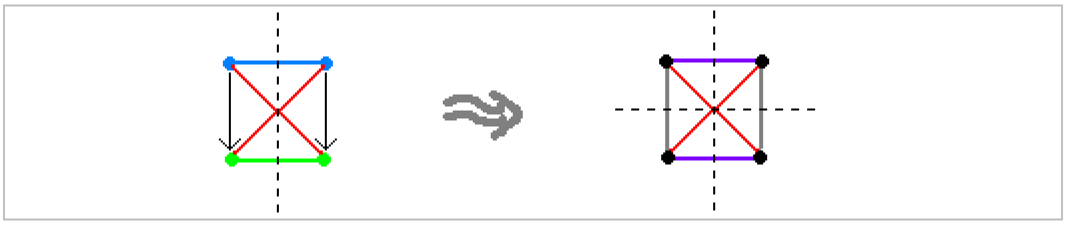


(6) He then shows how the theory of quaternality, applied to different notions, gives different kinds of square structures (as for instance the classical quantificational and modal squares).



(7) He lastly proves that, "more generally, any algebraic system with an involution gives rise to a theory of quaternality" (of this we will not speak).

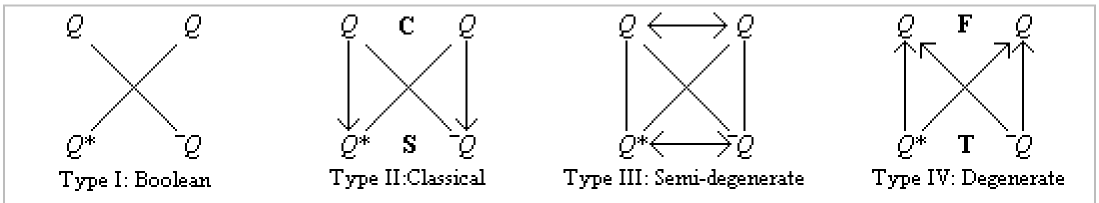
So, this theory is very fine: it throws some light over the mysterious square of opposition. Remark however that the asymmetric subalternation arrow, as well as the two mutually dual notions of contrariety and subcontrariety, seem to be an emergent feature, exceeding Gottschalk's quaternarity principles. Accidentally, the Swiss psychologist Jean Piaget simultaneously proposed an alternative but equivalent version of it, named "INRC group" (cf. ch. 6 *infra*). The obvious remark, from the point of view that interests us here, is that the object summarizing this theory (the square of quaternality) has a top-bottom symmetry (similar to the classical left-right one, but orthogonal to it), absent in the logical square.



We will come back to Gottschalk in ch. 8 (more precisely on § 08.03.03) *infra*.

05.03.02. Generalised quantifiers theory: there are four kinds of squares

Another interesting attempt to have a more general notion of the logical square has been made by M. Brown (1984)[104]. According to the typology originated by the theory of the generalised quantifiers, there are four (and only four) kinds of squares of opposition, ranging from "Boolean" to "degenerate".

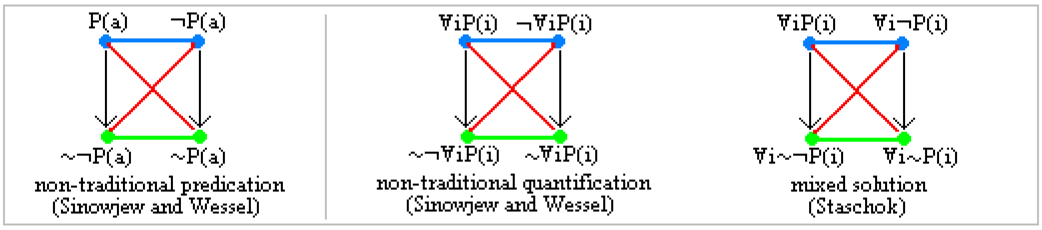


the four types of square of opposition

We recognise easily, in the first two figures, what we called (ch. 4 *supra*) Terence Parsons' "X-scheme" (the logical square as reduced by Boolean algebra) and the "square of opposition". We will not discuss the following two, which are degenerate cases.

05.03.03. Squares of non-traditional predication and quantification

Two logicians working on "non-traditional predication" and on "non-traditional quantification", Sinowjew and Wessel, have proposed some new readings of the logical square (at the left hand of the figure, an alternative proposed for the non-traditional quantification by M. Staschok)[105].



Non-traditional predication is supposed to allow to deal with vague predicates. Regarding non-traditional quantification, the question is more debated. Staschok for instance claims that in the case of quantification, there are no intuitive arguments justifying the exportation of the solutions acceptable in the case of predication, whence her proposal of an alternative model.

Notice that the shape of the square itself does not change. What does change is the logical decoration of its four vertices: there are changes in the use of negation (of all this we will talk again in ch.9).

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[104] Cf. M. Brown, "Generalised Quantifiers and the Square of Opposition", *Notre Dame Journal of Formal Logic*, Vol. 25, No. 4, October 1984.

[105] For all the references to this, cf. M. Staschok, "Non-traditional Squares of Predication & Quantification", (submitted); cf. A. Becchi, "La "teoria non tradizionale della predicazione" di Horst Wessel: negazione "esterna" e negazione "interna"", (draft, available on the net).

Having seen how important the logical square is for contemporary science (at least in mathematics and logic), we will now turn to some of its known applications outside mathematics and logic.

06.
SOME EXTRA-LOGICAL APPLICATIONS OF THE LOGICAL SQUARE : STRUCTURALISM

06.01. Introduction

In this chapter, partly by contrast to the previous one, we want to show how the structure of the square of opposition changes (partly) according to the extra-logical (extra-mathematical) fields where it is used. As a matter of fact, many people have used the square of opposition to build their own theoretical proposals in the so-called “humanities”. We cannot mention all the existing proposals (most probably we are not yet aware of many of them). However, three such uses have already become classical, deserving thus to be recalled here as paradigms of the possible extra-logical applications of the square. These are Piaget’s “I.N.R.C. group”, Greimas’ “semiotic square” and Lacan’s “sexuation formulae”.

A few words on each before going into some detail. Piaget’s theory is not so clearly related to the logical square, for the INRC group, though dealing with logic (from a psychological point of view – it formalises how the young human acquires logic capabilities) and square-shaped, seems to lack lateral arrows, a fundamental feature of the square of opposition. On the contrary, Greimas’ semiotic square – a general theory of meaning – seems to be perhaps the most important of the three applications, for it offers a structure both appealing *per se* and deeply related to the logical square. Not without problems however, as we are going to see (this time the arrows go in the wrong direction). As for Lacan’s structure – a formalisation of the “psychological sex (or gender)” –, it is a brilliant psychoanalytical attempt to show how Freud’s discovery, the human unconscious part of the mind, has a profound structure – grounded on clinical data – which is exceptionally puzzling for classical rationality (as embodied by standard logic, that is, again, the logical square). However, it is still not clear, scientifically (logically) speaking how Lacan’s proposal can be translated properly (if it can) into intelligible (modal) logic (or any other formalism).

The rest of the chapter is concerned with other extra-logical uses of the square of opposition in different fields of humanities. Remark (to this point we will come back at the end of the chapter, and on ch. 24 *infra*) that the proposals recalled here often belong to one of two thinking schools: (1) the so-called “structuralism” and (2) the so-called “system theory”.

06.02. Recalling some basic notions on "structuralism"

"Structuralism" is a school of thought which was very fashionable some decades ago. Its actual oblivion in the "world market" of theories must not be taken as a sufficient reason of criticism in itself (this market, as all markets, is stupid and dangerous if left uncontrolled and assumed – lazily and irresponsibly – as a pure Norm in itself), for this family of theories, despite the superficial appearances, still bears a notable conceptual weight (it is heuristically fertile), and – this is no difficult prophecy – new trends will come (as superficial as the actual ones) putting it again in the spotline. An important remark is that in some sense structuralism is a general model of how humanities could become "scientific": in this respect, structuralism is hated (more than ignored) by analytical philosophy, which aims at the same thing (the same supremacy) by "slightly" different means (structuralism invoked abstract mathematics, analytical philosophy invokes mathematical *logic*). It is interesting for us to recall structuralism here because several of its chief representatives did use versions of the logical square in their own theories (whereas most analytical philosophers, with the exception of Strawson, did criticise and discard it)[106]. In the rest of this chapter we will concentrate on some of these structuralist thinkers, mainly Piaget, Greimas, Lacan and (to a minor extent) Lévi-Strauss, but also on some theoreticians who are not (or not clearly) structuralist (Austin, Parsons, Latour, Seuren).

06.02.01. The main ideas of structuralism in general (brief recall)

Historically speaking, structuralists recognise their general common model in the work of the Swiss linguist Ferdinand de Saussure (1857-1913), considered by many still nowadays as the father of scientific linguistics. Saussure suggested very convincingly that behind the historicity of the evolution of human languages as a whole there is a formal, elegant and powerful "synchronic-diachronic" model to be seized. Language is then understood as a holistic system of signs, ruled by a small number of very abstract mathematical operations[107]. From this model of conceptual clarity (Saussure's ideas are still nowadays the standard in linguistics), so-called structuralist theories arose first in linguistics, but then in many other fields of human thought (in the formal sciences as well as in the humanities).

[106] On this last point cf. Terence Parsons, "The Traditional Square of Opposition", *Stanford Encyclopedia of Philosophy*, (internet), 2006 (1997) (Terence Parsons must not be confused here with Talcott Parsons, cf. *infra*).

According to the Swiss psychologist Jean Piaget, himself a leading figure in the structuralist movement (cf. *infra*), because of the great multiplicity of its concrete theoretical expressions, the spirit of structuralism can only be grasped if one distinguishes between two problems: (a) the positive ideal of finding an underlying structure and (b) the critical intentions that each such discovered structure has generated toward the previous reigning tendencies. In this respect, there is, for all structuralist approaches, a common ideal of (searched) intelligibility, whereas their critical intentions can vary indefinitely:

> "for some, as in mathematics, structuralism is opposed to the systematic separation of the heterogeneous chapters [of mathematics] and finds back unity thanks to some isomorphisms; for others, as in the successive generations of linguists, structuralism has mainly put itself at distance with respect to the diachronic researches dealing with isolated phenomena, so to find global systems in functional dependence to synchronicity; in psychology structuralism has mainly fought against the "atomistic" tendencies which tried to reduce the wholes to [bare] associations of pre-existing elements; in the ordinary discussions one can see structuralism attack historicism, functionalism and sometimes even all forms of use of the human subject in general" (pp.5-6)[108].

If one concentrates on the positive aspects of the idea of structure there will be, then, two aspects common to any structuralism: (1) an ideal or some hopes of intrinsic intelligibility, based on the postulate that a structure is self-sufficient and needs, in order to be seized, no appeal to elements foreign with respect to its nature (p. 6); (2) and the concrete realisations, in so far one indeed succeeded in obtaining some working structures, whose use reveals some general characters, necessary and independent from the variety of these structures (p.6). Piaget defines the central notion of structuralism in the following way:

> "In a first approximation, a structure is a system of transformations which has some *systemic* laws (by contrast with the properties of the *elements*) and which conserves or enriches itself by means of its own very transformations, without these letting their result falling out of its frontiers, or without these summoning some exterior elements. In a word, a structure comprises thus the three characteristics of totality, transformations and self-regulations" (pp.6-7).

The second approximation of the notion of structure, coming (eventually) later, is the realisation of a formalisation.

I myself would suggest the further idea that, historically speaking, structuralism can be seen (at least partly) as linked to a methodology theorised by Plato (if not even by someone before him): especially in his dialogue *Parmenides*, where he makes the old Parmenides teach both ironically and seriously Socrates the idea that the real philosopher (and real logician-dialectician) is one not afraid of the sometimes tedious examination of all possible cases ("mind the hair and the dirt"). This means two things: (1) a method is needed to order at best the (combinatorially infinite) field of the things to be checked conceptually; (2) one must not be afraid of questionning things apparently not worthy to be questioned (or investigated).

[107] Cf. F. de Saussure, *Cours de linguistique générale*, Payot, Paris, (1915†).

These two considerations, I maintain, can be seen as very close to the spirit of structuralism, at they both mean (a) the importance of having a structure (linked to some kind of combinatorics) as guide-line and (b) the importance of the "empty places" suggested by the leading-structure, despite the fact that *prima facie* the empty places may be conceptually repulsive (appearing meaningless and unworthy): well conceived and well applied, abstract structures reveal conceptual places still theoretically empty, the vector of future growing knowledge and understanding.

06.02.02. A partial list of the diversity of the classic structuralist theories

Direct followers of Saussure's structuralism in linguistics were G. Guillaume, L. Tesnière, R. Jakobson and L. Hjelmslev. N. A. Chomsky himself (the reference of computational linguists), with his "transformational grammar", is counted as a structuralist[109].

In mathematics, this gave rise to the so-called (outstanding) Bourbakian school (the elite of French mathematics), who tried to reform contemporary mathematics by re-writing all existing mathematical theories in strict set-theoretical style. Bourbakian mathematics became a world standard, only recently overcome by the so-called category theory, which gave a new foundation to mathematics, more powerful than set theory (a conservative extension of it).

In Anthropology, structuralism was embodied mainly by Claude Lévi-Strauss' theory of the "elementary structures of parenthood", which also became a world standard[110].

In sociology, not to speak of the older generation (Durkheim), structuralism gave the impressive theory of Pierre Bourdieu (whose theory is defined as a "genetic structuralism") and the not less impressive one of Niklas Luhmann (whose theory, previously "neo-functionalist", became after 1984 strongly structuralist after the adoption of the bio-mathematical concept of autopoiesis as fundamental structure of the human)[111].

In logic it gave the enquiries of R. Blanché, who (co-) discovered the "logical hexagon" (a conservative extension of the logical square) and developed upon it a

[108] Cf. J. Piaget, *Le structuralisme*, PUF, Paris, 1992 (1968) (we translate).

[109] Cf. G. Guillaume, *Prolégomène à la linguistique structurale*, (2 vol.), Presses de l'Université de Laval, Québec, 2003† and 2004†; L. Tesnière, *Eléments de syntaxe structurale*, Klinsieck, Paris, 1969 (1959†); R. Jakobson, *Essais de linguistique générale*, Seuil, Paris, 1963; S. Badir, *Hjelmslev*, Les Belles Lettres, Paris, 2000; N. Chomsky, *Syntactic Structures*, Mouton & Co, La Haye, 1957; on Chomsky as a structuralist linguist cf. G. C. Lepschy, *La linguistica strutturale*, Einaudi, Torino, 1966 and J. Piaget, *Le structuralisme*, PUF, Paris, 1968, pp.68-74.

[110] C. Lévi-Strauss, *Anthropologie structurale*, Plon, Paris, 1974 (1958).

[111] On Bourdieu's genetic structuralism cf. Addi, L., *Sociologie et anthropologie chez Pierre Bourdieu*, La découverte, Paris, 2002, ch. 6; on Luhmann cf. Becker F. and Rheinhardt-Becker E., *Systemtheorie. Eine Einführung für die Geschichts- und Kulturwissenschaften*, Campus, Frankfurt/New York, 2001.

philosophical proposal (which he called “reflexive logic”), based on the idea that the fundamental *structure* of human thought is related to *logical opposition* (of this we are going to speak to some extent in ch. 8 *infra*)[112]. One can also mention the proposal of the French-Swiss logician Jean-Yves Béziau of developing a “Universal Logic”, conceived as a methodic study of the purely logical invariants (mother structures, in the sense of Bourbaki) emergent from the systematic study of all conceivable non-standard systems (this idea was already expressed by the Russian logician and philosopher N. A.Vasil’ev, cf. ch. 7 *infra*; for Béziau’s position cf. ch. 10 *infra*).

In biology this gave, according to us, the theory of autopoiesis of the epidemiologists and cognitive-scientists Humberto Maturana and Francisco Varela. Autopoiesis is proposed as a model of the qualitative structure embodied by any living phenomenon.

In psychoanalysis, this gave Jacques Lacan’s theory (cf. § 06.05 *infra*), according to which the “unconscious” is structured as a language (Lacan’s theory is based on Saussure’s theory of the linguistic “sign”, which claims the primacy of the phonological over the written mental image). Lacanian psychoanalysis remained however isolated after Lacan’s exclusion from the International Psychoanalytical Association in the fifties, not to mention those simply hostile to any kind of psychoanalysis. Despite this isolation (somehow prisoner of the French tongue) the Lacanian stream became very powerful. Nowadays it is considered as one of the two major schools in the world (the other being the Kleinian).

In psychology, this gave Jean Piaget’s genetic epistemology, seemingly an ancestor of cognitive science, and in particular it gave his INRC group (of which we are going to speak in a while – cf. § 06.03 *infra* – to some extent)[113].

In Semiotics (a specialised branch of linguistics) it gave Algirdas Julien Greimas’s “narratology” (a formal theory of meaning for texts of more than two sentences) and in particular his “semiotic square” (of these we will speak in a while), which became world standard in semiotics. His theory of narratology seems to generalise Blanché’s intuition of the fundamental role of logical oppositions for constructing concepts or meanings[114].

In litterary critics (and semiology) this gave Roland Barthes (of whom, however, we know no direct link with the square of opposition).

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[112] Cf. R. Blanché, *Structures intellectuelles. Essai sur l’organisation systématique des concepts*, Vrin, Paris, 1966; R. Blanché, *Raison et discours. Défense de la logique réflexive*, Vrin, Paris, 1967.

[113] Cf. J. Piaget, *L’épistémologie génétique*, PUF, Paris, 1970; G. Lerbet, *Piaget*, Editions Universitaires, Paris, 1970; J.-M. Dolle, *Pour comprendre Jean Piaget*, Dunod, Paris, 1999 (1974); R. Mucchielli, *Introduction à la psychologie structurale*, Dessart, Bruxelles, 1966.

[114] Cf. J.A. Greimas, *Du sens*, Seuil, Paris, 1969; one of the best introductions to the theory of Greimas is: Groupe d’Entrevernes, *Analyse sémiotique des textes*, PUL, Lyon, 1979.

In political theory this gave Louis Althusser's structuralist reading of Marx's philosophy. Remark that this junction with Marxism was, because of its ideological commitment (in a sphere which was not allowed to remain private and free, as it must), rather bad for structuralism in the long run (although, of course, not all structuralist were Marxist *ipso facto*). Moreover, even the orthodoxy of the understanding of Marxist fundamental theses seems to be problematical in Althusser, as testify the criticisms of it made by the non-Marxist distinguished political philosopher Raymond Aron[115]. Althusser (and his fanatic political worshippers – among whom the then very young Alain Badiou) has possibly embodied one of the major long-term hindrances to the actual receptions of structuralism.

06.02.03. Contemporary structuralists?

Remark that some major continental French philosophers (as Deleuze, Foucault, Derrida) are considered by some as belonging to some kind of "neo-structuralism"[116]. This attribution is a critical one, for it changes the spirit of structuralism, opening it to a substantial abandoning of the search of abstract structures (the element kept of the original idea is only the idea of abandoning the concept of a substantial "subject", but this is definitely not enough)[117]. A less-known living French philosopher who used to claim to be structuralist is Michel Serres[118]. In a much more justified sense, a real neo-structuralist thinker must be seen in the impressive French philosopher Alain Badiou, who, no surprise (retrospectively, from this point of view), worked very hard (and brillantly) in order to rethink the central notions of (continental) philosophy by means of contemporary mathematics (set theory and category theory) instead of logic (cf. § 06.06.06 and 17.03.02 *infra*). But even outside France some major thinkers (alive or recently dead) can (and must) be considered structuralist: such as the Italian continental (very important) philosopher Emanuele Severino, whose main work (1957) bears the title "The originary structure", and as the American analytical (very important too)

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[115] Cf. R. Aron, "Althusser ou la lecture pseudo-structuraliste de Marx", in: R. Aron, *Marxismes imaginaires. D'une sainte famille à l'autre*, Gallimard, Paris, 1970.

[116] The case of Deleuze is complex : he wrote explicitly a paper on the notion of structuralism (I owe this point to the French phenomenologist philosopher Grégori Jean, whom I thank). The German philosopher Manfred Frank as attacked this alleged "neo-structuralist" position in his book *Qu'est-ce que le néo-structuralisme?*, (French translation of the German original), Cerf, Paris, 1989 (1984). Other "attacks" to the then (still) trendy notion of structuralism are: Rosset, C. (under the name of Roger Crémant), *Les matinées structuralistes*, Robert Laffont, Paris, 1969 (a sarcastic-surrealist comedy written by the then very young French philosopher); Descombes, V., *Le même et l'autre. Quarante-cinq ans de philosophie française*, Minuit, Paris, 1986 (1979).

[117] In the case of Foucault, people spoke of "a structuralism without structures" (cf. Piaget, *Le structuralisme*, op. cit., pp.108-115); in the case of Deleuze, his (and Guattari's) "structuralist elements" (if there are any) may come from the influence of L. Hjelmslev's theory (cf. Badir, *Hjelmslev*,op. cit., p. 22-23).

philosopher and logician David Lewis, whose "modal realism" (1973) is clearly some kind of "holistic metaphysics" where formal powerful considerations (as in Severino) push their conceptual servant to admit (and communicate) metaphysical-ontological theses shocking but highly convincing and innovative[119].

06.03. Piaget's psycho-logical square (the "I.N.R.C. group")

Jean Piaget's theoretic background (i.e. his intellectual main contribution), as we already mentioned when speaking about the theories of concepts (ch. 2 *supra*), is the so-called "genetic epistemology". This theory, a theory of the development of knowledge, can be said to have two parts: first, it gives a model of the natural growth of human intelligence for each individual (starting from early childhood), second, it applies the biological-psychological insights so gained to general (classical) epistemology (i.e. to the understanding of the different branches of science, as well as to the possible interactions among them). We will be concerned here, relatively to the square of opposition, with the first part of the theory. It is here that Piaget introduces, as a model of the capability of reasoning logically, his well-known "I.N.R.C. group", a square figure.

06.03.01. The 4 main phases of the development of human intelligence

The development of intelligence goes through several steps, the sensory-motor levels, the first level of the pre-operative thought (2-4 years), the second level of the pre-operative thought (5-6 years), the first level of the stage of the "concrete operations" (7-8 years), the second level of the concrete operations (9-10 years) and, finally, the formal level (11-12 years). It is at this last level that, according to Piaget, the child, who has constructed many abilities by using his motricity, develops the first elements pertaining clearly to the attitude of thinking logically (or scientifically).

[118] On Serres cf. Delcò, A., *Morphologies. À partir du premier Serres*, Kimé, Paris, 1998; Crahay, A., *Michel Serres. La Mutation du Cogito. Genèse du Transcendantal Objectif*, De Boeck, Bruxelles, 1988.

06.03.02. Piaget's square, called "the I.N.R.C. group"

The first move by which the child enters psychologically the realm of logic consists in "playing with abstraction and non-existence":

> "By operating concretely on more and more complex data, with which he/her cannot cope by the methods he/her uses, the child comes progressively and unconsciously to build with necessity a new operating structure.
>
> Instead of studying the complex links between two phenomena X and Y such that one of the two interferes on the other, troubles it, he/she will, by some kind of inversion of the direction, separate abstractly one of the two (or he/she will dissociate them) and will therefore neutralise the perturbations which are its own, in order to see what happens. This abstraction, a new objective assimilation which gives him/her, correlatively, a new apprehension of the world, relies on an operative structure more "distant" with respect to the objects. It allows him/her to integrate *what* is in *what could be* by means of the game of all the possible combinations"[120].

In other words, the child plays mentally with the following simple combinatorial structure

	p	$\bar{p}$
q	$p \cdot q$	$\bar{p} \cdot q$
$\bar{q}$	$p \cdot \bar{q}$	$\bar{p} \cdot \bar{q}$

Now, the hidden "formal ontology" of this kind of game is, notoriously, the so-called propositional calculus. Effectively, this combinatorics ("of two elements p and q taken to form a couple, consider the possible truth or falsity of each") generates 16 outcomes (the 16 binary connectives). The child can conceive, for a given abstract object like "p and not q" different kinds of "changes": one will be that leading to "not p and q" and this corresponds to a kind of mental activity (or operation) which is called "reciprocal" (or reciprocation); another operation will be that leading to "q→p", operation which is called, this time, "inverse" (or inversion). The point is that

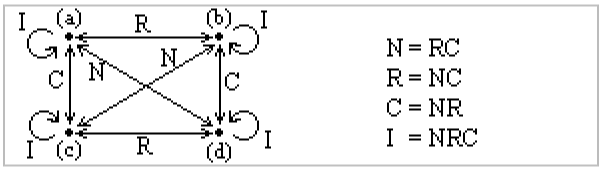
> "Whereas at the precedent period the child was unable to link the inversion-negation with the reciprocal, now he/she can. Each operation has therefore an inverse and a reciprocal" (*ibid.*, p. 76)

Now, the instinctive (empirical) way the child will find (in order) to coordinate coherently these possible fundamental "reversal operations" related to abstraction and negation (i.e. inversion and reciprocation), possibly interacting between them, happens to be ruled by a commutative group, which is isomorphic with a Klein group, and which Piaget calls "INRC group" (of mental transformations).

---

[119] Cf. Severino, E., *La struttura originaria*, Adelphi, Milano, 1981 (1957); Lewis, D., *Counterfactuals*, Blackwell, Oxford UK and Malden MA, 2001 (1973); Lewis, D., *On the Plurality of Worlds*, Blackwell, Oxford UK and Malden MA, 2001 (1986).

[120] Lerbet, G., *Piaget*, Paris, Editions Universitaires, 1970, p. 73-74.



The four letters I, N, R and C are acronyms for four basic operations, where "I" is identity, "N" is invertion (*alias* negation), "R" is reciprocation (some kind of "mirror image") and "C" is correlation. As shown by the previous (respectively: next) picture, the laws of this group can be expressed by a square (respectively: a diagram), the main property of which is, so to say, to show very clearly that in it "doing R", and then "doing N (over N)" is the same as "doing C" (and so on). One sees by Piaget's square (previous figure) that in fact the INRC group is a structure at least isomorphic to the one we saw in Gottschalk's case (cf. ch. 05 *supra*).

I = identité
N = inversion (négation)
R = réciprocation
C = corrélation

	I	N	R	C
I	I	N	R	C
N	N	I	C	R
R	R	C	I	N
C	C	R	N	I

This would mean that it should be possible, in some sense, to read the logical square as somehow deeply related to the logical square (for Gottschalk's square is deeply related, if not identical, to the logical square). And as in Gottschalk's theory of quaternality, in Piaget's case one can see the four operators I, N, R and C as ways to transform a truth table into another one.

06.03.03. Comparative remarks on the INRC group and Aristotle's square

Remark that in Piaget's theory the INRC group is, in some sense, the acme of the growing of reason. From this point of view the square of opposition, even if slightly different from the INRC group (in the sense already discussed in the case of Gottschalk, cf. ch. 5 *supra*), appears to be clearly something very important (for a Piagetian psychological point of view): it is the key to rationality for growing young people. Nevertheless, Piaget's square is very strange (and mysterious): it has negations but no (vertical) arrows. So its "R" operation must, some way or other, be different from the Aristotelian relation of contrariety: for we saw in § 04.04.02 *supra* that, given a contrariety relation between two elements, one can (and must, at least implicitely) immediately build the relative square of opposition for these two elements and their respective negations. On the question of better understanding how these square objects (the one of Gottschalk and the one of Piaget and all the ones isomorphic to them) can be related to the logical square we will come back on ch. 22 *infra*.

06.04. Greimas' "semiotic square" for narratology

The French-Lithuanian linguist Algirdas Julien Greimas (1917-1992) is one of the last recognisable figures of French structuralism. One of his greatest contributions to science consists in having built an impressive theoretical model of "large scale meaning" (the meaning of texts made of more than one sentence) by way of some kind of calculus of symmetric oppositions: that is narratology (a formal study of the meaning of the concatenated sets of sentences). But, related to this, he has also proposed, at a "micro" level, a similar oppositional approach, based on a particular structure, very similar but not identical to the logical square, which he has named "semiotic square".

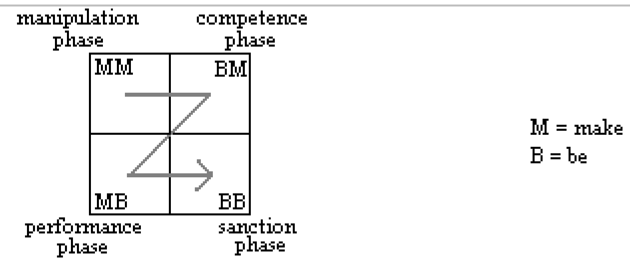
06.04.01. Greimas' theory of "narratology" (and its logical squares)

The theory of meaning developed by A.J. Greimas is very impressive, it seems to be possibly counted among the greatest achievements of structuralism. As a theory of "narratology" (or "narratologic semiotics"), i.e. as a general theory of meaning for any narrative text (written or other), studying the transitions and the transformations between opposed descriptive states, it uses, in its different parts, quite a large amount of different instances of logical squares (or squares of opposition). More particularly, as we will see, it relies, in its theoretical (semiotic) heart, on the original notion of "semiotic square", conceived by Greimas as the fundamental and inescapable model of all possible signification (cf. *infra*). This move, studying meaning ("macro" or "micro") as a differential phenomenon, is grounded on Saussure's intuitions (his theory of the linguistic sign). Despite the great interest (and success) of the Greimasian theory, it may seem rather puzzling that, astonishingly enough, forty years later it is still not completely clear (at least for logicians) how this semiotic square is supposed to be related to the logical square – of this we will speak back later (cf. §06.04.03 and ch.18 *infra*).

In this theory of narratology, the square of opposition intervenes at many places. In this respect, one can say that Greimas' theory is one of the most important non-logical applications of the square of opposition. Aristotle's structure (the square) takes place in two ways: first, as such (i.e. without changes), in many different parts of the theory (under a modal reading similar to the ones we described in ch. 05 *supra*); second, more fundamentally, it takes place in Greimas' most ambitious invention, the so-called "semiotic square", which is meant to be the theory's pivotal notion, the conceptual heart of all the narratologic enterprise.

Narratology is a strong generalisation of Propp's famous theory of the structure of the Russian fairy novels. It has different levels: a narrative (superficial) one, an intermediate one and a semiotic (deep) one.

The narrative level (i.e.the most visible level of narrations) is ruled by a "narratologic scheme", which comprises four abstract phases (manipulation, competence, performance and sanction: MM, BM, MB, BB – in the French original: E and F, "être" and "faire"). These phases, as an articulated whole, are ingredients (or moments) of the emergence of a "subject" (or actor) (this is structuralism: the "subject" is a result and not a transcendental condition of possibility). A full "actor" (or agent) of a narration is an individual (abstract or concrete) which goes through the four phases[121].

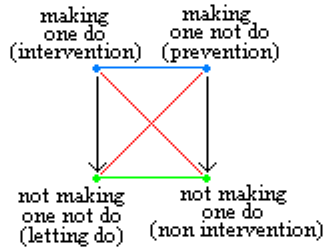


the four phases of the "narrative sequence"

These abstract phases can occur concretely according to different orders in a narration (for instance MM-BM-MB-BB, or MM-MB-BB-BM, etc.). Some of them can even be omitted, in some particular kinds of narratives (they will remain implicit). The different kinds of orderings (of phases) will be characteristic of different kinds of narrations or of different literary styles. For instance, scientific narrations (i.e. scientific papers) will emphasise phases (generally the BB one) different from the one emphasised by fantastic novels (which stress the MB one) or criminal thrillers (which may stress the MM phase, etc.). The important point for the theory is that, at least implicitly, all the phases are present in any conceivable narration (this quaternary scheme is conceived as being some kind of "Platonic idea of narration or textual meaning" – the abstract notion of "text" coming from Hjelmslev) if this narration is supposed to be meaningful. Let us have a quick overlook.

The first phase (i.e. the MM) of the narratologic scheme, the one where a possible subject is given a mission or a duty, is represented by a "square of manipulation".

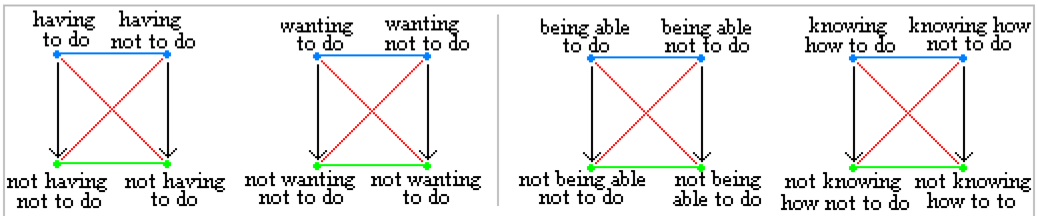
[121] A good introduction to the mainlines of Greimas' theory is: D. de Geest, "La sémiotique narrative de A.J. Greimas", *Image & Narrative. Online Magazine of the Visual Narrative*, Issue 5. The Uncanny (2003) (internet)



(the Greimasian square of the "phase of manipulation")

This is just an application of the modal square to the modality “making one do something”.

The second phase (i.e. the BM) of the narratologic scheme, consisting in giving to the possible (future) “subject” competences (in order to, later, fulfil her/his/its task), also makes use of the logical square.



(the Greimasian modalities of virtuality) (the Greimasian modalities of actuality)

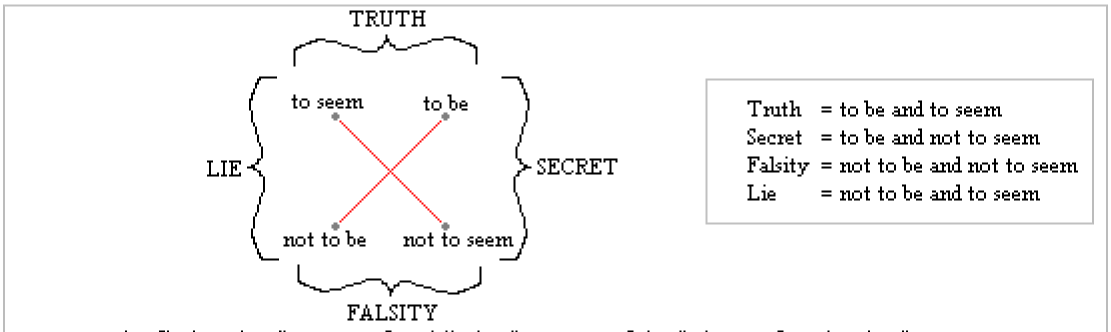
(in Greimas’ view, the three moments of the “competence phase” are virtuality, actuality and reality)

As for the third phase (i.e. the MB), where the possible-subject tries to achieve her/his/its task, Greimas gives no representation of it in terms of logical squares. He gives instead general formulae of narrative action (i.e. of the transformations carried on by the “actor”):

$f(S_1) \Rightarrow [(S_2 \vee O) \rightarrow (S_2 \wedge O)]$ and $f(S_1) \Rightarrow [(S_2 \wedge O) \rightarrow (S_2 \vee O)]$

The first of these two formulas expresses that an action “*f*” carried out by an actor “ S_1” unifies (symbolically: “$\wedge$”) an actor “ S_2” to an object “O” which “ S_2” previously lacked; the second one expresses that an action “f” carried out by an actor “ S_1” separates (symbolically: “$\vee$”) a an actor “ S_2” from an object “O” which “ S_2” previously had. Remark that the abstractness of the uniting/separating opposition is supposed to allow a recursive use of these formulas and therefore nests of nested such basic narratologic actions.

The fourth phase (i.e. the BB) of the narratologic scheme, consisting in making possible an evaluation of the performance of the third phase, is, again, represented by Greimas by means of some quaternary structure, the “square of veridiction”. Remark that this is not a logical square (we will come back extensively on this veridiction square in ch. 17 *infra*).

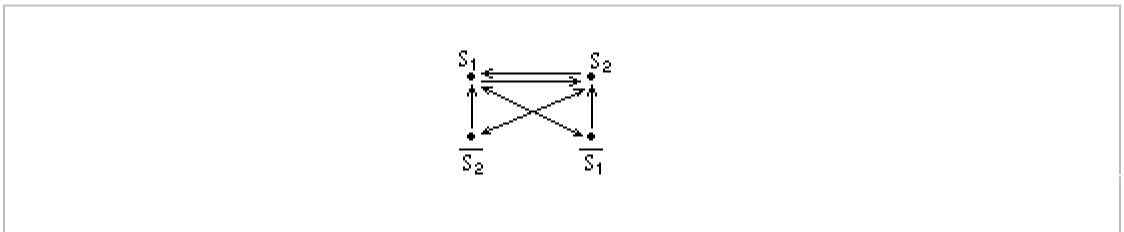


the Greimasian "square of veridiction", proper of the "phase of evaluation"

The Greimasian principle unifying all these four phases in a meaningful whole will be that any narration is structured by oppositions taken between these four structuring phases, themselves two by two opposed. The narratological technique will consist in finding back (in a way partly similar to the logical analysis of sentences) the hidden leading oppositions of any texts, these being often partly "fractal" (inside a given opposition many smaller ones can be hidden). The transition between the smallest units of the narratological analysis, however, deserves a further technical device, to which we come now.

06.04.02. Greimas' "semiotic square" of meaning

According to Greimasian semiotics there are minimal semiotic units (of meaning), the "semes". One of the structuralist fundamental principles tells that, in order to mean, they must support some "difference". The system of the articulations of such differences is said "structure". Of this structure, abstractly conceived, a concrete model should, if possible, be given. A first very important remark (one present, for instance, in Freud) is that (in ancient (archaic) languages (and in old fragments of our own languages) and in clinical data (psychopathological and onirical), meaning is almost always given by couples of opposed (in fact "contrary") terms: "old" can be meaningful only provided there is a "young" meaning, and so on. It is in order to express formally such fundamental meaning-relations (based on couples of contrary terms) that Greimas proposes his modified version of the square of oppositions.



Greimas' semiotic square

So, the model of the abstract semiotic structure is embodied, according to Greimas, by this "semiotic square".

Remark that Greimas' theory seems very important in order to try to make clear the complex notion of "signifiant" (invented by Saussure – meaning by him "acoustic mental image" – and widely used thereafter, slightly modified, by Lacan).

06.04.03. Remarks on Greimas' square compared to the logical one

As with Piaget (i.e. by reference to the very universal properties recalled by us in § 04.04.02 *supra*), one must remark here that something strange is going on. For, up to a certain extent, Greimas' square is *very* similar to Aristotle's one. But the "small differences" present in it (with respect to the classical square of opposition) are sufficient to make Greimas' square look crazy (logically speaking): its arrows go the wrong way. Apparently, one must get rid of the temptation to read such arrows logically, for they clearly have a linguistic-semiotic meaning instead. Remark also that, in this respect, one of the most famous interpreters of Greimas, Jean Petitot, has said that, truly speaking, Greimas's square – which is very much meaningful for many disciplines, *in primis* semiotics – is meaningless for logic (where, if taken seriously, it would collapse into nonsense). On this we are speaking again in ch. 9 and 17 *infra*.

06.05. Lacan's disruption of logic through "sexuation formulae"

If not exempt from raging polemics (or stupid idolatry) and accusations (against him and his school), Jacques Lacan (1901-1981) is generally recognised, even by his enemies, as being one of the most brilliant psychoanalysts ever[122]. He bears a strange explicit relation to the logical square.

06.05.01. Lacan's theory of psychoanalysis in general

As already recalled, his main conceptual framework consists in considering, along the structuralist model (Saussure, Jakobson, Hjelmslev), that "the unconscious is structured as a

[122] As is well known, the main intra-psychoanalytical accusation led against him (the one for which he was banned from the International Psychoanalytical Association, making the psychoanalytic actual world as divided as the philosophical one) is not based on his theory (of the unconscious and of the therapy) – recognised by his strongest opponents (as, paradigmatically, by D. Anzieu) as very deep and serious –, but on his reform of the way to train (and select) future psychoanalysts (the machinery called by Lacan "la passe").

language": the "subject" is not primary, it is an emerging order resulting from a psychogenesis submitted to formal constraints ruled both by the very structure of language and by the psychanalytical constraints discovered by Freud (Oedipus' complex, etc.). One of the main structuralist intuitions in linguistics having struck the young Lacan seems to have been that of the existence of a duality (like in "kernel" and "filter" in algebra, "$\forall$" and "$\exists$" in analysis or "$\wedge$" and "$\vee$" in logic): the duality of "metaphor" and "metonymy". One of Lacan's main projects (and realisations) will be to reconduct 2 over the 5 "axioms" of Freud's metapsychology ("displacement" and "condensation", considered as mutually dual) to these mutually dual notions of the structuralist approach to linguistics. Among many other theoretical novelties introduced by him with respect to classical psychoanalysis, he took the Freudian notion of "partial object" to a direction different from that leading Melanie Klein and her school (in the English-speaking world) to the theory of "object relations"[123]. Instead of this, Lacan developed a theory of the "*a* object" (this being the formal structure of human desire as based on the unconscious) by means of various topological models of the unconscious mind. In particular, he used the logical square (in a quite strange way) in order to express some puzzling features of "sexuation theory", the theory of the "mental sex"[124].

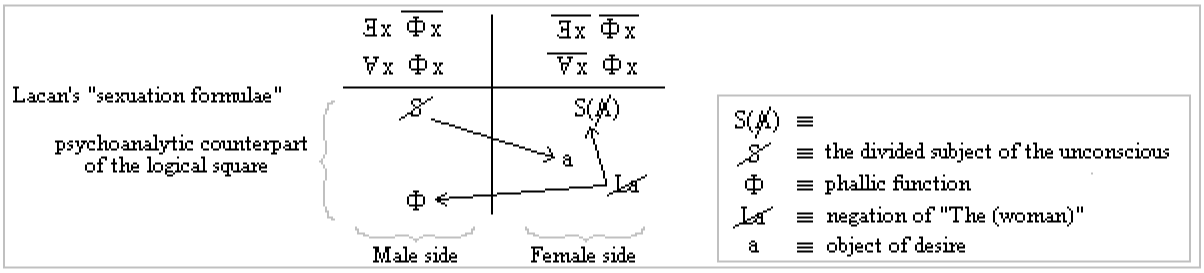
06.05.02. Lacan's "sexuation formulas" and the disruption of the square

The main idea here is to show, along a typical Freudian way, that unconscious phenomena testify structures coherent but very far from standard rationality. In doing that, Lacan starts from Freud's study *Totem and Taboo* (1912), one of Freud's most hypothetical writings (one of the most risky, deprived of empirical-clinical feedback), where the Austrian psychoanalyst admittedly speculates *una tantum* beyond clinical experience in order to make general considerations about human culture. Lacan resumes Freud's hypothesis of a myth of the original murder of the tribe-father. The French psychoanalyst wants to express the "logic" of the human fantasies about mental gender. His aim is to show the link between "sexual", "gender" and "enjoyment" (*jouissance* in French, term close to "orgasm") notions with respect to notions usually proper of logic, such as those of "universality" and "existence".

[123] On the Kleinian tradition, cf. L. Gomez, *An Introduction to Object Relations*, Free Association Books, London, 1997. On the possible relations between the Kleinian and the Lacanian approaches to psychoanalysis, cf. B. Burgoyne and M. Sullivan (eds.), *The Klein-Lacan dialogues*, Rebus Press, London, 1997.

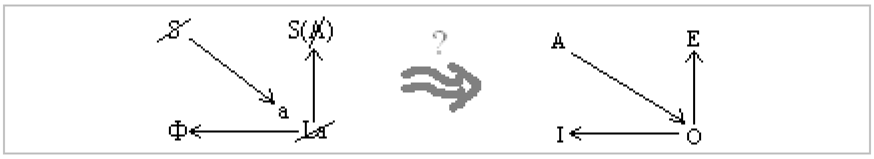
[124] On Lacan in general, cf. J. Dor, *Introduction à la lecture de Lacan - 1. L'inconscient structuré comme un langage*, Denoël, Paris, 1985 and id. *Introduction à la lecture de Lacan – 2. La structure du sujet*, Denoël, Paris, 1992. On the use of topology for the theory of the partial objects, cf. J.-D. Nasio, *Les yeux de Laure. Transfert,*

The starting point is to admit that there seems to be a difference between female and male sexual pleasure (orgasm). But, paradoxically, this difference is not only related to anatomy (and physiology), but also (and mainly) to the mental construction of gender and desire. In other words, there is some kind of “logic” of the construction of mental sexual identity (gender) related to Freud’s myth of the original murder (of the tribe-father). Accordingly, there is an understandable difference in the way of experiencing sexual final pleasure (orgasm) related to the notion of “totality” and “otherness”. The “female” abstract position is characterised by the absence of limit (the myth of the absolute Otherness) whereas the “male” abstract position is characterised by an overwhelming consciousness of the limit (i.e. “castration”). This gives, if one follows Lacan’s model, the “sexuation formulae” and some kind of psychoanalytical (Lacanian) counterpart of the logical square.



06.05.03. Lacan’s square compared to the logical square

Remark that this elaboration by Lacan surely has a relation to the logical square. This is understandable: Lacan himself elaborated it with constant reference to the Fregean version of the square (and also with references to some logical reflections of the logician and philosopher C.S. Peirce). But the relation of this “psychoanalytical square” to the “logical square” is strange: it is a destruction of it, making it hard to build real comparisons. This appears in different ways, notably by the use Lacan makes of the arrows (cf. figure).



For those, like myself, taking seriously the spirit and the starting intuition of psychoanalytical research and theory (which does not rule out the possibility for psychoanalysis to sometimes being mistaken, building wrong models, plainly false – to be spotted and corrected!), an interesting challenge would be to give a logically clearer model of

objet a *et topologie dans la théorie de J. Lacan*, Flammarion, Paris, 1987. On Lacan and mathematics in general, cf. N. Charraud, *Lacan et les mathématiques*, Anthropos, Paris, 1997.

this destruction of the square (a clearer model of the “unconscious square”, if there is one). For instance, Lacan’s arguments seem to request special new kinds of non-standard quantification and predication (Lacan says that in the topics he is speaking of – the theory of sexuation – the scope of negation is not the same as in classical modal logic). So it could be interesting to try to use new formalisms, perhaps somehow similar to those developed by Sinowjew and Wessel (cf. ch. 05 *supra*), in order to better understand the kind of rationality hidden (if any) behind Lacan’s strange uses of logic (motivated by him in order to explain positively psychopathology and onirism).

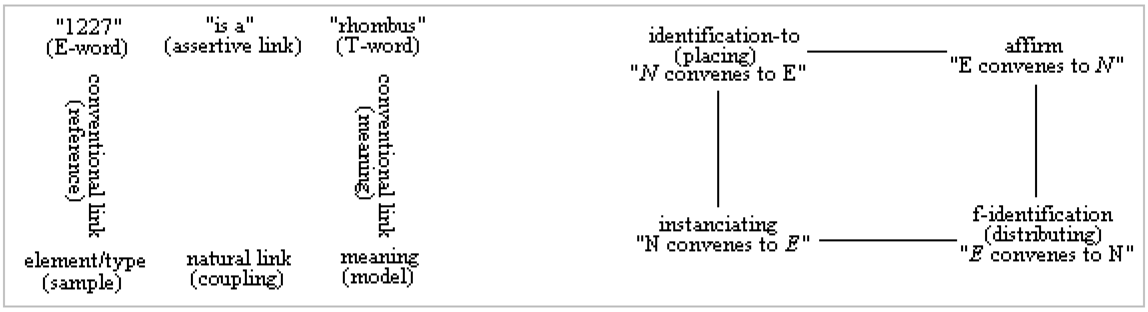
Remark also that, formally speaking, Lacan’s “Freudian disruption of logic” (i.e. of rationality) can be challenged by Matte Blanco'’ theoretical proposal of a bi-logic (in which we have strange algebrical symmetry phenomena due to a complex interplay between *n*-dimensional mental spaces)

06.06. Other extra-logical applications of the square

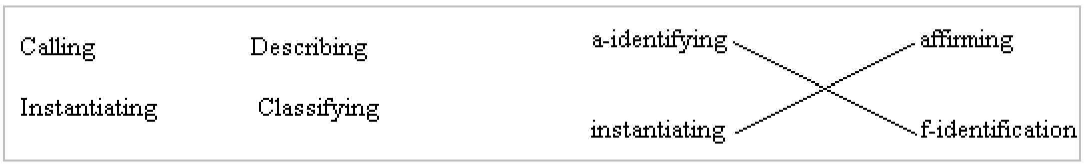
Another group of instances of square extra-logical formalisation, although clearly far from the square of opposition, seems to bear some interest in so far it is related with the attempt to think something more dynamical than just a static opposition. <…>

06.06.01. Austin (philosophy of ordinary language)

The British philosopher Austin has put forward a puzzling and interesting model of some linguistic phenomena, which seems close enough to the square of opposition.



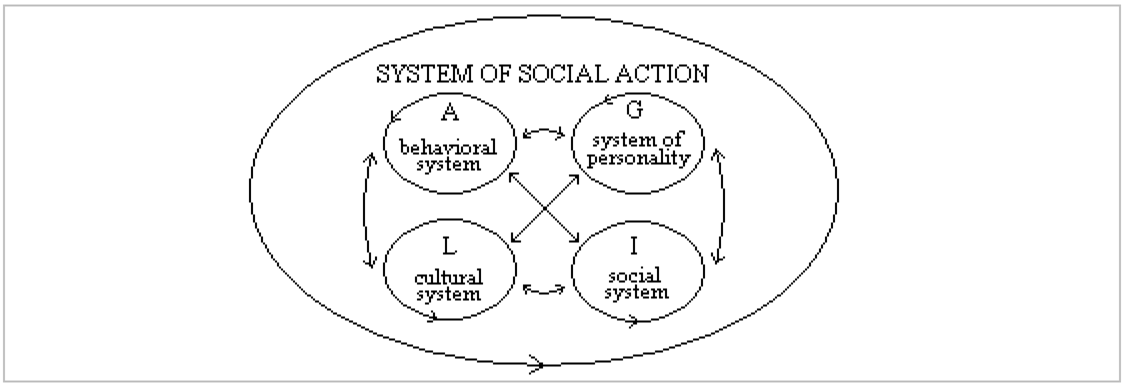
This use of a quaternary scheme intervenes several times in the same paper.



We leave open the question of investigating the relevance of its oppositions, for *prima facie* it is not clear that, for instance, in two terms linked by a same diagonal, one is the negation of the other. Remark however that in the second one the two "left terms" seem to stay in some kind of "reciprocation relation" (to speak like Piaget, cf. *supra*) with respect to the two "right terms"[125].

06.06.02. Sociology: Talcott Parsons' functionalist "AGIL group"

The American Talcott Parsons is one of the greatest sociologists of the twentieth century. He is also important for having inspired one of his pupils, the German jurist and sociologist Niklas Luhmann, himself one of the greatest theoreticians of the twentieth century (the theory of the latter is said to be neo-functionalist and systemic-autopoietic). In Parsons' sociological theory (whose method is said "structural-functionalist")[126], which is a multi-dimensional non-dualistic theory of action, the fundamental scheme is a square one. It is the well-known AGIL scheme (a fractal scheme, admitting an isomorphism between its global shape and that of each of its sub-parts).



Nevertheless, a closer geometrical examination seems to show that the real geometry implicit in the AGIL is not a two-dimensional scheme, but rather a three-dimensional tetrahedron (for: all the relations among the four "atoms" – i.e. the relations between any two of the A, G, I and L elements – are on a same plan)[127].

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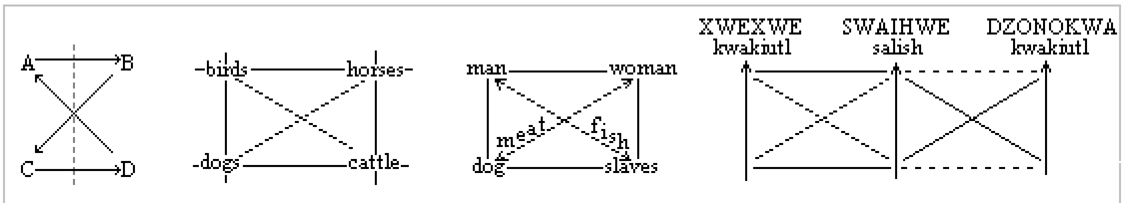
[125] The four schemes belong to a same paper: "Comment parler. Quelques façons simples", in Austin, J. L., *Ecrits philosophiques*, Paris, Seuil, 1994 (1961), p. 118, 123, 129, 133 (this is the French translation of a paper appeared in the *Proceedings of the Aristotelian Society*, 1952-1953).
[126] Cf. J. Piaget, *Le structuralisme*, op. cit., pp. 85-86 (Piaget seems to consider him a structuralist).
[127] Cf. N. Addario, *Azione e condizione umana. Talcott Parsons teorico dell'azione e interprete della modernità*, Rubbettino, Soveria Mannelli, 1999; cf. R. Prandini (ed.), *Talcott Parsons. La cultura della società*, Mondadori, Milano, 1998.

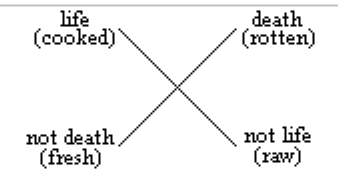
06.06.03. Claude Lévi-Strauss (anthropology)

Claude Lévi-Strauss, one of the leaders of the structuralist movement, has put forward some squares in his impressive anthropological work. We know of at least four such instances of formalisations apparently near to Aristotle's square. The first one (1952) is in *Anthropologie structurale*, the second one (1962) is in *La pensée sauvage*, p. 250, the third and the fourth ones (1979) are in *La voie des masques*, pp. 22 and 106 respectively (cf. figure).



squares by Lévi-Strauss

The first one reminds of Greimas' dynamic lecture of his semiotic square (cf. ch.24 *infra*). The second and third do not seem to be real squares of opposition, for there are no vertical arrows in them. The fourth one is rather unclear. Another square of him is the following[128].

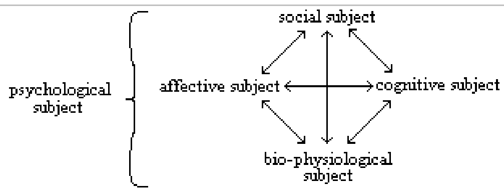


another opposition square by Lévi-Strauss

Remark that Lévi-Strauss is famous for having produced very successful graphs in order to explain his theory of the "elementary structures of parenthood" (such graphs are a flexible symbolic notation). But these ones bear no relation to the logical square. Remark also that this fifth square apparently influenced Greimas in his own way of conceiving the square (cf. *supra*).

06.06.04. Psychology: Jean-Marie Dolle's dynamic square

In a study on Freud and Piaget, the psychologist Jean-Marie Dolle has proposed two original instances of square relations[129]. The first one is the following (p.31).



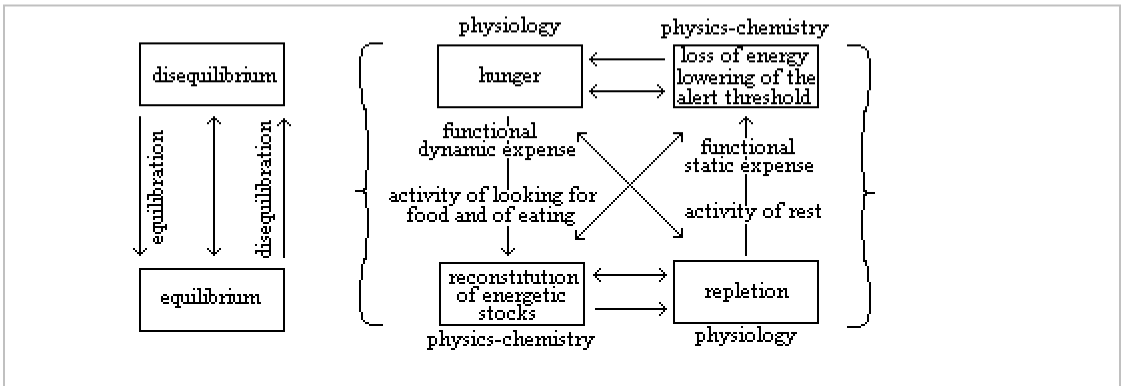
a square by Dolle

[128] Cf. C. Lévi-Strauss, *Le Cru et le Cuit*, Paris, Plon, 1964.

[129] Cf. J.-M. Dolle, *Au-delà de Freud et Piaget. Jalons pour de nouvelles perspectives en psychologie*, Privat, Toulouse, 1987.

This "square" seems to be rather à tetrahedron: all the possible relations between the four points are on a same plan, no qualitative distinction is made between diagonals and sides.

But a more interesting square formalisation appears later in the same book (p. 80).

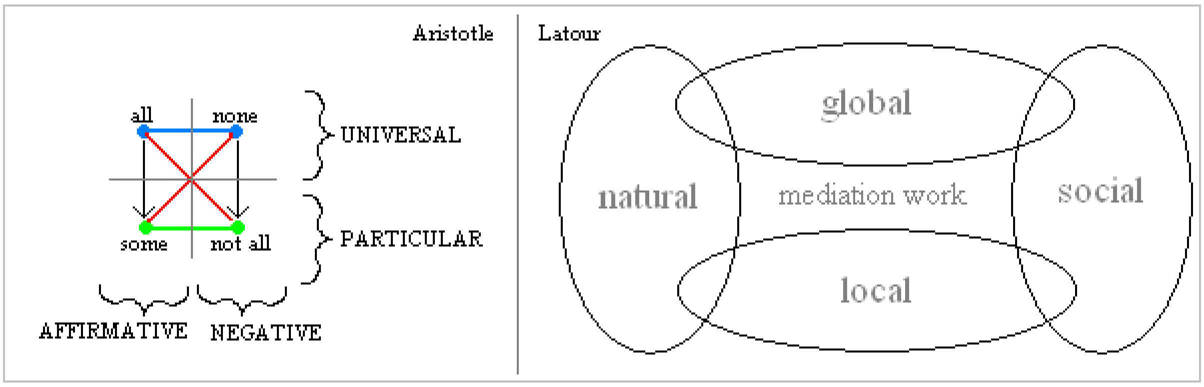


a square by Dolle

Its interest in our eyes lies in the fact that it is dynamical, and makes, this time, a quite clear distinction between the diagonals of the square, which are classically enough negations ("repletion" is the negation – or the contrary? – of "hunger", "loss of energy" is the negation – or the contrary? – of "reconstitution of energetic stocks"), whereas the vertical sides of the square are arrows. The problem this time (with respect to the square of opposition) is that the two vertical arrows do not share the same direction (one goes down, the other goes up).

06.06.05. Bruno Latour's implicit anthropologic-political square

The French philosopher and anthropologist Bruno Latour, generally known as a "sociologist of science" (because he spent years studying the Western labs from inside as if they were exotic African tribes), has developed a theoretical position which he calls "symmetric anthropology". His view, mistakenly taken as being "just a sociology of science", is in fact a rather political one (a little bit in the sense of the so-called "no global" theories), consisting in thinking (and criticising) the ways in which the Western rationality tends to become "conceptually imperialist": Latour studies the ways in which the concept of "universal" (in "universal cultural values", and so on) constructs implicitly a cultural theoretical paradigm (in the sense of Kuhn's theory) which remains unquestioned (because unseen) and thus very effective (in its worst bearings). More precisely, he claims that the Western anthropology is ruled by a fundamental (undue) asymmetry: (a) when it studies the non-Western cultures, it does it by investigating the "heart" of them; (b) but when it studies our own culture, it does it by limiting itself to the study of some "primitive borders" (like regionalist features, or under-cultured people, or Sunday hunters or fishers, etc.): being used to study “minor cultures” (minor with respect to our glorious Western one), it keeps this habit by studying (only) “minor phenomena” of our own culture, that is, systematically avoiding the study of the things which are the most important for us in our own time (our “living mythology”), that is Western science (in all its anthropological complexity). So he proposes an alternative method of anthropological enquiry, which he calls by contrast to the standard (asymmetric) one, “symmetric anthropology”. Now, it can be shown that some of the main categories of this view rely on a conceptual disposition which is almost isomorphic with that of the logical square, almost: for the form of the logical square is exactly what is criticised, relatively to its implicit horizontal and vertical strict bi-partition, inside anthropology.



(standard anthropology admits nothing neither between global and local, nor between natural and social – Latour fights this view)

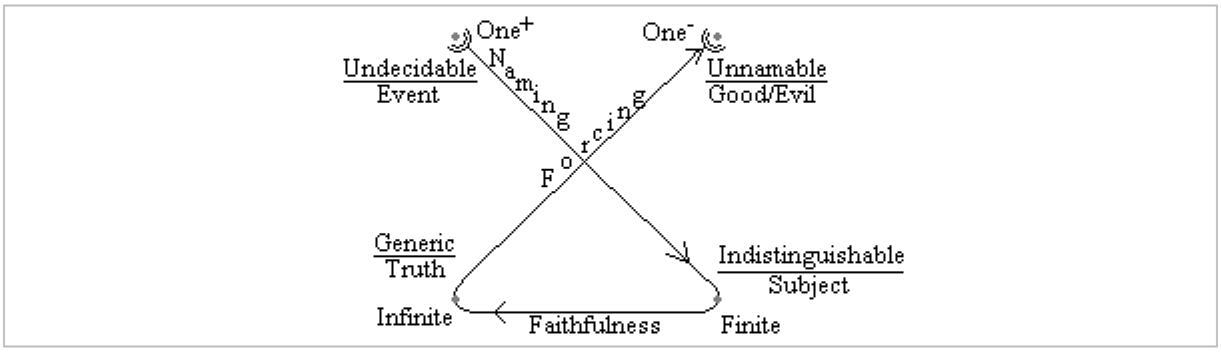
So, in Latour’s “square” the missing elements are the subalternation arrows. On the contrary, the extra element is the intermediate field (between global, local, natural and social), the “mediation work” (*travail de médiation*, in French)[130].

06.06.06. Alain Badiou’s “gamma scheme” for event theory

Alain Badiou (cf. § 06.02.03 *supra*) is actually the best-known (and seemingly the greatest) living French philosopher. His philosophical “continental” theory is very much influenced by contemporary mathematics (category theory). His general theory of “event” is expressed by a model saying that the correlate development of collective subjects, new truths and rare events always follows (when it doesn’t die beforehand) a conceptual path made of four positions (each position expresses a general structure of ontology revealed by

[130] Cf. B. Latour, *Nous n’avons jamais été modernes. Essai d’anthropologie symétrique*, La découverte, Paris, 1991 (the scheme we reproducted – right part – is at p. 167); a more light introduction is given by B. Latour, *Un monde pluriel mais commun. Entretiens avec François Ewald*, L’aube, Paris, 2003.

mathematics). Such a path is called "gamma path" (because of its "γ" shape) and the model expressing it is therefore called "gamma scheme"[131].

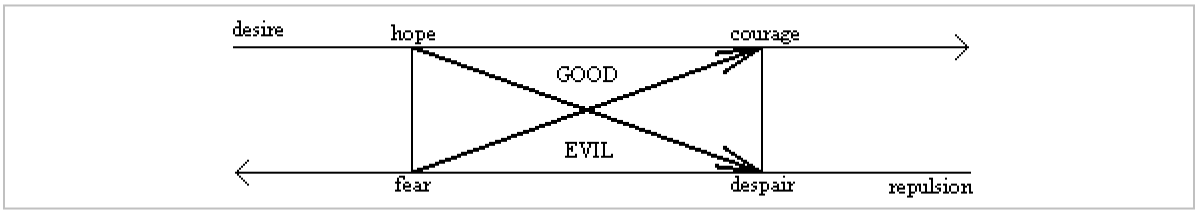


At first sight this "square" is far from the square of opposition. However, the structuralist concern for such "transcendental" structures is very present in Badiou (an heir of French structuralism, mainly through Lacan who is one of his major inspiration source – Badiou takes from Lacan both the idea that the "subject" is not transcendental but emergent and the idea that mathematics are the structure of ontology). Links with the square can appear if one focuses on the "quantities" pinned up at the four corners of the gamma scheme: the excedent unit, the unreachable unit, the finite many, the (actual) infinite many.

We will try to come back to Badiou in ch. 17 *infra*.

06.06.07. Michel Meyer's "rhetoric square" for argumentation theory

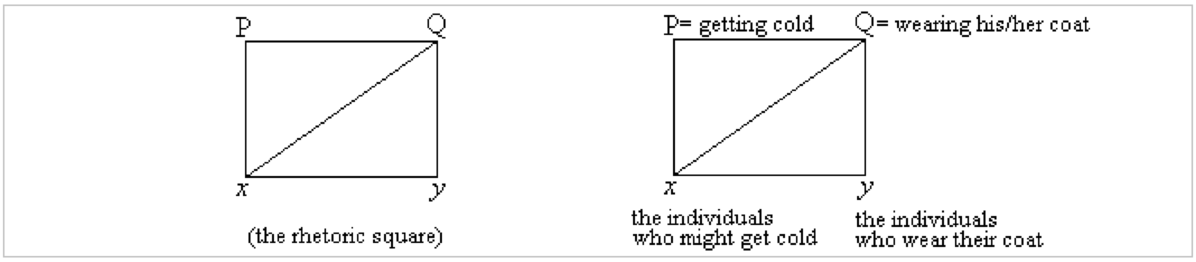
Inside the so-called "Brussels school", the Belgian philosopher Michel Meyer, a pupil of Chaïm Perelman (one of the founders, in 1950, of contemporary argumentation theory) studies the relevance of passions and emotions for the neo-rhetoric underlying structure of general argumentation theory. In this respect he proposed, in 1991, a small model of the "four passions of the upset" (in a discussion of St. Thomas' doctrine on this subject)[132].



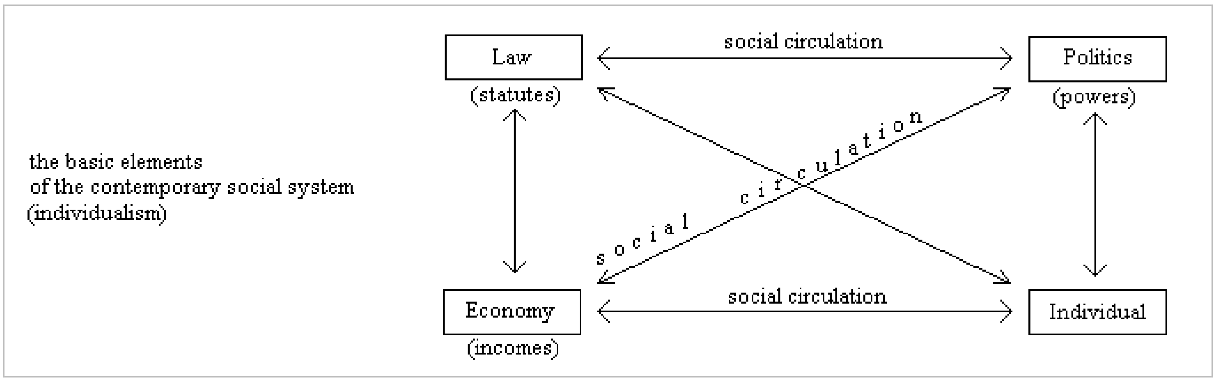
[131] For the gamma scheme, cf. A. Badiou, "Conférence sur la soustraction", in: *Conditions*, Seuil, Paris, 1992. Badiou's main treatises are: *L'être et l'événement*, Seuil, Paris, 1988 and *Logiques des mondes*, Seuil, Paris, 2006. A good introduction to his thought is his *Second manifeste pour la philosophie*, Fayard, Paris, 2009.

[132] Cf. M. Meyer, *Le Philosophe et les passions. Esquisse d'une histoire de la nature humaine*, Librairie Générale Française, Paris, 1991, p. 122. Meyer is known for his general theory of "problematology", cf. M. Meyer, *De la problématologie. Philosophie, science et langage*, Mardaga, Bruxelles, 1986.

This might have been influenced by the structuralist influence of the French speaking philosophy (Lévi-Strauss, Lacan, Greimas). Anyway, it seems to be a recurrent feature of Meyer's approach, for he reiterates such models several times in his work. A quite interesting one – if not for the direct similarity for the logical square, at least for the ambition of its program – is given by his recent "rhetoric square" for his general argumentation theory.



Let us mention, before quitting him, another instance of formal square by Meyer, one similar to Parsons' AGIL one for sociology (cf. § 06.06.02 *supra*)[133].



06.06.08. Pieter Seuren's "victorious square" (of psycholinguistics)

A special mention is deserved by the Dutch logician and linguist Pieter Seuren, who defends the view that a "natural logic", quite similar to Aristotle's one, has been selected during the evolution leading to mankind and that, parallel to it, some kind of natural set theory is already available. From this respect, Seuren studies in a very technical and precise way the history of the different versions of the logical square in ancient and medieval times, as well as their underlying precise logic (which he brings to light very methodically and clearly, using also contemporary logical technologies, particularly B. Van Fraassen's "valuations-space modelling")[134].

[133] These two last schemes can be found in M. Meyer, *Principia Rhetorica. Une théorie générale de l'argumentation*, Fayard, Paris, 2008, pp. 103-104 and 220. Another square scheme (dynamical) is at p. 231.

As a conclusion for this chapter, remark that such applications seem to belong mainly to two major trends of intellectual enquiry: (1) structuralism (Piaget, Lacan, Lévi-Strauss, Greimas, …) and (2) system theory (Piaget, Parsons, Dolle, …).

[134] Cf. P. Seuren, *The Victorious Square. A Study of Natural Predicate Logic*, (draft, forthcoming) (2007). Cf. also P. Seuren, “Psychology Regained in Natural Logic”, (draft, forthcoming) (2008).

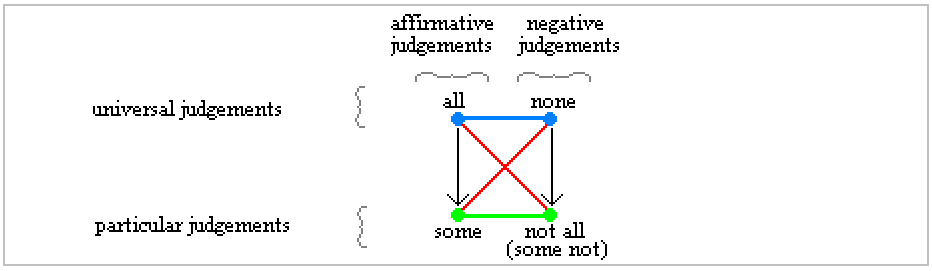
07.
VASIL'EV'S REMARKABLE ATTEMPT TO THINK OPPOSITION BEYOND THE SQUARE

The fact that Aristotle oscillates strangely between two models of the possible, one, "unilateral", giving the square (of opposition) and the other, "bilateral", giving some kind of mysterious triangle (of contrarieties), has not aroused enough attention until two French logicians (Sesmat and Blanché), around 1950, about whom we will speak in the next chapter. There is one notable exception, however: that of the Russian logician and philosopher N.A. Vasil'ev (1880-1940) who, starting from that remark (Aristotle's unresolved hesitation) and relying on non-Euclidean geometries (which revealed the possibility of incredible conceptual revolutions), tried to build a geometry-inspired series of "*n*-dimensional non-Aristotelian imaginary logics". Because his logical systems allow a non-standard treatment of the "principle of non-contradiction", Vasil'ev is considered by some, together with A. Meinong and L. Wittgenstein (and in some sense J. Łukasiewicz), as a forerunner of "paraconsistent logics" (the logics nowadays exploring the possible treatment of "non-trivial inconsistency", cf. ch.1 and ch.9); and because his logical systems allow a shift from the "principle of the excluded third" to the "principle of the excluded fourth" (and then of the excluded n-th), he is considered by all as one of the main forerunners (with C.S. Peirce and J. Łukasiewicz) of many-valued logics (the logics admitting more than 2 truth-values). In this chapter we want to recall Vasil'ev's very profound remarks on opposition, for they will turn out rather prophetic, if not fully normative, for the subsequent development of the theory of logical opposition.

07.01. Vasil'ev's first study: on the "triangle of contrarieties"

In 1910, the Russian logician N.A. Vasil'ev, professor of philosophy at the university of Kazan (the university of Lobachevski), enters the realm of logical investigation by way of a study on "some" (некоторые, *nekotorye*), a notion traditionally named (in Aristotelian, syllogistic terms) "particular judgement" (частное суждение, *chastnoe suzhdenie*)[135].

[135] Vasil'ev, N.A., "O chastnykh suzhdenijakh, o treugol'nike protivopolozhnostei, o zakone iskliuchennego cetvertogo" (1910), ("On particular judgements, on the triangle of oppositions, on the law of the excluded fourth"), in Vasil'ev, N.A., *Voobrazhaemaja logika. Izbrannye trudy*, Moscow, Nauka, 1989 (*Imaginary logic.*

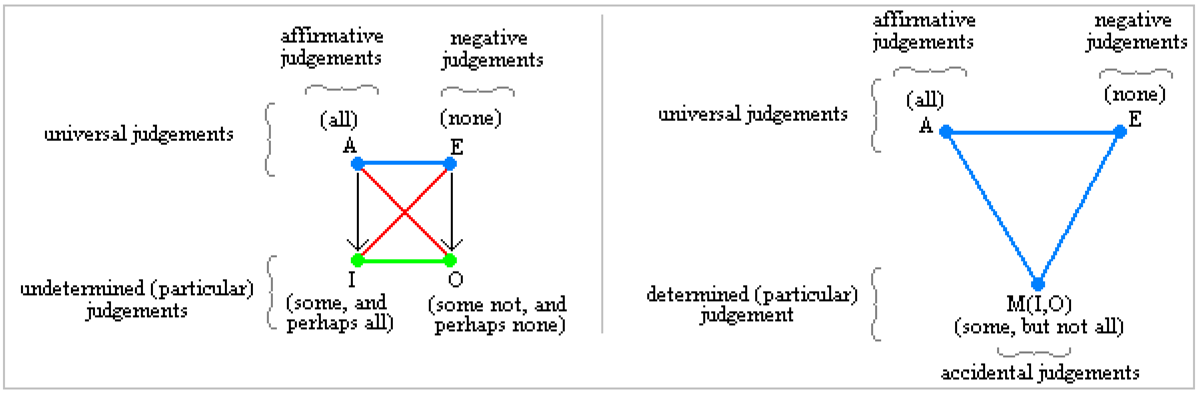


He remarks that this concept is highly ambiguous, meaning alternatively “some, and perhaps all” (некоторые, а может быть и все, *nekotorye, a mozhet byt’ i vse*) (or: “at least some”, по крайней мере некоторые, *po krainei mere nekotorye*) and “only some” (только некоторые, *tol’ko nekotorye*) (or: “some, but not all”, некоторые, а не все, *nekotorye, a ne vse* – the latter are traditionally called “accidental judgements”). While natural language (even in science) almost always sticks to the second meaning (“some, and not all”), which is much more precise, and which is the logically good one according to Vasil’ev, logical research until his time had quite often undertaken a use of “some” according to the first meaning (as in the logical square); a meaning which, according to Vasil’ev, is more psychological, heuristic and dynamical than logical[136]. But “some, and perhaps all”, because of the “perhaps” element (explicit or implicit), is a so-called “undetermined judgement” (неопределенные суждения, *neopredelonnye suzhdenija*): and here Vasil’ev follows Sigwart (1830-1904), for whom such undetermined judgements are not judgements (суждения, *suzhdenija*), but *attempted* judgements (попытки суждениий, *popytki suzhdenii*) (their value is dynamical, heuristic, not purely logical). So Vasil’ev shows that this use of “some” (i.e. “some, and perhaps all”) is conceptually dangerous, in so much as it loses the fact that, logically speaking, not one but *two* heterogeneous logical ways are possible here. And there is no reason to avoid logically distinguishing the two (and also studying the second). So, the limit of the square of opposition, based on “some, and perhaps all”, is that it is only fit for undetermined judgements (which are, epistemologically speaking, more empirical than scientific). The explanation for this accepted confusion (and oblivion) relies, according to him, on the fact that, fundamentally, there are *two different models of logical “opposition”* (противоположность, *protivopolozhnost’*), having each a different geometrical shape: a square one, the traditional “square of oppositions” (квадрат противоположностей, *kvadrat protivopolozhnostei*) (the logical square), which gives a logic of the “excluded third” (логика

Selected works) (in Russian). I wish to thank here Professor Sergei Odintsov, who helped me obtain the Russian original texts of Vasil’ev. I also am grateful to Professor Vladimir Vasyukov for several exchanges.

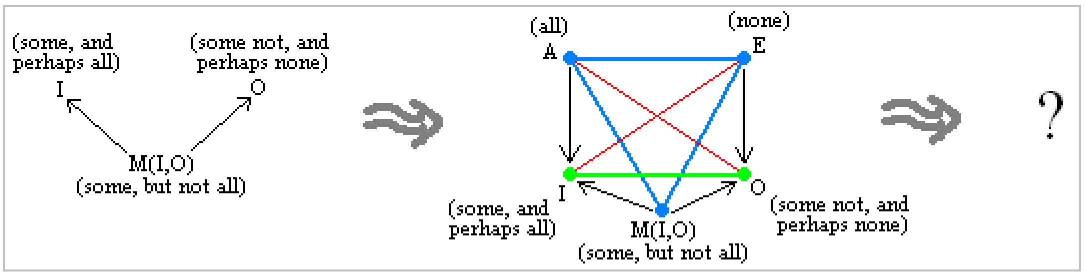
[136] Though this meaning (“some, and perhaps all”) is the one it has even now in contemporary logic and mathematics (i.e. ∃).

исключенного третьего, *logika iskljuchennogo tret'ego*) (this is standard Aristotelian logic), and here affirmative particulars and negative particulars *are distinguished*, but they both are only undetermined judgements (cf. next figure); and a triangular one, the "triangle of oppositions" (треугольник противоположностей, treugol'nik protivopolozhnostei), where affirmative particulars (частноутвердительные, *chastnoutberditel'nye*) and negative particulars (частноотрицательные, *chastnootritzatel'nye*) *are one and the same* (Vasil'ev gives a demonstration of the fact that the opposition involved in the triangle is contrariety). But if the square model, as we said, gives a logic of the excluded third (or "excluded middle"), the triangular model, where no couple of the three vertices (of the triangle) can be true at the same time, and where always one among the three must be true, gives a logic of the "excluded fourth" (логика исключенного четвертого, *logika iskljuchennogo chetvertogo*) (this logic being totally unknown at Vasil'ev's time).



Vasil'ev claims that the first of these two logics is the logic of the "judgements on facts" (фактические суждения, *fakticheskie suzhdenija*), while the second one is the logic of the "judgements on concepts" (суждения о понятиях, *suzhdenija o ponjatijakh*)[137]. This strange point will turn out to be very important from the point of view of Vasil'ev's further philosophy of logic.

An essential and strange point, capital for our purpose, is that Vasil'ev further remarks that the new logical position (i.e. "M(I,O)") introduced by him implies each of the previous ones (i.e. "I" and "O") in the logical square.



[137] On this subject, cf. our paper on the Vasil'evian logic IL2 (2007).

But here one must remark – this point is important – that whereas the square (of opposition) gives a logical model, the triangle (of contrarieties) as such gives no new geometrical-logical model, at least in Vasil'ev's own work (of this we will talk in more detail in ch. 8 *infra*).

In the rest of this chapter, we will recall the use Vasil'ev makes of this first discovery of his.

07.02. "Non-Aristotelian imaginary logic"[138] (1912-1913): Vasil'ev *sive* Lobachevski

Starting from the observation that there is no scientific proof that logic is one (and one only), and in order to clarify what increasingly seems to him, after his study on "some", an exciting new start in logic, he will have, in his most famous paper, the (bright) idea of interpreting such strange, new non-Aristotelian possibilities of logic as being closely analogous to the paradoxical non-Euclidean possibilities discovered less than a century earlier (around 1826), in geometry, by Gauss, Lobachevski and Bolyai. Vasil'ev's logical-geometrical analogy will be very serious – not just a metaphor –, taking three progressive shapes: an analogy of form, of content and of interpretation.

07.02.01. The analogy of form. The philosophical Vasil'evian project of an "imaginary, non-Aristotelian logic"

Vasil'ev's analogy of form says that in the same way as there are, in geometry, new possibilities beyond Euclid's geometry, there are in logic new possibilities beyond Aristotle's logic. Relying on the formal model of non-Euclidean geometries (mainly the one of Lobachevski, his compatriot), Vasil'ev tried, yet remaining inside the syllogistic pre-Frege-Russell frame, to break from Aristotelian logic – a logic bound, among others, to the principles of identity, of non-contradiction, of the excluded third and of sufficient reason (закон тождества, противоречия, исключенного третьего и достаточного основания, *zakon tozhdestva, protivorechija, iskljuchennogo tret'ego i dostatochnogo osnovanija*)[139]. Such logics (the "imaginary logics", воображаемая логика, *voobrazhaemaja logika*, or, later, the "logical *n*-dimensional system", логический систем n-измерений, *logicheskii sistem n-izmerenii*) were meant to portray realms that are different from the "real world" we live in (a

[138] Vasil'ev, N.A., "Voobrazhaemaja (nearistoteleva) logika" (*"Imaginary (non-Aristotelian) logic"*), in *Selected Works*, *op. cit.*

[139] Truly speaking the principle of identity also comes from the propositional logic of the Stoics: $p \to p$.

world ruled, Vasil'ev acknowledges it, by Aristotelian logic). Such are, among others, the worlds of imagination, dream, and fiction in general (and there are more)[140]. Again, remarking that, as was the case for geometry, there is no decent proof showing that our logic is *the only one*, Vasil'ev thinks of the imaginary logic he hypothetically explores as being irreducibly (i.e. axiomatically) incompatible with our ordinary, normal logic. More generally, he thinks that from the point of view of each of the yet unknown but possible logical worlds, the other ones must be false (not real), but that logically speaking each of these points of view could be endorsed (in a suitable "imaginary world", воображаемый мир, *voobrazhaemyi mir*)[141]. So, considering the principle of non-contradiction as being the logical analogue to the Euclidean geometrical (problematic) fifth postulate of the parallels (i.e. an arbitrary presupposition, axiomatically independent from the other fundamental geometrical axioms and thus avoidable), he builds, in 1912, an "imaginary (non-Aristotelian) logic" (воображаемая (неаристотелева) логика, *voobrazhaemaja (nearistotelava) logika*), a logic ruled – among others – by the "principle of the excluded fourth" (as he discovered it possible in his previous study on "some") and in which the principle of non-contradiction is not a universally valid axiom or law anymore. He will later claim that this is further generalizable to a series of logical systems of the "excluded *n*-th", the "imaginary *n*-dimensional logics". But let us see now more precisely the starting version of imaginary logic.

07.02.02. The analogy of content. 'Indifferent' judgements and syllogisms

Vasil'ev's analogy of content says that, exactly in the same way as, from Euclidean to non-Euclidean geometry, one passes from a geometry of two possible kinds of straight lines (parallel or secant to a given straight line) to a geometry of three possible kinds of straight lines (parallel, secant, or not parallel *but* not secant to a given straight line), from Aristotelian to non-Aristotelian (i.e. imaginary) logics, one passes from a logic of two possible judgements (affirmative and negative) to a logic of three possible judgements (affirmative, negative and

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[140] This view shares interesting parallels with I. Matte Blanco's theory of the Freudian unconscious in terms of a "bi-logic" based on mental *n*-dimensional spaces.

[141] This sort of "indexical relativism" reminds of D. Lewis' "modal realism" ("in every possible world – there are many! – people think that their own world is the only real one, the other being only possible in fiction"). More than this, we agree on this point ("non-normal points") with G. Priest ("Vasil'ev and Imaginary logic", in: *History and Philosophy of Logics*, N. 21, 2000, p.144), according to whom Vasil'ev could be seen as a true precursor of the nowadays very popular idea that there are "impossible possible worlds" (an issue much debated between D. Lewis, who rejects such worlds, and some paraconsistent philosophers and logicians, among which notably Priest himself, cf. *Notre Dame Journal of Formal Logic* **38**, n.4, Fall 1997, Special Issue on Impossible Worlds. I wish to thank Alessandro Facchini for the reference. For a more technical introduction to the

"indifferent")[142]. One of the most impressive features, from our point of view, of imaginary logic is thus that it is a logic which is also able, in some sense, to cope logically with real contradictions (противоречащие, *protivorechashchie*) (and therefore, in a way, speaking a tongue unknown to Vasil'ev, a "paraconsistent" logic). In Vasil'ev's mind, building syllogistic systems of imaginary logics is not only a strange possibility, but rather a scientifically *necessary* and exciting move in order to determine what can be changed in classical logic, without loss of "logicity": this unchangeable residual part will be the so-called "meta-logic" (металогика, *metalogika*). In order to seize this invariant part of logic, Vasil'ev advocates infinite variations of the accepted axioms. The result will be made of some positive kind of logic (a logic without negations, i.e. ruled by the "principle of the excluded second") as well as of a (small) hypothetical invariant part of Aristotelian logic (a part of the logic of the excluded third), which is unchangeable (as "Absolute Geometry" is with respect to all possible geometries)[143]. For these reasons, and especially for having connected the idea of exploring the "logically unchangeable" by means of systematically developing the "logically non-standard", Vasil'ev is rightly considered as one of the very first forerunners of "deviant logics" in general and, possibly (the question remains open), of paraconsistent and multi-valued logics in particular[144].

Remark, however, that if Vasil'ev's achievement has been to successfully reject (i.e. without loss of "logicity") in his imaginary logic the "principle of non-contradiction" (закон непротиворечия, *zakon neprotivorechija*) – thus proving the logical relativity of such a "principle" (logic can speak about *contradictory objects*) – he keeps as metalogical (i.e. absolute) the "principle of non *self*-contradiction" (закон несамопротиворечия, *zakon nesamoprotivorechija*) (logic can't formulate judgements in a *contradictory way*)[145].

"impossible possible worlds" (of Possible Worlds Semantics) cf. the excellent book: Priest, G., *An Introduction to Non-Classical Logic*, Cambridge, CUP, 2001 (cf. especially ch. 4).

[142] Indifferent judgements are sometimes also called "contradictory" judgements.

[143] This point, i.e. determining what Vasil'evian "metalogic" exactly is, is debated, cf. Smirnov, "Logicheskie idei N.A. Vasil'eva i sovremennaja logika" (1989), in: Vasil'ev, N.A., *Voobrazhaemaja logika. Izbrannye trudy*, Moscow, Nauka.

[144] In some sense, the Vasil'evian metalogic could be seen as a kind of precursor to approaches like the one of J.-Y. Béziau, i.e. "Universal Logic" (or "UL", cf. ch.10), although UL is something much more pluralistic, much more dynamic. But the spirit of it, i.e. the attraction to the non-standard, motivated as being an access to the more profound, is very akin to UL. Vasil'ev discusses such a point in his way while criticising Husserl's *Logische Untersuchungen* (cf. Vasil'ev, N.A., "Logika i metalogika", p. 95-98).

[145] This can be seen as somehow reflecting the fact that in contemporary logics, and paradigmatically in the da Costa's systems C_n, $1 \leq n \leq \omega$, paraconsistency is often obtained only at the object level, whereas at the meta-level of the paraconsistent system, one still uses classical non-paraconsistent logic. In "N.A. Vasil'év: a forerunner of paraconsistent logic" (in: *Philosophia Naturalis*, vol. 21, no. 2-4, 1984†), Arruda discusses the possible relations of Vasil'ev's thought with the different kinds of (then) existing paraconsistent systems (namely those of N.C.A. da Costa, R. Routley and R.K. Meyer, etc.).

For the working logician, all this leaves the problem open of having an intuitive representation of such logical, coherent but strange possibilities.

07.02.03. The "analogy of interpretation" as a further refinement of the project: the Vasil'evian experimental 3 self-interpretations of his own non-standard logic in terms of old standard logic

As we saw, Vasil'ev achieved for syllogistics what Gauss, Lobachevski and Bolyai did for geometry. He showed that, paradoxically, some logical "fundamental" laws can be changed without loss of "logicity"[146], and that this fact opens the doors to quite strange realms. Now, going back to the geometrical source of revolutionary inspiration, because non-Euclidean geometry can be expressed, by a suitable translation, inside Euclidean geometry itself, as did Beltrami (who linked Lobachevski's geometry with the "pseudo-sphere") and Poincaré (who linked Riemann's geometry with the "Euclidean sphere"), Vasil'ev thinks there should be as well, in the logical plan, a way to express non-Aristotelian imaginary logic inside Aristotelian logic[147]. And, indeed, by (a third successful) analogy with the aforementioned geometrical interpretations of non-Euclidean geometry in terms of Euclidean geometry, Vasil'ev *successfully* gives three interpretations of his non-Aristotelian "non-earthly" (i.e. imaginary) logic in terms of normal (Aristotelian) "earthly" logic.

The main idea of this is that, while Aristotelian logic is the "logic of things", imaginary logic, in our world, is embodied by the "logic of concepts", which is different from the logic of facts (or things), as revealed by his first study on "some" (cf. *supra*). A concrete spatiotemporal fact in the world can only be necessary, impossible *or contingent* (excluded fourth). Of this paradoxical embodiment (imaginary logic plunged into the real world!), Vasil'ev proposes three forms, all related to the conceptual abstract plan. The three "earthly" (or realist) interpretations Vasil'ev gives of his non-earthly (or imaginary) logic pertain to: (1) modal logic, (2) the logic of absolute resemblance and difference and (3) the general logic of concepts, of which he gives very interesting, precise semantic elements. About the latter two

[146] This fact was rather unknown at that time (and until recently, many logicians used to laugh ignorantly at those speaking of non-standard features like paraconsistency). Only Łukasiewicz had in some way demonstrated the same thing (i.e. the fact that the principle of non-contradiction is, logically speaking, an independent axiom) in his 1910 study, *Ozasadzie sprzecznosci u Arystotelesa* (*On the principle of contradiction by Aristotle*).

[147] Needless to say, such formal translations are very important. They were related, as it later appeared, to consistency-proofs and completeness considerations premonitory of some future tricky proof strategies, such as Gödel's 1931 one (on all this cf. the beautiful study: Nagel E. and Newman J.R., *Gödel's Proof*, New York, New York University Press, 1958).

logics, Vasil'ev remarks that their axiomatic system is in fact very similar to, but slightly different from, that, canonical, of the fundamental abstract version of imaginary logic[148].

However, one must understand that while the "earthly" interpretations of imaginary logic will give the impression that we remain inside old standard logic (just changing the object of application, now less abstract, but keeping the old logic at the metalevel), this impression is highly misleading: axiomatically speaking, imaginary logic is and remains *highly counterintuitive* and plainly non-standard, exactly as much as non-Euclidean geometry remains non-standard despite Beltrami's and Poincaré's interpretations. Vasil'ev's shocking (but coherent) axioms and theorems can be used as such, without any "earthly" make-up. In our aforementioned study on these topics, we show how the standard geometrical Euler-Venn diagrams must be "conceptually twisted" in order to make the geometrical expression of logic function again. For short, because one has, there, the law "$\alpha \rightarrow \beta \Rightarrow \neg\alpha \rightarrow \neg\beta$" (instead of "$\alpha \rightarrow \beta \Rightarrow \neg\beta \rightarrow \neg\alpha$"), one has to graphically express negation not by the "complement" (i.e. the "other side", the "outside" of the negated circle) but by a "mirror-image" (and this implies several changes).

07.03. Some remarks on the importance of Vasil'ev's thought

The first thing to be said is that his logical ideas, grounded on a strong philosophical reflection (that of the existence of worlds other than ours), were very interesting, but they were largely ignored or underestimated by other logicians[149]. This was mainly due to Vasil'ev's formal attachment to Aristotelian and Scholastic syllogistics. The Russian logician "forgot", despite an explicit project he had in that direction, to reformulate his thought in the logical language of modernity, the then young language of Boole-Frege-Russell-Whitehead[150].

Another reason for being almost forgotten by the history of logic (this big injustice has been repaired in the last few years only)[151] is the pure historical contingency of the political

[148] We studied in some detail the third form of Vasil'evian expression of imaginary logic in terms of earthly logic in our aforementioned paper on the Vasil'evian logic IL2. There we give our own geometrical translation – a graphical decision procedure – of Vasil'ev's logic.

[149] A reflection, of course, pioneered since long in the Western thought and nowadays brilliantly represented by thinkers like H. Everett and D. Lewis (on this argument cf. A. Moretti, "Trois ontologies extrêmales: E. Severino, D. Lewis et H. Everett", (to be submitted)).

[150] This is related by Smirnov.

[151] Thanks to the work of some researchers, notably the work of V. Bazhanov (as for making Vasil'ev known outside Russia). One must also mention the precious, technically valuable work of a group of Russian logicians, who work systematically on the rigorous translation, in logical contemporary terms, of the different syllogistic systems of Vasil'ev (cf. Markin, Zaitzev, …). For an historical look on all this, cf. Moretti, A., "A graphical

events: Vasil'ev wrote in Russian and published his papers in Russia, at a period (the years 1910-1913) quickly followed by the Russian revolution (1917) and the creation of the Soviet Union (1917-1991). Vasil'ev lived during the Soviet period and died in 1940 during World War II. Until now, few people (except the Russian philosophers and logicians) have *really* read him, important pieces of his work are still not translated.

An important remark from a conceptual point of view is that his idea of relying on geometry (that is: on the geometrical paradigm revolutions) in order to make a revolution in logic was simply brilliant. But, as we are going to see in the next chapter, he seemingly happened to choose a mistaken part of geometry (non-Euclidean geometry) among all those (topology, *n*-dimensional geometry, …) that could have been invoked at his times (some decades later, a new geometrical revolution will be the birth of "fractal geometry")[152].

Finally, his stress on the triangle of contrarieties and on "some, but not all" was a very good remark, potentially full of consequences for logic (as we are going to see in ch. 8 *infra*). Incidentally, Lacan's innovative concept "not all" (cf. ch. 6, "la femme n'est pas toute") could probably be expressed in some way using Vasil'ev's terms. But he missed a small geometrical remark that, although seemingly trivial, was *essential* to go further into the exploration of the space of opposition, avoiding thus the complete discovery of something logically very important (a successor of the logical square), as we are now going to see in the next chapter.

decision procedure and some paraconsistent theorems for the Vasil'evian logic IL2", J.-Y Béziau and W.A. Carnielli (eds.), *Handbook of Paraconsistency*, Elsevier, 2007.

[152] It is known that J. Hintikka sees his "IF logic", based on game theory and game-theoretical semantics, as a logic of "order 2 minus something" (i.e. a logic of fractal order!).

08

SESMAT'S AND BLANCHE'S PUZZLING DISCOVERY: THE "LOGICAL HEXAGON (OF OPPOSITION)"

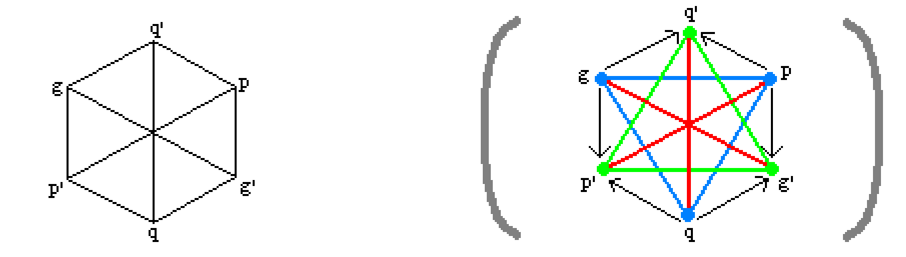
In this chapter we recall briefly the first major change of opposition theory since Aristotle, that is the "logical hexagon" (or "hexagon of opposition"), which is the successor of the logical square. Its discovery, made independently (and almost simultaneously) by the French logicians Augustin Sesmat (1951) and Robert Blanché (1953) is not as well known by contemporary logicians and philosophers as it would deserve. This amazing and very elegant structure is a conservative extension of the logical square, much more interesting than the latter from the mathematical point of view, for it bears more symmetries (three instead of only one). As we will show (in ch. 9), all that can be expressed by a logical square can be expressed by a logical hexagon, which makes the latter much more interesting. However, the appearance of this structure remains somewhat mysterious, and generates many new questions on the deep, formal nature of opposition.

08.01. Augustin Sesmat's pioneering work (1951)[153]

As we saw in ch. 7 *supra*, Vasil'ev had moved from Aristotle in order to accept more clearly the existence of an alternative to the square approach to opposition, namely a triangular approach (the triangle of contrariety). But this hadn't led him too far, despite the excitement for his "imaginary logic", with regards to elaborating a clear geometrical-logical model. Vasil'ev invoked geometry as a paradigm (Aristotle *sive* Euclid, Vasil'ev *sive* Lobachewski), but geometry did not really play a big part in his systems, as we recalled.

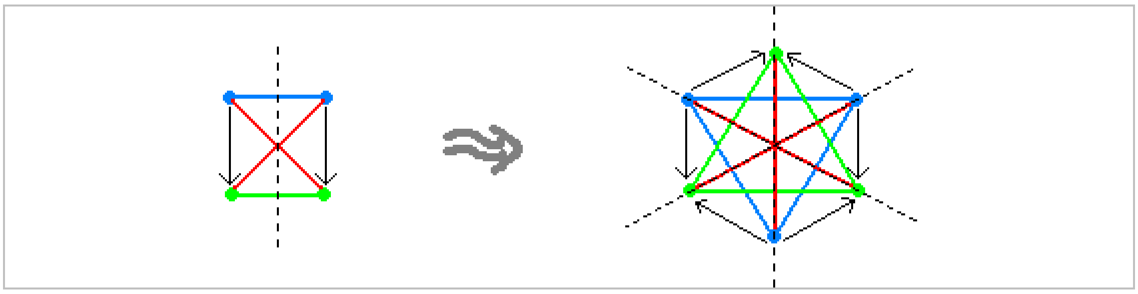
The French logician Augustin Sesmat, in 1951 (seemingly ignoring Vasil'ev, only readable in Russian), proposed a structure which, containing the logical square and the triangle of opposition, completed them in an elegant "logical hexagon"[154]. This was made by adding to the logical square not one, but two new positions: Vasil'ev's bottom one (called by the latter "M(I,O)") plus a top one, the contradictory negation of "M(I,O)".

[153] I owe to J.-Y. Béziau the reference to the works of Sesmat, of which I was totally unaware at the time of my first study on the geometry of opposition (2004), where I only knew part of Blanché's works. I also owe to the French mathematician René Guitart some illuminating discussions about the complex work of Sesmat.
[154] Cf. A. Sesmat, *Logique – II. Les Raisonnements, La Logistique*, Paris, Hermann, 1951.

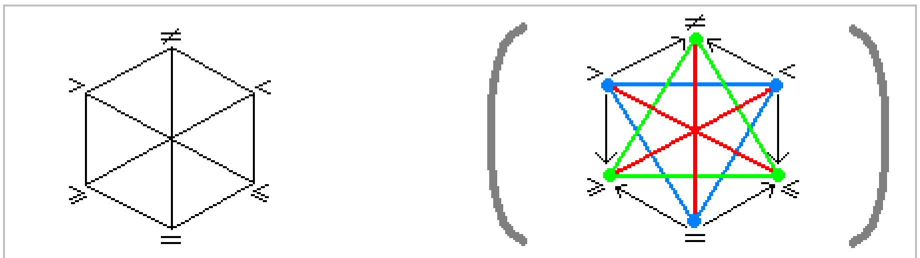


Sesmat's original 1951 figure (adapted from Sesmat)

The justification of this double introduction is clear, *a posteriori*: because, by taking into consideration the triangle of contrarieties, we introduce a fifth point (with respect to the four constituting the square of opposition), we can and must introduce a dual “triangle of subcontrarieties”, made of the two subcontrary points of the square (its basis) plus a new one, the sixth opposition point. The move is simple. But the result is amazing, for this structure not only contains all the previous elements but integrates them in a mathematical form which has three symmetry axes (whereas the logical square had just one).



Of this new oppositional structure Sesmat gives some examples of application (p. 412).

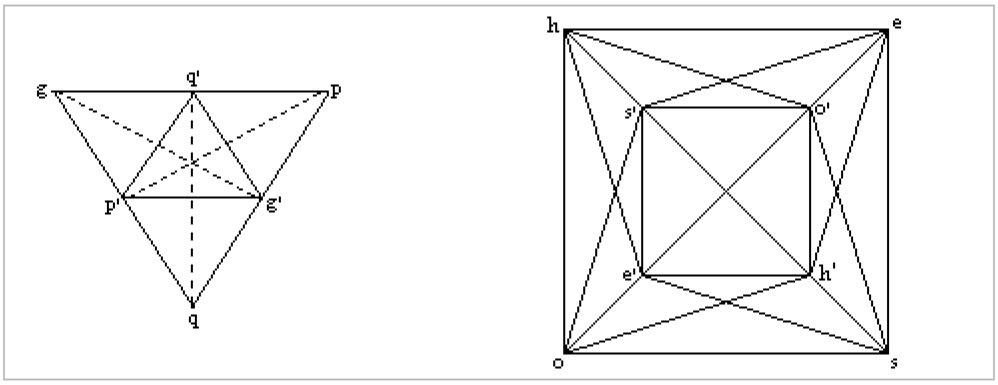


Sesmats hexagon of comparisons (adapted from Sesmat)

Vasil’ev having missed the introduction of the triangle of subcontrarieties, more than 2000 years after Aristotle, it is Sesmat who thus seems to be the first person to have thought out a real alternative (a real geometrical-logical model) to the logical square, that is the logical hexagon.

Notice that by an historical injustice, this structure is almost always referred to as being “Blanché’s hexagon” (or even “Blanché’s star”). Both authors (as Vasil’ev before them) are victims of the fact that they published their discoveries in a language that, some decades later, would have turned out to be “non-scientific” (i.e. French). However, Sesmat suffered a deeper injustice, his work being often ignored even by the French-speaking people (this is probably due to the fact that his work is seemingly contained in only one book, whereas Blanché’s mentions of the logical hexagon are spread over several books and papers, some of them in international journals). Another possible reason is that Sesmat’s figures are less clear than Blanché’s (Sesmat does not represent the arrows).

A very important point is that Sesmat calls his logical hexagon a “bi-triangular scheme” (p. 438) (cf. left side of the next figure). So, notice that, additionally, Sesmat also thinks of a “bi-tetrahedric scheme” (cf. next figure, on the right), for reasons that will appear later, of this we will speak extensively in ch. 11.



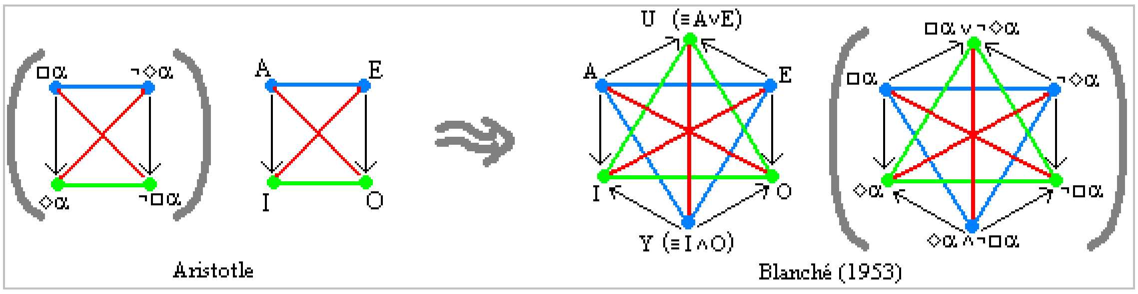
His commentary to that is that “unfortunately such a bi-tetrahedric scheme would need a tri-dimensional model” (p. 445).

This could have seemed to be a possible open door for further research (general extensions of the logical square). But at this point Sesmat writes (p. 446): “As for the schemes proper to the systems with more than 4 positive [formulae], if we can (possibly) conceive of them, we can in no way imagine them, and even less use them”.

Having recalled very quickly some of Sesmat’s valuable logical discoveries and innovations (there would be more to say), let us now turn to the other discoverer of the logical hexagon.

08.02. Robert Blanché’s classical discovery (1953)

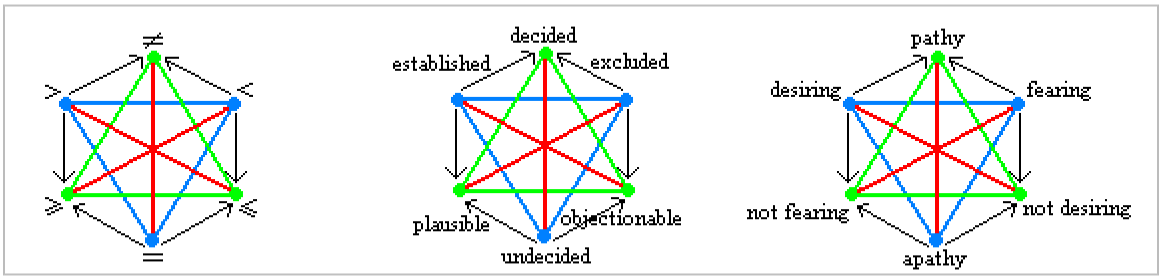
Whereas Vasil’ev (1910) and Jespersen (1917) refer to the mysterious (not further developed by them) “triangle of contrarieties” in order to, somehow, fight against the logical square (but they provide no new geometrical-logical model for doing this desired logical revolution), the French logician and philosopher Robert Blanché, independently from Sesmat, but in a much similar way, discovers in 1953 that it is possible to combine the traditional logical square and the mysterious triangle of contrarieties into a new “logical hexagon” (or “hexagon of contrarieties”): for this, one only has to deduce, from the triangle of contrarieties, its dual “triangle of subcontrarieties” (this is the new move)[155]. The logical hexagon is simply the figure uniting these two triangles, with the emergence of subalternation (i.e. implication) arrows between any contrary term and the two subcontrary terms not symmetrical to it by central symmetry (these arrows of subalternation constitute the hexagon’s perimeter).



Again, as in the case of Sesmat’s discovery of the same geometrical-logical object, one remarkable feature is that this logical hexagon contains the logical square as one of its particular cases. And, again, whereas the logical square has only one symmetry axis (the vertical one, making the left side symmetrical to the right side), the logical hexagon has three symmetry axis: the vertical one (of the logical square), plus two new oblique ones (belonging to two new logical squares).

Remark that such symmetries are clear on Blanché’s representation, not so in Sesmat’s (Sesmat omits to represent the subalternations by arrows – more generally Sesmat omits to distinguish by visual devices the different qualities of opposition).

Like Sesmat slightly before him, Blanché discovers some interesting decorations of the logical hexagon (notably Sesmat’s one about orders, plus others about more daily notions, such as “deciding”, “obligation”, “resolution” and “fear”).



Remark that these cover both very formal logical-mathematical notions and natural language common notions.

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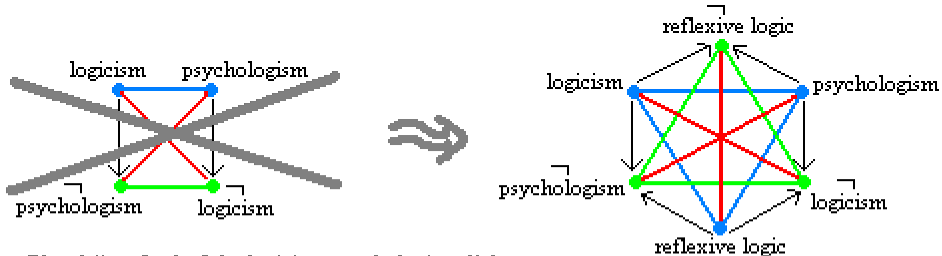
[155] Cf. R. Blanché, “Sur l’opposition des concepts”, *Theoria*, 19 (1953); “Opposition et négation”, *Revue Philosophique*, 167 (1957).

All this led Blanché to develop general philosophical considerations on the nature of opposition, conceived as a very good key to understand conceptual networks in general (Blanché, in this sense, seems related to the French "structuralist" movement, pioneered by the Swiss linguist Saussure and leading, in mathematics, to Bourbaki, cf. ch.06.02 *supra*)[156]. Nevertheless, as for logic itself, Blanché defended a philosophical, original approach to it (mainly against plain Anglo-American "logicism") which he called "reflexive logic" ("reflexive" being understood in philosophical rather than in algebraic terms). Blanché claims that this structure, the logical hexagon, is some kind of transcendental condition of human thought[157]. It is funny (or touching) to see that Blanché explained his own position in a way that could (but hasn't) be described by a logical hexagon: as a matter of fact, he criticises (in a constructive, polite way) the "logicist" position (which favours the mainly mathematical approach to logic) but stresses that this does not make of him an adept of "psychologism". In other words, Blanché criticises the classical dichotomy opposing (since Frege) logicians to psychologists, by summoning the existence of a third position (his "reflexive logic"). And this is exactly the kind of conceptual clarification a logical hexagon can perform very directly.



Blanché's refusal of the logicism-psychologism dichotomy (1966)

Another element of reflection for him on this subject, still in the aforementioned structuralist vein, is the remark that over the many decorations of the logical hexagon that he proposes, there are often empty places with respect to natural language. Logical hexagons can be seen as a way of clarifying our conceptuality beyond the necessary partial randomness (due to its historical genesis) of the human language.

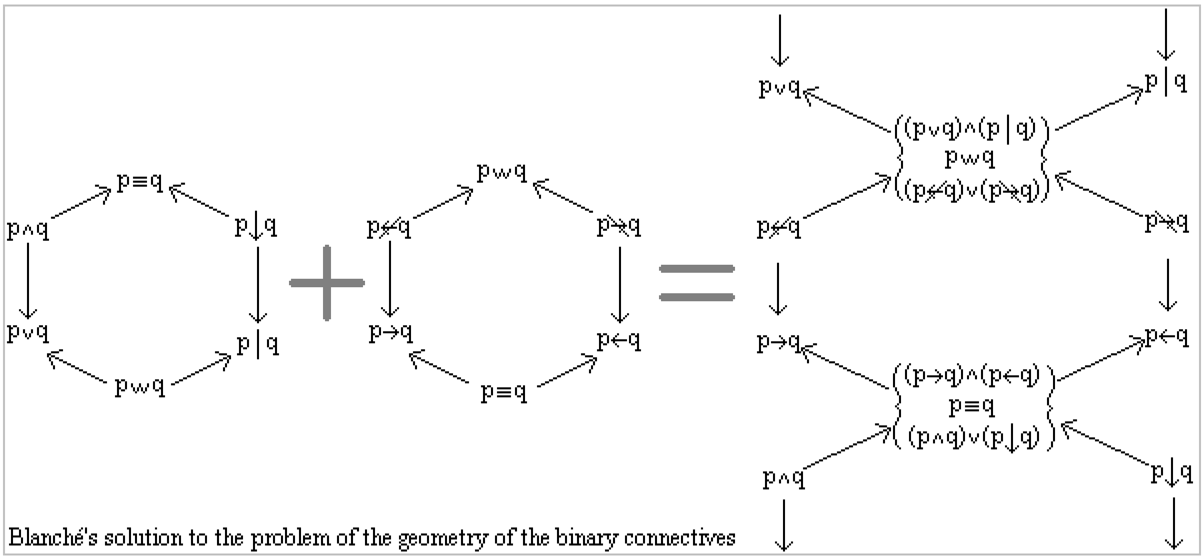
As for Blanché's own applications (i.e. decorations) of the logical hexagon, one in particular deserves specific attention. It concerns the binary connectives of the propositional calculus.

[156] In this respect one will remark the title of his main book on opposition: R. Blanché, *Structures intellectuelles. Essai sur l'organisation systématique des concepts*, Paris, Vrin, 1966.
[157] Cf. R. Blanché, *Raison et discours. Défense de la logique réflexive*, Paris, Vrin, 1967.

p	q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
T	T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F
		⊤	∨	←	p	→	q	≡	∧	|	w	¬q	↛	¬p	↚	↓	⊥

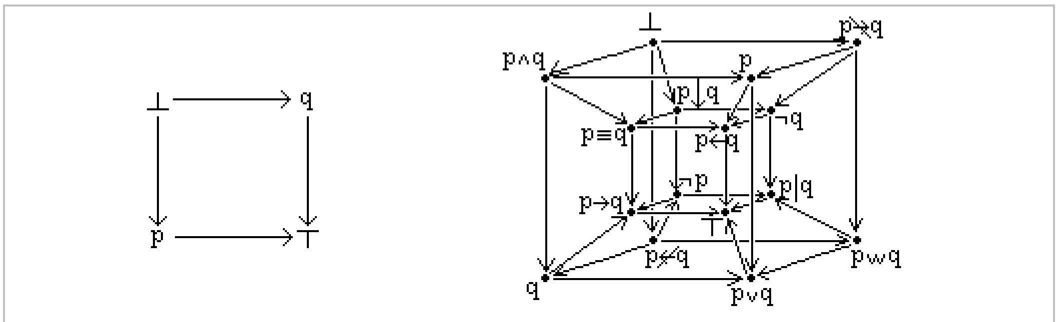
The complete list of the binary (bivalent) propositional connectives

As a matter of fact, two particular decorations, relative to 10 out of the sixteen binary connectives of the propositional logic ("the 10 major connectives", according to him), gave Blanché the idea, because these two hexagons have two elements in common, to join them into some kind of chain.



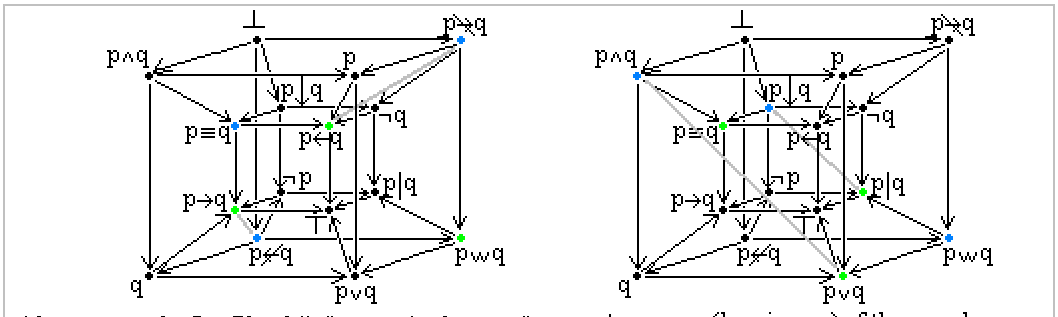
Blanché's solution to the problem of the geometry of the binary connectives

By this move he (quite rightly) claimed to be able to offer a better geometrical-logical understanding of the logical space of the binary connectives, of which previous models had been given, as we saw (cf. ch.5 *supra*), by Bochenski, Piaget, Gottschalk and Woodger.



Woodger's model for the binary connectives

If Blanché's solution has the defect, by comparison with that of Woodgers (to which many contemporary logicians come again and again by re-discoveries), that it takes into account only 10 of the 16 connectives (but, again, he claims these are the 10 most important ones), Blanché's solution is nevertheless (much) better in so far as it is able to express *all* the opposition relations at hand between these 10 (Woodger's model is radically unable to do it, as the following figures show).



with respect to the first Blanché's "connective hexagon" two arrows (here in grey) are forgotten

two arrows (here in grey) of the second Blanché' "connective hexagon" are missing here

On this topic we will come back later (cf. ch. 17) [158].

One last remark: in his fundamental 1953 paper, Blanché mentions Sesmat's 1951 book. But Blanché does not seem to ever discuss (unless we badly read him) the possibility of having, following Sesmat's bi-tetrahedric scheme, some kind of 4-opposition (or higher). Maybe he did not really read him (though this seems somehow hard to believe).

08.03. Final remarks

In order to go from the logical square to the logical hexagon, what has apparently been an obstacle during more than 2000 years is the "U" position (according to Blanché's terminology). So, before comparing Sesmat's and Blanché's approaches, let us have a quick look at the possible reasons for such a blockade, at least in the cases of Aristotle himself and of Vasil'ev.

08.03.01. Possible reasons why Aristotle didn't see the "U" position

In his remarkable study[159], Gardies, after having presented Blanché's discovery, gives an element of possible philosophical-historical explanation of the fact that Aristotle seemingly (at least according to the writings of his that survived until our time) remained in some kind of indecision with respect to the formal definition of the notion of "possibility" (unilateral and bilateral, cf. ch. 4 *supra*). The main point seems to be that, of the two logical positions to be integrated in order to pass from the square to the hexagon, one was fully compatible with

[158] I owe to J.-Y. Béziau the reference to Woodgers' discovery (notice however that Blanché does not mention Woodgers).

[159] J.-L. Gardies, *Essai sur la logique des modalités*, Paris, PUF, 1979. A book to which we are much indebted. It is there that we learned for the first time of the existence (and the way of functioning) of the "logical hexagon".

Aristotle's philosophy: this was the "Y" position, the place of "indeterminism" (free bilateral choice). But the other logical position, the "U" one, being the place of "necessity" (or "determinism"), may have seemed fully incompatible with Aristotle, who refused it. For, in the very act of creating the fundamental logical laws of the logical square, he was fighting against the necessitarianism (i.e. the determinism) of the Megarian school. So one can conjecture (as Gardies) that Aristotle took just one of the two twin notions (the bilateral possible and the determined position, Y and U, so to say): so that he was obliged to remain in the irresolution of having the logical square and a triangle of contrariety, but not the triangle of subcontrariety: the road to the logical hexagon was blocked[160].

08.03.02. Possible reasons why Vasil'ev didn't see the "U" position

In some sense, we may say that Vasil'ev had almost everything in hand to discover the logical hexagon, everything except the "U" position (following Blanché's terminology). The fact that he didn't find this logical position can perhaps be explained by remembering that he used the Aristotelian logic, which, as we said, is a fragment of predicate calculus, whereas what could have better directed him, for this problem, would have been the propositional calculus (again: "U" and "Y" are typically logical propositional objects, the former being a disjunction and the latter being a conjunction).

One interesting remark is that according to Vasil'ev the "undetermined judgements" (i.e. "some and perhaps all", etc.) give a dynamic character to the logical thought (they ask for further information). This point deserves to be underlined, for it possibly says (beyond Vasil'ev's own words) that the logical hexagon may have a certain amount of mixture of static and dynamic character. To this point we will return later (cf. ch. 9, on Gallais, and 24, on "opposition dynamics").

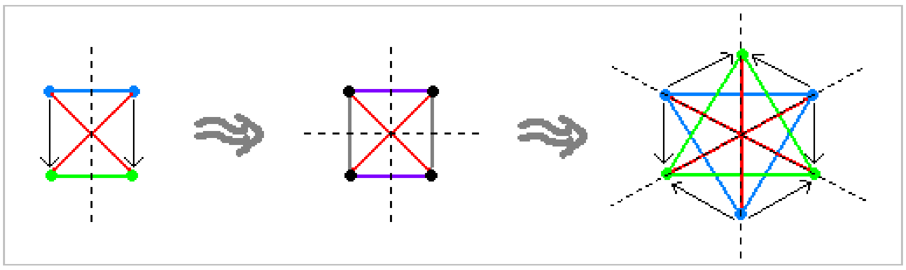
08.03.03. Did Gottschalk (co-)discover the logical hexagon in 1953?

There is in fact another candidate for the title of (co-)discoverer of the logical hexagon. As a matter of fact, on page 195 of his 1953 paper founding the theory of quaternality (cf. § 05.03.01 *supra*) Gottschalk, unexpectedly, extends all of a sudden the construction of a logical square for modalities – conceived by him, as we saw, as a particular

[160] Of course, another reason is that the "Y" and the "U" positions are in some sense essentially propositional (the former being a disjunction whereas the latter is a conjunction) and Aristotle's logic is predicative, and not propositional.

case of the more abstract structure of “square of quaternality” – by adding to it 2 points and 4 arrows to it. He says: “above the common terminal point of rising convergent arrows from the upper vertices [i.e. of the square] write: □*p*∨□¬*p*, *p* is noncontingent. Below the common initial point of rising divergent arrows to the lower vertices [i.e. of the square] write: ◊*p*∧◊¬*p*, *p* is contingent. Additional entries in this square are the same as in the square of quaternality for quantifiers together with “if lower is true than upper is true” along the oblique arrows”[161]. Gottschalk himself offers no drawings but gives hints to make the reader draw them himself/herself. It should look like the following (right side).

Gottschalk says nothing more on this addition of two extra points and four extra arrows. So the question raised is to know whether he presents there a discovery of his own, or whether he relies on the results of Sesmat and/or Blanché. In any case, one should remark at least four things: (1) there is no mention, in this paper about “quaternality-like relations”, touching the two extra points, other than the arrow-relations; (2) such a supposedly consciously discovered logical hexagon would fit rather badly with the theory of quaternality: it breaks the quaternary beauty of Gottschalk’s allegedly universal structure; (3) this hexagonal extension is only mentioned in the case of the application of the quaternality square to modalities; (4) he never mentions the gain of a third symmetry axis. In the light of these four considerations I would propend to think that: (i) he didn’t assume the logical hexagon as such, not having spoken of the C- or R-relations (Gottschalk’s terminology, cf. § 05.03.01 *supra*) possibly characterising it, he never even wrote (in that paper) the word “hexagon”; (ii) if he was conscious of the possibility of this new structure (which is not sure), he seemingly did not want to consider the issue, possibly frightened by its fitting not so well with the quaternary universe of his new theory.



[161] Gottschalk, W. H., “The Theory of Quaternality”, *The Journal of Symbolic Logic*, Vol. 18, No. 3, Sept. 1953.

Of course, one should read more materials of and on Gottschalk's thought, which we will not do here.

08.03.04. Comparing Sesmat and Blanché

Sesmat seems to look further away than Blanché, regarding the possibility of augmenting the number of opponents in a formalised opposition. In this respect he proposes the bi-tetrahedric scheme (opposition for four terms), but he immediately closes this door, judging it geometrically not viable. Blanché seems, on his end, to be more likely to defend the idea that the logical hexagon is THE transcendental form of opposition as such (for it is the final perfection of the logical square, which was already almost the "transcendental" form of rationality). On the other hand, Blanché seems to be a little bit more far-reaching than Sesmat regarding the idea of composing tri-dimensionally logical hexagons (he did it for two, even if not exactly tri-dimensionally, in his chain, made of two logical hexagons, for "the 10 most important binary connectives"). But he does not go beyond this idea of two hexagons (giving a "two-rings" chain). On these questions we will come back on ch. 11 *infra*.

09.
SOME KNOWN APPLICATIONS OF THE LOGICAL HEXAGON

Having recalled the main intended meaning of the logical hexagon (or hexagon of opposition) according to its two creators (Sesmat and Blanché), and having recalled the applications they imagined of it, we want to try to see if other possible applications have been found so far by other people since then. And in fact, it happens that there are some such other known applications of the logical hexagon.

09.01. Applying the logical hexagon. Introductory remarks

There seem to be relatively few applications of the logical hexagon so far. Historically speaking, in the case of this logical-geometrical structure (by comparison with the square of opposition) there has been less time to find applications. And, furthermore, the logical hexagon is still not very well known: not that many people in the world do read French (or look for old French-speaking journals and books), and neither Sesmat nor Blanché's works have been translated in English so far. Nevertheless, there are some applications of it. An interesting point is that in the case of the hexagon, the applications seem to be more pertinent then in the case of the square (with the square, many "applications" were mislead, due to the great simplicity of the geometry of this figure).

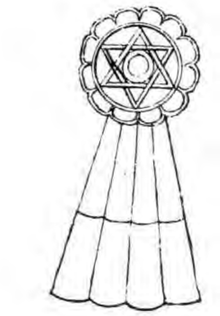
Remark, along with what has just been said, that most of the works analysed here are in French. This is probably simply due to the fact, already mentioned, that the seminal works of Sesmat and Blanché are in French and that they still have not been translated into English (with the notable exception of a standard, widely read work of L. Horn in English on the subject).

In what follows we present schemes (logical hexagons) drawn by us but faithful to the original ones, with only a translation into English of the non-English terms (again, most of the time in French).

09.02. Historical symbolic (pre- / para- philosophic) antecedents

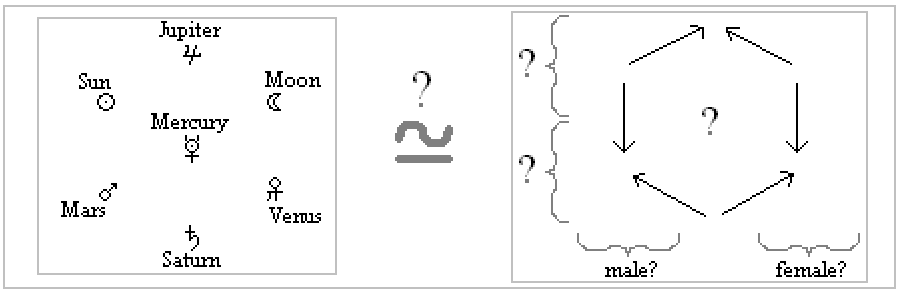
As in the case of the representations of the square oppositions (cf. ch. 4 *supra*), it must be remarked here that the hexagonal geometry of opposition is not unknown at the level of non-strictly philosophical and logical culture. We all remember the horrible images of Nazi persecution of the Jews (who had to wear a "yellow star", a yellow "star of David", made of two interlaced equilateral triangles). This tragic use of this geometrical symbol is however recent, while it was used in many other ways over the past millenia in human history and culture.

The logical hexagon in fact resembles at least three distinct cultural occurrences of this geometrical, very abstract symbol (we surely forget more possible known examples; the three we keep however suffice to make our point clear). The first is the so-called "Seal of Solomon" (in other cases: "Magen David", David's shield). The second is a symbol from American Indians before the Spanish conquest. The third is a Western symbol of esotericism, a symbol of "becoming" (cf. figure)[162].




(Seal of Solomon) (solar symbol in Uxmal, Yucatan) (G. Postel's "pentacle" [sic])

A standard traditional display of the astrological "system of thought" differs from the logical hexagon (despite apparent similarities – it uses an hexagon, or intersects three circles) for it deals with 7 objects (the seven traditional planets)



The central presence of Mercury in it seemingly breaks the possibility of having a logical hexagon (moreover, even without Mercury, one should find "astrological" interpretations for the contrariety triangle {Sun, Moon, Saturn}, for the subcontrariety triangle {Jupiter, Venus,

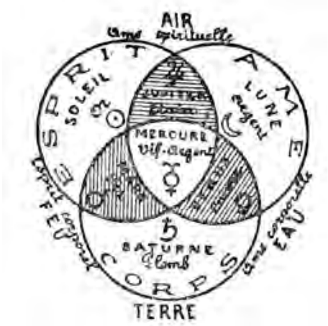
[162] The three images come respectively from: With, O., *Le tarot des imagiers du moyen âge*, Paris, Claude Tchou, 1966 (1924) (ch. III); Goblet Comte d'Alviella, E., *La migration des symboles*, Bruxelles, Louis Musin, 1983 (1891); Wirth, O., *Le symbolisme astrologique*, Paris, Dervy. The next 3 figures also come from the latter.

Mars} and for the six alternated arrows constituting the perimeter of the hypothetical logical hexagon.

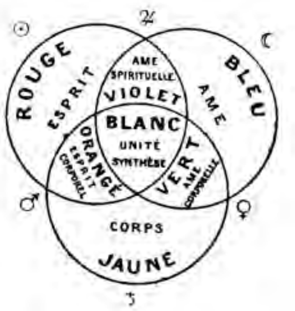
From the Babylonians, *via* the Egyptians, the Greco-Roman ancient world, the Middle Ages to the Renaissance, Astrology has long been considered a science. To play this part, it proposed a (qualitative) calculatory use of special ternary oppositions.

Another famous ternarity, quoted even by Descartes, is the one of Paracelsus, at the very frontier of alchemy and mineral medicine (a discipline introduced by him), in his "system of the three elements" constituted of "mercurius (liquor) – sulphur (ignis) – sal (balsamus)"[163].

We could be more complete on the subject if we mentioned the astrological theory of the typical deviations of the human (soul, spirit and body), but as this seems to be related, as mentioned, to an "hexagon with seven points", we leave it as not relevant.



(the "musical" tripartition of the human)



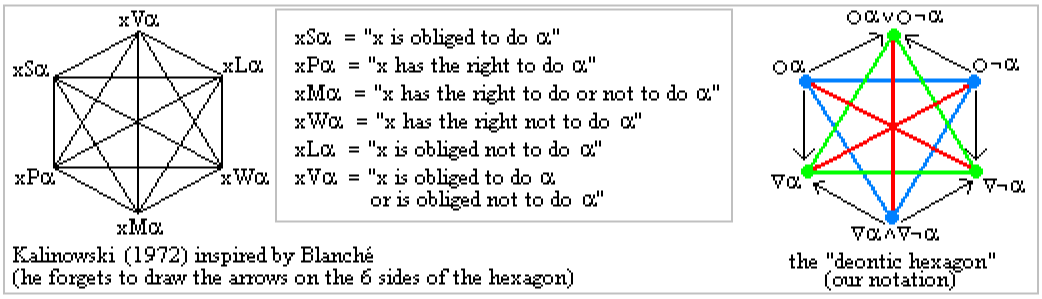
(the art of the heraldic colours)

Having briefly mentioned this strange and yet culturally luxuriant domain of the astrological opposition, let us now turn to the real applications of the logical hexagon.

09.03. Kalinowski's "hexagon of norms" (1972)

The Polish and French philosopher and logician Jerzy (Georges) Kalinowski (1916-2000), working on the logic of norms (i.e. the mathematical formalisation of juridical issues, cf. § 05.02.03 *supra*), put forward in 1972 a deontic version of the Sesmat-Blanché's hexagon, the "hexagon of norms".

[163] I thank Professor Daniel Schulthess for this reference.



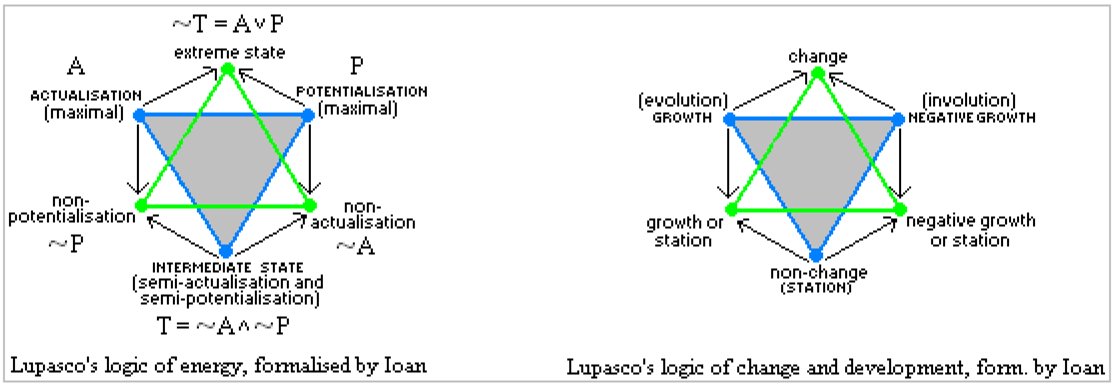
Kalinowski (1972) inspired by Blanché
(he forgets to draw the arrows on the 6 sides of the hexagon)
the "deontic hexagon"
(our notation)

We'll come back to some extent to deontic issues (cf. ch. 17 *infra*).

09.04. Petru Ioan hexagonalising Lupasco's "logic of energy"

Logically speaking, the French-Romanian physicist and philosopher Stéphane Lupasco (1900-1988) may seem disappointing, for his strong philosophical claims (about abandoning the principle of non-contradiction) are not followed by convincing logical technicalities, especially by comparison with the flourishing technical work of the paraconsistent logicians (seemingly unknown to him although almost his contemporaries). But, truly speaking, Lupasco's thought is interesting, for he explores in a new way a quite classical and important view: the metaphysics of change. Around 1935, Lupasco wants to join philosophy to quantum mechanics. He thinks, consequently, that Aristotle's logic (with its principles of Identity, Non-Contradiction and Excluded Middle) is not adequate anymore. This gives birth to his project of an "antagonist logic of energy" (1951).

The Romanian philosopher and logician Petru Ioan has proposed an interesting formalisation of Lupasco's logic in terms of Sesmat-Blanché's logical hexagon (Ioan forgets to mention his sources, however)[164].



Lupasco's logic of energy, formalised by Ioan
Lupasco's logic of change and development, form. by Ioan

Further, Ioan proposes, in the same philosophical spirit, some interesting examples of possible hexagons, 2 based on modal logic and 2 on Aristotle's theory of virtue.

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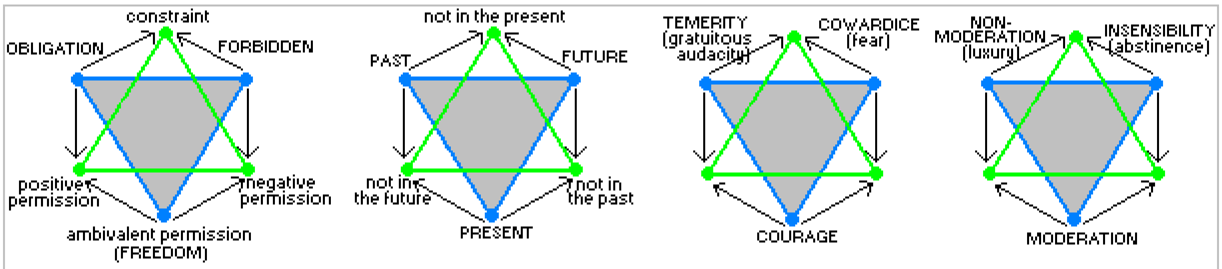
[164] Ioan, P., "Stéphane Lupasco et la propension vers le contradictoire dans la logique roumaine", in: Badescu H. and Nicolescu B. (eds.), *Stéphane Lupasco. L'homme et l'oeuvre*, Monaco, Éditions du Rocher, 1999.



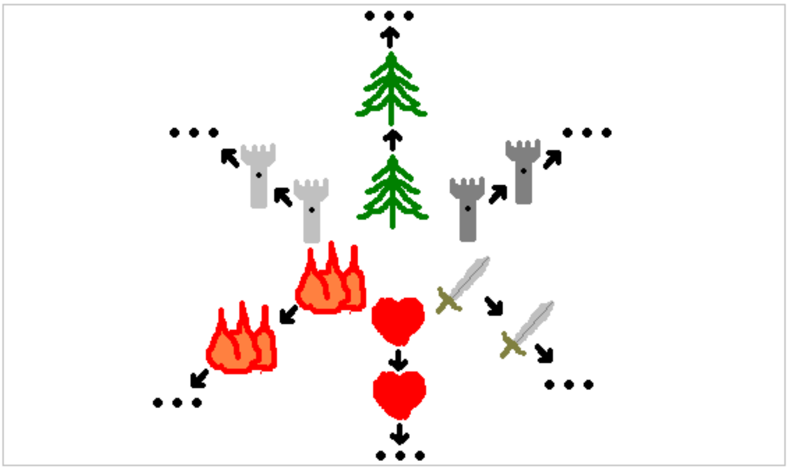
Petru Ioan' hexagonal formalisation of tempoal logic (deontic and temporal), and of Aristotle's theory (on vertue and moral)

The common feature seems to be always the same: a middle position (a vital one) exists between two extremes. This family of hexagons could be very promising, for G. Lerbet and P. Gallais (cf. *infra*) have shown the existence of two related interesting research fields.

09.05. The logical hexagons turned into spirals: Pierre Gallais' "dialectic hexagon of novels" (1982)

In 1982, the French literary critic Pierre Gallais (1929-2001) proposed a formal model intended to capture surprisingly well the essential features of some kinds of narrative. According to Gallais, one of the main features of medieval novels is the presence of multiple meaningful repetitions for the hero (ancestor theories of this one, obviously, are those of Propp and Greimas' narratology). The repeated meaningful situations, leading the hero to develop by introspection his/her initiation path, belong to a fixed small set[165].

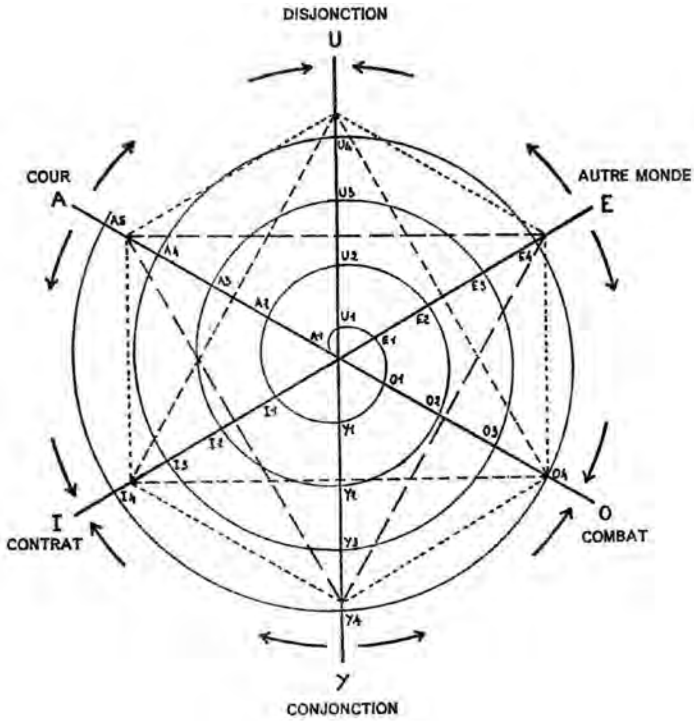
Each situation, by its repetitions, gives rise to a "ray".



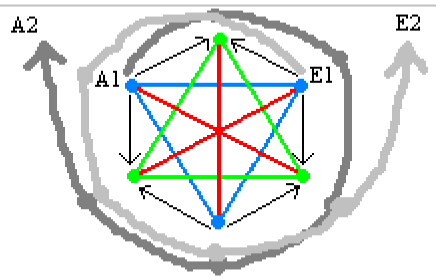
The plot develops some such centrifuge rays. According to Gallais, these elementary narrative situations happen to be six (cf. previous figure).

[165] Gallais, P. , *Dialectique du récit médiéval (Chrétien de Troyes et l'hexagone logique)*, Amsterdam, Rodopi, 1982.

The main point, quite astonishing, is that this model, very bright, convincing and surprising (despite its *prima facie* “craziness”), is said to rely on Sesmat’s and Blanché’s logical hexagon (cf. figure, from Gallais’ book).



As the logical hexagon seems to be “ice-cold” (with respect to life and plots), Gallais’ major move for making this *static*, highly symmetrical logical structure useful for formalising narrative *dynamics* (a typically structuralist approach) consists in joining it with a dynamic spiral, made of circular paths of nested logical hexagons: the model is thus supposed to capture the narrative cycles, each one isomorphic with a logical hexagon, where the personae of the plot regularly come back to the finite set (of six elements) of abstract narrative positions during their initiation path. Remark that a same hexagon can be read as containing two antagonist “hero’s paths”: one hero will have her/his “starting home” in A1 and will try to circulate clockwise, the other, antagonist to the first, will have her/his starting home in E1 and will try to circulate in the reverse circular sense.



two symmetrical antagonist points of view (the one of A1 and the one of E1)

The hero (or Subject) starting from A1 must be able to reach A2, whereas the hero (or Subject) starting from E1 must reach E2. As the path concerns more the interior evolution of the Subject than his/her external deeds, it is possible that both the antagonists simultaneously achieve their path.

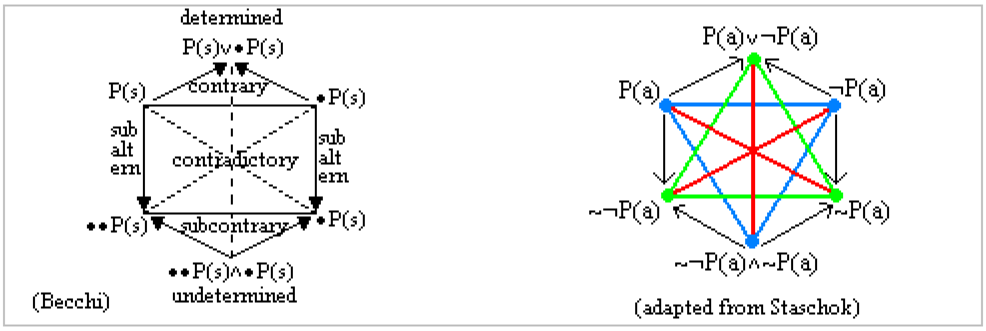
One of the amazing points is the use Gallais makes of all the logical relations holding elegantly between the six positions (the traditional A, U, E, O, Y, I). In his rich treatise, Gallais convincingly develops such an interpretation. Among others, Gallais mentions Lupasco's ideas on "becoming" (cf. § 09.04 *supra*) in order to explain somehow his own model.

This model is supposed to concern mainly (if not only) medieval (or far Eastern classic) literature, in so far as only the latter respects some constraints of narration, those specifically captured by Gallais' hexagonal spiral (the paradigm is taken in the literary work of the medieval French writer Chrétien de Troyes). But this model can be seen (says its author) as much more general than this, if one keeps in mind Gallais' sensible arguments possibly leading to think that this kind of literature is a kind of rational *prototype* (of human imagination) to which all divergent kinds of Western (and worldly) narratives can be reduced: medieval novels do perfectly espouse a scheme that can be shown to be common to all "initiation activities", initiation being itself a symbolised existential metaphor of the human condition in general.

09.06. Sinowjev and Wessel's "non-traditional predication" and "non-traditional quantification" hexagonalised (2008)[166]

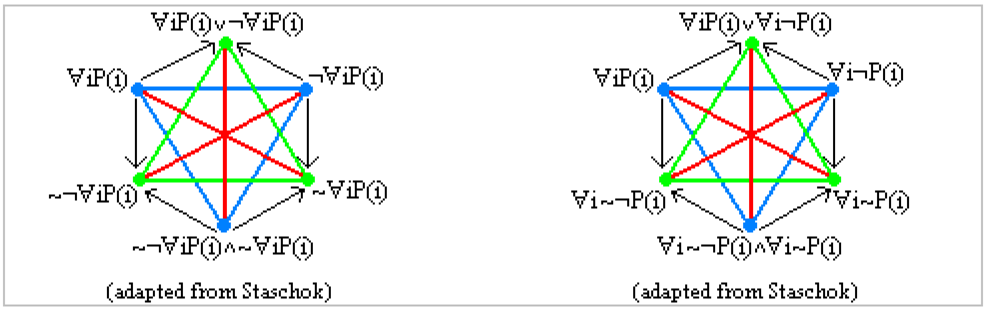
As we saw previously (cf. § 05.03.03 *supra*), first the Russian logician Sinowjev, and then the German logician Wessel proposed around the seventies a "non-traditional approach" to predication. Later they tried to extend these results, which concerned logical negation, to quantification. Despite the interesting criticism against the move from non-traditional predication to non-traditional quantification (it is argued by Mireille Staschock that from the former to the latter, negation changes radically), this gives new models of decorated logical hexagons.

[166] Cf. M. Staschok, "Non-traditional Squares of Predication & Quantification", *Logica Universalis*, 2, 1, 2008; cf. A. Becchi, "La "teoria non tradizionale della predicazione" di Horst Wessel: negazione "esterna" e negazione "interna"", (draft).



(Becchi) (adapted from Staschok)

Regarding the first half of the theory (non-traditional predication), we first have a hexagon of non-traditional predication (in the previous figure we give two drawings of it). If one accepts to extend this in order to have a “non-traditional quantification”, one can try to represent a corresponding logical hexagon. But here we have two possibilities.



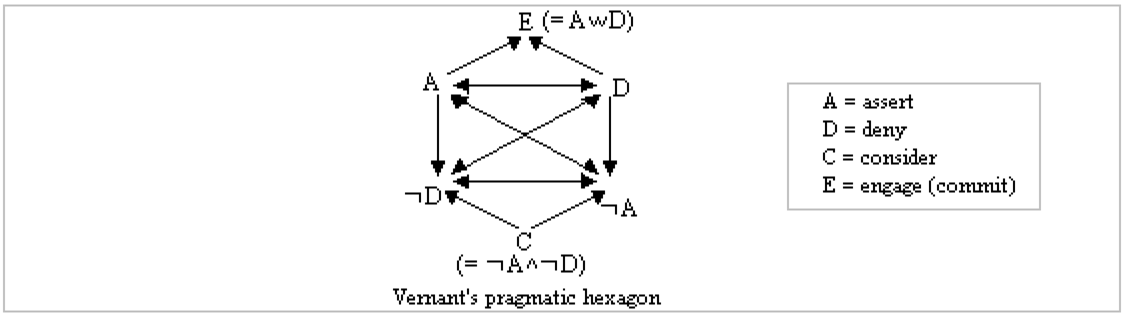
(adapted from Staschok) (adapted from Staschok)

(the first one is that endorsed by Sinowjev and Wessel, the second one is the suggested correction by Staschok).

As we are going to see in the next section, the first of these hexagons has been rediscovered from the point of view of pragmatics by Vernant.

09.07. Vernant’s “pragmatic hexagon” (2003)

Discussing within the “speech-act theory” framework the necessarily non-logical features of ordinary language, considered especially from the point of view of a hopefully (partially) formalised theory of assertion and denial, the French philosopher and logician Denis Vernant proposes (2003) to put conceptual order within these notions (and in particular with respect to the neglected concept of “denial”) by extending to them the formalism of the Sesmat-Blanché hexagon. He thus proposes what he calls a “pragmatic hexagon”.



Vernant's pragmatic hexagon

It must be stressed that Vernant's philosophical analysis deals mainly (and rightly) with the non-logical side of the subject matter. In this respect he argues the necessity of using a "dialogical" approach, in the sense of the German mathematician and philosopher Kuno Lorenz and of the French philosopher F. Jacques. His formalisation is (rightly) supposed to be only a fragment of the desired complete theory which will presumably never "simply end" in a mere logical calculus[167].

We propose the following three secondary remarks on this small logical side of the theory of assertion and denial as proposed by Vernant. First, in his original paper Vernant makes a graphic typo (rather typical in the literature quoting the logical hexagons), inverting the direction of the two upper arrows (thus going from E to, respectively, A and E). He later recognises that this is a mistake (personal communication).

Second, he forgets to draw the vertical double arrow (i.e. the vertical contradiction), as well as the double arrows (subcontrariety relations) linking "E" to "¬D" and "¬A", and those (subcontrariety relations) linking "C" to "A" and "D" (he does not distinguish graphically contrariety, contradiction and subcontrariety, representing all these relations with the same double arrow). This is probably due to the fact that he wants to stress that this figure is an expansion of the logical square (he wants to stress the ternarity of the "assertion-denial" opposition). So he just adds the four new implicative arrows (from "C" to "¬D" and "¬A", and from "A" and "D" to "E") to the original square.

Third, he defines the "E" position as the *exclusive* disjunction of the adjacent terms "A" and "E" (he writes "AwD" instead of "A∨D", thus quitting the Sesmat-Blanché's use). This is due to the fact that Vernant fears the implosion of the logical hexagon if one of its positions (in this case Vernant's "E") is contradictory (personal written communication). This fear is unjustified, the logical hexagon does not implode logically for having one of its logical *positions* ("E") expressing the strange possibility of the inclusive disjunction of two contrary terms (in a personal written communication Vernant finally and lastly agrees on this point).

Remark finally that one could think that Blanché already thought something similar (*Structures intellectuelles*, p.102). But a closer examination shows that Blanché's hexagon in question is one about "taking decisions" (cf. ch. 8 *supra*). Vernant's one is thus original – eventually parallel to one discovered by Staschok, using Sinowjev and Wessel's logic (cf.

[167] Vernant, D., "Pour une logique dialogique de la dénégation", in: Armengaud F., Popelard M.-D. and Vernant D. (eds.), *Du dialogue au texte. Autour de Francis Jacques*, Paris, Kimé, 2003.
.

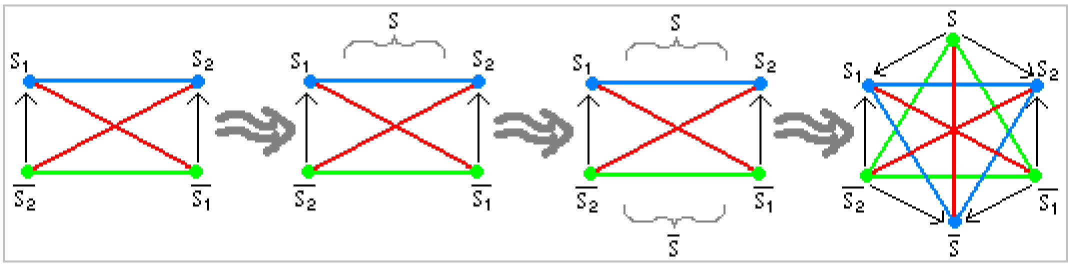
supra) – and very interesting (he offers opposition theory a new field of application). We will come back to this (ch. 18 *infra*).

09.08. Hexagonalising Greimas? A “semiotic hexagon”?

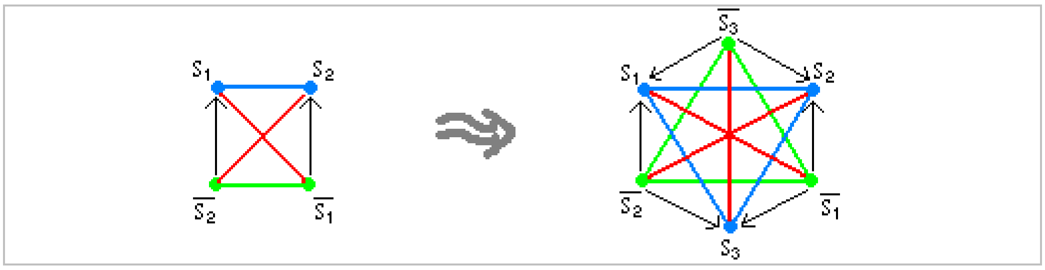
The idea of hexagonalising Greimas’ semiotic square, conceiving thus some kind of “semiotic hexagon” is *prima facie* a natural one, and could turn out to be useful in later developments. In his aforementioned 1982 book, Gallais declares he is astonished to see that neither Greimas nor his school did use (or mention) the logical square. Nevertheless, it could also seem an arbitrary point: asking such a strange question (“may we hexagonalise Greimas?”) may seem useless and time-wasting. However, investigating Greimas’ “square (semiotic) ideas” from the point of view (apparently ignored by him) of the Sesmat-Blanché’s logical hexagon could be very interesting at least for two simple but sensible reasons: (1) it could just be the case that semiotic hexagons do exist, in which case there is no reason why we should refrain from discovering them. (2) Even if semiotic hexagons could not exist (we still do not know for sure), it would be very instructive to understand *the reason why* they cannot: such an understanding would throw some clear light on the very concept of semiotic square, a concept still a bit obscure now, as we saw (ch. 6 *supra*). One thus sees that the question raised here is worth spending some time on.

But how to proceed then? So far there seem to be at least two ways of conceiving some kind of expansion of the semiotic square into an hypothetical semiotic hexagon:

[1] it could make sense to formalise just the semiotic square as it is (i.e. as semiotically binary) by adding an *internalised* expression of the “S” and “not S” terms (“S” is common to S_1 and S_2, whereas “not S” is common to ‘not S_1” and “not S_2”), by way of the “head” and “tail” of the logical hexagon.



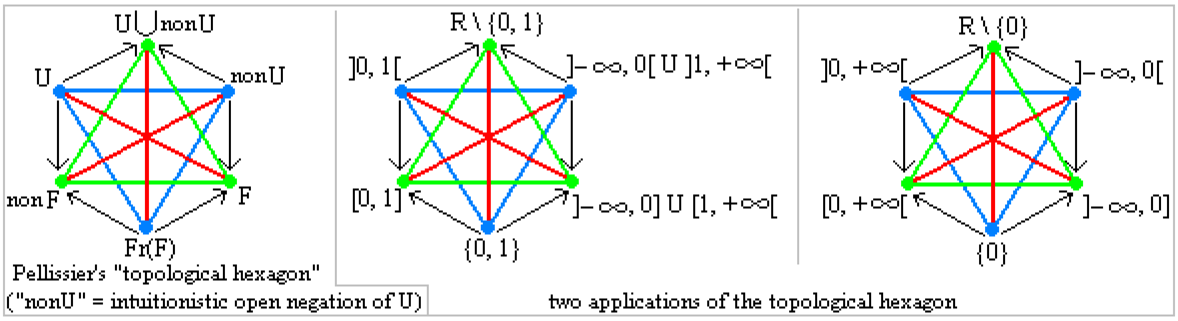
[2] It could make sense to conceive of semiotic meaning as being ruled by an inner *ternary* opposition (instead of a binary one, complemented or not by a head-tail couple).



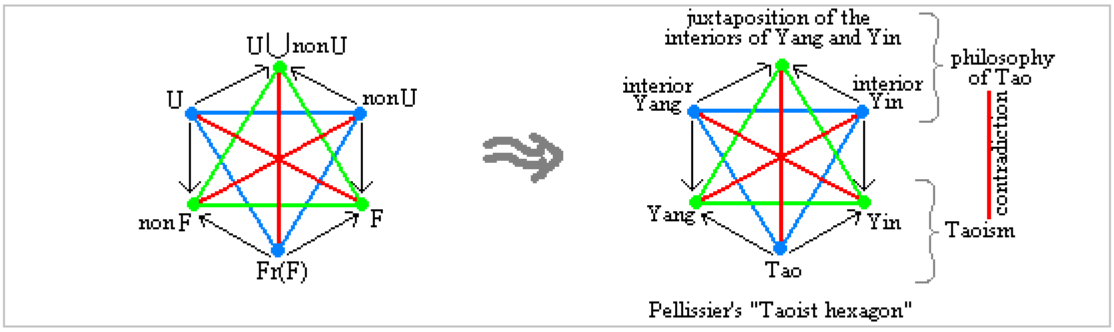
Note that if both solutions turned out to be viable (after a semiotic examination), these two hypotheses (internalising and ternarising) could suggest that trying to combine them could be interesting. But this would require the expression of a mysterious quaternary still unknown opposition at this stage (like in Kant's case for the three-dimensional incongruent counterparts, cf. ch. 3 *supra*). On this Greimasian topic we will return later (ch. 17 *infra*).

09.09. Pellissier's "topological" and "Taoist" hexagons (2008)

The French mathematician and logician (and philosopher of religions) Régis Pellissier (cf. ch. 12 *infra*) has proposed, besides important theoretical elements we are going to discuss in Part II, two remarkable logical hexagons[168]. They belong to very distant fields: abstract mathematics (topology) for the first, Eastern theology for the second. The first one is in fact a "topological hexagon", meant to investigate paraconsistent and paracomplete (i.e. intuitionistic) topological features from the point of view of category theory.



The second one, the "Taoist hexagon" is an amazing application of the topological hexagon to the fundamental concepts of this ancient Chinese religion/philosophy.



[168] Pellissier, R., "2-opposition and the topological hexagon", (forthcoming).

Relying on it, Pellissier highlights a structurally necessary conflict between the Taoist religion and the Taoist philosophy. The application works perfectly because Taoism uses the notions (topological in their essence) of frontier and interior (in a similar way, Pellissier is studying the oppositional ontology underlying the Zoroastrian religion and, more generally, monotheistic systems of thought and belief). We will come back to this in § 17.03.02 *infra*.

09.10. René Thomas' bio-mathematical hexagon (1999)

A very interesting or at least puzzling instance of logical hexagon comes from "bio-mathematics" (the mathematical approach to biology). For 20 years, the Belgian biologist René Thomas has been successfully applying mathematical logic to biology. The idea is to study biological complexity (in the sense of "chaos theory"), reducing, when possible, continuous mathematics (analysis) to discrete mathematics (logic – in 1979 Thomas developed "kinetic logics", a special kind of temporal logic useful for biology). In several cases this works, making research much easier. In one of his papers (1999), Thomas exhibits a set of solutions to a complex system ruled by a "strange attractor" (Rössler's attractor)[169].

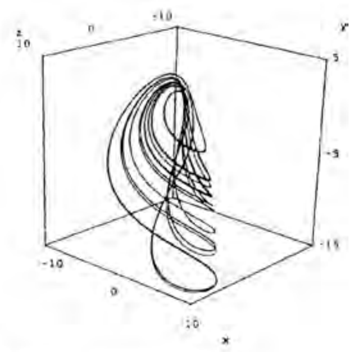
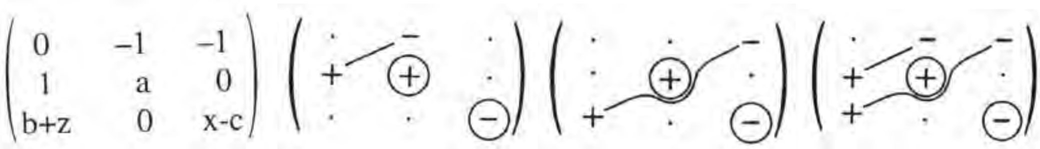


Figure 1. *Système (2)*:

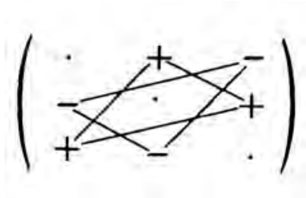
$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = bx^2 - cz$$

I reproduce here some of the solutions in terms of the "Jacobian matrices" of the system studied by Thomas in his paper: in the following picture the first one takes classical values corresponding to the linear system; the last three are matrices interpreted according to Thomas' own method. Lines represent "retroaction circuits"; "+" signs represent matrix elements superior to 0; "-" signs represent matrix elements inferior to 0. Positive circuits (i.e. made of +) imply the existence of "multistationarity"; negative (i.e. made of -) circuits imply "stable periodicity".



[169] R. Thomas, "Analyse et synthèse de réseaux de régulation en termes de boucles de rétroaction", in: B. Feltz, M. Crommelinck and Ph. Goujon (eds), *Auto-organisation et émergence dans les sciences de la vie*, Bruxelles, Ousia, 1999.

Now, among all these solutions displayed on Jacobian matrices 3x3, one (on page 280) much resembles our, by now familiar, logical hexagon.

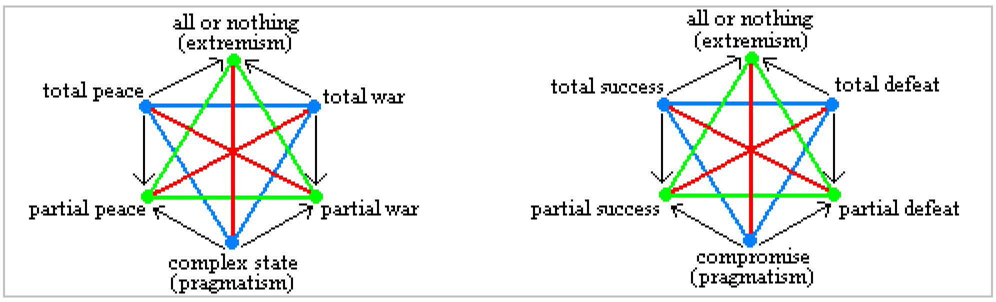


Is the resemblance fortuitous? Or is this somehow related to opposition theory? Can one imagine, for instance, "arrows" between positive and negative values of the Jacobian matrix? Which, of + and – signs, could correspond to contrariety, which to subcontrariety? As Thomas' research concerns homeostasy, and as homeostasy is one of the main ingredients of Maturana and Varela's "theory of autopoiesis" (an abstract structuralist mathematical definition of life as such), and as the latter theory is widely used by N. Luhmann (1984) in his very impressive (structuralist) global theory of humanities (the "theory of the social systems"), such a link with opposition theory, if it were confirmed, could be very interesting. We will try to come back on this topic later (ch. 17)[170].

09.11. Two "war hexagons" inspired by Clausewitz (1832†)

In fact, constructing logical hexagons (i.e. decorating them), if not always straightforward, is all in all quite easy. Human languages do not always have enough terms (common nouns) to fill all the positions of a hexagon. But when places are empty, this suggests – that's the main point with structuralism – the implicit presence of a still unnoticed possible concept.

As a last example, in order to illustrate such an easiness, we propose here two "war hexagons" (inspired to us by the reading of von Clausewitz's classic treatise *On war*).



[170] For more technical presentations, cf. R. Thomas and M. Kaufman, "Multistationarity, the basis of cell differentiation and memory – I. Structural conditions of multistationarity and other nontrivial behavior", *Chaos*, Vol. 11, No. 1 (2001) and "II. Logical analysis of regulatory networks in terms of feedback circuits", *ib*. I wish to thank Professor Thomas for his comments in a stimulating (and still open) e-mail exchange on the subject.

These do not contain interesting “empty places”, but they could turn out useful if one wanted to try to draw connections between, say, game theory and opposition theory (to this topic we will return later, cf. ch 24 *infra*).

09.12. Final remarks on the known applications

Of course, we cannot pretend we have shown either all the possible or all the already existing applications of the logical hexagon. But, as one can see, despite the small number of known applications, these are rather interesting and go from humanities to the formal sciences. Clearly, the logical hexagon adds some vital information to the one that was available with the logical square: there are many situations where the logical square is not enough of a model of opposition, whatever the investigated field is.

As a concluding remark on Pellissier’s proposal, we can observe how powerful the theory of opposition can be, for relying on existing opposed terms, the oppositional formalism (here the logical hexagon) allows to inescapably predict the presence, in the concerned conceptual field, of conceptual empty places, needing to be filled with new names. This happens in both cases: the topological hexagon helps to think new things on paraconsistency (cf. Pellissier’s paper), whereas the Taoist hexagon helps to understand an opposition between Taoist religion and philosophy, which would have been hardly noticeable otherwise.

As a last remark, notice the appearance, from time to time, of the question of having a simultaneous multiplicity of related logical hexagons, as it is clearly the case, for instance (but not only), with Blanché’s “chain made of two logical hexagons” (for expressing “the 10 main binary connectives”). How can one think out this kind of multiplicities?

10.
BEZIAU'S DISCOVERY OF FURTHER HEXAGONS WHILE DEFENDING PARACONSISTENCY: HIS IDEA OF A HIGHER-DIMENSIONAL ORDER OF THE OPPOSITIONS

In this chapter we mainly recall Jean-Yves Béziau's discoveries in the field of oppositional geometry. They are very important for two reasons: (1) they generated many new discoveries by other researchers (among which the ones that make up the heart of the present work): one can say that Béziau's studies worked as a wake-up call, drawing back attention – after 50 years of oblivion – on the existence of the logical hexagon; (2) they showed that the notion of logical negation (and hence the whole philosophy of logics) gains by being studied geometrically. As these discoveries originated in the context of a dispute with Hartley Slater over the very possibility of having "paraconsistent logics" in mathematics, we start here by recalling the notion of paraconsistency, then Slater's sharp and disturbing criticisms against it and finally Béziau's complex but convincing reaction. We end by mentioning how other researchers (mainly Hans Smessaert and myself) reacted positively to Béziau's discoveries, which will lead us to the Part II of this essay.

10.01. Paraconsistent logics and Universal Logic

Here we recall briefly what paraconsistent logics are and why they are very important (for they are). In addition we recall the further development of "universal logic", a new science (of the *logical* extreme possibilities) that seems to be quite a natural generalisation of the global revolution brought out mainly (if not only) by the paraconsistent research.

10.01.01 Paraconsistent logics as the "arena" of major logical exploration

Intuitively, contradiction is "$A \wedge \neg A$" ("A and not A"): it is the simultaneous presence (or truth) of something and its "negation". The classical view of mathematics (and logic) on contradiction is that "there can be no calculus of contradiction": contradiction is logically and mathematically meaningless, contradiction is the "negative transcendental" of these two sciences, their condition of impossibility (if there is contradiction, there are no mathematics and no logic). Logic (and mathematics) is a global negation (a global refusal) of contradiction: it is, so to say, human speech and human thought made free of contradictions. This state of the art is nevertheless clearly a bit frustrating: situations with no contradiction (at all) are very, very rare (if they exist at all) in real life. It would, instead, be useful to be able to pursue formal calculations even in situations where some nasty, embarrassing (but unavoidable) contradiction does appear. Historically, Hegel (and some others) tried to establish the view that contradiction is the most important feature in the world (both of concrete and of abstract things), because it rules the process of becoming. But this led to no convincing formal calculus: the marriage of Hegel's "scientific philosophy" and science never happened, contradiction was only possible in natural-language philosophical speech, hence suspected to be sophistic (i.e. conceptually tricky and dishonest, or at least very badly mistaken). On the contrary, as we already said in § 01.07 *supra*, serious science, based on mathematics, held to the mathematical understanding of contradiction, saying that in all conceivable formal languages, if a contradiction appears, it "kills" the whole formal system. Again, technically speaking, this is called "logical explosiveness": the very small logical law, called "*ex contraditione sequitur quodlibet*" (from any contradiction, everything follows), saying that "$A \wedge \neg A \rightarrow B$", shows that any local contradiction gives rise to a global contradiction, contradiction being to logic (and mathematics) as – sorry for the ugly metaphor – an incurable cancer. This logical law (or axiom) seeming to be proper of any conceivable formal system (it derives from the usual meaning of the basic ingredients of logic), the matter seemed settled once and for all ("any formal system has logical explosiveness, and therefore contradiction can never be accepted without logical death").

But as logic (and mathematics) became fully axiomatic (and less intuitive), the idea grew that every axiom could in principle be abandoned or slightly modified, so that small, wise changes could produce interesting global effects on the considered logical system. Around 1950, the Polish logician Jaskowski's and the Brazilian logician N.C.A. da Costa's proposed, independently, to build formal (mathematical) systems allowing to make calculations beyond the possible emergence of frightening and troubling contradictions inside of them: in these systems, one can, during some logical calculation, derive some theorem of the form "$A \wedge \neg A$" (or: first some theorem "A" and then its negation "$\neg A$") without logical horror (i.e. without having to shout things like "oh, my system is turning into rubbish, gosh I have to throw it away!"); under certain circumstances, made possible by the special axiomatisation, the calculation may go on without being trivial (without deducing everything). The idea of "paraconsistency" (i.e. "going beyond consistency", consistency being the property of having no contradiction) was born, the latter being defined as the possibility of having “inconsistent non-triviality”. In order to do that, the whole problem was to neutralise somehow the property of logical explosiveness. It turned out that it was possible to do it in different ways: the huge number of such possible axiomatic modifications or alternative formal constructions is still not exhausted to this day, and therefore there are lots of different systems of paraconsistent logic. The three main families of paraconsistent logics seem to be the Brazilian (da Costa’s school), the Australian (Routley, Meyer and others’ school) and the Belgian (Batens’ school), having each specialised itself in some of the main strategies of neutralisation of the logical explosiveness.

The philosophical ancestors of paraconsistent research are in fact many. One can distinguish broadly two classes at least:

(1) the far ancestors: Lao Zi, Heraclitus, Protagoras, Hegel, Marx, …;

(2) the near ancestors: Meinong, Vasil’ev (cf. ch.7 *supra*), Łukasiewicz, Wittgenstein.

The research in paraconsistent logic yielded many valuable technical results. But one of its structural open problems seems to be the lack of major philosophers involved in it. A huge philosophical contribution was being made by Richard Routley (alias Richard Sylvan), both an outstanding logician and philosopher. He was able, answering a severe criticism, to build an adequate formal semantics in a place where a demonstration seemed to exclude this possibility. But with his death, paraconsistent philosophers seemed to reduce to epistemologists, often brilliant (as A. Bobenrieth), but lacking a vast philosophical (and not epistemological) project. G. Priest has taken Routley’s role, and indeed he develops an original and ambitious (if not foolhardy) philosophy of contradiction; however, he changed the name of his logical-philosophical adventure, preferring “dialetheias” to paraconsistency, and this seems symptomatic of the fact that despite the quite large success of Priest among the non-logician readers, not that many working logicians seem to have followed him as a research-school inspirer (if we except the valuable work he himself is doing on “hyper-contradictions”, which inspires such logicians as Schramko and Wansing, cf. ch.23 *infra*)[171].

The point we touch here (the question of the existence of a convincing paraconsistent general *philosophy*) is not a moral one, consisting in (us) giving evaluation marks, but one concerning the very meaning of the whole paraconsistent enterprise. And this point is very sensitive, making the absence of a critical thought rather delicate, for some believe that

[171] One among some possible exceptions is constituted by the Spanish philosopher Lorenzo Peña. A young, promising paraconsistent true philosopher is possibly Alexandre Costa-Leite, for he masters both the logical technical side of paraconsistent logic and he seems to have philosophical ambition and culture. And recently the paraconsistent community has been joined by the talented pupil of Severino, Francesco Berto.

paraconsistency bears the big risk of giving hold to profound conceptual phantasmagorias. Paraconsistent research is potentially dangerous in terms of self-deception. From a (forgive me, kind reader!) psychoanalytical point of view – that is one taking seriously into consideration the hypothesis of the existence of some kind of autonomous unconscious thought, tied to unsuspected unconscious emotive disturbance sources – paraconsistency may seem to be highly connoted: it makes one *dream*[172]. Pushing the image very far, one may dare say that it could be seen as some kind of "realised logical incest": for, at stake we have some kind of pleasure, possibly felt for the achieved refusal of the logical taboo[173]. The promise to be able to cope mathematically with contradictions is no light one, it easily attracts people. So it is very important not to let the dream overcome reason.

From this point of view, a quite important figure of the current paraconsistent movement seems to be the French-Swiss logician and philosopher Jean-Yves Béziau, pupil of da Costa. His importance comes mainly from the coexistence in him of two seemingly antinomic elements: in his research line he both holds rough hardness with respect to paraconsistency (which he regularly questions and criticises) and astonishing openness with respect to radical logical experimentation (he discovered many paraconsistent results). His main results so far are:

(1) the discovery of extensions of the classical da Costa paraconsistent systems C_n;

(2) the discovery of paradoxes of the "inter-logical" translation (somehow similar to the ones Galileo found to hold between the set of natural and the set of even numbers);

(3) the proposal of "Universal Logic" as a general (powerful) research framework.

To this last point we now come more precisely (but briefly).

10.01.02. Béziau's proposal of Universal Logic as a general framework

There have been at least four "generalisations" of paraconsistent logic. One is the development of the so-called "relevant logics" (mainly developed in Australia and in the USA): its main philosophical aim is to look for some natural-looking substitute to the "material implication". Remark that, technically speaking, every relevant logic is paraconsistent (not the other way round). Another one is the development (by G. Priest) of the so-called "dialetheism". This philosophical position, due to Priest's own talent, is very

[172] The links between the subversions of logic and the oniric world are known, a paradigmatic case being the narrative (very oniric) work of the logician Lewis Carroll (Alice's adventures). For a high-level philosophical analysis of it, cf. G. Deleuze, *Logique du sens*, Paris, Éditions de Minuit, 1969.

[173] Remember Dana Scott's joke on "paraconsistency being to logic what pornography is to true love" (ref. lost).

interesting (Priest has a special gift for explanation). However, it may seem to lack logical strength: it does not seem to have gained widespread acceptance among logicians (with the exception, again, of the very interesting notion of hyper-contradiction). A third one is the development of what is now called “substructural logic”, a variation on some deep properties (the “structural properties”) of the “sequent calculus” approach to logic (we will come back to this issue on ch.23 *infra*). Béziau proposed a fourth one, “Universal Logic”. Despite its strange name, seemingly a bit naive (almost a pleonasm: isn’t logic *always* universal?), this approach is both very ambitious and very interesting. Béziau’s bet, after a deep study of the algebraic characterisation of logic given by the Polish school, is at least double: (1) there is a formal specificity of logic with respect to the classical structuralist (i.e. Bourbakian) parts of mathematics (i.e. algebraic structures, topological structures, order structures): logic is a new family of mathematical structures, only partially (or “clumsily”) reducible to algebra (as is done usually); (2) logical notions are open to infinite variation. The supposed “transcendental” elements of logic can only be grasped by a hard, long-run systematic study of all possible deviancies, whereas many lovers of logic regard deviances as a flaw[174]. Universal Logic, which was conceived by Béziau in reference to Universal Algebra, is not a unique, all-encompassing logical system (as some contemporary Chinese interprets of Universal Logic mistakenly say and hold), but, on the contrary, according to him, a science of all the logical multiplicity and diversity.

So, there are exciting openings for formal logic, at least as proposals. Have they been accepted?

10.02 Slater against paraconsistent logics. And some reactions

In a very short paper (4 pages including footnotes and bibliography!) the Australian philosopher of logic Hartley Slater dismisses paraconsistent logic as a whole as oxymoronic (i.e. built mistakenly over an untenable basis, a house of cards). This attack is very problematic for the paraconsistentist because the underlying reasoning may seem, at first glance, unobjectionable and the conclusion is more than harsh. He seems to have done against paraconsistency what Diodorus Cronus did against the concept of freedom with his Master Argument (cf. ch. 4 *supra*).

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[174] This is the famous position of the American philosopher and mathematician W. V. Quine, mainly in ch. 6 of his *Philosophy of Logic*, Cambridge MA and London UK, Harvard University Press, 1994 (1970).

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Slater makes a quick (6 lines!) and abstract reasoning (implicitly involving the notions of contradiction and subcontrariety) showing the impossibility of paraconsistent logic. He then exhibits a concrete example of this, Priest's logic LP. He recalls how this paraconsistent (and "dialethic") logic functions (its axiomatics), then he exhibits a major problem with it (which embodies the abstract starting problem). Further, Slater recalls that Priest is aware *in abstracto* of this problem, for he himself has criticised it harshly (with the help of Routley) for da Costa's system C_1 (thus condemned by Priest and Routley) and has precisely conceived LP in order to avoid this problem. This means two things, Slater suggests: that LP is a total failure (it misses its main explicit target – the one already missed unconsciously by da Costa) and that consequently *any* paraconsistent logic is destined to fail in the same way (no axiomatic or semantic self-reform will be able to truly avoid the problem stated abstractly by Slater at the beginning of his reasoning). In order to definitely prove both points (by figuring out and refuting Priest's possible answers to this criticism), Slater makes an excursus over a similar past debate where Copeland criticised relevant logic (the aforementioned famous sub-family of paraconsistent logic). The comparison of the two debates, because Priest (unlike the past relevantist) showed an *explicit* intention of defending the point problematic in both cases (in the first it was unclear and related to confusions) and because his deepest arguments (on 'truth' and 'contradiction') are untenable (Slater relies on the authority of Tarski, Montague and Goodstein), shows the definitive failure, with Priest and beyond Priest, of the whole paraconsistent project. Truly speaking (i.e. without fallacious conceptual face-lifts) "there are no paraconsistent logics".

We will consider here two ways of re-stating Slater's reasoning. One is philosophical, using a transcendental argument (again, we mean by "transcendental" a "necessary condition of possibility"): something (a transcendental structure, which cannot be demonstrated directly) is demonstrated indirectly by showing that anyone who tries to think beyond it is destined to contradict himself. Slater implicitly recalls that paraconsistency is defined (by its partisans) as the possibility of having a non-trivial but true contradiction (A and not A, both true). He claims that by virtue of the logical principles and operators embodied by the logical square (the transcendental heart of standard logic), paraconsistency as such (i.e. a "true contradiction") is just impossible. Paraconsistency, truly speaking (i.e. in its real axiomatic instances, as C_1 or LP), deals with "subcontrariety", not with "contradiction". In some sense Slater's argument can be reformulated geometrically by saying that "we cannot do whatever we want with a square, coloured structure" (as is Aristotle-Apuleius' "logical square").

A second possible way of restating Slater's criticism is purely logical (axiomatic). We take it from Francesco Paoli, for it seems both very clear and true to the facts (Paoli discussed it with Slater, and scholars debating this issue seem to find a consensus on this reconstruction)[175]. It says that Slater's reasoning is made of four premises (explicit or implicit):

(1) contradictories cannot be true together;

(2) a sentence and its negation are contradictories;

(3) if L is a paraconsistent logic, then in the semantics for L, there are "inconsistent" valuations that assign both A and $\neg$A a designated value, for some formula A;

(4) if A and B both receive a designated value, under some valuation v, in the semantics for L, then A and B can be true together according to L.

From these, two consequences follow (inside Slater's argument), with no particular deduction rule other than the simple and usual ones (like Modus Ponens):

(5) in paraconsistent logics, A and $\neg$A may not be contradictories (from (1), (3) and (4));

(6) thus, paraconsistent "negations" are not negations (from (2), (5)).

These two consequences lead (if accepted) to the negation of paraconsistent logics. So, because the deduction is a very simple one, anyone wanting to refuse Slater's consequences will have to reject at least one of the four premises.

Again, Slater's criticism is astonishing: in a few lines it denies (in some sense) the work of quite a lot of bright people (mostly professional mathematicians) over several decades now, published in official scientific journals. The only possible understanding of this could be that these works have (maybe) some value, but surely not the one they are claiming to have! They are not researches in "paraconsistency" (for there is no such thing as "paraconsistent logics"…).

There have been at least 6 reactions to Slater. We will speak of Béziau's ones in a while. The other five have been elaborated by 4 logicians (Restall, Priest, Brown and Paoli) and by a philosopher of logic (Duthil Novaes). The logical answers each consist in focussing on one of Slater's four premises (the first is rejected by Priest, the third by Paoli, the fourth by Restall) and in showing that there is a family of paraconsistent logics refusing that very premise. The philosophical paper discusses the distinction to be made between negation and

[175] Cf. F. Paoli, "Quine and Slater on paraconsistency and deviance", *Journal of Philosophical Logic*, **32**, pp.531-548, 2003.

contradiction. In what follows we will focus on the rejection by Béziau of Slater's second premise[176].

10.03. Béziau's defence of paraconsistency against Slater

Béziau wrote mainly two papers against Slater around 2002. These two form a single reasoning, articulated into two parts (one more logical, the other more philosophical). First (2006), he analyses in a logically technical and precise way (using his, Alves' and da Costa's "bivaluation theory") what is going on with Slater (but also with da Costa and Priest): by using very precise and powerful logical definitions of "contradiction" and formal translation-rules between logical systems, and with a powerful general logical theorem (over paraconsistent systems in general), he shows how the problem must be restated (against Slater and his confusing confusions). Once he has given such a general precise analysis, he tackles the problem in a new way (2003), this time from a more philosophical (and linguistic) point of view (the problem being to defend and explain the significance of subcontrariety), by summoning a "geometry of the logical oppositions" and by making some claims about the relations between the opposition-forming operators and the kinds of logical negations of contemporary logic (among which the paraconsistent one). Béziau's "geometrical answer" (the one developed in his second paper) may seem logically strange to some (the arguments there are of a new kind), but it has the merit of going right to the heart of Slater's criticism (the notion of subcontrariety). Moreover, as we will see in Part II, partly involuntarily but very coherently with his general research line (UL), he has thus opened a new fundamental branch of pure logic, one that seems to show that logic is in fact an autonomous (so to say Bourbakian) new family of abstract structures (or mother structures), parallel to topology, algebra, etc. (and not just a sub-section of algebra). And this, as we will see, has a big relevance with respect to the Slater debate.

10.03.01. Béziau's first defence line: pointing Slater's confusions[177]

Béziau's first answer complies anticipatively with Duthil Novaes' posterior remark: it makes the effort of giving a new (non-classical) definition of "contradiction". He starts by

[176] We give a more detailed discussion of the whole issue in our paper A. Moretti, "The Critics to Paraconsistency and to Many-Valuedness and the Geometry of Oppositions", submitted for the *Proceedings of the Fourth World Congress on Paraconsistency* held in Melbourne, July 2008.
[177] This line of thought is developed in J.-Y. Béziau, "Paraconsistent logic! (A reply to Slater)", *Sorites*, 17, p. 17-25, 2006.

showing that Slater, in his paper, used wrong (i.e. incomplete, truncated) definitions (for instance, Slater defines contradiction as “the impossibility of being true together”, whereas the true traditional definition is “the impossibility of being true together *and the impossibility of being false together*”). The main thesis of Béziau’s paper is that if one uses good definitions instead (which implies to co-define contrariety and subcontrariety, to stress their mathematical symmetry), then Slater’s claims are either false or, at best, tautological. But preliminarily, by means of considerations over the translatability conditions between logical systems, Béziau recalls that paraconsistent logic is not “just switching names” (as Slater’s accusation against paraconsistency harshly states): it is instead the emergence of a genuinely new phenomenon (comparable to the emergence of non-Euclidean geometry, where the mathematical meaning of “straight line” changed drastically): the meaning of several classical fundamental notions of logic does change indeed. Once this provable and proven essential point recalled and restated, and after having recalled the classical opposition theory (and its explicit and complete definitions of contradiction, contrariety and subcontrariety), Béziau starts analysing the opposition definitions inside the framework of da Costa’s system C_1: he shows that its negation operator is a subcontrariety-forming operator (as Priest and Slater reproachfully claimed) but *relative to the semantics (given by “bivaluation theory”) of this system* (in this respect, Slater is totally wrong – he totally missed this essential technical point). He then analyses the opposition relations inside Priest’s system LP and shows that Slater’s claim thereupon (“Priest’s paraconsistency deals with subcontrariety”) is partly false and partly true. It is generally false because LP’s negation operator (as C_1’s negation operator) is relative to *that* logic (LP) and cannot be translated simply so into classical logic (as Slater implicitly and very mistakenly does): so LP’s negation operator is not the *classical* subcontrariety-forming operator (like Slater claims), it is an *LP* subcontrariety-forming operator. Nevertheless, Slater is right when he points out the presence of an *illicit* “trick” in Priest’s logic: an imprecision, due to a play with the “designated values”, in the use of the concept of truth (for Priest in some sense truth is “1” and in some sense truth is “1 or ½”) and this cannot be. This means that either truth is “1 or ½” (as LP’s set of designated values states), in which case the system LP is paraconsistent but its negation is (only) a subcontrariety-forming operator (Slater is right) *from the point of view of LP* (Slater is nevertheless partially wrong); or truth is (only) “1” (as LP’s explicit definition of its truth predicate says), in which case the negation operator of LP is a contradiction-forming operator (as claims Priest) *from the point of view of LP*, but then LP is not paraconsistent (so Slater is, in some sense – which he only confusedly perceived –, right). Having examined with precise definitions the opposition relations inside (1) classical logic, (2) da Costa's system C_1 and (3) Priest's system LP, Béziau can now express what he thinks to be the real question at stake: that is, knowing whether a paraconsistent negation *in general* can be a contradiction-forming operator *from the point of view of its own semantics*. The previous analysis has already shown that neither da Costa's C_1 nor Priest's LP can. But Béziau shows more generally, by a powerful theorem, that it is not possible for a paraconsistent negation operator to be a contradiction-forming operator (as Slater provokingly asks) *from the point of view of its own semantics*. Remark – this point will turn out important later – that Béziau reaches his theorem (i) by using, against Priest, Suszko's remarks over many-valued logics (his ideas on the binarity of the designated-undesignated subsets of the sets of all truth-values of a given system), (ii) by using a very general definition of "logic" (one based on Béziau's notion of "universal logic") and (iii) by using a general Béziau-Dacostian definition of contradiction (i.e. one in terms of "bivaluation theory"): he recognises that, following Malinowski's anti-Suszko's strategy (i.e. "*q*-logics", cf. ch. 23 *infra*) we could escape Suszko's restriction and thus his own theorem would be less general: so it could be seemingly possible to look for paraconsistent systems with a contradiction-forming negation from the point of view of their own semantics (this would be another anti-Slater strategy, one accepting Slater's second premise). But he leaves this case aside as being very remote and very non-standard (Béziau sticks, in his anti-Slater strategy, to refuting the second premise). Which means that, in some sense, Béziau acknowledges Slater's idea that "paraconsistent" negations can only be subcontrariety-forming operators. As a first corollary, only classical negation is a contradiction-forming operator (hence the tautological, uninteresting value of Slater's thesis according to Béziau). But then, if we additionally followed Slater's drastic anti-paraconsistent philosophical criteria (i.e. equating negation to contradiction), we should say – which is commonsensically absurd – that not even the intuitionist negation – a fully recognised one! – is a "negation" (because intuitionist negation is a contrariety-forming operator). As a second corollary (this concerns the "Brazilian-Australian cold war"), the negation operator of Priest's LP system has no superiority over the one of da Costa's C_1 system. So Slater is wrong, but a bit right nevertheless (!): there is indeed a strong link between paraconsistency and subcontrariety. So, refusing Slater's second premise (the one equating negation and contradiction), the remaining problem for Béziau will be a philosophical one: taking subcontrariety (and hence paraconsistency) seriously in general.

10.03.02. Paraconsistent logic from a modal point of view[178]

By analogy with an old known result by K. Gödel, proving that intuitionist negation has the same properties as "¬◊", Béziau draws the attention on the fact that "¬□" has the same behaviour as paraconsistent negation, both from the point of view of the negative requirements and from that of the positive requirements. This result is not too surprising, since "¬◊" and "¬□" are dual modalities, in the same way that paraconsistency and paracompleteness (i.e. intuitionism) are dual.

Additionally, resuming an old remark made by several scholars in different disciplines (logics, linguistic, law, …), Béziau points out that in the logical square the modality "¬□" generally has no name, no matter the language. Béziau finds it interesting to see that it is the paraconsistent modality (i.e. paraconsistent negation), among all the possible ones, that bears no specific name in natural language.

10.03.03. Béziau's second defence line of paraconsistency

As we saw, the first defence line does not end up on an established result. It only proves that Slater's argument is unreliable in so far as it is based on imprecise definitions of what an opposition is. But Slater's point according to which paraconsistency has no real relation to the (classical) notion of contradiction seems founded. The main idea behind Béziau's second defence line consists in pursuing the examination of the links between possible negations and possible oppositions, by having a special look at the logical hexagon (forgotten by Slater, who only refers to the logical square).

Béziau's starts from his previous discovery, that is from the fact that the modality "¬□" is in fact logically isomorphic to a paraconsistent negation, and by the old remark, which he reactualises, according to which the "O" corner of the logical "AEIO" square (precisely that of the "¬□" modality) is unlexicalised (i.e. has no traditional name of its own).

Now, according to Béziau, the fact that the corner corresponding to paraconsistent negation is not lexicalised is interesting for several reasons:

(1) this could be used, *prima facie*, against the very notion of paraconsistent negation ("having no natural name, this notion is not natural at all");

[178] This is mainly developed in J.-Y. Béziau, "Paraconsistent logic from a modal viewpoint", *Journal of Applied Logic*, 3, p. 7-14, 2005.

(2) but, on the other hand, even without a name, the fact that this notion appears explicitly in the square of opposition seems, on the contrary, a proof of its formal naturality (this is a structuralist line of thought): paraconsistent negation seems quite natural if we observe that the "O" corner can be interpreted as a paraconsistent negation;

(3) the paraconsistent view of the "O" corner fits well even in Sesmat-Blanché's hexagon;

(4) we can even construct a more sophisticated geometrical object (still unknown) that explains quite well the relations between modalities and negations.

So, clearly, on the one hand Béziau accepts Slater's idea that paraconsistent negation is subcontrary opposition (as we saw in his first defence line), but on the other hand he proposes to remark a striking similarity between the following three orders of considerations:

1) the 3 Aristotelian combinatorial definitions of opposition, that is: contradiction, contrariety and subcontrariety;
2) the following 3 new definitions of opposition-making operators, that is: the contradictory operator, the contrary operator and the subcontrary operator;
3) Miro Quesada's classical definitions of non-standard logical notions: classical negation, proper paracomplete (i.e. intuitionist) negation, proper paraconsistent negation ("proper" means "non-alethic").

Then, with these three parallel levels we have a nice one-to-one correspondence between three kinds of negation and the three Aristotelian notions of opposition *via* the 3 related notions of logical operators. Slater is against this kind of move, i.e. he is against the fact of taking seriously such correspondences, because he thinks that there is only one kind of negation, the classical negation. For Slater there are no paraconsistent and no paracomplete negations. "Paraconsistent "negation" cannot be a negation because it is only a subcontrary-forming operator" (Slater, modified). On the contrary, Béziau claims that paraconsistent negations *are* negations just *because* they are subcontrary operators and thus they are opposition operators. "This last claim is based on the idea that it is difficult to dissociate negation from opposition, that the background of negation is opposition and therefore if there are three kinds of oppositions there must also be three kinds of negations" (Béziau).

Béziau sums up as follows:

1 – there is a conceptual danger in restricting unduly the notion of negation;

2 – there is actually no general pluralist theory of negation;

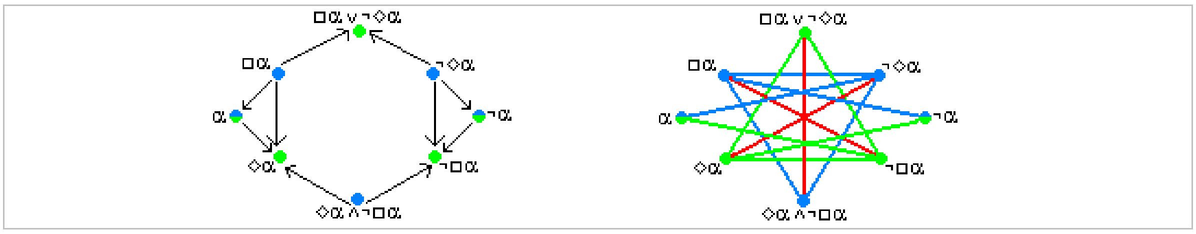
3 – there are advantages to using Aristotle's theory of opposition

However, one must be conscious of some possible drawbacks in Aristotle's theory: among others, he does not explicitly defines subcontraries (cf. ch. 4 *supra*); he considers only two kinds of oppositions, contradiction and contrariety. But Béziau argues, from the point of view of contemporary logic, that all notions have a symmetric (or dual) counterpart; so Aristotle's asymmetry, when he favours the principle of non-contradiction over the principle of the excluded third, is not acceptable. In other words, it makes no sense to say that contraries are oppositions but subcontraries are not.

However, it really makes no sense, according to Béziau, to say that subalternation is an opposition relation. So, while studying (after Slater) opposition for the sake of "negation theory", we can – according to Béziau – put away the subalternation arrows in Sesmat-Blanché's hexagon and better represent it by a star ("Sesmat-Blanché's star"). Doing this (i.e. rejecting subalternation out of the scope of opposition) Béziau criticises Avi Sion's definition of opposition (for Sion "the various relations of opposition make up a continuum")[179]. Béziau invokes the popular definition of opposition, as for instance in the Longman Dictionary, which says that "opposition is based on difference, but on strong difference". Otherwise, pursues Béziau, we identify opposition with simple difference.

10.04. Béziau's discoveries about the logical hexagons

Now, Béziau remarks that the "null modalities" are not taken into account by Sesmat and Blanché[180]. Remark that this is another structuralist move ("don't forget the 'meaningless terms'!"). And this despite the fact that *p* (as its negation, ¬*p*) does have opposition relations with respect to its paraconsistent and paracomplete (i.e. intuitionist) negations (i.e. ¬□*p* and ¬◊*p*). How to reform the logical hexagon then? One possible solution would be simply to add, to the logical hexagon, these two null modalities. This would give some kind of irregular "octagon of modalities". But this is not elegant.

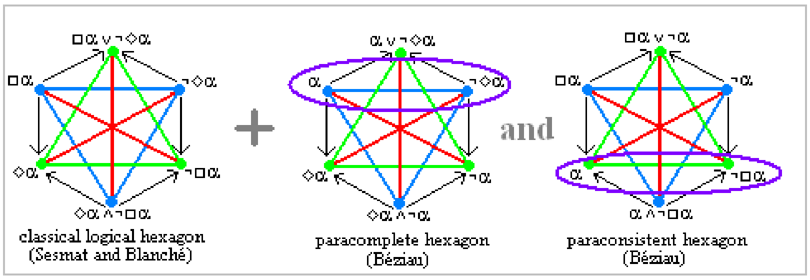


So, this rather means that there are in fact two new hexagons, strangely forgotten (one for paraconsistency, the other for intuitionism): one is obtained by expressing the opposition relations between *p* (and its negation) and the paracomplete negation "¬◊p", the other by

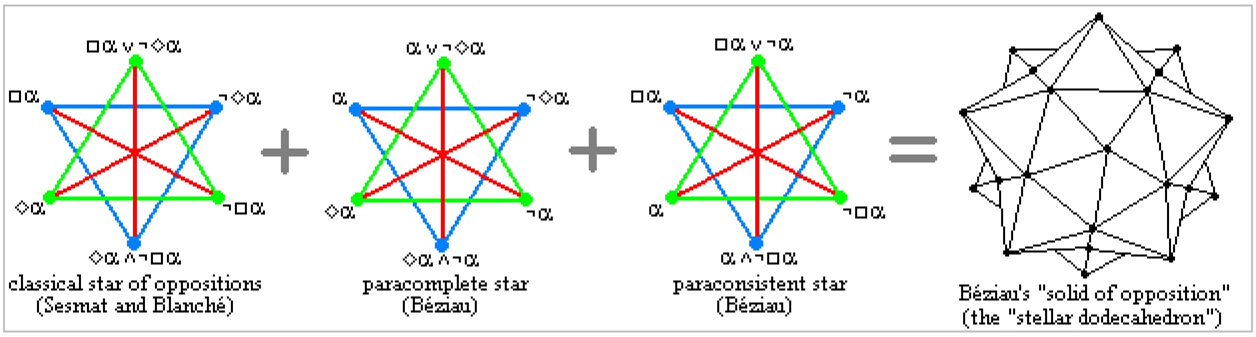
[179] Sion, A., *Future logic*, Geneva, 1996 (quoted by Béziau).

[180] Truly speaking he does not name them that way. I take this terminology, which I find very clear and appropriate, from W.A. Carnielli and C. Pizzi, *Modalità e multimodalità*, Milano, Franco Angeli, 2001, p. 11.

expressing the opposition relations between *p* (and its negation) and the paraconsistent negation “¬□p”.



Béziau remarks that among the 18 vertices given by the three logical hexagons (the Sesmat-Blanché one plus his own two new ones), 6 of them appear twice (for instance, *p* and ¬*p* appear both in the paracomplete and in the paraconsistent hexagon). So he has the (brilliant) idea of joining the common vertices, and thus looking for some global structure uniting these three logical hexagons. Again, Béziau stresses that, as subalternation arrows do not count as oppositions, it is useless to draw them, trying to combine the three “stars” is sufficient. As there are 12 different vertices, the best solution seems to him to be a three-dimensional structure, which he believes appropriate to identify with the one mathematically ordering at best 12 vertices, that is, mathematics say, the “stellar dodecahedron” (12 vertices and 60 triangular faces).



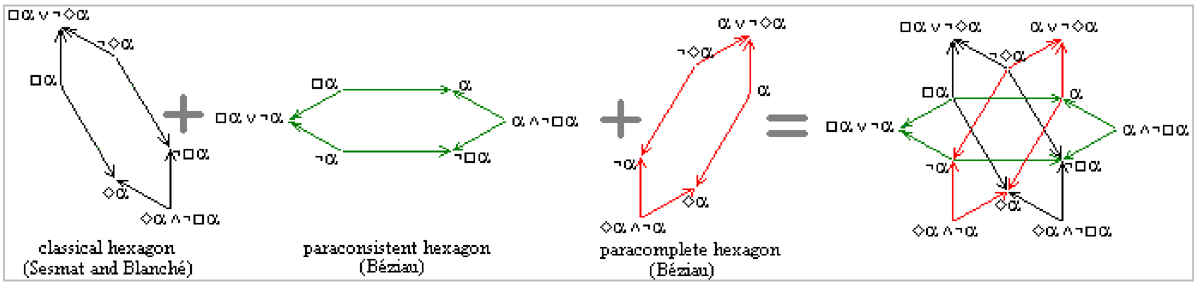
By reaching such a geometrical-logical figure, Béziau believes he makes an important step in answering Slater’s challenge: “The Stellar Dodecahedron of Opposition permits to have a better understanding of the concept of negation in its plurality and of its relation with possibility and necessity, by presenting *the full oppositions* between 12 basic unary connectives” (the italics are ours). Remark however that Béziau never gives (and has never given so far) a representation of this figure (the one in the previous figure is ours).

Supposing the stellar dodecahedron is true, Béziau would then seemingly be the second person to have had the idea of looking for a tri-dimensional composition of logical hexagons, the first being Blanché with his study on the chain made by the two hexagons of binary connectives – at least if one credits Blanché of the remark of such a 3-dimensionality, which is not sure, his drawing being “infinite” (cyclic) but 2-dimensional (cf. ch. 8 *supra*). Remark also that these strange but stimulating novelties (i.e. the existence of an unsuspected “geometry of oppositions”) makes (potentially) much sense from the point of view of universal logic. For, if confirmed, it seems to verify Béziau’s intuition of a specificity of the logical with respect to its “sisters”, that is the topological, the algebraic, and the order-theoretical domains of mathematics.

And this is where, thanks to these pioneering discoveries by Béziau, my own oppositional adventure begins.

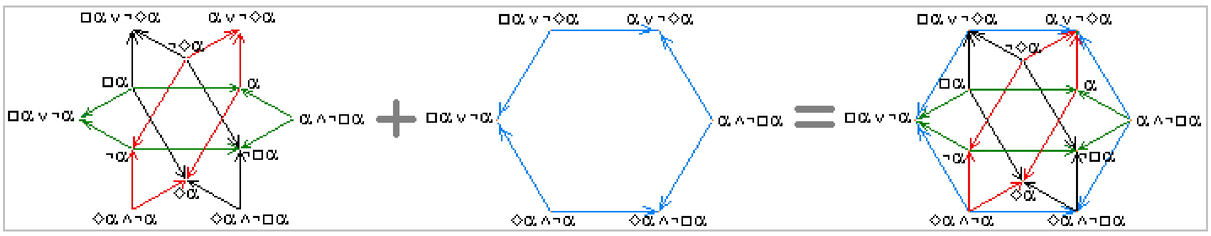
10.05. Moretti’s and Smessaert’s reactions. Béziau and Moretti’s “logical cuboctahedron”

In his paper (the draft as well as, later, the published one) Béziau – let me stress it again – gave no representation of his “stellar dodecahedron of oppositions”. So, having his draft in my hands (I happened to know the work of Blanché since 1995, a work which I admired) and being unsatisfied by the lack of explicit visual representation, I built one for myself with matches (big matches, spaghetti had failed!). With a small difference however: instead of building at once what Béziau said (a stellar dodecahedron), I took into consideration hexagons instead of stars (I took into consideration the subalternation arrows – to me there were logical hexagons, not “stars”).



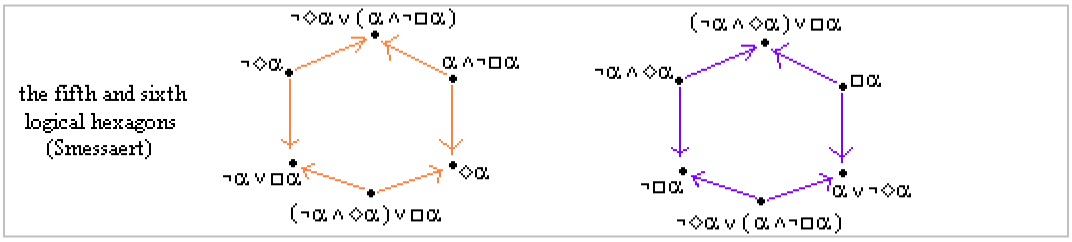
classical hexagon (Sesmat and Blanché)
paraconsistent hexagon (Béziau)
paracomplete hexagon (Béziau)

This led me to two unexpected remarks: (1) the solid in question let emerge a fourth logical hexagon, and (2) this solid had not 12, but 14 faces: it was a tetradecahedron, not a dodecahedron, and it was not stellar (i.e. it had no “spikes”).



I immediately wrote an e-mail to Béziau, thinking this (not mentioning a fourth, emergent hexagon) was just some kind of typo in his draft paper. But he answered me with a very dry "no, there is no mistake at all in what I wrote: the structure has 12 faces". In honour of Béziau, who had had the brilliant ideas (1) to remember the existence of the logical hexagon, (2) to take into consideration null modalities and then (3) to try to combine spatially the thus obtainable hexagons I baptised this arrowed structure – which I seemingly was the first to contemplate visually and conceptually – the "Béziau-Moretti tetradecahedron" (14 faces). Later I learned in a book that this solid (arrows aside) is an "Archimedean solid" and is more precisely called "cuboctahedron".

The Belgian linguist and logician Hans Smessaert, reading Béziau's draft almost at the same time, made partly similar remarks: he didn't see the cuboctahedron (he didn't use matches !) but by purely combinatorial means applied over the underlying logic, he noticed like me (independently and simultaneously) that there was a fourth hexagon, and in fact he discovered that there are even two more logical hexagons. In total: three more hexagons than what Béziau thought, in other words, 6 logical hexagons.



10.06. The need of a new framework for oppositional geometry

To me such new discoveries meant a clear (exciting) sign of the fact that something was going on: these objects were mere fragments of some much bigger theory waiting to be discovered. Philosophically speaking, they seemed to me at first glance to signify some kind of Platonic revenge over Aristotelism: the infinite (i.e. the mathematical) taking back over the finite (i.e. the logical).

As for Smessaert's two hexagons, having not understood how Smessaert reached them, I had not understood that there would be no other beyond (Hans had). So I feared a not mastered explosion of logical hexagons, making things harder to grasp. So, as a pragmatic choice (not to despair in front of an infinity of hexagons possibly popping up everywhere by propositional combinations), I decided to ignore them "as being ugly"![181]

[181] In which judgement, in some sense, I was uglily wrong.

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Part II

THE NEW THEORY OF OPPOSITION: THE LOGICAL BI-SIMPLEXES OF DIMENSION *M*

11.
“*N*-OPPOSITION THEORY” (2004): THE “LOGICAL BI-SIMPLEXES OF DIMENSION *M*”

In this chapter we recall the basic tenets of the “theory of *n*-opposition” (for short NOT), our generalisation of Aristotle’s theory (seen as a theory of 2-opposition) to any number *n* ($n \in \mathrm{N}$, $n \geq 2$) of opposed terms. This theory (which we proposed in 2004) is based on the notion of “logical bi-simplex (of dimension *m*)” and on the distinction made between the “modal graph” and the “oppositional structures” of any system of modal logic (although the theory concerns quantification and opposition, more broadly than just modalities)[182].

11.01 The need of a general geometrical-logical framework

A question urges by now. What are, truly speaking, the logical square and the logical hexagon? What is the logical cuboctahedron? What are we speaking about? Of logic? Of geometry? Of something intermediate? As we have seen (cf. ch. 10 *supra*), because of Slater’s criticism opposing the square of opposition’s geometrical constraints to the general idea (and definition) of logical paraconsistency, and because of Béziau’s answering strategy recalling the discovery of the hexagon (rather than the square) of opposition, we need to take these few (by now 3, qualitatively speaking) strange logical-geometrical objects (square, hexagon and cuboctahedron) seriously. We have to understand what conceptual framework they really belong to.

11.02. Towards a theory of opposition for *n* terms (*n*-opposition)

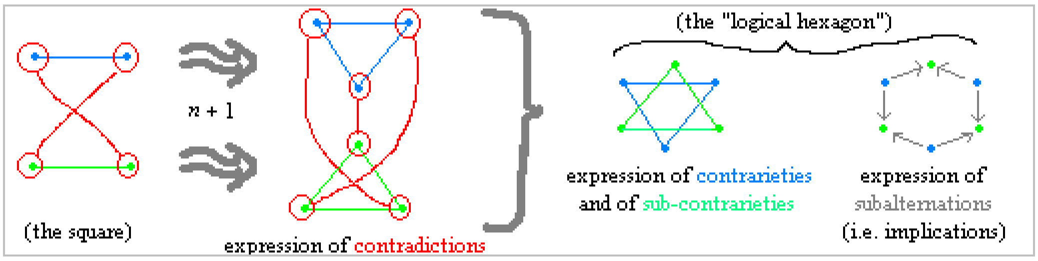
Now, we saw that the logical square exhibits oppositions between two terms. Sesmat and Blanché showed us that by means of logical hexagons we can express (and better understand) opposition between three terms. The question arises naturally as to whether there is a limit to the expression of opposition between an arbitrary number of terms: will this generate new logical-geometrical objects? Which shape will they have? Which laws will

[182] For all this chapter, cf. A. Moretti, “Geometry for Modalities? Yes: Through *n*-Opposition Theory”, in: Béziau J.-Y., Costa-Leite A. and Facchini A. (eds.) *Aspects of Universal Logic*, Cahiers de logique - Université de Neuchâtel, December 2004.

govern them? To these first basic questions, *n*-opposition theory (NOT) gives a precise answer.

11.02.01. From square to hexagon: what happened ?

Let us try to express more clearly what is going on when passing, à la Sesmat-Blanché, from the logical square to the logical hexagon. As we know (cf. ch. 8), the (blue) segment of contrariety becomes a (blue) triangle of contrariety (as Vasil'ev already knew) and, simultaneously, the (green) segment of subcontrariety becomes a (green) triangle of subcontrariety, the two triangles being thus mutually logically dual (this is the point missed by Vasil'ev, cf. ch. 7).

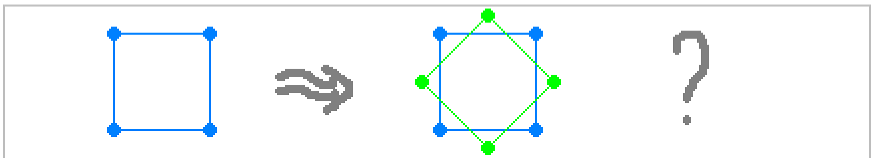


As shown by the previous figure (and according to Sesmat's and Blanché's discovery), the (blue) triangle of contrariety and the (green) triangle of subcontrariety form together a logical hexagon. But apparently on three conditions: (1) the two dual triangles are so interlaced that their contradictory points (each blue point is contradictory to one and only one green point) are symmetrical two by two, with respect to a central symmetry; (2) the contradiction relations are thus the (red) diagonals, intersected in the centre of symmetry of the figure (this does not appear here in the left part of our pedagogical representation); and (3) *all* subalternation arrows are expressed: there is exactly one going from each blue point to each green point not contradictory (i.e. not centrally symmetrical) to the starting blue point (the coherence of all these thus expressed opposition relations can be checked, as Sesmat and Blanché did).

11.02.02. Opposition for 4 terms: the "logical cube" of opposition

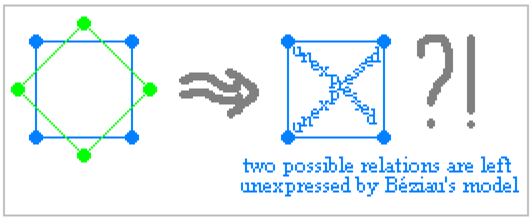
Having reinvested (or resuscitated) the field of the logical hexagons, Béziau tried to argue (in a convincing, somewhat Vasil'evian, way) that oppositions, in some sense, are instances of "dichotomies" (as in the logical square) and "tri-tomies" (as in the logical hexagon): they "divide" (something). The main change in passing from the logical square to the logical hexagon consists in showing that division needs not to be binary. So this vision inspiring Béziau in his interest, as a logician, for a geometrisation of the foundations of the discipline originates in his struggle for a pluralist vision. From this point of view, the question arises as to the limit of such a pluralisation of the initially double (a pluralisation of logic).

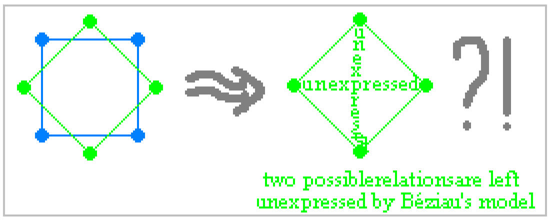
Hence he got the idea of looking for some kind of further step, some kind of quadri-tomy. Is it indeed possible to logically oppose neither 2 nor 3, but 4 terms? Béziau's sketched proposal was that of a "logical quadri-tomy" (the blue and the green squares interlaced).


Béziau's idea of looking for "logical quadritomy"

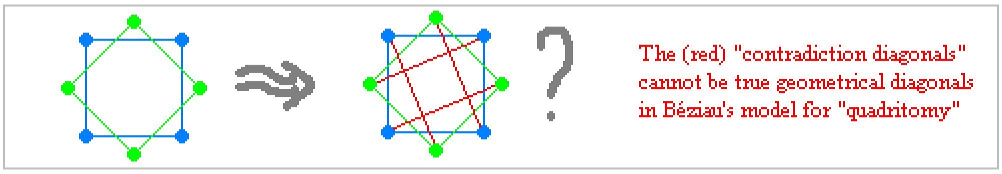
But, as I pointed out, there are at least three problems with this kind of solution:

(1) it does not respect graphically the equidistancy (for: the four points should be on a same logical plane: their distance, two by two, should be the same – at least so it happens, with the success that we know, in the logical hexagon); moreover, not all contrariety and subcontrariety relations are expressed if one uses blue and green squares;

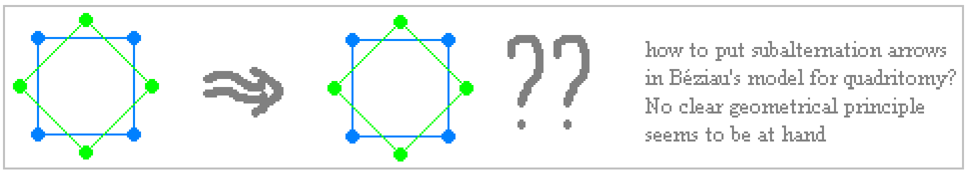




(2) it does not express by symmetry (red diagonals) contradictory negations: with Béziau's model, each point of the (blue) square of contrariety is centrally symmetric (but not contradictory!) to another point of the same square, and each point of the (green) square of subcontrariety is centrally symmetric (but not contradictory!) to another point of the same square;

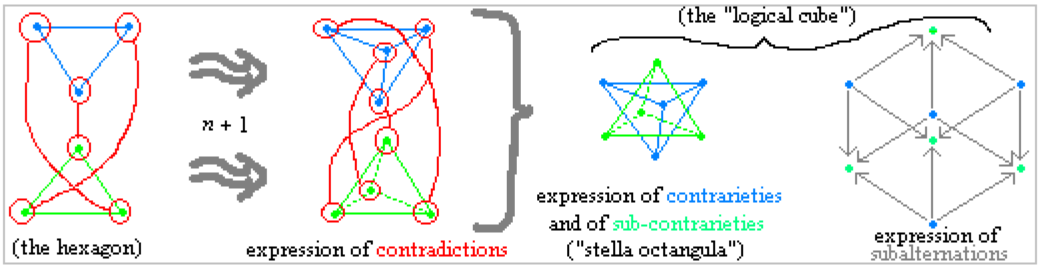


(3) and one does not see how to draw (easily) the subalternation arrows (as it so happens, Béziau was not interested in arrows, preferring the stars).



So, if one considers that the main problem with Béziau's extension of logical tri-tomy to logical quadri-tomy is that the equidistance condition (for each couple of contraries) is lost, the most natural step seems to be to realise that the geometrical discrete space for expressing equidistance (of the contraries) for 4 points is not the square, but the (tri-dimensional) tetrahedron, in which any couple of its 4 points (vertices) is indeed equidistant. And the tetrahedron expresses simply but perfectly *all* the possible couples of contrary terms in a given group of 4.

So we followed that idea, we took a blue tetrahedron. The second very natural step was to build a dual structure for subcontrariety, always keeping the analogy with the cases of the logical square and hexagon: this gave a green tetrahedron of subcontrariety (the key intuitive principle remaining central symmetry). This logical bi-tetrahedron, once furnished with its arrows (of subcontrariety), gives – oh surprise – a "logical cube".



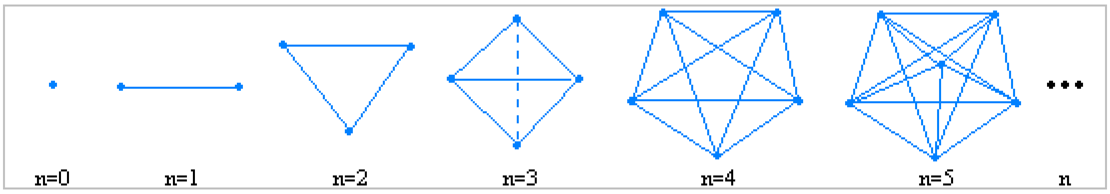
So the solution to Béziau's problem of having logical quadri-tomy is a logical cube of opposition[183].What's next? What is the geometry of the equidistance for 5 points? Can it be 3-dimensional (like the tetrahedron)? Mathematically speaking, it turns out that such a solution to the problem of having a logical penta-tomy cannot be 3-dimensional. One could imagine, as logicians after Vasil'ev often do, that it could be non-Euclidean. But it seems difficult to imagine what such a non-Euclidean solution could be like.

11.02.03. The general solution is *beyond Kant*: the "logical bi-simplexes"

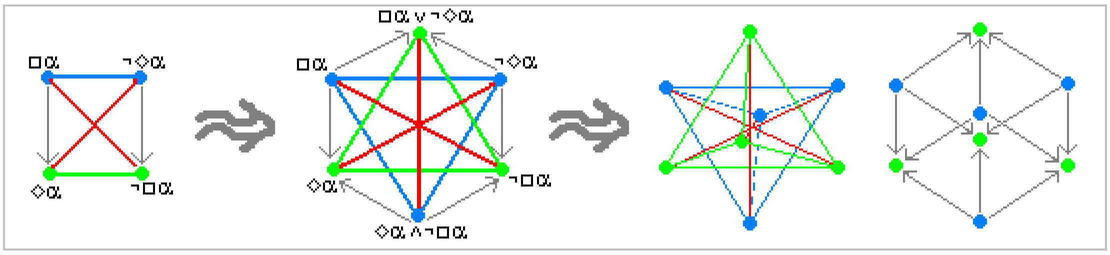
In fact, there is a standard mathematical answer to this problem of finding a geometrical expression of *n* points so deployed in space as to be 2 by 2 equidistant: it is given by the theory of the "geometrical simplexes" (or simplices), of which I luckily happened to

[183] Three remarks here: (1) Sesmat, in 1951, had already thought of a "bi-tetrahedric scheme" (cf. ch. 8 *supra*). (2) But Sesmat, who draws the subalternation arrows badly (he draws them as simple straight lines, instead of as implication arrows) apparently did not notice that the bi-tetrahedron of 4-opposition, once "arrowed", becomes (or reveals itself) a "logical cube" (of opposition). (3) Sesmat thinks that the next figures (i.e. those hypothetically following the "bi-tetrahedric scheme") are neither easy to conceive nor *a fortiori* easy to represent (cf. ch. 8 *supra*).

have some (very basic) notions[184]. This "goes beyond Kant", in the sense that it postulates the mathematical existence of geometrical spaces furnished with arbitrarily (but finite) high dimensionality. It is embodied by a series of very basic mathematical objects, more precisely geometrical figures, where each element of the series is obtained from the previous one by adding one point and by linking this point to all previous ones, the series starting with one point alone. Each term is said to be a simplex of dimension *m*, with *m* being the number of dimensions of the space to which the considered simplex belongs (the point is 0-dimensional, the segment is 1-dimensional, the triangle is 2-dimensional, and so on).



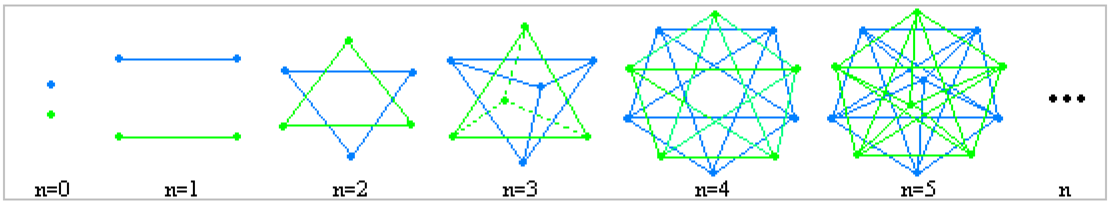
This series seems to possibly be the relevant one for us in so far as the segment, triangle and tetrahedron (the main ingredient of, respectively, the logical square, hexagon and cube) are precisely terms of it: they are respectively the geometrical simplexes of dimension 1, 2 and 3.



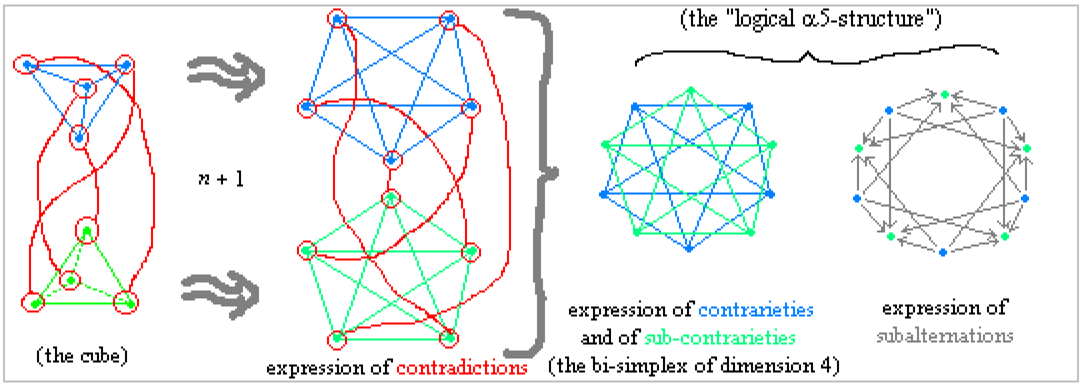
In other words, the logical square, the logical hexagon and the logical cube are, in some sense, "logical bi-simplexes": they are, respectively, a logical bi-segment, a logical bi-triangle and a logical bi-tetrahedron.

In a logical bi-simplex of dimension *m* we have, for any *m* ($m \in$ N, $m \geq 1$) one (blue) simplex of contrariety and its dual (= centrally symmetric) (green) simplex of subcontrariety. The crazy idea will be of generalising our {2, 3, 4}-opposition (logical square, hexagon and cube) for any *n* (going, philosophically speaking, let say it again, "beyond Kant", cf. ch. 3 *supra*). Indeed, now it should be possible to build further "geometrical bi-simplexes".

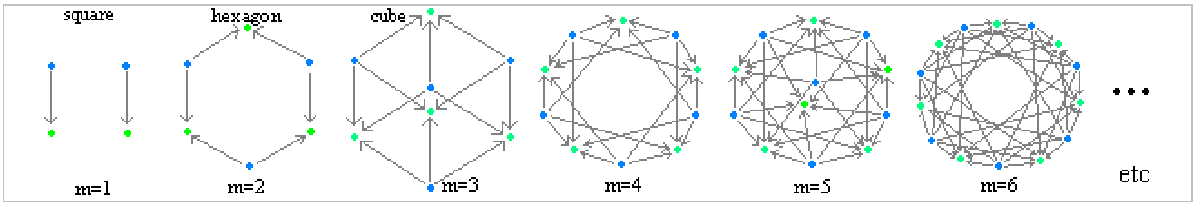
[184] As a funny or astonishing detail, it was the study of the psychoanalytical model of the mind given by I. Matte Blanco (1975), a pupil of M. Klein, which had forced me to have some notions of what a 4-dimensional object can be, and T. F. Banchoff's book on the fourth dimension (*Beyond the Third Dimension: Geometry, Computer Graphics, and Higher Dimensions*, Scientific American Library Series, 1990) who had taught me some facts about the series of the geometrical simplexes. On I. Matte Blanco, cf. *The Unconscious as Infinite Sets, op. cit.*, and E. Reyner, *Unconscious Logic. An Introduction to Matte-Blanco's Bi-Logic and its Uses*, London, Routledge, 1995.



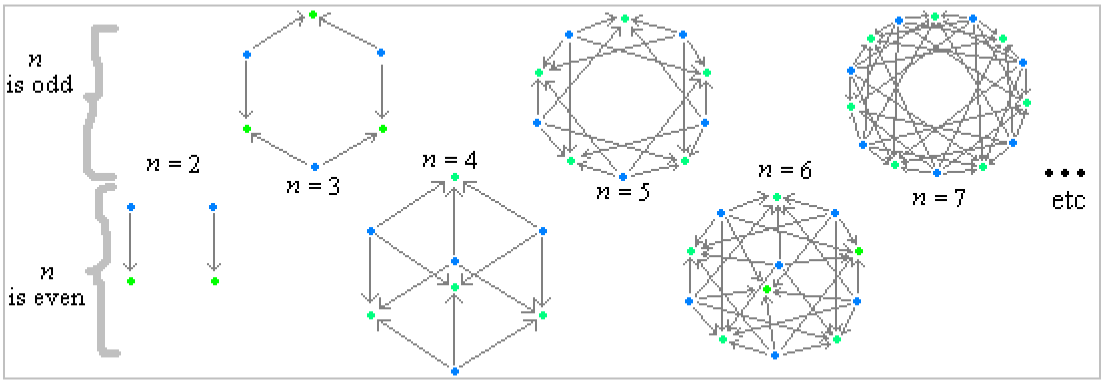
As an illustration of this general algorithm, and in order to test our conjecture, let us try to do it beyond the logical cube.



We can easily see that, geometrically speaking, this works. Hence, apparently, the series of the "*logical* bi-simplexes".



Remark that graphically speaking there are two cases, the odd one and the even one. In the odd one, the logical bi-simplexes are kinds of circles, in the even one they add two points inside the previous circle (not the square, which has reasons to have a particular behaviour).



For reasons to appear soon, we call the logical bi-simplexes "α*n*-structures" (the β*n*-structures will be the gatherings of α*n*-structures, like the logical cuboctahedron is some kind of gathering of logical hexagons)

Remark that the logical bi-simplexes each have a different geometrical dimension. In spite of appearances, the logical square is a one-dimensional figure (for other geometrical representations, cf. ch. 12 *infra*, where we will examine Smessaert's own discoveries).

11.03. Distinguishing modal graphs and oppositional structures

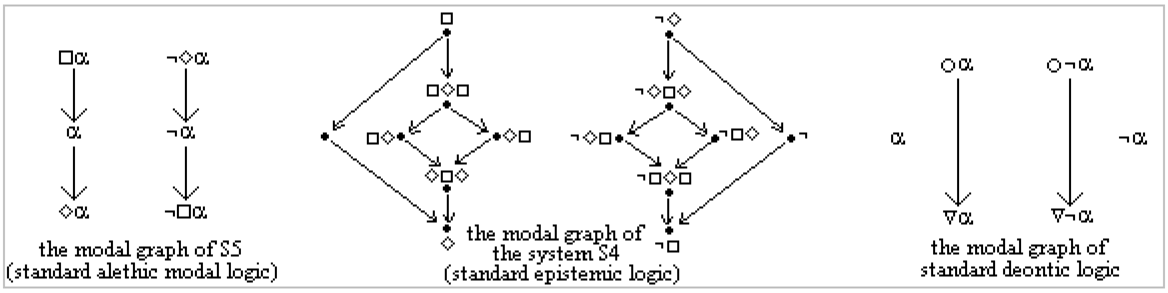
The NOT had so far given a general theory of the "naked" oppositional structures (the α*n*-structures): it had given a simple theory of the logical bi-simplexes (of dimension *m*), that is, a generalisation of the logical square (the α2-structure, or logical bi-simplex of dimension 1, or logical bi-segment) and of the logical hexagon (the α3-structure, or logical bi-simplex of dimension 2, or logical bi-triangle). But the avatars of the square and hexagon so granted, that is, the logical cube (the α4-structure, or logical bi-simplex of dimension 4, or logical bi-tetrahedron) and its successors, were not yet granted a modal decoration: nothing proved, so far, that these nice solids of *n*-opposition could receive, in each of their 2*n* blue or green points, logical (and more precisely modal) values. The oppositional skeletons needed modal flesh. So, the second move of the theory consisted in trying to find a way to furnish some kind of general decorating principle[185]. The simplest way seemed to me to reason by analogy, starting from what was already known (the Lewis system S5 of classical logic, and also the system S4) and then by extrapolating.

11.03.01. The paradigmatic case(s) of S5 (and S4): "basic modalities" and "modal graphs" (i.e. γ-structures)

As is well known, the Lewis system S5 is the modal logical counterpart of classical contemporary logic. In other words, when dealing with classical geometrical-logical solids (like the logical square and hexagon), one is apparently walking implicitly through S5. What do we know about this system? Standard treatises on modal logic, as for instance Chellas'[186], teach us that each modal system has some kind of unmistakable passport (or modal signature): this is (a) the set of its "basic modalities" and (b) the oriented graph expressing two kinds of possible relations (implication and negation) among such basic modalities. A "basic modality" of a modal system is one which cannot be reduced *logically* to smaller ones (for instance, in S5, which has 6 basic modalities, "□□p" is not a basic modality, because in S5 we

[185] I took this notion of "decoration", perhaps deforming it a little bit, from my reading of Etchemendy and Barwise, *The Liar. An Essay on Truth and Circularity*, Oxford - New York, OUP, 1989 (1987).

have “□□p↔□p”, and “□p” is smaller than “□□p”, etc.)[187]. As these graphs seemingly had no name (and no explicit theory: apparently they were only pedagogical devices), I called them “modal graphs”. Modal graphs seemingly present the following features: (1) for any system, they show (by explicit arrows) the logical implications between the basic modalities of this system; (2) for each system, they express (by central symmetry) the contradictory negations between couples of basic modalities of this system; (3) as a construction rule, modal graphs are built so that the arrows always tend to go downwards: so the implying basic modalities are at the top, while the implied basic modalities are at the bottom; (4) in the modal graphs of the known modal systems, the only negation expressed (by central symmetry) is the classical one: this means that all known modal graphs have two “sides”, a left one (for the positive basic modalities) and a right one (for the contradictory negations of the positive basic modalities). These two sides apparently having no names in the literature, I called them “modal columns”. Standard modal graphs (i.e. the known modal graphs) have only 2 modal columns (left and right). Here are for instance two very well known modal graphs: the one of S5, the one of S4 and the one of standard deontic logic.



the modal graph of S5 (standard alethic modal logic)
the modal graph of the system S4 (standard epistemic logic)
the modal graph of standard deontic logic

Remark that the first modal graph is linear, whereas the second one is branching and the third one has “isolated points” (we will come back on this important distinction in ch. 16 *infra*).

Now, the crucial remark is that in the case of S5 we have two seemingly similar, but in fact strictly distinct things: (A) its “modal graph”, which is very simple (6 basic modalities and 4 arrows) and bi-dimensional, and (B) the gathering of all its known decorations of oppositional structures (the logical cuboctahedron, containing 4 logical hexagons), which is more complex than the modal graph and tri-dimensional[188].

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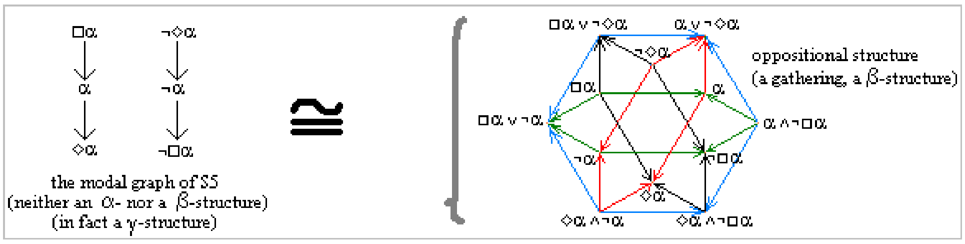
[186] Chellas, B. , Modal Logic. An Introduction, Cambridge, CUP, 1980.

[187] The system S4 has 14 basic modalities. Some systems, as S1 and S2, have an infinite number of basic modalities. Notice that if a modality is a basic modality, its negation is also a basic modality.

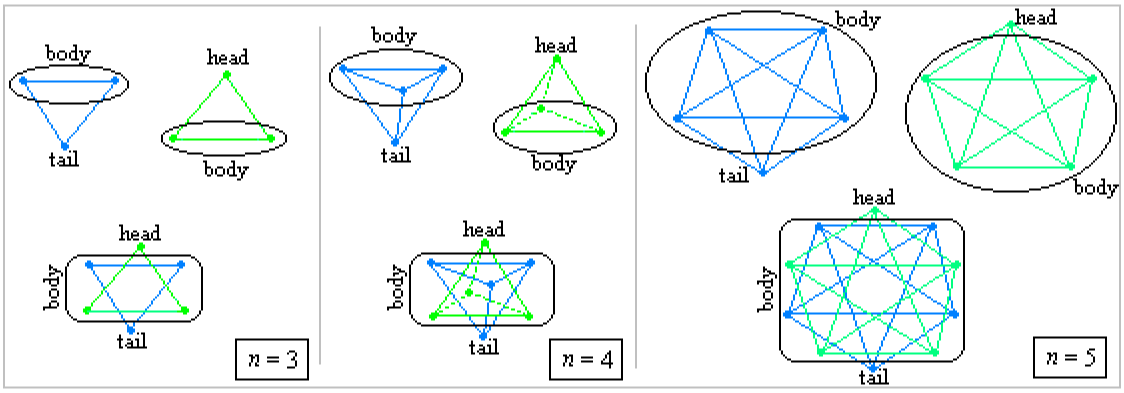
[188] Here I omit for simplicity’s sake Smessaert’s 2 logical hexagons, which in fact do belong as well to S5.



So it seems important, in the case of S5, to distinguish the two geometrical-logical objects: (i) the modal graph and (ii) the oppositional structures. It seems natural to suspect that this distinction is very important in general. Remark that sooner or later some kind of theoretical bridge linking systematically modal graphs and oppositional structures will be necessary (cf. ch. 16 *infra*).

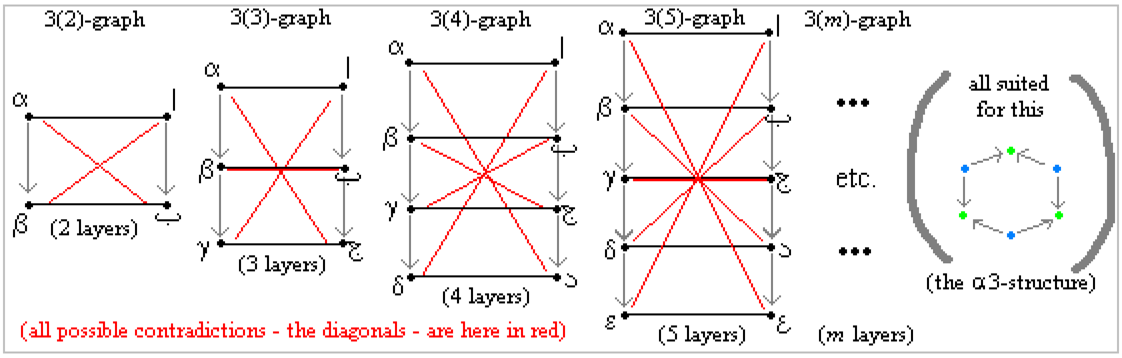
11.03.02. Decorating: the theory of the "heads and tails"

Our problem being that of decorating the logical bi-simplexes, let us look once more at what happened when we passed from the logical square to the logical hexagon, this time from the point of view of their decoration. In a sense, it happened that we added to the square one "head" term (the logical disjunction of the two contrary vertices of the square) and one "tail" term (the logical conjunction of the two subcontrary vertices of the square). And the starting vertices were simple (in fact basic) modalities, whereas the head and the tail are composed terms. So, if we think of each possible logical bi-simplex dealing with *n*-opposition as being constituted of *n*-1 basic contrary vertices plus one head (the logical disjunction of the *n*-1 contrary vertices) and one tail (the logical conjunction of the *n*-1 subcontrary vertices), by analogy with the case of the decoration of the logical hexagon, it seems possible, in principle, to grant a decoration to any conceivable logical bi-simplex (no matter how big *n* is), provided that we are able to modally decorate the *n*-1 basic contrary vertices: the 2 remaining vertices will be decorated as head and tail, that is, simply by logical composition (disjunction or conjunction of already decorated vertices).



11.03.03. From S5's modal graph to the "linear modal *n*(*m*)-graphs"

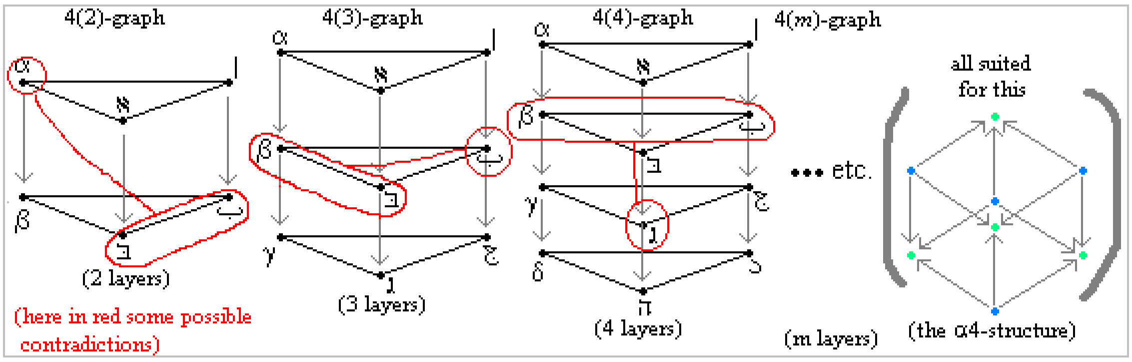
So, the idea was to have modal graphs (i.e. γ-structures) with an arbitrary number of modal columns. Is it possible? For terminology's sake, we first named the simple extensions of the modal graph of S5 (which is linear). We did it as shown in the following picture:



It represents the "modal 3(*m*)-graphs: "3" means that we deal with linear graphs suitable for 3-opposition – i.e. with α3-structures –, while *m* means the number of "layers" of the modal graph (here we have two modal columns, one of Greek terms on the left side, another one of Arabic terms on the right side. Each Greek term is the contradictory negation of one and only one Arabic term, and *vice versa*. Each point is contradictory to another point, by central symmetry).

So, we already know that the 3(3)-graph suffices for decorating (at least) 4 logical hexagons (those of the cuboctahedron): the 3(*m*)-graph should (and does indeed) furnish the decoration of many more logical hexagons (this can be proven, cf. Moretti 2004). As I saw no way of decorating logical cubes with such 3(*m*)-graphs (but of this I gave no proof), I resolved to turn to modal graphs with 3 instead of 2 modal columns, in order to decorate the freshly discovered (but not yet decorated) logical cube.

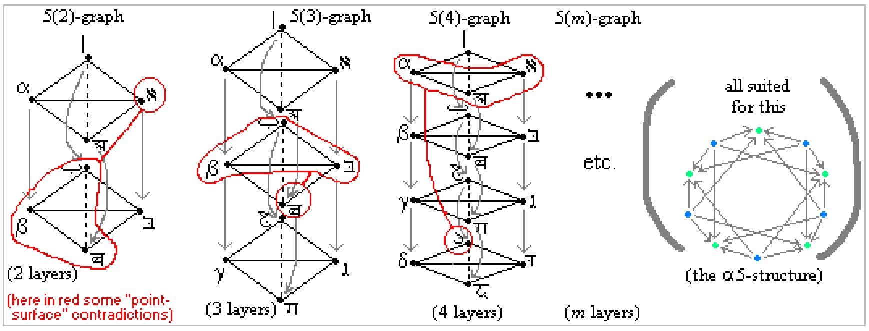
Here is, graphically sketched, what I proposed to be the series of the "modal 4(*m*)-graphs".



How does it work? First of all, we add one modal column to the standard two, so we have three, one column of Greek terms, another one of Arabic terms and a third one of Hebrew terms. We entered a space where, in some sense, negation is not binary, but ternary. Then the principle is nevertheless simple: it keeps all the mentioned features of the standard modal graphs and in particular the fact of expressing negation by central symmetry. But here, points are symmetric not to other points (as in the standard modal graphs), but to segments. Then the semantic interpretation we proposed was to read the segments as the logical disjunction of the two points delimiting them. So, each term of any modal column is logically contradictory to the disjunction of the two terms (belonging to the other two modal columns) delimiting the segment centrally symmetric to the starting term (as shown in red in the previous figure).

So, it is possible to decorate a logical cube using a 4(m)-graph: one has to take one term[189] in each of the three modal columns of the 4(m)-graph and give it respectively to three out of the four blue terms of the logical cube; by central symmetry inside the cube, three out of the four green points will be immediately decorated in turn, and the two remaining points will be decorated as head (for the fourth green point) and tail (for the fourth blue point). This algorithm suffices for decorating logical cubes, provided one has 4(m)-graphs.

Similarly, we can introduce so-called "modal 5(m)-graphs" in order to decorate the logical α5-structure (the logical bi-simplex successor of the logical cube). The important point here (analogous to the previous one) is that we have 4 modal columns (Greek, Arabic, Hebrew and Indian) and that contradictory negation here, always expressed by central symmetry, links points neither to points, nor to segments, but to surfaces (triangles): such surfaces (triangles) are interpreted as the logical disjunction of the three points delimiting the triangular surface. So here each point of each modal column will be the contradictory negation of (or contradictory negated by) the disjunction of the three points (each belonging to a modal column different from that of the starting point) delimiting the triangular surface centrally symmetric to the starting point.



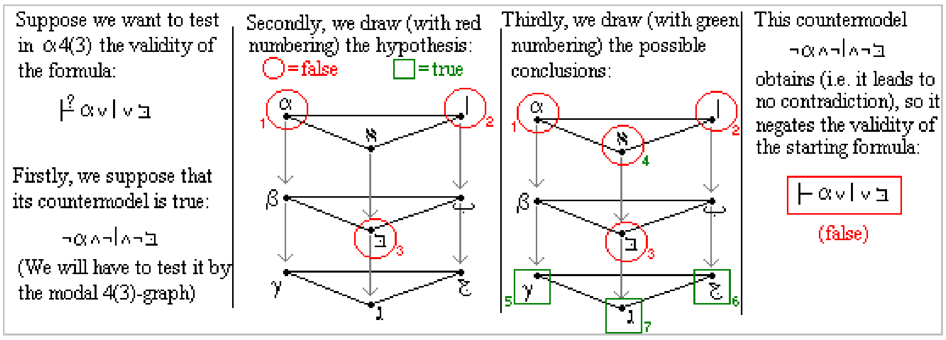
[189] When I use the word "term", I do it without any philosophical interpretation (like is done traditionally when speaking about a "logic of terms"). By "term" I just mean "object" (or "item", or "point", or "vertex", etc.).

This device suffices to decorate modally any α5-structure: four of its 5 blue points will be decorated directly with values taken from each of the four modal columns of the modal 5(*m*)-graph, while the fifth blue point will be a tail (dually for the other 4 green points and last blue point). Because it also relies (as do the bi-simplexes) on the notion of geometrical simplex (I called "gems" the black simplexes of the previous figures, the "arrowed" stakes of which constitute the modal *n*(*m*)-graphs), this algorithm can be easily generalised to any modal *n*(*m*)-graph, making it possible to modally decorate in this way any α*n*-structure.

11.03.04. Using the linear modal graphs as modal oppositional truth tables

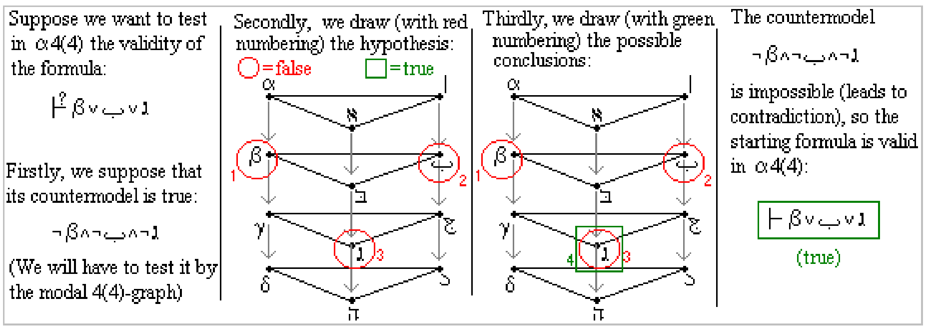
Having no metatheorem for my theory yet, I pursued investigating a bit empirically its bare practical aspects (i.e. its viability). One good point of the modal *n*(*m*)-graphs was that they apparently allowed coherent modal calculations (about oppositions).

Let us consider first an example of refutation in the logical space of the modal 4(3)-graph (thus a universe of, so to say, ternary negation). Let us check the validity of the abstract modal formula "α∨ا ∨ב".



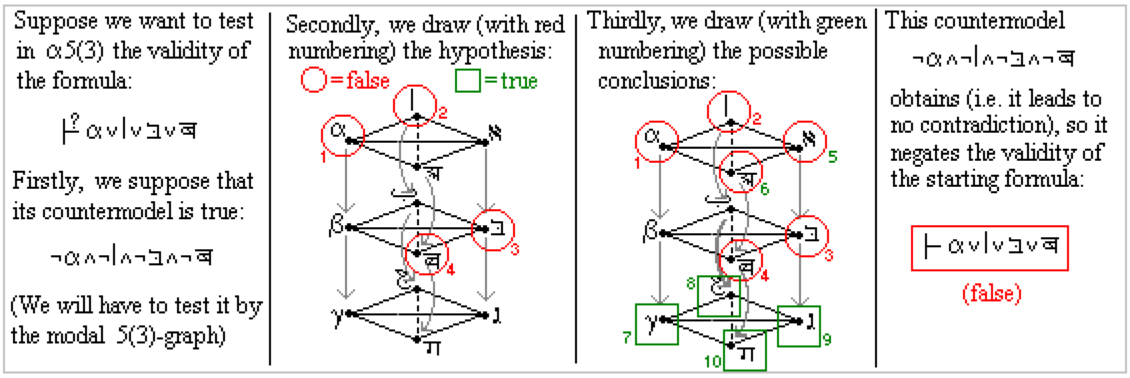
So, the abstract modal formula "α∨ا ∨ב" turns out to be invalid in the abstract logical space of the modal 4(3)-graphs. Remark that this result is totally general! It holds for any modal value put in place of the modal variables α, β, γ, ا, ب, ج, א, ב, ג. And this shows that it is possible to falsify formulas using modal 4(3)-graphs.

Let us consider now an example of demonstration inside the abstract logical space of the modal 4(4)-graph. We want to check the abstract modal formula "β∨ ب ∨ ג". Is it a valid formula of the modal 4(4)-graph?



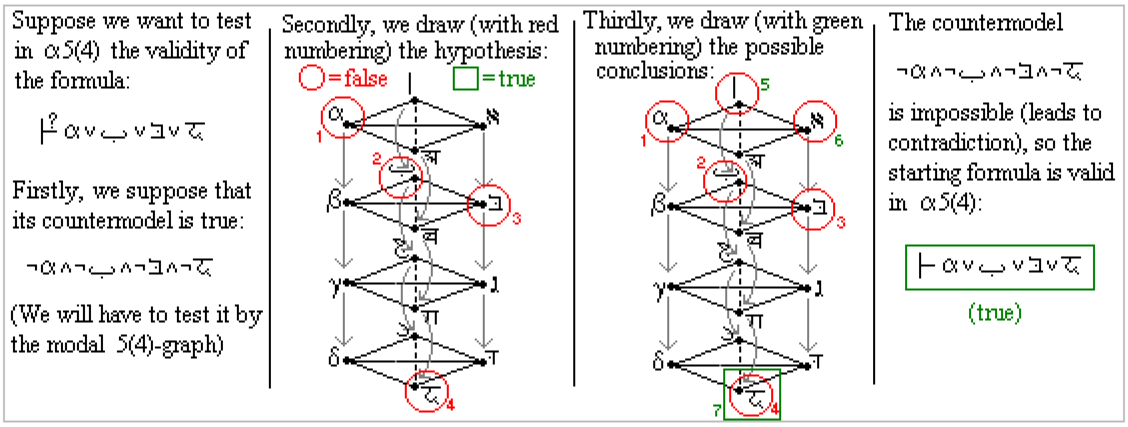
Yes: the modal 4(4)-graph allowed us to demonstrate a theorem of this abstract logical space, in it the abstract modal formula “β∨ ب ∨ ג” is true, whatever modalities are put in place of the modal variables α, β, γ, δ, ا, ب, ج, د, א, ב , ג, ד.

Another example of refutation is the following, relative to the modal 5(3)-graph (a universe of quaternary negation). In it we want to test the validity of the abstract modal formula “α∨ا ∨ב∨✚”.



In this case, the modal 5(3)-graph falsifies the formula, showing that it is not a theorem of this abstract space. Again, the formula being constituted of empty modal variables (the α, β, γ, δ, ا, ب, ج, א, ב, ג, ◘, ✚, ♦), it can be instantiated by *any* modalities (alethic, epistemic, deontic, temporal, hybrid, …) provided their mutual relations respect the abstract structure of the modal 5(3)-graph.

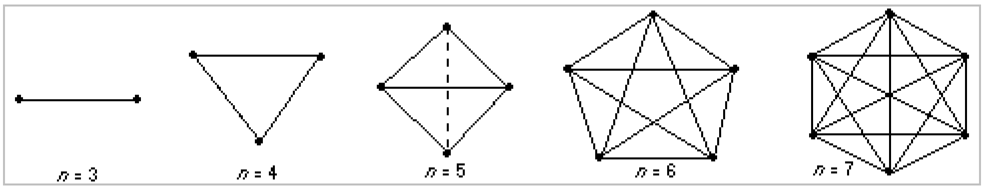
One last example of demonstration, inside the logical space of the modal 5(4)-graph, in order to check the validity of the abstract modal formula “α∨ ب ∨ ב ∨ +”.



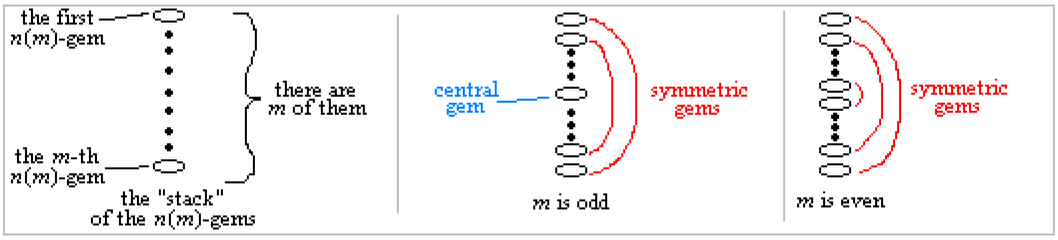
Here the modal graph shows that the starting abstract modal formula is a theorem of this logical space.

11.03.05. General properties of the linear modal graphs

So, supposing (yet without metatheoretical proof) that the modal *n*(*m*)-graph machinery was really the needed tool, it seemed natural to investigate its properties more abstractly. The geometrical simplexes did rescue us once more. As we said, for any modal *n*(*m*)-graph its (black) “gems” are simplexes of dimension *n*–2 (in the following picture *n* is the parameter of the modal *n*(*m*)-graphs of which they are the gems).



My own characterisation, in 2004, viewed them quite empirically as possible abstract stacks. The question of having stacks of qualitatively different gems still remains open. I also gave some simple theorems helping to make quicker truth-table-like calculations over the gems. This implied establishing a suitable nomenclature of the kinds of gems according to their position in the stack or to their symmetries.



For an interesting, unexpected application of these generalised linear modal *n*(*m*)-graphs, cf. a paper by Luzeaux, Sallantin and Dartnell, as well as ch. 17 *infra*)[190]. In that paper I also gave some theorems, among which a rather general one, for handling the gems quickly and making oppositional calculations easier.

11.04. Distinguishing the α*n*-structures from the β*n*-structures

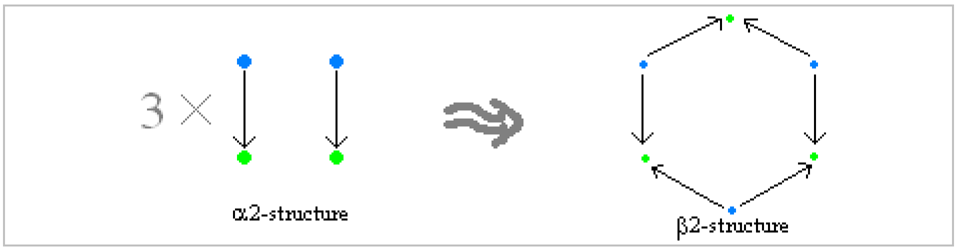
So far we have only dealt with what we called the α-structures of opposition (i.e. the logical bi-simplexes, the successors of the beginning series “logical square, logical hexagon, logical cube, …”). But we saw (cf. ch. 8 and ch. 10 *supra*) that there are also three examples of “gatherings” of oppositional structures: the logical hexagon (made of 3 logical squares),

[190] Remark that in the same paper of 2008, Luzeaux, Sallantin and Dartnell gave a more precise mathematical definition of the modal *n*(*m*)-graphs as being the Cartesian product in the category of graphs of a (*n*-2)-simplex

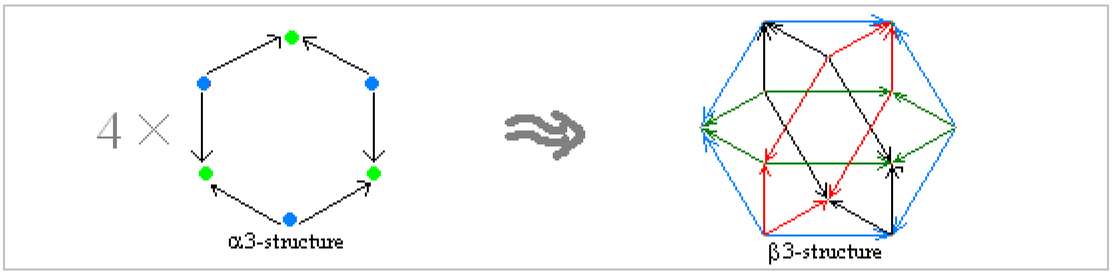
Blanché's chain (made of two hexagons) and our "logical cuboctahedron" (made of 4 hexagons). We call such (now) mysterious gatherings "β-structures". It is about these that we want to say a few words now.

11.04.01. The problem

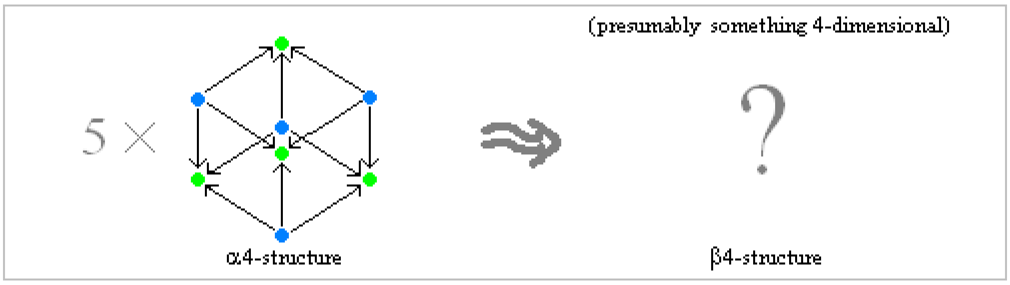
It is useful to come back to Sesmat and Blanché. We saw that there is a gathering of logical squares, that is the logical hexagon, which contains exactly 3 logical squares.



We also saw that there is a gathering of logical hexagons, that is the logical cuboctahedron, which is made exactly of 4 logical hexagons.



Is there, in a similar way, a gathering of logical cubes? How many of them would such a gathering collect? Would it be, by analogy with the previous cases, 4-dimensional? Would it contain, by analogy, 5 logical cubes?



Is there, more generally, a gathering for each of the α*n*-structures? Is there, in other words, a series of β*n*-structures? In 2004 we conjectured that there is indeed such a series of β*n*-structures. It is in order to try to test this conjecture that we proposed, surprised by some results, the notion of "logical hyper-flower".

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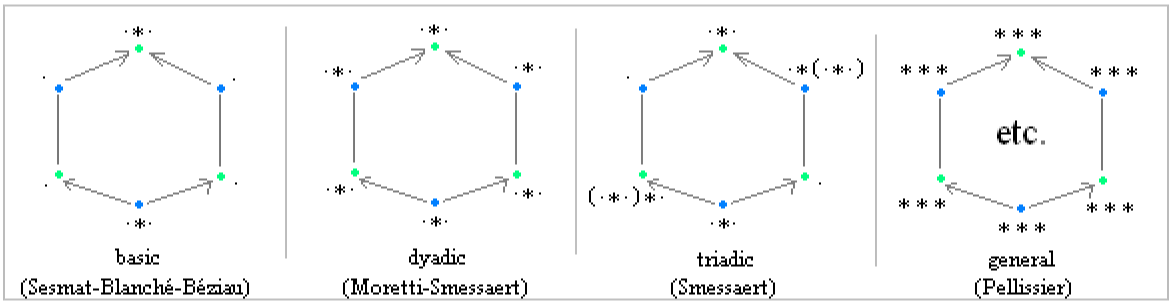
by the oriented graph 1→2→…→*m*, cf. Luzeaux D., Sallantin J and Dartnell C., "Logical Extensions of Aristotle's Square", *Logica Universalis*, 2, 1, 2008, p. 170.

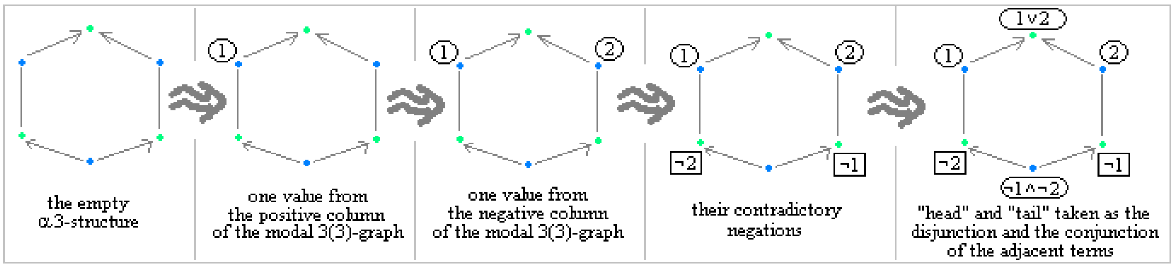
11.04.02. A nice game: the series of the "logical hyper-flowers"[191]

In 2005, taking at last seriously in consideration the specificity of the 5th and 6th logical hexagons of S5 (discovered by Smessaert, cf. ch. 10 *supra*), I proposed a sketch of qualitative typology of logical hexagons. As a matter of fact, one can consider the "propositional logical form" of the modalities (simple or complex) decorating a logical hexagon: the 6 logical hexagons of S5 already suggest the following typology (points represent simple modalities, stars represent logical binary connectives).



One can see that the Sesmat-Blanché (classical) hexagon and the two Béziau (paraconsistent and paracomplete) hexagons of S5 are "basic", whereas the Moretti-Smessaert hexagon is "dyadic" and the two Smessaert hexagons are each a "triadic" hexagon. And there seems to be no limit to the growth of compositional complexity.

Still guided by analogy, I proposed a decoration principle, based on what we knew about the decorations relative to the modal 3(3)-graph (the one of S5).

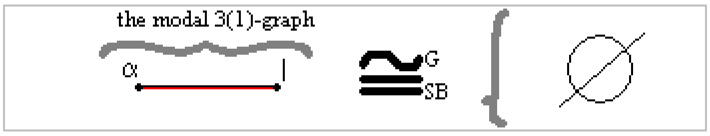


This method allows to go from an arbitrary modal 3(*m*)-graph to a set of decorated logical hexagons, simply by checking all the possible combinations of couples of modalities, by taking the first in the first modal column, the second in the second modal column of the modal 3(*m*)-graph.

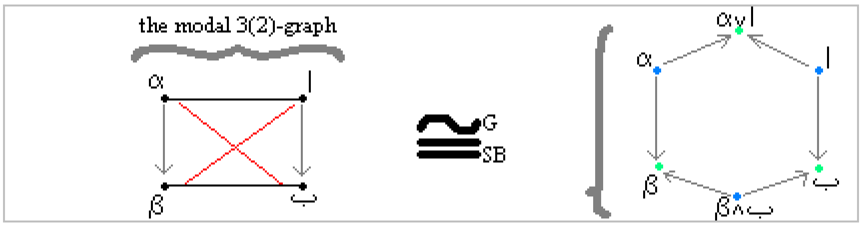
As a next methodical step, I started examining all the increasing modal 3(*m*)-graphs, starting from *m*=1 (for *m* increasing). The result, amazing to me, was the following.

[191] This § 11.04.02 is a summary of a paper of mine still unpublished, "Logical 'hyper-flowers'. The 'logical cuboctahedron' belongs to an infinite (fractal) series of *n*-dimensional solids". It corresponds to a talk I gave at

The modal 3(1) graph generates no logical hexagon.

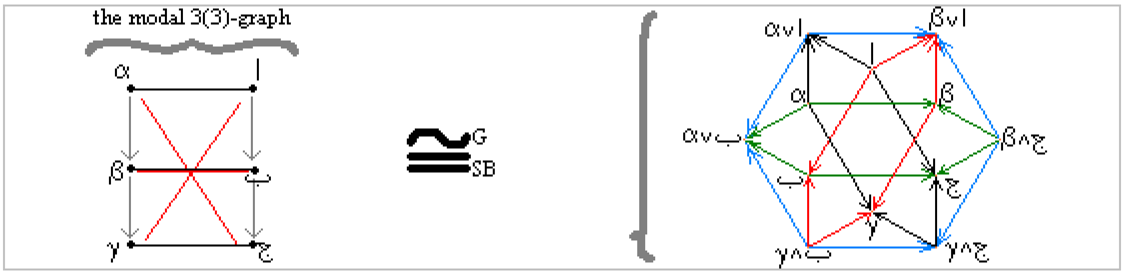


The modal 3(2)-graph generates just one logical hexagon.



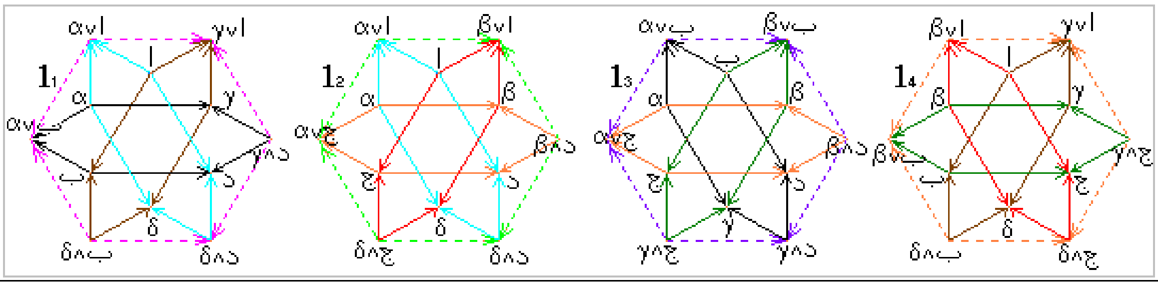
(it is a "basic" one)

The modal 3(3)-graph (the one of the Lewis system S5) generates 6 hexagons: there are 3 "basic" ones, plus a "dyadic" one, plus 2 "triadic" ones. But we know by experience that if we remain concentrated on the "basic" ones, once interconnected, they let the fourth (blue) one emerge (the one I co-discovered with Smessaert).



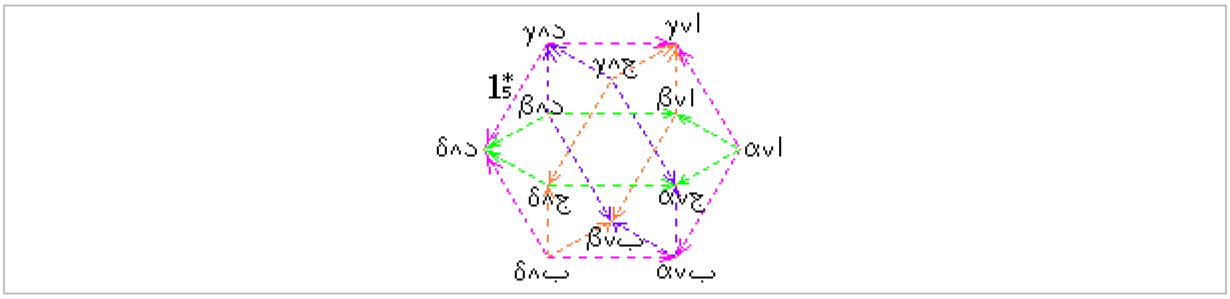
So we take it as being a possible construction rule: remaining concentrated on the "basic" hexagons and, beside them, accepting only the hexagons emerging from the triples of the "basic" ones (so we will refuse all hexagons "à la Smessaert" or more complex). Let us try to play this game further, by letting the 3(*m*)-graphs get longer and longer.

The modal 3(4)-graph generates firstly 6 logical "basic" hexagons (we do not consider the other ones). These can be combined (by triples) so to generate exactly 4 (no less, no more) logical cuboctahedra, provided one accepts (as one has to, according to the rules of our game) 4 new "emergent" logical "dyadic" hexagons (depicted here in scattered lines).

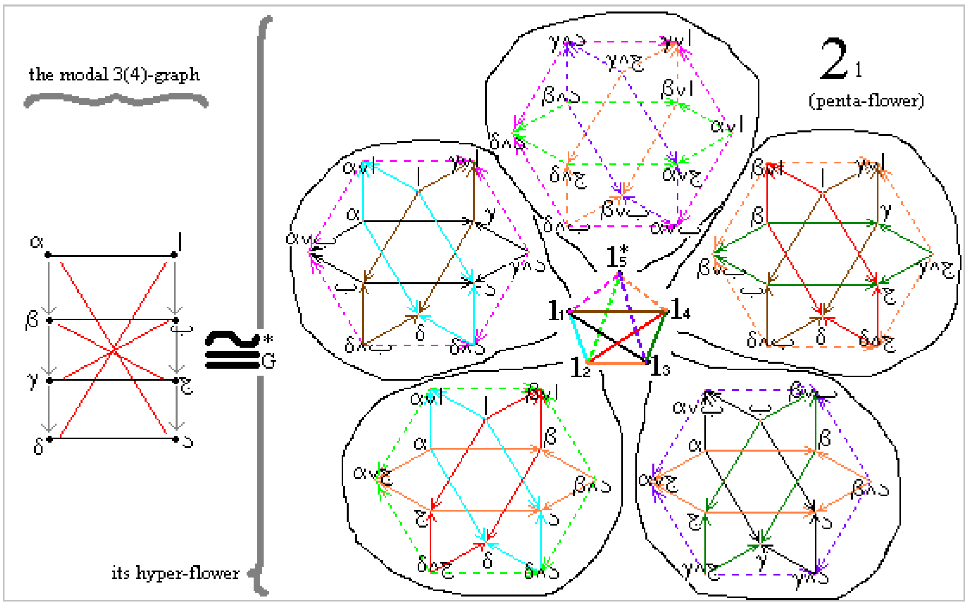


the UNILOG 2005 in Montreux, Switzerland, March 2005 (it is dedicated to the Belgian philosopher and logician Paul Gochet).

But these 4 new emergent logical dyadic hexagons (the scattered ones) do in fact combine together perfectly into a new "emergent" logical cuboctahedron (which we call 1_5^*)!

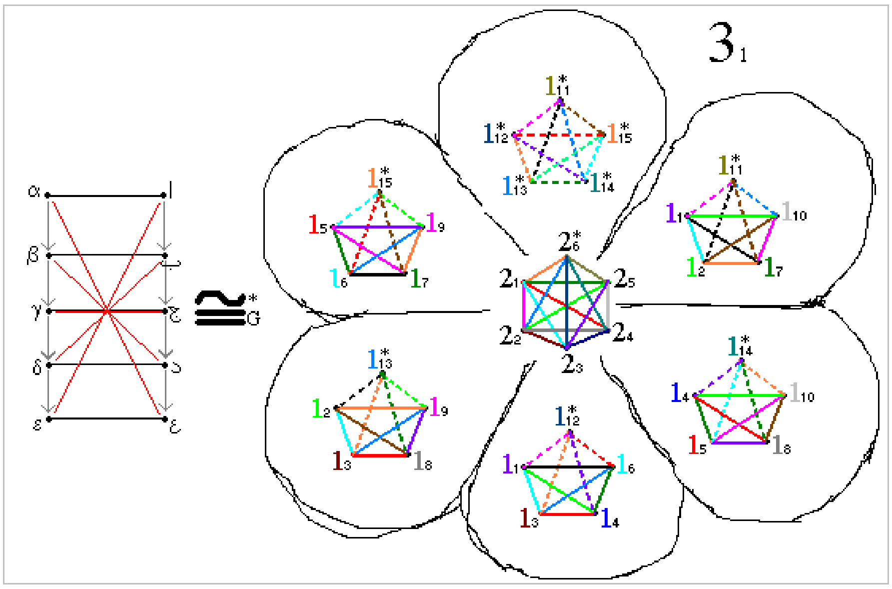


So, finally (according to our starting formation rule), the modal 3(4)-graph generates what we call, because of its very shape, a "logical penta-flower" (a 4-dimensional structure made of 5 "petals", which are each a logical cuboctahedron – the 5 petals being linked two by two, each time by a different common logical hexagon).



The same line of reasoning gives, for the modal 3(5)-graph, 10 logical dyadic hexagons (again, we forget the more complex logical hexagons). A combinatorial examination shows that they form 10 triples of mutually coherent dyadic hexagons, each triple letting a dyadic hexagon emerge, thus giving 10 logical cuboctahedra (let us call "internal" cuboctahedra the cuboctahedra made of triples of basic hexagons). But the 10 dyadic hexagons (those which emerged from the triples) can combine together in 4-tuples of coherent hexagons: a combinatorial examination shows that this gives exactly 5 further logical cuboctahedra (let us call "external" the cuboctahedra made of four dyadic hexagons). Now, a combinatorial exploration shows that the 10 inner cuboctahedra combine into 5 hyper-cuboctahedra (or penta-flowers), each of them isomorphic to the one we got as a final result in the case of the modal 3(4)-graph (let us call them "inner" hyper-cuboctahedra). A similar combinatorial exploration shows that the remaining 5 external cuboctahedra combine in turn into one more hyper-cuboctahedron (we call it “internal” hyper-cuboctahedron, because it emerged from internal cuboctahedra). But now one can see that these 6 hyper-cuboctahedra (5 inner and 1 external) are such that each of them shares each of its 5 “petals” (i.e. each of its five constitutive cuboctahedra) with one of the other hyper-cuboctahedra, with perfect distribution, leaving no unused elements and no overlapping. This means the emergence of an all-encompassing solid, with 6 petals, that we therefore call a logical (5-dimensional) hexa-flower (or logical hyper-hyper-cuboctahedron)[192].

Again, each of its 6 petals is isomorphic to the previous logical penta-flower. And in fact this

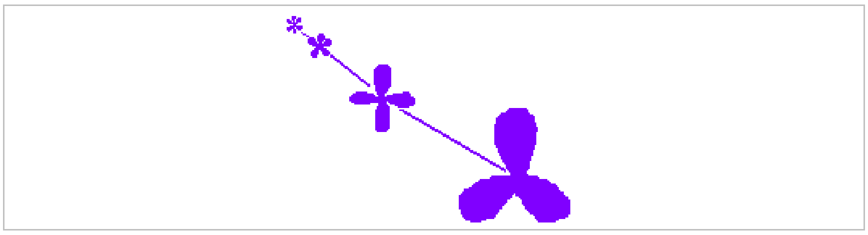


behaviour is stable: adding two more (parallel) arrows at the bottom of the linear modal graph generates a hyper-flower of higher order, with no finite limit. And each hyper-flower is made of petals such that they are isomorphic to the previous hyper-flower (hence the slightly fractal behaviour – a finite fractality in the top-down direction, but infinite in the bottom-up direction). Remark that the logical cuboctahedron can retrospectively be seen as a tetra-flower, in which the 4 petals are its 4 constitutive hexagons. Similarly, the logical hexagon is a tri-flower, its 3 petals being its 3 constitutive logical squares.

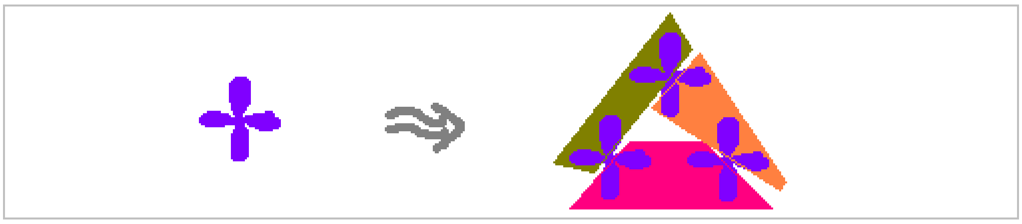
11.04.03. Apparently, a “logical hyper-jungle” at stake

[192] We prove all this combinatorially, with many more pictures, in our paper “Logical ‘hyper-flowers’: the ‘logical cuboctahedron’ belongs to an infinite (fractal) series of *n*-dimensional solids” (draft)

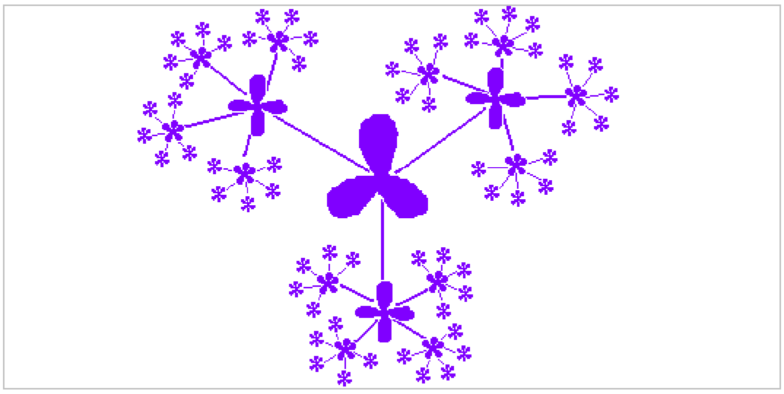
If we now try to represent the whole of the hyper-flowers we saw, one possibility is the following "branch", where each "logical *n*-flower" is represented as a *n*-lobe (i.e. a kind of violet corolla made of *n* petals). The biggest element represents the "logical 3-flower" (that is, the logical hexagon made of three logical squares, generated by the modal 3(2)-graph). From it sprouts the "logical 4-flower" (i.e. the logical cuboctahedron, made of 4 logical hexagons), from which sprouts the "logical 5-flower" (the logical hyper-cuboctahedron, made of 5 logical cuboctahedra), and so on. We depict the logical *n*-flowers smaller in proportion with *n* (as *n* increases, the size of its flower decreases) in order to make the global drawing graphically manageable.



Now, as we are going to see in ch. 12 *infra*, the whole story about logical hexagons is more complex: thanks to Smessaert, we already know that in S5, there is not only the logical hyper-cube (the logical tetra-flower) with its 4 logical hexagons, but also two more logical hexagons (the Smessaert ones). Moreover, it will turn out that there are not one, but three logical cuboctahedra, such that they partially overlap: each cuboctahedron shares two of its 4 hexagons with each of the other two cuboctahedra, so that they form together a more complex solid (still tri-dimensional, cf. § 12.01.04 and 12.02.02 *infra*).



Consequently, if we duly take into account this new branching behaviour, but (temporarily) ignore the aforementioned overlapping (which appears not only at the level of the tetra-flower, but more generally at each level *k* of the *k*-hyper-flowers), the structure of the whole series of logical hyper-flowers can roughly be represented as follows:



So there is some kind of “hyper-flowers jungle” for the logical cuboctahedra. Actually, this gives us some first (rough) overview of the conjectured series of the β*n*-structures. It clearly is still fragmentary (we know for instance that the overlaps are not yet expressed), and we are still to identify the “rules of the game” (analogous to the one we adopted: considering only the “basic” hexagons) generating the other branches of logical hyper-flowers. Despite all these shortcomings, this still unruly jungle seems nevertheless to show that something ordered exists in that direction (the direction of the conjecture on the existence of the β*n*-structures), deserving to be better investigated.

11.04. Four problems left open by *n*-opposition theory (2004)

In the end, there seemed to be at least four problems with the *n*-opposition theory proposed by us in 2004: (1) it does not take into account Smessaert’s two strange hexagons (cf. ch. 10 *supra*); (2) it does not fully explain, mathematically speaking, what the modal *n*(*m*)-graphs are; (3) it remains linear (what about the non-linear modal graph of the Lewis system S4, containing bifurcations, for instance?); and (4) it still does not yield access to structures higher than (but similar to) the logical cuboctahedron, the structures I called, conjecturally, β*n*-structures, for instance the β4-structure, which should be one (four-dimensional?) structure gathering together (five?) logical cubes.

All four problems were destined to be solved soon, as we are going to see in the next chapters of this Part II.

12.
PELLISSIER: N.O.T. AS SET-PARTITION[193] (β3-STRUCTURE = "LOGICAL TETRAICOSAHEDRON") SMESSAERT'S "RHOMBIC DODECAHEDRON" LUZEAUX'S DISCOVERIES AND PROPOSALS

In this chapter I will briefly recall some (but not all!) of the most important points of the whole theory of *n*-opposition. They were not made by me, but, independently, by Régis Pellissier (a mathematician) and Hans Smessaert (a linguist and logician). A fourth very interesting contribution to NOT has been made by Dominique Luzeaux (a mathematician and robotician). Such discoveries make it possible to establish *n*-opposition as a real, well-established, fertile science. Among (many) other things, they make it possible to study the series of the β*n*-structures (the kind of structure instantiated, at least as a fragment, by the logical cuboctahedron) in a very precise way. At the same time, the plurality of the approaches to NOT let emerge, in some points, the possibility of having to face difficult choices.

12.01. Pellissier's discoveries[194]

Pellissier's discoveries comprise different items: (1) he first discovered the existence of a new category of logical hexagon, the "weak hexagons", never discovered before (this lead to the introduction, inside S5, of 4 new hexagons, with respect to the 6 known ones); (2) he then discovered a general technique allowing to decorate any α*n*-structure using only modal 3(*m*)-graphs (no need of more "modal columns" than the classical two – the "positive" and the "negative"); (3) thanks to this general decorating method he could definitely establish the exact shape of the β-structure corresponding to the modal graph of S5: in other words, he discovered the true shape of the β3-structure, which happens to be a "logical

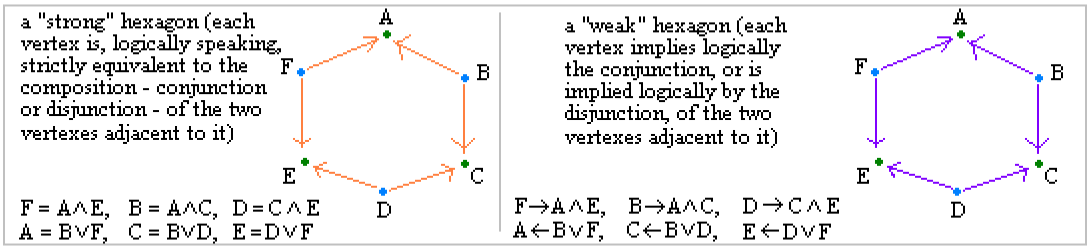
[193] In what follows I will concentrate on Pellissier's result, which I'm more familiar with, and which I will use to a large extent as a tool in the rest of this study. Cf. R. Pellissier, " 'Setting' *n*-opposition", *Logica Universalis*, **2**, 2, 2008.

[194] In 2004, reading my work (I needed a rigorous eye before giving my final manuscript for publication), Régis Pellissier made the very important discoveries of which we are going to speak. As an anecdote, the categorist mathematician he was used, at that time, to judge me, with a friendly frankness, as "wasting my time which such puerile small drawings" (he sometimes was upset for this!). Being a friend, he nevertheless agreed to read my manuscript. After a few days, immediately after having read it, he told me that he had had many difficulties to understand it, and that in order to figure it out, he had had to think of it in another way, which led him to "some discoveries" of his own... Since that moment he stopped thinking that such "small drawings " are a puerile waste of time (not all experience is useless in life!). Far from there, in Belgium, Hans Smessaert worked alone in a rather similar direction.

tetraicosahedron” (24 faces), of which our logical cuboctahedron is just a large fragment. The question of generalising the notion of β3-structure to that of β*n*-structure ($n \in \mathbb{N}$, $n \geq 3$) will be tackled in the chapters 13-15 *infra*.

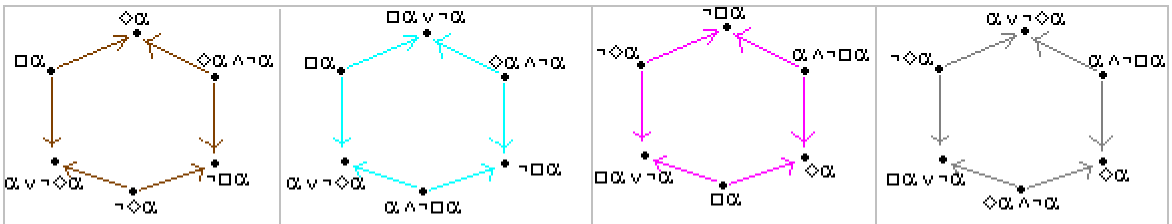
12.01.01. Logical hexagons can be weak or strong

Studying the notion of logical hexagon, Pellissier noticed something about logical hexagons that nobody else had before [195]: there is no need, for the blue (contrariety) vertices of a logical hexagon, to be logically equivalent to the conjunction of the logical values of their adjacent green vertices; and, dually, there is no need for the green (subcontrariety) vertices of a logical hexagon to be logically equivalent to the disjunction of the logical values of their adjacent blue vertices. As a matter of fact, they can, respectively, imply the conjunction of the two adjacent vertices or be implied by the logical disjunction of the two adjacent vertices.



So Pellissier introduces the terminological distinction between the “weak hexagons”, i.e. those logical hexagons whose vertices are not equivalent to disjunctions or conjunctions of the two adjacent vertices, and “strong hexagons”, i.e. those, like the classical Sesmat-Blanché, Béziau, Moretti-Smessaert and Smessaert ones, where the blue vertices must be the logical conjunctions of the adjacent green vertices and the green vertices must be the logical disjunctions of the adjacent blue vertices.

After having discovered this possibility, Pellissier was able to exhibit concretely (inside S5) the following weak hexagons.



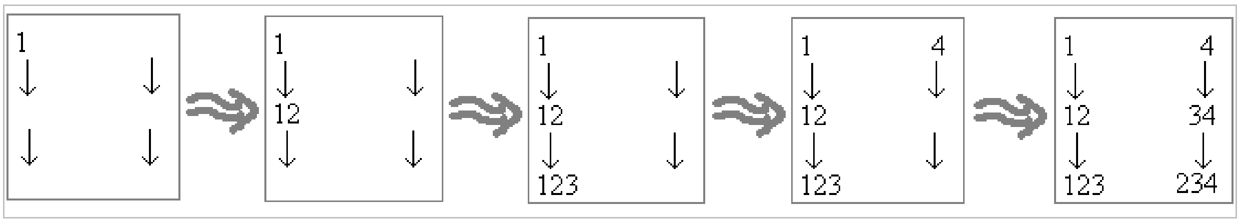
the 7th hexagon (Pellissier) the 8th hexagon (Pellissier) the 9th hexagon (Pellissier) the 10th hexagon (Pellissier)

But how to be sure that nothing had been forgotten? Have we got all of S5’s modalities?

12.01.02. Pellissier's "setting technique" for the modal *n*(*m*)-graph

The problem being that of extracting *all* the modalities of any given modal graph (and notably those of the modal graph of S5), Pellissier's idea was to try to use set theory in order to characterise symbolically what I called the modal *n*(*m*)-graphs. First of all, he wanted to be able to have a full exhibition of all the possible cases, and at the same time he wanted to be able to shift from a symbolic notation, such as mine, where one used different alphabets to differentiate the different modal columns (one Greek, one Arabic, one Hebrew, etc., cf. ch. 11 *supra*), to a symbolic notation more rooted in some set-theoretical device. He found the following, very powerful one.

One of the very first prerequisites of a modal graph, whatever its shape, consists in dealing with "basic modalities" (that's the very function of a modal graph). So, Pellissier wants to set-theoretically grasp the notion of "basic modality". He characterises a given "logical position" (i.e. a modality, simple like $\Box\alpha$ or composed like $\Box\alpha\vee\Box\neg\alpha$) by a string of characters (say: a string made of decimal numbers). In order to express that this logical position, atomic, logically implies another one, you characterise symbolically this second logical position by a string of characters made of the first character plus a new one. In Pellissier's notation, we may for instance have "1→12" for expressing that "the basic modality 1 implies the basic modality 12" ("12 is irreducible to 1 because it has one more symbol"). Logically speaking, this presents the advantage of letting one express logical conjunction by means of an intersection of strings and logical disjunction as a concatenation of strings. To take an example, if in S5 we have that $\Box p \rightarrow \Diamond p$, and if we express it, in terms of strings, by 1→12, we can accordingly express the conjunction of these two terms: $\Box p\wedge\Diamond p \leftrightarrow \Box p$ corresponds, as for the string expression, to $1\wedge 12 \leftrightarrow 1$ (that is: "$1\cap 12 = 1$"). For example, the encoding of the modal 3(3)-graph (2 columns, 3 layers) proceeds as follows:

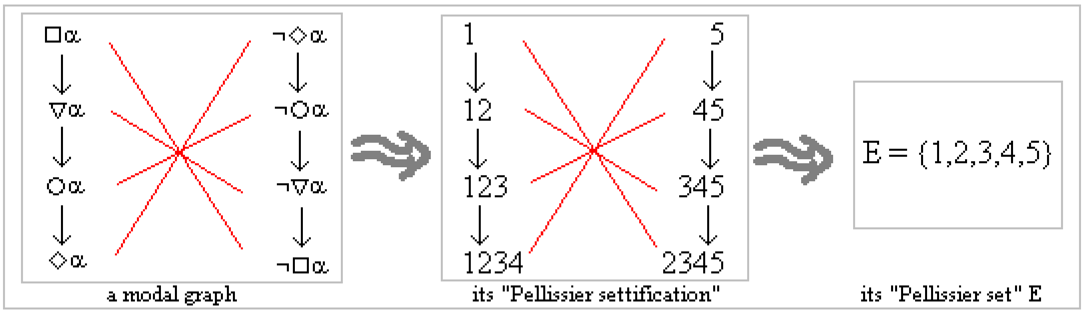


Remark here visually the following properties: (1) in the left column, the lower nodes add symbols to the higher ones; (2) the last bottom-left node determines the first top-right node (as being a string composed of a single new symbol); (3) the right column behaves like the left

[195] At least neither Sesmat, nor Blanché, nor Béziau, nor myself, nor Smessaert had noticed it.

one; (4) the two columns are related by the fact that the contradictory terms of the modal graph (i.e. the couple of terms determining its diagonals) have, from the point of view of their strings, an empty intersection and a total union ($\forall x \forall y \in E$, $x \cap y = \varnothing$ and $x \cup y = E$).

Take the example of the modal 3(4)-graph. This translated modal graph only gives us its 6 starting basic modalities. But all the other ones can be found by playing algebraically with the basic strings.



Indeed, Pellissier proves mathematically that if we characterise a given modal graph by the set E of its maximal string (like "12345" in the previous example), this set E expresses, in two moves, all the underlying logic (from the point of view of the logical oppositions): (i) all the possible modalities (the basic ones and all those obtainable by composition) coincide exactly with the possible non-empty strict substrings of the maximal string (i.e. E); (ii) all the possible oppositions (i.e. all the possible logical bi-simplexes, strong or weak) can be obtained by means of all the possible partitions of this string characterised by E.

Pellissier's general formula for "settifying" (i.e. giving a set "E" to) the modal $n(m)$-graphs is the following (two cases, depending on whether n is even or odd, with $n, k \in \mathbb{N}$):

$\alpha n(2k) \rightarrow (\mathrm{Card}(E_{\alpha 3(4)}) = (n\text{-}1)k+1)$

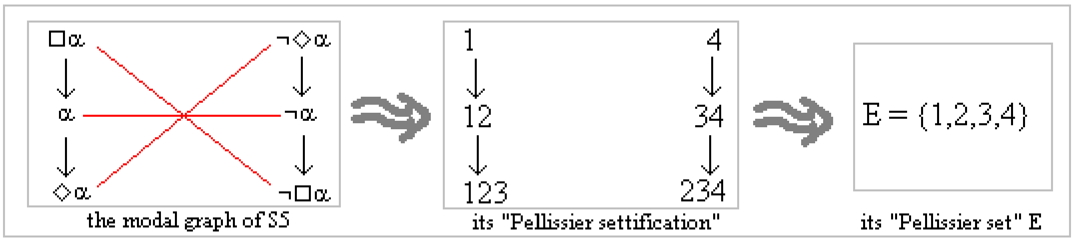
$\alpha n(2k+1) \rightarrow (\mathrm{Card}(E_{\alpha 3(4)}) = (n\text{-}1)(k+1))$

In other words, this method, mathematically founded, allows us to obtain, from any modal $n(m)$-graph, the set of all the possible logical oppositions it generates: again, this is done by obtaining, *via a suitable numbering*, a set E and then by extracting from it all its non-empty strict subsets and all its partitions (the n-oppositions are the n-partitions).

Notice an exception: the partition technique does not work for obtaining the logical squares: the 2-partitions (which one could expect to correspond to the logical squares, which are the solids for 2-opposition) only give the contradictions, not the contrarieties (which form the squares). This is due to the fact that, as Pellissier's discovers and demonstrates,[196] the logical square is in fact the weak part of 2-opposition, the strong one (i.e. the real "logical bi-segment") being contradiction (a segment, and not a square). Nevertheless, in order to get the logical squares (all of them) of a given modal system *via* its modal graph and *via* Pellissier's technique, you simply have to calculate (by 3-partition) its logical hexagons and then deduce from each of them 3 logical squares (there is no overlapping between squares belonging to different logical hexagons).

12.01.03. The complete study of the modal 3(3)-graph (the system S5)

As an example of this general method, Pellissier applies it to the case of the Lewis system S5, that is the modal 3(3)-graph (duly decorated with □ and ◊).



Its settification, as shown by a previous figure, gives E = {1,2,3,4}

Starting from this, he first determines all the possible vertices of the β3-structure (i.e. all the possible modalities, simple or composed): these are given by all the non-empty strict substrings of the maximal string E (or by the subsets of the set E). This gives these 14:

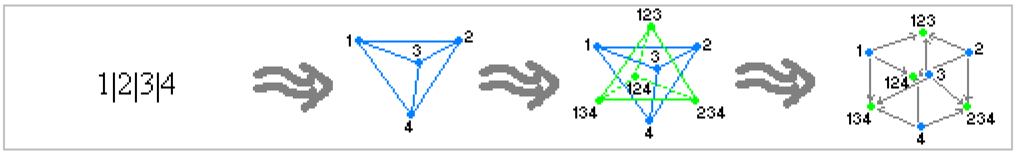
- 1, 2, 3, 4;
- 12, 13, 14, 23, 24, 34;
- 123, 124, 134, 234.

He then exhibits all the possible *n*-oppositions of this modal logic: the strong bi-simplexes will be the partitions of the set E, whereas the weak bi-simplexes will be the partitions of the strict subsets of E. Let us concentrate on the strong partitions.

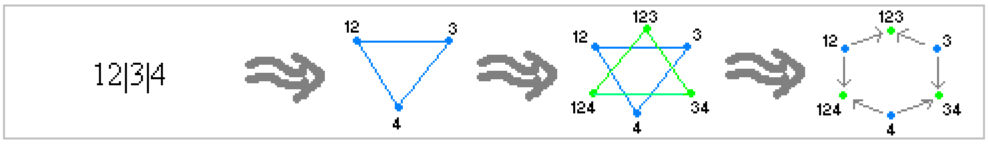
This gives the following:

- the largest possible partition of this 4-symbols-string is the 4-partition: 1|2|3|4. To this corresponds a tetrahedron of contrarieties, the logical values of which are expressed respectively by 1, 2, 3 and 4. The logical values of 1 and 4 are known by construction: they are □α and □¬α. The value expressed by 2 can be calculated: it is equal to 2 = 12∩234, that is (logically speaking) to α ∧ ¬□α. The value expressed by 3 can be calculated as well: it is equal to 3 = 123∩34, that is (logically speaking) to ◊α∧¬α. So, finally, this gives a tetrahedron of contrariety of which the four vertices are: □α, □¬α, α ∧ ¬□α, ◊α∧¬α (of course, this gives a logical cube once the arrows are added).

[196] Cf. R. Pellissier, "2-opposition and the topological hexagon", (forthcoming).



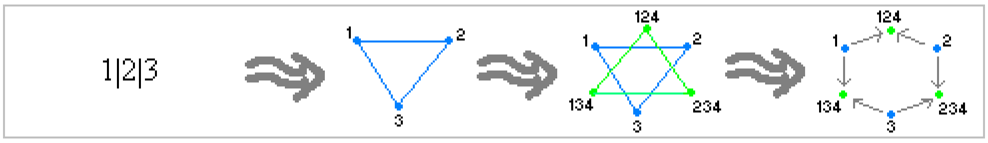
- the next partition is the 3-partition (of the set E relative to S5). This gives the following 6 possible partitions: <1> 12|3|4, <2> 13|2|4, <3> 14|2|3, <4> 1|23|4, <5> 1|24|3, <6> 1|2|34.



Once translated in a way similar to the preceding one, this gives 6 triangles of contrariety, which, with their 6 dual triangles of subcontrariety, give 6 logical hexagons. In fact, these are exactly the traditional 6 logical hexagons (Sesmat-Blanché, Béziau 1 and 2, Moretti-Smessaert, Smessaert 1 and 2).

- as for the next partition, as we said, Pellissier demonstrated that this method does not yield, by 2-partition of E, the logical squares. In order to obtain them, one has to refer directly to the logical hexagons: each of them contains exactly 3 logical squares (as in the classical case of the Sesmat-Blanché logical hexagon with respect to Aristotle's logical square), and it has been proven that there are no overlaps of logical squares. So, in our case, because there are exactly 6 logical (strong) hexagons we know immediately that there are exactly 18 logical squares.

Similar considerations over the possible partitions of all the possible strict subsets of E show that the only other weak figures are 4 weak hexagons: 1|2|3, 1|2|4, 1|3|4, 2|3|4.

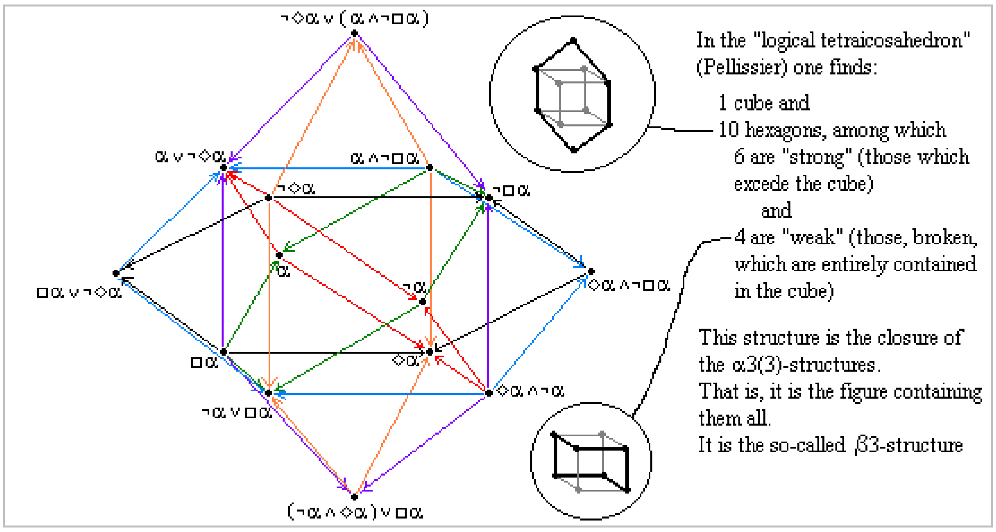


No other partition of E is possible. We have all the oppositions.

But how to view all this in space? Is there a gathering of all these oppositions of S5?

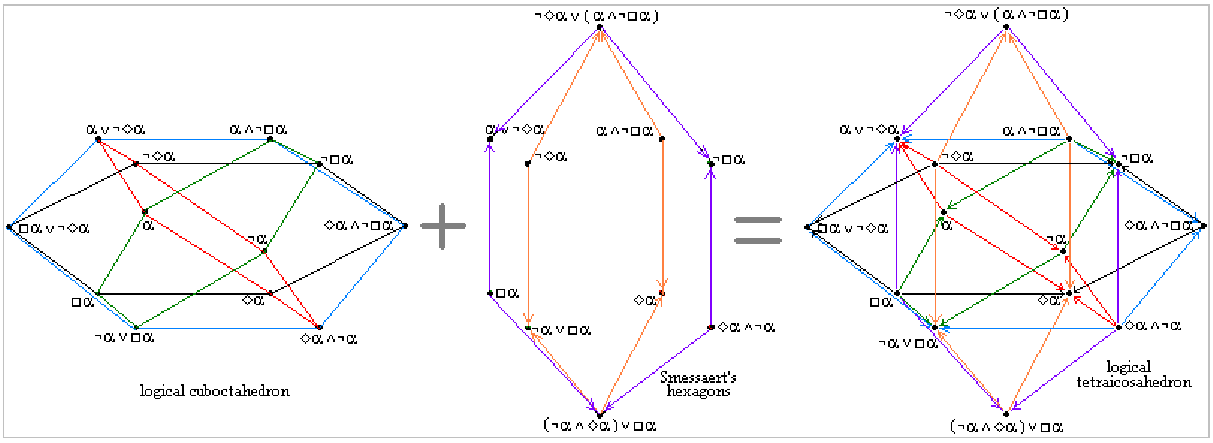
12.01.04. The β3-structure is Pellissier's "logical tetraicosahedron"

Having found the complete list of all the possible logical oppositions generated (or supported) by the modal graph of S5 (that is, all the α-structures of S5) – which consists in a logical cube, 6 strong logical hexagons, 4 weak logical hexagons and 18 logical squares – Pellissier tries to understand what can be the geometrical form of their gathering (their β-structure). The logical cuboctahedron was only sufficient to gather the first 4 known strong hexagons. But we know now that there are all in all, corresponding to the modal graph of S5, a logical cube, six strong hexagons and 4 weak ones. How are we to modify or extend the



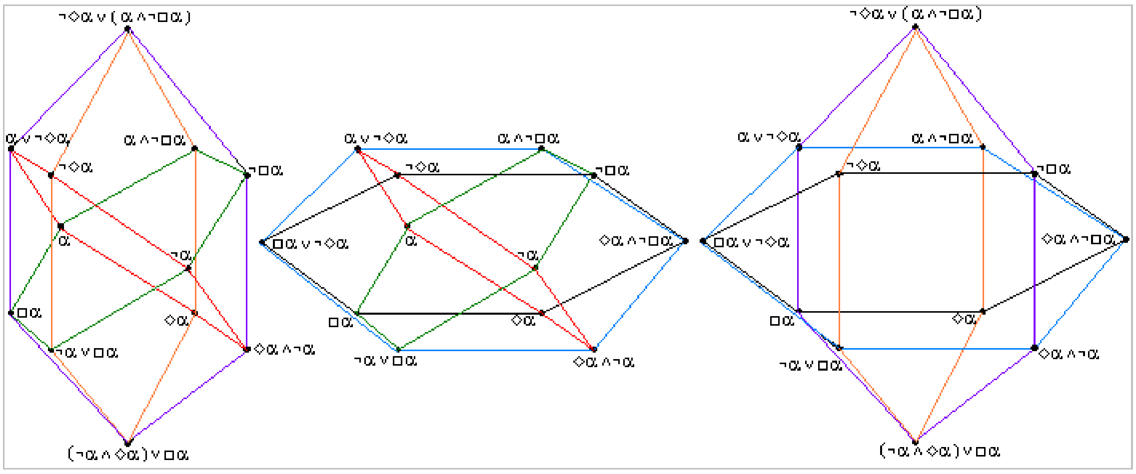
logical cuboctahedron in order to include all these new elements, if at all possible?

Pellissier finally found that the solution, that is the β3-structure, happens to be a "logical tetraicosahedron" (i.e. a solid with 24 triangular faces)[197]. As one can see, all the 6 strong hexagons and the 4 weak hexagons fit perfectly in it (the 6 strong hexagons are "planar", whereas the 4 weak ones are "broken"). In some sense the logical tetraicosahedron can be obtained from the logical cuboctahedron simply by adding to it the two Smessaert hexagons.



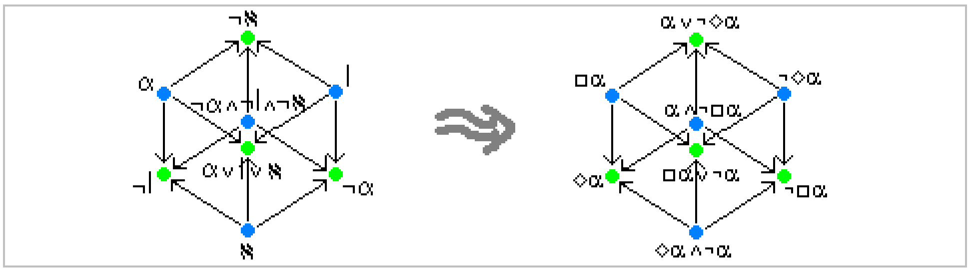
Notice as well (as I did, when examining Pellissier's logical tetraicosahedron with my cuboctahedronal eye) that inside the logical tetraicosahedron one can read not one, but 3 logical cuboctahedra (partially but regularly overlapping).

[197] As it seems, the codified scientific name for this solid (outside logic) sounds like "polyhedral development of a symmetric hyper-pyramid with cubic base" (cf. Banchoff, T., *Beyond the Third Dimension*, *op. cit.*, ch. 2).



This is what motivated our idea of a "hyperflower jungle" (CF; § 11.04.03 *supra*).

Incidentally, Pellissier also happened to give the first known "classical" (i.e. not involving strange modal columns, committing one to ternary negations) decoration of the logical cube (right side), emptily discovered by me in 2004 (left side of next figure).

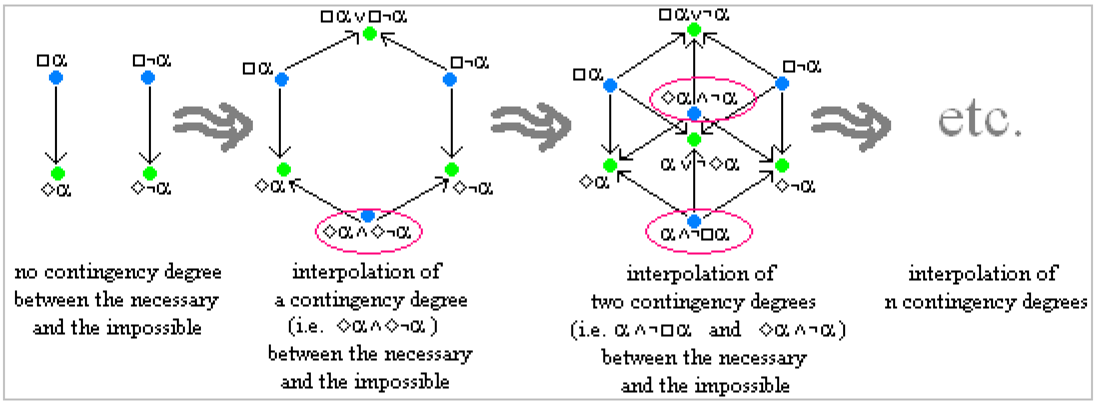


12.01.05. The reduction theorem for the modal *n*(*m*)-graphs

Pellissier also gave a powerful and very important reduction theorem of NOT for the β-structures, showing that (and how) all *n*(*m*)-graphs, *from the point of view of their mapping into β-structures*, can be reduced to 3(*k*)-graphs (for a big enough "*k*"). Among other things, this very important result shows that there are classes of equivalencies with respect to the β*n*-structures: to this β*n*-structure correspond not just one, but infinitely many modal graphs, although these do differ from one another: they are the same, modulo the given β-structure. It also shows that the universe of the β-structures is smaller than what we could have expected, for there is not a series of series of them, but just a series, the one indexed over *n*, admitting a bijection on the series of the modal 3(*n*)-graphs. In other words, contrary to my initial conjecture, the β-structures are not β*n*(*m*)-structures, but just β*n*-structures.

12.01.06. Pellissier's interpretation: contingency degree's interpolation

To his technical discovery, Pellissier adds a philosophical remark. The four contrary (blue) modalities of the logical cube can be seen, in fact, as constituted on one side of two modal extremes (necessity and impossibility) and on the other side of two "interpolated contingency degrees" (comprised between the first two ones). And this is a complexification of what the logical hexagon already did with respect to the logical square (it added the contingent value "◊α∧◊¬α", by interpolating it, as a new contrary, between "□α" and "□¬α"). So one can show that this behaviour is constant: for any α*n*-structure, it adds – with respect to the previous one – new contingency degrees between the extreme non-contingent modalities, that is "□α" and "□¬α" ("necessity" and "impossibility").



This remark, possibly not totally clear yet for the reader, will later turn out quite important for the general interpretation of NOT (cf. ch. 25 *infra*).

12.02. Smessaert's method and (similar) discoveries: the "logical rhombic dodecahedron"[198]

Hans Smessaert, whom we already mentioned in ch. 10 *supra* for some of his early discoveries on oppositional geometry (among which three logical hexagons), made thereafter very important discoveries similar (and parallel) to those of Pellissier, but by a different formal, very interesting and elegant technique, seemingly more akin to "generalised quantifiers theory" (some sort of algebraic treatment rooted in linguistic considerations, inspired by the American logician and linguist Richard Montague). In particular, by this means he proposed a very regular solid, the "rhombic dodecahedron", corresponding to the oppositions between all the modalities of S5, a solid comprising as parts all the known 6

[198] Cf. Smessaert, H., "On the 3D visualisation of logical relations", *Logica Universalis*, 3, 3, 2009.

(strong) hexagons of S5 (as already mentioned, he was the first discoverer of the last two of them, around 2002). This solid is in fact very close to the logical tetraicosahedron of Pellissier, except it lacks 12 of the 36 arrows present in the logical tetraicosahedron (more precisely: the 12 arrows of the logical cube constituting the “heart” of the β3-structure). This difference of the two solids is highly instructive (it can tell much on the complex relations between NOT and algebra) and may tell us much about the very special essence and identity of NOT.

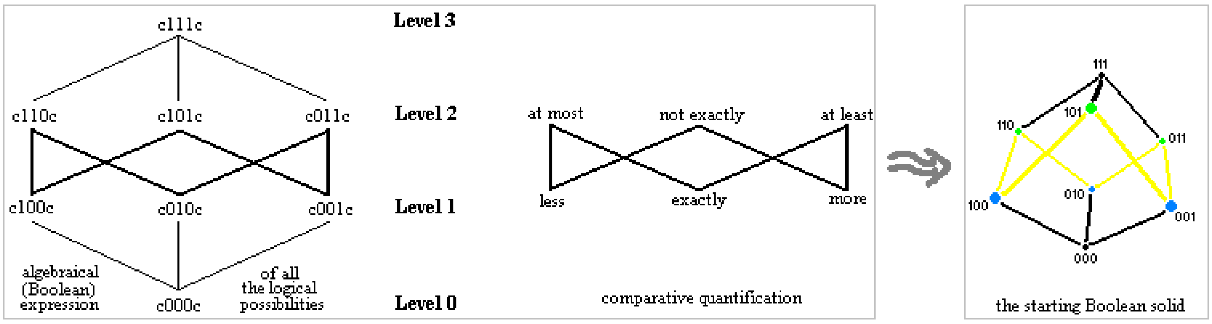
12.02.01. Smessaert’s aims and strategy

Smessaert’s explicit aim, when studying the Aristotelian kinds of opposition relations, is to approach them from a double, coordinated point of view: that of Boolean algebras and that of the geometry of the so-called Platonic and Archimedean solids[199]. Moreover, as a linguist, Smessaert endorses the point of view of the theory of the Generalised Quantifiers, one allowing to study linguistic issues through the use of quantification (this is also akin to Richard Montague’s intensional logic). As for his strategy, Smessaert will construct in three steps a geometrical representation of modal logic systems of increasing complexity. In each of these three steps, he will first develop a reasoning over Generalised Quantifiers, producing a certain amount of them; then, considering their closure by means of their possible combinations through the Boolean operators (“and”, “or”, “not”), he will introduce in each case their Quantifier Algebra; then he will take in consideration the so-called Hasse Diagram of this algebra, a lattice that gives a standard solid representation of the structured totality of the modalities, and shows how to make this solid representation better visually speaking (for this Smessaert will propose two very interesting criteria). This strategy will aim at leading from a fragment to the totality of the possible oppositions of standard modal logic (that is, the modal oppositions of S5), the whole starting from the ontologically underlying algebra.

12.02.02. The geometrical results on oppositions

[199] The interesting point here consists in playing with “geometrical dualities”. First, the (well-known) dualities of the five Platonic regular solids (tetrahedron, cube, octahedron, dodecahedron, icosahedron), of which two couples (cube-octahedron and dodecahedron-icosahedron) are mutually dual, whereas one of them (the tetrahedron) is self-dual. Second, the dualities of the Archimedean solids, which display the dualities of Platonic solids in another way (for instance: the Archimedean cuboctahedron is related both to the cube and to the octahedron).

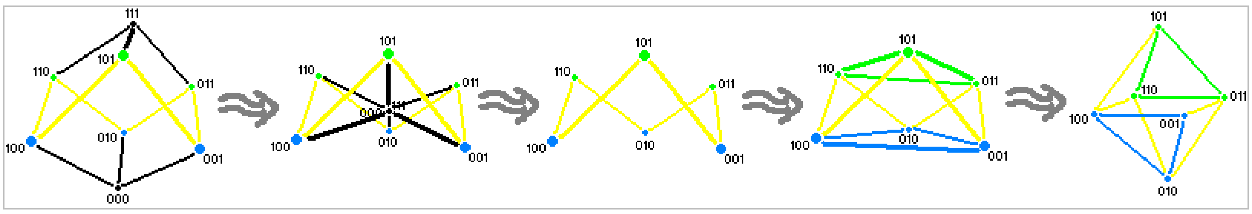
The application of his sophisticated method to a small *ad hoc* logical universe called CQA (Comparative Quantification Algebra) gives in the end a Hasse diagram (left side of the next figure) of the relative algebra constituted by 8 elements (generalised quantifiers) displayed over a tri-dimensional cubic lattice. Over the 8 elements, 6 are "normal", whereas two are trivial (the trivial quantifiers "c000c" and "c111c", meaning respectively the trivially false empty set and the trivially true power set of the universe, cf. middle of the next figure). We reproduce here adaptations of Smessaert's original graphs, where (right side of the next figure) the "normal" implications (of the lattice) are depicted in yellow, whereas the



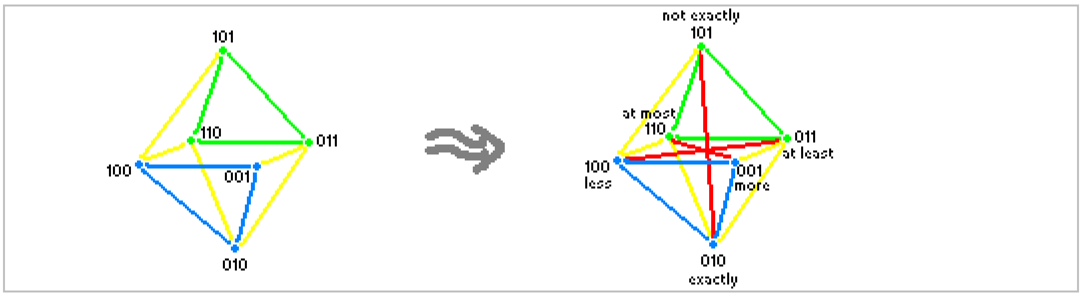
implications from the trivial 000 (implying all) or to the trivial 111 (implied by all) of the two trivial quantifiers are depicted in black.

Starting from this yellow-black cube (right side of the previous figure), which is a faithful image of the underlying Boolean algebra (left side) expressing explicitly only the implications (i.e. subalternations), Smessaert investigates how the Aristotelian relations of opposition (contradiction, contrariety, subcontrariety) can be expressed for this system CQA and looks for an alternative 3D visualisation. Now, here comes Smessaert's important (philosophical-methodological) move (partially inspired by the philosophy of language of Pieter Seuren, cf. § 06.06.08 *supra*), he puts forward two requirements: (1) "First of all, it [= the new visualisation] should render the trivial entailment relations [= the black edges on the right of the figure] […] visually less prominent. Secondly, it should render (at least some of the) Aristotelian relations of Opposition more prominent" (p. 7-8).

In order to face the first requirement, Smessaert's hits a big strike: he has the amazing idea of internalising the two algebraic extrema: he makes the two, despite their mutual contradictoriness, *coexist* in a same point right at the centre of the solid, its centre of symmetry (second on the left of next figure). Remark that this is geometrically very elegant, the black arrows arriving to the point 111 (from the green points) and those leaving the point 000 (and reaching the blue points) make, two by two, a coherent bigger arrow (for instance, one "same" arrow goes from 011 to 111, and then from 000 to 100).

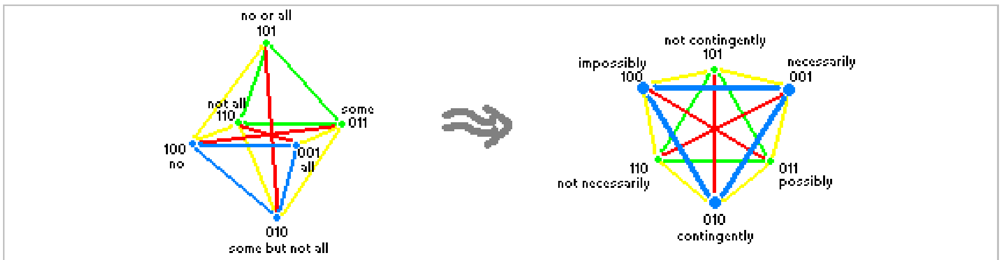


In order to fulfil his second requirement, Smessaert first gives an expression of the opposition relations in terms of the language of the particular system CQA. Then, with this expression of the oppositions in hand, he can determine for any two quantifiers, say “110” and “010” (or the like), which opposition relation links the two of them (for instance, “100” and “011” are contradictory, as well as “010” and “101”, etc.). Then, after having erased the black arrows (middle of the previous figure), he expresses graphically (by adding coloured segments) the Aristotelian relations (subcontrariety in green and contrariety in blue) in what remains of the cube deprived of its two trivial points (now internalised). This gives a blue triangle of contrariety on the bottom and a green triangle of subcontrariety on the top (last two elements on the right of the previous figure). Once all is done, the result is a “logical octahedron” (left of next figure), which, from the point of view of the expression of the oppositions, is a simpler (i.e. better) representation than the starting Hasse cube diagram. In order to represent the third, still missing Aristotelian opposition relation, that is contradiction, it suffices to draw (in red) the diagonals of the octahedron (right side of next figure).



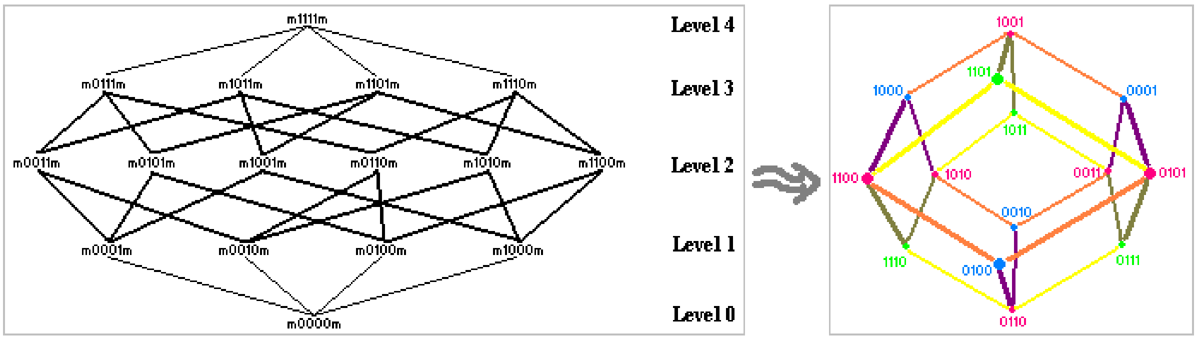
This representation seems much more satisfactory (it deals with non-trivial meanings and puts forward all the opposition relations), and close to the spirit of Aristotle’s theory of opposition (and its logical square).

In the second step, Smessaert constructs two further algebras, SQA and MQA (Standard Quantification Algebra and Modal Quantification Algebra), the first dealing with the classical quantifiers of first order logic, the second with the classical modalities of modal logic. As, at the end, these two systems also give quantifier algebras made of 8 terms each (again, 6 “normal” and 2 trivial), these two algebras are isomorphic with the previous (i.e. CQA) and therefore have the same geometrical expression: the logical octahedron. They differ for the decoration: quantificational (left side of next fig.) or modal (next fig., right side).

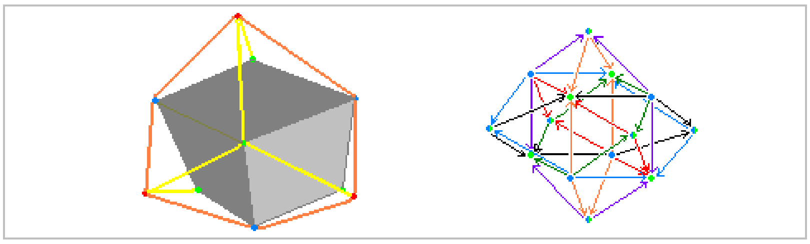


Remark that the coloured geometry of this octahedron is such that duly projected on a plan it can give Sesmat's and Blanché's logical hexagon (interpreted modally); other projections give the three classical logical squares contained in the Sesmat-Blanché's hexagon. This means that Smessaert, starting from the relative algebra, has given a satisfactory tri-dimensional model of the classical logical hexagon. "Put it differently, the octahedron can be considered as the 3D Boolean algebraic foundation of the 2D Blanché stars" (p. 16).

The third and last step will consist in targetting, for determining its satisfactory oppositional geometry, the whole of standard modal logic (i.e. the system S5). Now, because of Béziau's pioneering revolution inside the oppositional geometry, which originated the discovery of a plurality of modal hexagons instead of Sesmat's and Blanché's unique one, was brought by the structuralist gesture consisting in taking into account *also* the zero-degree modalities (that is "*p*" and "¬*p*"), Smessaert as well needs now, if he wants to follow Béziau's example, to take into account the null modalities of modal logic, that is, from the point of view of his quantificational approach, the "actual world". This motivates the passage from the previous logic MQA to a richer version of it, MQA2 (= MQA + the actual world). As previously, by a reasoning based on the use of suitable generalised quantifiers, Smessaert deduces, this time, a set not of 6 (as previously) but of 14 quantifier denotations. Once more, while looking for its operational closure (through the Boolean operators "and", "or" and "not") in order to obtain a structure of Boolean algebra suitable for them, he needs to add to the 6 generalised quantifiers a trivially false quantifier (m0000m) and a trivially true quantifier (m1111m). Having done this, there are now not 14 but 16 elements. So, this Boolean algebra MQA2 gives the following Hasse diagram (in fact a four-dimensional hypercube), which Smessaert, as previously, will have to make oppositionally-visually "sexier" (following his two methodological-philosophical prescriptions). The Hasse diagram, freed of its two trivial quantifiers (left side of the next figure, colourless), gives the following tri-dimensional solid (right side of the next figure, coloured).



Smessaert explains that this new solid, the “rhombic dodecahedron” (en passant, the dual of the logical cuboctahedron of ch. 11 *supra*), made of 12 faces having each a form of rhombus (i.e. shaped as lattices) and having, of course, 14 vertices is the needed solution (the needed solid in order to express perfectly all the modalities of S5). As a matter of fact, according to Smessaert, once given the needed *ad hoc* translation rules for determining the kind of opposition between any two given generalised quantifiers of MQA2, the Aristotelian relations of opposition in MQA2 work pretty well with this expression in terms of a rhombic dodecahedron of oppositions. Notably, it contains and expresses fairly well all the 6 logical hexagons of S5. So Smessaert judges that he has reached his aim.



Remark that this solid (left of previous figure) is said to be “very close to Pellissier’s logical tetraicosahedron” (right of previous figure), with which it can fully coincide if only one folds each of the rhombic faces along its smaller diagonal ...

12.02.03. NOT-Remarks on Smessaert’s approach

Smessaert’a approach is highly interesting, well-thought and well-constructed. It represents for NOT a considerable theoretical enrichment and stimulation. Nevertheless, beside many (very) good points, some elements could seem problematic and require discussions, with perhaps theoretical decisions at stake for the future theoretical evolution of NOT.

In my view there are at least five (main) good points for NOT in Smessaert’s approach. (1) Smessaert’s way offers a clear description of the geometrical dualities (with the Platonic and the Archimedean solids), whereas previously we handled, when it happened (as it happened to me), Archimedean solids (like the logical cuboctahedron) without even knowing the many possible relations (of duality) to other similar (not yet logical) solids. It seems very interesting to deepen this kind of reflection, so to say injecting more (classical) solid geometry into NOT. Linking logical duality with geometrical duality seems a very fascinating and stimulating perspective. (2) Coherently with the previous point, Smessaert’s approach offers a clear improvement with respect to my logical cuboctahedron: all 6 strong hexagons of S5 are present in the rhombic dodecadron, not so in the former (which contains only four of them). (3) Another good point, more pragmatically speaking, is that this approach since the beginning seems to open to possible powerful linguistic applications, mediated by the common reference (of Smessaert's oppositional geometry and of contemporary linguistics) to the theory of the generalised quantifiers. (4) Still in a pragmatical vein, Smessaert's approach offers to NOT, otherwise very abstract modally speaking (cf. § 11.03.03 *supra* for instance), a more direct link to the accepted modal-logical standard of Possible World Semantics (as one sees in his construction of the systems MQA and MQA2). (5) Smessaert's idea of "internalising the two extrema" (i.e. the trivial elements), that is, considering the two missing (in NOT) trivial modalities (the empty set and power set of the universe) as conjuncted (paradoxically) in the oppositional solid's symmetry centre, is simply *brilliant*. In my opinion it tells NOT not to worry about the possible reproach of having forgotten "God only knows where, and why" such two points[200]. Such a reproach could be frightening for us, claiming to prove the radical incompleteness of NOT (NOT would be algebra badly understood by the auto-proclaimed NOT-scholars). On the contrary, Smessaert's idea seems to suggest very strongly that NOT is a formalism deeply linked with the starting move of neutralising (*without ignoring*) the two trivial elements. With this respect, Smessaert's move gives back its "structuralist dignity" to NOT[201].

As for some (possibly) critical points, four at least seem to deserve to be discussed. (i) Smessaert's approach says nothing about Pellissier's four "weak hexagons" (maybe it can, if used differently? Then how?). In any case, Pellissier's method seems to give, on the contrary, all of Smessaert's results (the only Smessaertian things it does not give are the already mentioned immediate linguistic interface, the link to possible world semantics and the explicit reference to the solid potential dualities). (ii) In some sense Smessaert's approach, at least as embodied by the logical rhombic dodecahedron, seems to overlook (i.e. to miss) 12 logical arrows, those belonging to the inner cube, itself exhibited *graphically* but not *logically* by Smessaert. Taking into account the "smaller diagonals" of the 12 faces of the rhombic dodecahedron makes the complete solid of oppositions have not 12 rhombic faces (Smessaert), but 24 triangular faces (Pellissier): so something (geometrical) seems to be

[200] A similar reproach has seemingly been made to NOT (more precisely: to the logical tetraicosahedron) by Luzeaux *et alii* in their paper "Logical Extensions of Aristotle's Square", p. 172 (cf. § 12.03 and § 17.02.03 *infra*).

[201] Paradoxically enough (poor me! I must confess I'm really not fond of phenomenology, usually), it does it by some kind of genial "structuralist-phenomenological" gesture: by a suitable, tricky "reduction" (εποχη, *epokhe*), neutralising some NOT-unuseful aspect of algebra (its trivial extrema), it lets emerge the new plan of "oppositional geometry" (i.e. NOT), otherwise hindered and hidden by the standard format of algebra.

possibly simply wrong with the rhombic dodecahedron's claim to be a graphical closure of S5. This point should be verified, checking if Smessaert's actual method can – perhaps it can – produce by itself the *natural* expression of that "inner logical cube" until now problematic (as does, very naturally, the logical tetraicosahedron), whose edges are precisely the 12 missing arrows. (iii) In the same vein, it is not clear whether Smessaert's method, which, as said, is very sensible to geometrical dualities in general, is able to see that in S5 there are 3 partially overlapping logical cuboctahedra (cf. § 12.01.04 *supra*). Now, because the tetraicosahedron itself is, traditionally speaking, a "polyhedral development of a symmetric hyper-pyramid with cubic base", this could mean that the "solid-duality dimension" of Smessaert's approach should be enriched by taking into account, beside the Platonic and the Archimedean families of solids, the class of the hyper-pyramids (or something similar): so if this view holds, the Smessaertian approach to NOT would be good (and enriching), but still incomplete[202]. (iv) So, it would be tempting to consider that *geometrically speaking* the real oppositional closure of S5 is the one showed by the "setting method" in terms of β-structures (it better complies to Smessaert's second programmatic requirement: "rendering the Aristotelian relations of opposition visually more prominent"). Remark, again, that this "perfection" of the "setting-NOT" (it gives *all* the relations, *all* the arrows and all the oppositional subfigures) is granted *in fundo* by Smessaert's himself, by his aforementioned crucial (and brilliant) idea of seeing the symmetry centre as containing (paradoxically but successfully) both the algebraic trivial extrema: with that very precious Smessaertian aid the fundamental (debated) independency of NOT with respect to Boolean algebra (from which it *emerges*) seems to be better understandable[203].

On this question of the relations between hypercubic Boolean algebras and NOT-theoretical tetraicosahedra we will come back in § 17.02.03 *infra*.

[202] A visual reference to (and a theorical contextualisation of) the "polyhedral development of a symmetric hyper-pyramid with cubic base", i.e. the solid rediscovered by Pellissier with his "logical tetraicosahedron", can be found in Banchoff, T., *Beyond the Third Dimension*, *op. cit.*, ch. 2. To my eyes the intuitive, still unmathematised geometrical difference between the tetraicosahedron and the rhombic dodecahedron seems to be the fact that the former is some kind of "tri-dimensional star", whereas the latter is a convex, "spikeless" solid. This difference could have a more precise geometrical meaning in terms of traditional families of (hyper-)solids.

[203] Remark that this "flaw" (the absence of the cube), if confirmed, could be linked with Smessaert's adoption of Béziau's deliberate choice not to consider subalternations among the opposition relations. Looking at the logical *hexagons* as if they were *stars* hinders the complete exploration of the geometry of oppositions, cf. § 25.01 *infra*.

12.03. Luzeaux, Sallantin, Dartnell *et alii*'s discoveries[204]

In two papers, one published jointly with M. Afshar, C. Dartnell, J. Sallantin and Y. Tognetti (in 2007), the other published jointly with J. Sallentin and C. Dartnell (in 2008), the French mathematician and robotician Dominique Luzeaux made very nice discoveries starting from the papers of Béziau, Pellissier and myself on opposition theory (seemingly, Luzeaux made his first discoveries without knowing Pellissier's results). In particular – but their main results also concern new uses of the modal *n*(*m*)-grahs, as well as other applications and new perspective suggested (as the very interesting one of taking into account asymmetric modal graphs, like the ones generated by an intuitionist metalanguage of modal logic) – they showed a very nice result about the binary connectives of propositional logic (14 over the 16 form a logical tetraicosahedron – we will speak of this again in chapter 17 below).

12.03.01. Aristotle's square as a philosophical starting point

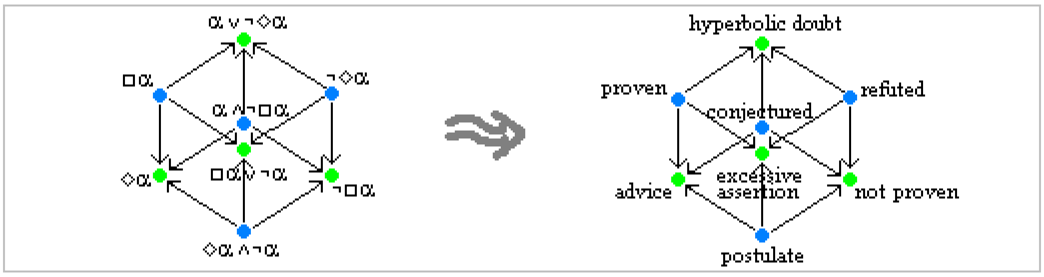
The starting point of Luzeaux's studies on the geometry of oppositions seems to be an informal one: wanting to build models (for robotics and artificial intelligence), he gets interested in Aristotle's logic and therefore in the square of opposition (this structure bearing interesting deep formal properties for modeling). So, wanting to reintroduce actively (not just as old historical fossiles) such old structures in the contemporary "hi-tech" modelisations, he (and his joint authors) faces, from an epistemological point of view, the problems of the square (like undue existential import, etc., cf. § 04.05.03 *supra*) with respect to the new logic of the XIXth century (and thereafter) which overwhelmed Aristotle's logic and occasioned a strong criticism and a progressive abandon of the square. That is, Luzeaux's position will be of methodological distrust with respect to the standard logical interpretation of Aristotle's syllogistics: in a way partly similar (and parallel) to that of Pieter Seuren (cf. § 06.06.08 *supra*) he will see Aristotle as extremely interesting for contemporary AI (and possibly for robotics), in virtue of the very *natural* character of his logic, that is, the theory of syllogism. Of course, it is this "naturality" which imposes constraints (as, for instance, the non-emptiness of the quantification domain) felt as being excessive by pure mathematical logic. But keeping as valuable scientific tools syllogistics (and the square which is at the heart of it), as Luzeaux

[204] In this paragraph we will discuss the results contained in the two papers: Afshar M., Dartnell C., Luzeaux D., Sallantin J. and Tognetti Y., "Aristotle's Square Revisited to Frame Discovery Science", *Journal of Computers*, Vol. 2, No. 5, July 2007, p. 54-66; Luzeaux D., Sallantin J. and Dartnell C., "Logical Extensions of Aristotle's Square", *Logica Universalis*, Vol. 2, No. 1, 2008, p. 167-187.

wishes, requires a clearer epistemological move, that is, a theoretical justification; like perceiving that, paradoxically, the syllogistics (and the square) has to do *not with mathematical logic* (in the sense praised by many analytical philosophers), but rather *with a theory of (natural) reasoning and with the geometrical expression of such (natural) reasoning*. The authors say: "Indeed Aristotle's theory of syllogism and all the Medieval syllogistic corpus developed until the 19^{th} century should not be seen as a formal theory of logic, but as a theory of reasoning […] Therefore the square of opposition appears as a geometrization of the inference process" ([2008] p. 168). So, the next step for them in order to fortify and deepen this project is seeing if there are, available "in the market", logical (or geometrical) extensions of the logical square: the result of this "buying promenade" will be adopting (among others) NOT, as being an interesting tool for generalising the starting tool called "Aristotle's square". An interesting point for them (always keeping things like robotics or AI in mind) is that the modal logic underlying the ancient and medieval square is the so-called S5 system. Now, as we saw before in this chapter, NOT tells us that the logical tetraicosahedron (a β*n*-structure) is the geometrical closure of S5 and that the logical cube (an α*n*-structure), the "heart" of the tetraicosahedron, is itself the 3-dimensional generalisation of the starting logical square.

12.03.02. First applications of elements of NOT for AI models (2007)

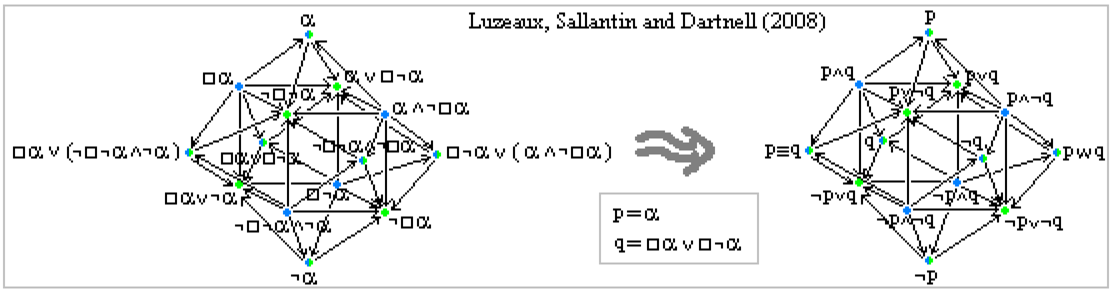
A first direct application of NOT will consist, in the 2007 paper, in using the logical cube for supervising by a simple model the process of "theory formation", aiming at "a formal framework to capture the entire process of scientific discovery including hypothesis, formation, reasoning, identifying contradictions, peer reviewing, reformulating and so on" (p. 54). This is done, while studying a concrete case of drug invention in a pharmaceutical group, on top of an interpretation of this theoretical invention process in terms of two main dynamics: a "personal dynamic" and a "social dynamic" (cf. [2007] p. 54-55). The interest in proposing a model using the logical cube is that the latter contains 6 logical hexagons (instead of just one), which allows to express 6 notions inherent to the process of scientific discovery (the square, again, is for several reasons a much praised ingredient of Luzeaux's project) ; as for the logical tetraicosahedron (which expands the cube, its heart, by six spikes or pyramids) contains all in all 18 logical squares, but Luzeaux is concentrated on squares and *cubes*, rather than hexagons. So they propose a decoration of the logical cube in terms of epistemological concepts, obtained *via* a discussion of the meaning of the values at the 8 corners of the modal cube.



This is only the starting point. The further introduction of two new suitable logical squares motivates the introduction of a "hyper-cube of oppositions" (never drawn …), which is supposed to be a very powerful model for rational agency, able to define or substantiate any step of the discovery process (to this we will try to return in ch. 17 *supra*).

12.03.03. Luzeaux's discoveries and openings in (and for) NOT (2008)

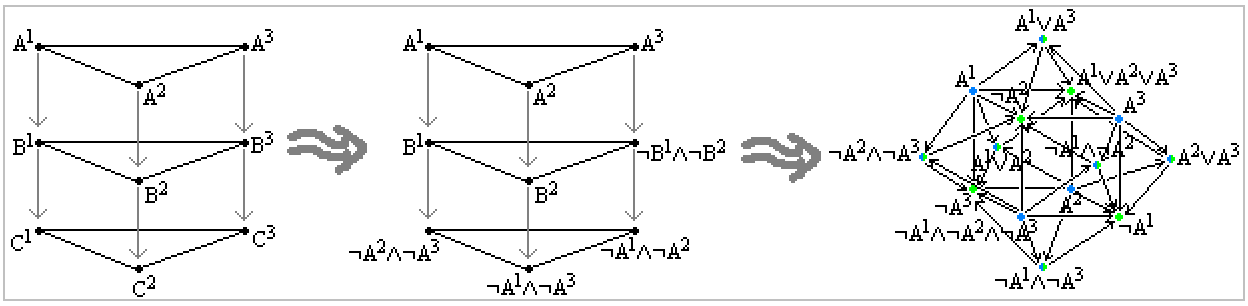
A very nice discovery by Luzeaux concerns the fact that, by a suitable translation (i.e. by taking in it p = α and q = $\Box\alpha\vee\Box\neg\alpha$), Pellissier's modal decoration of the logical tetraicosahedron (i.e. the modal decoration of the β3-structure) can be turned – thus generalising Blanché's own model made of 2 chained logical hexagons (cf. ch. 8 *supra*) – into a tetraicosahedron of the propositional binary connectives.



Luzeaux uses this possibility in order to argue that it is mistaken to restrict the tetraicosahedron of opposition to its modal use (cf. *infra*).

A second discovery uses another translation of the logical tetraicosahedron, this time semantical (the previous translation was syntactic): embedding S5 within a four-valued logic Luzeaux *et alii* get a translation of the tetraicosahedron decorated modally into a tetraicosahedron decorated with substrings of a string of length 4: this boils down to re-discovering (by another strategy, one involving a many-valued logic and a possible worlds semantics with two worlds) Pellissier's "settifying" technique.

A third discovery is very interesting for it uses a NOT-concept (that of modal *n*(*m*)-graph with *n*>3) which had not been used anymore since my 2004 study, for it had been strongly relativised since Pellissier's discoveries (i.e. his reduction theorem of § 12.01.05 *supra*). The discovery consists in showing that the β-structure corresponding to the modal 4(3)-graph is, astonishingly, still the logical tetraicosahedron (as such this would be totally strange, for the tetraicosahedron is already the β-structure of the modal 4(2)-graph), but with the amusing feature that some of the modalities (in fact those composed of "B") are expressed not as usual by the vertices of the β-structure, but by its edges!



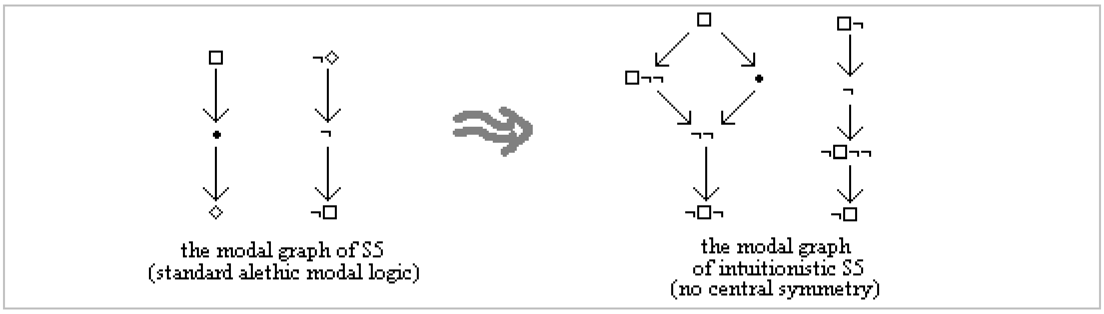
The (complex) demonstration of this consists in showing that the terms (or points) of the lower gem (i.e. the "C") can be expressed by way of the terms of the upper gem (this is straightforward), and then by demonstrating that the Boolean closure of the "A" terms of the starting modal 4(3)-graph (and therefore the terms of the upper and of the lower gems of this modal graph) decorate exactly one logical tetraicosahedron. Then to the question of knowing what would have happened of the middle gem (i.e. of the "B" terms) if this particular tetraicosahedron were really the (complete) β-structure of the starting modal 4(3)-graph, the astonishing answer is that the "B" terms (more precisely: their Boolean closure) is indeed present in this tetraicosahedron, under the unusual species of its … arrows! The exact meaning of this discovery is not totally clear to me, but it is clear that this idea (the arrows as the expression of particular modalities), by now proven, is new (and seemingly quite meaningful).

Let us leave the mention of further, very interesting results (concerning artificial intelligence) of Luzeaux *et alii*, more related to *applications* of NOT (obtained *via* some very tricky suitable translations), to ch. 17 *infra*.

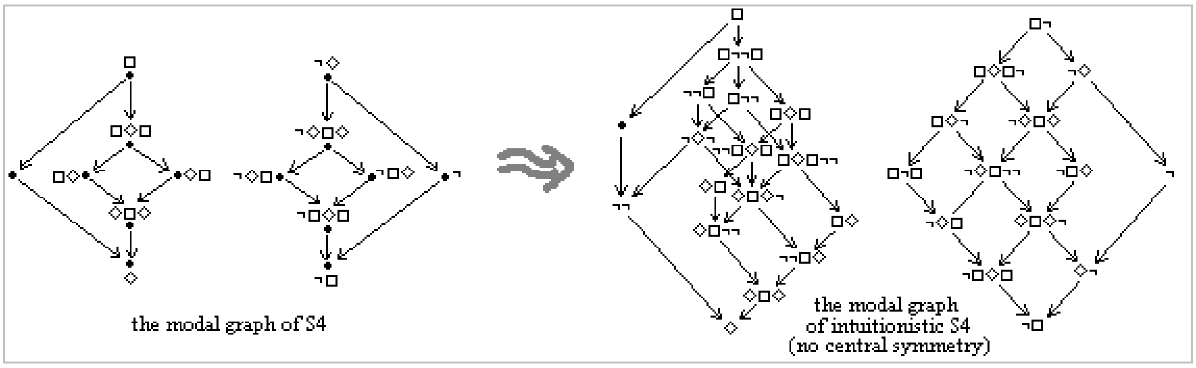
Lastly, Luzeaux starts from a study of the Spanish logician J. M. Font on intuitionist modal logic in order to propose a very interesting possible future expansion of NOT[205]. The study by Font shows that the known systems of modal logic, even when they have, at the

[205] Cf. Font, J. M., "Modality and possibility in some intuitionistic modal logics", *Notre Dame Journal of Formal Logic*, 27-4, 1986, p. 533-546.

object level, some non-classical properties (in this respect S4 has some intuitionistic features), they nevertheless always have a classical logic at their metalevel. So the study shows that adopting a non-classical logic at the metalevel gives as a result modal systems where the set of basic positive modalities is not bijective with respect to that of the basic negative modalities (in simpler terms: 2-dimensional modal graphs are no more centrally symmetric). In the case of S5 with intuitionistic metalevel, for instance, there will be 5 positive and 4 negative modalities.



In the case of S4 there will be 17 positive and 14 negative modalities.

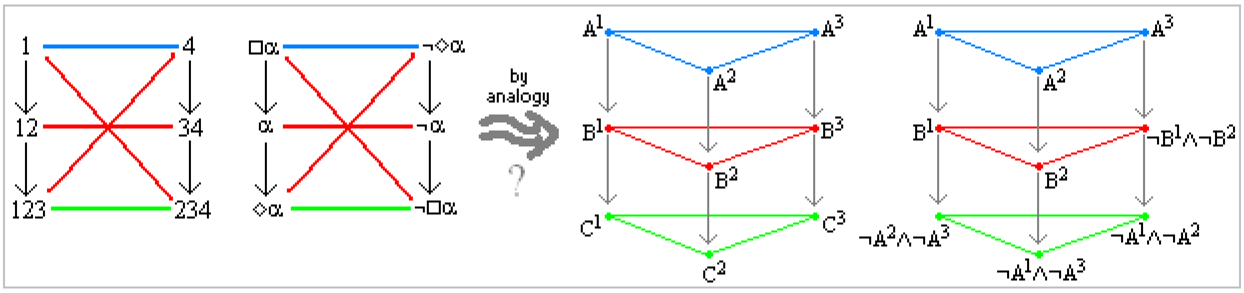


Therefore, Luzeaux (*et alii*) rightly points out that it should be possible to enquire questions like those handled so far by NOT in the case of modal systems with a classical metalevel in this new case of the modal systems obtained (starting by the classical ones) by substituting at their metalevel the classical with a non-classical logic (intuitionist or other). Needless to say, this suggestion made to NOT by Luzeaux *et alii* seems very promising and stimulating: it could open a new field of NOT.

12.03.04. Points of discussion (and decision) with Luzeaux

At least in four points Luzeaux, in his very interesting investigations and proposals, makes claims (about NOT) more or less problematic from my point of view. Needless to say, I take it as a critical stimulation for getting new enriching viewpoints over the subject matter.

First, he uses colours when dealing with the modal *n*(*m*)-graphs or γ-structures (in their black and white 2008 paper this is done with different kinds of dotted or scattered black lines). He does this at least with two such γ-structures: the modal 3(3)-graph (the one of S5, p. 171 and 173) and the modal 4(3)-graph (p. 179).



This use is not totally false: there really are such Aristotelian relations, at least for the modal 3(3)-graph: a green subcontrariety segment at the bottom "gem" (as I call it, cf. ch. 11 *supra*), a red contradiction segment at the middle gem and a blue contrariety segment at the top gem of this linear modal graph. This can be checked logically. But I claim that this use is highly misleading: putting such colours could: (1) make one forget that what we are speaking of is a modal graph and (2) expose to the illusion of having in hand, by virtue of such colouring, all the oppositions, whereas the NOT teaches (cf. ch. 12) that the only way of having them all (i.e. having all the possible oppositions of a given logical space, modal or not) consists in *getting the associated βn-structure*. This last one will offer, "like in a silver plate" (as the French say), *all* the existing oppositional configurations, that is all the α*m*-structures. (3) Moreover, one sees on the previous figure that the use of colours (i.e. of the 4 Aristotelian colours: red, blue, green , black [or grey]) would be highly problematic: how to draw red contradictions in the modal *n*(*m*)-graphs with *n*>3? For, in such graphs, as we know, the contradictions relate points with surfaces, volumes, hyper-volumes, etc. (but no more with points), except in the modal 3(*m*)-graph. For all these reasons we claim that the γ-structures must be monochrome: the only acceptable distinction (if there must be one) is the one separating the gems (in black) from the arrows (in grey); or: there can be some use in representing true points in green (or inside green circles) and false points in red (or inside red circles). But no opposition relation.

Second, when dealing with the logical tetraicosahedron, Luzeaux (*et alii*) claims that, as such (i.e. as presented by Pellissier and, successively, by myself), it is incomplete: mathematically speaking, because it has to do (implicitly) with a Boolean algebra, one should not forget two over its 16 elements (the so-called trivial ones). But the logical tetraicosahedron has 14 elements (its vertices) and the two missing ones (the trivial ones) are

⊥ and T (or, alternatively, the empty set and the power set of the universe): so Luzeaux proposes the notion of "completed tetraicosahedron" ([2008] p. 172). I claim that an assertion like that may let one think that NOT is a badly thought fragment of "serious logic" (serious logic being, of course, the well-known standard algebraic tools). But this is not so! NOT offers a new point of view, which is complete! And as for the question of the two "missing boolean possibilities", Smessaert's move (cf. § 12.02 *supra*) of uniting them in the symmetry centre of the oppositional solids (he does it successfully first for the logical octagon and second for the rhombic dodecahedron) shows that NOT – coherently with its (neo-) structuralist spirit – in fact does not forget these two points. NOT is a new branch of mathematics, and one allowing to express much better (in fact: perfectly) the opposition relations.

Third, when proposing (very interestingly) to take into consideration γ-structures asymmetrical (with respect to the usual central symmetry) because of the use of a non-classical metalanguage (as he suggests, for instance, an intuitionist one) he adds the final remark that the interesting point won't be that of studying the geometries stemming from that, but rather their applications. The reason for such a suggested lack of interest for a study up to now totally unknown is that it will be "only a matter of combinatorics". This point, partly understandable for me (combinatorial explorations are a rather mechanical part of mathematics, one which generally can be left to computers), seems to me to be excessive (and dangerously misleading): for, in a sense, I would say (*reductio ad absurdum contra* Luzeaux): "in the long run all mathematics are "just" a matter of combinatorics". And the history of mathematics shows that many important discoveries depended on "stupid" repeated identical calculations or on "useless" direct (visual) representation[206].

Fourth, Luzeaux says that the logical tetraicosahedron is modal only in so far it can be decorated with the modalities of S5. But because it can also be decorated with the binary connectives (the 14 non-trivial ones) really speaking it is not specifically modal. I would say two things: (1) in principle NOT does not say anything else, being a general theory of opposition: as Blanché's own decorations testify, the *structure* of the hexagon of oppositions applies to concepts (which is more general than modalities or logical sentences); (2) nevertheless, one should also remark that the tetraicosahedron of the binary connectives can

[206] I rely on Mandelbrot, B., "Fractals and the Rebirth of Experimental Mathematics", foreword to Peitgen H.-O., Jürgens H. and Saupe D., *Fractals for the Classroom – Part One. Introduction to Fractals and Chaos*, New York, Springer, 1992; and also on Patras, F., "Sur la vérité mathématique", (forthcoming).

also be read, in a formally very elegant way, as a particular modal logic, one with two "zero-degree modalities", the atoms *p* and *q* (we will come back on this issue on § 17.02.03 *infra*).

12.04. Conclusive remarks

Béziau gave the impulse for looking for interesting uses of the known extensions of the logical square, to which he himself contributed very originally (cf. ch. 10 *supra*). This impulse was hosted by Smessaert and myself, in particular with the elaboration of a generalisation of opposition theory in terms of logical bi-simplexes (this idea, implicitely, was already present, at least *in nuce*, by Sesmat). The other big family of structures of this generalised theory, the β*n*-structures, have been pioneered implicitly by Blanché, explicitly by Béziau, and clearly conjectured by myself. It was only with Pellissier's results, however, that it was possible to investigate it properly, as we will do in the next three chapters. Smessaert has given a method alternative (but partly similar) to Pellissier's one, a method related to "generalised quantification theory" and possibly related to computational linguistics. Beside having as well given (later, not independently) an alternative version of Pellissier's technique (cf. § 12.03.03 *supra*), Luzeaux, Sallantin and Dartnell have given new interpretations of some of these recently discovered formalisms, in particular with application to cognitive science and AI. Luzeaux's proposal of studying inside NOT the case of the asymmetric bi-dimensional modal graphs (due to non-classical metalevel) is very interesting and could open a rather new page of NOT. As far as we are concerned here, the urgent next task seems to be that of fortifying the knowledge acquired until now (on the α-, β- and γ-structures), so to be able later to pursue the enquiry further on all these suggested directions, and possibly by asking more radical questions on the essence of static opposition.

13.
THE EXISTENCE OF THE β4-STRUCTURE: THE "LOGICAL HYPER-TETRAICOSAHEDRON"[207]

In this chapter we demonstrate that, in correspondence with modal 3(4)-graphs, there exists an elegant β4-structure, that is an arrowed 4-dimensional solid gathering nicely a four-dimensional α5-structure, several three dimensional α4-structures (or logical cubes) and several two-dimensional α3-structures (or logical hexagons). This very complex and rather elegant logical-geometrical structure, which we call "logical hyper-tetraicosahedron" because it follows Pellissier's logical tetraicosahedron, besides being highly instructive in itself, constitutes by its mere existence a further step towards the recursive definition of a general series of *n*-dimensional logical hyper-tetraicosahedra (i.e. the closure of the β*n*-structures), a series until now only conjectured (cf. ch. 11), but now seemingly close to exact recursive definition.

13.01. Our ancient conjecture: there should be a "β4-structure"

In 2004 we proved that there is a series of α-structures (the logical bi-simplexes, generalising the logical square and the logical hexagon) but we also conjectured a series of β-structures (generalising the logical cuboctahedron). This conjecture remained invincible (except for the series of the hyper-flowers, cf. ch. 11 *supra*) until Pellissier's elaboration of his powerful setting technique for determining all the α-structures contained in any finite linear modal *n*(*m*)-graph.

13.01.01. Utility of the β3-structure: it is a useful complete gathering

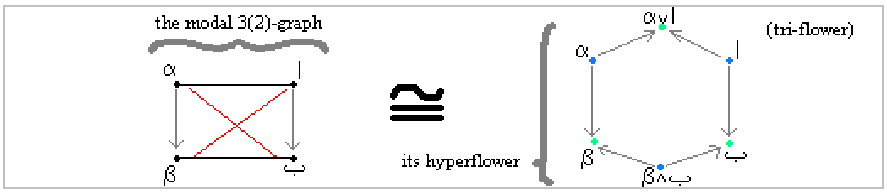
The utility of the β3-structure (Pellissier's tetraicosahedron) consists in being a complete list of all the possible oppositions of the logical space of the modal 3(3)-graph, which is among others the modal graph of S5. We saw that before Pellissier's proof, one could not be prevented from the discovery of new unsuspected oppositions (as testified recently his discovery of the four "weak hexagons" of S5). So, if we could be able to

[207] Some elements of the reflexion of this chapter are the result of a coming common work with Régis Pellissier.

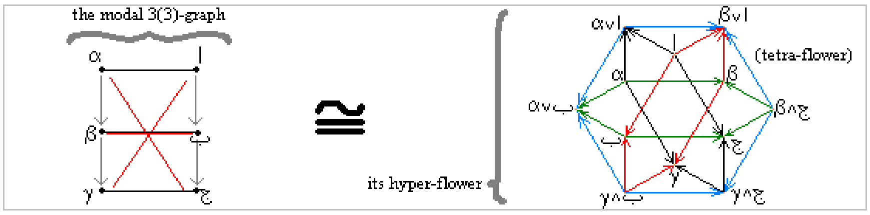
determine the β-structure of the other modal graphs (the γ-structures, of which there is an infinite number), we would presumably be allowed to master all the possible logical oppositions proper to each of them (cf. ch. 16 *infra*). For instance, being able to determine the modal graph of the system S4 would allow us to specify the geometry of the logical oppositions of that famous system (by now we are not able to do it, but we will try, cf. ch. 17).

13.01.02. Going beyond the β3-structure: beyond the hyper-flowers

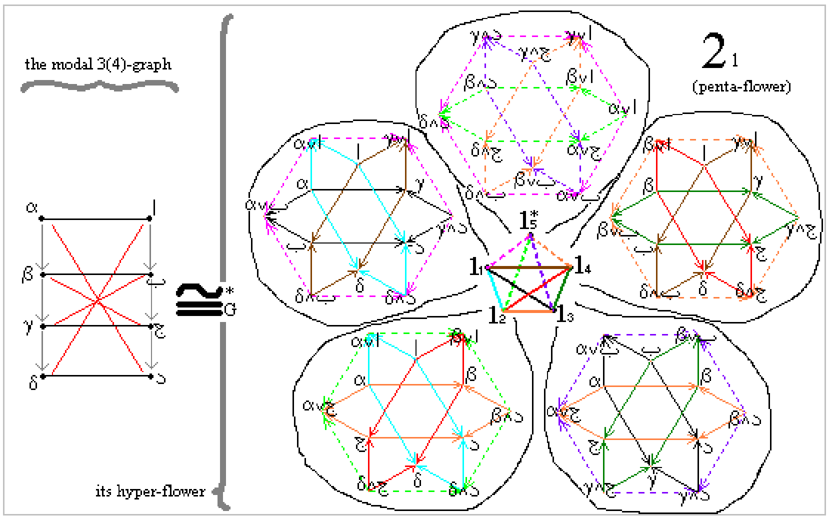
Remember that the elements of the tetraicosahedron are: 1 cube, 6+4 hexagons and 18 squares. The structure of the tetraicosahedron is a cube with 6 pyramids, its surface being made of 24 logical triangles. Now, in order to clarify ideas, let us turn back to the logical cuboctahedra and to the series it belongs to, that of the hyperflowers. The case of the 3(2)-graph is unambiguous: it gives one (and only one) logical hexagon (the tri-flower).



Not so for the modal 3(3)-graph: as we saw it corresponds to the logical cuboctahedron (the tetra-flower) but only under certain restrictions, that is if one truncates some kinds of solutions, i.e. if one accepts only the "basic" hexagons (cf. § 11.04.02 *supra*).

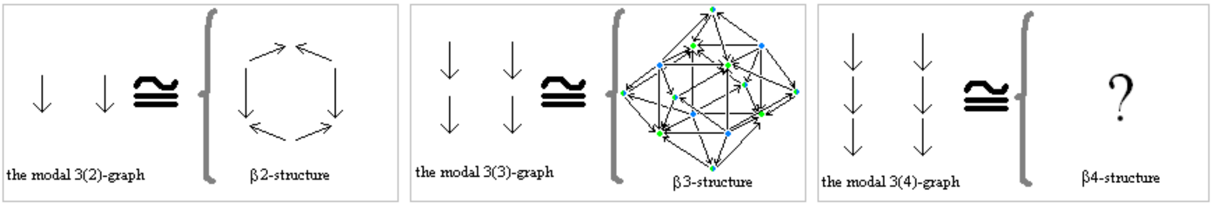


We also saw that, under the same restrictions (the same truncations), the modal 3(4)-graph corresponds then to the "logical hyper-cuboctahedron" (the logical penta-flower).

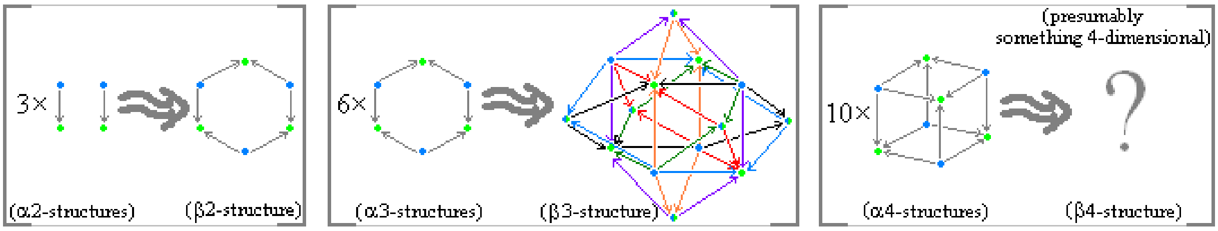


And we claimed (we prove it elsewhere) that this behaviour is stable: there is an infinite series of growing logical hyper-flowers. And apparently, when one takes other playing rules, i.e. other kinds of restrictions, one should get to some kind of "hyper-jungle" (cf. § 11.04.03 *supra*).

Now, we know after Pellissier (cf. ch. 12 *supra*) that the full geometrical-oppositional translation of the modal 3(3)-graph is, in fact, the logical tetraicosahedron (of which the logical cuboctahedron is just a regular fragment, present three times). Which will be the full, Pellissier's style geometrical-oppositional translation of the modal 3(4)-graph?



What the hyper-cuboctahedron (i.e. the penta-flower) seems to say to us is that there is at least a fragment (indeed beautiful) of the unknown hypothetical β4-structure (this fragment is 4-dimensional, and has a pentagonal structure). And apparently the still mysterious β4-structure should be a four-dimensional solid made, in some sense, of logical cubes.



13.02. Determining the β4: number and quality of its elements

We will apply Pellissier's setting technique to the modal 3(4)-graph. After having determined its characteristic set E_p, we will at first count the basic constituting elements of the β4-structure: that is, the possible subsets (this will give all the vertices of the β4-structure) and *p*-partitions, strong or weak (this will give the strong and weak α-structures), of its representing set $E_{\alpha 3(4)}$.

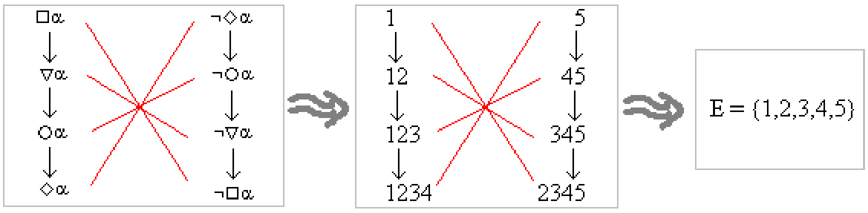
13.02.01. Determining E_p for α3(4): the vertices of the β4-structure

As we mentioned (cf. ch. 12) Pellissier's formula for "settifying" the modal *n*(*m*)-graphs is the following (there are two cases, according to whether *m* is even or odd, with *n* and $k \in \mathbb{N}$):

the case $\alpha n(2k)$ (i.e. when m is even) gives $(\mathrm{Card}(E_{\alpha 3(4)}) = (n\text{-}1)k+1)$

the case $\alpha n(2k+1)$ (i.e. when m is odd) gives $(\mathrm{Card}(E_{\alpha 3(4)}) = (n\text{-}1)(k+1))$

So, calculating E_p for the logical space of α3(4), where n=3 and m=4 (and thus k=2) gives the result: $E_p(\alpha 3(4)) = E_{\alpha 3(4)} = \{1, 2, 3, 4, 5\}$ (that is, p = 5). This, in turn, gives the following decoration of the modal 3(4)-graph (i.e. the linear modal graph with 2 columns and 4 rows).



(at the left of the figure we give an intuitive modal decoration, even if the meaning of ∇ and O is left totally undetermined by us).

The subsets of E_p give the points: any vertex of the geometric figure (the β4-structure) is one and only one possible unordered concatenation of numbers such that: (a) there are no repetitions ("11" is the same as "1"), (b) there must be at least a number in each concatenation (" " , the empty symbol, is not a concatenation), (c) there are maximally 4 numbers ("12345", the maximal string, is not an acceptable concatenation), (d) concatenations are written putting the given numbers in a numerical order ("12" is the same as "21", but "21" must therefore be written "12"). Clearly, P, the number of parts, is given by the formula:

$P = \wp(E_p) - 2$ (we exclude the empty and the maximal string)

Given that strings here can have 1 to 4 characters, it is easy to check that the combinatory gives the following 30 points or "vertexes" (there are no others):

-	1,	2,	3,	4,	5					
-	12,	13,	14,	15,	23,	24,	25,	34,	35,	45
-	345,	245,	235,	234,	145,	135,	134,	125,	124,	123
-	2345,	1345,	1245,	1235,	1234					

(the first row is, with respect to E_p, complementary – i.e. contradictory – with the fourth, the second is complementary – i.e. contradictory – with the fourth).

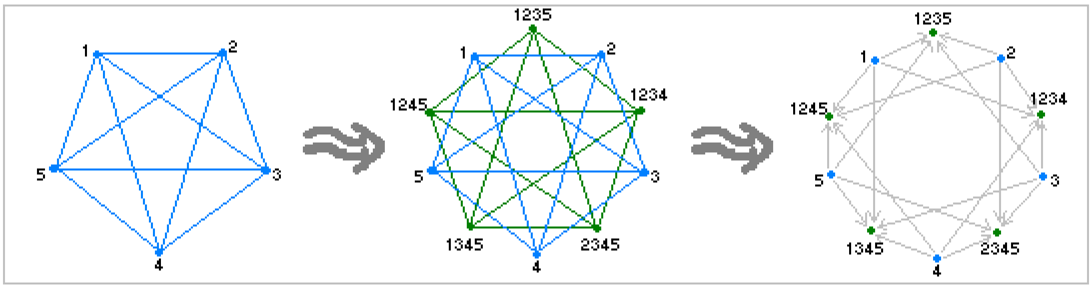
Remember that, logically speaking, and keeping in mind the modal decoration proposed above (with still uninterpreted modal operators: boxes, diamonds, nablas and circles), these numbered 30 abstract modalities correspond respectively to:

- □α, ∇α∧¬□α, Oα∧¬∇α, ◊α∧¬Oα, ¬◊α
- ∇α, □α∨(Oα∧¬∇α), □α∨(◊α∧¬Oα), □α∨¬◊α, Oα∧¬□α,
(∇α∧¬□α)∨(◊α∧¬Oα), (∇α∧¬□α)∨¬◊α, ◊α∧¬∇α, (Oα∧¬∇α)∨¬◊α, ¬O α
- ¬∇α, (∇α∧¬□α)∨¬◊α, (Oα∧∨¬□α)∨¬◊α, ◊α∧¬□α, □α∨¬Oα,
□α∨(Oα∧¬∇α)∨¬◊α, □α∨(◊α∧¬∇α), ∇α∨¬◊α, ∇α∨¬(◊α∧¬Oα), Oα
- ¬□α, □α∨¬∇α, ∇α ∨¬Oα, Oα∨¬◊α, ◊α

But in what follows we keep the numbered notation, easier to use (because shorter).

13.02.02. The “heart” of the β4-structure has the shape of the α5-structure

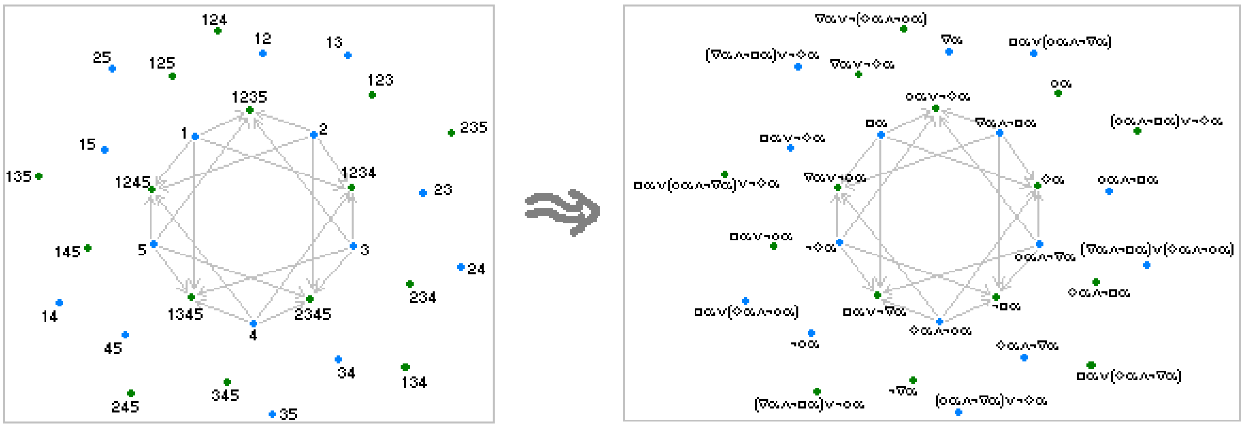
We saw that the β3-structure, Pellissier’s logical tetraicosahedron, has a “heart” which is in fact none other than the α4-structure, a logical cube (it is its unique largest partition). Is there a “heart” in the β4-structure we are now investigating? Yes. The maximal partition of E_p(α3(4)) is its unique 5-partition, that is “ 1 | 2 | 3 | 4 | 5 ”. Which means that inside the logical space of α3(4) (i.e. inside the logical space generated by the modal 3(4)-graph), we have one (and one only) viable decoration of the α5-structure, the structure expressing 5-opposition (a 4-dimensional blue-green logical bi-simplex, with grey implicational arrows, the blue simplex expressing contrariety, the green one sub-contrariety cf. ch. 11).



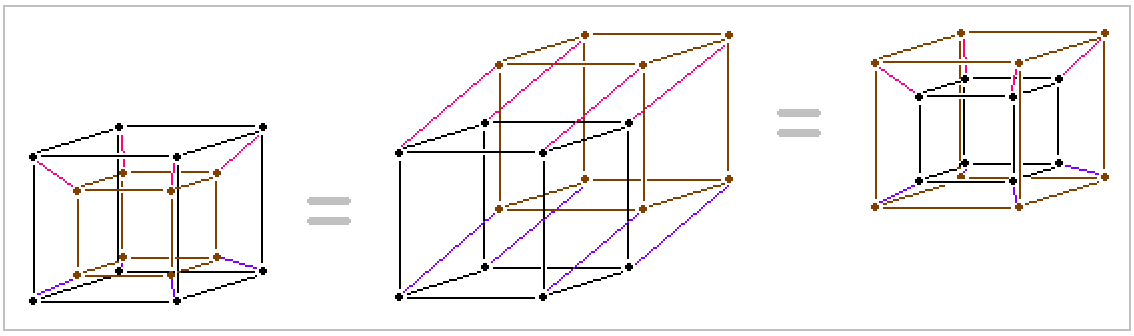
Again, in this case as well, to keep the analogy with the β3-structure, we call this bi-simplex, corresponding to the unique 5-partition of E(α3(4)), its “heart”.

Now, because we know that there are 30 points in the β4-structure, and because the heart here has only 10 points, we know that there are 20 more points to be expressed, in the global β4-structure that we are investigating, apart from its heart (let’s call them “cloud points”). Graphically speaking, for reasons to appear soon, they can be represented “outside” the heart just depicted (all this is nevertheless conventional, the space we are handling being

4-dimensional, cf. *infra*). Which gives the following provisory figure (heart surrounded by cloud).



Again, a warning is necessary: because we are handling a 4-dimensional space, “inner” and “outer” can be deceitful, as show the following standard pictures of 3 equivalent 3-dimensional representations of a 4-dimensional hyper-cube.



(despite the appearances, the 3 figures are totally equivalent, “inner” and “outer” are changeable). In other words, the representation of the “cloud” with respect to that of the “heart” could be alternatively outside or inside of it (with no exclusion of a third possibility)

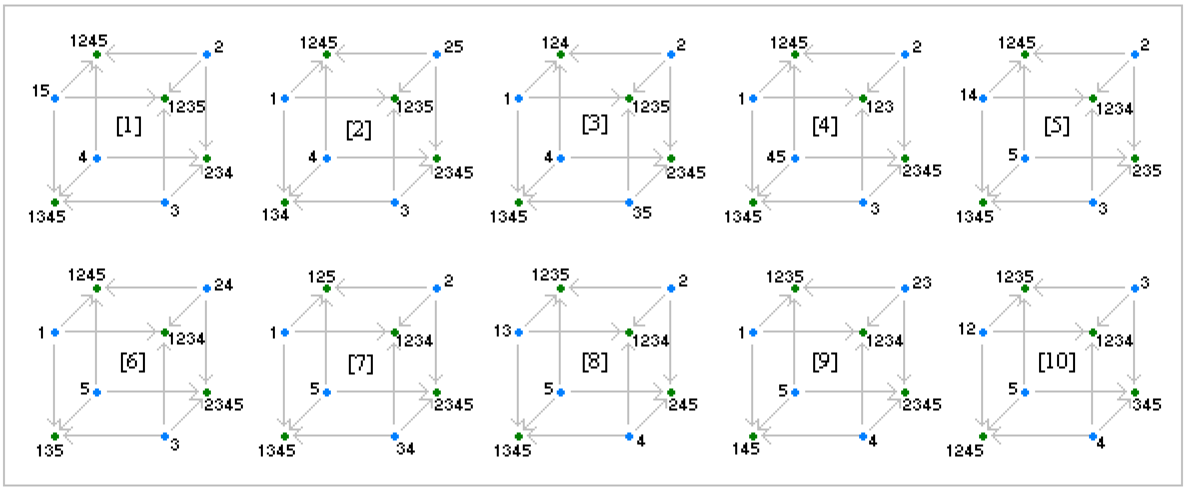
So, the next step will consist in finding how to relate to this heart the structures corresponding to the 4-partitions (which are, we know it in advance, logical cubes), that is, fundamentally, how to draw all existing grey implicational arrows (subalternation relations). In particular, we will have to understand how the 20 cloud points outside the heart are related to one another.

13.02.03. Looking for all the logical cubes of the β4-structure

We want now to find the logical cubes of the β4-structure. The ‘setting’ decorating technique gives us the following list of 4-partitions (logical cubes). There are 10 “strong” cubes (ten 4-partitions of E), which are the following:

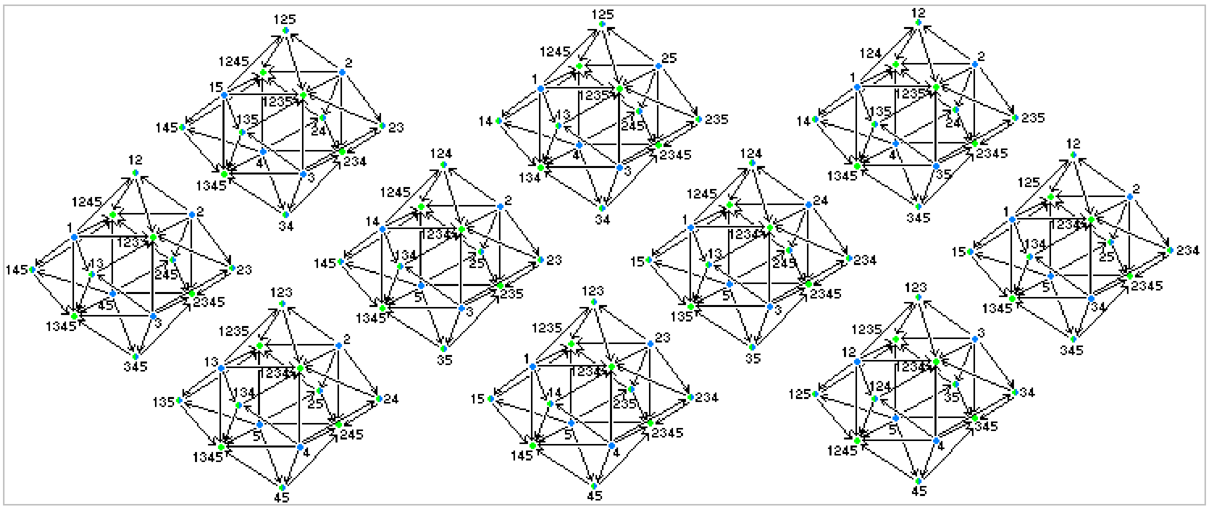
[1]	15 | 2 | 3 | 4	[2]	1 | 25 | 3 | 4	[3]	1 | 2 | 35 | 4
[4]	1 | 2 | 3 | 45	[5]	14 | 2 | 3 | 5	[6]	1 | 24 | 3 | 5
[7]	1 | 2 | 34 | 5	[8]	13 | 2 | 4 | 5	[9]	1 | 23 | 4 | 5
[10]	12 | 3 | 4 | 5				

Remember: the four points in each of these 4-partitions are the blue ones, whence the four green ones are immediately deduced by contradictory negation of each of them. The two kinds of tetrahedra, blue (for contrarieties) and green (for subcontrarieties) are mutually dual. Consequently, in order to decorate the green vertexes of each cube one has to take for them the value complementary (with respect to E) to the corresponding blue vertex (the union of each blue with its green must be exactly equal to E, with no redundancy and no lack of elements – say, for example: “13 and 245” but not “13 and 2345”). This gives the following 10 logical (strong) cubes (we restrict ourselves to the sole expression of subalternations; we name these logical cubes [1]-[2]).



We will see later that each of such strong cubes exceeds the “heart” by 2 of its 8 vertices (the other 6 belonging to the heart)

It is easy, and nevertheless crucial, to see that each of these 10 cubes can be expanded, within the space of the β4-structure, to a logical tetraicosahedron: just by remarking that, in each of its six sides, the logical conjunction of the two green is equivalent to the logical disjunction of the two blue (as an example: in the upper face of the cube [1] we have: $15 \cap 2 = 1245 \cup 1235$). So we can add two arrows going from the two blue vertices to their disjunction, and from this point we can draw two arrows going to the two green points of which it is the conjunction, which gives the following list of tetraicosahedra of the β4-structure:

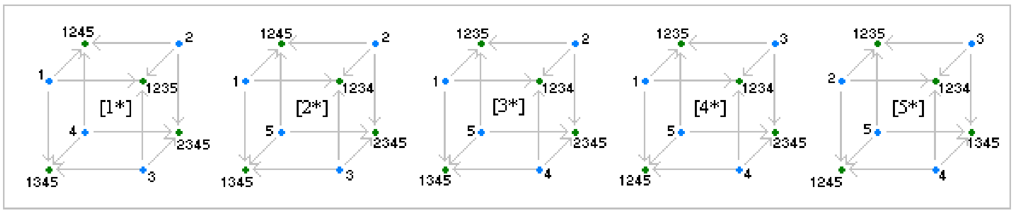


As for the “weak” cubes, Pellissier’s setting method proves that there are exactly 5 of them (that is, five 4-partitions of proper subsets of E):

[1*] 1 | 2 | 3 | 4 [2*] 1 | 2 | 3 | 5 [3*] 1 | 2 | 4 | 5

[4*] 1 | 3 | 4 | 5 [5*] 2 | 3 | 4 | 5

Which gives us the following 5 logical (weak) cubes (we name them [1*]-[5*]).



But now we see that these five weak logical cubes, differently from the 10 previous strong ones, do not develop into logical tetraicosahedra: for each of the five cubes in each of its six faces the logical product of the two green is not equivalent to the logical sum of the two blue.

This means that, as we will see later, all vertices of each one of these 5 weak cubes belong to the “heart”. The weak logical cubes are totally intrinsic to the heart. In other words, whereas normal logical cubes exceed the heart, the “non-standard” logical cubes, being distorted, belong entirely to the heart (weak logical cubes are distorted in a way similar to that by which weak hexagons are broken, cf. ch. 12). This behaviour of strong and weak cubes here keeps the analogy with what happens in the β3-structure – the tetraicosahedron – to strong and weak hexagons with respect to the logical cube, the heart of that structure.

13.02.04. The 25 logical hexagons (or 3-partitions of E) inside the β4-str.

A similar reasoning can be brought out for the logical hexagons of α3(4) (i.e. its 3-partitions, strong and weak). This will give the following 25 tri-partitions of E.

<1> 1 | 234 | 5 <2> 2 | 134 | 5 <3> 3 | 124 | 5

<4> 4 | 123 | 5 <5> 12 | 34 | 5 <6> 13 | 24 | 5

<7> 14 | 23 | 5 <8> 15 | 2 | 34 <9> 1 | 25 | 34

<10> 1 | 2 | 345 <11> 15 | 3 | 24 <12> 1 | 35 | 24

<13> 1 | 3 | 245 <14> 15 | 4 | 23 <15> 1 | 45 | 23

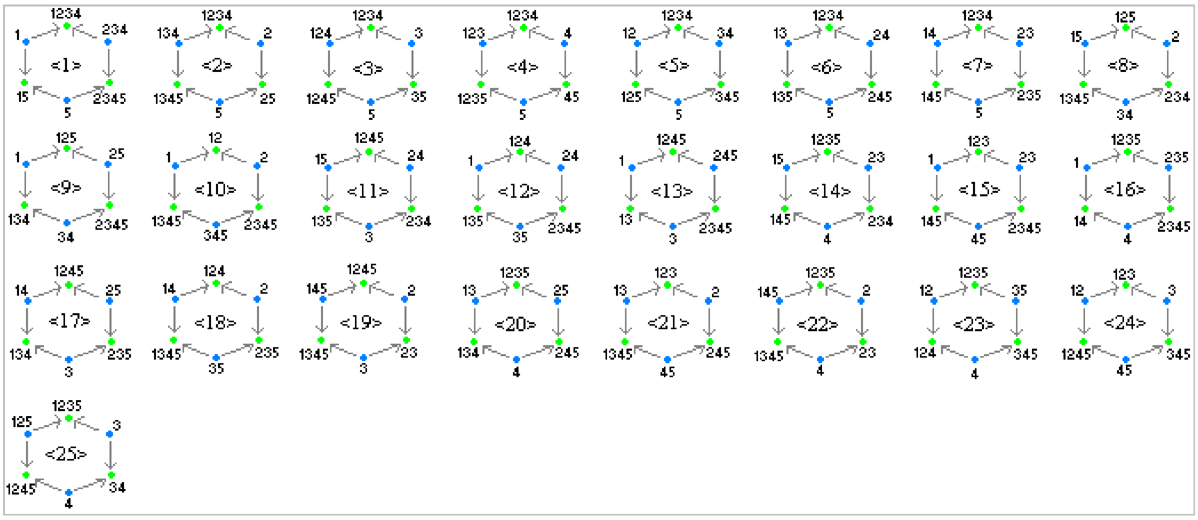
<16> 1 | 4 | 235 <17> 25 | 3 | 14 <18> 2 | 35 | 14

<19> 2 | 3 | 145 <20> 25 | 4 | 13 <21> 2 | 45 | 13

<22> 2 | 4 | 135 <23> 35 | 4 | 12 <24> 3 | 45 | 12

<25> 3 | 4 | 125

which correspond to 25 strong logical strong hexagons.



These hexagons, being strong, are planar (cf. ch. 12 *supra*).

As for the weak logical hexagons, Pellissier's method gives the following 3-partitions of proper subsets of E.

<1*> 12 | 3 | 4, <2*> 13 | 2 | 4, <3*> 14 | 2 | 3,

<4*> 1 | 23 | 4, <5*> 1 | 24 | 3, <6*> 1 | 2 | 34,

<7*> 12 | 3 | 5, <8*> 13 | 2 | 5, <9*> 15 | 2 | 3,

<10*> 1 | 23 | 5, <11*> 1 | 25 | 3, <12*> 1 | 2 | 35,

<13*> 12 | 4 | 5, <14*> 14 | 2 | 5, <15*> 15 | 2 | 4,

<16*> 1 | 24 | 5, <17*> 1 | 25 | 4, <18*> 1 | 2 | 45,

<19*> 13 | 4 | 5, <20*> 14 | 3 | 5, <21*> 15 | 3 | 4,

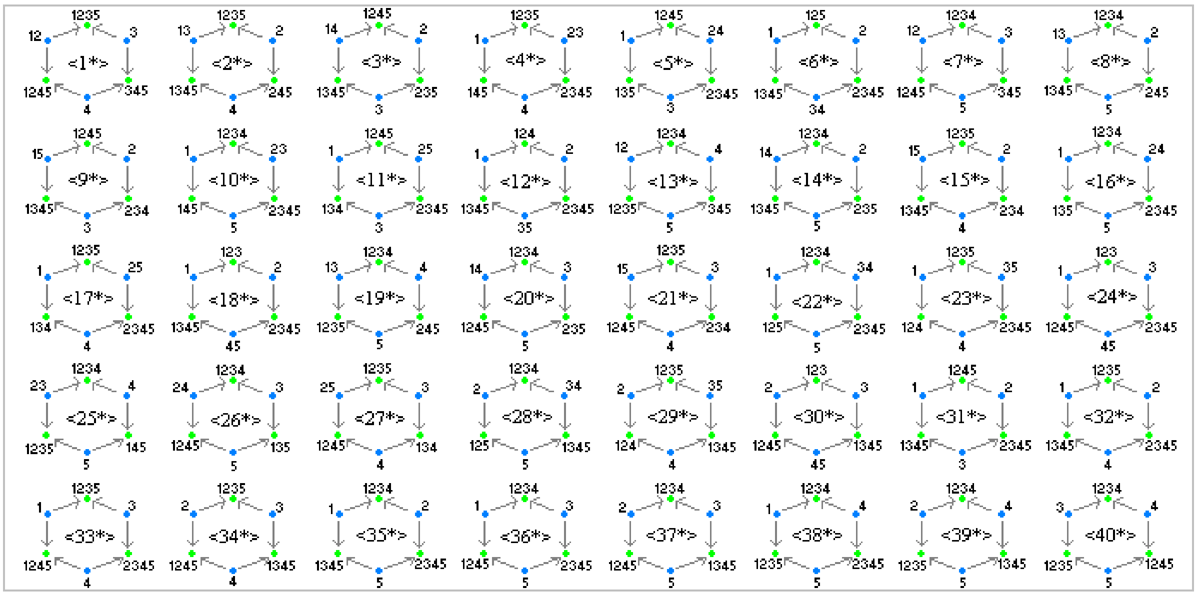
<22*> 1 | 34 | 5, <23*> 1 | 35 | 4, <24*> 1 | 3 | 45,

<25*> 23 | 4 | 5, <26*> 24 | 3 | 5, <27*> 25 | 3 | 4,

<28*> 2 | 34 | 5, <29*> 2 | 35 | 4, <30*>2 | 3 | 45,

<31*> 1 | 2 | 3, <32*> 1 | 2 | 4, <33*> 1 | 3 | 4,
<34*> 2 | 3 | 4, <35*> 1 | 2 | 5, <36*> 1 | 3 | 5,
<37*> 2 | 3 | 5, <38*> 1 | 4 | 5, <39*>2 | 4 | 5,
<40*> 3 | 4 | 5.

So, it turns out that there are 40 of them, corresponding to 40 weak logical hexagons.



This is coherent with the fact that each strong logical cube contains exactly four weak hexagons (cf. ch. 12 *supra*) and that the β4-structure contains, as we saw, 10 strong logical cubes (cf. *supra*).

As for the logical squares, as Pellissier's theory shows (cf. ch. 12 *supra*), they cannot be reached through the 2-partition of E: for, such a 2-partition corresponds in fact to the simple contradiction among any two couples of contradictory vertices of the β4-structure (i.e. of E). But the logical squares can be obtained simply through the strong logical hexagons, each one of these last containing exactly three logical squares (we will not represent them).

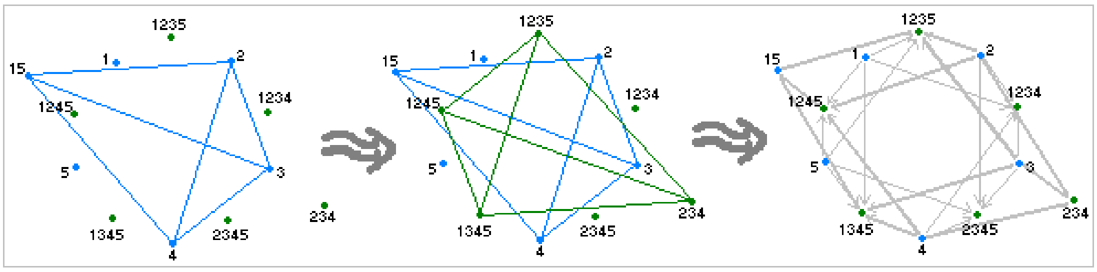
In what follows, in order to explore the geometrical structure it makes more sense to concentrate on the largest *n*-partition available, the unique 5-partition and the fourteen 4-partitions.

13.03. Determining the β4-structure (II): shape, overall structure

Having seen the different constituting elements (at least the most important – we will see others later) we now have to understand the way they combine together. We will try to determine the geometrical shape of the β4-structure, by investigating how such constitutive basic elements relate to one another according to complex but regular geometrical relations.

13.03.01. The 10+5 cubes of the β4-structure with respect to its “heart”

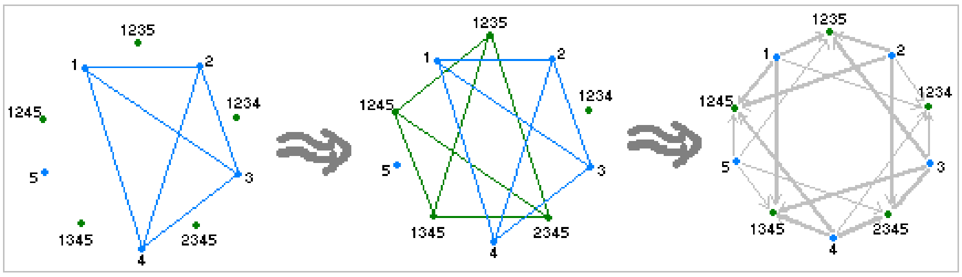
In order to make things clearer about the strong cubes of the β4-structure, we will take the example of just one of them. First of all, let us exhibit the inner structure of this logical cube, that is its two oppositional tetrahedra, blue (for contrarieties) and green (for subcontrarieties), forming together the relevant logical bi-simplex (here of dimension 3 – tetrahedra being 3-dimensional geometrical entities).



As the picture shows, following the usual construction principle (cf. ch. 11 *supra*) we see the emergence of an external envelope (constituted of grey arrows) of the logical bi-tetrahedron, in this case that is the “logical cube” itself. Clearly, a similar strong cube can be built for any of the 10 couples of contradictory pairs of vertices (one blue, the other green) outside the “heart” (i.e. couples of vertices belonging to the cloud – here we saw the couple 15-234).

Summarising what we saw so far, logical strong cubes “exceed” (geometrically) the heart of the β4-structure by two (mutually contradictory) of their 8 vertices, the remaining six belonging to the heart.

In a similar way, for similar reasons, let’s now take the example of a weak logical cube (we saw there are five in β4-structure). Let us exhibit, again, the inner structure of this logical cube, that is, its two oppositional tetrahedra, blue (for contrarieties) and green (for subcontrarieties), forming together a bi-simplex of dimension 3 (a logical bi-tetrahedron).



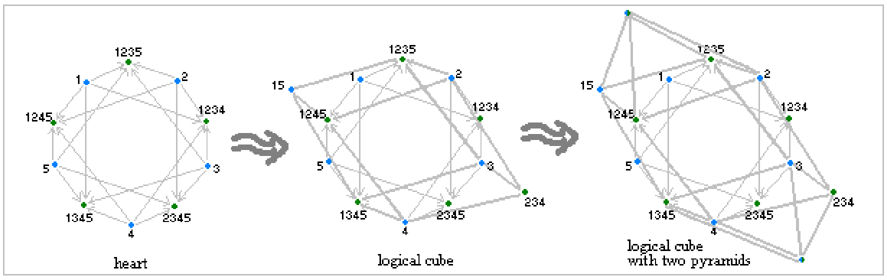
Here as well, as the picture shows, following the usual construction principle (cf. ch. 11 *supra*) we see the emergence of an external implicative envelope (constituted of grey arrows) of the bi-tetrahedron, that is, the “logical cube” itself. But this time the cube is graphically “strange”. That is, this cube is “under pressure” (geometrically speaking).

Summarising what we saw so far, the logical weak cubes remain entirely comprised ("under pressure") into the heart of the β4-structure, which means that, in each case, all of their 8 vertices belong to the set of the 10 vertexes of the heart.

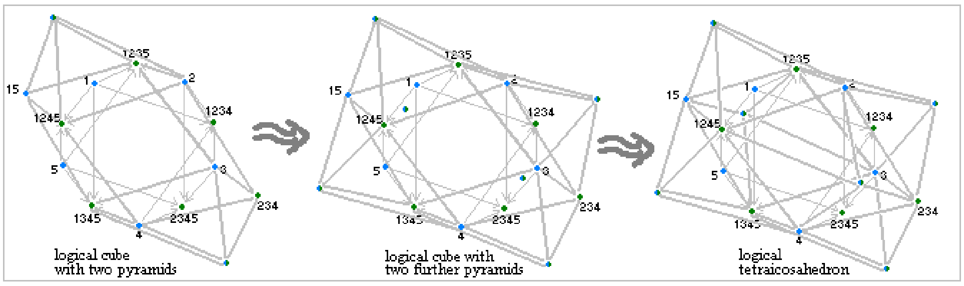
Remark that here any weak logical cube is singularised graphically by the fact of excluding two among the ten mutually contradictory vertices of the heart.

13.03.02. The logical hexagons combined with the cubes: tetraicosahedra

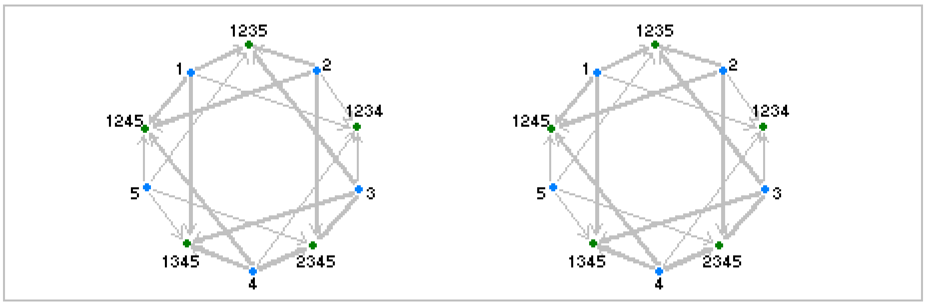
How shall we order all the 25 strong hexagons available in β4? Apparently, by analogy with the β3-structure, into tetraicosahedra, when Pellissier investigating the β3-structure built it by "diamondifying" its heart, the logical cube, i.e. by adding on top of each of its six square faces an Egyptian pyramid (i.e. a pyramid with a square basement). So, similarly, we should look, in β4, for a 4-dimensional analogue of Pellissier's 3-dimensional diamondification in β3. Here we will do the same diamondification of the ten strong cubes: and indeed, apparently, there are 10 such tetraicosahedra which are tied to the 10 strong cubes. We can give a first representation of the "Egyptian pyramids" resting on each of the six square faces of any strong logical cube. In a first step we construct a logical cube over the heart and then we "pyramidise" 2 of its 6 two by two opposite faces.



Then we "pyramidise" two other opposite faces, and then the last two.



This leads, starting from any given strong logical cube and adding to it its 6 Egyptian pyramids, to the emergence of a logical tetraicosahedron {“diamondation” of weak cubes} {Explain: here several diamondifications are possible (explain with a DRAWING). In fact there are 5x(2x2x2)= 40} We can give a first representation of the “Egyptian pyramids” resting on each of the six square faces of each weak logical cube.



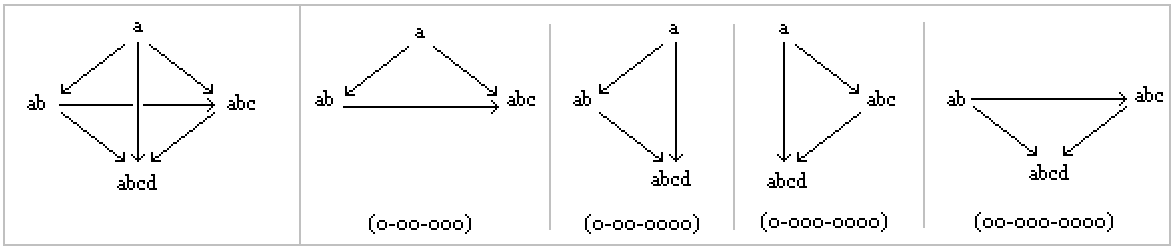
This leads, starting from a given weak cube and adding to it its 6 <diamondations>, to the emergence of a logical tetraicosahedron. Having enquired the “flesh” of the β4-structure, it is time to have a look at its “skin”.

13.03.03. The “surface” of the β4 is made of 120 “logical tetrahedra”

One of the final elements to be mentioned is the “tiling” constituting the β4-structure’s surface. Remember that in the case of the β3-structure (the logical tetraicosahedron) the solid was tri-dimensional, its surface was 2-dimensional, and it was made of 24 “logical triangles” (i.e. triangles whose three sides are implication arrows).

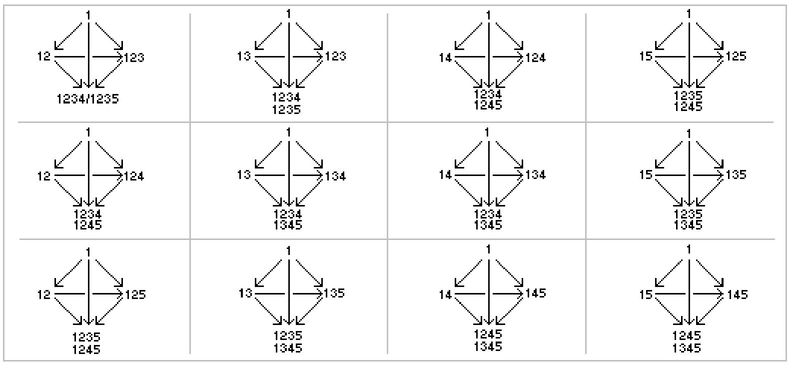
Now, in the present case, the β4-structure seems to be 4-dimensional (because of its “heart”, which has that dimensionality). So we can expect, by analogy, its “surface” to be 3-dimensional. As a matter of fact, the smallest 3-dimensional units are “logical tetrahedra”.

In order to determine their exact number in the β4-structure we must stick to their construction principle (their arrows) by a combinatorial exploration. This will be the following.



The number of “o” represents the number of symbols in a string. The previous picture shows, combinatorially,the possible kinds of “surface triangles”. Remark, indeed, that here as well there are, beyond the logical tetrahedra and constituting them, logical triangles (seemingly the simplicial basic unit of surfaces).

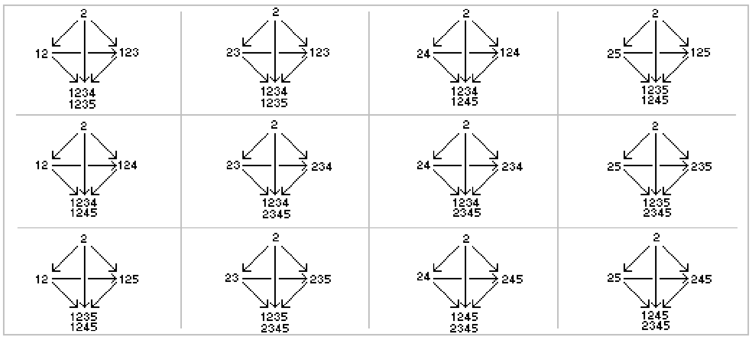
The possible arrows arrangements of four terms (constituting a logical tetrahedron) are distributed in the following five series. The first is one admitting "1" as a value for the first position (the other 3 terms are given by all their possible decorations).



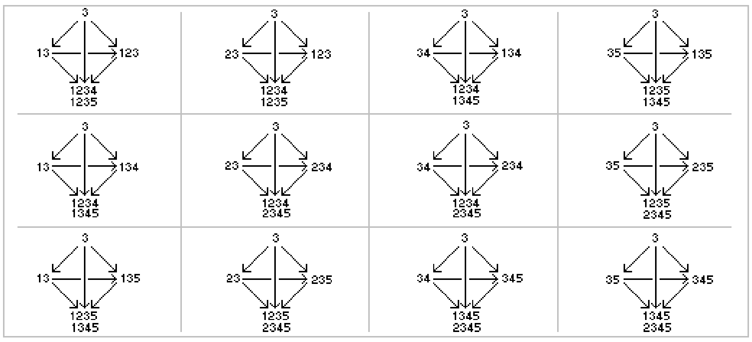
Remark that for brevity's sake we depicted the fourth, bottom position of each logical tetrahedron by two (instead of one) string of numbers: so, each of these logical tetrahedra represents in fact two different ones (according to the string chosen at its bottom).

A second series of 24 tetrahedra is generated by the value "2" of the first position.

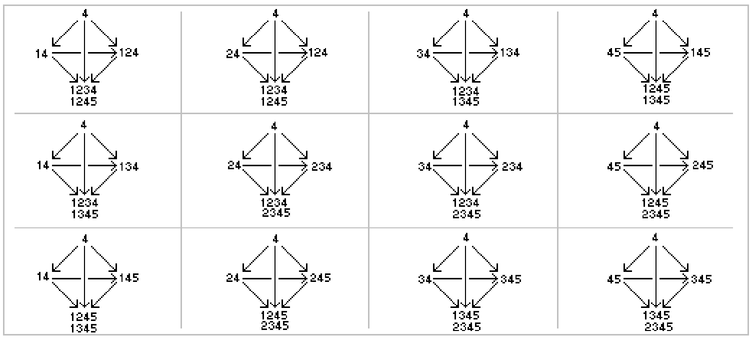
(here as well the positions other than the first one embody a possible combinatorial decoration)



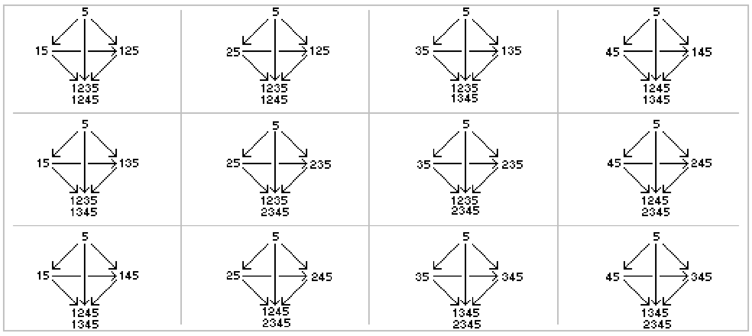
A third series of 24 tetrahedra is generated by the value "3" of the first position.



and similarly with the value "4".



and, finally, the same with the value "5".



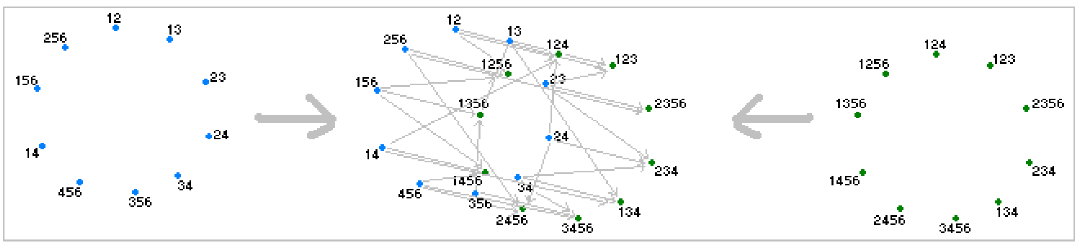
which closes the counting. So at the end it appears that in β4 there are 120 such logical tetrahedra. They constitute its "surface volume" (or, more poetically, its "wall") (as in β3, the logical tetraicosahedron, the 24 two-dimensional faces were triangles).

So the β4-structure seems to be – from the point of view of its 2-dimensional surface – a 4-dimensional logical "icoekatonhedron" (120 faces). But because it is also the successor of the logical tetraicosahedron, we will rather call it the "logical hyper-tetraicosahedron".

13.03.04. Looking for the global geometrical shape of the β4-structure

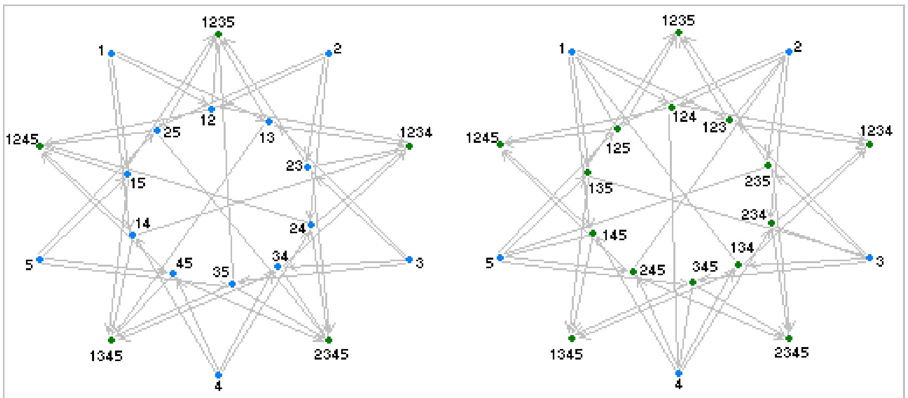
As already hinted at (with the example of the equivalent ways of expressing 3-dimensionally a 4-dimensional hyper-cube) there are problems for representing our 4-dimensional β4-structure.

It turns out that the simplest way to characterise geometrically the whole β4-structure seems to be the following. First, the cloud can be divided into its blue and its green parts. Let us call these two parts "ears" (the left, blue one and the right, green one).



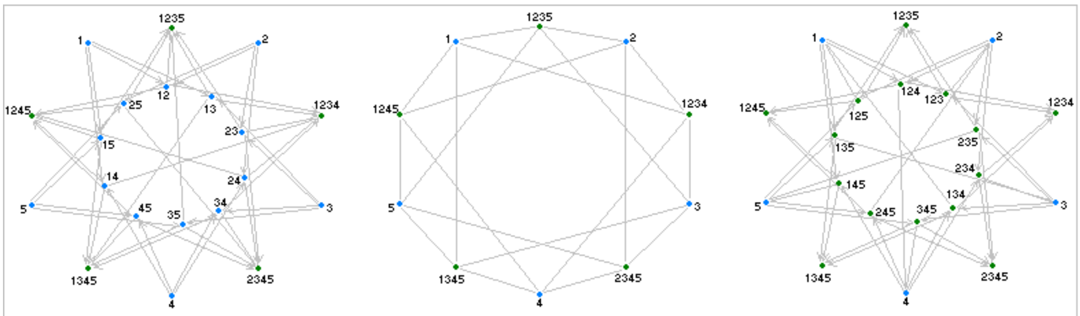
The left and the right ears are nicely connected by arrows: they form some kind of "drum" (or "ear-drum").

Each of these 2 parts (the ears), if introduced inside the 10 vertices of the heart (of which here we do not express the own arrows) bears a rather nice geometrical regularity of the arrows (some kind of duality). There is a nice "stellar inscription" of each of the ears inside the heart.



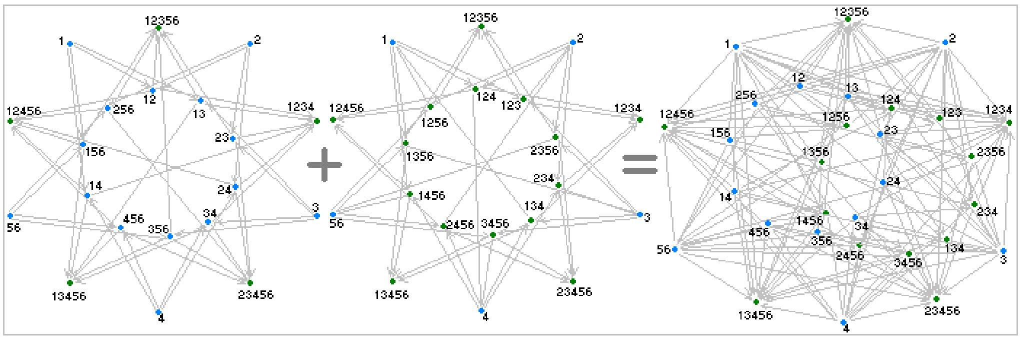
distributing the "cloud" at the two sides of the "heart" of the β4-structure

As the heart is an element of the β-structures easy to grasp, we would like to keep it fixed. So we add it between the two ears, the problem being then of unifying these three geometrical redundant fragments of the β4-structure.

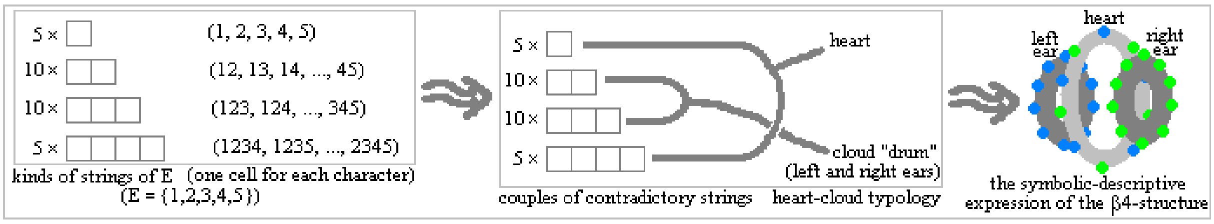


towards a global representation of the β4-structure

The final result of this offers an interesting "double stellar inscription of the ears-drum in the heart", which can be represented like this (this is a complete representation of the4-dimensional β4-structure and of all its arrows!).



But this representation is a very complex one, and probably not fit to be generalised as such for the expression of its hypothetical successors (the β5-structure and so on). Then, looking for something easier to handle and generalise (but still geometrically faithful), if we rely on a qualitative vision of the kinds of vertices, the β4-structure can in fact be expressed, in a slightly simplified way, as follows.



We call this a "symbolic-descriptive" figure of the β4-structure (in the rest of this study this will bear some importance, cf. especially ch. 14, 15 and 17 *infra*).

Now, retrospectively, this way, not too bad, of expressing something of the overall geometry is related to the specificities of the vertices of the β4-structures. As a matter of fact, indeed, these vertices divide entirely into three categories (like the three "doughnuts" of our symbolic-descriptive representation):

- (a) the vertices expressed by a 1-character string (like "1", "2", etc.);
- (b) the vertices expressed by a 2-characters string (like "12", "13", "23", etc.);
- (c) the vertices expressed by a 3-characters string (like "123", "124", "234", etc.).

Now, the maximal string is a 4-character string (i.e. the "1234"). This means that the (a) elements must be symmetric (i.e. contradictory) with the (c) elements, that the (b) elements are symmetric (i.e. contradictory) to other (b) elements. So, the simplified geometrical model we proposed here respects and expresses this kind of features.

In the next chapters we will try to progressively generalise this kind of representation of the β*n*-structures, to have some kind of graphical "β-alphabet" for NOT.

13.04. Are the β*n*-structures *n*-dim. hyper-tetraicosahedra?

The β4-structure, the hyper-tetraicosahedron, has affinity to its fragment, the hyper-cuboctahedron (it has a decagonal basis, which is akin to the pentagonal basis of the hyper-cuboctahedron, cf. ch. 11 *supra*).

So, the existence of the series of the hyper-cuboctahedra (or series of the hyper-flowers) seems a good reason to maintain the conjecture of the existence of a series of β*n*-structures as series of the hyper-tetraicosahedra.

The newly discovered β4-structure seems to share interesting regularities with the β2-structure (the logical hexagon) and the β3-structure (the logical tetraicosahedron), showing possible invariants leading, in the future, to a precise definition of the general notion of β*n*-structure

14.
BEYOND THE HYPER-TETRAICOSAHEDRON (I.E. BEYOND THE β4-STRUCTURE): FURTHER USEFUL β*N*-STRUCTURES (*N* = 5, 6, 7)

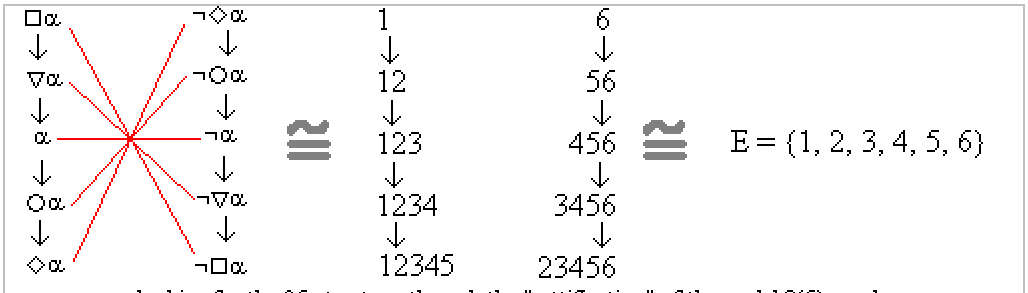
In this chapter we exhibit the three further β*n*-structures which follow the β4-structure studied in detail in the previous chapter. The examination may seem tedious, but, as long as avoiding tediousness sometimes prevents from discovering or preparing future discoveries, we accept to undertake here such a rather clumsy review of combinatorial results. If the analysis of the different components is rather easy, the major problem, because of combinatorial explosiveness, seems to be that of the *geometrical* (global) representation of these β*n*-structures ($n \in \{5, 6, 7\}$). We try in each case to give a useful "trick" in order to benefit from a partly symbolic and partly descriptive representation.

14.01. Preliminary remarks on the β(5-7)-structures

For reasons that will appear later, the somehow tedious combinatorial exploration of these β5- to β7-structures will turn out useful. About this future utility, roughly speaking one first aim of ours will be that of looking for generalities on the global series of the β*n*-structures (cf. ch. 15 *infra*). But more precise information on precise small sized β*n*-structures will also turn out useful for concrete applications of NOT (cf. ch. 16 and ch. 17 *infra*). Another use of it is to show concretely how Pellissier's powerful method can be used (it serves to familiarise the reader with "the handling of NOT").

14.02. The 5-dimensional β5-structure

Because we have studied the β4-structure, corresponding to the modal 3(4)-graph, in the previous chapter, we start here from its successor, the modal 3(5)-graph.



looking for the β5-structure: through the "settification" of the modal 3(5)-graph

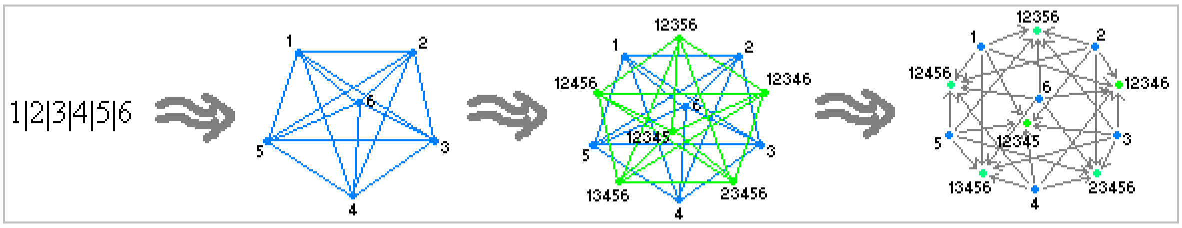
How many "points" (vertices) are there? Given that strict non-empty sub-strings of E here can have 1 to 5 characters, this gives all in all the following 62 (= 2^6-2) points or "vertices":

- a: 1, 2, 3, 4, 5, 6 (strings with one character);
- b: 12, 13, 14, 15, 16, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56 (strings with two characters);
- c: 123, 124, 125, 126, 134, 135, 136, 145, 146, 156, 234, 235, 236, 245, 246, 256, 345, 346, 356, 456 (strings with three characters);
- d: 1234, 1235, 1236, 1245, 1246, 1256, 1345, 1346, 1356, 1456, 2345, 2346, 2356, 2456, 3456 (strings with four characters);
- e: 12345, 12346, 12356, 12456, 13456, 23456 (strings with five characters).

One must remember that the number of vertices of E is given by the formula "$\wp$(E)-2", because we must exclude two over all the possible strings: the empty one and the total one (it is of this move that Smessaert gives a very elegant geometrical reading). The row (a) is, with respect to E_p, complementary (i.e. contradictory) with the row (e), the row (b) is complementary (i.e. contradictory) with the row (d), the row (c) is complementary (i.e. contradictory) with itself.

Now that we know the number and the modal quality of the points (the vertices) of the overall structure, we need the partitions (i.e. the possible geometrical reunions of points: squares, hexagons, cubes, …)[208].

What is the largest possible *n*-partition of E, with E = {1,2,3,4,5,6}? The answer is clear: it is the 6-partition (an α6-structure), giving the following result: 1|2|3|4|5|6. This will be the heart of the β5-structure (later we will need to know something about its "cloud", i.e. that part of the β-structure which is not the heart).

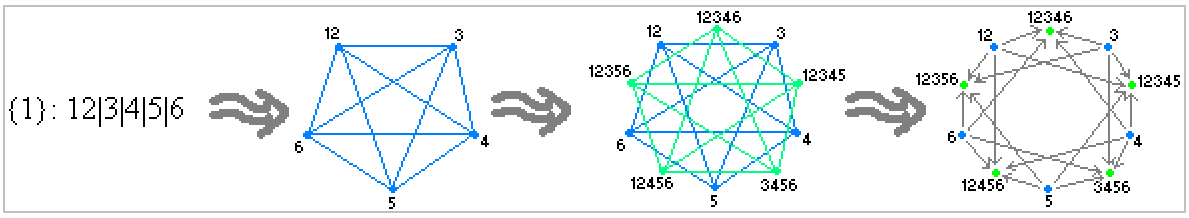


Are there then smaller *m*-partitions of E? Yes. How many 5-partitions (i.e. α5-structures)? There are 15, namely the following:

{1} 12|3|4|5|6, {2} 13|2|4|5|6, {3} 14|2|3|5|6, {4} 15|2|3|4|6, {5} 16|2|3|4|5, {6} 1|23|4|5|6, {7} 1|24|3|5|6, {8} 1|25|3|4|6, {9} 1|26|3|4|5, {10} 1|2|34|5|6, {11}1|2|35|4|6, {12} 1|2|36|4|5, {13} 1|2|3|45|6, {14} 1|2|3|46|5, {15} 1|2|3|4|56.

[208] In what follows we will uniquely deal with the *strong* α-structures contained in the β5-structure.

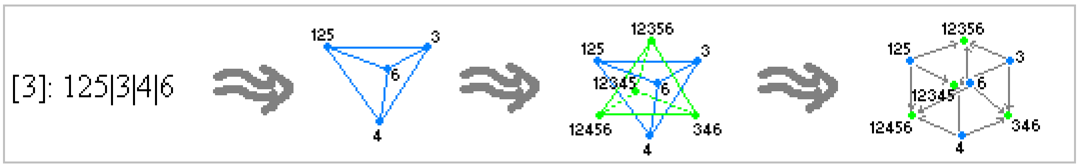
Each of these gives a 5-oppositio, i.e. a blue simplex of contrariety (of dimension 4). And from this one derives (cf. ch. 11 *supra*) a logical bi-simplex (of dimension 4). Here we give a graphical example for one of them, the one called {1}.



How many 4-partitions (i.e. α4-structures, or "logical cubes") are there in the β5-structure? There are 65:

[1] 123|4|5|6, [2] 124|3|5|6, [3] 125|3|4|6, [4] 126|3|4|5, [5] 134|2|5|6, [6] 135|2|4|6, [7] 136|2|4|5, [8] 145|2|3|6, [9] 146|2|3|5, [10] 156|2|3|4, [11] 1|234|5|6, [12] 1|235|4|6, [13] 1|236|4|5, [14] 1|245|3|6, [15] 1|246|3|5, [16] 1|256|3|4, [17] 1|2|345|6, [18] 1|2|346|5, [19] 1|2|356|4, [20] 1|2|3|456, [21] 12|34|5|6, [22] 12|35|4|6, [23] 12|36|4|5, [24] 12|3|45|6, [25] 12|3|46|5, [26] 12|3|4|56, [27] 13|24|5|6, [28] 13|25|4|6, [29] 13|26|4|5, [30] 13|2|45|6, [31] 13|2|46|5, [32] 13|2|4|56, [33] 14|23|5|6, [34] 14|25|3|6, [35] 14|26|3|5, [36] 14|2|35|6, [37] 14|2|36|5, [38] 14|2|3|56, [39] 15|23|4|6, [40] 15|24|3|6, [41] 15|26|3|4, [42] 15|2|34|6, [43] 15|2|36|4, [44] 15|2|3|46, [45] 16|23|4|5, [46] 16|24|3|5, [47] 16|25|3|4, [48] 16|2|34|5, [49] 16|2|35|4, [50] 16|2|3|45, [51] 1|23|45|6, [52] 1|23|46|5, [53] 1|23|4|56, [54] 1|24|35|6, [55] 1|24|36|5, [56] 1|24|3|56, [57] 1|25|34|6, [58] 1|25|36|4, [59] 1|25|3|46, [60] 1|26|34|5, [61] 1|26|35|4, [62] 1|26|3|45, [63] 1|2|34|56, [64] 1|2|35|46, [65] 1|2|36|45|.

Each of these give a blue tetrahedron of contrariety and hence (cf. ch. 11 *supra*) a logical cube (i.e. a 4-opposition). Here we represent one of them (the one called [3]).

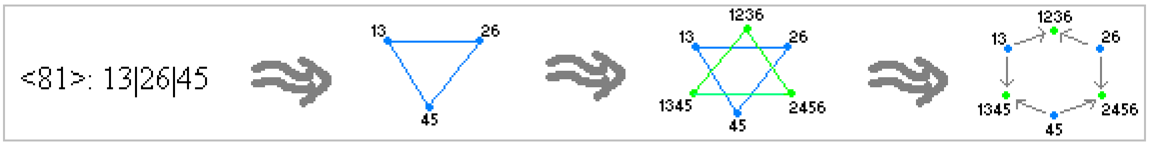


Now, how many 3-partitions (α3-structures, logical hexagons) are there in the β5-structure? Exactly 90:

<1> 1234|5|6, <2> 1235|4|6, <3> 1236|4|5, <4> 1245|3|6, <5> 1246|3|5, <6> 1256|3|4, <7> 1345|2|6, <8> 1346|2|5, <9> 1356|2|4, <10> 1456|2|3, <11> 1|2345|6, <12> 1|2346|5, <13> 1|2356|4, <14> 1|2456|3, <15> 1|2|3456, <16> 123|45|6, <17> 123|46|5, <18> 123|4|56, <19> 124|35|6, <20> 124|36|5, <21> 124|3|56, <22> 125|34|6, <23> 125|36|4, <24> 125|3|46, <25> 126|34|5, <26> 126|35|4, <27> 126|3|45, <28> 134|25|6, <29> 134|26||5, <30> 134|2|56, <31> 135|24|6, <32> 135|26|4, <33> 135|2|46, <34> 136|24|5, <35> 136|25|4, <36> 136|2|45,

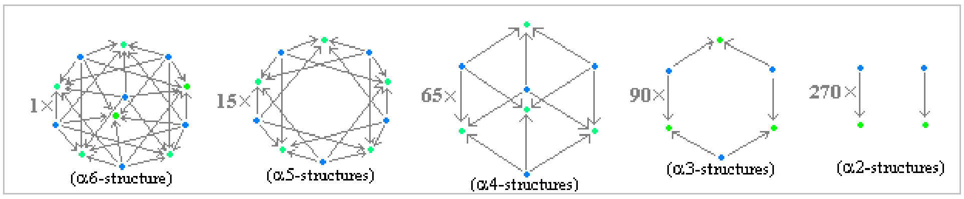
<37> 145|23|6, <38> 145|26|3, <39> 145|2|36, <40> 146|23|5, <41> 146|25|3, <42> 146|2|35, <43> 156|23|4, <44> 156|24|3, <45> 156|2|34, <46> 15|234|6, <47> 16|234|5, <48> 1|234|56, <49> 14|235|6, <50> 16|235|4, <51> 1|235|46, <52> 14|236|5, <53> 15|236|4, <54> 1|236|45, <55> 13|245|6, <56> 16|245|3, <57> 1|245|36, <58> 13|246|5, <59> 15|246|3, <60> 1|246|35, <61> 13|256|4, <62> 14|256|3, <63> 1|256|34, <64> 12|345|6, <65> 16|2|345, <66> 1|345|26, <67> 12|346|5, <68> 15|2|346, <69> 1|346|25, <70> 12|356|4, <71> 14|2|356, <72> 1|356|24, <73> 12|3|456, <74> 13|2|456, <75> 1|23|456, <76> 12|34|56, <77> 12|35|46, <78> 12|36|45, <79> 13|24|56, <80> 13|25|46, <81> 13|26|45, <82> 14|23|56, <83> 14|25|36, <84> 14|26|35, <85> 15|23|46, <86> 15|24|36, <87> 15|26|34, <88> 16|23|45, <89> 16|24|35, <90> 16|25|34.

Each of these gives a blue triangle of contrariety and hence (cf. ch. 8 *supra*) a logical hexagon, i.e. a 3-opposition. Here we represent one of them, the one called <81>.

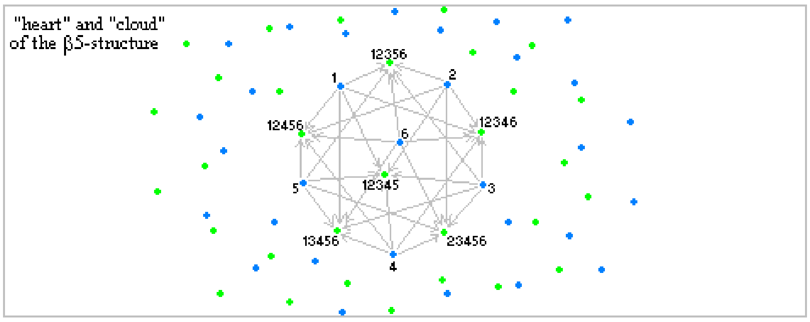


Finally, how many 2-partitions (i.e. α2-structures, or logical squares) are there? For logical squares, as we already said and saw (cf. ch. 12 *supra*), the setting method does not give direct simple partitioning results (it gives a complex formula), the 2-partition of E giving just the possible couples of contradictory terms (not the squares). These can nevertheless be deduced easily from our knowledge about logical hexagons: each logical hexagon contains exactly three logical squares. Hence we get that the β5-structure has 90x3= 270 logical squares!

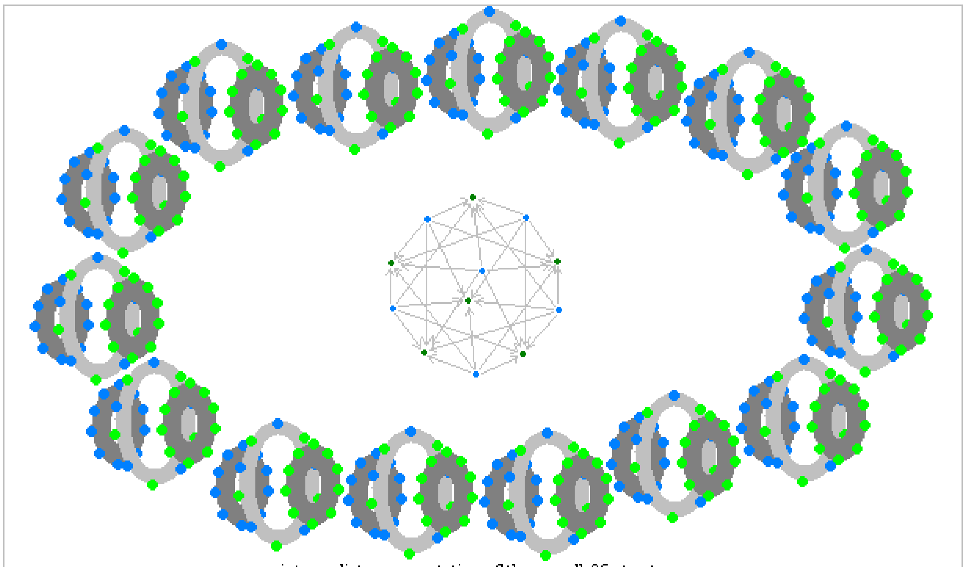
So, here is a qualitative list of *all* the α*n*-structures belonging to the β5-structure : one α6-structure, fifteen α5-structures, sixty-five α4-structures (or logical cubes), ninety α3-structures (or logical hexagons) and two-hundred-seventy α2-structures (or logical squares).



Now, so far the geometry of the whole β5-structure will consist of the heart (i.e. the α6-structure) and its ordered cloud.



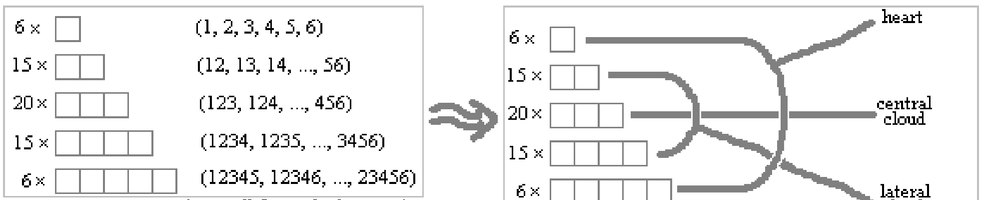
A final sketched (very imperfect) representation of the β5-structure could be the following



an intermediate representation of the overall β5-structure
(the "logical five-dimensional hyper-tetraicosahedron")

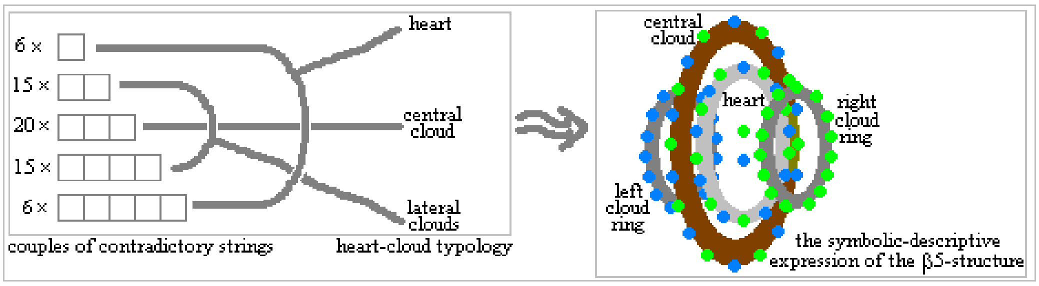
(by which we express that on the heart of the β5-structure 15 structures isomorphic to the logical β4-structure are "seated" (cf. ch. 13 *supra*)

But in fact a better representation of the β5-structure, half symbolic and half descriptive, is rather the following. It takes in consideration the *quality* of its vertices (more precisely: the number of characters of their respective Pellissier string). So let us consider again these qualities, related to the length of the respective substrings of E representing the 62 vertices of the β5-structure.

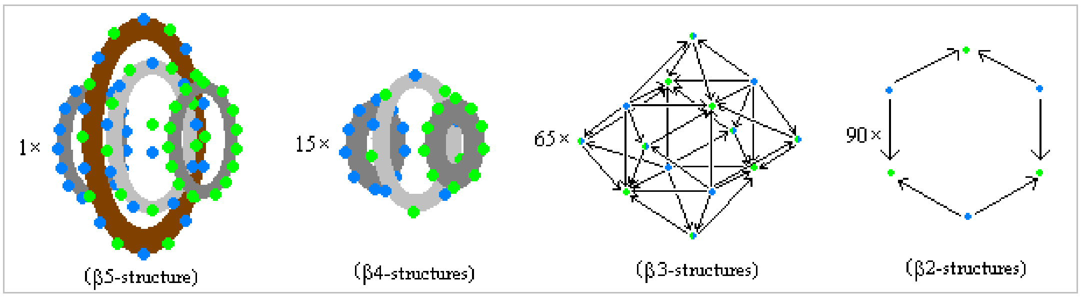


kinds of strings of E (one cell for each character)
(E = {1,2,3,4,5,6})
couples of contradictory strings heart-cloud typology

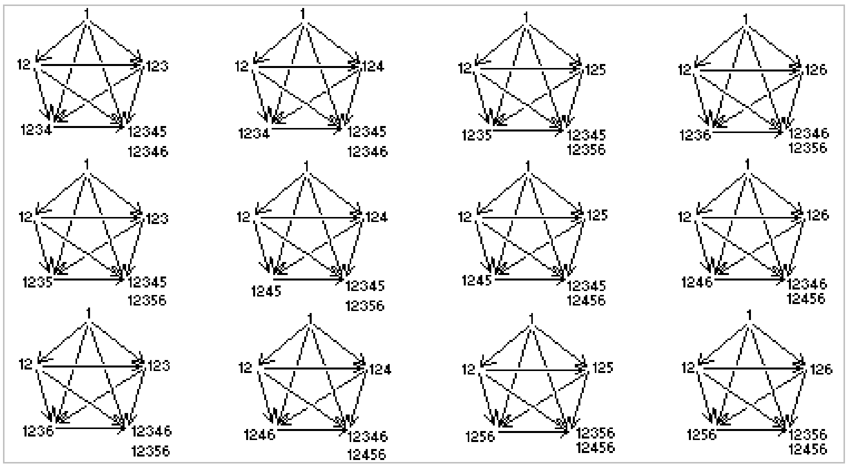
The right part of the previous scheme means that in the case of the β5-structure, this gives the heart (constituted by strings of minimal and maximal length), and two forms of cloud: a "central cloud" and some "lateral clouds" (this last is like the "ear drum" of the β4-structure, cf. ch. 13 *supra*).



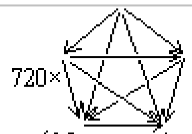
So, the synthetic qualitative list of all the β*n*-structures (2 ≤ 5) belonging to the β5-structure is the following.



As for the β4-structure (cf. ch. 13 *supra*), one last point about the β5-structure concerns its "surface". We saw that the "surface" of the Sesmat-Blanché's logical hexagon (the β2-structure) is its perimeter, made of 1-dimensional arrows (6 of them, hence the name of the polygon); similarly, we saw that the surface of Pellissier's logical tetraicosahedron (the β3-structure) is made of 2-dimensional arrow-triangles (24 of them, hence the name of the polyhedron, cf. ch. 12 *supra*) and that the "surface" of the logical hyper-tetraicosahedron (the β4-structure) is made of 3-dimensional arrow-tetrahedra (120 of them, cf. ch. 13 *supra*). What about the 5-dimensional hyper-tetraicosahedron (the β5-structure) we are now dealing with? Its "bricks" have in fact the shape of a geometrical simplex of dimension four (a 4-dimensional composition of 5 tetrahedra). A simple combinatorial examination can reveal easily how many of them there can be.



As there are 5 ways of changing “12” (“1” being fixed), and 6 ways to change “1” (including the “no change” option), finally there are 24×5×6=720 cases.



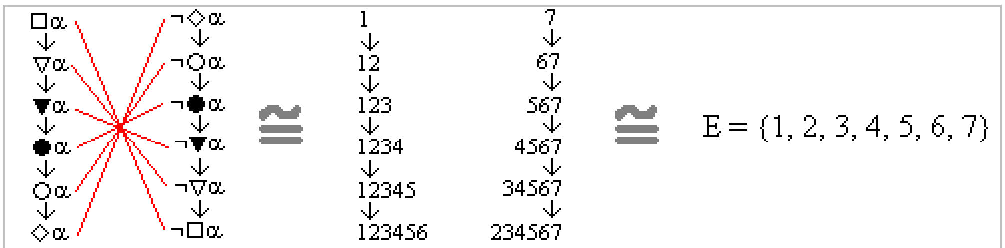
(β5-structure)

So, it turns out that the analogy with the previous instances of β*n*-structures can be kept: the β5-structure has a “surface” made of 4-dimensional arrow-penta-simplexes (720 of them).

Knowing enough about the β5-structure, we can step to the next β*n*-structure.

14.03. The 6-dimensional β6-structure

We know (cf. ch. 12 *supra*) that the β6-structure is generated paradigmatically (if not only) by the modal 3(6)-graph. Here are the modal 3(6)-graph and its Pellissier-style translation.



looking for the β6-structure: through the "settification" of the modal 3(6)-graph

Its characteristic set is E = {1, 2, 3, 4, 5, 6, 7}.

The number of its vertices is 2^7-2 = 128-2 = 126. Here is their complete list:

- there are seven vertices with just one character:

1, 2, 3, 4, 5, 6, 7

- there are 21 vertices with 2 characters in their string:

12, 13, 14, 15, 16, 17, 23, 24, 25, 26, 27, 34, 35, 36, 37, 45, 46, 47, 56, 57, 67

- there are 35 vertices with 3 characters in their string:

123, 124, 125, 126, 127, 134, 135, 136, 137, 145, 146, 147, 156, 157, 167, 234, 235, 236, 237, 245, 246, 247, 256, 257, 267, 345, 346, 347, 356, 357, 367, 456, 457, 467, 567

- there are 35 vertices with 4 characters in their string:

1234, 1235, 1236, 1237, 1245, 1246, 1247, 1256, 1257, 1267, 1345, 1346, 1347, 1356, 1357, 1367, 1456, 1457, 1467, 1567, 2345, 2346, 2347, 2356, 2357, 2367, 2456, 2457, 2467, 2567, 3456, 3457, 3467, 3567, 4567

- there are 21 vertices with 5 characters in their string:

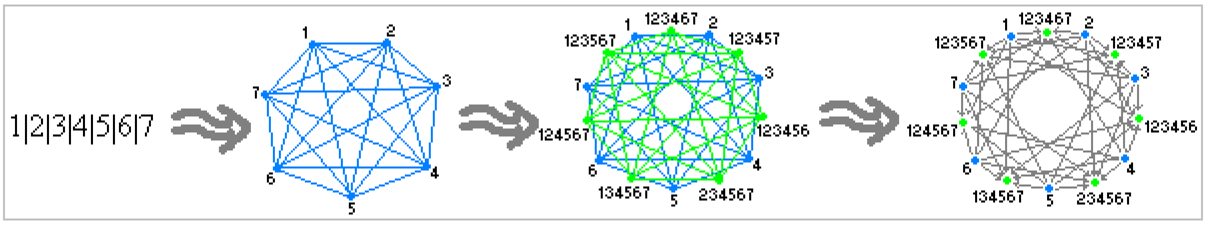
12345, 12346, 12347, 12356, 12357, 12367, 12456, 12457, 12467, 12567, 13456, 13457, 13467, 13567, 14567, 23456, 23457, 23467, 23567, 24567, 34567

- there are 7 vertices with 6 characters in their string:

234567, 134567, 124567, 123567, 123467, 123457, 123456.

Having seen the vertices of the β6-structure (here and in the following we omit to give their modal translation), we want now to know something about its geometrical inner architecture. To achieve this, we must collect information about its constituting α*n*-structures and β*n*-structures. As usual, we begin by looking for the inner α*n*-structures, reachable by studying the partitions of the set E.

What is the biggest possible partition of E = {1, 2, 3, 4, 5, 6, 7}? Clearly, it is the unique 1|2|3|4|5|6|7. This is an α7-structure (a 7-opposition) and it is the heart of the β6-structure.

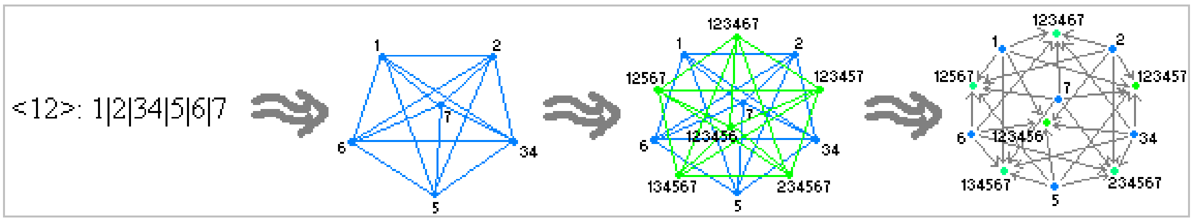


So, the next step is to know things about the rest, that is the cloud of the β6-structure.

How many 6-partitions (or α6-structures) are there inside the β6-structure? Exactly 21. Here we give them.

<1> 12|3|4|5|6|7, <2> 13|2|4|5|6|7, <3> 14|2|3|5|6|7, <4> 15|2|3|4|6|7, <5> 16|2|3|4|5|7, <6> 17|2|3|4|5|6, <7> 1|23|4|5|6|7, <8> 1|24|3|5|6|7, <9> 1|25|3|4|6|7, <10> 1|26|3|4|5|7, <11> 1|27|3|4|5|6, <12> 1|2|34|5|6|7, <13> 1|2|35|4|6|7, <14> 1|2|36|4|5|7, <15> 1|2|37|4|5|6, <16> 1|2|3|45|6|7, <17> 1|2|3|46|5|7, <18> 1|2|3|47|5|6, <19> 1|2|3|4|56|7, <20> 1|2|3|4|57|6, <21> 1|2|3|4|5|67.

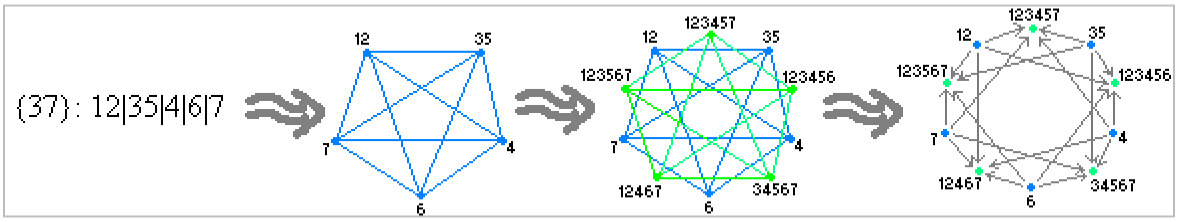
Here we give an example of such 6-partitions (or 6-oppositions) of E (the partition <12>).



How many 5-partitions (or α5-structures) are there inside the β6-structure? There are exactly 140 of them.

{1} 123|4|5|6|7, {2} 124|3|5|6|7, {3} 125|3|4|6|7, {4} 126|3|4|5|7, {5} 127|3|4|5|6, {6} 134|2|5|6|7, {7} 135|2|4|6|7, {8} 136|2|4|5|7, {9} 137|2|4|5|6, {10} 145|2|3|6|7, {11} 146|2|3|5|7, {12} 147|2|3|5|6, {13} 156|2|3|4|7, {14} 157|2|3|4|6, {15} 167|2|3|4|5, {16} 1|234|5|6|7, {17} 1|235|4|6|7, {18} 1|236|4|5|7, {19} 1|237|4|5|6, {20} 1|245|3|6|7, {21} 1|246|3|5|7, {22} 1|247|3|5|6, {23} 1|256|3|4|7, {24} 1|257|3|4|6, {25} 1|267|3|4|5, {26} 1|2|345|6|7, {27} 1|2|346|5|7, {28} 1|2|347|5|6, {29} 1|2|356|4|7, {30} 1|2|357|4|6, {31} 1|2|367|4|5, {32} 1|2|3|456|7, {33} 1|2|3|457|6, {34} 1|2|3|467|5|, {35} 1|2|3|4|567, {36} 12|34|5|6|7, {37} 12|35|4|6|7, {38} 12|36|4|5|7, {39} 12|37|4|5|6, {40} 12|3|45|6|7, {41} 12|3|46|5|7, {42} 12|3|47|5|6, {43} 12|3|4|56|7, {44} 12|3|4|57|6, {45} 12|3|4|5|67, {46} 13|24|5|6|7, {47} 13|25|4|6|7, {48} 13|26|4|5|7, {49} 13|27|4|5|6, {50} 13|2|45|6|7, {51} 13|2|46|5|7, {52} 13|2|47|5|6, {53} 13|2|4|56|7, {54} 13|2|4|57|6, {55} 13|2|4|5|67, {56} 14|23|5|6|7, {57} 14|25|3|6|7, {58} 14|26|3|5|7, {59} 14|27|3|5|6, {60} 14|2|35|6|7, {61} 14|2|36|5|7, {62} 14|2|37|5|6, {63} 14|2|3|56|7, {64} 14|2|3|57|6, {65} 14|2|3|5|67, {66} 15|23|4|6|7, {67} 15|24|3|6|7, {68} 15|26|3|4|7, {69} 15|27|3|4|6, {70} 15|2|34|6|7, {71} 15|2|36|4|7, {72} 15|2|37|4|6, {73} 15|2|3|46|7, {74} 15|2|3|47|6, {75} 15|2|3|4|67, {76} 16|23|4|5|7, {77} 16|24|3|5|7, {78} 16|25|3|4|7, {79} 16|27|3|4|5, {80} 16|2|34|5|7, {81} 16|2|35|4|7, {82} 16|2|37|4|5, {83} 16|2|3|45|7, {84} 16|2|3|47|5, {85} 16|2|3|4|57, {86} 17|23|4|5|6, {87} 17|24|3|5|6, {88} 17|25|3|4|6, {89} 17|26|3|4|5, {90} 17|2|34|5|6, {91} 17|2|35|4|6, {92} 17|2|36|4|5, {93} 17|2|3|45|6, {94} 17|2|3|46|5, {95} 17|2|3|4|56, {96} 1|23|45|6|7, {97} 1|23|46|5|7, {98} 1|23|47|5|6, {99} 1|23|4|56|7, {100} 1|23|4|57|6, {101} 1|23|4|5|67, {102} 1|24|35|6|7, {103} 1|24|36|5|7, {104} 1|24|37|5|6, {105} 1|24|3|56|7, {106} 1|24|3|57|6, {107} 1|24|3|5|67, {108} 1|25|34|6|7, {109} 1|25|36|4|7, {110} 1|25|37|4|6, {111} 1|25|3|46|7, {112} 1|25|3|47|6, {113} 1|25|3|4|67, {114} 1|26|34|5|7, {115} 1|26|35|4|7, {116} 1|26|37|4|5, {117} 1|26|3|45|7, {118} 1|26|3|47|5, {119} 1|26|3|4|57, {120} 1|27|34|5|6, {121} 1|27|35|4|6, {122} 1|27|36|4|5, {123} 1|27|3|45|6, {124} 1|27|3|46|5, {125} 1|27|3|4|56, {126} 1|2|34|56|7, {127} 1|2|34|57|6, {128} 1|2|34|5|67, {129} 1|2|35|46|7, {130} 1|2|35|47|6, {131} 1|2|35|4|67, {132} 1|2|36|45|7, {133} 1|2|36|47|5, {134} 1|2|36|4|57, {135} 1|2|37|45|6, {136} 1|2|37|46|5, {137} 1|2|37|4|56, {138} 1|2|3|45|67, {139} 1|2|3|46|57, {140} 1|2|3|47|56.

Here we give an example of such 5-partitions of E (the partition number {37}).

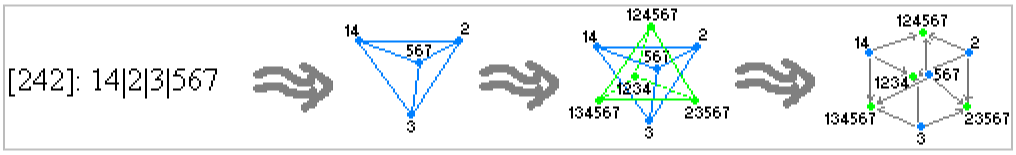


How many 4-partitions (i.e. α4-structures, or logical cubes) are there inside the β6-structure? There are exactly 350.

[1] 1234|5|6|7, [2] 1235|4|6|7, [3] 1236|4|5|7, [4] 1237|4|5|6, [5] 1245|3|6|7, [6] 1246|3|5|7, [7] 1247|3|5|6, [8] 1256|3|4|7, [9] 1257|3|4|6, [10] 1267|3|4|5, [11] 1345|2|6|7, [12] 1346|2|5|7, [13] 1347|2|5|6, [14] 1356|2|4|7, [15] 1357|2|4|6, [16] 1367|2|4|5, [17] 1456|2|3|7, [18] 1457|2|3|6, [19] 1467|2|3|5, [20] 1567|2|3|4, [21] 1|2345|6|7, [22] 1|2346|5|7, [23] 1|2347|5|6, [24] 1|2356|4|7, [25] 1|2357|4|6, [26] 1|2367|4|5, [27] 1|2456|3|7, [28] 1|2457|3|6, [29] 1|2467|3|5, [30] 1|2567|3|4, [31] 1|2|3456|7, [32] 1|2|3457|6, [33] 1|2|3467|5, [34] 1|2|3567|4, [35] 1|2|3|4567, [36] 123|45|6|7, [37] 123|46|5|7, [38] 123|47|5|6, [39] 123|4|56|7, [40] 123|4|57|6, [41] 123|4|5|67, [42] 124|35|6|7, [43] 124|36|5|7, [44] 124|37|5|6, [45] 124|3|56|7, [46] 124|3|57|6, [47] 124|3|5|67, [48] 125|34|6|7, [49] 125|36|4|7, [50] 125|37|4|6, [51] 125|3|46|7, [52] 125|3|47|6, [53] 125|3|4|67, [54] 126|34|5|7, [55] 126|35|4|7, [56] 126|37|4|5, [57] 126|3|45|7, [58] 126|3|47|5, [59] 126|3|4|57, [60] 127|34|5|6, [61] 127|35|4|6, [62] 127|36|4|5, [63] 127|3|45|6, [64] 127|3|46|5, [65] 127|3|4|56, [66] 134|25|6|7, [67] 134|26|5|7, [68] 134|27|5|6, [69] 134|2|56|7, [70] 134|2|57|6, [71] 134|2|5|67, [72] 135|24|6|7, [73] 135|26|4|7, [74] 135|27|4|6, [75] 135|2|46|7, [76] 135|2|47|6, [77] 135|2|4|67, [78] 136|24|5|7, [79] 136|25|4|7, [80] 136|27|4|5, [81] 136|2|45|7, [82] 136|2|47|5, [83] 136|2|4|57, [84] 137|24|5|6, [85] 137|25|4|6, [86] 137|26|4|5, [87] 137|2|45|6, [88] 137|2|46|5, [89] 137|2|4|56, [90] 145|23|6|7, [91] 145|26|3|7, [92] 145|27|3|6, [93] 145|2|36|7, [94] 145|2|37|6, [95] 145|2|3|67, [96] 146|23|5|7, [97] 146|25|3|7, [98] 146|27|3|5, [99] 146|2|35|7, [100] 146|2|37|5, [101] 146|2|3|57, [102] 147|23|5|6, [103] 147|25|3|6, [104] 147|26|3|5, [105] 147|2|35|6, [106] 147|2|36|5, [107] 147|2|3|56, [108] 156|23|4|7, [109] 156|24|3|7, [110] 156|27|3|4, [111] 156|2|34|7, [112] 156|2|37|4, [113] 156|2|3|47, [114] 157|23|4|6, [115] 157|24|3|6, [116] 157|26|3|4, [117] 157|2|34|6, [118] 157|2|36|4, [119] 157|2|3|46, [120] 167|23|4|5, [121] 167|24|3|5, [122] 167|25|3|4, [123] 167|2|34|5, [124] 167|2|35|4, [125] 167|2|3|45, [126] 15|234|6|7, [127] 16|234|5|7, [128] 17|234|5|6, [129] 1|234|56|7, [130] 1|234|57|6, [131] 1|234|5|67, [132] 14|235|6|7, [133] 16|235|4|7, [134] 17|235|4|6, [135] 1|235|46|7, [136] 1|235|47|6, [137] 1|235|4|67, [138] 14|236|5|7, [139] 15|236|4|7, [140] 17|236|4|5, [141] 1|236|45|7, [142] 1|236|47|5, [143] 1|236|4|57, [144] 14|237|5|6|, [145] 15|237|4|6, [146] 16|237|4|5, [147] 1|237|45|6, [148] 1|237|46|5, [149] 1|237|4|56, [150] 13|245|6|7, [151] 16|245|3|7, [152] 17|245|3|6, [153] 1|245|36|7, [154] 1|245|37|6|, [155] 1|245|3|67, [156] 13|246|5|7, [157] 15|246|3|7, [158] 17|246|3|5, [159] 1|246|35|7, [160] 1|246|37|5, [161] 1|246|3|57, [162] 13|247|5|6, [163] 15|247|3|6, [164] 16|247|3|5, [165] 1|247|35|6, [166] 1|247|36|5, [167] 1|247|3|56, [168] 13|256|4|7, [169] 14|256|3|7, [170] 17|256|3|4,

[171] 1|256|34|7, [172] 1|256|37|4, [173] 1|256|3|47, [174] 13|257|4|6, [175] 14|257|3|6,
[176] 16|257|3|4, [177] 1|257|34|6, [178] 1|257|36|4, [179] 1|257|3|46, [180] 13|267|4|5,
[181] 14|267|3|5, [182] 15|267|3|4, [183] 1|267|34|5, [184] 1|267|35|4, [185] 1|267|3|45,
[186] 12|345|6|7, [187] 16|2|345|7, [188] 17|2|345|6, [189] 1|26|345|7, [190] 1|27|345|6,
[191] 1|2|345|67, [192] 12|346|5|7, [193] 15|2|346|7, [194] 17|2|346|5, [195] 1|25|346|7,
[196] 1|27|346|5, [197] 1|2|346|57, [198] 12|347|5|6, [199] 15|2|347|6, [200] 16|2|347|5,
[201] 1|25|347|6, [202] 1|26|347|5, [203] 1|2|347|56, [204] 12|356|4|7, [205] 14|2|356|7,
[206] 17|2|356|4, [207] 1|24|356|7, [208] 1|27|356|4, [209] 1|2|356|47, [210] 12|357|4|6,
[211] 14|2|357|6, [212] 16|2|357|4, [213] 1|24|357|6, [214] 1|26|357|4, [215] 1|2|357|46,
[216] 12|367|4|5, [217] 14|2|367|5, [218] 15|2|367|4, [219] 1|24|367|5, [220] 1|25|367|4,
[221] 1|2|367|45, [222] 12|3|456|7, [223] 13|2|456|7, [224] 17|2|3|456, [225] 1|23|456|7,
[226] 1|27|3|456, [227] 1|2|37|456, [228] 12|3|457|6, [229] 13|2|457|6, [230] 16|2|3|457,
[231] 1|23|457|6, [232] 1|26|3|457, [233] 1|2|36|457, [234] 12|3|467|5, [235] 13|2|467|5,
[236] 15|2|3|467, [237] 1|23|467|5, [238] 1|25|3|467, [239] 1|2|35|467, [240] 12|3|4|567,
[241] 13|2|4|567, [242] 14|2|3|567, [243] 1|23|4|567, [244] 1|24|3|567, [245] 1|2|34|567
[246] 12|34|56|7, [247] 12|34|57|6, [248] 12|34|5|67, [249] 12|35|46|7, [250] 12|35|47|6,
[251] 12|35|4|67, [252] 12|36|45|7, [253] 12|36|47|5, [254] 12|36|4|57, [255] 12|37|45|6,
[256] 12|37|46|5, [257] 12|37|4|56, [258] 12|3|45|67, [259] 12|3|46|57, [260] 12|3|47|56,
[261] 13|24|56|7, [262] 13|24|57|6, [263] 13|24|5|67, [264] 13|25|46|7, [265] 13|25|47|6,
[266] 13|25|4|67, [267] 13|26|45|7, [268] 13|26|47|5, [269] 13|26|4|57, [270] 13|27|45|6,
[271] 13|27|46|5, [272] 13|27|4|56, [273] 13|2|45|67, [274] 13|2|46|57, [275] 13|2|47|56,
[276] 14|23|56|7, [277] 14|23|57|6, [278] 14|23|5|67, [279] 14|25|36|7, [280] 14|25|37|6,
[281] 14|25|3|67, [282] 14|26|35|7, [283] 14|26|37|5, [284] 14|26|3|57, [285] 14|27|35|6,
[286] 14|27|36|5, [287] 14|27|3|56, [288] 14|2|35|67, [289] 14|2|36|57, [290] 14|2|37|56,
[291] 15|23|46|7, [292] 15|23|47|6, [293] 15|23|4|67, [294] 15|24|36|7, [295] 15|24|37|6,
[296] 15|24|3|67, [297] 15|26|34|7, [298] 15|26|37|4, [299] 15|26|3|47, [300] 15|27|34|6,
[301] 15|27|36|4, [302] 15|27|3|46, [303] 15|2|34|67, [304] 15|2|36|47, [305] 15|2|37|46,
[306] 16|23|45|7, [307] 16|23|47|5, [308] 16|23|4|57, [309] 16|24|35|7, [310] 16|24|37|5,
[311] 16|24|3|57, [312] 16|25|34|7, [313] 16|25|37|4, [314] 16|25|3|47, [315] 16|27|34|5,
[316] 16|27|35|4, [317] 16|27|3|45, [318] 16|2|34|57, [319] 16|2|35|47, [320] 16|2|37|45,
[321] 17|23|45|6, [322] 17|23|46|5, [323] 17|23|4|56, [324] 17|24|35|6, [325] 17|24|36|5,
[326] 17|24|3|56, [327] 17|25|34|6, [328] 17|25|36|4, [329] 17|25|3|46, [330] 17|26|34|5,
[331] 17|26|35|4, [332] 17|26|3|45, [333] 17|2|34|56, [334] 17|2|35|46, [335] 17|2|36|45,
[336] 1|23|45|67, [337] 1|23|46|57, [338] 1|23|47|56, [339] 1|24|35|67, [340] 1|24|36|57,

[341] 1|24|37|56, [342] 1|25|34|67, [343] 1|25|36|47, [344] 1|25|37|46, [345] 1|26|34|57, [346] 1|26|35|47, [347] 1|26|37|45, [348] 1|27|34|56, [349] 1|27|35|46, [350] 1|27|36|45; Here we give an example of such 4-partitions of E (the partition number [242]).



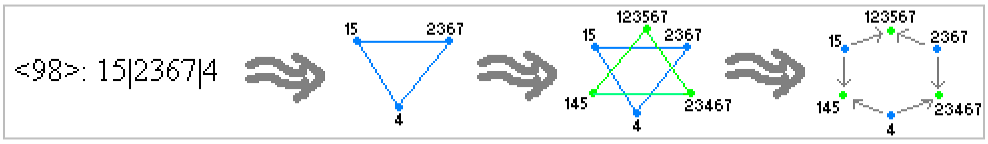
How many 3-partitions (i.e. α3-structures, or logical hexagons) are there inside the β6-structure? There are exactly 301.

<1> 12345|6|7, <2> 12346|5|7, <3> 12347|5|6, <4> 12356|4|7, <5> 12357|4|6, <6> 12367|4|5, <7> 12456|3|7, <8> 12457|3|6, <9> 12467|3|5, <10> 12567|3|4, <11> 13456|2|7, <12> 13457|2|6, <13> 13467|2|5, <14> 13567|2|4, <15> 14567|2|3, <16> 1|23456|7, <17> 1|23457|6, <18> 1|23467|5, <19> 1|23567|4, <20> 1|24567|3, <21> 1|2|34567, <22> 1234|56|7, <23> 1234|57|6, <24> 1234|5|67, <25> 1235|46|7, <26> 1235|47|6, <27> 1235|4|67, <28> 1236|45|7, <29> 1236|47|5, <30> 1236|4|57, <31> 1237|45|6, <32> 1237|46|5, <33> 1237|4|56, <34> 1245|36|7, <35> 1245|37|6, <36> 1245|3|67, <37> 1246|35|7, <38> 1246|37|5, <39> 1246|3|57, <40> 1247|35|6, <41> 1247|36|5, <42> 1247|3|56, <43> 1256|34|7, <44> 1256|37|4, <45> 1256|3|47, <46> 1257|34|6, <47> 1257|36|4, <48> 1257|3|46, <49> 1267|34|5, <50> 1267|35|4, <51> 1267|3|45, <52> 1345|26|7, <53> 1345|27|6, <54> 1345|2|67, <55> 1346|25|7, <56> 1346|27|5, <57> 1346|2|57, <58> 1347|25|6, <59> 1347|26|5, <60> 1347|2|56, <61> 1356|24|7, <62> 1356|27|4, <63> 1356|2|47, <64> 1357|24|6, <65> 1357|26|4, <66> 1357|2|46, <67> 1367|24|5, <68> 1367|25|4, <69> 1367|2|45, <70> 1456|23|7, <71> 1456|27|3, <72> 1456|2|37, <73> 1457|23|6, <74> 1457|26|3, <75> 1457|2|36, <76> 1467|23|5, <77> 1467|25|3, <78> 1467|2|35, <79> 1567|23|4, <80> 1567|24|3, <81> 1567|2|34, <82> 16|2345|7, <83> 17|2345|6, <84> 1|2345|67, <85> 15|2346|7, <86> 17|2346|5, <87> 1|2346|57, <88> 15|2347|6, <89> 16|2347|5, <90> 1|2347|56, <91> 14|2356|7, <92> 17|2356|4, <93> 1|2356|47, <94> 14|2357|6, <95> 16|2357|4, <96> 1|2357|46, <97> 14|2367|5, <98> 15|2367|4, <99> 1|2367|45, <100> 13|2456|7, <101> 17|2456|3, <102> 1|2456|37, <103> 13|2457|6, <104> 16|2457|3, <105> 1|2457|36, <106> 13|2467|5, <107> 15|2467|3, <108> 1|2467|35, <109> 13|2567|4, <110> 14|2567|3, <111> 1|2567|34, <112> 12|3456|7, <113> 17|2|3456, <114> 1|27|3456, <115> 12|3457|6, <116> 16|2|3457, <117> 1|26|3457, <118> 12|3467|5, <119> 15|2|3467, <120> 1|25|3467, <121> 12|3567|4, <122> 14|2|3567, <123> 1|24|3567, <124> 12|3|4567, <125> 13|2|4567, <126> 1|23|4567, <127> 123|456|7, <128> 123|457|6, <129> 123|467|5, <130> 123|4|567,

<131> 124|356|7, <132> 124|357|6, <133> 124|367|5, <134> 124|3|567, <135> 125|346|7, <136> 125|347|6, <137> 125|367|4, <138> 125|3|467, <139> 126|345|7, <140> 126|347|5, <141> 126|357|4, <142> 126|3|457, <143> 127|345|6, <144> 127|346|5, <145> 127|356|4, <146> 127|3|456, <147> 134|256|7, <148> 134|257|6, <149> 134|267|5, <150> 134|2|567, <151> 135|246|7, <152> 135|247|6, <153> 135|267|4, <154> 135|2|467, <155> 136|245|7, <156> 136|247|5, <157> 136|257|4, <158> 136|2|457, <159> 137|245|6, <160> 137|246|5, <161> 137|256|4, <162> 137|2|456, <163> 145|236|7, <164> 145|237|6, <165> 145|267|3, <166> 145|2|367, <167> 146|235|7, <168> 146|237|5, <169> 146|257|3, <170> 146|2|357, <171> 147|235|6, <172> 147|236|5, <173> 147|256|3, <174> 147|2|356, <175> 156|234|7, <176> 156|237|4, <177> 156|247|3, <178> 156|2|347, <179> 157|234|6, <180> 157|236|4, <181> 157|246|3, <182> 157|2|346, <183> 167|234|5, <184> 167|235|4, <185> 167|245|3, <186> 167|2|345, <187> 1|234|567, <188> 1|235|467, <189> 1|236|457, <190> 1|237|456, <191> 1|245|367, <192> 1|246|357, <193> 1|247|356, <194> 1|256|347, <195> 1|257|346, <196> 1|267|345, <197> 123|45|67, <198> 123|46|57, <199> 123|47|56, <200> 124|35|67, <201> 124|36|57, <202> 124|37|56, <203> 125|34|67, <204> 125|36|47, <205> 125|37|46, <206> 126|34|57, <207> 126|35|47, <208> 126|37|45, <209> 127|34|56, <210> 127|35|46, <211> 127|36|45, <212> 134|25|67, <213> 134|26|57, <214> 134|27|56, <215> 135|24|67, <216> 135|26|47, <217> 135|27|46, <218> 136|24|57, <219> 136|25|47, <220> 136|27|45, <221> 137|24|56, <222> 137|25|46, <223> 137|26|45, <224> 145|23|67, <225> 145|26|37, <226> 145|27|36, <227> 146|23|57, <228> 146|25|37, <229> 146|27|35, <230> 147|23|56, <231> 147|25|36, <232> 147|26|35, <233> 156|23|47, <234> 156|24|37, <235> 156|27|34, <236> 157|23|46, <237> 157|24|36, <238> 157|26|34, <239> 167|23|45, <240> 167|24|35, <241> 167|25|34, <242> 15|234|67, <243> 16|234|57, <244> 17|234|56, <245> 14|235|67, <246> 16|235|47, <247> 17|235|46, <248> 14|236|57, <249> 15|236|47, <250> 17|236|45, <251> 14|237|56, <252> 15|237|46, <253> 16|237|45, <254> 13|245|67, <255> 16|245|37, <256> 17|245|36, <257> 13|246|57, <258> 15|246|37, <259> 17|246|35, <260> 13|247|56, <261> 15|247|36, <262> 16|247|35, <263> 13|256|47, <264> 14|256|37, <265> 17|256|34, <266> 13|257|46, <267> 14|257|36, <268> 16|257|34, <269> 13|267|45, <270> 14|267|35, <271> 15|267|34, <272> 12|345|67, <273> 16|27|345, <274> 17|26|345, <275> 12|346|57, <276> 15|27|346, <277> 17|25|346, <278> 12|347|56, <279> 15|26|347, <280> 16|25|347, <281> 12|356|47, <282> 14|27|356, <283> 17|24|356, <284> 12|357|46, <285> 14|26|357, <286> 16|24|357, <287> 12|367|45, <288> 14|25|367, <289> 15|24|367, <290> 12|37|456, <291> 13|27||456, <292> 17|23|456, <293> 12|36|457, <294> 13|26|457, <295> 16|23|457,

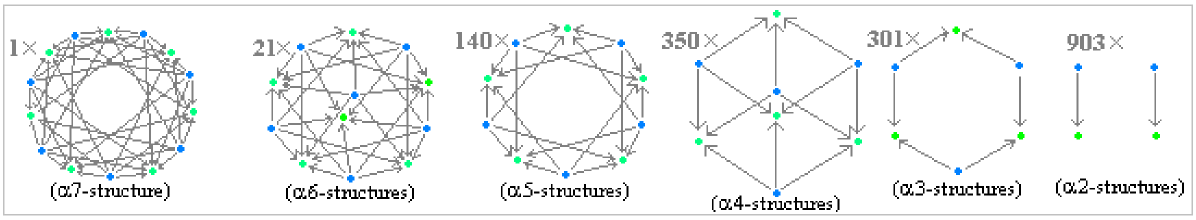
<296> 12|35|467, <297> 13|25|467, <298> 15|23|467, <299> 12|34|567, <300> 13|24|567, <301> 14|23|567.

Here we give an example of such 3-partitions of E (the partition number <98>).

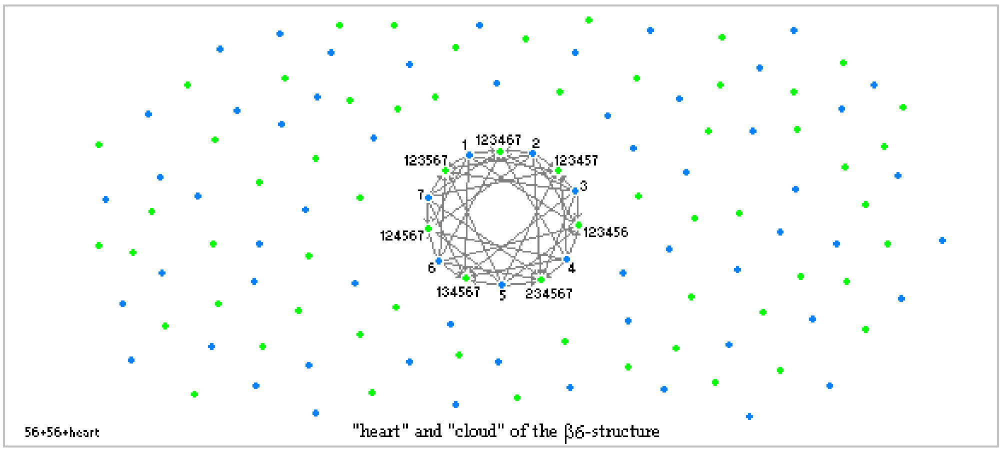


As usual, the list of the logical squares (of the β6-structure) is implicit inside the series of the logical hexagons (of the β6-structure). As each logical hexagon contains 3 logical squares, without overlaps with the ones generated by the other logical hexagons, the number of them is simply three times the number of the logical hexagons. So there are <903> α2-structures (logical squares) in the β6-structure.

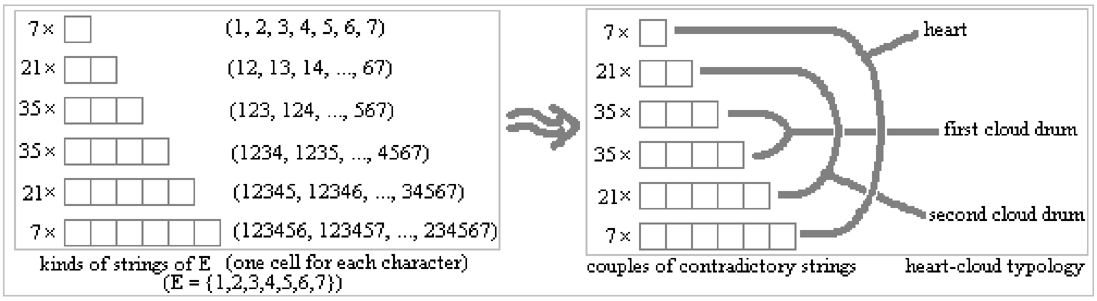
So, we have now the complete list of the α*n*-structures belonging to the β6-structure, list which is synthetically the following.



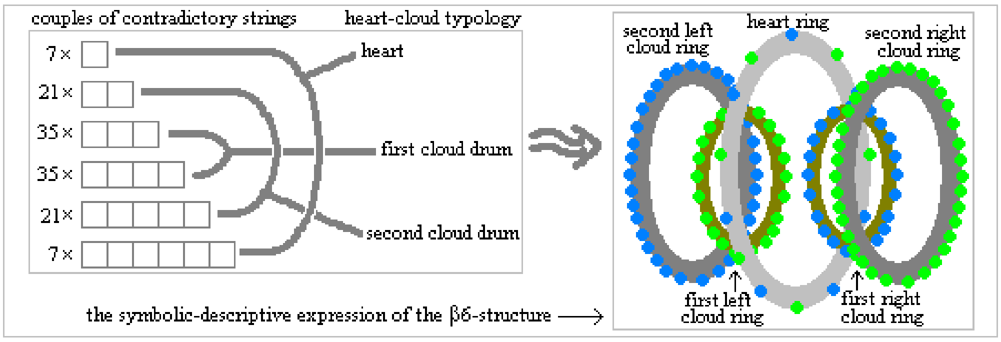
So, we know that the β6-structure is constituted of a heart, which is isomorphic to a α7-structure, and a cloud. We want to know something about the shape of the cloud.



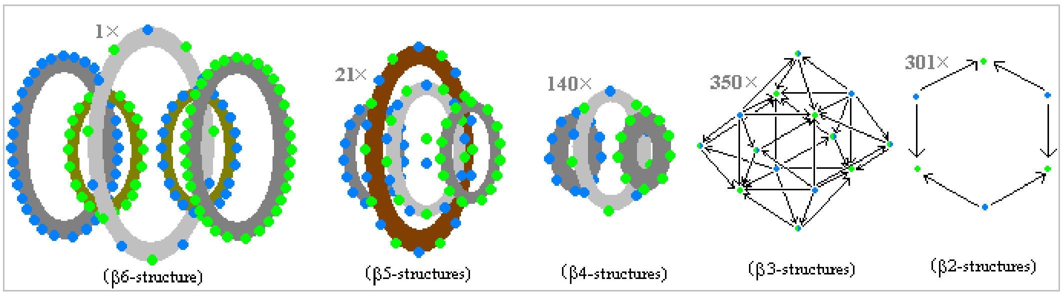
Here as well, working out the concrete geometry of the β6-structure seems to be really very hard, for this object belongs to a 5-dimensional space and is rather complex. Nevertheless, by reference to the qualitative distribution of its vertices we can here as well offer half a symbolic and half a descriptive representation of it (as in the β4- and in the β5-structures). First we consider the existing kinds of strings of characters inside the E of the β6-structure.



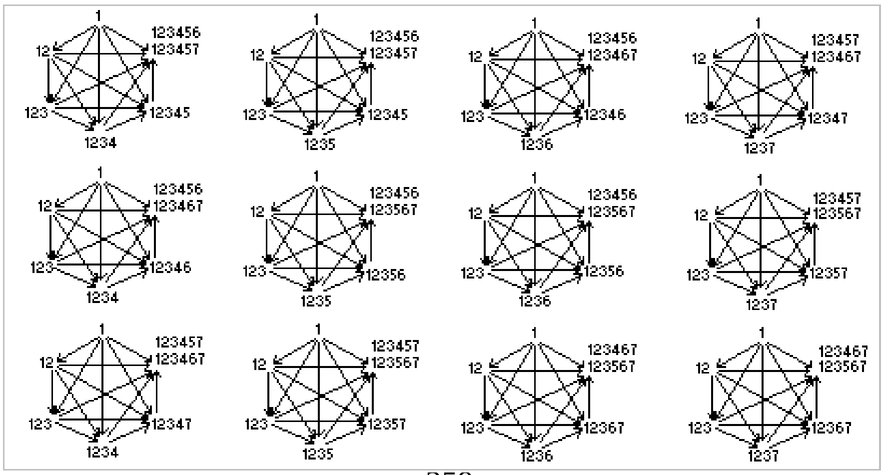
Second, knowing thus that there are a heart and two cloud drums in the β6-structure, we have the following symbolic-descriptive representation of the whole β6-structure.



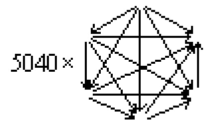
From this, knowing, as we do, how each β*n*-structure is constructed on top of an α(*n*+1)-structure (its heart), we can draw the complete qualitative list of the β*n*-structures belonging to the β6-structure. It is the following:



Finally, as for the "surface" of the β6-structure, a combinatorial examination similar to the previous ones shows that it is made of 5040 logical elements shaped as arrowed simplexes of dimension 7.



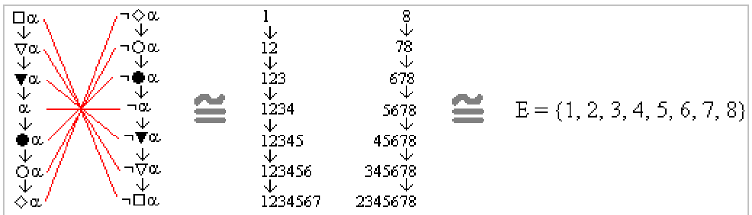
As there are 6 ways to change "12" ("1" being fixed), 5 ways to change "123" ("12" being fixed) and 7 ways to change "1" (including the "no change" option), all in all there are 7×5×6×24=5040 elementary bricks.



Having finished our check-up of the β6-structure, we can finally step to the study of the β7-structure.

14.04. The 7-dimensional β7-structure

Once again, to study the a β*n*-structure, we must start from its modal 3(*n*)-graph. The modal 3(7)-graph, with its Pellissier-translation, is the following.



looking for the β7-structure: through the "settification" of the modal 3(7)-graph

Its characteristic set is E = {1, 2, 3, 4, 5, 6, 7, 8}

The points of the β7-structure are the following:

- a: 1, 2, 3, 4, 5, 6, 7, 8 (8 terms),
- b: 12, 13, 14, 15, 16, 17, 18, 23, 24, 25, 26, 27, 28, 34, 35, 36, 37, 38, 45, 46, 47, 48, 56, 57, 58, 67, 68, 78 (28 terms),
- c: 123, 124, 125, 126, 127, 128, 134, 135, 136, 137, 138, 145, 146, 147, 148, 156, 157, 158, 167, 168, 178, 234, 235, 236, 237, 238, 245, 246, 247, 248, 256, 257, 258, 267, 268, 278, 345, 346, 347, 348, 356, 357, 358, 367, 368, 378, 456, 457, 458, 467, 468, 478, 567, 568, 578, 678 (56 terms),
- d: 1234, 1235, 1236, 1237, 1238, 1245, 1246, 1247, 1248, 1256, 1257, 1258, 1267, 1268, 1278, 1345, 1346, 1347, 1348, 1356, 1357, 1358, 1367, 1368, 1378, 1456, 1457, 1458, 1467, 1468, 1478, 1567, 1568, 1578, 1678, 2345, 2346, 2347, 2348, 2356, 2357, 2358, 2367, 2368, 2378, 2456, 2457, 2458, 2467, 2468, 2478, 2567, 2568, 2578, 2678, 3456, 3457, 3458, 3467, 3468, 3478, 3567, 3568, 3578, 3678, 4567, 4568, 4578, 4678, 5678 (70 terms),
- e: 12345, 12346, 12347, 12348, 12356, 12357, 12358, 12367, 12368, 12378, 12456, 12457, 12458, 12467, 12468, 12478, 12567, 12568, 12578, 12678, 13456,

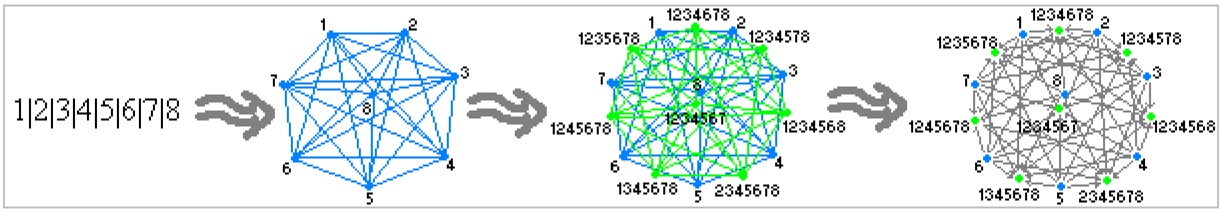
13457, 13458, 13467, 13468, 13478, 13567, 13568, 13578, 13678, 14567, 14568, 14578, 14678, 15678, 23456, 23457, 23458, 23467, 23468, 23478, 23567, 23568, 23578, 23678, 24567, 24568, 24578, 24678, 25678, 34567, 34568, 34578, 34678, 35678, 45678 (56 terms),

- f: 123456, 123457, 123458, 123467, 123468, 123478, 123567, 123568, 123578, 123678, 124567, 124568, 124578, 124678, 125678, 134567, 134568, 134578, 134678, 135678, 145678, 234567, 234568, 234578, 234678, 235678, 245678, 345678 (28 terms),
- g: 1234567, 1234568, 1234578, 1234678, 1235678, 1245678, 1345678, 2345678 (8 termes).

So all in all there are 8+28+56+70+56+28+8 = 254 points in the β7-structure…

Which α*n*-structures (and then which β-structures) are there in the β7-structure?

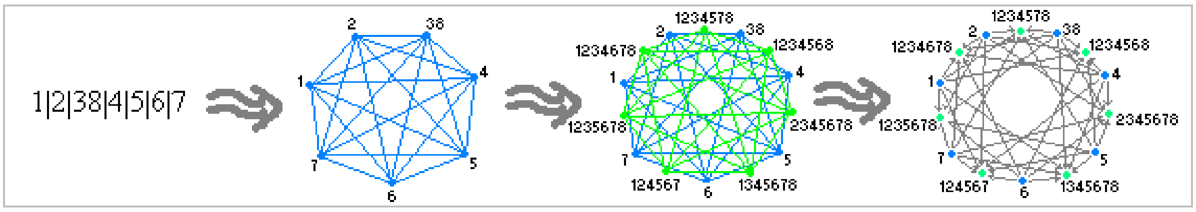
The biggest possible partition (a logical α8-structure) of E here: the 8-partition 1|2|3|4|5|6|7|8.



How many 7-partitions (i.e. α7-structures)? There are 28 possible 7-partitions (or logical α7-structures) of E:

1{ 12|3|4|5|6|7|8, 2{ 13|2|4|5|6|7|8, 3{ 14|2|3|5|6|7|8, 4{ 15|2|3|4|6|7|8, 5{ 16|2|3|4|5|7|8, 6{ 17|2|3|4|5|6|8, 7{ 18|2|3|4|5|6|7, 8{ 1|23|4|5|6|7|8, 9{ 1|24|3|5|6|7|8, 10{ 1|25|3|4|6|7|8, 11{ 1|26|3|4|5|7|8, 12{ 1|27|3|4|5|6|8, 13{ 1|28|3|4|5|6|7, 14{ 1|2|34|5|6|7|8, 15{ 1|2|35|4|6|7|8, 16{ 1|2|36|4|5|7|8, 17{ 1|2|37|4|5|6|8, 18{ 1|2|38|4|5|6|7, 19{ 1|2|3|45|6|7|8, 20{ 1|2|3|46|5|7|8, 21{ 1|2|3|47|5|6|8, 22{ 1|2|3|48|5|6|7, 23{ 1|2|3|4|56|7|8, 24{ 1|2|3|4|57|6|8, 25{ 1|2|3|4|58|6|7, 26{ 1|2|3|4|5|67|8, 27{ 1|2|3|4|5|68|7, 28{ 1|2|3|4|5|6|78.

Here we give an example of such 7-partitions of E (the partition number "18{").



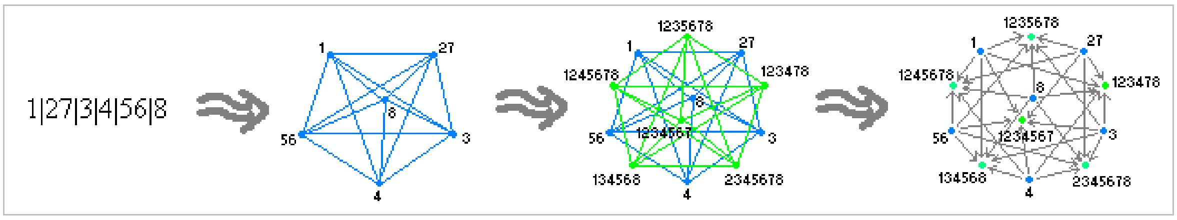
How many 6-partitions (i.e. α6-structures)? There are 266 possible 6-partitions (or logical α6-structures) of E:

1} 123|4|5|6|7|8, 2} 124|3|5|6|7|8, 3} 125|3|4|6|7|8, 4} 126|3|4|5|7|8,
5} 127|3|4|5|6|8, 6} 128|3|4|5|6|7, 7} 134|2|5|6|7|8, 8} 135|2|4|6|7|8,
9} 136|2|4|5|7|8, 10} 137|2|4|5|6|8, 11} 138|2|4|5|6|7, 12} 145|2|3|6|7|8,
13} 146|2|3|5|7|8, 14} 147|2|3|5|6|8, 15} 148|2|3|5|6|7, 16} 156|2|3|4|7|8,
17} 157|2|3|4|6|8, 18} 158|2|3|4|6|7, 19} 167|2|3|4|5|8, 20} 168|2|3|4|5|7,
21} 178|2|3|4|5|6, 22} 1|234|5|6|7|8, 23} 1|235|4|6|7|8, 24} 1|236|4|5|7|8,
25} 1|237|4|5|6|8, 26} 1|238|4|5|6|7, 27} 1|245|3|6|7|8, 28} 1|246|3|5|7|8,
29} 1|247|3|5|6|8, 30} 1|248|3|5|6|7, 31} 1|256|3|4|7|8, 32} 1|257|3|4|6|8,
33} 1|258|3|4|6|7, 34} 1|267|3|4|5|8, 35} 1|268|3|4|5|7, 36} 1|278|3|4|5|6,
37} 1|2|345|6|7|8, 38} 1|2|346|5|7|8, 39} 1|2|347|5|6|8, 40} 1|2|348|5|6|7,
41} 1|2|356|4|7|8, 42} 1|2|357|4|6|8, 43} 1|2|358|4|6|7, 44} 1|2|367|4|5|8,
45} 1|2|368|4|5|7, 46} 1|2|378|4|5|6, 47} 1|2|3|456|7|8, 48} 1|2|3|457|6|8,
49} 1|2|3|458|6|7, 50} 1|2|3|467|5|8, 51} 1|2|3|468|5|7, 52} 1|2|3|478|5|6,
53} 1|2|3|4|567|8, 54} 1|2|3|4|568|7, 55} 1|2|3|4|578|6, 56} 1|2|3|4|5|678,
57} 12|34|5|6|7|8, 58} 12|35|4|6|7|8, 59} 12|36|4|5|7|8, 60} 12|37|4|5|6|8,
61} 12|38|4|5|6|7, 62} 12|3|45|6|7|8, 63} 12|3|46|5|7|8, 64} 12|3|47|5|6|8,
65} 12|3|48|5|6|7, 66} 12|3|4|56|7|8, 67} 12|3|4|57|6|8, 68} 12|3|4|58|6|7,
69} 12|3|4|5|67|8, 70} 12|3|4|5|68|7, 71} 12|3|4|5|6|78, 72} 13|24|5|6|7|8,
73} 13|25|4|6|7|8, 74} 13|26|4|5|7|8, 75} 13|27|4|5|6|8, 76} 13|28|4|5|6|7,
77} 13|2|45|6|7|8, 78} 13|2|46|5|7|8, 79} 13|2|47|5|6|8, 80} 13|2|48|5|6|7,
81} 13|2|4|56|7|8, 82} 13|2|4|57|6|8, 83} 13|2|4|58|6|7, 84} 13|2|4|5|67|8,
85} 13|2|4|5|68|7, 86} 13|2|4|5|6|78, 87} 14|23|5|6|7|8, 88} 14|25|3|6|7|8,
89} 14|26|3|5|7|8, 90} 14|27|3|5|6|8, 91} 14|28|3|5|6|7, 92} 14|2|35|6|7|8,
93} 14|2|36|5|7|8, 94} 14|2|37|5|6|8, 95} 14|2|38|5|6|7, 96} 14|2|3|56|7|8,
97} 14|2|3|57|6|8, 98} 14|2|3|58|6|7, 99} 14|2|3|5|67|8, 100} 14|2|3|5|68|7,
101} 14|2|3|5|6|78, 102} 15|23|4|6|7|8, 103} 15|24|3|6|7|8, 104} 15|26|3|4|7|8,
105} 15|27|3|4|6|8, 106} 15|28|3|4|6|7, 107} 15|2|34|6|7|8, 108} 15|2|36|4|7|8,
109} 15|2|37|4|6|8, 110} 15|2|38|4|6|7, 111} 15|2|3|46|7|8, 112} 15|2|3|47|6|8,
113} 15|2|3|48|6|7, 114} 15|2|3|4|67|8, 115} 15|2|3|4|68|7, 116} 15|2|3|4|6|78,
117} 16|23|4|5|7|8, 118} 16|24|3|5|7|8, 119} 16|25|3|4|7|8, 120} 16|27|3|4|5|8,
121} 16|28|3|4|5|7, 122} 16|2|34|5|7|8, 123} 16|2|35|4|7|8, 124} 16|2|37|4|5|8,
125} 16|2|38|4|5|7, 126} 16|2|3|45|7|8, 127} 16|2|3|47|5|8, 128} 16|2|3|48|5|7,

129} 16|2|3|4|57|8, 130} 16|2|3|4|58|7, 131} 16|2|3|4|5|78, 132} 17|23|4|5|6|8,
133} 17|24|3|5|6|8, 134} 17|25|3|4|6|8, 135} 17|26|3|4|5|8, 136} 17|28|3|4|5|6,
137} 17|2|34|5|6|8, 138} 17|2|35|4|6|8, 139} 17|2|36|4|5|8, 140} 17|2|38|4|5|6,
141} 17|2|3|45|6|8, 142} 17|2|3|46|5|8, 143} 17|2|3|48|5|6, 144} 17|2|3|4|56|8,
145} 17|2|3|4|58|6, 146} 17|2|3|4|5|68, 147} 18|23|4|5|6|7, 148} 18|24|3|5|6|7,
149} 18|25|3|4|6|7, 150} 18|26|3|4|5|7, 151} 18|27|3|4|5|6, 152} 18|2|34|5|6|7,
153} 18|2|35|4|6|7, 154} 18|2|36|4|5|7, 155} 18|2|37|4|5|6, 156} 18|2|3|45|6|7,
157} 18|2|3|46|5|7, 158} 18|2|3|47|5|6, 159} 18|2|3|4|56|7, 160} 18|2|3|4|57|6,
161} 18|2|3|4|5|67, 162} 1|23|45|6|7|8, 163} 1|23|46|5|7|8, 164} 1|23|47|5|6|8,
165} 1|23|48|5|6|7, 166} 1|23|4|56|7|8, 167} 1|23|4|57|6|8, 168} 1|23|4|58|6|7,
169} 1|23|4|5|67|8, 170} 1|23|4|5|68|7, 171} 1|23|4|5|6|78, 172} 1|24|35|6|7|8,
173} 1|24|36|5|7|8, 174} 1|24|37|5|6|8, 175} 1|24|38|5|6|7, 176} 1|24|3|56|7|8,
177} 1|24|3|57|6|8, 178} 1|24|3|58|6|7, 179} 1|24|3|5|67|8, 180} 1|24|3|5|68|7,
181} 1|24|3|5|6|78, 182} 1|25|34|6|7|8, 183} 1|25|36|4|7|8, 184} 1|25|37|4|6|8,
185} 1|25|38|4|6|7, 186} 1|25|3|46|7|8, 187} 1|25|3|47|6|8, 188} 1|25|3|48|6|7,
189} 1|25|3|4|67|8, 190} 1|25|3|4|68|7, 191} 1|25|3|4|6|78, 192} 1|26|34|5|7|8,
193} 1|26|35|4|7|8, 194} 1|26|37|4|5|8, 195} 1|26|38|4|5|7, 196} 1|26|3|45|7|8,
197} 1|26|3|47|5|8, 198} 1|26|3|48|5|7, 199} 1|26|3|4|57|8, 200} 1|26|3|4|58|7,
201} 1|26|3|4|5|78, 202} 1|27|34|5|6|8, 203} 1|27|35|4|6|8, 204} 1|27|36|4|5|8,
205} 1|27|38|4|5|6, 206} 1|27|3|45|6|8, 207} 1|27|3|46|5|8, 208} 1|27|3|48|5|6,
209} 1|27|3|4|56|8, 210} 1|27|3|4|58|6, 211} 1|27|3|4|5|68, 212} 1|28|34|5|6|7,
213} 1|28|35|4|6|7, 214} 1|28|36|4|5|7, 215} 1|28|37|4|5|6, 216} 1|28|3|45|6|7,
217} 1|28|3|46|5|7, 218} 1|28|3|47|5|6, 219} 1|28|3|4|56|7, 220} 1|28|3|4|57|6,
221} 1|28|3|4|5|67, 222} 1|2|34|56|7|8, 223} 1|2|34|57|6|8, 224} 1|2|34|58|6|7,
225} 1|2|34|5|67|8, 226} 1|2|34|5|68|7, 227} 1|2|34|5|6|78, 228} 1|2|35|46|7|8,
229} 1|2|35|47|6|8, 230} 1|2|35|48|6|7, 231} 1|2|35|4|67|8, 232} 1|2|35|4|68|7,
233} 1|2|35|4|6|78, 234} 1|2|36|45|7|8, 235} 1|2|36|47|5|8, 236} 1|2|36|48|5|7,
237} 1|2|36|4|57|8, 238} 1|2|36|4|58|7, 239} 1|2|36|4|5|78, 240} 1|2|37|45|6|8,
241} 1|2|37|46|5|8, 242} 1|2|37|48|5|6, 243} 1|2|37|4|56|8, 244} 1|2|37|4|58|6,
245} 1|2|37|4|5|68, 246} 1|2|38|45|6|7, 247} 1|2|38|46|5|7, 248} 1|2|38|47|5|6,
249} 1|2|38|4|56|7, 250} 1|2|38|4|57|6, 251} 1|2|38|4|5|67, 252} 1|2|3|45|67|8,
253} 1|2|3|45|68|7, 254} 1|2|3|45|6|78, 255} 1|2|3|46|57|8, 256} 1|2|3|46|58|7,

257} 1|2|3|46|5|78, 258} 1|2|3|47|56|8, 259} 1|2|3|47|58|6, 260} 1|2|3|47|5|68, 261} 1|2|3|48|56|7, 262} 1|2|3|48|57|6, 263} 1|2|3|48|5|67, 264} 1|2|3|4|56|78, 265} 1|2|3|4|57|68, 266} 1|2|3|4|58|67.

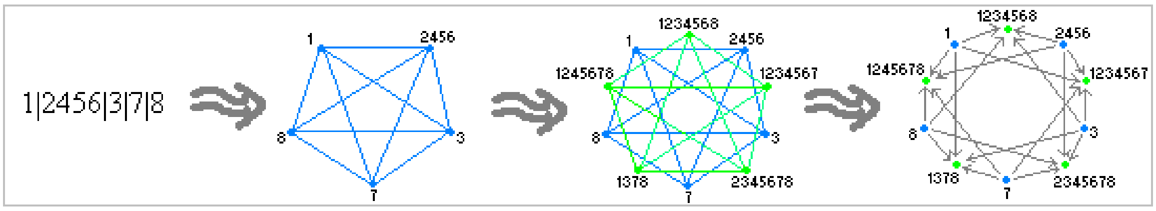
Here we give an example of such 6-partitions of E (the partition number 209}).



How many 5-partitions (i.e. α5-structures) of E are there? Stirling's formula tells us that there are exactly 1050 of them (ordered here in 150 lines of 7 strings each: 7×150=1050).

1234|5|6|7|8, 1235|4|6|7|8, 1236|4|5|7|8, 1237|4|5|6|8, 1238|4|5|6|8, 1245|3|6|7|8, 1246|3|5|7|8, 1247|3|5|6|8, 1248|3|5|6|7, 1256|3|4|7|8, 1257|3|4|6|8, 1258|3|4|6|7, 1267|3|4|5|8, 1268|3|4|5|7, 1278|3|4|5|6, 1345|2|6|7|8, 1346|2|5|7|8, 1347|2|5|6|8, 1348|2|5|6|7, 1356|2|4|7|8, 1357|2|4|6|8, 1358|2|4|6|7, 1367|2|4|5|8, 1368|2|4|5|7, 1378|2|4|5|6, 1456|2|3|7|8, 1457|2|3|6|8, 1458|2|3|6|7, 1467|2|3|5|8, 1468|2|3|5|7, 1478|2|3|5|6, 1567|2|3|4|8, 1568|2|3|4|7, 1578|2|3|4|6, 1678|2|3|4|5, 1|2345|6|7|8, 1|2346|5|7|8, 1|2347|5|6|8, 1|2348|5|6|7, 1|2356|4|7|8, 1|2357|4|6|8, 1|2358|4|5|7, 1|2367|4|5|8, 1|2368|4|5|7, 1|2378|4|5|6, 1|2456|3|7|8, 1|2457|3|6|8, 1|2458|3|6|7, 1|2467|3|5|8, 1|2468|3|5|7, 1|2478|3|5|6, 1|2567|3|4|8, 1|2568|3|4|7, 1|2578|3|4|6, 1|2678|3|4|5, 1|2|3456|7|8, 1|2|3457|6|8, 1|2|3458|6|7, 1|2|3467|5|8, 1|2|3468|5|7, 1|2|3478|5|6, 1|2|3567|4|8, 1|2|3568|4|7, 1|2|3578|4|6, 1|2|3678|4|5, 1|2|3|4567|8, 1|2|3|4568|7, 1|2|3|4578|6, 1|2|3|4678|5, 1|2|3|4|5678, 123|45|6|7|8, 123|46|5|7|8, 123|47|5|6|8, 123|48|5|6|7, 123|4|56|7|8, 123|4|57|6|8, 123|4|58|6|7, 123|4|5|67|8, 123|4|5|68|7, 123|4|5|6|78, 124|35|6|7|8, 124|36|5|7|8, 124|37|5|6|8, 124|38|5|6|7, 124|3|56|7|8, 124|3|57|6|8, 124|3|58|6|7, 124|3|5|67|8, 124|3|5|68|7, 124|3|5|6|78, 125|34|6|7|8, 125|36|4|7|8, 125|37|4|6|8, 125|38|4|6|7, 125|3|46|7|8, 125|3|47|6|8, 125|3|48|6|7, 125|3|4|67|8, 125|3|4|68|7, 125|3|4|6|78, 126|34|5|7|8, 126|35|4|7|8, 126|37|4|5|8, 126|38|4|5|7, 126|3|45|7|8, 126|3|47|5|8, 126|3|48|5|7, 126|3|4|57|8, 126|3|4|58|7, 126|3|4|5|78, 127|34|5|6|8, 127|35|4|6|8, 127|36|4|5|8, 127|38|4|5|6, 127|3|45|6|8, 127|3|46|5|8, 127|3|48|5|6, 127|3|4|56|8, 127|3|4|58|6, 127|3|4|5|68, 128|34|5|6|7, 128|35|4|6|7, 128|36|4|5|7, 128|37|4|5|6, 128|3|45|6|7, 128|3|46|5|7, 128|3|47|5|6, 128|3|4|56|7, 128|3|4|57|6, 128|3|4|5|67, 134|25|6|7|8, 134|26|5|7|8, 134|27|5|6|8, 134|28|5|6|7, 134|2|56|7|8, 134|2|57|6|8, 134|2|58|6|7, 134|2|5|67|8, 134|2|5|68|7, 134|2|5|6|78, 135|24|6|7|8, 135|26|4|7|8, 135|27|4|6|8, 135|28|4|6|7, 135|2|46|7|8, 135|2|47|6|8, 135|2|48|6|7, 135|2|4|67|8, 135|2|4|68|7, 135|2|4|6|78, 136|24|5|7|8, 136|25|4|7|8, 136|27|4|5|8, 136|28|4|5|7, 136|2|45|7|8, 136|2|47|5|8, 136|2|48|5|7, 136|2|4|57|8, 136|2|4|58|7, 136|2|4|5|78, 137|24|5|6|8, 137|25|4|6|8, 137|26|4|5|8, 137|28|4|5|6, 137|2|45|6|8, 137|2|46|5|8, 137|2|48|5|6, 137|2|4|56|8, 137|2|4|58|6, 137|2|4|5|68, 138|24|5|6|7, 138|25|4|6|7, 138|26|4|5|7, 138|27|4|5|6, 138|2|45|6|7, 138|2|46|5|7, 138|2|47|5|6, 138|2|4|56|7, 138|2|4|57|6, 138|2|4|5|67, 145|23|6|7|8, 145|26|3|7|8, 145|27|3|6|8, 145|28|3|6|7, 145|2|36|7|8, 145|2|37|6|8, 145|2|38|6|7, 145|2|3|67|8, 145|2|3|68|7, 145|2|3|6|78, 146|23|5|7|8, 146|25|3|7|8, 146|27|3|5|8, 146|28|3|5|7, 146|2|35|7|8, 146|2|37|5|8, 146|2|38|5|7, 146|2|3|57|8, 146|2|3|58|7, 146|2|3|5|78, 147|23|5|6|8, 147|25|3|6|8, 147|26|3|5|8, 147|28|3|5|6, 147|2|35|6|8, 147|2|36|5|8, 147|2|38|5|6, 147|2|3|56|8, 147|2|3|58|6, 147|2|3|5|68, 148|23|5|6|7, 148|25|3|6|7, 148|26|3|5|7, 148|27|3|5|6, 148|2|35|6|7, 148|2|36|5|7, 148|2|37|5|6, 148|2|3|56|7, 148|2|3|57|6, 148|2|3|5|67, 156|23|4|7|8, 156|24|3|7|8, 156|27|3|4|8, 156|28|3|4|7, 156|2|34|7|8, 156|2|37|4|8, 156|2|38|4|7, 156|2|3|47|8, 156|2|3|48|7, 156|2|3|4|78, 157|23|4|6|8, 157|24|3|6|8, 157|26|3|4|8, 157|28|3|4|6, 157|2|34|6|8, 157|3|36|4|8, 157|2|38|4|6, 157|2|3|46|8, 157|2|3|48|6, 157|2|3|4|68, 158|23|4|6|7, 158|24|3|6|7, 158|26|3|4|7, 158|27|3|4|6, 158|2|34|6|7, 158|2|36|4|7, 158|2|37|4|6, 158|2|3|46|7, 158|2|3|47|6, 158|2|3|4|67, 167|23|4|5|8, 167|24|3|5|8, 167|25|3|4|8, 167|28|3|4|5, 167|2|34|5|8, 167|2|35|4|8, 167|2|38|4|5, 167|2|3|45|8, 167|2|3|48|5, 167|2|3|4|58, 168|23|4|5|7, 168|24|3|5|7, 168|25|3|4|7, 168|27|3|4|5, 168|2|34|5|7, 168|2|35|4|7, 168|2|37|4|5, 168|2|3|45|7, 168|2|3|47|5, 168|2|3|4|57, 178|23|4|5|6, 178|24|3|5|6, 178|25|3|4|6, 178|26|3|4|5, 178|2|34|5|6, 178|2|35|4|6, 178|2|36|4|5, 178|2|3|45|6, 178|2|3|46|5, 178|2|3|4|56, 15|234|6|7|8, 16|234|5|7|8, 17|234|5|6|8, 18|234|5|6|7, 1|234|56|7|8, 1|234|57|6|8, 1|234|58|6|7, 1|234|5|67|8, 1|234|5|68|7, 1|234|5|6|78, 14|235|6|7|8, 16|235|4|7|8, 17|235|4|6|8, 18|235|4|6|7, 1|235|46|7|8, 1|235|47|6|8, 1|235|48|6|7, 1|235|4|67|8, 1|235|4|68|7, 1|235|4|6|78, 14|236|5|7|8, 15|236|4|7|8, 17|236|4|5|8, 18|236|4|5|7, 1|236|45|7|8, 1|236|47|5|8, 1|236|48|5|7, 1|236|4|57|8, 1|236|4|58|7, 1|236|4|5|78, 14|237|5|6|8, 15|237|4|6|8, 16|237|4|5|8, 18|237|4|5|6, 1|237|45|6|8, 1|237|46|5|8, 1|237|48|5|6, 1|237|4|56|8, 1|237|4|58|6, 1|237|4|5|68, 14|238|5|6|7, 15|238|4|6|7, 16|238|4|5|7, 17|238|4|5|6, 1|238|45|6|7, 1|238|46|5|7, 1|238|47|5|6, 1|238|4|56|7, 1|238|4|57|6, 1|238|4|5|67, 13|245|6|7|8, 16|245|3|7|8, 17|245|3|6|8, 18|245|3|6|7, 1|245|36|7|8, 1|245|37|6|8, 1|245|38|6|7, 1|245|3|67|8, 1|245|3|68|7, 1|245|3|6|78, 13|246|5|7|8, 15|246|3|7|8, 17|246|3|5|8, 18|246|3|5|7, 1|246|35|7|8, 1|246|37|5|8, 1|246|38|5|7, 1|246|3|57|8, 1|246|3|58|7, 1|246|3|5|78, 13|247|5|6|8, 15|247|3|6|8, 16|247|3|5|8, 18|247|3|5|6, 1|247|35|6|8, 1|247|36|5|8, 1|247|38|5|6, 1|247|3|56|8, 1|247|3|58|6, 1|247|3|5|68, 13|248|5|6|7, 15|248|3|6|7, 16|248|3|5|7, 17|248|3|5|6, 1|248|35|6|7, 1|248|36|5|7, 1|248|37|5|6, 1|248|3|56|7, 1|248|3|57|6, 1|248|3|5|67, 13|256|4|7|8, 14|256|3|7|8, 17|256|3|4|8, 18|256|3|4|7, 1|256|34|7|8, 1|256|37|4|8, 1|256|38|4|7, 1|256|3|47|8, 1|256|3|48|7, 1|256|3|4|78, 13|257|4|6|8, 14|257|3|6|8, 16|257|3|4|8, 18|257|3|4|6, 1|257|34|6|8, 1|257|36|4|8, 1|257|38|4|6, 1|257|3|46|8, 1|257|3|48|6, 1|257|3|4|68, 13|258|4|6|7, 14|258|3|6|7, 16|258|3|4|7, 17|258|3|4|6, 1|258|34|6|7, 1|258|36|4|7, 1|258|37|4|6, 1|258|3|46|7, 1|258|3|47|6, 1|258|3|4|67, 13|267|4|5|8, 14|267|3|5|8, 15|267|3|4|8, 18|267|3|4|5, 1|267|34|5|8, 1|267|35|4|8, 1|267|38|4|5, 1|267|3|45|8, 1|267|3|48|5, 1|267|3|4|58, 13|268|4|5|7, 14|268|3|5|7, 15|268|3|4|7, 17|268|3|4|5, 1|268|34|5|7, 1|268|35|4|7, 1|268|37|4|5, 1|268|3|45|7, 1|268|3|47|5, 1|268|3|4|57, 13|278|4|5|6, 14|278|3|5|6, 15|278|3|4|6, 16|278|3|4|5, 1|278|34|5|6, 1|278|35|4|6, 1|278|36|4|5, 1|278|3|45|6, 1|278|3|46|5, 1|278|3|4|56, 12|345|6|7|8, 16|2|345|7|8, 17|2|345|6|8, 18|2|345|6|7, 1|26|345|7|8, 1|27|345|6|8, 1|28|345|6|7, 1|2|345|67|8, 1|2|345|68|7, 1|2|345|6|78, 12|346|5|7|8, 15|2|346|7|8, 17|2|346|5|8, 18|2|346|5|7, 1|25|346|7|8, 1|27|346|5|8, 1|28|346|5|7, 1|2|346|57|8, 1|2|346|58|7, 1|2|346|5|78, 12|347|5|6|8, 15|2|347|6|8, 16|2|347|5|8, 18|2|347|5|6, 1|25|347|6|8, 1|26|347|5|8, 1|28|347|5|6, 1|2|347|56|8, 1|2|347|58|6, 1|2|347|5|68, 12|348|5|6|7, 15|2|348|6|7, 16|2|348|5|7, 17|2|348|5|6, 1|25|348|6|7, 1|26|348|5|7, 1|27|348|5|6, 1|2|348|56|7, 1|2|348|57|6, 1|2|348|5|67, 12|356|4|7|8, 14|2|356|7|8, 17|2|356|4|8, 18|2|356|4|7, 1|24|356|7|8, 1|27|356|4|8, 1|28|356|4|7, 1|2|356|47|8, 1|2|356|48|7, 1|2|356|4|78, 12|357|4|6|8, 14|2|357|6|8, 16|2|357|4|8, 18|2|357|4|6, 1|24|357|6|8, 1|26|357|4|8, 1|28|357|4|6, 1|2|357|46|8, 1|2|357|48|6, 1|2|357|4|68, 12|358|4|6|7, 14|2|358|6|7, 16|2|358|4|7, 17|2|358|4|6, 1|24|358|6|7, 1|26|358|4|7, 1|27|358|4|6, 1|2|358|46|7, 1|2|358|47|6, 1|2|358|4|67, 12|367|4|5|8, 14|2|367|5|8, 15|2|367|4|8, 18|2|367|4|5, 1|24|367|5|8, 1|25|367|4|8, 1|28|367|4|5, 1|2|367|45|8, 1|2|367|48|5, 1|2|367|4|58, 12|368|4|5|7, 14|2|368|5|7, 15|2|368|4|7, 17|2|368|4|5, 1|24|368|5|7, 1|25|368|4|7, 1|27|368|4|5, 1|2|368|45|7, 1|2|368|47|5, 1|2|368|4|57, 12|378|4|5|6, 14|2|378|5|6, 15|2|378|4|6, 16|2|378|4|5, 1|24|378|5|6, 1|25|378|4|6, 1|26|378|4|5, 1|2|378|45|6, 1|2|378|46|5, 1|2|378|4|56, 12|456|3|7|8, 13|2|456|7|8, 17|2|456|3|8, 18|2|456|3|7, 1|23|456|7|8, 1|27|456|3|8, 1|28|456|3|7, 1|2|456|37|8, 1|2|456|38|7, 1|2|456|3|78, 12|457|3|6|8, 13|2|457|6|8, 16|2|457|3|8, 18|2|457|3|6, 1|23|457|6|8, 1|26|457|3|8, 1|28|457|3|6, 1|2|457|36|8, 1|2|457|38|6, 1|2|457|3|68, 12|458|3|6|7, 13|2|458|6|7, 16|2|458|3|7, 17|2|458|3|6, 1|23|458|6|7, 1|26|458|3|7, 1|27|458|3|6, 1|2|458|36|7, 1|2|458|37|6, 1|2|458|3|67, 12|467|3|5|8, 13|2|467|5|8, 15|2|467|3|8, 18|2|467|3|5, 1|23|467|5|8, 1|25|467|3|8, 1|28|467|3|5, 1|2|467|35|8, 1|2|467|38|5, 1|2|467|3|58, 12|468|3|5|7, 13|2|468|5|7, 15|2|468|3|7, 17|2|468|3|5, 1|23|468|5|7, 1|25|468|3|7, 1|27|468|3|5, 1|2|468|35|7, 1|2|468|37|5, 1|2|468|3|57, 12|478|3|5|6, 13|2|478|5|6, 15|2|3|478|6, 16|2|3|478|5, 1|23|478|5|6, 1|25|3|478|6, 1|26|3|478|5, 1|2|35|478|6, 1|2|36|478|5, 1|2|3|478|56, 12|3|4|567|8, 13|2|4|567|8, 14|2|3|567|8, 18|2|3|4|567, 1|23|4|567|8, 1|24|3|567|8, 1|28|3|4567, 1|2|34|567|8, 1|2|38|4|567, 1|2|3|48|567, 12|3|4|568|7, 13|2|4|568|7, 14|2|3|568|7, 17|2|3|4|568, 1|23|4|568|7, 1|24|3|568|7, 1|27|3|4|568, 1|2|34|568|7, 1|2|37|4|568, 1|2|3|47|568, 12|3|4|578|6, 13|2|4|578|6, 14|2|3|578|6, 16|2|3|4|578, 1|23|4|578|6, 1|24|3|578|6, 1|26|3|4|578, 1|2|34|578|6, 1|2|36|4|578, 1|2|3|46|578, 12|3|4|5|678, 13|2|4|5|678, 14|2|3|5|678, 15|2|3|4|678, 1|23|4|5|678, 1|24|3|5|678, 1|25|3|4|678, 1|2|34|5|678, 1|2|35|4|678, 1|2|3|45|678, 12|34|56|7|8, 12|34|57|6|8, 12|34|58|6|7, 12|34|5|67|8, 12|34|5|68|7, 12|34|5|6|78, 12|35|46|7|8, 12|35|47|6|8, 12|35|48|6|7|, 12|35|4|67|8, 12|35|4|68|7, 12|35|4|6|78, 12|36|45|7|8, 12|36|47|5|8, 12|36|48|5|7, 12|36|4|57|8, 12|36|4|58|7, 12|36|4|5|78, 12|37|45|6|8, 12|37|46|5|8, 12|37|48|5|6, 12|37|4|56|8, 12|37|4|58|6, 12|37|4|5|68, 12|38|45|6|7, 12|38|46|5|7, 12|38|47|5|6, 12|38|4|56|7, 12|38|4|57|6, 12|38|4|5|67, 12|3|45|67|8, 12|3|45|68|7, 12|3|45|6|78, 12|3|46|57|8, 12|3|46|58|7, 12|3|46|5|78, 12|3|47|56|8, 12|3|47|58|6, 12|3|47|5|68, 12|3|48|56|7, 12|3|48|57|6, 12|3|48|5|67, 12|3|4|56|78, 12|3|4|57|68, 12|3|4|58|67, 13|24|56|7|8, 13|24|57|6|8, 13|24|58|6|7, 13|24|5|67|8, 13|24|5|68|7, 13|24|5|6|78, 13|25|46|7|8, 13|25|47|6|8, 13|25|48|6|7, 13|25|4|67|8, 13|25|4|68|7, 13|25|4|6|78, 13|26|45|7|8, 13|26|47|5|8, 13|26|48|5|7, 13|26|4|57|8, 13|26|4|58|7, 13|26|4|5|78, 13|27|45|6|8, 13|27|46|5|8, 13|27|48|5|6, 13|27|4|56|8, 13|27|4|58|6, 13|27|4|5|68, 13|28|45|6|7, 13|28|46|5|7, 13|28|47|5|6, 13|28|4|56|7, 13|28|4|57|6, 13|28|4|5|67, 13|2|45|67|8, 13|2|45|68|7, 13|2|45|6|78, 13|2|46|57|8, 13|2|46|58|7, 13|2|46|5|78, 13|2|47|56|8, 13|2|47|58|6, 13|2|47|5|68, 13|2|48|56|7, 13|2|48|57|6, 13|2|48|5|67, 13|2|4|56|78, 13|2|4|57|68, 13|2|4|58|67, 14|23|56|7|8, 14|23|57|6|8, 14|23|58|6|7, 14|23|5|67|8, 14|23|5|68|7, 14|23|5|6|78, 14|25|36|7|8, 14|25|37|6|8, 14|25|38|6|7, 14|25|3|67|8, 14|25|3|68|7, 14|25|3|6|78, 14|26|35|7|8, 14|26|37|5|8, 14|26|38|5|7, 14|26|3|57|8, 14|26|3|58|7, 14|26|3|5|78, 14|27|35|6|8, 14|27|36|5|8, 14|27|38|5|6, 14|27|3|56|8, 14|27|3|58|6, 14|27|3|5|68, 14|28|35|6|7, 14|28|36|5|7, 14|28|37|5|6, 14|28|3|56|7, 14|28|3|57|6, 14|28|3|5|67, 14|2|35|67|8, 14|2|35|68|7, 14|2|35|6|78, 14|2|36|57|8, 14|2|36|58|7, 14|2|36|5|78, 14|2|37|56|8, 14|2|37|58|6, 14|2|37|5|68, 14|2|38|56|7, 14|2|38|57|6, 14|2|38|5|67, 14|2|3|56|78, 14|2|3|57|68, 14|2|3|58|67, 15|23|46|7|8, 15|23|47|6|8, 15|23|48|6|7, 15|23|4|67|8, 15|23|4|68|7|, 15|23|4|6|78, 15|24|36|7|8, 15|24|37|6|8, 15|24|38|6|7, 15|24|3|67|8, 15|24|3|68|7, 15|24|3|6|78, 15|26|34|7|8, 15|26|37|4|8, 15|26|38|4|7, 15|26|3|47|8, 15|26|3|48|7, 15|26|3|4|78, 15|27|34|6|8, 15|27|36|4|8, 15|27|38|4|6, 15|27|3|46|8, 15|27|3|48|6, 15|27|3|4|68, 15|28|34|6|7, 15|28|36|4|7, 15|28|37|4|6, 15|28|3|46|7, 15|28|3|47|6, 15|28|3|4|67, 15|2|34|67|8, 15|2|34|68|7, 15|2|34|6|78, 15|2|36|47|8, 15|2|36|48|7, 15|2|36|4|78, 15|2|37|46|8, 15|2|37|48|6, 15|2|37|4|68, 15|2|38|46|7, 15|2|38|47|6, 15|2|38|4|67, 15|2|3|46|78, 15|2|3|47|68, 15|2|3|48|67, 16|23|45|7|8, 16|23|47|5|8, 16|23|48|5|7, 16|23|4|57|8, 16|23|4|58|7, 16|23|4|5|78, 16|24|35|7|8, 16|24|37|5|8, 16|24|38|5|7, 16|24|3|57|8, 16|24|3|58|7, 16|24|3|5|78, 16|25|34|7|8, 16|25|37|4|8, 16|25|38|4|7, 16|25|3|47|8, 16|25|3|48|7, 16|25|3|4|78, 16|27|34|5|8, 16|27|35|4|8, 16|27|38|4|5, 16|27|3|45|8, 16|27|3|48|5, 16|27|3|4|58, 16|28|34|5|7, 16|28|35|4|7, 16|28|37|4|5, 16|28|3|45|7, 16|28|3|47|5, 16|28|3|4|57, 16|2|34|57|8, 16|2|34|58|7, 16|2|34|5|78, 16|2|35|47|8, 16|2|35|48|7, 16|2|35|4|78, 16|2|37|45|8, 16|2|37|48|5, 16|2|37|4|58, 16|2|38|45|7, 16|2|38|47|5, 16|2|38|4|57, 16|2|3|45|78, 16|2|3|47|58, 16|2|3|48|57, 17|23|45|6|8, 17|23|46|5|8, 17|23|48|5|6, 17|23|4|56|8, 17|23|4|58|6, 17|23|4|5|68, 17|24|35|6|8, 17|24|36|5|8, 17|24|38|5|6, 17|24|3|56|8, 17|24|3|58|6, 17|24|3|5|68, 17|25|34|6|8, 17|25|36|4|8, 17|25|38|4|6, 17|25|3|46|8, 17|25|3|48|6, 17|25|3|4|68, 17|26|34|5|8, 17|26|35|4|8, 17|26|38|4|5, 17|26|3|45|8, 17|26|3|48|5, 17|26|3|4|58, 17|28|34|5|6, 17|28|35|4|6, 17|28|36|4|5, 17|28|3|45|6, 17|28|3|46|5, 17|28|3|4|56, 17|2|34|56|8, 17|2|34|58|6, 17|2|34|5|68, 17|2|35|46|8, 17|2|35|48|6, 17|2|35|4|68, 17|2|36|45|8, 17|2|36|48|5, 17|2|36|4|58, 17|2|38|45|6, 17|2|38|46|5, 17|2|38|4|56, 17|2|3|45|68, 17|2|3|46|58, 17|2|3|48|56, 18|23|45|6|7, 18|23|46|5|7, 18|23|47|5|6, 18|23|4|56|7, 18|23|4|57|6, 18|23|4|5|67, 18|24|35|6|7, 18|24|36|5|7, 18|24|37|5|6, 18|24|3|56|7, 18|24|3|57|6, 18|24|3|5|67, 18|25|34|6|7, 18|25|36|4|7, 18|25|37|4|6, 18|25|3|46|7, 18|25|3|47|6, 18|25|3|4|67, 18|26|34|5|7, 18|26|35|4|7, 18|26|37|4|5, 18|26|3|45|7, 18|26|3|47|5, 18|26|3|4|57, 18|27|34|5|6, 18|27|35|4|6, 18|27|36|4|5, 18|27|3|45|6, 18|27|3|46|5, 18|27|3|4|56, 18|2|34|56|7, 18|2|34|57|6, 18|2|34|5|67, 18|2|35|46|7, 18|2|35|47|6, 18|2|35|4|67, 18|2|36|45|7, 18|2|36|47|5, 18|2|36|4|57, 18|2|37|45|6, 18|2|37|46|5, 18|2|37|4|56, 18|2|3|45|67, 18|2|3|46|57, 18|2|3|47|56, 1|23|45|67|8, 1|23|45|68|7, 1|23|45|6|78, 1|23|46|57|8, 1|23|46|58|7, 1|23|46|5|78, 1|23|47|56|8, 1|23|47|58|6, 1|23|47|5|68, 1|23|48|56|7, 1|23|48|57|6, 1|23|48|5|67, 1|23|4|56|78, 1|23|4|57|68, 1|23|4|58|67, 1|24|35|67|8, 1|24|35|68|7, 1|24|35|6|78, 1|24|36|57|8, 1|24|36|58|7, 1|24|36|5|78, 1|24|37|56|8, 1|24|37|58|6, 1|24|37|5|68, 1|24|38|56|7, 1|24|38|57|6, 1|24|38|5|67, 1|24|3|56|78, 1|24|3|57|68, 1|24|3|58|67, 1|25|34|67|8, 1|25|34|68|7, 1|25|34|6|78, 1|25|36|47|8, 1|25|36|48|7, 1|25|36|4|78, 1|25|37|46|8, 1|25|37|48|6, 1|25|37|4|68, 1|25|38|46|7, 1|25|38|47|6, 1|25|38|4|67, 1|25|3|46|78, 1|25|3|47|68, 1|25|3|48|67, 1|26|34|57|8, 1|26|34|58|7, 1|26|34|5|78, 1|26|35|47|8, 1|26|35|48|7, 1|26|35|4|78, 1|26|37|45|8, 1|26|37|48|5, 1|26|37|4|58, 1|26|38|45|7, 1|26|38|47|5, 1|26|38|4|57, 1|26|3|45|78, 1|26|3|47|58, 1|26|3|48|57, 1|27|34|56|8, 1|27|34|58|6, 1|27|34|5|68, 1|27|35|46|8, 1|27|35|48|6, 1|27|35|4|68, 1|27|36|45|8, 1|27|36|48|5, 1|27|36|4|58, 1|27|38|45|6, 1|27|38|46|5, 1|27|38|4|56, 1|27|3|45|68, 1|27|3|46|58, 1|27|3|48|56, 1|28|34|56|7, 1|28|34|57|6, 1|28|34|5|67, 1|28|35|46|7, 1|28|35|47|6, 1|28|35|4|67, 1|28|36|45|7, 1|28|36|47|5, 1|28|36|4|57, 1|28|37|45|6, 1|28|37|46|5, 1|28|37|4|56, 1|28|3|45|67, 1|28|3|46|57, 1|28|3|47|56, 1|2|34|56|78, 1|2|34|57|68, 1|2|34|58|67, 1|2|35|46|78, 1|2|35|47|68, 1|2|35|48|67, 1|2|36|45|78, 1|2|36|47|58, 1|2|36|48|57, 1|2|37|45|68, 1|2|37|46|58, 1|2|37|48|56, 1|2|38|45|67, 1|2|38|46|57, 1|2|38|47|56.

Here we give an example of such 5-partitions (i.e. a 5-opposition, an α5-structure) of E (we give the 46th 5-partition, i.e. the 4th string on the 7th row).



How many 4-partitions (i.e. α4-structures, or logical cubes)? Stirling's formula tells us there are exactly 1701 (due to the size, here we will give only a part of them).

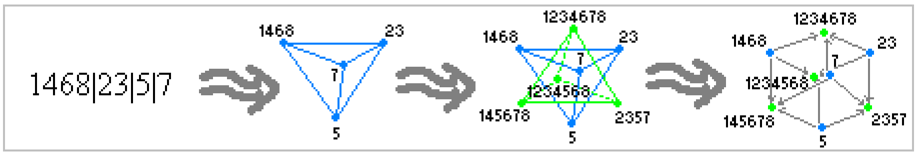
12345|6|7|8, 12346|5|7|8, 12347|5|6|8, 12348|5|6|7, 12356|4|7|8, 12357|4|6|8, 12358|4|6|7,
12367|4|5|8, 12368|4|5|7, 12378|4|5|6, 12456|3|7|8, 12457|3|6|8, 12458|3|6|7, 12467|3|5|8,
12468|3|5|7, 12478|3|5|6, 12567|3|4|8, 12568|3|4|7, 12578|3|4|6, 12678|3|4|5, 13456|2|7|8,
13457|2|6|8, 13458|2|6|7, 13467|2|5|8, 13468|2|5|7, 13478|2|5|6, 13567|2|4|8, 13568|2|4|7,
13578|2|4|6, 13678|2|4|5, 14567|2|3|8, 14568|2|3|7, 14578|2|3|6, 14678|2|3|5, 15678|2|3|4,
1|23456|7|8, 1|23457|6|8, 1|23458|6|7, 1|23467|5|8, 1|23468|5|7, 1|23478|5|6, 1|23567|4|8,
1|23568|4|7, 1|23578|4|6, 1|23678|4|5, 1|24567|3|8, 1|24568|3|7, 1|24578|3|6, 1|24678|3|5,
1|25678|3|4, 1|2|34567|8, 1|2|34568|7, 1|2|34578|6, 1|2|34678|5, 1|2|35678|4, 1|2|3|45678,
1234|56|7|8, 1234|57|6|8, 1234|58|6|7, 1234|5|67|8, 1234|5|68|7, 1234|5|6|78, 1235|46|7|8,
1235|47|6|8, 1235|48|6|7, 1235|4|67|8, 1235|4|68|7, 1235|4|6|78, 1236|45|7|8, 1236|47|5|8,
1236|48|5|7, 1236|4|57|8, 1236|4|58|7, 1236|4|5|78, 1237|45|6|8, 1237|46|5|8, 1237|48|5|6,
1237|4|56|8, 1237|4|58|6, 1237|4|5|68, 1238|45|6|7, 1238|46|5|7, 1238|47|5|6, 1238|4|56|7,
1238|4|57|6, 1238|4|5|67, 1245|36|7|8, 1245|37|6|8, 1245|38|6|7, 1245|3|67|8, 1245|3|68|7,
1245|3|6|78, 1246|35|7|8, 1246|37|5|8, 1246|38|5|7, 1246|3|57|8, 1246|3|58|7, 1246|3|5|78,
1247|35|6|8, 1247|36|5|8, 1247|38|5|6, 1247|3|56|8, 1247|3|58|6, 1247|3|5|68, 1248|35|6|7,
1248|36|5|7, 1248|37|5|6, 1248|3|56|7, 1248|3|57|6, 1248|3|5|67, 1256|34|7|8, 1256|37|4|8,
1256|38|4|7, 1256|3|47|8, 1256|3|48|7, 1256|3|4|78, 1257|34|6|8, 1257|36|4|8, 1257|38|4|6,
1257|3|46|8, 1257|3|48|6, 1257|3|4|68, 1258|34|6|7, 1258|36|4|7, 1258|37|4|6, 1258|3|46|7,
1258|3|47|6, 1258|3|4|67, 1267|34|5|8, 1267|35|4|8, 1267|38|4|5, 1267|3|45|8, 1267|3|48|5,
1267|3|4|58, 1268|34|5|7, 1268|35|4|7, 1268|37|4|5, 1268|3|45|7, 1268|3|47|5, 1268|3|4|57,
1278|34|5|6, 1278|35|4|6, 1278|36|4|5, 1278|3|45|6, 1278|3|46|5, 1278|3|4|56, 1345|26|7|8,
1345|27|6|8, 1345|28|6|7, 1345|2|67|8, 1345|2|68|7, 1345|2|6|78, 1346|25|7|8, 1346|27|5|8,
1346|28|5|7, 1346|2|57|8, 1346|2|58|7, 1346|2|5|78, 1347|25|6|8, 1347|26|5|8, 1347|28|5|6,
1347|2|56|8, 1347|2|58|6, 1347|2|5|68, 1348|25|6|7, 1348|26|5|7, 1348|27|5|6, 1348|2|56|7,
1348|2|57|6, 1348|2|5|67, 1356|24|7|8, 1356|27|4|8, 1356|28|4|7, 1356|2|47|8, 1356|2|48|7,
1356|2|4|78, 1357|24|6|8, 1357|26|4|8, 1357|28|4|6, 1357|2|46|8, 1357|2|48|6, 1357|2|4|68,
1358|24|6|7, 1358|26|4|7, 1358|27|4|6, 1358|2|46|7, 1358|2|47|6, 1358|2|4|67, 1367|24|5|8,
1367|25|4|8, 1367|28|4|5, 1367|2|45|8, 1367|2|48|5, 1367|2|4|58, 1368|24|5|7, 1368|25|4|7,
1368|27|4|5, 1368|2|45|7, 1368|2|47|5, 1368|2|4|57, 1378|24|5|6, 1378|25|4|6, 1378|26|4|5,
1378|2|45|6, 1378|2|46|5, 1378|2|4|56, 1456|23|7|8, 1456|27|3|8, 1456|28|3|7, 1456|2|37|8,
1456|2|38|7, 1456|2|3|78, 1457|23|6|8, 1457|26|3|8, 1457|28|3|6, 1457|2|36|8, 1457|2|38|6,
1457|2|3|68, 1458|23|6|7, 1458|26|3|7, 1458|27|3|6, 1458|2|36|7, 1458|2|37|6, 1458|2|3|67,
1467|23|5|8, 1467|25|3|8, 1467|28|3|5, 146|7|2|35|8, 1467|2|38|5, 1467|2|3|58, 1468|23|5|7,
1468|25|3|7, 1468|27|3|5, 1468|2|35|7, 1468|2|37|5, 1468|2|3|57, 1478|23|5|6, 1478|25|3|6,

1478|26|3|5, 1478|2|35|6, 1478|2|36|5, 1478|2|3|56, 1567|23|4|8, 1567|24|3|8, 1567|28|3|4,
1567|2|34|8, 1567|2|38|4, 1567|2|3|48, 1568|23|4|7, 1568|24|3|7, 1568|27|3|4, 1568|2|34|7,
1568|2|37|4, 1568|2|3|47, 1578|23|4|6, 1578|24|3|6, 1578|26|3|4, 1578|2|34|6, 1578|2|36|4,
1578|2|3|46, 1678|23|4|5, 1678|24|3|5, 1678|25|3|4, 1678|2|34|5, 1678|2|35|4, 1678|2|3|45,
16|2345|7|8, 17|2345|6|8, 18|2345|6|7, 1|2345|67|8, 1|2345|68|7, 1|2345|6|78, 15|2346|7|8,
17|2346|5|8, 18|2346|5|7, 1|2346|57|8, 1|2346|58|7, 1|2346|5|78, 15|2347|6|8, 16|2347|5|8,
18|2347|5|6, 1|2347|56|8, 1|2347|58|6, 1|2347|5|68, 15|2348|6|7, 16|2348|5|7, 17|2348|5|6,
1|2348|56|7, 1|2348|57|6, 1|2348|5|67, 14|2356|7|8, 17|2356|4|8, 18|2356|4|7, 1|2356|47|8,
1|2356|48|7, 1|2356|4|78, 14|2357|6|8, 16|2357|4|8, 18|2357|4|6, 1|2357|46|8, 1|2357|48|6,
1|2357|4|68, 14|2358|6|7, 16|2358|4|7, 17|2358|4|6, 1|2358|46|7, 1|2358|47|6, 1|2358|4|67,
14|2367|5|8, 15|2367|4|8, 18|2367|4|5, 1|2367|45|8, 1|2367|48|5, 1|2367|4|58, 14|2368|5|7,
15|2368|4|7, 17|2368|4|5, 1|2368|45|7, 1|2368|47|5, 1|2368|4|57, 14|2378|5|6, 15|2378|4|6,
16|2378|4|5, 1|2378|45|6, 1|2378|46|5, 1|2378|4|56, 13|2456|7|8, 17|2456|3|8, 18|2456|3|7,
1|2456|37|8, 1|2456|38|7, 1|2456|3|78, 13|2457|6|8, 16|2457|3|8, 18|2457|3|6, 1|2457|36|8,
1|2457|38|6, 1|2457|3|6|8, 13|2458|6|7, 16|2458|3|7, 17|2458|3|6, 1|2458|36|7, 1|2458|37|6,
1|2458|3|67, 13|2467|5|8, 15|2467|3|8, 18|2467|3|5, 1|2467|35|8, 1|2467|38|5, 1|2467|3|58,
13|2468|5|7, 15|2468|3|7, 17|2468|3|5, 1|2468|35|7, 1|2468|37|5, 1|2468|3|57, 13|2478|5|6,
15|2478|3|6, 16|2478|3|5, 1|2478|35|6, 1|2478|36|5, 1|2478|3|56, 13|2567|4|8, 14|2567|3|8,
18|2567|3|4, 1|2567|34|8, 1|2567|38|4, 1|2567|3|48, 13|2568|4|7, 14|2568|3|7, 17|2568|3|4,
1|2568|34|7, 1|2568|37|4, 1|2568|3|47, 13|2578|4|6, 14|2578|3|6, 16|2578|3|4, 1|2578|34|6,
1|2578|36|4, 1|2578|3|46, 13|2678|4|5, 14|2678|3|5, 15|2678|3|4, 1|2678|34|5, 1|2678|35|4,
1|2678|3|45, 12|3456|7|8, 17|2|3456|8, 18|2|3456|7, 1|27|3456|8, 1|28|3456|7, 1|2|3456|78,
12|3457|6|8, 16|2|3457|8, 18|2|3457|6, 1|26|3457|8, 1|28|3457|6, 1|2|3457|68, 12|3458|6|7,
16|2|3458|7, 17|2|3458|6, 1|26|3458|7, 1|27|3458|6, 1|2|3458|67, 12|3467|5|8, 15|2|3467|8,
18|2|3467|5, 1|25|3467|8, 1|28|3467|5, 1|2|3467|58, 12|3468|5|7, 15|2|3468|7, 17|2|3468|5,
1|25|3468|7, 1|27|3468|5, 1|2|3468|57, 12|3478|5|6, 15|2|3478|6, 16|2|3478|5, 1|25|3478|6,
1|26|3478|5, 1|2|3478|56, 12|3567|4|8, 14|2|3567|8, 18|2|3567|4, 1|24|3567|8, 1|28|3567|4,
1|2|3567|48, 12|3568|4|7, 14|2|3568|7, 17|2|3568|4, 1|24|3568|7, 1|27|3568|4, 1|2|3568|47,
12|3578|4|6, 14|2|3578|6, 16|2|3578|4, 1|24|3578|6, 1|26|3578|4, 1|2|3578|46, 12|3678|4|5,
14|2|3678|5, 15|2|3678|4, 1|24|3678|5, 1|25|3678|4, 1|2|3678|45, 12|3|4567|8, 13|2|4567|8,
18|2|3|4567, 1|23|4567|8, 1|28|3|4567|, 1|2|38|4567, 12|3|4568|7|, 13|2|4568|7, 17|2|3|4568,
1|23|4568|7, 1|27|3|4568, 1|2|37|4568, 12|3|4578|6, 13|2|4578|6, 16|2|3|4578, 1|23|4578|6,
1|26|3|4578, 1|2|36|4578, 12|3|4678|5, 13|2|4678|5, 15|2|3|4678, 1|23|4678|5, 1|25|3|4678,
1|2|35|4678, 12|3|4|5678, 13|2|4|5678, 14|2|3|5678, 1|23|4|5678, 1|24|3|5678, 1|2|34|5678,

123|456|7|8, 123|457|6|8, 123|458|6|7, 123|467|5|8, 123|468|5|7, 123|478|5|6, 123|4|567|8, 123|4|568|7, 123|4|578|6, 123|4|5|678, 124|356|7|8, 124|357|6|8, 124|358|6|7, 124|367|5|8, 124|368|5|7, 124|378|5|6, 124|3|567|8, 124|3|568|7, 124|3|578|6, 124|3|5|678, 125|346|7|8, 125|347|6|8, 125|348|6|7, 125|368|4|7, 125|378|4|6, 125|3|467|8, 125|3|468|7, 125|3|478|6, 125|3|4|678, 126|345|7|8, 126|347|5|8, 126|348|5|7, 126|357|4|8, 126|358|4|7, …

(we stop here the calculation)

Here we give an example of such 4-partitions of E (we give the 231^{st} 4-partition, i.e. the rightmost string on the 33^{rd} row).

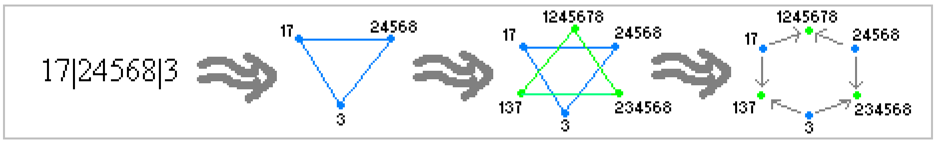


How many 3-partitions (i.e. α3-structures, or logical hexagons)? Stirling's formula tells us there are exactly 966 of them (again, we will not give their complete list).

123456|7|8, 123457|6|8, 123458|6|7, 123467|5|8, 123468|5|7, 123478|5|6, 123567|4|8, 123568|4|7, 123578|4|6, 123678|4|5, 124567|3|8, 124568|3|7, 124578|3|6, 124678|3|5, 125678|3|4, 134567|2|8, 134568|2|7, 134578|2|6, 134678|2|5, 135678|2|4, 145678|2|3, 1|234567|8, 1|234568|7, 1|234578|6, 1|234678|5, 1|235678|4, 1|245678|3, 1|2|345678, 12345|67|8, 12345|68|7, 12345|6|78, 12346|57|8, 12346|58|7, 12346|5|78, 12347|56|8, 12347|58|6, 12347|5|68, 12348|56|7, 12348|57|6, 12348|5|67, 12356|47|8, 12356|48|7, 12356|4|78, 12357|46|8, 12357|48|6, 12357|4|68, 12358|46|7, 12358|47|6, 12358|4|67, 12367|45|8, 12367|48|5, 12367|4|58, 12368|45|7, 12368|47|5, 12368|4|57, 12378|45|6, 12378|46|5, 12378|4|56, 12456|37|8, 12456|38|7, 12456|3|78, 12457|36|8, 12457|38|6, 12457|3|68, 12458|36|7, 12458|37|6, 12458|3|67, 12467|35|8, 12467|38|5, 12467|3|58, 12468|35|7, 12468|37|5, 12468|3|57, 12478|35|6, 12478|36|5, 12478|3|56, 12567|34|8, 12567|38|4, 12567|3|48, 12568|34|7, 12568|37|4, 12568|3|47, 12578|34|6, 12578|36|4, 12578|3|46, 12678|34|5, 12678|35|4, 12678|3|45, 13456|27|8, 13456|28|7, 13456|2|78, 13457|26|8, 13457|28|6, 13457|2|68, 13458|26|7, 13458|27|6, 13458|2|67, 13467|25|8, 13467|28|5, 13467|2|58, 13468|25|7, 13468|27|5, 13468|2|57, 13478|25|6, 13478|26|5, 13478|2|56, 13567|24|8, 13567|28|4, 13567|2|48, 13568|24|7, 13568|27|4, 13568|2|47, 13578|24|6, 13578|26|4, 13578|2|46, 13678|24|5, 13678|25|4, 13678|2|45, 14567|23|8, 14567|28|3, 14567|2|38, 14568|23|7, 14568|27|3, 14568|2|37, 14578|23|6, 14578|26|3, 14578|2|36, 14678|23|5, 14678|25|3, 14678|2|35, 15678|23|4, 15678|24|3, 15678|2|34, 17|23456|8, 18|23456|7, 1|23456|78, 16|23457|8, 18|23457|6, 1|23457|68, 16|23458|7, 17|23458|6, 1|23458|67, 15|23467|8, 18|23467|5, 1|23467|58, 15|23468|7, 17|23468|5, 1|23468|57, 15|23478|6, 16|23478|5, 1|23478|56, 14|23567|8, 18|23567|4, 1|23567|48, 14|23568|7, 17|23568|4, 1|23568|47, 14|23578|6, 16|23578|4, 1|23578|46, 14|23678|5, 15|23678|4, 1|23678|45, 13|24567|8, 18|24567|3, 1|24567|38, 13|24568|7, 17|24568|3, 1|24568|37, 13|24578|6, 16|24578|3, 1|24578|36, 13|24678|5, 15|24678|3, 1|24678|35, 13|25678|4, 14|25678|3, 1|25678|34, …

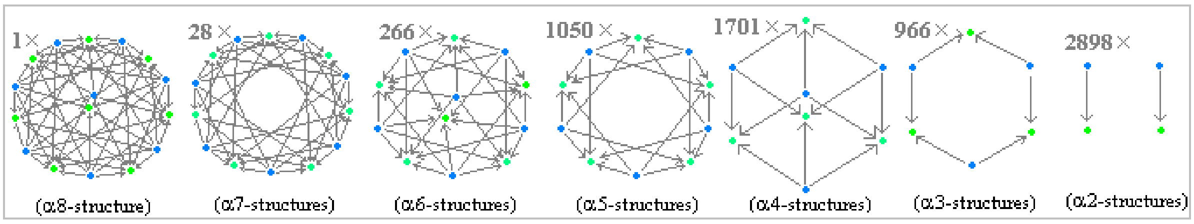
(we stop here the calculation).

Here we give an example of such 3-partitions of E (the 168th partition, i.e. the rightmost one in the 24th row).

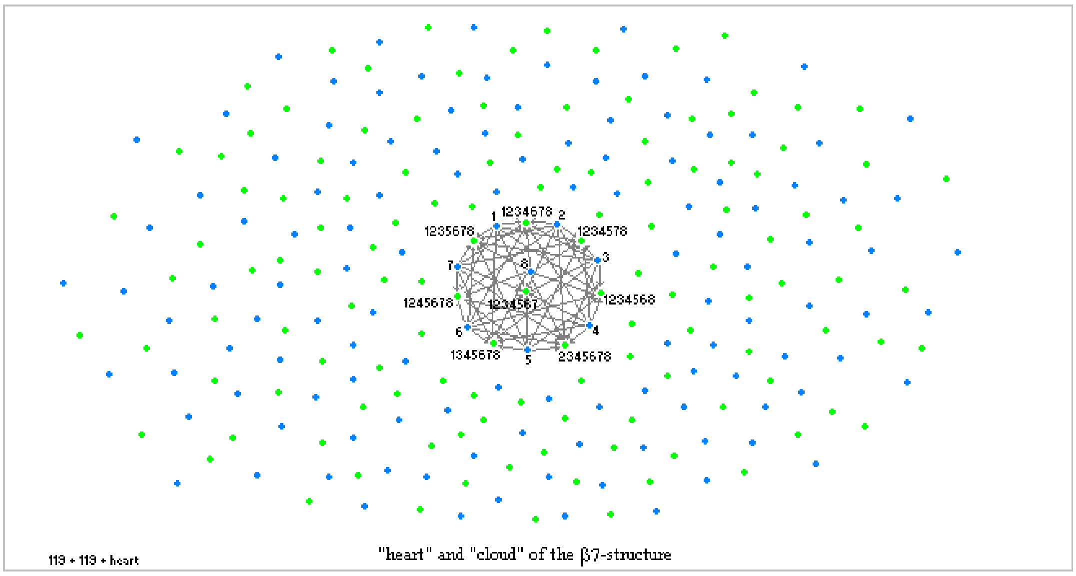


How many 2-partitions (i.e. α2-structures, or logical squares)? Pellissier's result, proving that each strong hexagon has exactly three logical square, without overlapping, tells us therefore that there are exactly 3×966=2898

So, here is finally the list of all the α*n*-structures belonging to the β7-structure

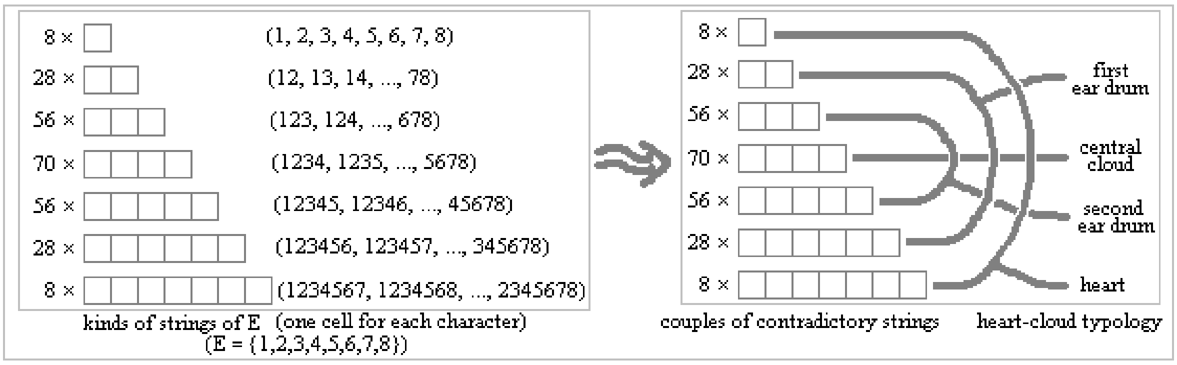


So, having seen its components, now we want to determine the shape of the cloud of the β7-structure with respect to its heart (which is an α8-structure).

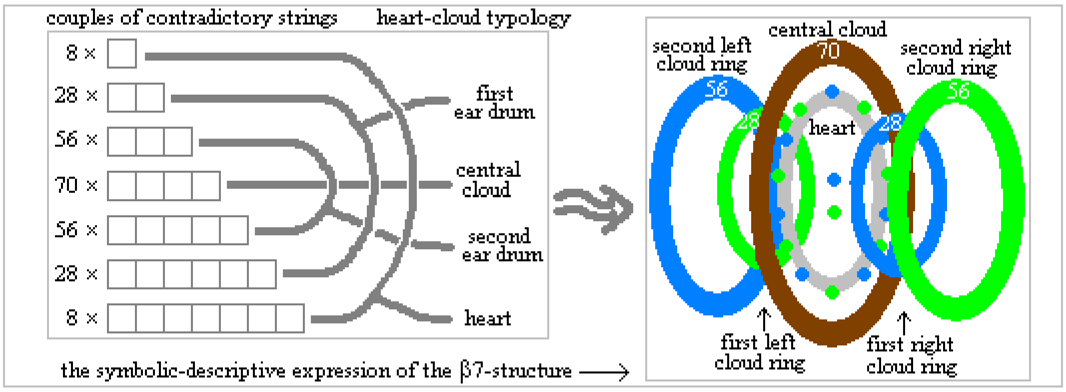


"heart" and "cloud" of the β7-structure

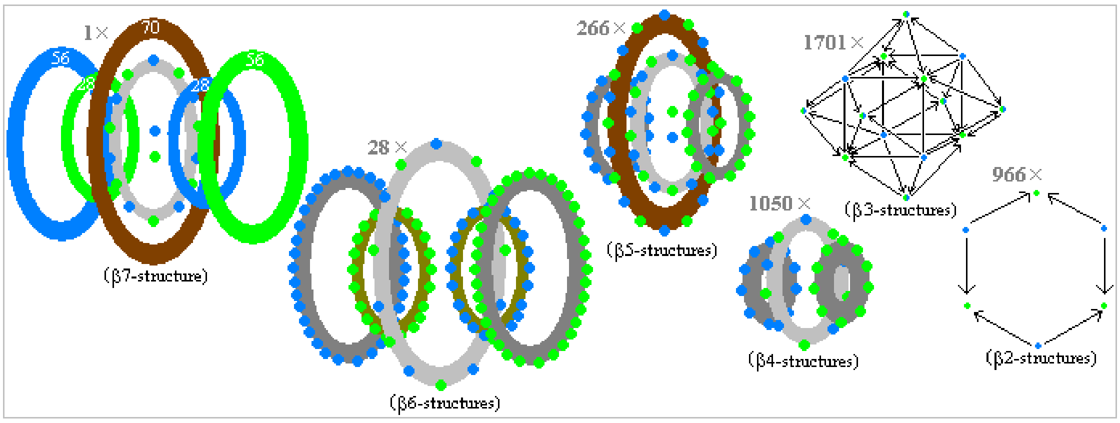
Again, this task being too complex from a geometrical point of view (we should master the complexity of a 6-dimensional space) we use our technique based on the typology of the strings.



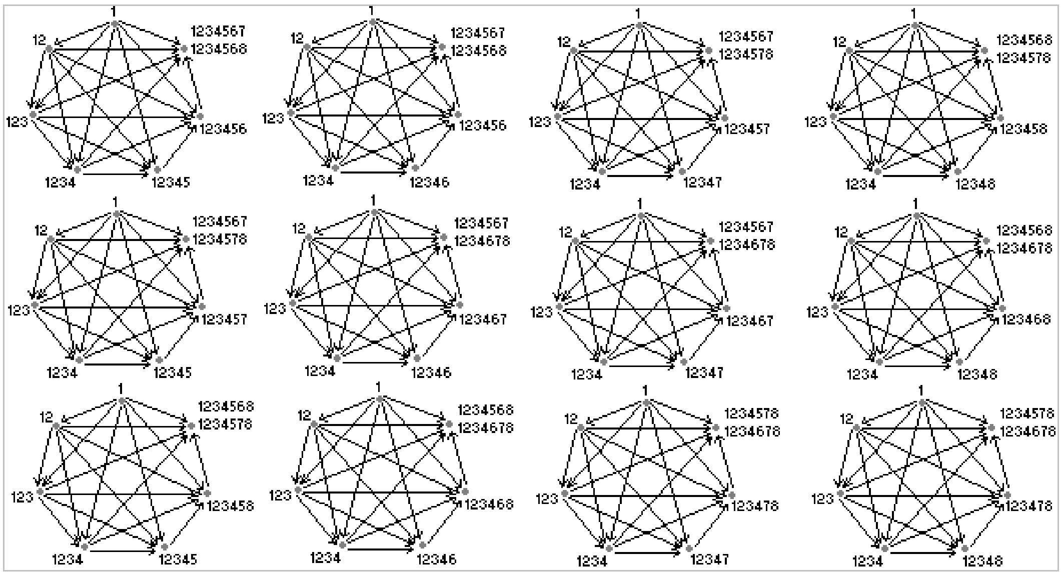
Knowing, now, that the β7-structure has a heart, a cloud ball and two cloud drums (each made of two cloud monochrome ears), we can give a half-symbolic and half-descriptive drawing.



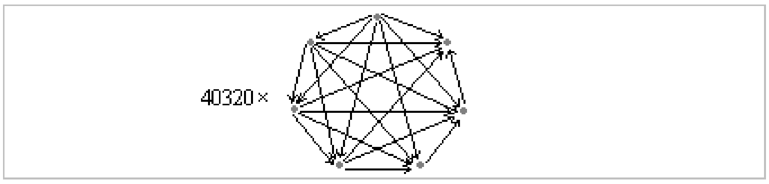
So, finally, the list of the β*n*-structures which are there inside the β7-structure is the following.



As for the "surface" of the β7-structure, a reasoning similar to those of the previous cases can be brought about.



As there can be 5 ways to make "1234" change (if "123" is kept unchanged), and as there are 6 ways to make change "123" (if "12" is kept unchanged) and 7 ways to make "12" change (if "1" is kept unchanged), and because there are 8 ways to change "1" (including the "no change option") there are finally 5x6x7x8x24= 40320 elementary bricks, each having the following shape.



So we ended the boring but instructive study of the three β*n*-structures following the β4-structure (in the next chapter we will use some of the so obtained numerical results).

14.05. Final remarks

The combinatorial examination made in this chapter may seem tedious. Its important purpose was nevertheless double: (1) to familiarise (the reader) with this universe of the β*n*-structures, which we will try to sum up in the next chapter and which will be used more concretely in chapters 16 and 17 *infra*; (2) to give hints as for the most important regularities to be seized. The major open problem seems to be that of finding, if possible, a useful graphical representation of the β*n*-structures ($n \geq 5$), some kind of graphical algorithm comparable to the one we proposed for the αn-structures. In this chapter we showed some elements going in this direction.

15.
THE SERIES OF THE β*N*-STRUCTURES: THE "LOGICAL *M*-DIM. HYPER-TETRAICOSAHEDRA" (EXHAUSTIVE GATHERINGS OF BI-SIMPLEXES)

The aim of this chapter is to give as much general information as possible over the series of the β*n*-structures. This aim is justified if one thinks, as we are going to show later (ch. 16 and 17 mainly), that all translations of given modal systems into oppositional geometry is a translation of the given system into one and one only β*n*-structure. Knowing the general behaviour of the β*n*-structures allows to master the logical space of the oppositions of modal logic. Technically speaking, there are at least two kinds of things which it would be desirable to master: (a) a symbolic knowledge and (b) a geometrical knowledge of the β*n*-structures. As we are going to see, the first one is already reachable by us, whereas the second one remains more difficult but can finally seemingly be reached as well. In conclusion we recall an already mentioned remark of Pellissier (cf. ch.12) which, compared to some results of Béziau (on the hidden presence of paraconsistent behaviours inside S5), helps to understand the general meaning of the the β*n*-structures and therefore of NOT as a whole.

15.01. The vertices and the dimension of the β*n*-structure

The first characterisation of any β*n*-structure is given by its vertices and its general geometrical dimensionality.

Generally speaking, the β*n*-structure is generated by the modal 3(*n*)-graph (but it can also be generated by more complex – but then shorter, from the point of view of the number of the layers of arrows, cf. ch.11 – modal *m*(*k*)-graphs, with $3 \leq m$, $k \leq n$). The Pellissier technique associates to it the characteristic set E = {1, 2, …, *n*, *n*+1}.

Because the possible vertices of a β*n*-structure are all and only the possible subsets of its characteristic set E (excluding ∅ and E itself), the number of vertices of the β*n*-structure is: $2^{n+1} - 2$. One sees that the number of vertices has an exponential growth.

15.02. The “heart” of the β*n*-structure

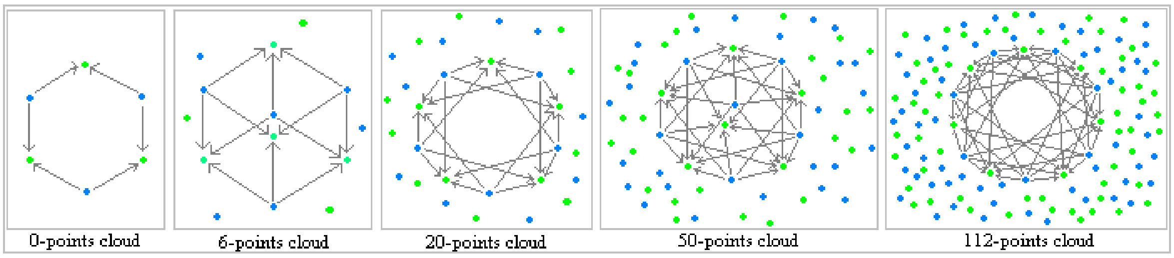
Because *n*-opposition is in some sense set-theoretical *n*-partition (Pellissier) and because each characteristic set E = {1,2,…,*n*,*n*+1} (representing a given β-structure) admits one and one only maximal partition (namely the (*n*+1)-partition “1|2|…|*n*|*n*+1”) we saw that there is a “heart” of the β*n*-structure, which is exactly its biggest α*m*-structure. More precisely, the heart is the α(*n*+1)-structure. And thus it has 2*n*+2 elements (2*n*+2 vertices), the elements of the α(*n*+1)-structure precisely.

Remark that the growth of the heart with respect to *n* is not exponential, for the formula of such a growth is the following: Card(Heart(β*n*))=2(*n*+1). So, the number of elements (i.e. vertices) of the *whole* β*n*-structure being exponential in growth, one sees that the bigger *n* is, the smaller is the β*n*-structure’s heart with respect to its globality. The β*n*-structures are, so to say, “giants with a very small heart”. So, the rule for hearts is very simple: for any *n*, the heart of the β*n*-structure is the α(*n*+1)-structure (i.e. the logical bi-simplex of dimension *n*).

What about the rest of the β*n*-structure, the cloud, then?

15.03. The cloud of the β*n*-structure

As we saw, the “cloud” is that part of any given β*n*-structure which is not its heart. The cloud of the β*n*-structure, being by definition the β*n*-structure minus its heart (i.e. the β*n*-structure minus the α(*n*+1)-structure), has thus $(2^{n+1} – 2) – (2n+2) = 2^{n+1} – 2(n– 2)$ elements (vertices). So, the first remark is that the bigger *n* is, the bigger the cloud with respect to the heart. Growing β*n*-structures are “giants *very cloudy* with very small hearts”, so to say.



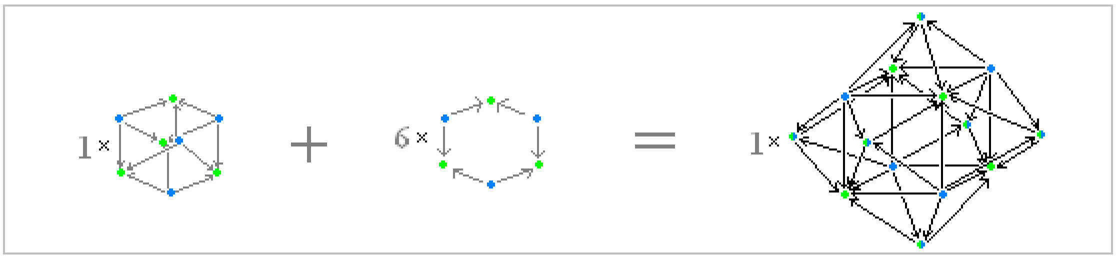
The shape of the cloud of the β*n*-structure can be characterised in at least three ways (a fourth one, purely geometrical will be elaborated in § 16.05 below): (1) as a gathering of α*m*-structures, (2) as a gathering of β*k*-structures (2≤*k*≤*n*) and (3) as something contained in a poly-dimensional “surface”.

15.03.01. It’s α*n*-structures

As for the first point, each β*n*-structure contains instances of all the α*m*-structures ($m \leq n+1$) smaller than the β*n*-structure’s heart. In the future it should be easy to determine a general formula for displaying the coefficients of the series of the α*m*-structures belonging to the β*n*-structure (we omit investigating this here).

15.03.02 Its β*m*-structures

Second, the heart of the β*n*-structure, which is an α(*n*+1)-structure, is the basis where are “seated” all the β(*n*-1)-structures contained in the β*n*-structure. But this “sitting behaviour” (with respect to the heart) is even more general: for, on each of the α*k*-structures contained in the β*n*-structure ($4 \leq k \leq n$) are “seated” α(*k*-1)-structures forming with it a β(*k*-1)-structure.

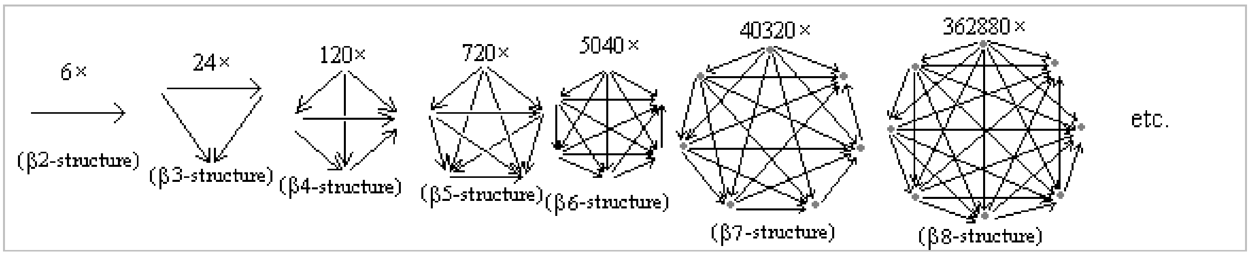


Again, in a near future it would be useful (and, presumably, easy) to determine a general formula for displaying the series of the β*p*-structures belonging to the β*n*-structure. Another future task could be that of expressing algorithmically something like the articulation ‘heart’ - ‘maximal internal β*m*-structure’.

15.04. The “surface” of the β*n*-structure

The third characterisation of the β*n*-structures concerns their multi-dimensional “envelope”. At this level, as we saw in chapters 13 and 14, their “walls” (or “skins”) are made of “bricks” (or “cells”) of growing complexity. As a matter of fact, each β*n*-structure has a “maximal brick”, these bricks belonging all to a series of geometrical simplexes whose edges are made of logical arrows (remember that in the logical bi-simplexes the arrows are *between* the couples of logical simplexes, but the logical simplexes themselves are not made of arrows).

The series of the "bricks" of the β*n*-structures is the following.



One can see easily that there is a clear arithmetical regularity with respect to the number of bricks in each β*n*-structure:

24=6×4, 120=24×5, 720=120×6, 5040=720×7, 40320=5040×8, 362880=5040×9, …

The surface of the β*n*-structure is made of *m* atomic surfaces such that (with W(x)= "the wall of x"):

$m = \text{Card}(W\beta_n) = (n+1)\times\text{Card}(W\beta_{n-1})$

This formula is simple. It could nevertheless be important, terminologically speaking, for one could name accordingly the different members of the series of the β*n*-structures relatively to the number of the "bricks" of their "wall".

In this respect, the bricks of the β2 are the arrows themselves, whereas its wall is the logical hexagon (restricted to its arrows – the hexagon's perimeter). The bricks of the β3 are the arrow-triangles, while its wall is the paving made of such arrow-triangles. The bricks of the β4 are the arrow-tetrahedra, whereas its wall is the composition of such arrow-tetrahedra. And so on. One general rule of this "β*n*-structural masonry" is that one never has, between two distinct points of the β*n*-structure, two arrows going in opposite directions (because then this would abolish the starting difference of the two considered points, which would consequently implode into one and one point only).

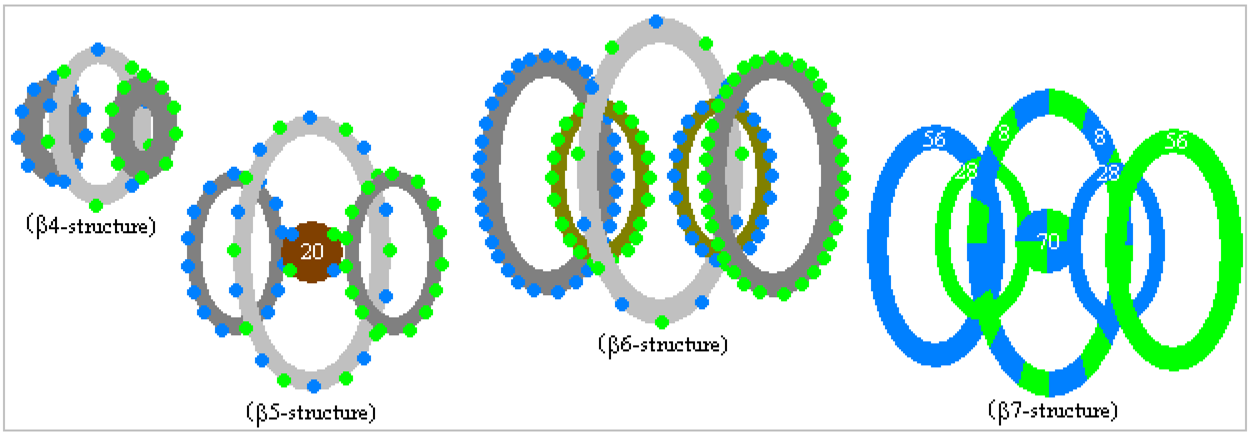
15.05. The global geometrical representation of the β*n*-structure

As we saw (cf. ch.13 and 14), it is hard to grasp and express geometrical global features of objects which are *n*-dimensional ($n\geq4$). We would nevertheless need some kind of mastery over the representation of these structures (the bare "β*n*-structure" name is not visually perceivable). But we saw that Pellissier's method allows, when determining the β*n*-structures' vertices by calculating the possible subsets (non-empty, non-maximal) of the E set characteristic of its modal *n*(*m*)-graph, the distinction of symmetric sets of such solutions.

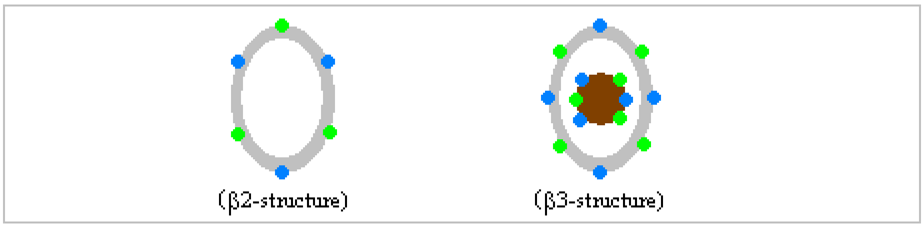
This is done with respect to the number of characters of each considered string (the strings being the subsets of E). So, a general method to have what we call “symbolic-descriptive” representations of the β*n*-structures consists in determining:

- the set of elements of the “heart ring”;
- (eventually) the set of elements of the “cloud ball”;
- the couples of complementary sets with respect to contradiction: that is, the couples of blue-green concentric “cloud rings” (or “ears”), forming two by two “cloud drums” (or “ear drums”).

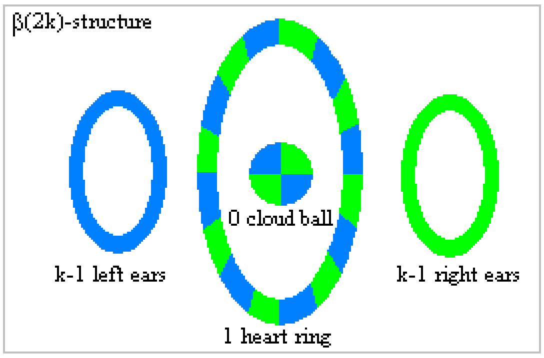
We saw this behaviour for some β*n*-structures (n=4, 5, 6, 7).

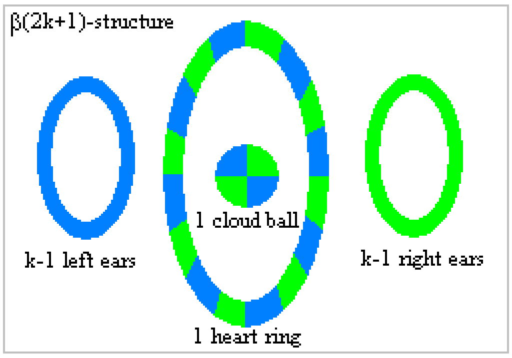


But this can be seen even in the smaller cases (i.e. those where a direct global representation of the β*n*-structure was available): n=2,3.



The general case, for any β*n*-structure, seems then to be the following.



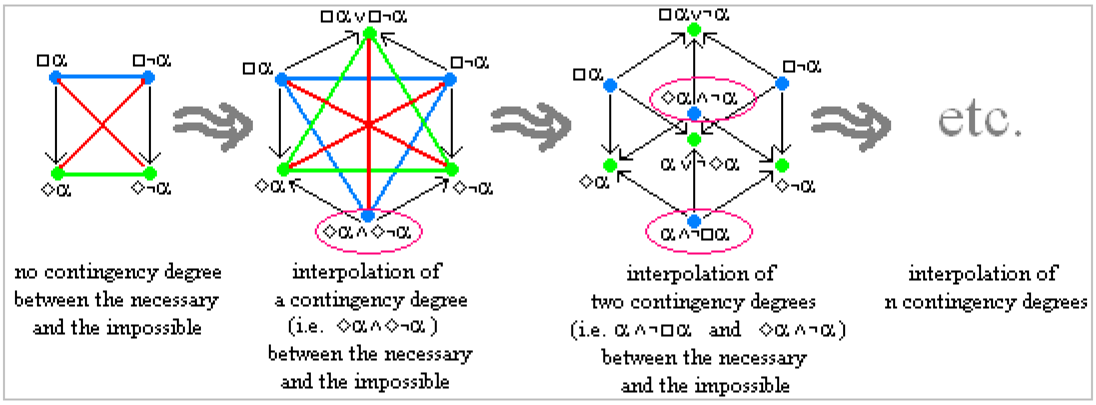


In a near future one could give explicit laws for the number of elements in each ‘heart-cloud’ sub-category (we have to omit doing it here).

With this device in hands (a graphical general algorithm), any β*n*-structure may in principle be represented somehow geometrically.

15.06. The interest of NOT: collecting *all* modalities (also the hidden ones)

So far, one could think that the main interest of NOT consists in allowing the expression of *n*-opposition (i.e. the α*n*-structures), for any finite integer *n*. This is not false, except that there are other considerable conceptual gains offered by NOT. In order to fully understand the usefulness of NOT, one has to come back here to some earlier works. As a matter of fact, in some papers apparently unrelated to our present concern, Béziau' discovered that, very paradoxically, in the universal (non-paraconsistent) Lewis system S5 there are hidden parts (familiar "uninteresting" modalities) which offer a clear paraconsistent behaviour. Béziau used this in order to claim, a bit shockingly, that "S5 is paraconsistent and so is first order logic". With the knowledge of some years thereafter (and with the teachings of NOT), we can perceive now that Pellissier's discovery that the intermediate (composed!) modalities of the logical cube are "interpolated contingency degrees" comprised between the "necessary" and the "impossible" (cf. ch.12 *supra*) is in fact akin to Béziau's one. In other words, this shows that *all* the propositional combinations of basic modalities – i.e. by use of ¬, ∧ and ∨ (as explored very consequently Smessaert, cf. ch.12) – are themselves (hidden) modalities (i.e. they bear meaningful modal meanings).



With this respect, the meaning of the β*n*-structures (and hence of NOT as a whole) is to show that there are more modalities than it seems (more than just the "basic" ones) and to give them *all* (the β*n*-structures being their *closure*). The β*n*-structures *are* the abstract structures of the possible distributions of modal meaning. Therefore they seem to be very important structures. In the next chapter we will try to bring light on the possible methods of applying NOT to most of the known logical systems (mainly the modal ones), by means of the study of the translation rules between modal graphs and β*n*-structures.

16.
GENERAL TRANSLATION RULES: FROM *N*-OPPOSITION TO (POSSIBLY) ALL MODAL SYSTEMS WITH FINITE GRAPHS

In this chapter we try to give general principles aiming at the translation, in the future, of any finite modal graph into solid oppositional geometry (i.e. β*n*-structures). However, there is so far no general algorithm for decorating in Pellissier style any given modal graph: in the case of non-linear modal graphs, their "Pellissierisation" must be produced skilfully (i.e. empirically, as we don't have so far a general algorithm for that). The search for generality indeed tells us that two cases must be dealt with as for translations inside NOT: modal graphs can be (1) linear (in which case their translation is straightforward – cf. ch.12 *supra*) or (2) non-linear (non-linearity being due to the presence of (a) bifurcations (i.e. forkings) or (b) isolations – isolated points or isolated arrows), in which case things are less easy. Here I propose some achieved Pellissierisations of some non-linear modal graphs, together with the empirical knowledge (about translating tricks) they provide. Extrapolating from that, I would propound to conjecture that all finite modal graphs, no matter how complex, can (in principle) be translated in Pellissier's style (i.e. can be associated to some characteristic set E), the sole problem being (seemingly) that of an exponential growth of Card(E) with respect with the growth of the complexity of the studied modal graph. A positive result, already underlined by Pellissier (2008), is that in this context, it seems that a new kind of concept of equivalence is available (in terms of β*n*-structures), allowing to classify and put into relations all modal systems generated by finite graphs (will come back to this in ch. 17 *infra*).

16.01. Introductory remarks on "oppositional translations"

As we have seen so far (cf. ch.12), we have one big result concerning the possible translations of linear modal *n*(*m*)-graphs into geometrical oppositions. As a matter of fact, (1) any (linear) modal 3(*m*)-graph can be mapped into one and one only β*m*-structure (the series of the hyper-tetraicosahedra, cf. ch.15); and (2) each linear modal *n*(*m*)-graph (with $n \geq 3$) can be shown to belong to one and one only β*r*-structure. In other words, fundamentally speaking, there are no β*n*-structures other than the ones generated by the linear modal 3(*m*)-graphs (and resumed in ch. 15 *supra*).

What about the modal graphs which are non-linear? The possible non-linear cases seem to be two: bifurcations (or forkings) and isolated points. Here as well, one great result seems to be at stake: for, as we will show in this chapter, we can have, for non-linear modal graphs, *local* results comparable to the *global* one we got relatively to the linear cases. Hence we will caress the hope of finding, sometime or other, translation rules for all finite modal graphs (linear or not).

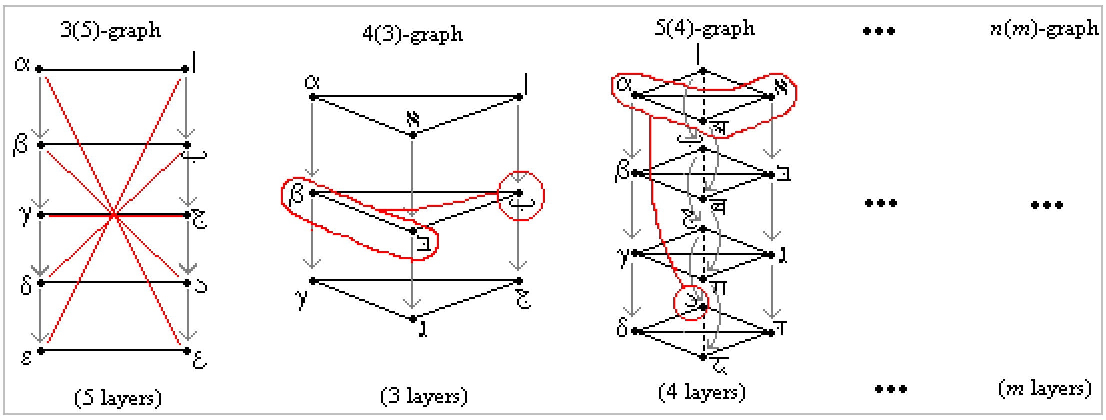
In order to make all this clearer, we will examine with some detail the first cases of non-linear modal graphs (i.e. those implying not too large oppositional structures). As a last consequence of all this, we have a class of geometrical equivalencies between apparently very different modal systems.

16.02. Trying to translate all finite modal graphs (linear or not)

Finite modal graphs seem to divide into two groups: the linear ones, which can be seen as modal *n*(*m*)-graphs (cf. ch.11), and the non-linear ones, which in turn divide into two sub-groups: the bifurcating graphs and the graphs with isolations (isolated points or arrows). Of course, there can be mixtures of these three "pure groups": there can be finite modal graphs admitting together linear chains, bifurcations and isolated points. Let us see the three "pure groups" one by one.

16.02.01. Translating linear modal graphs (Pellissier's method)

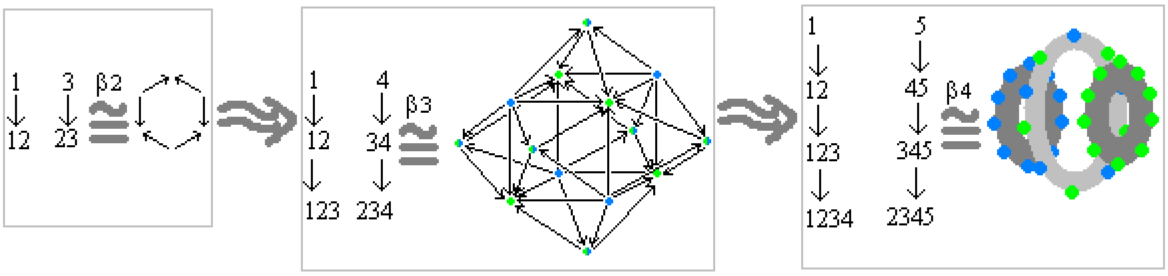
As we saw (cf. ch. 11 *supra*) linear modal *n*(*m*)-graphs are vertical series of downward arrows, relating *m* abstract layers, each layer being composed of a (black) "gem", which is geometrical simplex of dimension *n*-1 (cf. ch. 11).



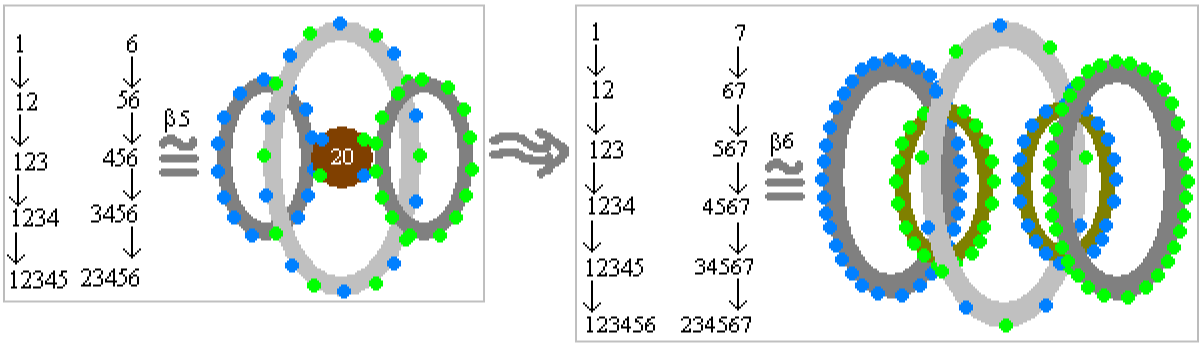
As we have seen (cf. ch. 12), Pellissier demonstrated two important things: (1) that my conjecture over the existence of a series of the β*n*-structures (one β*n*-structure for each modal

3(*n*+1)-graph) was right: for each E determined by a given modal *n*(*m*)-graph, its partition (i.e. the partition of its string into all the possible sub-strings) gives a gathering of geometrical oppositions; (2) that I was mistaken in believing that even for values of *n* other than 3 each modal *n*(*m*)-graph would have its own β*n*-structure: Pellissier showed that all β*n*-structures reduce to those generated by the 3(*m*)-graphs. Things being so, and having laws relating each *n*(*m*)-graph to a 3(*k*)-graph, all one needs is knowing the translation rules between the 3(*m*)-graphs and the β*n*-structures.

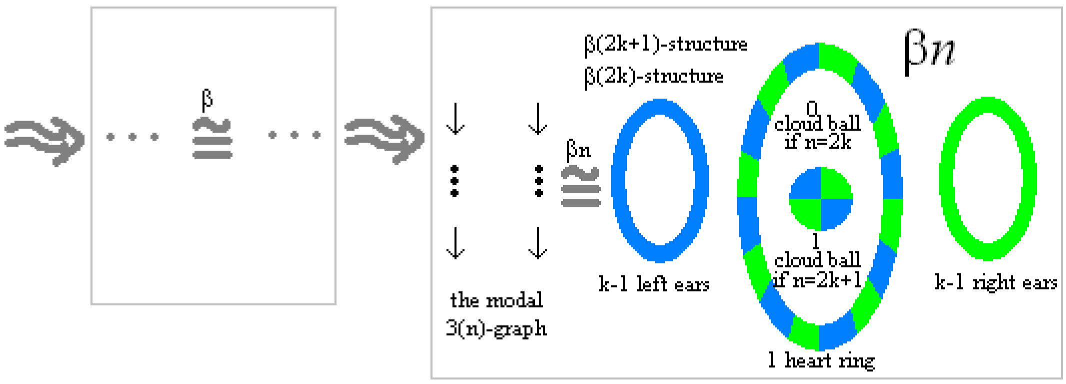
We first saw the translation bridging the modal linear 3(*m*)-graphs (*m*∈{2,3,4}) to their respective β*m*-structures.



Then we explored (cf. ch.14 *supra*) such translations for the next values of *n* (*n*=5,6,7).



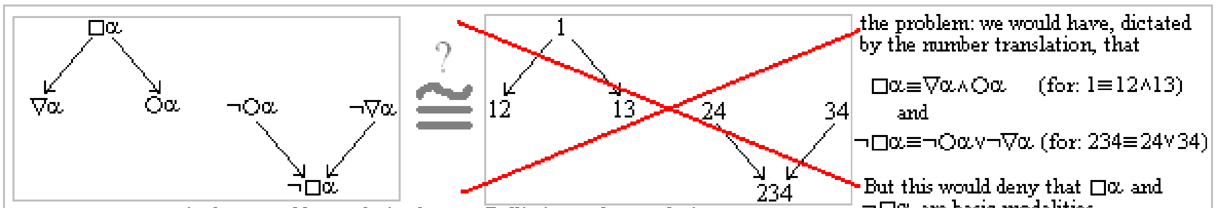
And we finally reached a general translation rule (cf. ch.15 *supra*).



There seems to be no problem at this level. But in all this we remained linear. What about the non-linear cases?

16.02.02. Non-linear modal graphs: bifurcations (Pellissier's method)

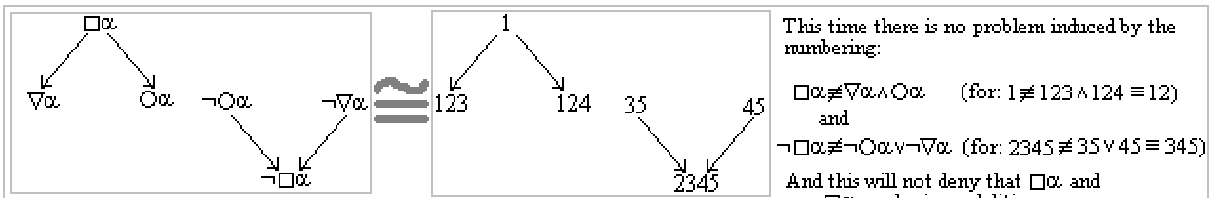
Pellissier has also thought about how to extend his method to the case, which I submitted to him, of the bifurcating modal graphs. This was not straightforward, for if one only applies plainly the linear method to the bifurcating cases, Pellissier's conventions (for linearity) would induce reading the nodes (the parent-modalities) of a starting bifurcation as being logically equivalent to the conjunction (∧) of the son-modalities and, similarly, to read the nodes (son-modality) reached by some arrows as being logically equivalent to the sum (∨) of the parent modalities. But this would have involved that some basic modalities of the studied modal system in fact are not *basic* modalities.



a seemingly natural but truly inadequate Pellissier-style translation

In other words, the problem to be solved here was to avoid having basic modalities equivalent to the propositional composition of other basic modalities (for this would have been a *contradictio in adiecto*). Again, the problem would have concerned diverging bifurcations as well as converging ones. But this cannot be, if we want to keep the notion, crucial for NOT, of "basic modality" (cf. ch. 11 *supra*).

Pellissier's solution is to add, for each bifurcation, one more (new) common character in the string of the two successors, so to make the logical conjunction of the two bifurcated successors different from the bare ancestor of the bifurcation[209]. This gives the following settification.



the correct application of Pellissier's technique to the bifurcations

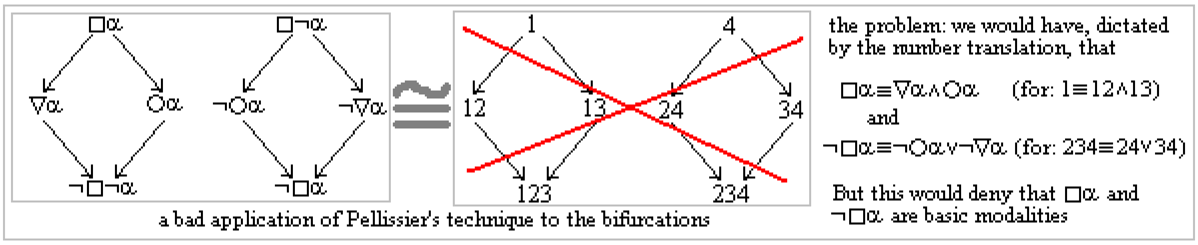
Now, this strategy can be easily generalised, except for more complication appearing progressively (and requiring finer translation rules).

Remark that, truly speaking, bifurcating modal graphs have to respect a top-bottom symmetry (unless one gives a logical rule to forbid constructing, from a top modality "X", its dual – and vertical symmetric – "¬X¬"). So we must beware: when we study these more

[209] Personal communication.

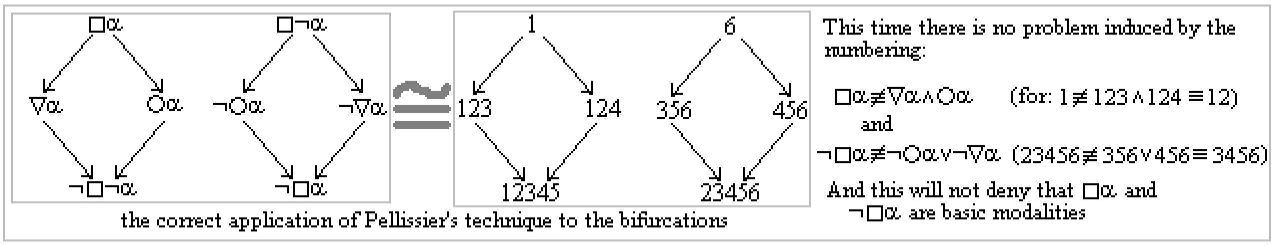
standard (bifurcating) modal graphs (i.e. those admitting a top-bottom symmetry of the arrows) we see a property which was already present in Pellissier's linear settifications, and which will turn out to be useful more generally (for getting translations): *the horizontally symmetric strings have same cardinality (same length)*. This is not the case in the previous scheme, therefore false: the string "123" has not the same length (= 3) as the string "45" (= 2).

Now, if we try to "Pellissierise" a standard bifurcation (i.e. one furnished with a top-bottom symmetry), a simple and straightforward adaptation of Pellissier's technique will give the following results. First, let us expose a way in which things cannot be.



a bad application of Pellissier's technique to the bifurcations

Again, one sees that here we have a problem: taken as such, this translation would imply that some basic modalities are logically reducible to the propositional composition of others (which is a *a contradictio in adjecto*). In order to have an acceptable translation (i.e. one that respects the starting axioms) one should rather give the following translation, mapping to the β5-structure[210].

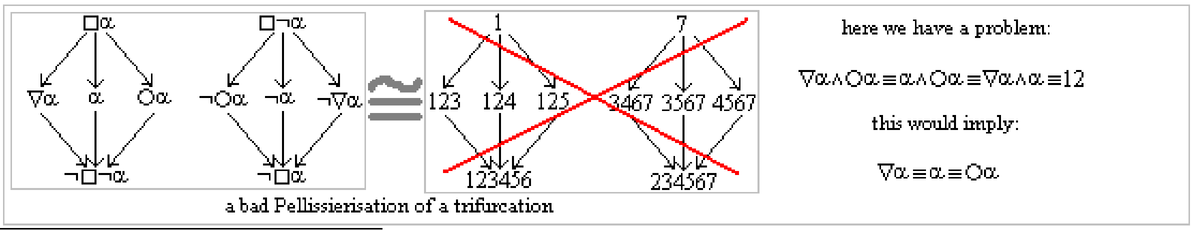


the correct application of Pellissier's technique to the bifurcations

(one sees that "1" has same length as "6", "123" has same length as "456", "12345" has same length as "23456": the horizontally symmetric terms have equal string-length)

So, it seems that simple bifurcations can be handled with no major harm. As we will see, however, things become more difficult when dealing with nested bifurcations (cf. *infra*).

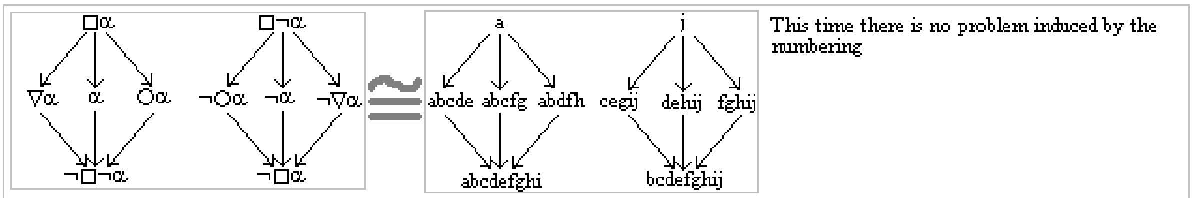
What if we have a "trifurcation"? (i.e. forking with 3 branches) Is the rule just the same? The answer is "no".



a bad Pellissierisation of a trifurcation

[210] Again, the first person to find this solution was Régis Pellissier at the time (2004-2005) when he invented his method (linear case) and when I submitted to him the problem of the existence of bifurcating graphs (mutual personal communications).

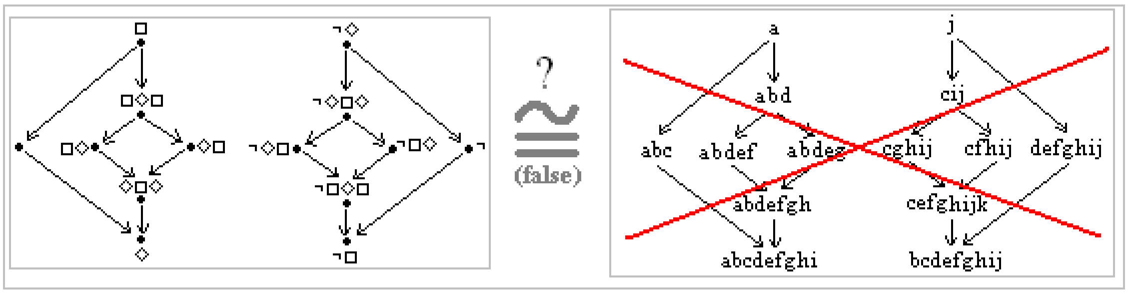
As we say, “just applying the same rule” does not suffice. There is the need of a new strategy, one integrating some new tricks (in order to avoid the undesired bad effects). The good Pellissierisation of the simple bifurcation seems to be the following, mapping it to the β9-structure.



the good Pellissierisation of a trifurcation

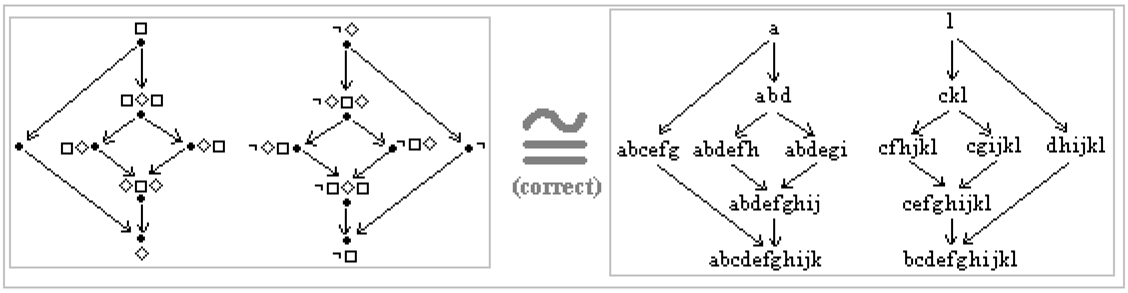
(here we used letters instead of numbers, because we clearly needed 10 symbols or more, which is uneasy with the 10 symbols of the usual decimal number system)

If we now take into consideration *nested* bifurcations things become even wilder. A classical “school case” for us can be that of the Lewis system “S4” (a rather important modal system, especially for epistemic logic). Here is a possible (bad) Pellissierisation of it, one which simply applies the so far known rule for bifurcations.



(one of the many problems here is that, for instance, the □◊ and the ◊□ modalities would collapse together, losing their mutual difference)

An empirical series of “tries and corrections” leads us to the following solution, where some precautions have been taken in order to avoid undesired logical equivalencies.



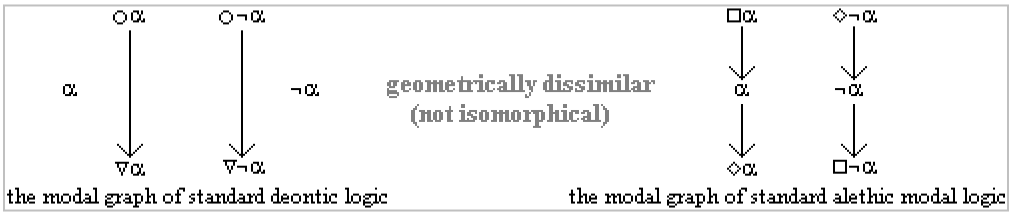
(this solution, conjectured, still has to be checked fully by combinatorial means)

As one sees, this says that the simplest nested bifurcation (as is the one of S4) needs an E set of cardinality 12 (which then maps it to the β11-structure), whereas the standard simplest (closed) bifurcation has an E of cardinality 5 (cf. *supra*).

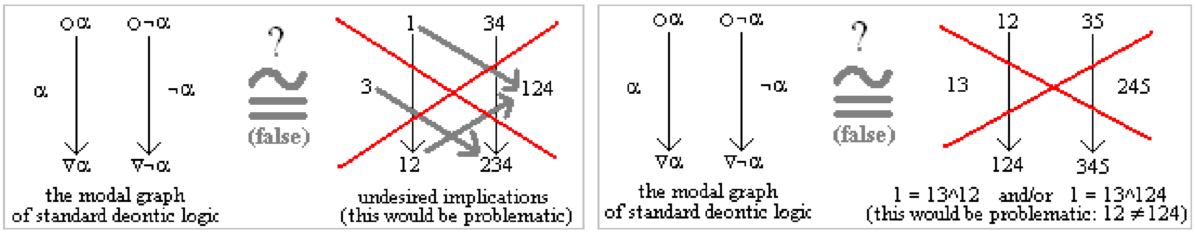
So, we see that the generalisation is problematic: it seems that if one tries to Pellissierise a particular new case of forking, there is a solution to be found, but adding or nesting more bifurcations seems to make the needed rule far from simple. Let us now turn to the second family of non-linear simple modal graphs.

16.02.03. Non-linear modal graphs: isolations (isolated points and arrows)

We define the "isolated points" as being points (i.e. logical positions – one positive and one negative) representing some basic modalities (at least two) which neither have arrows leaving them nor arrows reaching them. There are quite many possible kinds of them (according to their compositions). In the classical modal systems a clear and simple example of modal graph with isolated points is the one of standard deontic logic (Oα means "α is obligatory", ∇α means "α is permitted", O¬α means "α is forbidden", etc.).

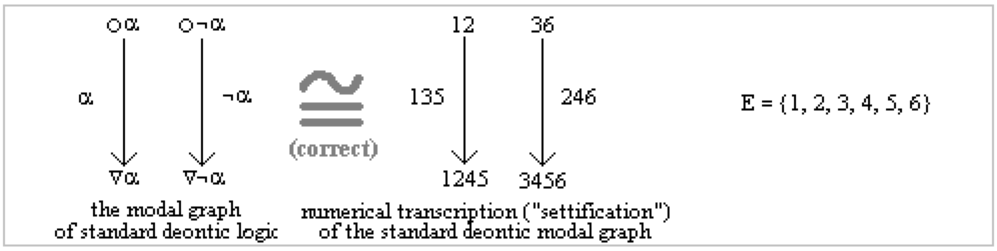


As one can easily see through a graphical examination, many pitfalls wait for those who would simply apply the so far known numbering methods. Here we show two of them (one, on the left, where E = {1, 2, 3, 4} and one, on the right, where E = {1, 2, 3, 4, 5}), both false.



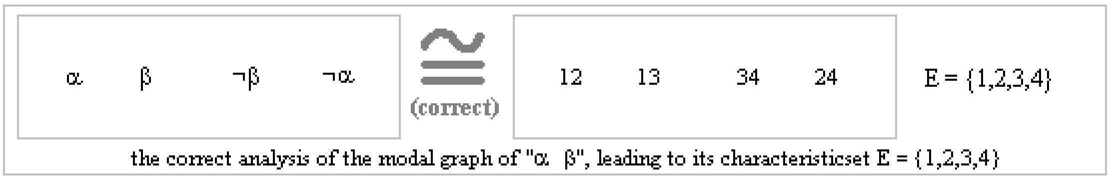
The problem with the first settification (equivalence on the left of the previous figure) is that it generates undesired implications (from 1 to 124, from 3 to 234 and from 12 to 124): these are logically shocking, for they say that some positive basic modalities do imply some negative basic modalities (in such a "deontic logic" law would always be "against reality"). The problem with the second settification (equivalence on the right of the previous figure) is that it generates ambiguities: the same modality (here "1") can be then equivalent to two different modalities (here "13∧12" and 13∧124", with "12" different from "124" – this would mean, deontically speaking: α∧Oα≡α∧∇α, "if something is true, it makes *no* difference whether this thing is obligatory or permissible", which is deontically and morally false!). So these two translations must be abandoned.

This case can be solved, as I proposed by guaranteeing a “1” in all left modalities, a “6” in all right modalities (this prevents undesired arrows), and by imposing to comparable left-right terms (i.e. terms belonging to a same horizontal row) to have the same string-length, in the following way[211].

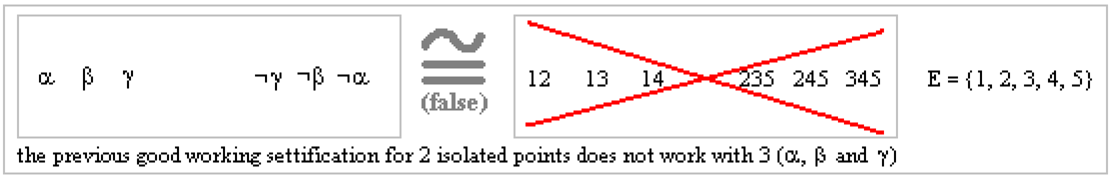


One sees here that neither of the two previous problems (bad arrows and ambiguities) is present now. However, the price paid for this is the assumption of a bigger Pellissier set: E = {1, 2, 3, 4, 5, 6} (whence, of course, a bigger corresponding β*n*-structure, namely a β5-structure).

How are we to generalise this treatment? It turns out that the generalised expression of the setting of the isolated points is less straightforward than the one of the bifurcations (and this bears consequences on the general treatment of the modal graphs). Let us consider other simple cases of modal graphs with isolated points. First two isolated points.



In this case, it seems things are quite analogous with our previous treatment of isolated points (the modal graph of standard deontic logic). But look at what happens if we add a further isolated point.



This does not work because we have once more the problem with ambiguity (here “1” is logically equivalent to three logically not equivalent conjunctions, “12∧13”, “12∧14” and “13∧14”). The solution for this case, as previously, is to adopt a larger E.

16.02.04. The general resisting problem

[211] Cf. Moretti, A., “The Geometry of Standard Deontic Logic”, *Logica Universalis*, 3, 1, 2009.

In fact it turns out that each small variation on a modal graph, with respect to a preceeding one which was mastered as for its translation, generally implies a considerable increase of complexity: new combinatory "tricks" are necessary, while decorating the modal graph with strings of symbols, in order to avoid admitting one of the forbidden logical consequences (violations of the basic modalities, ambiguity in the definition of a modality, undesired implications, etc.). So two facts seem to be interesting: (1) in principle, there should always be some way of decorating in a satisfactory way a given finite modal graph, no matter how big and/or complex; (2) however, in reality this task, seemingly, can quickly become *devilish*, for the complexity of the decoration seems to depend – for combinatorial reasons – on an *exponential* growth from the graphical complexity of the starting modal graph. Both points are only conjectures, to be tested and decided in a near future.

16.04. Toward a table of paradoxically equivalent modal systems

Pellissier's results shows a very interesting perspective: modal systems apparently that are apparently very different may happen to have, from the point of view of the geometry of their oppositions, the same solid (the same β*n*-structure). This means that we have a new kind of equivalence class in our hands. Hence the natural idea of classifying all known modal from this point of view. Put in another way, such an indexing of modal systems over the series of the β*n*-structures could give a new kind of measure of logical complexity.

The underlying idea is double: (1) it could be useful to have some kind of a "map" (a "treasure map" …) for wandering inside the geometrical jungle of modal logic; (2) it could be useful to put into explicit relation modal systems so far unrelated, using the fact that their respective oppositional geometries are isomorphic: this could bring new ideas as for translations of some families of modalities into other.

β2: Sesmat's and Blanché's logical hexagon

β3: the Lewis system S5 (standard modal logic)

β4:

β5: standard deontic logic

β6:

β7:

β8:

β9:

β10:

β11: the Lewis system S4 (conjecture to be tested)

β12:

β13:

…

β21:

We see that by now we know of no couple of different logical systems sharing the same β*n*-structure. Perhaps later (cf. ch. 17 *infra*) we will be able to exhibit one such couple.

16.05. Conclusive remarks

It is not easy to find a simple, all encompassing algorithm for translating all the possible finite modal graphs. Nevertheless, it seems reasonable to think that, in principle, it should be always possible to find a translation for a given finite modal graph, whatever its shape.

But if it turned out (as it seems to be the case) that the complexity of the solution increases exponentially with respect to the complexity of the modal graph, then we should accept the idea that in general it is hard to translate geometrically the modal systems characterised by a large modal graph. Possible but hard.

17.
KNOWN APPLICATIONS OF *N*-OPPOSITION THEORY

17.01. Preliminary remarks on applications of N.O.T.

We saw how the static notion of opposition (cf. ch.1 *supra*) can be taken to a more general stage, stepping from the classical 2-opposition (the logical square) to the general *n*-opposition (the logical bi-simplex of dimension *m*, with *m*=*n*-1). We saw that this implies distinguishing between three similar but in fact conceptually different kinds of structures: modal graphs (i.e. γ-structures, cf. ch. 11 *supra*), oppositional bi-simplexes (i.e. α*n*-structures) and gatherings of bi-simplexes (i.e. β*n*-structures, cf. ch.15 *supra*). And we saw that a general translation rule matches, at least in principle, each finite modal graph to some element of the series of the β*n*-structures (cf. ch.16 *supra*), which in turn gives the complete series of the α*n*-structures available for that modal graph. What can the use of all this be? It seems that the principal use of NOT consists in clarifying logically (oppositionally) conceptual situations. It does so by helping to find: (1) the good relations among the known terms and (2) the still invisible terms (and relations).

17.02. Examining anew some of the old cases

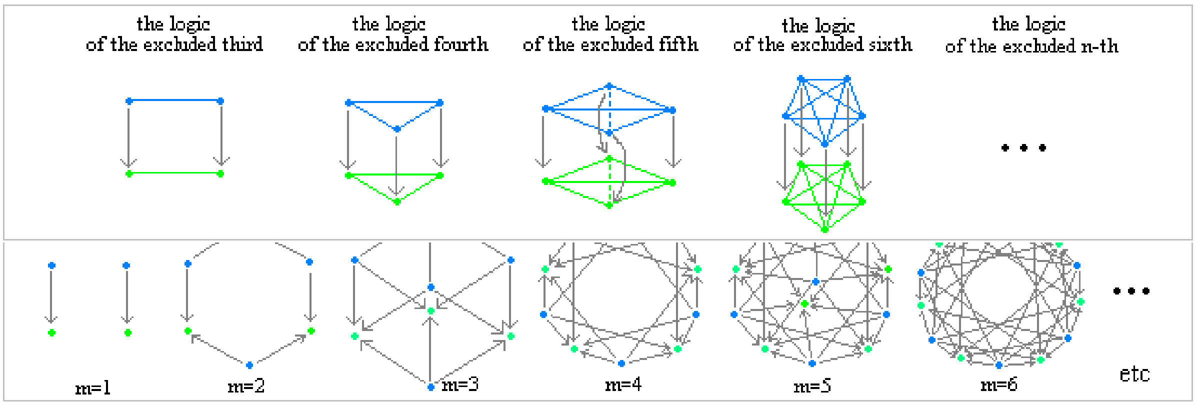
As for the possible applications of the geometry of the logical oppositions, we saw so far (cf. chapters 5, 6 and 9 *supra*) only the possible known applications of the logical square and of the logical hexagon. Here we examine if the notion of logical bi-simplex of dimension *n* and NOT in general do help us in pushing further the discovery of possible applications. It seems they do.

17.02.01. Back to Vasil’ev: his imaginary logic is not enough

Vasil’ev was very close to discovering the logical hexagon and he conjectured, of his imaginary non-Aristotelian logic, a possible extension in a theory of *n*-dimensional logic. However, the parameter “*n*” for him was not related to the *n*-dimensionality of the so-called hyper-space (his geometrical model being non-Euclidean geometry and not *n*-dimensional geometry), but to a purely logical generalisation of the “principle of the excluded third” to a “principle of the excluded *n*-th”. Of this, as we saw (cf. ch. 7 *supra*) he explored the first step, i.e. the move from a logic of the excluded third to a logic of the excluded fourth (1911). In a paper on the Vasil’evian systems of “imaginary non-Aristotelian logic” the Rusian (then Soviet) logician Smirnov tries to show in which way Vasil’ev’s claimed intuition could really lead to such a series of logical "*n*-valued" systems[212]. He didn't explicitly draw this series, but what he writes suggests something like this (except for the use of the simplexes: probably both Vasil'ev and Smirnov would have used simple bi-dimensional polygons, i.e. a square instead of a tetrahedron, etc).

But now, thanks to NOT (cf. ch. 11 *supra*) we can understand now that this is unduly restrictive. For, the real series (of successors of the logical square) is the following.

The difference is that Vasil'ev (and Smirnov with him), keeping a one-one arrow



correlation between contraries and subcontraries forgets many arrows. This is due to the (paradoxical) attachment of Vasil'ev to Aristotle: for, Vasil'ev remains attached to syllogistics and to the fact that he could not help himself by finding a help in the model of the logical hexagon.

Therefore, "Vasil'ev's series", if we name thus the series of the modal *n*(2)-graphs, is just a fragment of a much bigger theory. As a matter of fact, having studied Vasil'ev (around 2001) helped me in making my own discoveries (it gave me the practice with drawing temptative geometrical figures, especially logical 3-dimensional trihedra as the one of the system "IL2")[213].

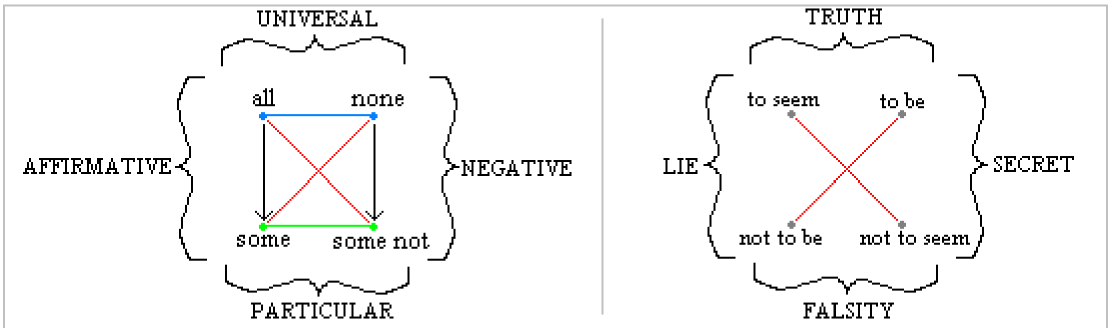
17.02.02. Back to Greimas: a whole series of semiotic bi-simplexes?

The NOT turns out useful for understanding and expanding Greimas at two levels at least: the square of veridiction and (more deeply) the semiotic square.

[212] Cf. Cf Smirnov, V.A., "Logicheskie idei N.A. Vasil'eva i sovremennaja logika" ("Vasil'ev's logical ideas and contemporary logic") (1989), contained in Vasil'ev, N.A., *Voobrazhaemaja logika. Izbrannye trudy*, (*Imaginary logic. Selected works*). An English translation is available in: J.E. Fenstad et alii (eds), *Logic, Methodology and Philosophy of Science VIII*, Elsevier, 1989. A French translation by us of the Russian original to be published in *Lec Cahiers de l'ATP*, (internet).

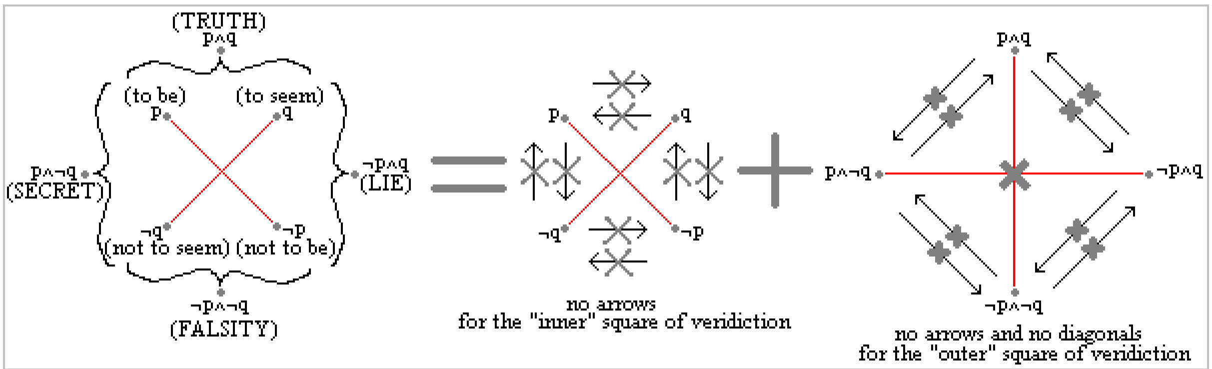
[213] Cf. A. Moretti, "A graphical decision procedure and some paraconsistent theorems for the Vasil'evian logic IL2", in: J.-Y. Béziau , W.A. Carnielli and D. Gabbay (eds.), *Handbook of Paraconsistency*, King's College, London, 2007.

First, it helps understanding the true oppositional nature of Greimas' "square of veridiction". We already saw (cf. ch. 6 *supra*) how this structure functions. Now, if it shares some similarities with Aristotle's square, it is evident that it is not fullly isomorphic with it: in Greimas' case, only the contradiction relations are present, not the three other kinds of oppositions (in the next figure, left side, Aristotle is made ambiguous by us but similar to Greimas).



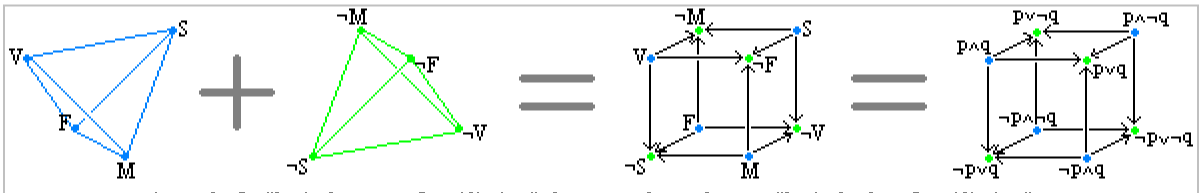
Aristotle's square, compared to one of its incomplete counterparts, the Greimasian "square of veridiction"

This analysis can in fact be made more precise, by separating the two distinct (and nested) squares present in Greimas' square of veridiction (for there is an inner and an outer one).



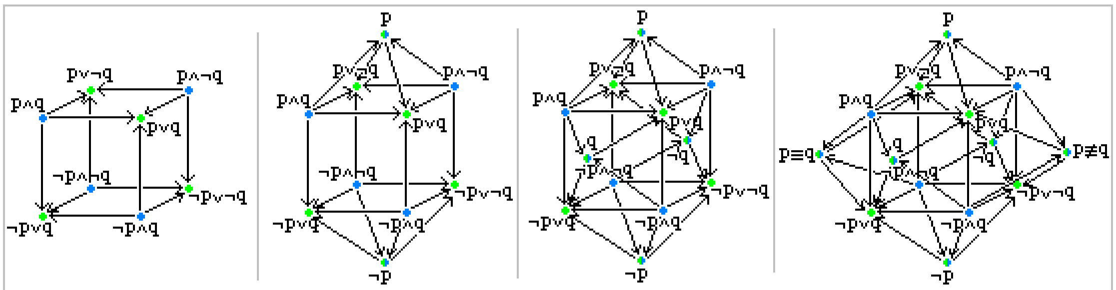
One can see that the inner square has only the contradiction relations, whereas the outer one (emerging from the first by means of logical conjunction) has not even this relation.

If we take the symbols V (*verum*), F (*falsum*), S (*secretum*) and M (*mendax*) for expressing the meanings "true", "false", "secret" and "lie" of Greimas' outer square, we can in fact construct (or put into evidence) the following "Greimasian cube" (a true logical cube decorated "à la Greimas").

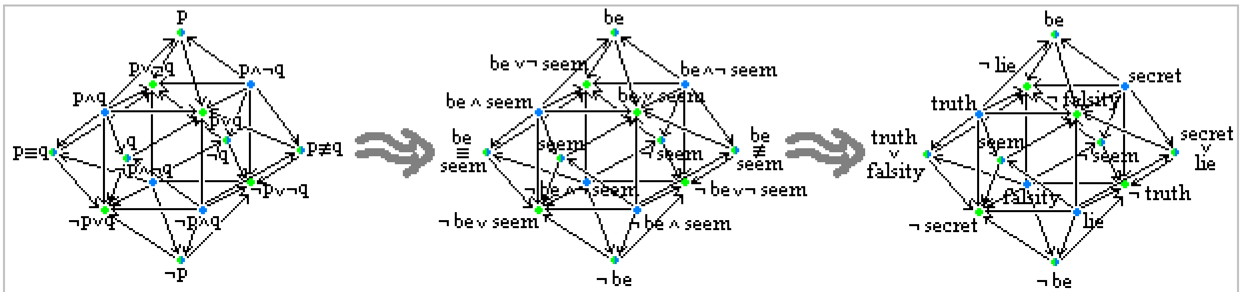


instead of a "logical square of veridiction" there must be, at least, a "logical cube of veridiction"

But the six faces of this cube let emerge, each, a “Pellissierian pyramid”, the whole letting emerge a logical tetraicosahedron displaying 14 over the 16 binary connectives of the propositional calculus (to this propositional result we return in a while with D. Luzeaux).

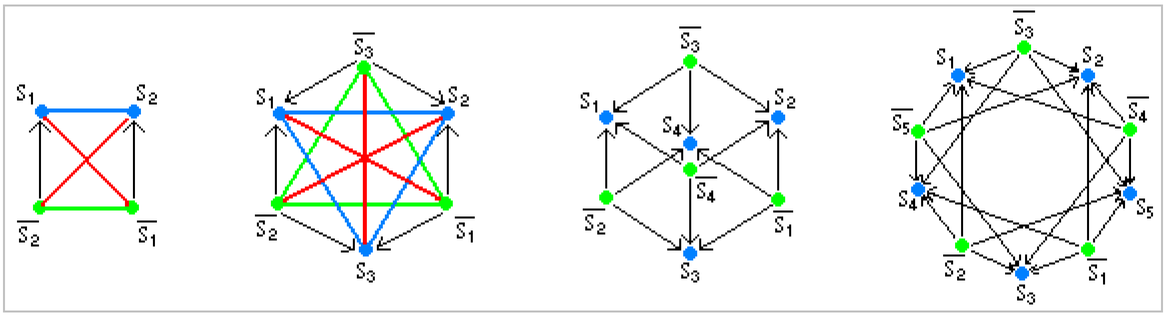

adding, face by face, natural new terms to the "logical cube of veridiction"

This gives, rightly translated, a “Greimasian tetraicosahedron of veridiction”.



an unsuspected "Greimasian tetraicosahedron of veridiction"

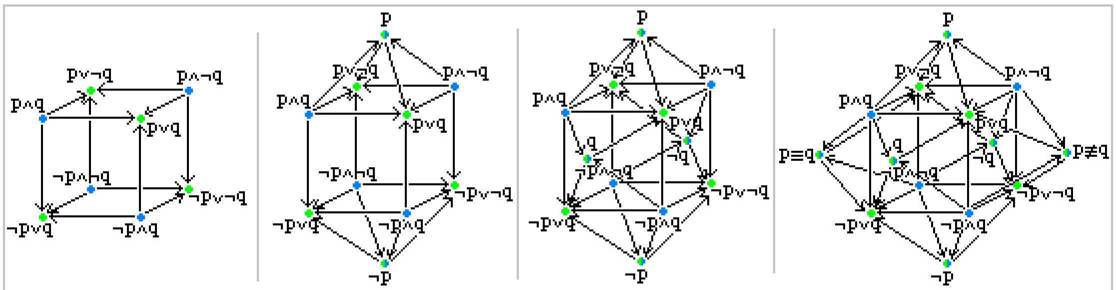
The second possible use of NOT for thinking Greimas newly concerns, this time, his “semiotic square”. We saw (cf. ch. 9 *supra*) that one could be tempted to express Greimas’ ideas not with a logical square (duly modified “à la Greimas”) but with a logical hexagon (possibly modified in an analogous way). Now, assuming this last move to be possible, one could be tempted to lead the extension further, introducing a whole series of “semiotic bi-simplexes” (obtained by modifying “à la Greimas” the whole series of the *logical* bi-simplexes).



At the level of this work, this can only be a suggestion, which will have to be verified more accurately in a coming specific work. It seems remarkable however, *a posteriori*, that in his impressive book on Greimas' theory[214], the French philosopher and mathematician Jean Petitot (a pupil of the famous French mathematician René Thom) *never* mentions Sesmat's and Blanché's logical hexagon, whereas Greimas does indeed[215]. This seems related to the disgrace of (logical) structuralism (cf. § 06.02 *supra*): Petitot tries to develop some kind of (neo-)neo-Kantian way (hence the subtitle of his book) based on Thom's catastrophe theory (continuous mathematics instead of discrete mathematics). But then he loses all the (very structuralist) possibilities that actually NOT seems to discover and develop.

17.02.03. Back to Blanché: the geometry of the binary connectives

In chapter 8 above we saw that Blanché used his discovery of the logical hexagon in order to give a new expression of the space of the logical relations relating the binary connectives of propositional calculus. He took in consideration "the 10 most important ones" and expressed all their reciprocal relations by means of a chain made of two logical hexagons. But in the previous paragraph we also saw that, developing Greimas' strange "square of veridiction", we get to a decoration of the logical tetraicosahedron which orders perfectly 14 over the 16 binary propositional connectives (excluding the "T" and the "⊥" – we saw, ch. 12, that Smessaert identified magistrally these two with the centres of symmetry of the solid).

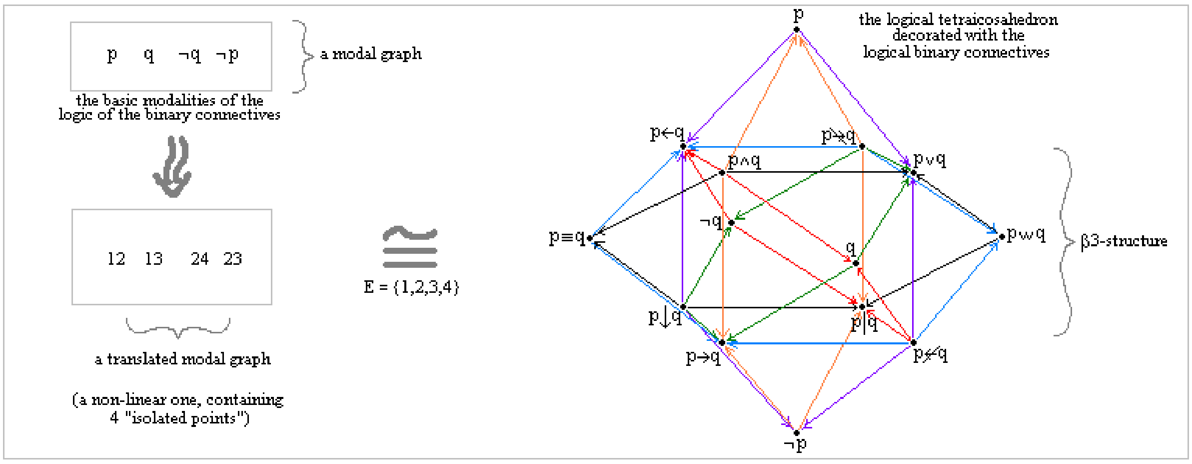


from the "cube of veridiction" to the "tetraicosahedron of binary connectives"

This result seems to be very interesting for three reasons at least. First, it gives a model of the geometry of the binary connectives which is better than Blanché's (in fact a conservative extension of it). Second, when observed from another point of view (the point of view of the general translation rules of NOT, cf. ch. 16 *supra*), it shows retrospectively how this problem could (should) have been treated systematically (thanks to the NOT methodology). As a matter of fact, if we consider propositional logic (as being the theory of the couples of logical

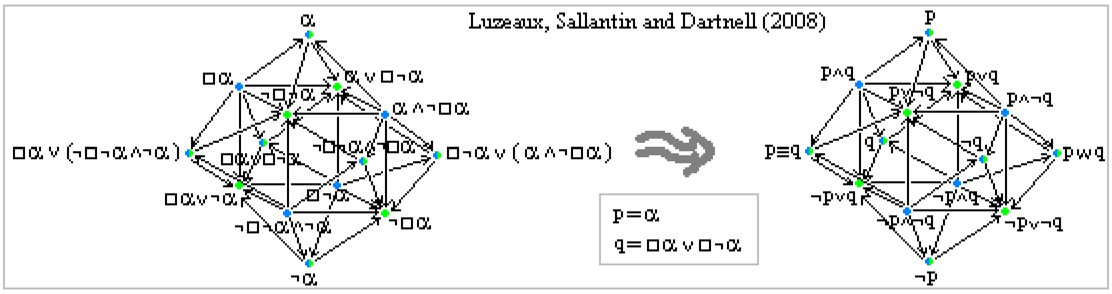
[214] Petitot-Cocorda, J., *Morphogenèse du sens – I. Pour un schématisme de la structure*, PUF, Paris, 1985.

atoms) from a modal point of view, that is looking for its *modal graph*, we see that such a modal graph is one made of two unrelated atoms (with their respective negations). The NOT general translation rules (cf. ch. 16 *supra*) tell us how to handle this.



Third, this shows us that Blanché's chain of two logical hexagons is in fact a fragment of the β3-structure. This means that, seemingly, Blanché was the first known person ever to have handled some kind of β-structure. This also shows, if it were still needed, the importance not to forget the null modalities: despite their non-modal appearance, the null modalities *are indeed* modalities!

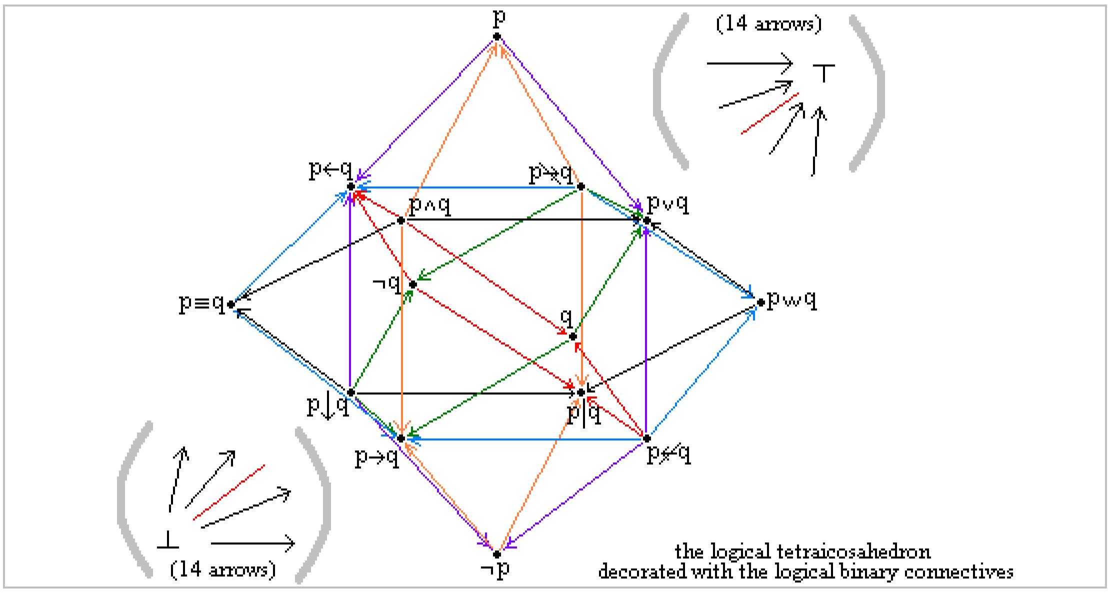
Remark that this tetraicosahedron of the binary connectives has been discovered independently by the French mathematician Dominique Luzeaux in a joint paper with the mathematician Jean Sallantin and the computer scientist Cristopher Dartnell[216].



They proceeded, however, in another way: starting from Pellissier's modal reading of the logical tetraicosahedron (cf. ch. 12 *supra*) and substituting "α" with "*p*" and "□α∨□¬α" with "*q*".

[215] On some relations of Greimas with Blanché (and his hexagon), cf. Bonfiglioli, S., "Aristotle's Non-Logical Works and the Square of Oppositions in Semiotics", *Logica Universalis*, 2, 1, 2008.
[216] Cf. Luzeaux D., Sallantin J. and Dartnell C., "Logical Extensions of Aristotle's Square", *Logica Universalis*, 2, 2008. The day before Sallantin was going to present, among others, this result of his joint paper with Luzeaux and Dartnell, I was able to show him my personal notes leading me independently to the same discovery.

Now, as already hinted to in § 12.02 *supra*, their approach, distinct from mine, gives rise to an important discussion here, which consists in asking the question: is this "binary connectives tetraicosahedron" imperfect? For this seems to be the position of Luzeaux, Sallantin and Dartnell.

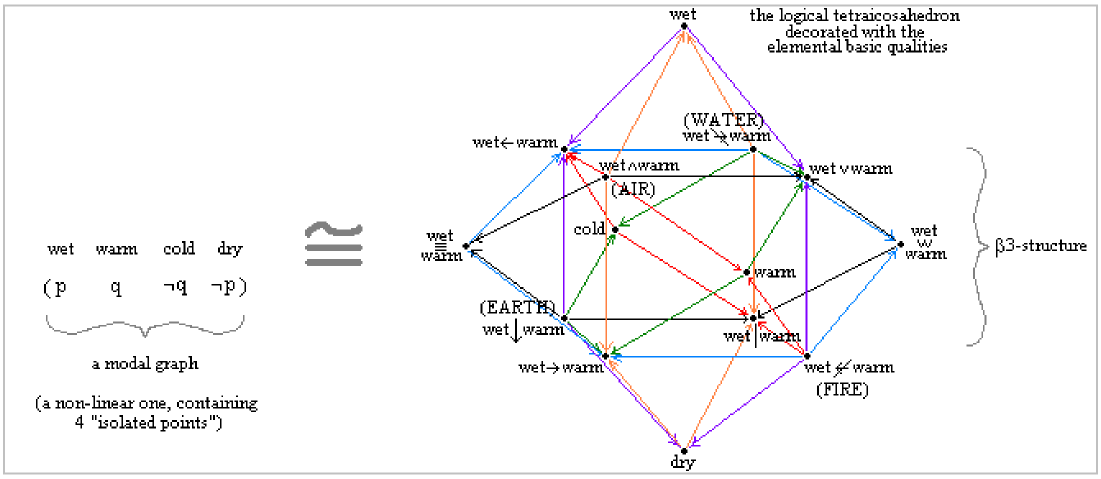


I strongly claim, Pellissier on my side, that it is mistaken not to take *n*-opposition theory as a regular approach of its own, even concerning the binary connectives. The tetraicosahedron of the binary connectives is superior to the other known approaches (like the hyper-cube of Woodgers and its avatars of nowadays) because it is able to exhibit clearly and explicitly *all* the existing opposition relations holding between any arbitrary couple of its 14 elements. The fact that two elements are absent from the model (the tautology "T" and the self-contradiction "⊥") is a misleading appearance. These two cases must not be forgotten – and thanks to Smessaert's construction (a model NOT has adopted) we know by now exactly where they are: right in the middle, in the symmetry centre of the oppositional solid –, but they are two extrema (they do not take part to the "regulated life" of the oppositions). The models integrating these two cases (i.e. out of the symmetry centre, i.e. letting them be two geometrically distinguished points like all others) are unable to express *many* of the existing relations between connectives. NOT does it better.

17.02.04. Back to the traditional "opposition of the four elements"

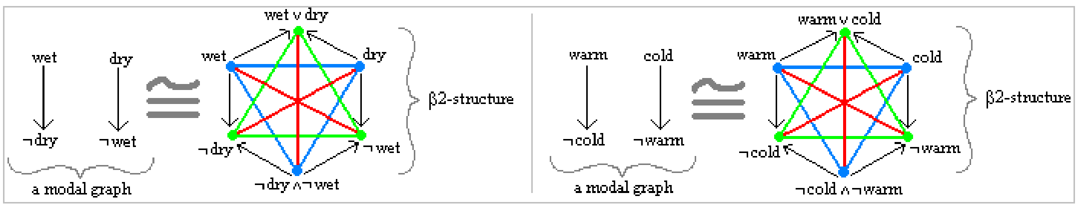
If we now come back to the puzzling traditional scheme of the "oppositions of the four elements (and of the four qualities)" (cf. ch. 04 *supra*), it seems that NOT helps us in understanding it better, and in better expressing the real (much more complex) geometry of its underlying logic.

As a matter of fact, one (first) easy way to understand it would be to put it as isomorphic to the two cases just seen (the "tetraicosahedron of veridiction" and the "tetraicosahedron of the binary connectives"). Which would give the following reading.

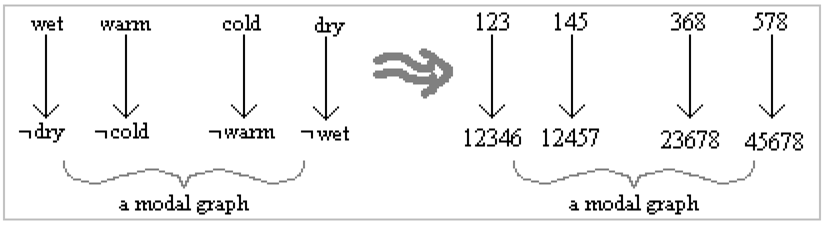


This reading seems faithful to the traditional meaning in so far as the result in it is that the four elements (air, water, fire, earth) form a tetrahedron of contrariety: they are, two by two, mutually contrary.

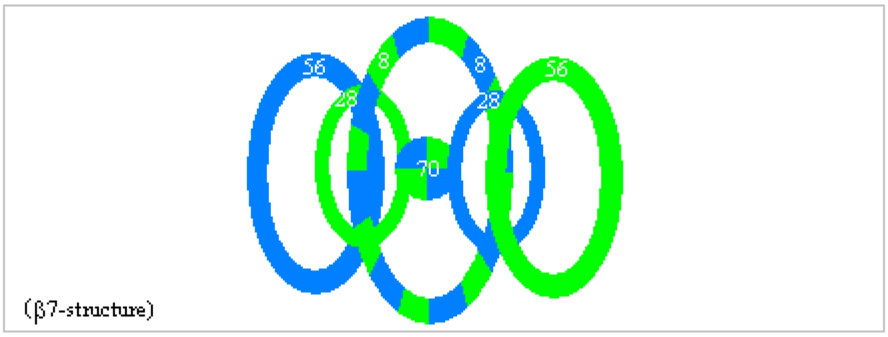
However, this construction holds only if one accepts to consider, as seemingly tradition (including Aristotle?) did, that "warm" is contradictory to "cold", and that "dry" is contradictory to "wet". But one, following logical common sense (especially after NOT), could consider that rather than *contradiction* the relevant opposition relation here is *contrariety*, which means that the opposition wet–dry as well as the opposition warm–cold form each a logical square (and in fact a logical hexagon).



In virtue of the properties of the combinations of modal graphs according to NOT (cf. ch. 16 *supra*), this will give the following global modal graph (and its Pellissier set E).



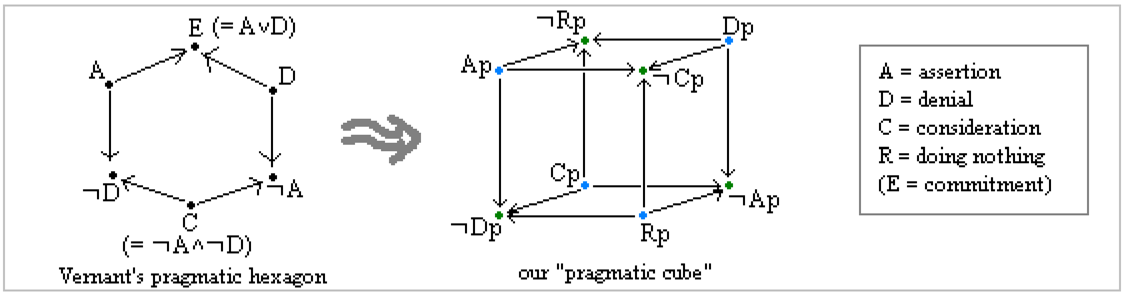
So, from this point of view, it is not the logical tetraicosahedron (the β3-structure) that we are going to find as a solution to the problem of the elemental opposition, but rather the β7-structure (for here we have E={1,2,3,4,5,6,7,8}), which we studied partly in ch.14 above.



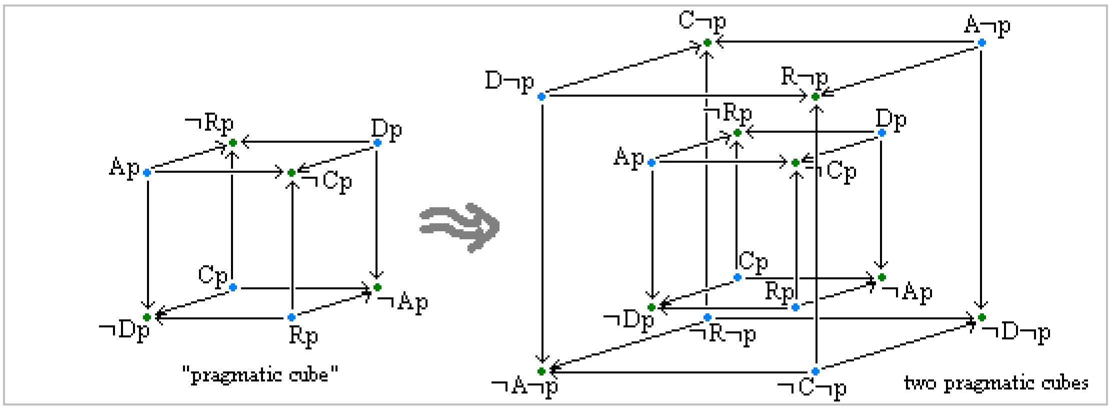
(β7-structure)

17.02.05. Back to Vernant: the pragmatic quasi-hyper-cube

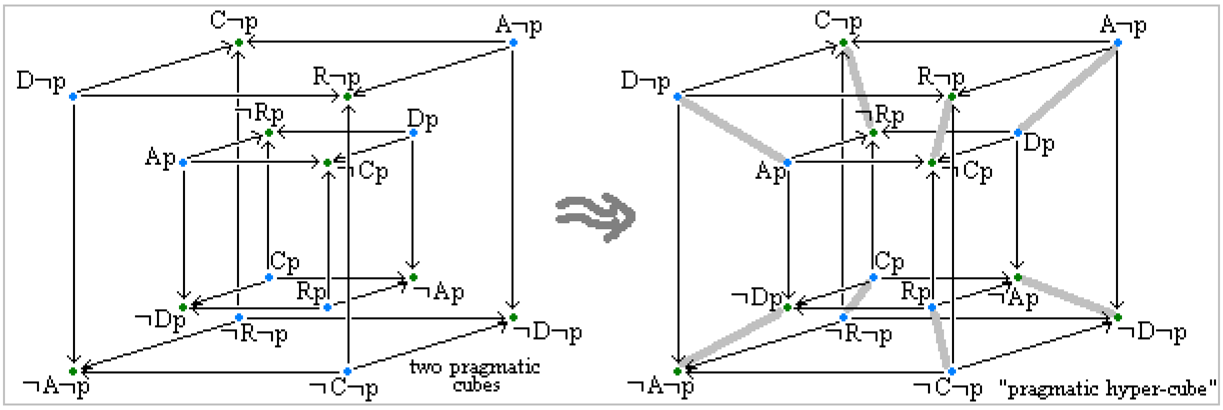
As we saw in chapter 9 above, Vernant (2003) has put forward a very interesting application of Sesmat-Blanché's logical hexagon to the concepts of pragmatics. Now, I argue that, in the same spirit, one could (and should) add a fourth term to Vernant's pragmatic 3-opposition, namely one denoting the "null action", that we will call "R" (from the French "rien", which means "nothing"). This gives a tetrahedron of contrarieties, relating assertion, denial, consideration *and* the "no move" term. This in turn gives, according to N.O.T., a logical cube of opposition which is an hypothetical "pragmatic cube".



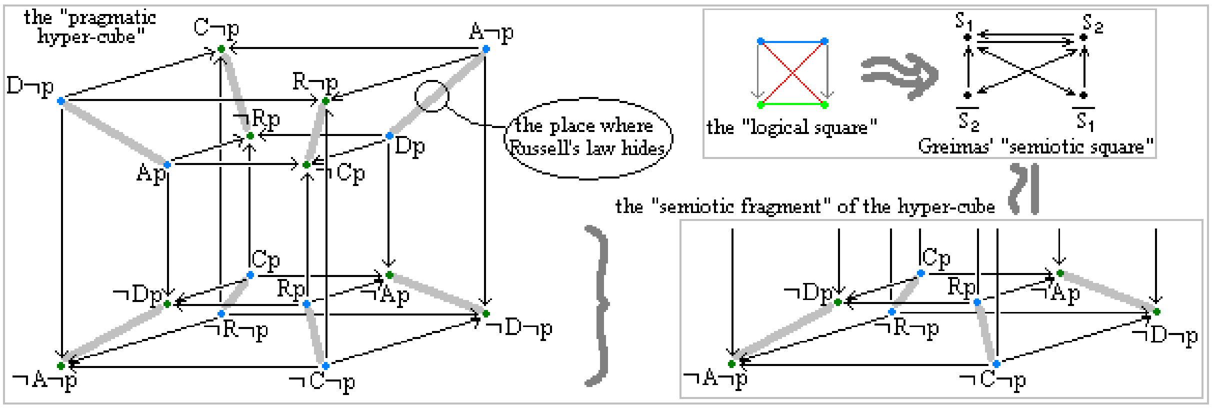
However, the very *pragmatic* nature of the opposition of "assertion" and "denial" prohibits some equivalencies usually available *logically*, as for instance Ap↔D¬p, or Dp↔A¬p. These equivalencies are weakened into conditionals, for, some implications intuitively (i.e. pragmatically) do hold, as "Dp→A¬p" ("Russell's law"), but not all of them. But this problem remains unclear not only for the pragmatic hexagon, but even for the "pragmatic cube": this structure (as well as Vernant's hexagon) totally avoids the question, by containing only "p" atoms (and not the "¬p" ones). And we cannot simply invoke 8-opposition (i.e. taking all Xp and X¬p modalities, with X∈{A,D,C,R}, as two by two contrary inside a logical bi-simplex of dimension 7) because this would imply abandoning useful things like Russell's law (for: between two contrary things – as would be "Dp" and "A¬p" – no implication could live). So, a new strategy is needed in order to carry on a formal (geometrical) analysis of this part of pragmatics. But then it seems natural to simply introduce a second pragmatic cube, for the "¬p" atoms this time.



From that, it turns out that these two "parallel" (or nested) pragmatic cubes can be combined in some kind of "pragmatic hyper-cube" (a 4-dimensional structure). In this new model (the hyper-cube) the 8 thicker grey edges, *which are not yet arrows* (this point is the one to be explored), express the psychological (linguistic, pragmatic) non purely logical relations holding between the 16 correlative terms of the pragmatic weakening (like Russell's law, etc).



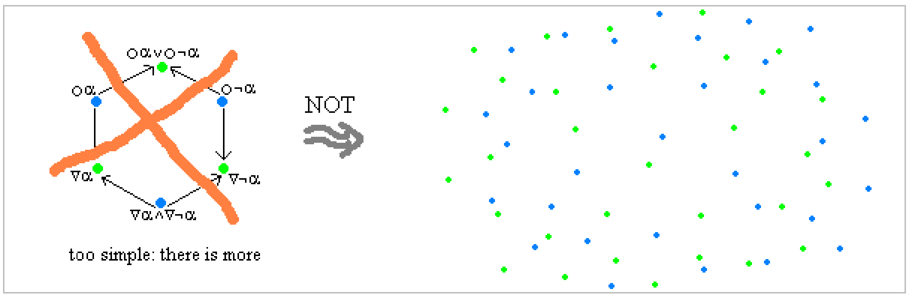
A further advantage with this model is that it contains, as a fragment, a seemingly Greimasian structure. As a matter of fact, the relation relying Cp and ¬R¬p (and similarly) seems to denote the semiotic notion of “co-implication” of Greimas’ theory (cf. ch. 06 *supra*), where this was tentatively expressed by his puzzling “semiotic square” (as a co-implication relation between S1 and S2). So our model seems to offer a possibly interesting basis for discussing some pragmatic extra-logical features of, so to say, “natural logic” (the logic of “real” thought and speech) and could serve as a field for trying to merge semiotics (in Greimas’ sense) and pragmatics.



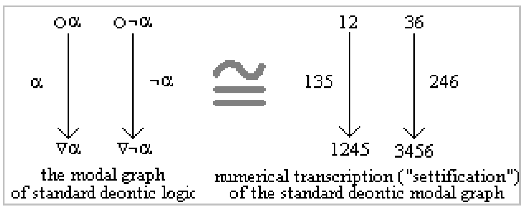
All this must be investigated further (especially the meaning of the thicker grey edges).

17.02.06. Back to Kalinowski: the geometry of standard deontic logic

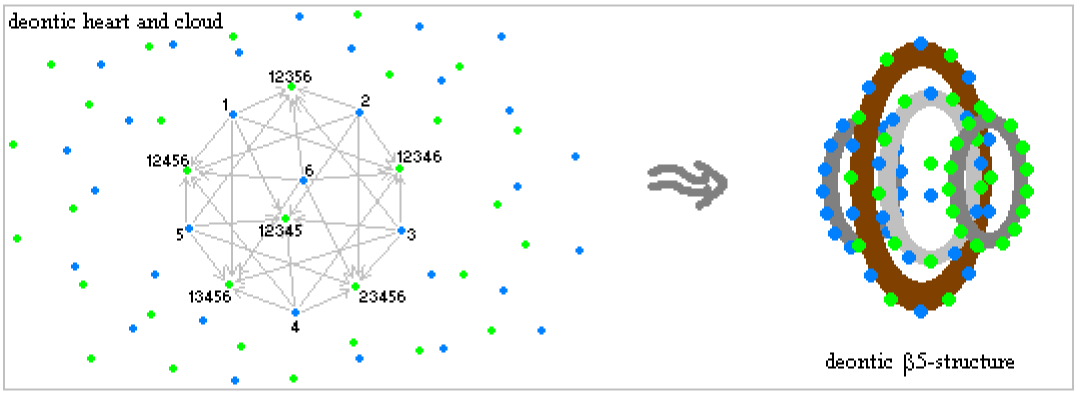
In ch. 09 above we mentioned the existence of an application of the logical hexagon to deontic logic by Kalinowski (1972). Now, NOT allows us to see that, even if this deontic hexagon in itself is perfectly correct, it is mistaken to believe (as seemingly Kalinowski did) that such a deontic hexagon expresses all the oppositions of standard deontic logic.



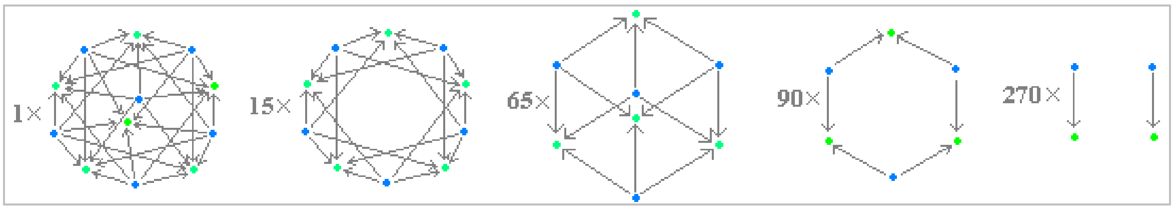
To do this, we have to come back to the modal graph of standard deontic logic.



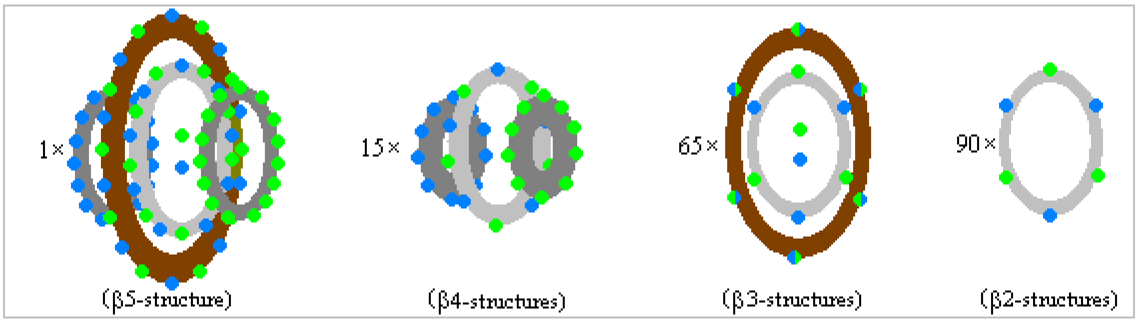
As we already saw (cf. ch. 16 *supra*) such modal graphs with "isolated modalities" can be handled by NOT. Because the Pellissier set of this modal graph is E = {1,2,3,4,5,6}, the corresponding β*n*-structure is the β5-structure, whose heart is a deontic α6-structure (with its cloud).



This β5-structure of standard deontic logic contains the following kinds and number of α*n*-structures (i.e. logical bi-simplexes).



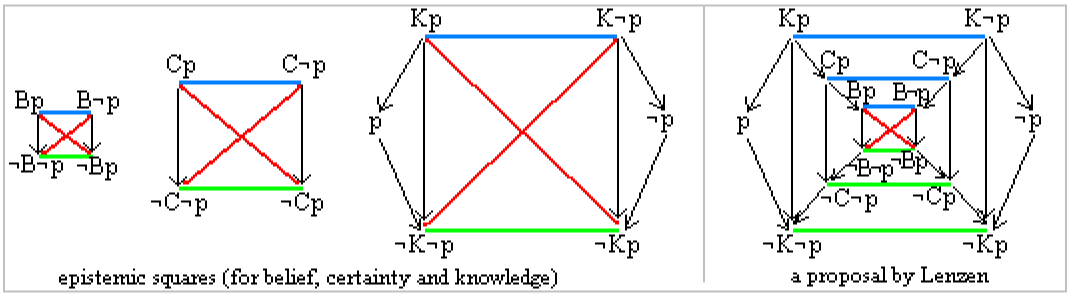
The kinds of β*n*-structures contained in standard deontic logic are, finally, the following.



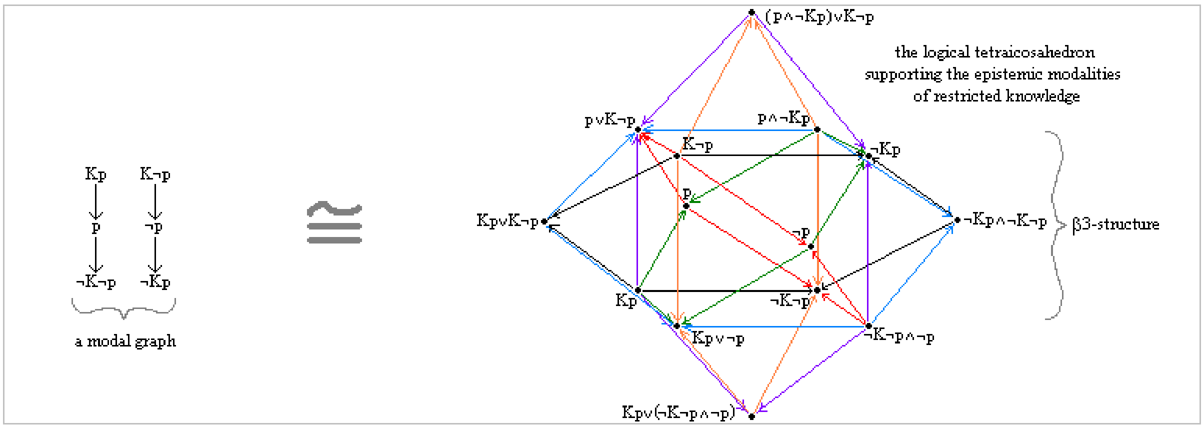
This is much more than Kalinowski's unique deontic hexagon (there are in fact 90 deontic hexagons!)[217].

17.02.07. Back to general epistemic logic (following a hint by W. Lenzen)

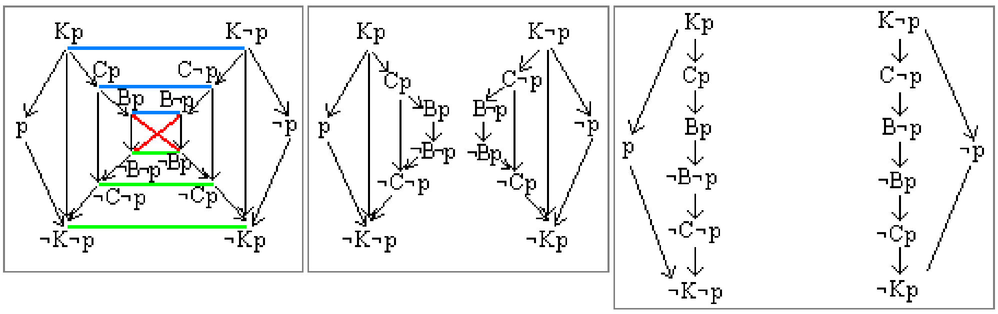
Another example of possible application of NOT concerns epistemic logic (a modal logic very important for artificial intelligence). As we saw (ch. 05 *supra*), following a suggestion of W. Lenzen, the main epistemic modalities can be displayed geometrically into some kind of "trice nested logical square".



Now, this can be analysed in several ways, the most intuitive of which being the separate analysis of each of the three (nested) epistemic squares. If we thus restrict ourselves to the "knowledge fragment" of epistemic logic (that is the one square also admitting two outer modalities, the null ones, i.e. "p" and "¬p"), than we have a modal graph which is isomorphic with the one of S5 (cf. ch. 12 *supra*) and thus we have, as geometrical solid counterpart of it, an "epistemic tetraicosahedron".

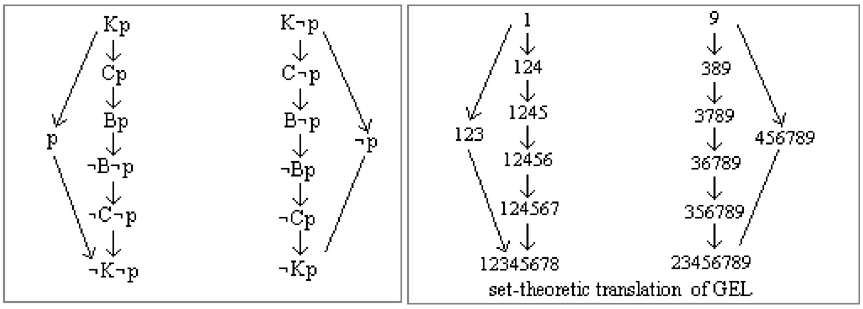


But this analysis is incomplete with respect to the "certainty" (Cp) and to the "belief" (Bp) modalities (and their respective squares). So, a better, more systematic technique seems to be one which respects the NOT methodology. To do this, we must first try to know which general modal graph (or γ-structure) we are speaking of. From this point of view, it turns out that what we could call "Lenzen's modal graph" for general epistemic logic (centre of the next figure) can be simplified, taking into account the property of transitivity of the arrows of the modal grtaphs (right side of the next figure).



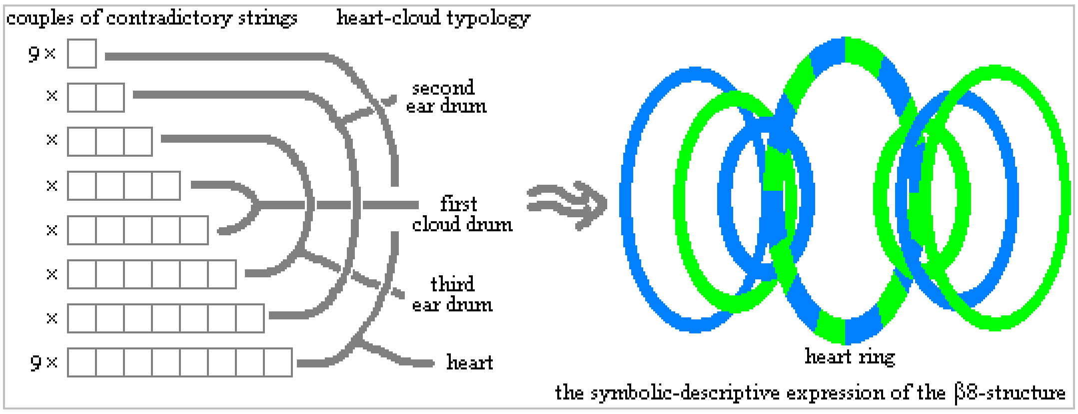
[217] Cf. our study: Moretti, A., "The Geometry of Standard Deontic Logic", *Logica Universalis*, 3, 1, 2009.

Now, having an acceptable modal graph (right side of the previous figure), that is one which cannot be further simplified (in terms of arrows), we can apply to it Pellissier's technique (as extended to the case of the bifurcating modal graphs, cf. ch. 16 *supra*), which gives the following result.



set-theoretic translation of GEL

Thus, here we have E = {1,2,3,4,5,6,7,8,9}, which means (cf. ch.15 *supra*) that the β*n*-structure corresponding to this logical space (the one of general epistemic logic, at least in the sense of Lenzen), provided our Pellissierisation is the good one (this has to be proven), is the β8-structure. According to the technique we proposed in ch. 15, this can be represented, half descriptively and half symbolically, like this.

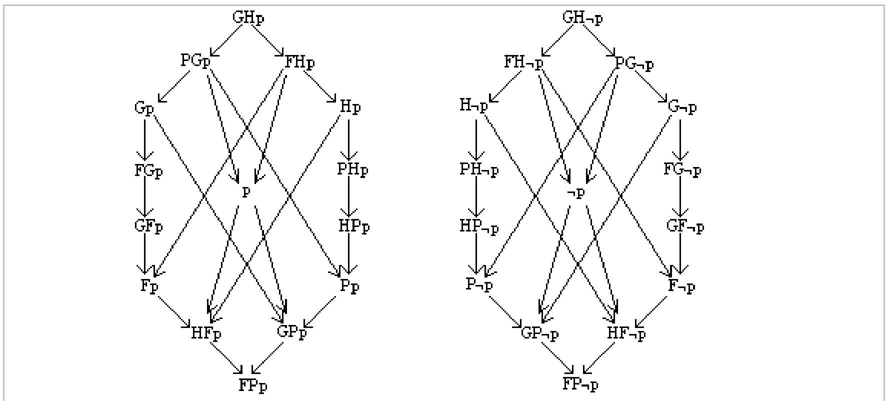


the symbolic-descriptive expression of the β8-structure

Of course, this is just the beginning of a possible NOT-study of general epistemic logic.

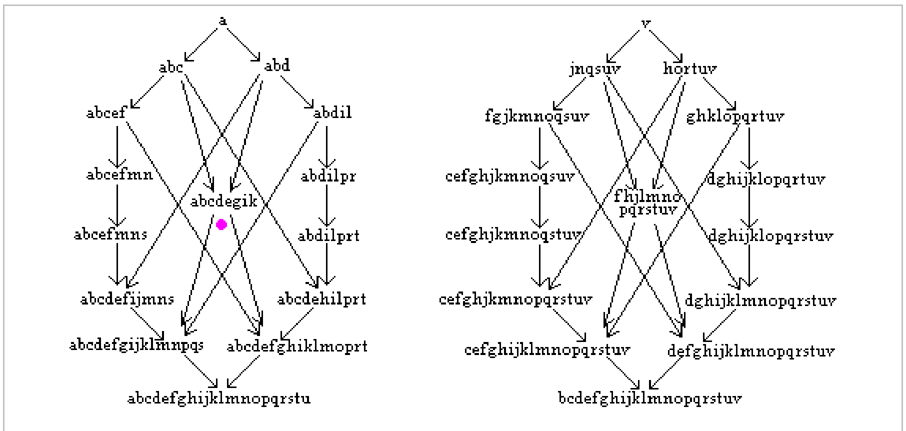
17.02.09. Back to tense logic (Hamblin and Prior revisited)

When speaking about the logical square, we mentioned that modal logic still uses it, and recalled how a double version of it is used by some (namely J.-L. Gardies) for modelling the modal logic of tense by way of two tense squares (ch. 5 *supra*). Here we want to show how, thanks to NOT, one can go much further. In 1965 Hamblin and Prior propose the following "implicative structure" (in fact a non-linear modal graph) for the non-metrical tense-operators[218].



Hamblin and Prior's "implicative structure" for non-metrical tense logic (1965)

According to N.O.T., this is the modal graph of the basic modalities of standard tense logic. It seems to give, once duly settified (with letters, numbers are not sufficient here), a set E of cardinality 22.



E = {a, b, c, d, e, f, g, h i, j, k, l, m, n, o, p, q, r, s, t, u, v}

This would mean (but such Pellissierisation still has to be checked) that, in order to have a complete knowledge of the possible oppositions one has to use the 21-dimensional β21-structure.

17.03. Examining some new cases through NOT

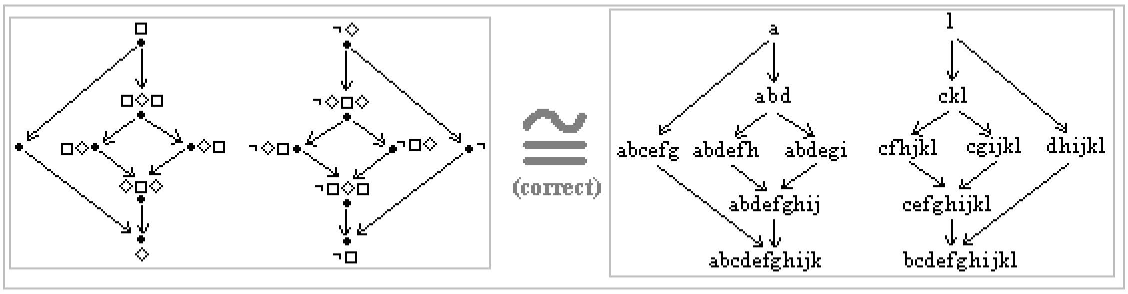
Until now, wanting to explore the possible applications of NOT, we came back to cases we had already seen, involving originally either logical squares (as in Greimas) or hexagons (as in Blanché, Kalinowski, Gallais, Vernant, …) or modal graphs (as in Lewis, Hamblin and Prior, …), or to fragments of β*n*-structures (as in Blanché). Now we turn to cases which are new, mainly so because either they concern oppositional structures (α*n*-structures) superior (as for the dimensionality of their simplexes) to the logical square and hexagon (as will be the case with Joerdens), or because they concern theories which nobody

[218] Øhrstrøm, P., "A.N. Prior's Rediscovery of Tense Logic", *Erkenntnis*, vol. 39, pp. 23-50, 1993.

had tried to study from an oppositional point of view (as with Badiou); another class which is new is the one of oppositional graphs which do not seem to be logical bi-simplexes at all (as will be with McNamara, Wessels, …). In all cases it seems that NOT brings clear light and order on the fundamental underlying oppositions.

17.03.01. The Lewis system S4

The system S4 historically is quite important for philosophy. It differs from S5 (the "universal system") in so far it better expresses some notions of epistemic logic. As we saw in ch. 16, the modal graph of S4 is made of two nested bifurcations, and its correct Pellissierisation seems to be the following one.

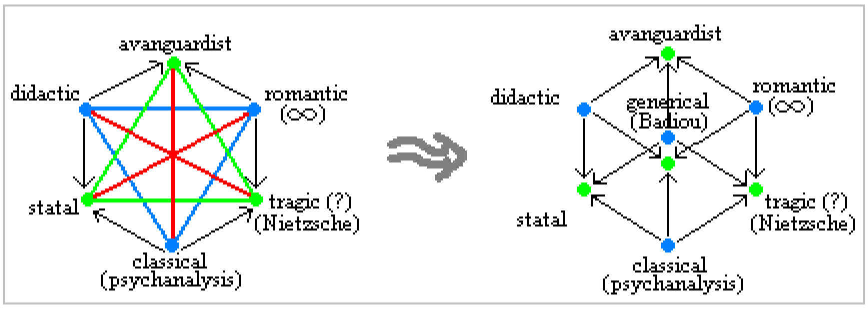


(as far as we know, no problem arises from this settification – no undue implications or logical equivalencies – but we cannot give any proof of this yet)

Because its characteristic set E seems to be equal to {a,b,c,d,e,f,g,h,i,j,k,l} – that is, it is of cardinality 12 – the resulting β*n*-structure is the β11-structure.

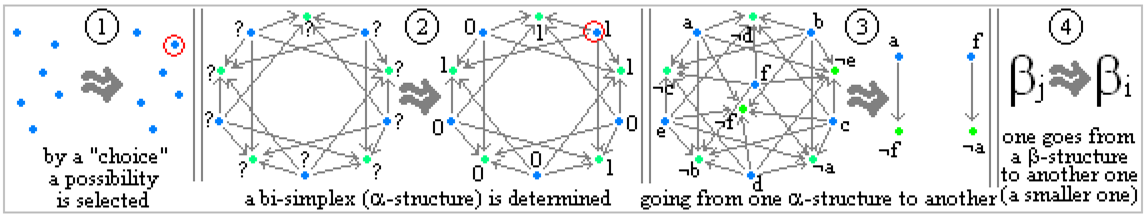
17.03.02. Formalising contemporary structuralist approaches with NOT

As already mentioned, the French Alain Badiou (cf. § 06.02.02 and § 06.06.06 *supra*) is an impressive contemporary ("continental") philosopher who proposes a large number of new models of classical philosophical topics (such as the notions of "event", "acting", "subject", …), models which he grounds, mathematically, in category theory. One of such models of him (one outrageously simple formally speaking, and with no links with category theory – just a partition) offers us a good opportunity for understanding how NOT can be useful to philosophy. But also it can help us understand what can be the limits of such a theory of opposition (limits in grasping changes). Badiou claims (and explains) that in the history of mankind (at least in the West) there have been three major paradigms for the philosophy of art: so we can represent it as a 3-opposition (i.e. a logical hexagon). Then he introduces, by a complex reasoning based on his philosophy of the event, his own new model of a theory of art: so we (= users of NOT) represent it as being alternative to the previous 3 choices, and thus we introduce as whole representation of this discussion a 4-opposition (i.e. a logical cube).



The question here is to judge if NOT is a good tool for expressing this kind of models. And what can be the utility here of our formalisation of Badiou's philosophical ideas? In my opinion the first utility is to show that, if Badiou is to be taken seriously, there are empty places left unoccupied by this conceptual scheme. In this sense, NOT can help get more complete. Badiou, being structuralist (i.e. believing that there are abstract, invisible but powerful formal constraints ruling the apparently free particular cases), could consider this point and try to see the possible meanings of the empty places[219].

Another interesting case of possible future application of NOT is given by the already quoted theory of the "social autopoietic systems" by Niklas Luhmann. This theory is clearly structuralist in so far it relies on one basic *structure*, the notion of "autopoietic system". More than this (and still in a clear structuralist vein), in this theory human actions (both social and cultural) are conceived as micro-relations ruled by a much bigger (blind) "subject", that is the autopoietic society, of which our minds (and bodies) are only small atoms. In his theory, one central concept is that of "communication" (of which the notion of "action" is a particular case). This concept is conceived in a clear structuralist way, akin to strategies of Saussure and Greimas. Here we propose four graphical ways in which the basic articulation of Luhmannian communication (a matter of choice between possibilities) could be formalised by NOT.



Remark, moreover, that an eventual confirmation of the existence of serious links between NOT and bio-mathematics (by now just an open hypothesis, both mine and of René Thomas,

219 In personal exchanges Badiou seems to say that he is highly interested in the development of such a "logical NOT-structuralism" and agrees in seeing it as possibly alternative to the analytical, non-structuralist use of logic.

cf. § 09.10 *supra*), if they were to give a NOT-intelligence of biological fundamental phenomena like multistationarity, could possibly have a relevance for the notions of the theory of autopoiesis and therefore offer a further NOT-access to Luhmann's theory.

A third future structuralist domain of application could arise by the enterprise of modelling the theories of argumentation. Argumentation theory tries to investigate, since 1950, the "logic" of human reasoning, accorded (as it nowadays is generally is) that this is not the least reducible to mathematical logic (this point has been proven with very solid arguments). We are thinking of authors such as the Belgian Chaïm Perelman and Lucie Olbrechts-Tyteca, who developed a powerful theory called "new rhetorics", or as the Swiss mathematician and philosopher Jean-Blaise Grize (a pupil of Piaget) who has been developing for decades a "natural logic"[220]. To these several other authors and new disciplines should be joined in order to approach a general (needed) unified standard model of argumentation[221]. The notion of argumentation has clear links with the notion of opposition. Just to keep reference to the two quoted theories, Perelman and Olbrechts-Tyteca stress the importance of the "dissociation of notions" (i.e. the introduction of a new binary opposition with respect to a given binary opposition) for the argumentation characteristic of philosophy[222], whereas Grize conceives (in some sense) argumentation as a sedimentation (or agglutination) of arguments either mutually strengthening one another, either opposing one another[223]. We will not give proposals of NOT-formalisations here, but this line of enquiry seems to us to be very interesting, as will also emerge when speaking of the reaserch line (for elaborating new internet tools for dealing interactively with public debates on blogs and the like), interacting with NOT, of the LIRMM lab in Montpellier (cf. § 17.03.06 *infra*).

A last possible future structuralist NOT-development which we would like to mention here is the systematic enquiry of the structure of religions (or comparable systems of values or of beliefs) in terms of their (inner) fundamental oppositions. In this respect (i.e. the one of a renewed structuralist study of human culture by means of NOT-tools) a first very convincing element is the study of Taoism by Pellissier, to which he is joining a similar systematic text-based and NOT-modelised analysis of Mazdeism (cf. § 09.09 *supra*).

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[220] To which one could add many other authors, like for instance Stephen Toulmin (juridical thinking), Frans H. van Eemeren and Rob Grootendorst (pragma-dialectics), J. Anthony Blair and John Woods (informal logic).

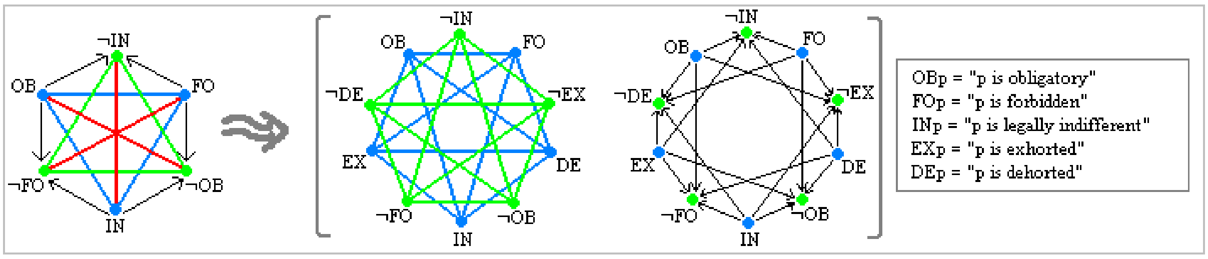
[221] Among the authors we are thinking of Matte Blanco, Luhmann and Badiou. As already mentioned (§ 06.06.07 *supra*) the Belgian philosopher Michel Meyer has recently proposed some interesting outlines for a possible candidate for becoming a general theory of argumentation in *Principia Rhetorica*, *op. cit.* (2008).

[222] Perelman C. and Olbrechts-Tyteca L., *Traité de l'argumentation. La nouvelle rhétorique*, Editions de l'Université de Bruxelles, Bruxelles, 2000 (1958), ch. IV; Perelman, C., *L'empire rhétorique. Rhétorique et argumentation*, Vrin, Paris, 2002 (1977), ch.XI.

[223] Grize, J.-B., *Logique naturelle et communications*, PUF, Paris, 1996, p. 106.

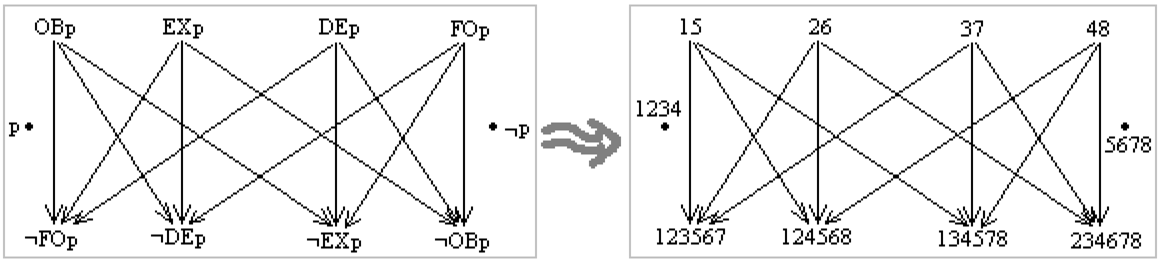
17.03.03. Hruschka and Joerden's deontic logic for supererogation (1987)

In order to better think the notion of "supererogation", that is, in order to internalise it into the formalism of deontic logic, two German authors (two lawmen) proposed in 1987 the following conservative extension of Kalinowski's deontic hexagon[224].

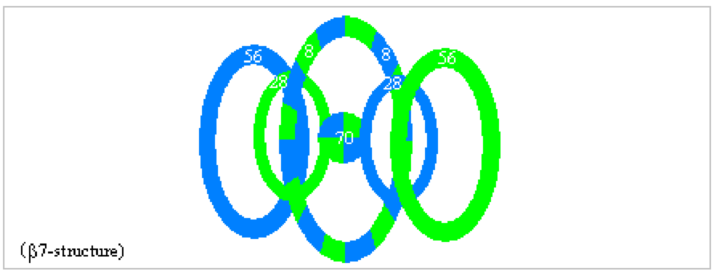


Note that this is remarkable: it is (so long) the first known instance of a logical bi-simplex of dimension superior to four. These authors (in fact Joerden) found it without the help of NOT.

Now, despite the big merit of having done this, NOT tells us that the two authors have forgotten many other solutions (for instance, they do not take into account the null modalities). In order to get them all, we must, as always, start by trying to fix the system's γ-structure (i.e. its modal graph), *via* its basic modalities. This is not so easy, in this case. One sensible proposal would seem to be the following.



However, this looks strange: we never saw a γ-structure with arrows going from left to right and *vice versa*. In any case, if we accept this reconstruction, the E set corresponding to Hruschka and Joerden's logic would seem to be {1,2,3,4,5,6,7,8}. So this would lead apparently to a β7-structure (cf. ch. 15 *supra*).



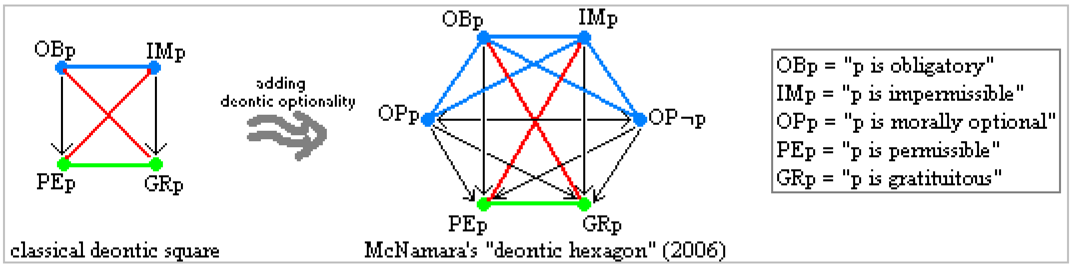
[224] Cf. J. Hruschka and J.C. Joerden, "Supererogation: Vom deontologischen Sechseck zum deontologischen Zehneck. Zugleich ein Beitrag zur strafrechtlichen Grundlagenforschung", *Archiv für Rechts und Sozialphilosophie*, 73, 1, 93-123, 1987. I wish to thank here Professor Joerden.

Hence, Hruschka and Joerden's deontic logic for supererogation would seem (but this must be enquired and checked much more systematically) to be isomorphic to the β7-structure (whereas standard deontic logic, as we saw, is isomorphic to the β5-structure).

17.03.04. McNamara's deontic geometries for supererogation (1996)

The American philosopher and logician Paul McNamara achieved to introduce the notion of supererogation into deontic logic[225]. Besides a deep philosophical discussion of this, he gave (together with Ed Mares) a proof of the soundness and correctness of such a conservative extension of deontic logic[226]. This logic has become a standard[227]. For instance, in the dispute between Joerdens and Wessels (about supererogation) both these authors claim McNamara's authority on their side. However, NOT allows to see that even the best scholars, when lacking a specific methodology for dealing with the geometry of oppositions, are exposed to systematic mistakes if, as McNamara does here, they try to give a geometrical (oppositional) expression of their correct ideas. The following is therefore highly instructive.

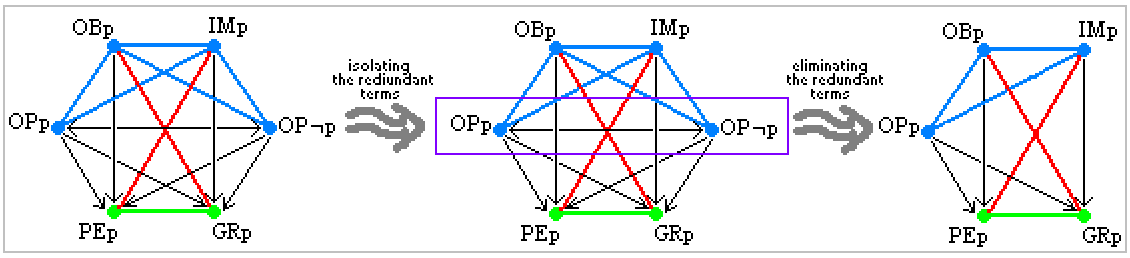
First (2006), in order to express geometrically standard deontic logic (so to highlight the changes he is going to introduce) McNamara proposes a "deontic hexagon" (seemingly new) which *prima facie* could seem to be different from Kalinowski's of 1972. As a matter of fact, one sees that McNamara's deontic hexagon is not a logical bi-simplex: the blue part (for contrariety) is bigger than, and not symmetrical to, the green part (for subcontrariety).



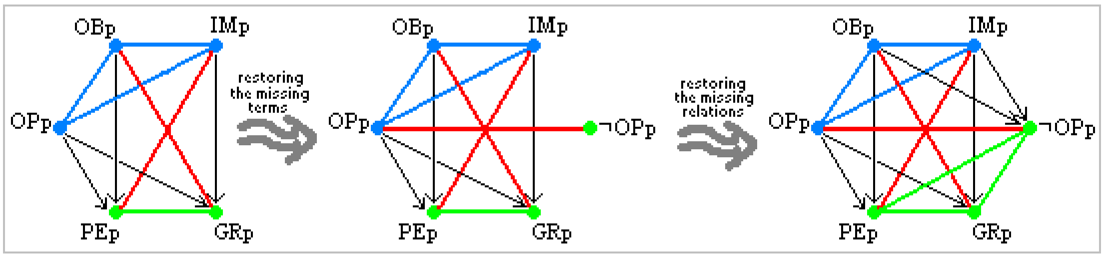
But McNamara's own hexagon suffers in fact, *from the point of view of NOT*, from redundancies. Two terms, OPp and OP¬p, are logically equivalent (so as are, then, the relations relied to them, as for instance OBp—OPp and OBp—OP¬p). Remark that McNamara is fully aware of that: he simply does not have the aim of expressing oppositions in a way perfectly analogous to that used by Aristotle (the square) and Sesmat-Blanché (the

[225] This was done mainly in: P. McNamara, "Making Room for Going Beyond the Call", *Mind*, Vol. 105, 419, July 1996 and in P. McNamara, 'Doing Well Enough: Toward a Logic for Common-Sesne Morality", *Studia Logica*, **57**, 167-192, 1996.
[226] Cf. E.D. Mares and P. McNamara, "Supererogation in Deontic Logic: Metatheory for DWE and Some Close Neighbours", *Studia Logica*, **57**, 397-415, 1997.
[227] McNamara is the author of the *Stanford Encyclopedia of Philosophy* entry "Deontic Logic" (2006).

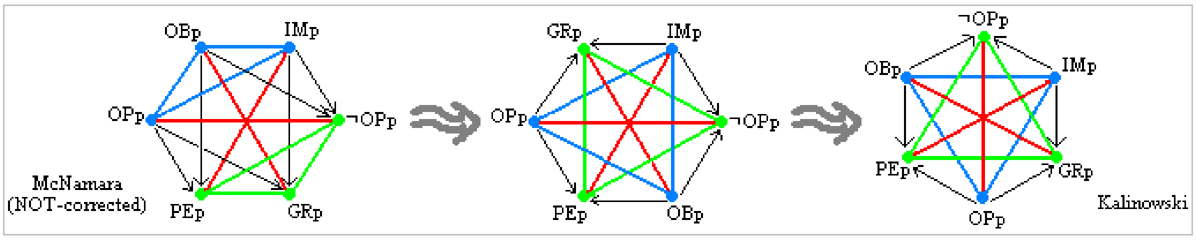
hexagon). For in the logical square and hexagon there are no redundancies. These redundancies of terms (like “OP¬p”) and relations (like “OP¬p→GRp”), from the point of view of NOT, are useless (so we correct them by erasing them).



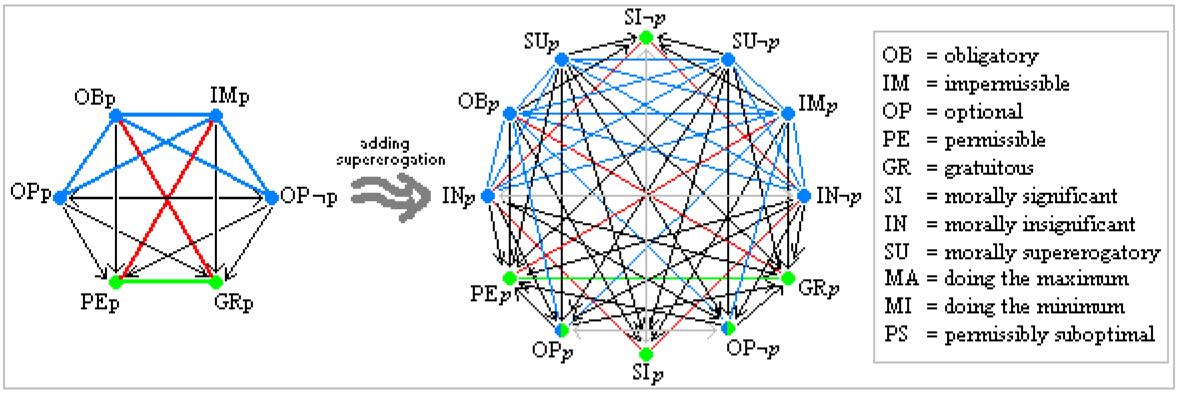
This hexagon also suffers from lacks, for the negations of some of its elements are missing from McNamara’s polygon. For instance, the term OPp is deprived of its negation, ¬OPp (the same can be said for the relations linked to these missing elements, as for instance the relation GRp—¬OPp). Again, McNamara is fully aware of it: he simply does not have this aim (i.e. NOT’s faithfulness to the spirit of the logical square and hexagon). According to NOT this must not be, so we correct it as well, this time by inserting the missing terms (i.e. some contradictory negations) and (their) relations.



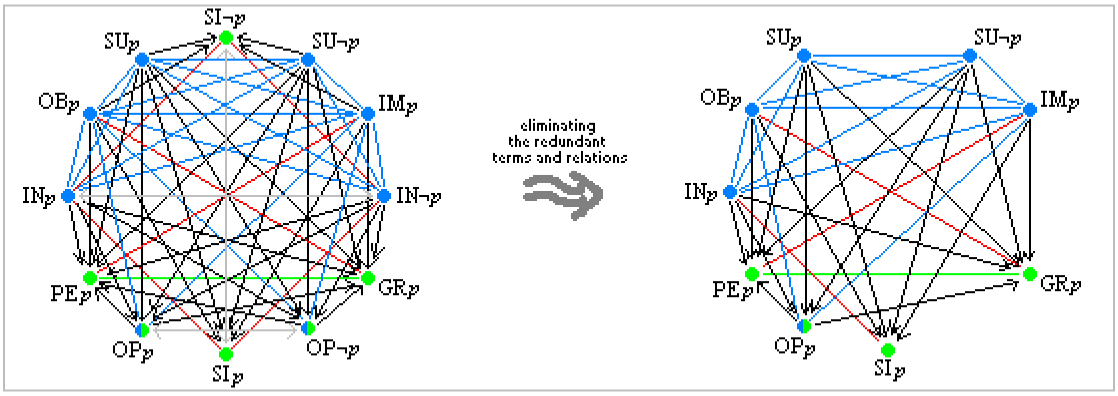
So, one sees that once the redundant elements are eliminated and the missing ones are restored, McNamara’s deontic hexagon (left side of next figure) is in fact exactly Kalinowski’s deontic hexagon (right side of next figure).



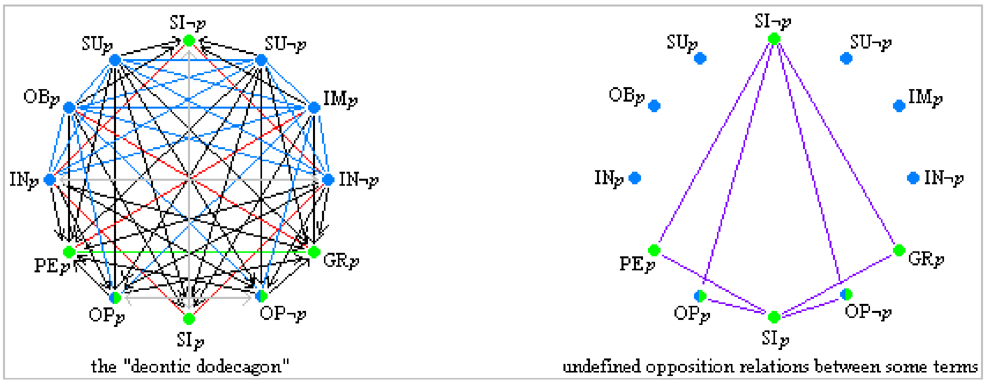
Second (1996), McNamara proposed a “deontic dodecagon” in order to represent the deontic logic obtained by him by adding some new modalities relative to “supererogation” (we do not enter here the philosophical argumentation of McNamara, which is very robust and accepted as a standard model for supererogation).



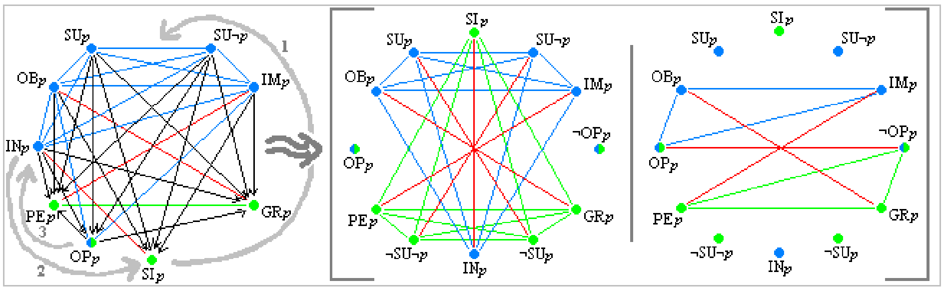
But this new polygon is incorrect from the point of view of NOT. There are some modalities which are redundant (like IN¬P, which is logically equivalent to INp (McNamara says it explicitly), so they should not be both present in the same figure, according to NOT). So, following NOT, we eliminate all the redundant terms and relations by erasing them.



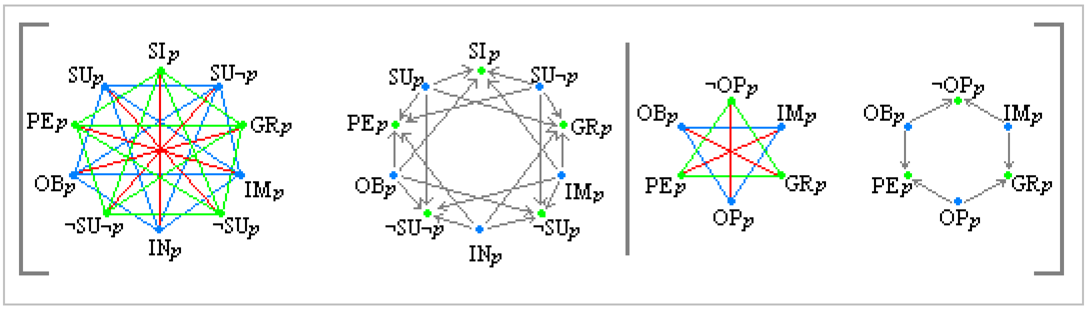
Moreover, as previously, some things are missing (they are undefined). In the next figure we highlight all the relations (right side of the picture) which are undefined in the deontic dodecagon (as for instance the OPp—SIp one).



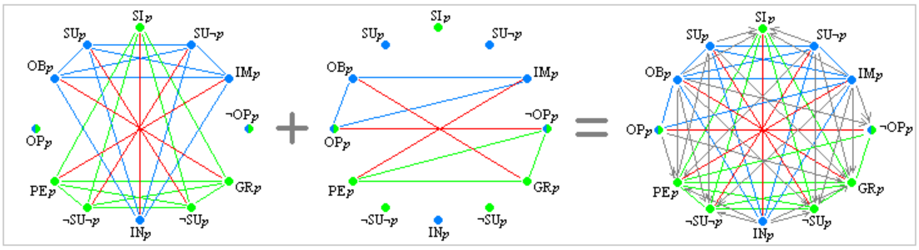
Following NOT, we correct this by restoring the missing elements (as ¬SUp) and relations (at least those which we can identify at this stage, as PEp—SIp, which is a subcontrariety), and by forcing the contradictory elements to be symmetric (by central symmetry) with one another, central symmetry of the contradictories being a fundamental construction principle of NOT (in the next figure, middle and right, we omit the arrows for simplicity).



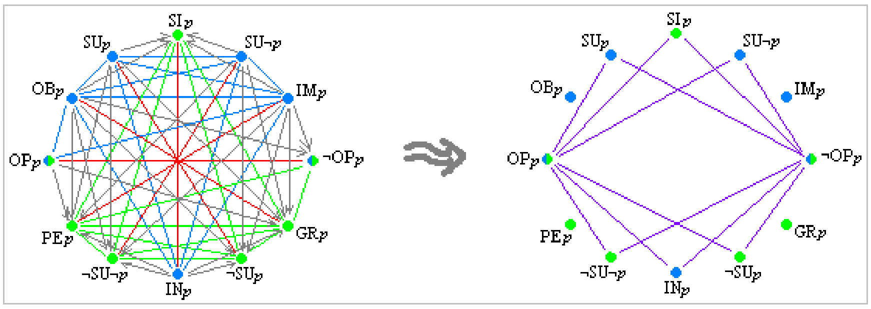
What we get thus is fully NOT-compatible: two logical bi-simplexes, of dimension 4 and 2.



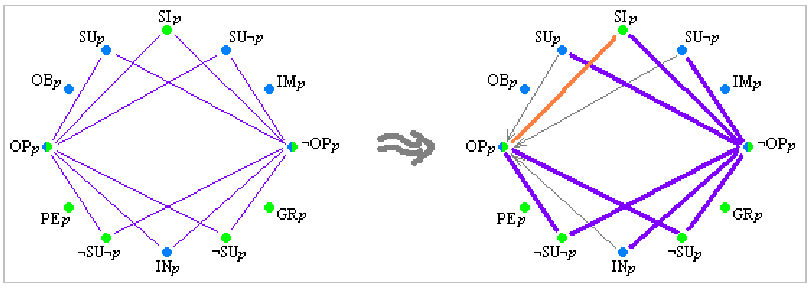
If we superpose these two components (representing this time also all the known arrows) we get some kind of “new McNamarian NOT-dodecagon” (right side of next figure).



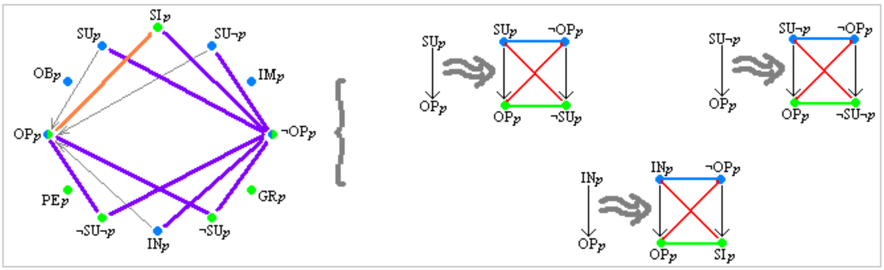
This figure is useful for it shows that so far this (new) polygon is still not complete from the point of view of NOT, for some relations are still undetermined (cf. right side of the next figure, in violet).



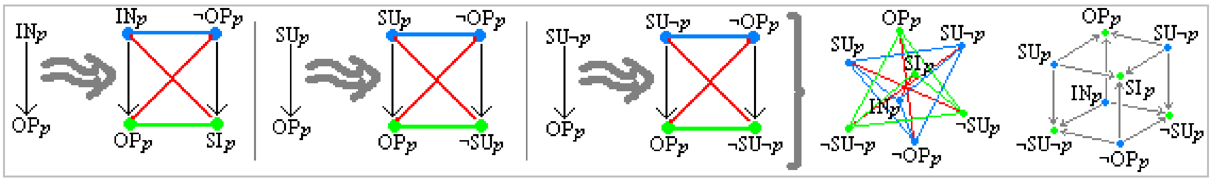
In particular, among the violet now missing relations we can see that one had been missing since the beginning (this is the orange one in the next figure, absent in the original deontic dodecagon). The remaining violet relations are new: they concern logical modalities (like “¬OPp” or “¬SU¬p”) which were absent in his starting polygon. On the contrary, some of these now missing relations were present before: this is the case with the three implications SUp→OPp, SUp→OPp and IMp→OPp (depicted in grey).



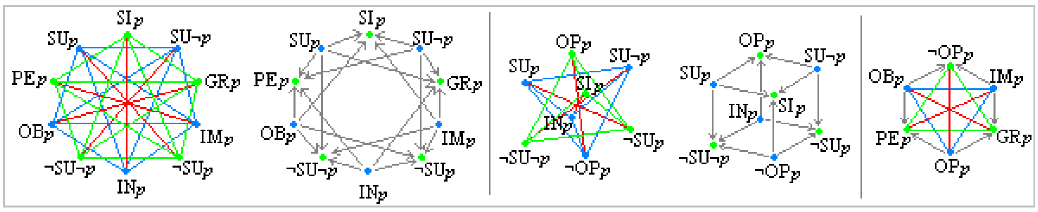
But, in fact, taking these three into account, one can easily see (by contraposition) that each generates a logical square (a deontic square).



And the three terms contrary to ¬OPp (i.e. the INp, SUp and SU¬p) form with ¬OPp a blue tetrahedron of contrariety, that is (by virtue of NOT, cf. ch. 11 *supra*), a logical cube (or deontic cube).

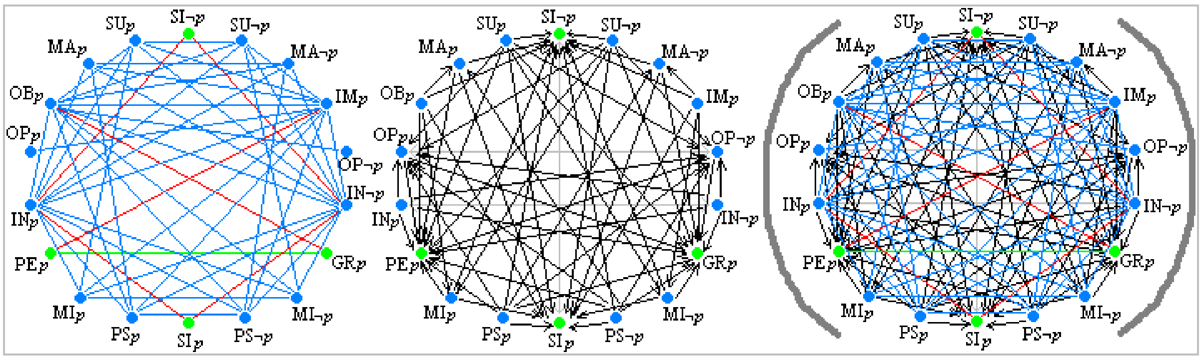


And now, as one can check, all elements and relations are now present. So, finally, the geometrical oppositions expressed by McNamara's first conservative extension of standard deontic logic are in fact better expressed in terms of the junction of three logical bi-simplexes (of dim 5, 3 and 2 respectively) than in terms of a (problematic within NOT) "deontic dodecagon".



Third (1996), in order to express geometrically some further enrichments of his logic for supererogation, McNamara proposed a "deontic octodecagon". Again, despite the fact that the underlying logical system is fully correct, the geometrical (oppositional) expression of it is, from the point of view of NOT, incorrect. These graphs are not mistaken but incomplete, and this is normal, provided their underlying logic is a very rich one and provided that by the time

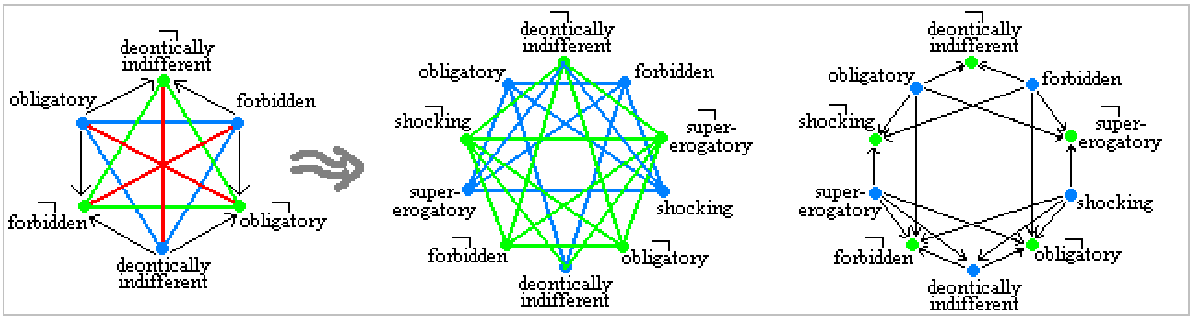
(1996) McNamara lacked, as everyone else, a scientific methodology for handling the geometrical expression of logical oppositions.



We will not analyse this last figure here, our results thereupon are still in progress. But clearly it will be interesting, sometime or other, to obtain a complete and stable geometrical-oppositional knowledge over the logic DWE. This will be some rather high β*n*-structure, collecting α*k*-structures and reflecting faithfully the γ-structure of McNamara's logic.

17.03.05. Wessels "deontic hexadecagon" for supererogation (2002)

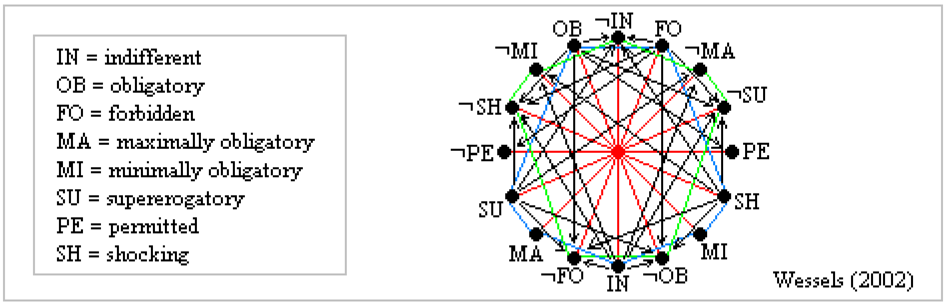
In 2002, in a study on the notion of supererogation, the German philosopher and logician Ulla Wessels proposed two deontic polygons[228]. The first one, which is a "deontic decagon", was intended to express geometrically Chisholm and Sosa's theory of supererogation of 1966 (and is similar to, but in fact different from, Hruschka and Joerdens' one)[229].



Wessels' second deontic polygon, based on McNamara's deontic octodecagon (cf. §17.03.04 *supra*), has been proposed by her for expressing her own philosophical ideas on supererogation.

[228] Cf. U. Wessels, *Die gute Samariterin. Zur Struktur der Supererogation*, Walter de Gruyter, Berlin, 2002. Cf. also U. Wessels, "Und es gibt doch Supererogationslöcher. Eine Erwiderung auf Jan C. Joerdens Besprechung von *Die gute Samariterin*", *Annual Review of Law and Ethics*, Band 12 (2004).

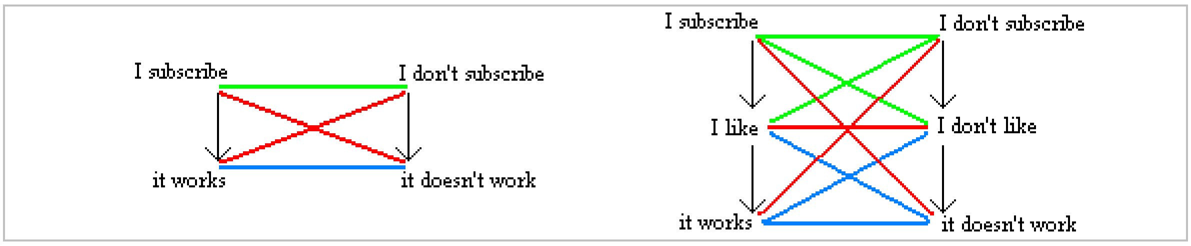
A close examination, which we skip here, reveals omissions, which means that, independently with respect to the intrinsic value of Wessels' investigations, their geometrical expression can be said to be incorrect from the point of view of NOT (and waiting for NOT's help).



17.03.06. Web applications of *n*-opposition theory: the Montpellier group

A French research group in Montpellier (belonging to the "LIRMM" lab), led by the already mentioned physicist and AI researcher Jean Sallantin, has recently tried to use NOT in order to produce a new family of "new generation" software applications for the internet. The main idea seems to be that of using a renewed theory of opposition and of argumentation (and new logical technologies in general – mainly paraconsistent and defeasible logics –, as well as new theories of the "public debates") in order to rethink in a new way the structure, say, of those web pages intended for canalising public debates and the like (political forums, campaign blogs, etc.). The idea is very stimulating, with possibly many credits at stake (the issue being actually politically "hot"). However, many of their graphs (which in their view are "oppositional graphs", allegedly inspired by NOT) are mistaken from the NOT point of view. Besides mistaking blue with green (which is secondary), one really big problem emerging from the examination of this interesting new project is that, seemingly, a very important point of NOT has been equivocated so far by the LIRMM group: the *essential* difference between modal graphs (= γ-structures) and oppositional structures (α*n*-structures). As a matter of fact, *one should avoid putting colours on the γ-structures (or modal graphs)*. But they do not avoid it, they put the 4 Aristotelian colours on γ-structures.

[229] Cf. R.M. Chisholm and E. Sosa, "Intrinsic Preferability and the Problem of Supererogation", *Synthese*, 16 (1966).



As a matter of fact, they represent such γ-structures (right side of the previous picture) with the colours classically used in order to distinguish the four Aristotelian relations (red, blue, green, black), whereas modal graphs are colourless! (well, they are nevertheless visible under a black cloth, ok) Conversely (in fact by the same movement), believing to be dealing with structures "of the same breed" as the logical square and hexagon, they feel free to produce such structures in arbitrary shapes (right side of the previous picture), whereas the true relatives of the logical square and hexagon (i.e. the α*n*-structures, left side of the previous picture) have very constrained shapes: they must be *logical bi-simplexes (of dimension* m*)*. Even the β*n*-structures are very constrained – they are hyper-tetraicosahedra, cf. ch. 13-15 *supra* –, the only oppositional geometries with free shape are … the colourless (black if you want) modal graphs, i.e. the γ-structures. Now, this point can lead to quite big mistakes, for the modal graphs deliver all their "oppositional secrets" only when translated into β*n*-structures. Putting colours on a γ-structure is equivalent to looking *randomly* (i.e. *not systematically*) for the "oppositions" of this modal graph: once so coloured it will give at best incomplete information, at worst false or slippery ones.

So, the Montpellier project (inside the LIRMM), excellent in spirit (im my opinion) and possibly destined to a successful future (in term of hi-tech software development for the world wide web), deserves to be developed, but at the inflexible price of using a better informed formal (NOT-) technology[230].

17.03.07. Luzeaux, Sallantin and Dartnell's applications of N.O.T. (2008)

As we already said (cf. § 12.03 *supra*), Luzeaux, Sallantin and Dartnell propose to see Aristotle's square not as a formal theory of logic, but as a theory of reasoning (a generalisation of the inferring process). In this respect they develop systems on the structures proposed by Pellissier and myself (and of course on the logical hexagon, notably the two proposed by Béziau). By doing this, that is, developing multi-agent systems considered as

[230] We consider here two studies: A. Seilles, "Modèles et outils de raisonnement argumentatif dans les communeautés et organisations virtuelles", *Master Thesis*, LIRMM - University of Montpellier 2, 2007; and A.

systems of oppositions they show that the logical cube, as decorated modally inside Pellissier's logical tetraicosahedron, can be interpreted with epistemic notions (one for each vertex) (for something similar, developed independently by Schang with my help – Schang had not read Luzeaux et alii – cf. ch. 22 *infra*). In some places of their paper they work in the direction of expanding NOT by looking at it from the point of view of category theory (this is done at least when producing a mathematical quick description of what I proposed to call a "modal *n*(*m*)-graph" in 2004), having in mind the further (very interesting) aim of formalising, inside a generalised theory of reasoning for many agents, an agent as a topos and the interaction between agents as an adjunction in the corresponding topos.

17.03.08. Smessaert's propositions of linguistic application of NOT

Let us only mention that very interesting linguistic applications of NOT are being investigated by Smessaert. Recently Béziau proposed to read the first geometrical structures he had discovered (2003) in a plain linguistic way, that is, taking quantifiers instead of modalities (which implies, in some sense, exploring linguistic ways of naming composed quantifiers that still have no clear name).

17.04. The table of interpreted β*n*-structures refreshed

The knowledge acquired in this chapter allows to add elements to our table of geometrically translated modal systems

β2: Sesmat's and Blanché's logical hexagon.

β3: the Lewis system S5 (standard modal logic); the space of the binary connectives; the traditional reading of the elemental logic.

β4:

β5: standard deontic logic.

β6:

β7: Hruschka and Joerden's supererogation logic (conjecture); our reading of the elemental logic.

β8: general epistemic logic (à la Lenzen).

Seilles, J. Sallantin, A. Gouaich, C. Douy and J.-B. Soufron, "Le carré des oppositions comme cadre de débats publics", (LIRMM, September 2007). I thank Jean Sallantin and Antoine Seilles for materials and discussions.

β9:

β10:

β11: the Lewis system S4 (conjecture to be verified)

β12:

β13:

…

β21: Hamblin and Prior's standard tense logic (conjecture).

17.05. Concluding remarks on the actual applications of N.O.T.

In this chapter we treated examples in most of the principal "flavours" of modal logic: epistemic, temporal, deontic, … We presented different kinds of possible intervention of NOT:

- we decorated existing modal graphs (as with Hamblin and Prior's tense logic);
- we modified existing modal graphs (refining them);
- we filled the gap left empty by oppositional structures which are correct but fragmentary (cf. Kalinowski, Joerden);
- we corrected the systematically (i.e. necessarily) mistaken geometrical expression of axiomatically and philosophically good logical systems (McNamara);
- we challenged existing founding oppositional models (cf. Vernant);
- we proposed possible unexpected developments (in terms of NOT) of well-established formalisms (Greimas);

Some previous known applications (of the logical square and/or of the logical hexagon) remain obscure at this stage: this concerns notably Piaget, Lacan, Gallais and Thomas.

Among the possible future evolutions we would like to mention (as a reasonable conjecture) the passionating but still mysterious challenge of formalising art theory and aesthetics.

But, in a sense, all these fine things are too static. Can we do better? Can we make them dynamic?

Part III

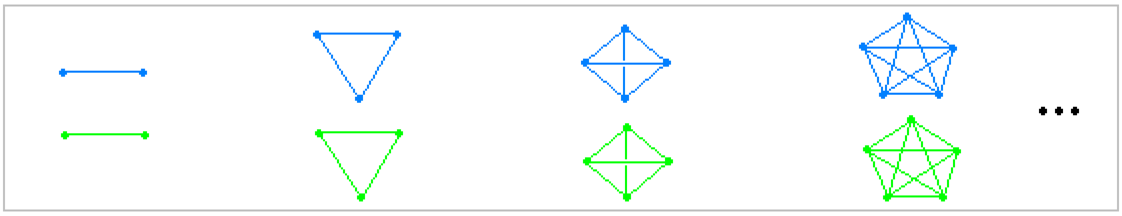
TOWARD A DYNAMIC (AND AN INTENSIVE) THEORY OF OPPOSITION: THE LOGICAL *P*-SIMPLEXES OF DIMENSION *M*

18.
RADICALISING *N*-OPPOSITION THEORY: "ARISTOTELIAN OPPOSITIONAL P^Q-SEMANTICS"

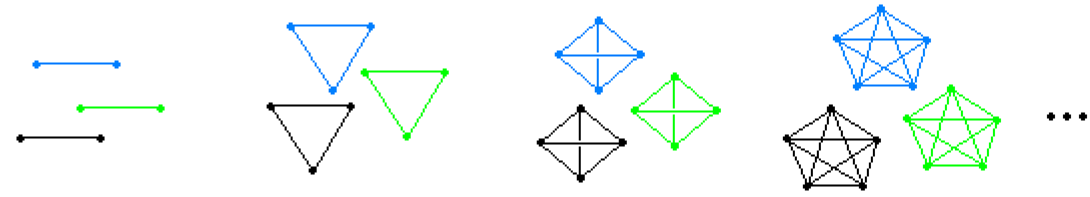
In this chapter we want to radicalise the notion of opposition. After recalling that *n*-opposition theory remains, globally speaking, inside Aristotle's quaternary framework (bi-simplexes still generate 4, and 4 only, kinds of opposition, 4 colours: blue, red, green and grey – this last is often also represented in black), we propose a new semantics, which we call "Aristotelian oppositional p^q-semantics" in honour to the Stagirite, of which it generalises the fundamental simple and bright intuition. This semantics, very simple in its principle (it varies the number of possible questions and possible answers), will turn out to be (in the next chapters of this Part III) the needed tool in order to move from the logical bi-simplexes to the logical *tri*-simplexes and beyond (the "logical *p*-simplexes", $p \geq 2$, $p \in \mathbb{N}$). And this move will change radically our understanding of opposition, leading us to a new extended theory (where the number of possible oppositions changes) of which *n*-opposition theory is just a fragment.

18.01. Must there be 4, and 4 only, kinds of opposition?

The notion of logical bi-simplex gives to opposition theory a domain larger than it had in Aristotle's classical theory: an arbitrary number of contrary "opponents" (two in the logical square, three in the logical hexagon, four in the logical cube, etc.). Nevertheless, it must be noted that something of Aristotle's theory is kept unchanged *with any logical bi-simplex*: that is, the number of qualitatively different kinds of opposition, which is and remains four (said with colours: contrariety, contradiction, subcontrariety, subalternation). But is this the only way we can have it? Is opposition necessarily a *quaternary* composition or structure? Is the quatern of the kinds of opposition a *transcendental* unchangeable structure? If the logical "bi-simplexes" are not enough in order to answer this question, one possible way to try to change this (in order to go further into the exploration of oppositional possibilities) is to take the "bi-" of the expression "logical bi-simplexes" as a parameter still to be exploited and inquired into.



In other words, we want to see if there can be “logical tri-simplexes”, and in case there are, if this brings a change with respect to the number of fundamental qualities of oppositions.



looking for "logical tri-simplexes of dimension n"

But how to (try to) build (or discover) such logical tri-simplexes? What could these last be?

18.02. Aristotle had in fact an “Aristotelian 2^2-semantics”

Because we actually have no clue about how to find out “logical tri-simplexes” (or whatever), a sensible way to proceed seems to go back to Aristotle’s starting intuition. How did he generate the logical square? (i.e. the first known logical bi-simplex) Truly speaking, it was by some kind of game of “asking and answering”, in the sense that he asked two possible “ontological” questions, and admitted two possible answers to each of them:

Question 1 (Q1):	“can two things (of which we test the mutual opposition ontology) be false together?”
Answer 1 (A1):	“X” (with $X \in$ {no, yes})
Question 2 (Q2):	“can two things (of which we test the mutual opposition ontology) be true together?”
Answer 2 (A2):	“Y” (with $Y \in$ {no, yes})

Now, this gave 4 possibilities (the first term X of each of the following four X-Y couples is A1, i.e. the answer to the Q1 question, the second term Y is A2, i.e. the answer to the Q2 question):

no-no, no-yes, yes-no, yes-yes.

And each of these four couples was exactly one of the four Aristotelian kinds of opposition:

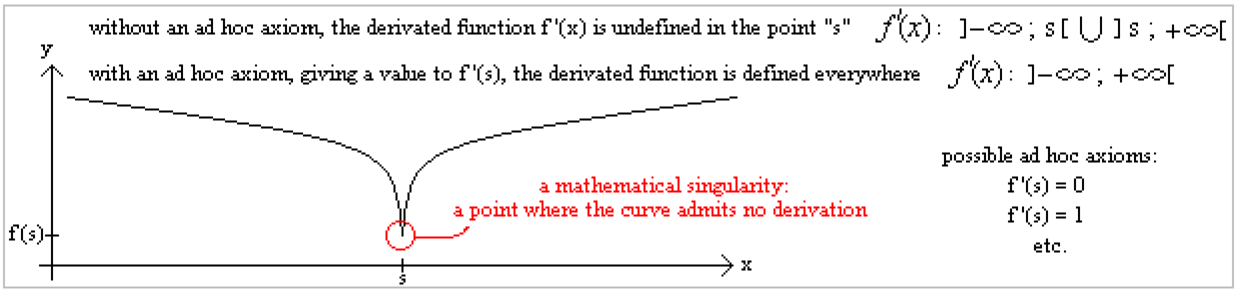
no-no: contradiction;

no-yes: subcontrariety;

yes-no: contrariety;

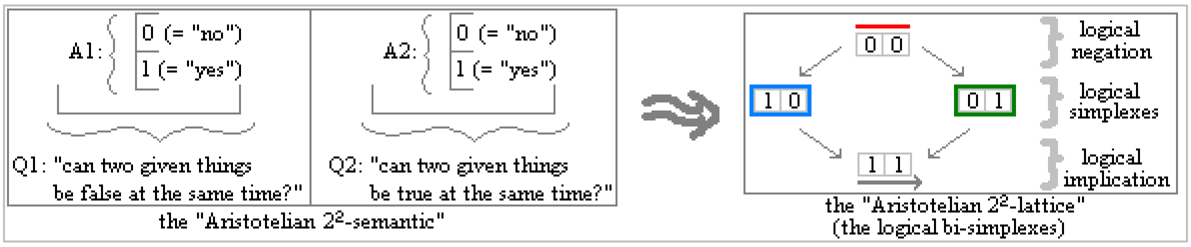
yes-yes: subalternation.

Now, the case of subalternation needs a discussion *per se*. As a matter of fact, this case concerns a “mathematical singularity” (a “singular point”), that is, a particular situation, more complicated than usual, needing a special formal *ad hoc* treatment involving arbitrary (but hopefully wise and fertile) choices (like in the paradigmatic case of the treatment of functional continuity in mathematical analysis, cf. figure).



The singular point here at stake is the [1|1] issue in so far it corresponds, intuitively, to logical *equivalence* (not to logical *implication*) and logical equivalence is a too much trivial case for an opposition relation. Geometrically speaking this is expressed by the fact that logical equivalence makes the geometrical objects collapse, whereas logical implication (a weakening of logical equivalence) both expresses a form of opposition (i.e. the violence of *order*, “first me, than you”) and avoids the geometrical collapse. So, in order to keep our formal construction balanced (which means here: in order to [1] restrict the relations to opposition relations and [2] to avoid geometrical collapse) we need to add to the simple question-answer device two further conditions saying (1) that the two considered objects are *ordered* and (2) that, as for the value [1|1], the corresponding logical equivalence is weakened by abolishing the arrow going from the second to the first.

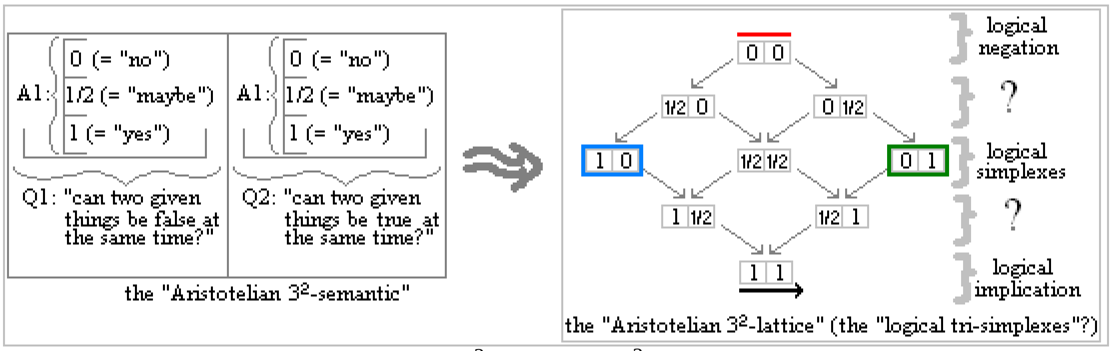
With all this in mind, let us depict Aristotle’s semantic game in terms of a two-terms “question-answer cell” (left side of the next figure) and of a four-terms lattice of the possible outcomes of answers, or lattice of the [*x*|*y*] cells (right side of the next figure).



As we are going to see, this is the needed (and sufficient) starting point of our inquiry on the generalisation of the logical bi-simplexes.

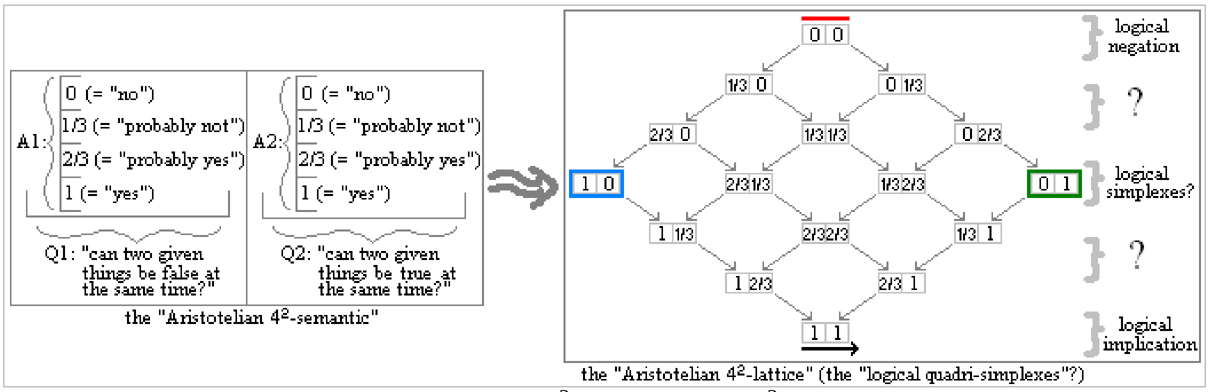
18.03. Aristotelian 3^2-semantics and 4^2-semantics

So, if our starting semantic analysis is correct, if we want to try to change the number of kinds of opposition, it could be useful to try to change Aristotle's game. If we call this still unnamed game "Aristotelian 2^2-semantics" (the first "2" being the number of possible answers [false, true], the second "2" representing the number of questions [Q1, Q2]), we can try doing it by generalising it to the notion of "Aristotelian p^q-semantics", meaning by that that, in principle, it should be possible to play similar "games of ontological opposition" with a different number q of questions and a different number p of possible answers. If we pass from 2 to 3 possible kinds of answers, we get an Aristotelian 3^2-semantic (3 kinds of answers, 2 kinds of questions).



One sees that this generates not 4 (= 2^2) but 9 (= 3^3) "opposition cells" and hence, seemingly, 9 kinds of opposition (instead of the 4 of Aristotle). So this seems to be a promising issue with respect to our starting question. Notice however that over the 9 kinds of opposition, 4 are the old ones, whereas 5 are new ones, still to be determined. But the game can even go further.

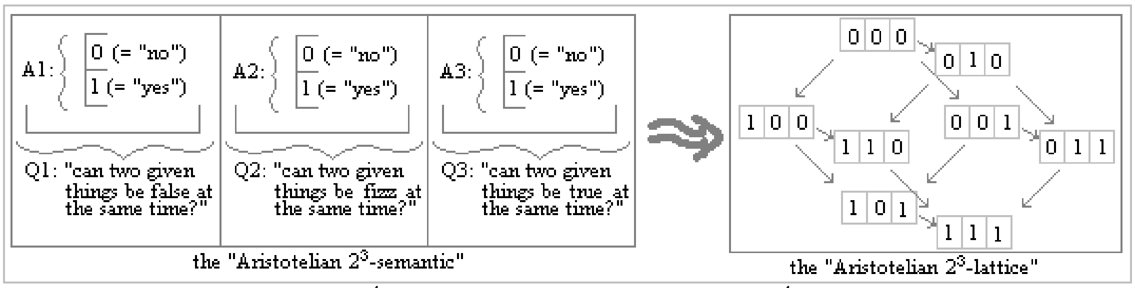
As a matter of fact, if we allow not 3, but 4 possible kinds of answers, we get instead an Aristotelian 4^2-semantic.



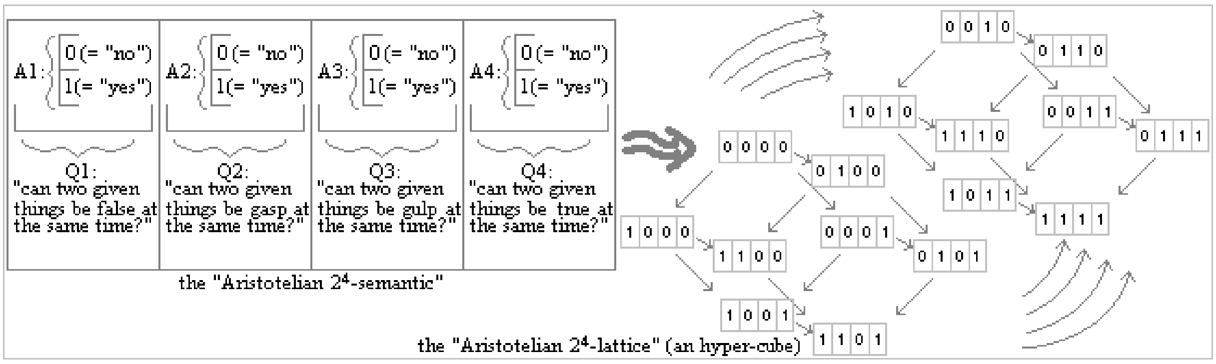
One sees that this generates not 4 (= 2^2) but 16 (= 4^2) "opposition cells" and hence, seemingly, 16 kinds of opposition (instead of the 4 of Aristotle). Here as well, notice that, over these 16 kinds of opposition, 4 are the old ones, whereas 12 are new ones, still to be determined. As one imagines, this kind of expansion seems to know no finite limit.

18.04. Aristotelian 2^3-semantics and 2^4-semantics

A similar game can be played if one changes the *q* parameter (from 2 to 3): this gives an Aristotelian 2^3-lattice ordering 8 (= 2^3) opposition cells (i.e. 8 kinds of opposition)[231].

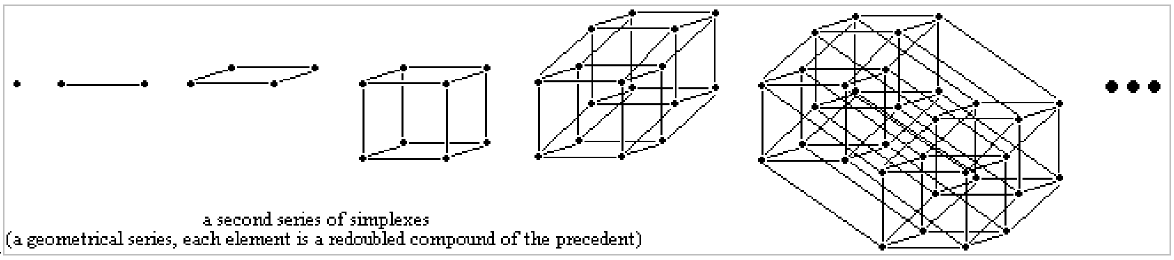


In a similar way, a 2^4-semantics will generate a 2^4-lattice, ordering 16 (= 2^4) opposition cells (i.e. 16 kinds of opposition).



Again, one sees here that this new structure may not be a conservative extension of the preceding one (for, the cells have changed in their inner structure). This point is not clear yet, it deserves an investigation of its own (which we cannot afford to do in the present study).

An important remark here is that, very seemingly, the series of the Aristotelian 2^q-lattices is isomorphic to the series of the geometrical cubic simplexes (point, line, cube, hyper-cube, …).

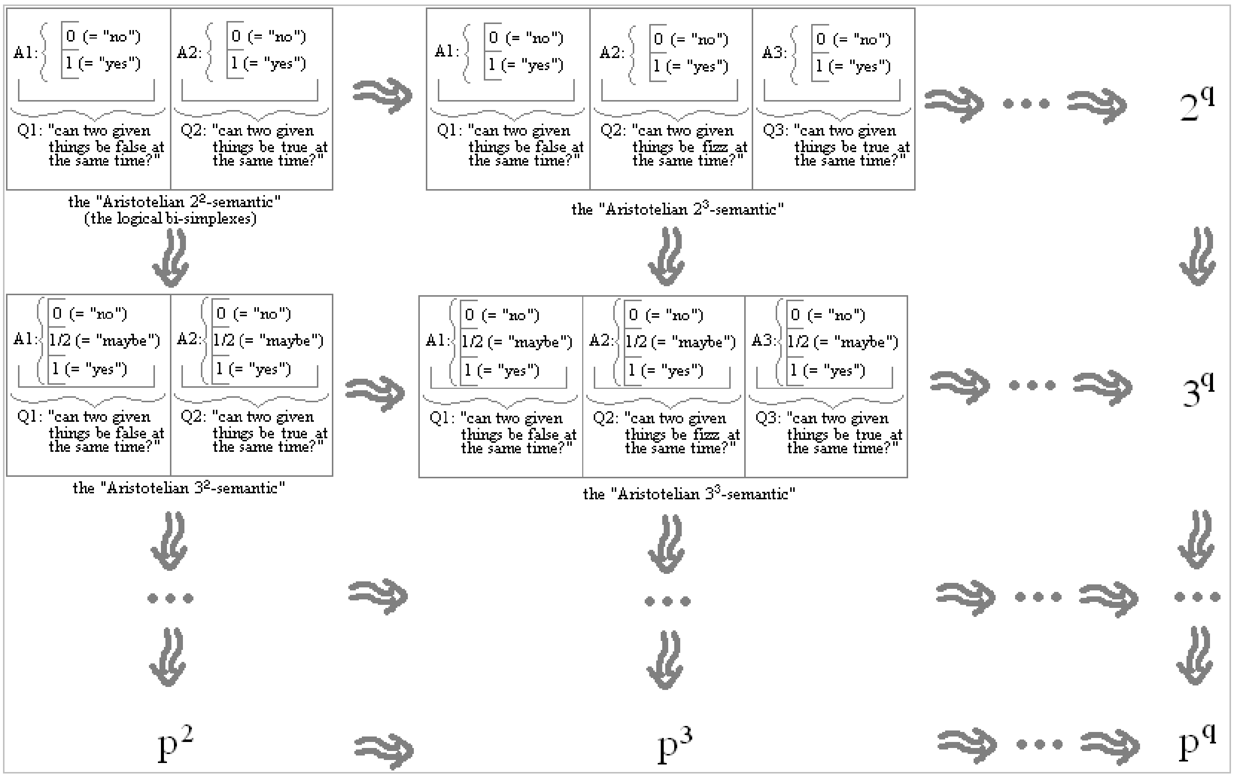


[231] One sees here at least two things: (a) the "opposition cells" generated have 3 parts (and not 2), which seems to mean that it is not clear if they are a conservative extension of the old ones; (b) the Aristotelian lattice is no more 2-dimensional, but 3-dimensional, and is not clearly a conservative extension of the previous (it is, seemingly, a redoubled image of it).

Again, this new field of opposition theory deserves a specific investigation which we will not get into here.

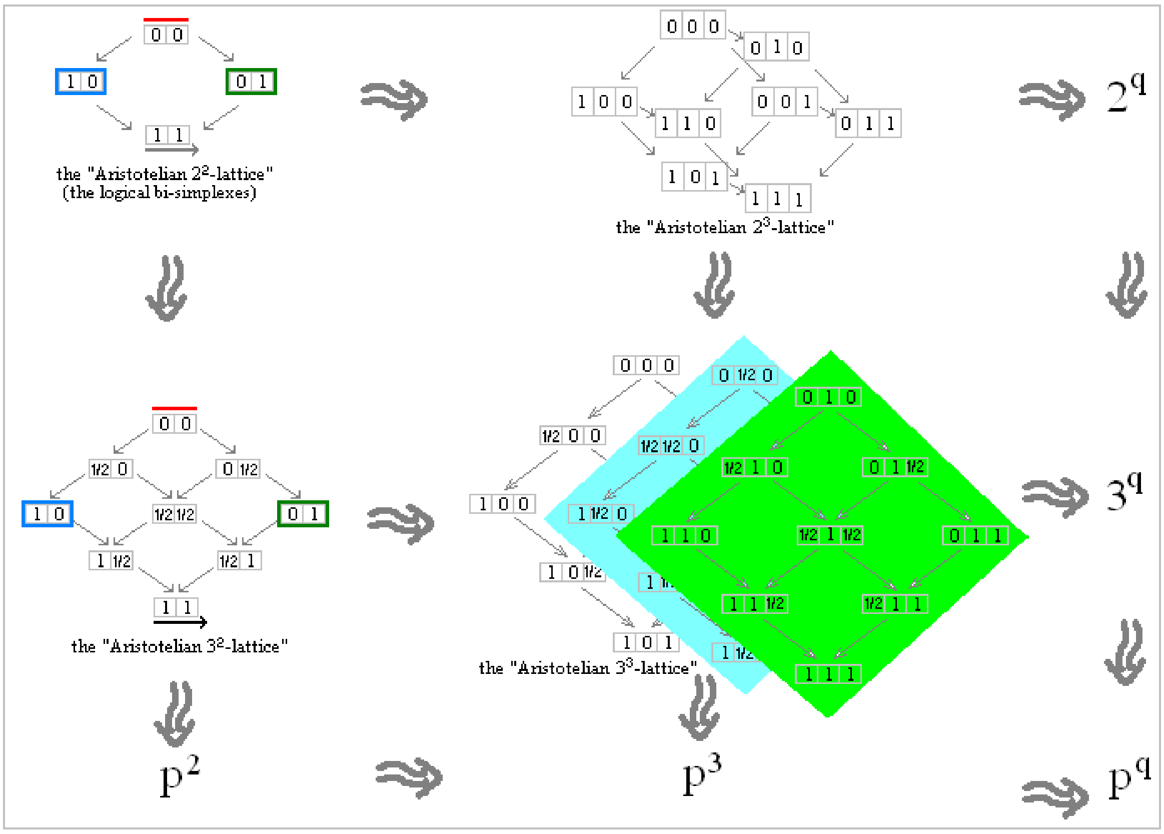
18.05. Towards Aristotelian p^q-semantics (and p^q-lattices)

The following picture (a bi-dimensional diagram) represents intuitively the display of the Aristotelian p^q-semantics.



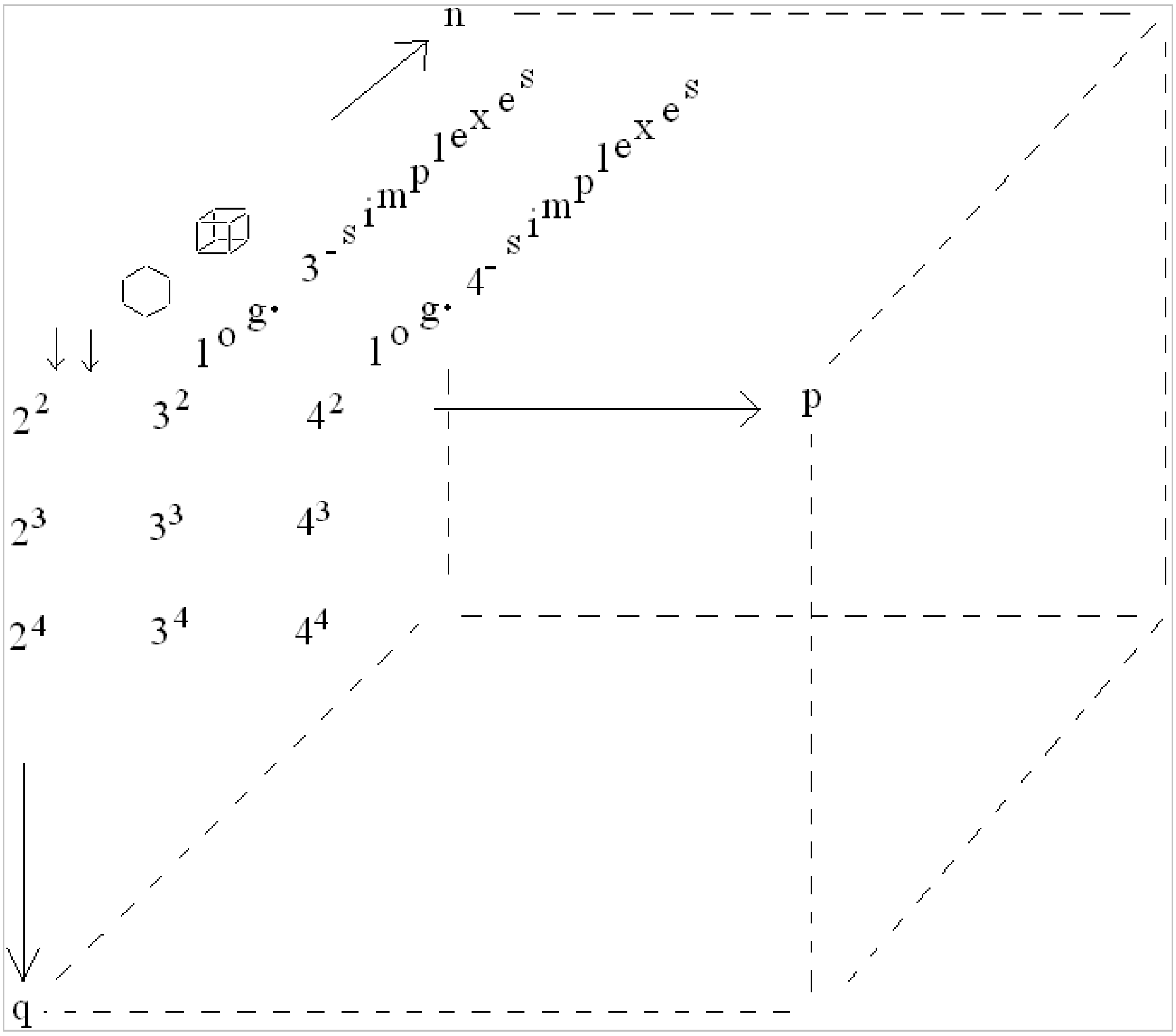
As shown by the previous picture, the expansion of the Aristotelian p^q-semantics forms some kind of "infinite rectangle" (of sides of length p and q).

In fact, corresponding to the changes of the Aristotelian p^q-semantics, we also have a change of the correlative Aristotelian p^q-lattices.



Here as well, the expansion of the Aristotelian p^q-lattice forms some kind of "infinite p^q-rectangle" (of sides p and q).

Now, supposing that each p^q-lattice can (as can indeed the 2^2-lattice) give rise to solids differing for their parameter n (squares, hexagons, cubes, …), the figure summing up all this seems to be one which we can call the "infinite (Aristotelian) n-p^q-parallelepiped" (or "n-p^q-box").



Remark that this structure (the n-p^q-box) is very important and deep: if confirmed, it seems to be the new "transcendental" of the general theory of opposition.

As there is some kind of parallelism or redundancy in the ways of introducing new truth values (for, one can do it either by changing p or by changing q)[232] it might be argued that it would be preferable to take first in consideration the p^q-semantics where $p=q$ (so to guarantee some kind of inner coherence of the logical space, some kind of coherence between the object level and the metalevel). We will not follow this restriction yet (we will authorise having $p \neq q$).

For reasons to appear later (in fact for simplicity's sake: exploring *p* seems easier than exploring *q*) in the rest of this study, in fact, we will mainly concentrate on the exploration of the changes of the *p* value.

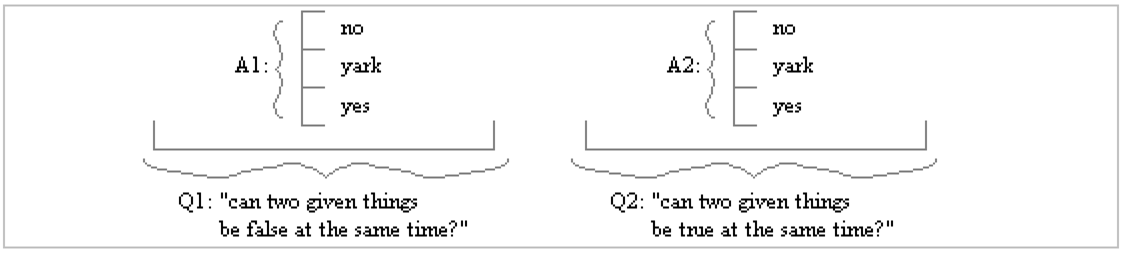
[232] It may seem to be the same to say (a) "can two things be ½ together?", "yes", or to say (b) "can two things be true together?", "1/2".

19.
THE "LOGICAL TRI-SIMPLEXES (OF DIMENSION *M*)": A MORE FINE-GRAINED OPPOSITION

In this chapter, which follows closely the previous one, we try to explore, relying on the Aristotelian 3^2-semantics (and its Aristotelian 3^2-lattice), the still unknown realm of the "logical tri-simplexes" (of dimension *m*). If one succeeds in showing that they do exist (as they seemingly do), many questions do arise about them. Let us mention the following: (1) the formal choices to be done (among the many constructing options available), (2) the philosophical interpretation of the logical tri-simplexes and that of their constituents, (3) their possible applications and (4) the perspectives the logical tri-simplexes open to thinking (at least to logic and philosophy). So we start with the first of these questions.

19.01. The key of the tri-simplexes: the Aristotelian 3^2-semantics

What is concretely an "Aristotelian 3^2-semantics"? It is the oppositional logic displayed by 2 questions (about the possible co-existence of two "objects", abstract or concrete) each admitting one answer over 3 possible ones (one more than by Aristotle). Here are, synoptically depicted, the questions (Q1,Q2) and the answers (A1, A2).



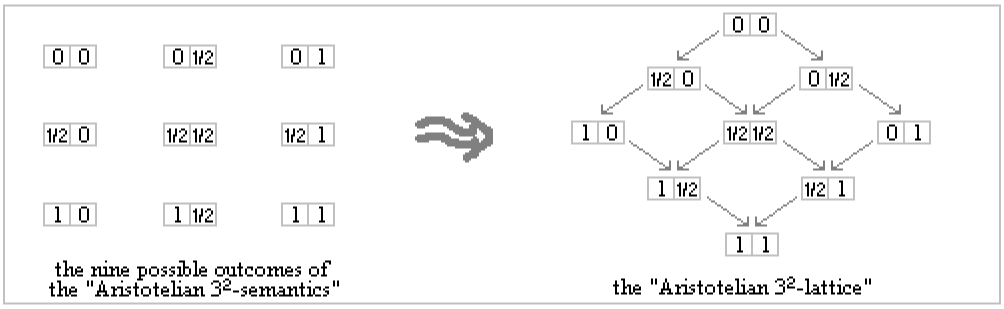
The 3^2-semantics' solution turns out to offer quite nice advantages for us. First it says that there are 3^2=9 possible cases (in each X—Y couple, or "[*x*|*y*] cell", X is A1 and Y is A2, "yark" is the new truth-value):

no-no, no-yark, no-yes, yark-no, yark-yark, yark-yes, yes-no, yes-yark, yes-yes.

Let us use numbers instead of "no, yark, yes": 0, ½, 1. The solution of the 3^2-semantics can then be rewritten thus:

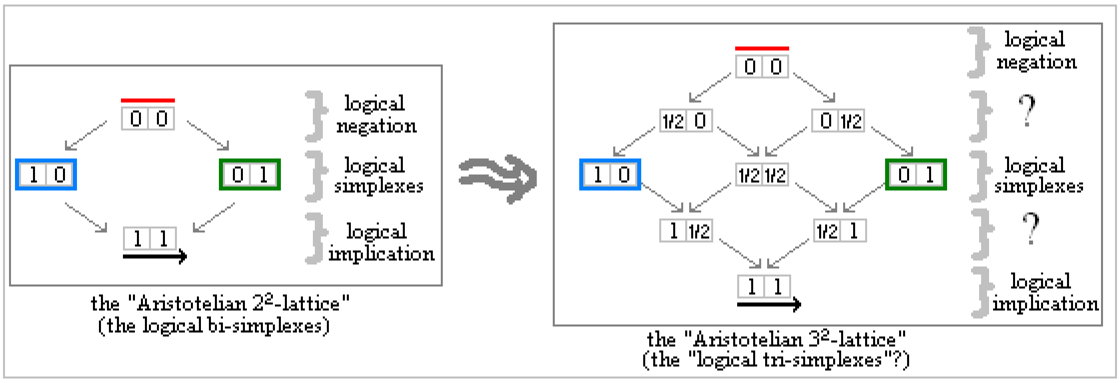
[0|0], [0|½], [0|1], [½|0], [½|½], [½|1], [1|0], [1|½], [1|1].

These are the nine possible ways for two things to be related (in terms of coexistence) by an Aristotelian 3^2-semantics. What happened? Let us represent it in terms of Aristotelian lattices (that is, a partial ordering of all the possible outcomes): the result of this is a new square (or lozenge) lattice (one with nine [x|y] cells), the "Aristotelian 3^2-lattice".



(remember that the arrows in this lattice are lattice orders and not logical implications)

If we now compare this new Aristotelian lattice with the one of Aristotle's own theory (i.e. the one with four [*x*|*y*] cells, ruling all logical bi-simplexes), we see that five new terms (5 new "opposition cells" [*x*|*y*]) are added to the traditional four. (in the picture we highlight with colours the four already known cases). These new five terms (the [½|0], [0|½], [½|½], [1|½], [½|1]) need now to be identified.



19.02. Shape and main (good) properties of the "3²-semantics"

Remark that the basic meaning of the change adopted here (i.e. shifting from the 2²-semantics to the 3²-semantics) is that there are now *three* truth-values (false, undetermined, true: 0, ½, 1) instead of two (false and true: 0, 1). This 3²-semantics happens to have several nice properties. Its "truth-table style tree" (i.e. the combinatory of the possible 3-valued valuations of 2 empty places), which is linear, is the following (again, in each line the first place stays for "possibly false at the same time" [◊FF], the second one for "possibly true at the same time" [◊TT]), made of nine cells:

	◊FF-◊TT	kind of opposition	
(1)	[1|1]	subalternation	(implication)
(2)	[1|½]	= ?	
(3)	[1|0]	contrariety	(logical simplex)
(4)	[½|1]	= ?	
(5)	[½|½]	= ?	
(6)	[½|0]	= ?	
(7)	[0|1]	subcontrariety	(logical simplex)
(8)	[0|½]	= ?	
(9)	[0|0]	contradiction	(logical – i.e. contradictory – negation)

A study of this Aristotelian lattice shows, again, that over the nine, four elements (i.e. [*x*|*y*] cells) – the 1, 3, 7 and 9 – are the old ones (subalternation, contrariety, subcontrariety, contradiction), five others (the 2, 4, 5, 6, 8) are new (their interpretation is so far unknown). So a first remark is that this 3^2-semantics seems to allow a conservative extension of the classical 2^2 one (it yields the known quaternary *n*-opposition theory – i.e. N.O.T. –, plus something else). How to read the remaining five combinations of truth values (i.e. the cells 2, 4, 5, 6, 8) ?

Let us go back (once more) to Aristotle's classical quaternary case of fundamental combinatory and our representation of it in terms of its associated Aristotelian lattice. One can do the following three remarks:

(a) "[0|1]" (subcontrariety) and "[1|0]" (contrariety) are, Aristotle didn't know it, "logical simplexes" (i.e. geometrical simplexes used in order to express logical oppositional relations – we call them so in so much they are the ingredients of which is made any logical bi-simplex);

(b) "[0|0]" (i.e. contradictory negation) can be seen as the result of "taking the minimum of each component of each of the two logical simplexes";

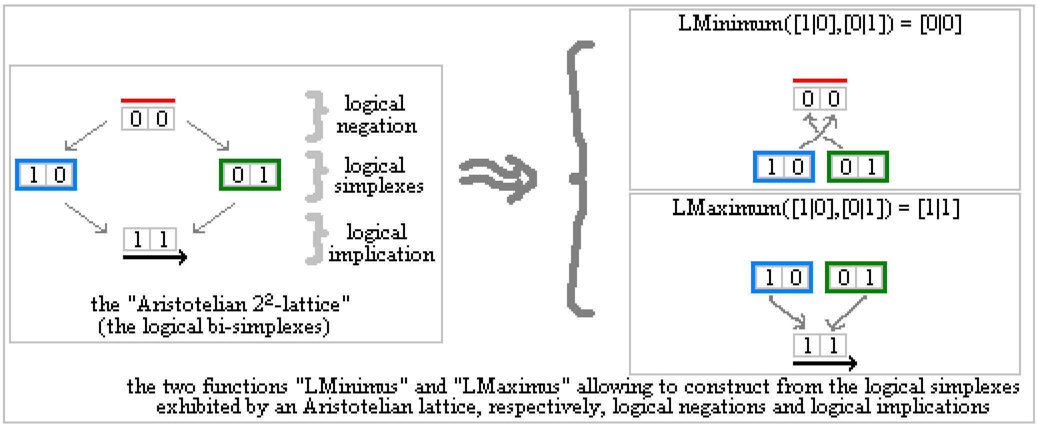
(c) "[1|1]" (i.e. implication) can be seen as the result of "taking the maximum of each component of each of the two logical simplexes".

Now, the remarks (b) and (c) above can be formalised by the introduction of two suitable new functions (both are defined as *f*: OC×OC → OC, where OC is the set of the opposition cells):

LMinimum([x|y],[w|z]) ≡ [Min(x,w)|Min(y,z)],

LMaximum([x|y],[w|z]) ≡ [Max(x,w)|Max(y,z)].

This can be expressed graphically as follows:



But this turns up to be a precious hint for us! For, if we dare (as it seems we can) generalise it, we have that arrows (in general) take the maximal elements of the simplexes, whereas negations (in general) take the minimal elements of the simplexes. Let us follow this intuition and see if it leads us to a real, good working generalisation.

This would give, if adapted to our 3^2-semantics for the hypothetical logical tri-simplexes:

(i) "[0|1]" (subcontrariety), "[1|0]" (contrariety) and "[½|½]" (indetermination) are *logical simplexes* (they lay on the "diagonal" – or middle lane – of the Aristotelian square lattice, their "sum" – i.e. the sum of the 2 values in each cell – is equal to 1). They are the "bricks" for composing the logical bi-simplexes and the logical tri-simplexes;

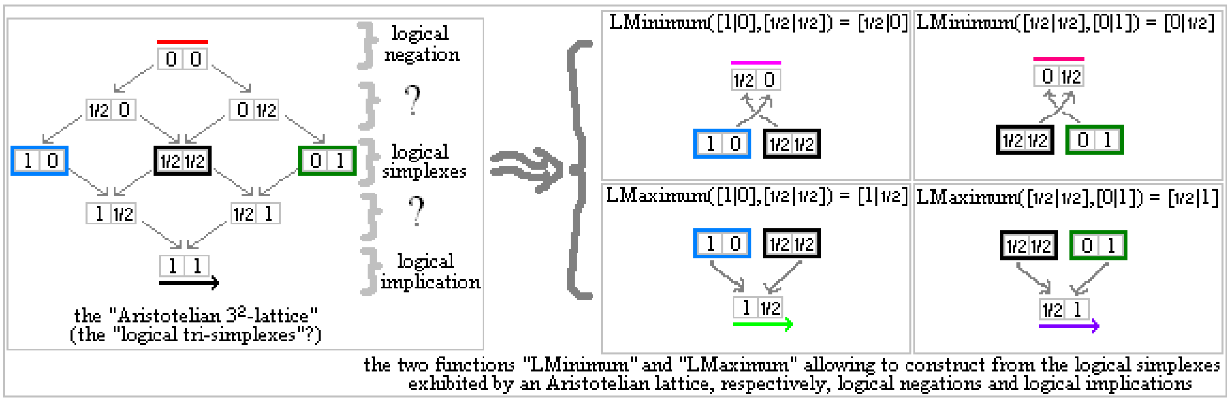
(ii) "[0|0]" and "[1|1]" are, respectively, the negation and the arrow between the "[0|1]" and the "[1|0]" simplexes (they are "top" and "bottom" of the Aristotelian lattice, the "sum" of the terms *x* and *y* of the negation cell [*x*|*y*] is strictly inferior to 1, the sum of the terms *x* and *y* of the implication cell [*x*|*y*] is strictly superior to 1);

(iii) "[0|½]" and "[½|1]" are, respectively the negation and the arrow between the "[½|½]" and the "[0|1]" logical simplexes;

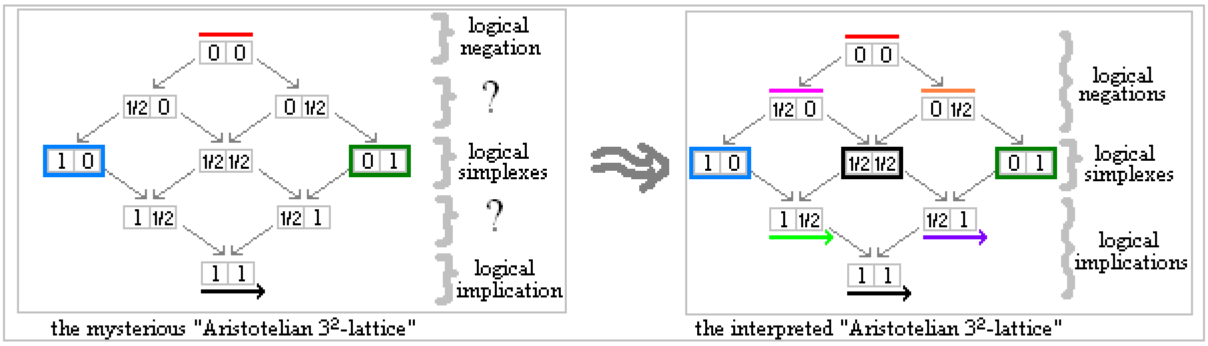
(iv) "[½|0]" and "[1|½]" are, respectively, the negation and the arrow between the "[1|0]" and the "[½|½]" logical simplexes.

(remark that we keep here the idea that negations have a "cell sum", i.e. the value of *x*+*y*, strictly inferior to 1, whereas implications have a "cell sum" strictly superior to 1)

Again, the previous remarks (iii) and (iv) can be represented, by our functions LMinimum and LMaximum.



If we adopt this line of interpretation we understand what are the 5 new cells and thus are authorised to fulfil (with arbitrary new colours) the general 3^2-Aristotelian lattice of the Aristotelian 3^2-semantics.



In other words, the relevant trick seems to be to consider tri-simplicial *n*-opposition as a matter of relations, two by two, between the three logical simplexes (the LMinimum function giving the negation operators, the LMaximum function giving the implication operators). Such hypothesis, if good, should, of course, be tested over the different values of *n* (we will do it in the rest of this chapter), thus looking for tri-simplicial equivalents of the bi-simplicial squares, hexagons, cubes, etc: not simply logical tri-simplexes, but logical tri-simplexes of dimension *m*.

19.03. How to interpret and use semantically the value “1/2”?

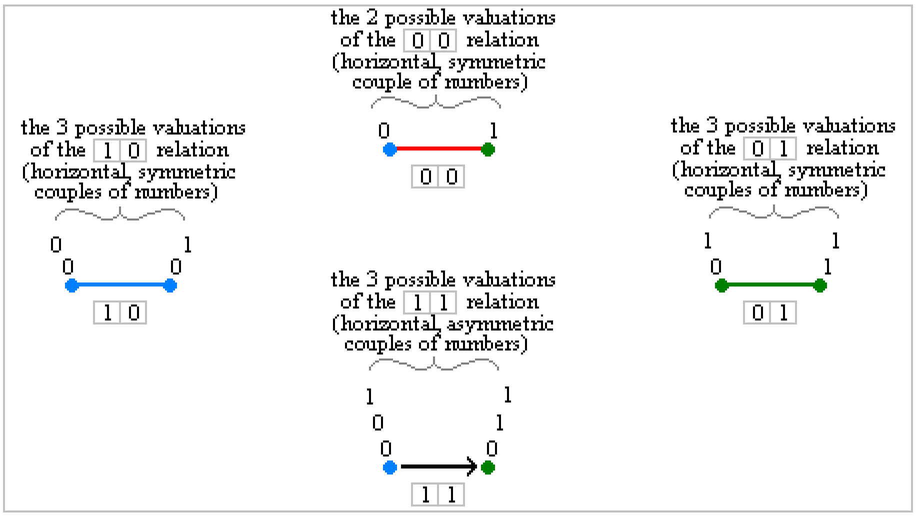
The next problem is that the geometrical-logical structures we are going to explore, the logical tri-simplexes of dimension *m*, are so far empty structures. So, we must face at least two problems before adopting them truly: (1) making sure that these structures are logically viable; (2) find a device in order to decorate them modally.

In Aristotle's classical case (the 2^2-semantic), in order to valuate (with 0 and 1) the vertices of the logical square embodying the four kinds of oppositions the following rule of pragmatic reading was implicitly followed:

0 = "we refuse completely" (the 0-valued configuration is totally rejected);

1 = "we accept completely" (the 1-valued configuration is totally accepted).

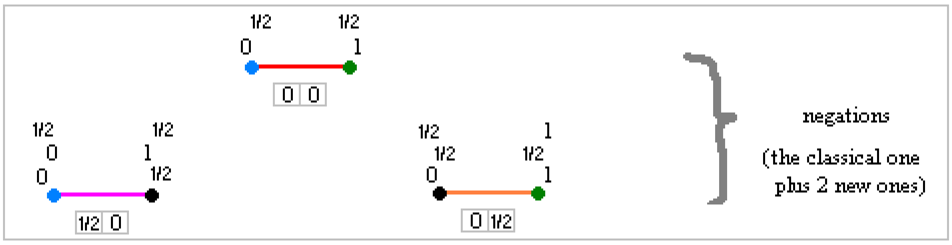
In order to make the things clearer in what follows, we propose to represent graphically this pragmatic behaviour with the following "table of semantic 2^2-valuations" of the referents of the [*x*|*y*] cells.



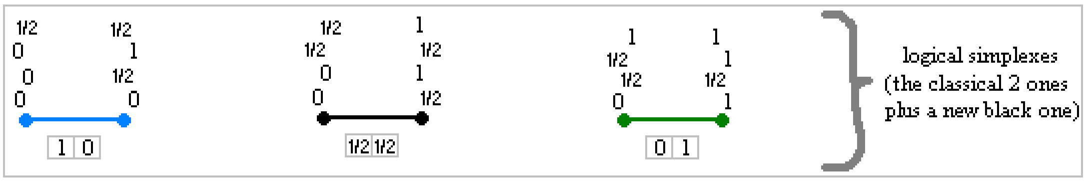
Remark that: (a) in the case of the symmetric relations (the first three segments, red, blue and green, those not oriented) the couples of numbers written above can (and must) be possibly reversed horizontally (we omit writing them twice here); (b) in the case of the asymmetric relation (the fourth one, implication) the written couple of numbers (above the arrow) are all and only the possible cases (no horizontal reversal is allowed here).

The horizontal couples of numbers above each of the four depicted relations must be read from bottom to top (0—0, than 0—1, than 1—1). For each of the four depicted relations, the series of horizontal couples of numbers above it is the complete list of valuations admitted by that relation. The first three relations are symmetrical, the fourth is not. To make things clearer, let us take two examples. (1) For the green [0|1] segment (the subcontrariety relation), the first horizontal couple of numbers above it says that "the 0 value is compatible with the 1 value" (and the other way round), whereas the second horizontal couple of numbers above the green bar says that "a 1 value is (also) compatible with another 1 value". All other possible coexistences (for instance: "a 0 with another 0") are excluded by the fact that they are not represented in the figure. (2) If we now look at the fourth relation, the black arrow (classical implication), the first horizontal couple of numbers above the black arrow tells us that "0 can imply 0”, the second horizontal couple of numbers above the arrow tell us that “0 can imply 1” (but we cannot symetrise here) and, finally, the third horizontal couple of numbers above the arrow tells us that “1 can imply 1”: here also, the valuations not depicted on the figure are *ipso facto* excluded, as is for instance the 1—0 valuation. All possible valuations can (and must) be read on the figure this way.

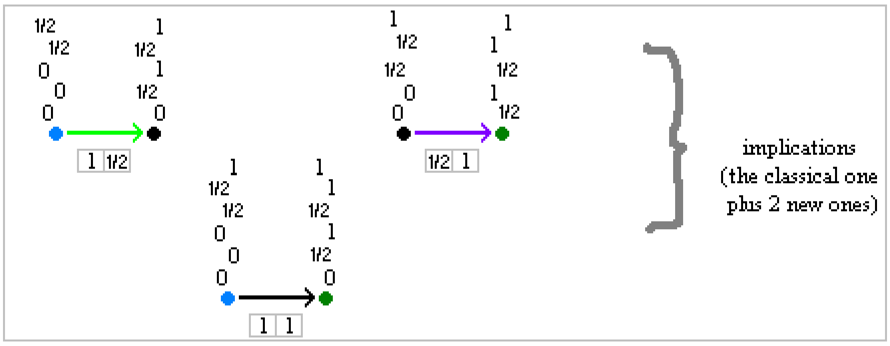
Now, relatively to the new, tri-simplicial case for valuating (with 0, ½ and 1) the 9 kinds of oppositions, we propose the following 3 tables for semantic 3^2-valuations. The first table is for the negations (here there are three negations instead of one).



The second table is for the logical simplexes (not to be confused with the logical *bi*-simplexes) (here there are three simplexes instead of two).



The third table is for the implications (here there are three implications instead of one).



The rule of pragmatic reading here is (the case “½” being new):

0 = “we refuse completely”

½ = “we accept some kind of compromise”

1 = “we accept completely”

The global table made up of the three will be the basis of our following examinations in this chapter, for it gives the fundamental properties of the logical tri-simplexes (if these do exist).

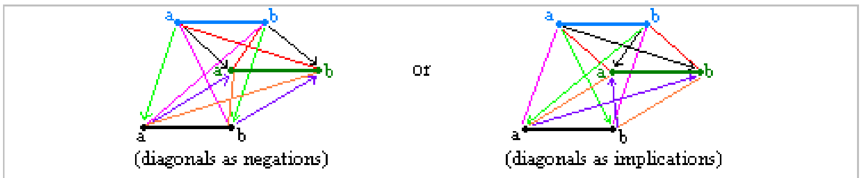
We must now try to see if, parallel to the known series of the logical bi-simplexes of dimension *m* there can be a series of the logical tri-simplexes of dimension *m*. In what follows we leave open the question of the possible existence (and shape) of a “logical tri-simplex of dimension 0” (a logical tri-point) and concentrate ourselves to the examinations of the dimensions *m* for $m \in N$, $m \geq 1$ (the logical tri-segment, tri-triangle, tri-tetrahedron …).

19.04. The logical 3-simplex of dimension 1 (the “tri-segment”)

So we begin by looking for a tri-simplicial analogue of the logical bi-simplex of dimension 1 (i.e. the logical square, or logical bi-segment). For graphical reasons to appear soon, it seems reasonable to call it “logical tri-segment” (of opposition).

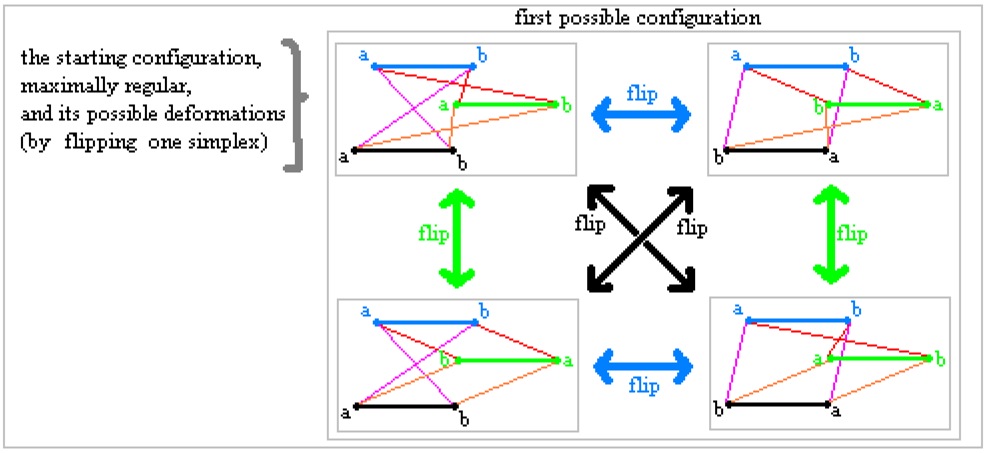
19.04.01. First problem: there are two alternative geometrical configurations

Right at the beginning we must face a problem: we have to choose between two possible geometrical configurations of the hypothetical tri-simplicial equivalent of the logical square, the first one resulting from putting the *negations* on the square’s diagonals (as is traditionally done with the logical bi-simplexes, cf. the left side of the figure), the second one resulting from putting rather on the diagonals the *implications* (cf. the right side of the figure). These two are not the same geometrically.

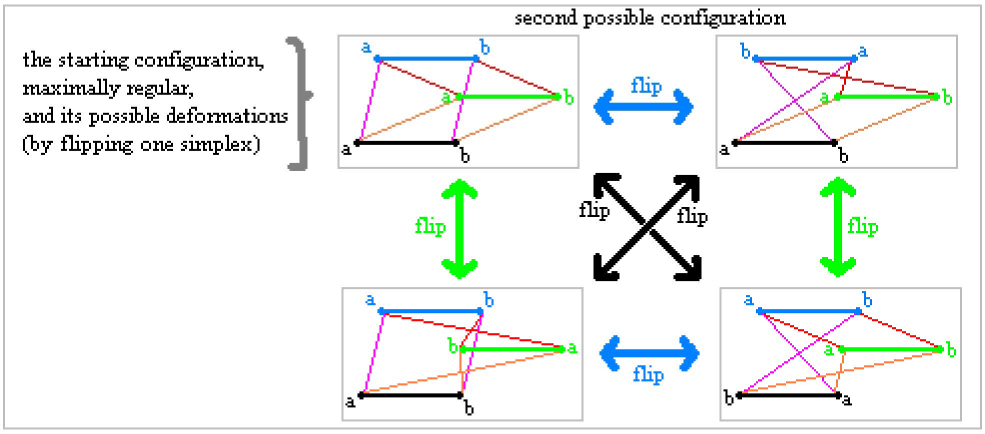


a choice to be made between two possible ways of conceive a logical tri-simplex of dimension 1

One could think it possible to have still more possible different configurations, by considering a figure where, for instance, some diagonals are negations whereas some other are implications. However, it can be proven geometrically that, in fact, all other possible geometrical configurations of our hypothetical “logical tri-square” are equivalent to one of the two mentioned ones. Here for instance are the four equivalent forms which the first configuration can take (for simplicity we omit here drawing the implication arrows of the tri-segment).

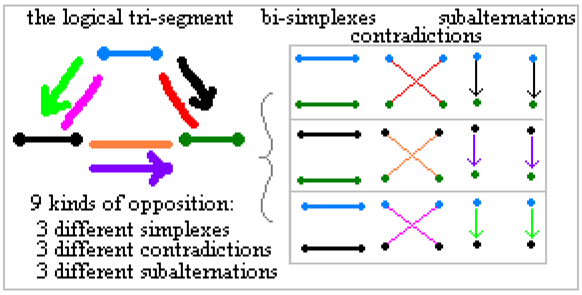


As a matter of fact, the previous and the next figures show that transforming configurations by switching one of their simplicial segments (just twisting it of 180° with respect to its middle point) leads to a set of eight possible spatial configurations, belonging to two distinct equivalence classes (the one on the previous figure and the one on the next figure). We name the vertices in two different ways: one in which diagonals (red, light green or violet) relate only analogous letters (“a” with, “a”, “b” with “b”) and one in which diagonals (red, light green or violet) relate only different letters (“a” with “b”, “b” with “a”, etc.).



19.04.02. The abstract decomposition of the logical tri-segment

Lacking for the moment of a precise knowledge about the right configuration to be chosen among the two possible ones must not prevent us to go further. For, if we now turn to the problem of having a useful representation of the global properties of the logical tri-simplex of dimension 1 (or “logical tri-segment”) the following model seems to be in any case acceptable. The model we are speaking of is the one showing the components of the tri-simplex of dimension 1.

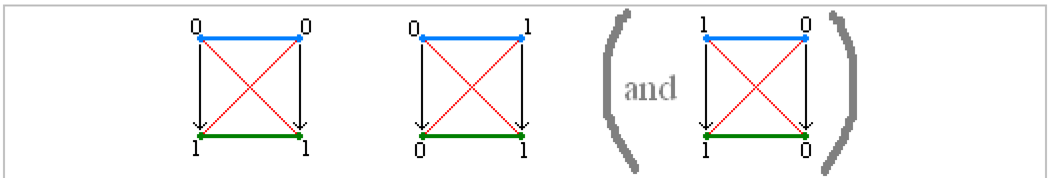


Remark that it looks like a classical logical square with, interpolated between its two constituting blue and green logical simplexes, a new, black logical simplex. The result seems to be that a logical tri-simplex of dimension 1 is made of three interlaced logical bi-simplexes of dimension 1 (the classical Aristotle's one plus two interpolated new ones).

19.04.03. Determining the possible valid valuations bottom-up

In order to try to find the shape of the tri-square, we face the problem by a bottom-top strategy: (1) we begin by looking for all (and only) the possible *valid decorations* (in terms of the three admitted truth-values: 0, ½, 1) of the previously seen constituents of the tri-simplex: that is, three logical bi-simplexes (the classical blue-green one plus two interpolated new ones, the orange-green and the blue-orange ones); (2) then, we look for the possible global combinations of these partial solutions: if they exist, the global valid decorations of the tri-simplex (of dimension 1) will be made of the valid decorations of the bi-simplexes (of dimension 1).

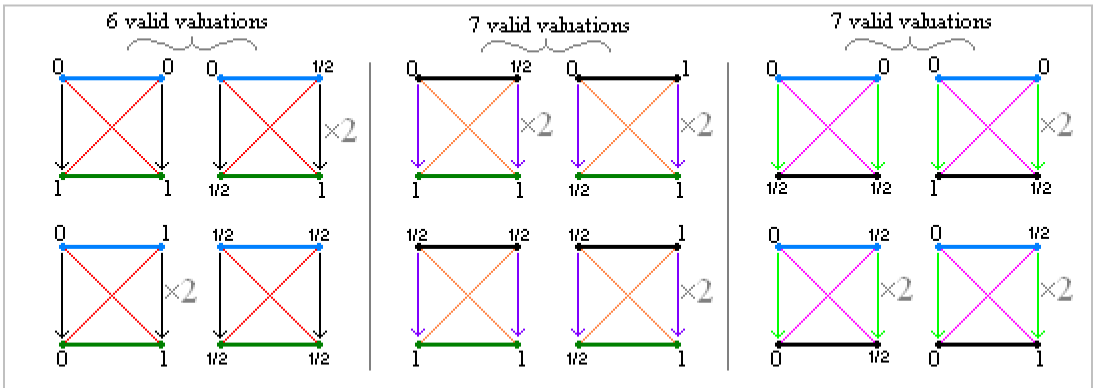
Just to fix ideas, remember that the valid valuations (i.e. the only possible valuations) of the logical square (the logical bi-segment) are the following three.



the 3 possible valid valuations of the global logical bi-segment (the second solution admits a valid mirror image)

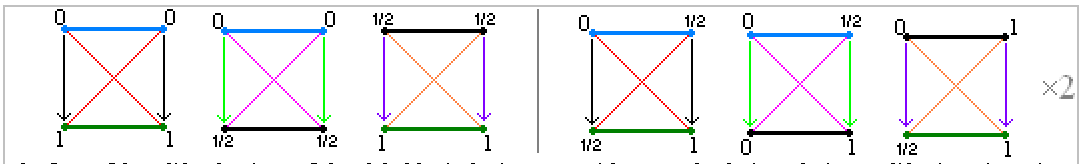
It is the equivalent of this, in the new, tri-simplicial framework, that we want to find.

A simple combinatorial examination, using the table of semantic 3^2-valuations (cf. *supra*) gives the following valid valuations of the three logical bi-simplexes constituting the logical tri-simplex (for simplicity, we omit to draw the solutions symmetric to those already given).



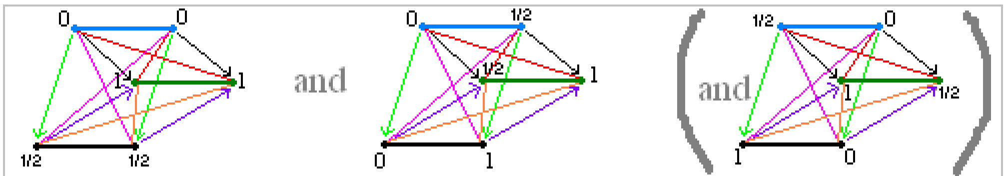
the possible valid valuations of each of the 3 logical bi-simplexes composing the logical tri-simplex (of dimension 1)

Knowing these partial solutions, we know that any eventual *global decoration of the logical tri-simplex* must be made of them. So we can look for the set of the valid valuations of the whole logical tri-simplex (of dimension 1) considered as any non-empty intersection of some valid valuations of its three components. It turns out that there are all in all 3 of them.



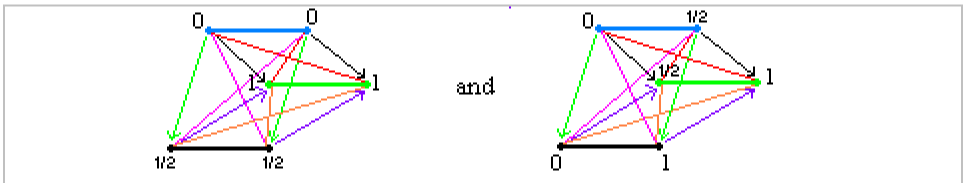
the 3 possible valid valuations of the global logical tri-segment (the second solution admits a valid mirror image)

The combinatorial examination shows accordingly that there are in fact three and three only possible such valid valuations of the tri-segment. Here is their global, 3D representation.



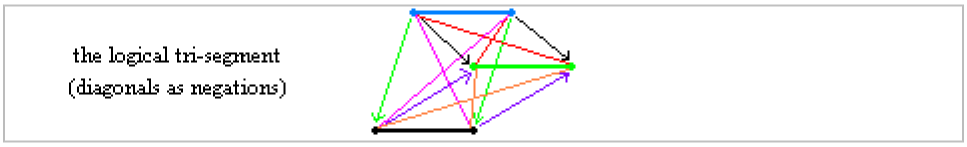
the 3 possible valid decorations of the logical tri-simplex of dimension 1

For simplicity's sake, we represent it like this, eliminating the redundancy due to the existence of symmetric solutions.



the 2 possible, non-redundant, valid decorations of the logical tri-simplex of dimension 1

The first one is so much symmetrical that any of the two previously seen geometrical configurations (i.e. diagonals as negations or diagonals as implications) does satisfy it; not so with the second one: it commands, to be effective, to adopt the geometrical configuration of the tri-segment where the three couples of diagonals are the three negations. So we adopt this configuration as being the good one.



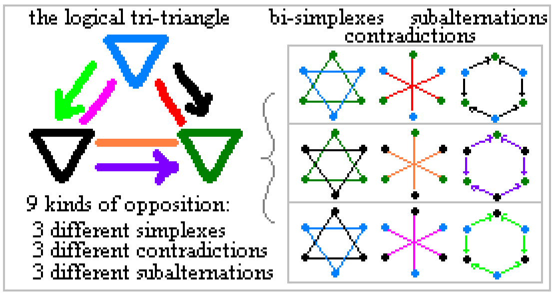
We must now verify if this geometrical solution (diagonals as negations) can be generalised (to the dimensions higher than 1): this seem to be the price for being allowed to say that the logical tri-simplexes of dimension *n* do exist. And *a priori* it isn't at all clear that this will be the case.

19.05. Logical 3-simplexes of dimension 2 (or tri-triangles)

If the logical tri-simplexes of dimension 1 are logical tri-segments, the logical tri-simplexes of dimension 2 are "logical tri-triangles". They are the tri-simplicial analog (in fact: expansion) of the bi-simplicial "logical hexagons". Let us see their main features.

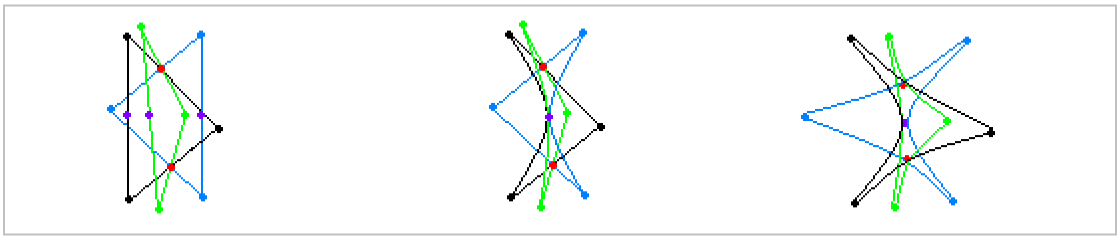
19.05.01. The abstract decomposition of the logical tri-triangle

We begin in an intuitively clear way by adapting our previous model of the decomposition of the logical tri-simplex (here: of dimension 2). So we have now three triangles (three geometrical simplexes of dimension 2), the classical blue (contrariety) and the classical green (subcontrariety) plus a new one, black (to be interpreted). Between the blue and the green simplex we have the classical red negation and black implication. Between the blue simplex and the newly interpolated black simplex we have the rose sub-negation and the light-green sub-implication. Finally, between the black interpolated simplex and the green one we have the orange sub-negation and the violet sub-implication.

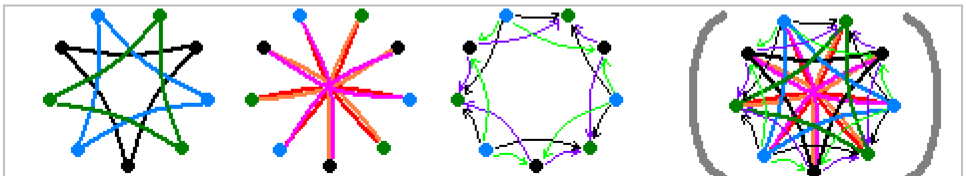


Again, as previously with respect to the logical square, remark here that it looks like if it just was the classical known logical bi-simplex of dimension 2 (Sesmat and Blanché's logical hexagon) with, interpolated between its constitutive blue and green logical simplexes, a new, black logical simplex, which finally gives two more logical bi-simplexes of dimension 2 (two more logical hexagons).

If we try to represent it globally, for instance inthe 3D space, it seems the result may look like this.



Remark that this looks a little bit like a “non-Euclidean” extension of the *n*-dimensional geometry of logical opposition: the triangles of the logical tri-simplexes here are such that the sum of their inner angles is inferior to 180°. But this 3-dimensional representation does not clearly respect central symmetry. In fact, it turns out that there is no need here to postulate a 3-dimensional representation: the following 2-dimensional representation suffice perfectly.

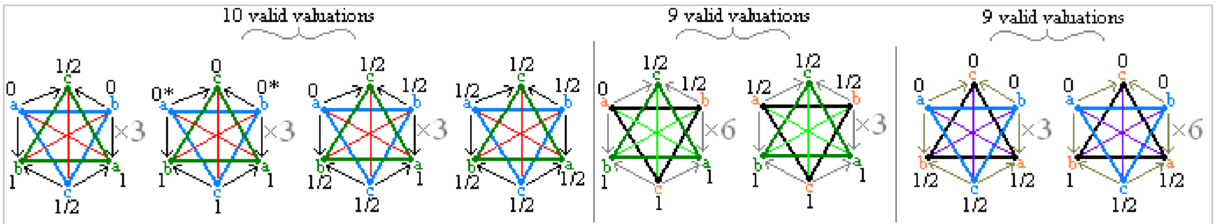


the logical tri-triangle (simplexes, negations, implications)

Notice, however, that this truly suggests, as hinted above, some kind of “non-Euclidean” feature, for the 3 triangles involved do not seem to have as sum of their inner angles the value 180°. The essential, pragmatic remark seems to be that this figure is very complex for the eye (but still visible).

19.05.02. Determining the possible valid valuations bottom-up

If we adopt the same bottom-top strategy as before, a simple (but tedious) combinatorial examination gives us the following result (we give just the result, the reader can check her/him/itself).



the possible valid valuations of each of the 3 logical bi-simplexes composing the logical tri-simplex (of dimension 2)

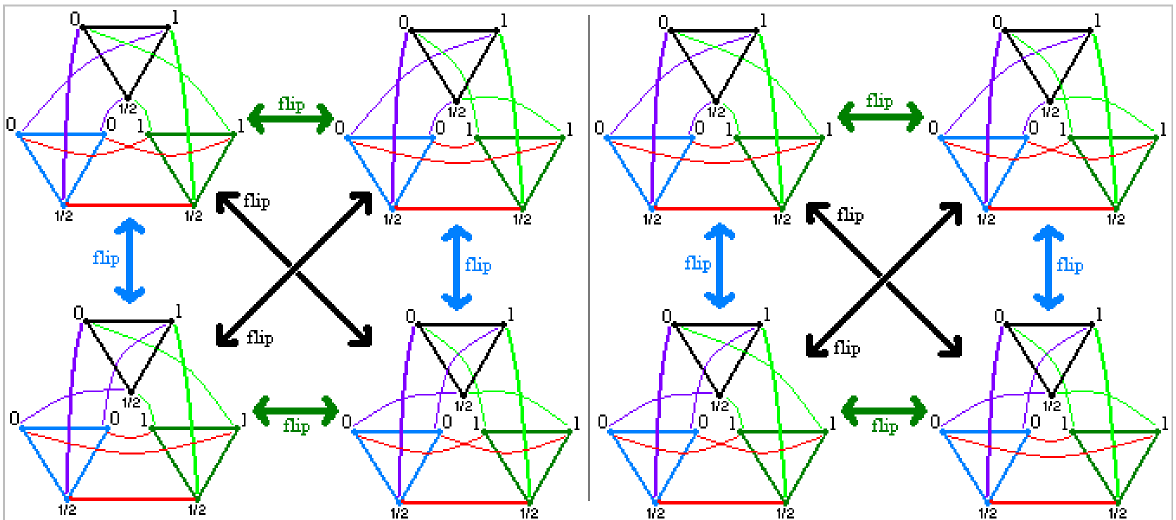
Note that, differently from the case of the logical square, here we must omit taking as a valid decoration the one in which all three blue vertices are given a “0” value: a logical square can have a true blue “0, 0” decoration (e.g.: it is both false that “all crows are black” and that “no crow is black”), whereas a logical hexagon cannot have a true blue “0, 0, 0” decoration (e.g.: if the two previous assertions on crows are both false, the third one must be true: “some crows are black and some crows are not black”). The special status of the logical square (or bi-segment) is due to its being a *weak* 2-opposition[233].

Now, if there are valid valuations of the global logical tri-simplex of dimension 2 (the logical tri-hexagon), these must be made of the valid valuations of the composing logical bi-simplexes of dimension 2 (for any global, valid decoration is equivalent to some non-empty intersection of some valid decorations of each of the three components of the tri-simplex).

Again, a combinatorial examination over the previous valid valuations of the logical bi-simplexes shows that there are in fact 2 and 2 only possible valid valuations of the logical tri-simplex of dimension 3. We show these 2 solutions, without drawing neither the negations nor the implications, in the following picture.

the 2 only possible valid valuations of the logical tri-simplex of dimension 2 (the logical tri-hexagon)

Having now to come back to the global geometrical structure of the tri-triangle we have to represent the diagonals, and this can be done in several ways. Here we represent the 8 possible ways of representing the diagonals (for graphical simplicity we omit representing the the implications – they follow the usual simple convention ruling the logical hexagons).

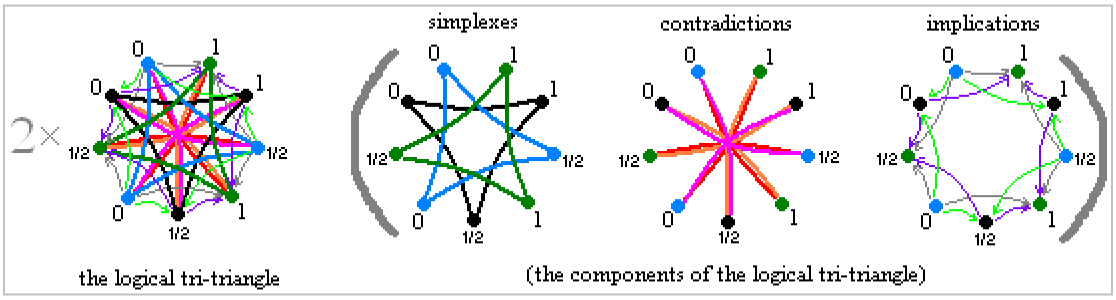


the two possible geometrical configurations of the logical tri-simplex of dimension 2 (the logical tri-hexagon)

233 Cf. Pellissier, R., “2-opposition and the topological hexagon” (forthcoming).

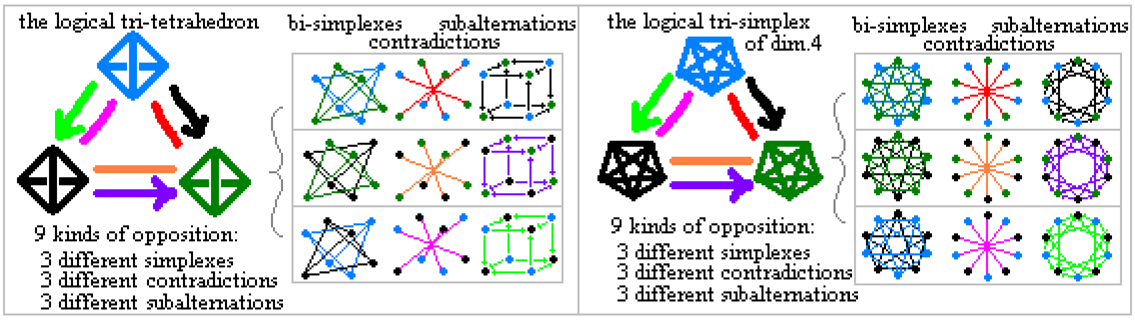
One sees that in fact there are redundancies: eliminating them we finally have, as previously, only two different kinds of possible geometrical configurations of the logical tri-triangle.

Hence, finally, the two possible (symmetric) global valid valuations of the logical tri-triangle (here we give only one of them).



19.06. Logical 3-simplexes of dimension 3 (3-tetrahedron) and 4

The same reasoning can be carried over the next cases, that is, for the logical tri-tetrahedron and for the logical tri-hyper-tetrahedron (or logical tri-simplex of dimension four).



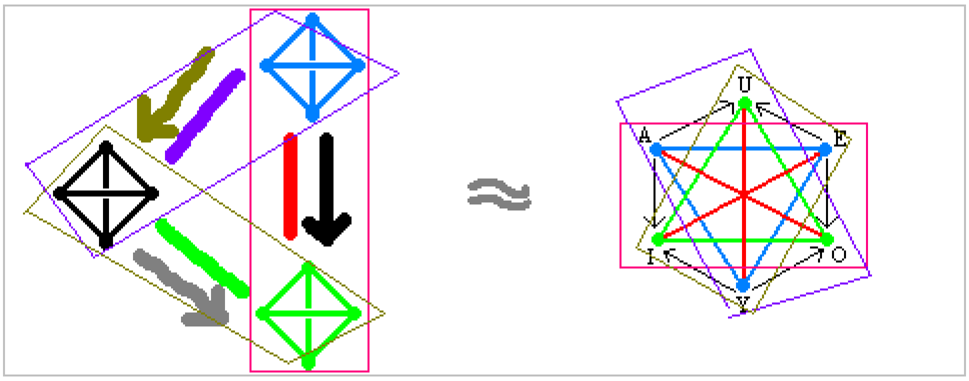
Semantic considerations analogue to those developed for the logical tri-simplexes of dimension 1 and 2 will provide the means for decorating these structures (with three possible truth-values) and evaluating them. Here as well (as for the tri-segment and the tri-triangle) it can be shown that there is only one valid global decoration (for each), if one eliminates the symmetric cases (otherwise there are two possible cases).

19.07. What are the logical tri-simplexes? Are they useful?

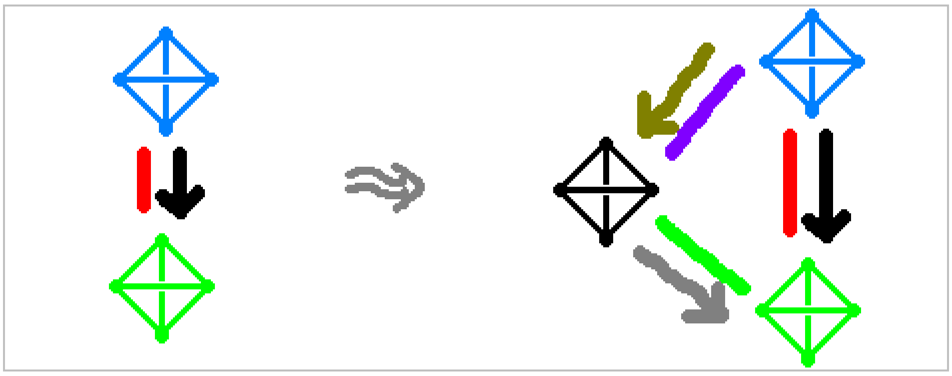
Having explored some basic elements of this structure which we propose to accept to the logical pantheon, it is time to try to see more closely what it does amount to, properly speaking. With this aim in mind, let us first recall briefly what we have seen so far.

19.07.01. What are, by the way, the logical tri-simplexes?

Each logical tri-simplex is made of three logical bi-simplexes (among which the classical Aristotelian one). In this respect, the discovery of the logical tri-simplexes shares some similarity with that, by Sesmat and Blanché, of the logical hexagon (this last containing three logical squares, among which the classical Aristotelian one cf. ch.8 *supra*).



One further important remark is that, consequently, the logical tri-simplexes can be seen as logical bi-simplexes with an interpolated (black) logical simplex.



The effect over NOT seems to be at least 4-tuple:

(1) there is the introduction of a third truth value (i.e. "½"), read by us (arbitrarily) as "undetermined";

(2) there is the introduction of a new logical simplex (the black one), which is a new opposition relation;

(3) we obtain the interpolation of two new logical bi-simplexes, and hence the interpolation of 2 new negations and 2 new implications.

(4) the logical tri-simplexes have not one, but three distinct symmetry centres[234].

[234] Cf. Moretti, A., "La complexité formelle de la "contradiction" dans la géométrie des oppositions", forthcoming in: *Actes des Ateliers sur la contradiction* (École des Mines, Saint-Etienne, 19-21 mars 2009).

19.07.02. Are the logical tri-simplexes useful?

A possible objection is that we do not need the tri-simplicial machinery in order to make opposition tri-valued (supposing we need to cope with such a task sometime or other). As a matter of fact, why don't we simply add to NOT the third truth-value, i.e. "½"?

There are at least three answers which we could give to this.

1) First, the logical tri-simplexes allow to express not only tri-valence, but also a new form of opposition (the "black" opposition) and, thereby, 2 new forms of negation end 2 new forms of implication. And inside them the simpler case of the classical logical bi-simplex, now tri-valued, is included. So why to renounce to the whole logical tri-simplexes?
2) The tri-simplex theory, *via* the theory of the Aristotelian p^q-semantics, offers a mathematical general framework for opposition: that is, this generalisation is nice as such.
3) Finally, it might be answered that it seems to be inherent to the very logic of opposition to generate a full combinatory of answers once the basic ingredients given. In other words, the Aristotelian 3^2-semantic (and what follows from it) is very natural and plausible (and unavoidable), once you introduce the third truth value ("1/2") into NOT. From this point of view, it seems natural to take the tri-simplicial machinery (or "package") as being the adequate expression of tri-valued opposition.

Again, the best apology for the logical tri-simplexes is to remember that they show that, mathematically speaking, the logical bi-simplexes are a particular case of something bigger (something still needing to be elucidated). This is already a very interesting result! Remember, for instance, that S5 is in some important sense contained in the logical bi-simplexes.

A much more profound question seems to be that of the philosophical interpretation of the logical tri-simplexes. This question does not seem to be easy. Seemingly, there still is a need of philosophical investigation thereupon. The problem is at least 4-tuple:

a) understanding the black logical simplex;
b) making sense of the two new interpolated logical bi-simplexes;
c) making sense of the 2 new weakened contradictions (classical and sub-classical negations);
d) making sense of the two new weakened subalternations (implications).

The black logical simplex, by comparison with its two blue and green mates, seems to be characterised by the fact of refusing "extremality": all opposition combinations are accepted by it, excepted the 0-0 one and the 1-1 one. This, of course, in a way different from that of the classical contradiction (which also excluded these two extreme cases).

As for the applications, we will discuss partly this topic in ch. 22 below.

19.08. Concluding remarks over the logical tri-simplexes

The price to be paid for going beyond the "transcendental bound" of the logical bi-simplexes (by way of the logical tri-simplexes) is to accept having more kinds of negations (3 instead of 1) and more kinds of implications (3 instead of 1).

A good point of the tri-simplexes, in a sense, is that this is a conservative extension (as we have seen, the logical tri-simplexes contain properly the logical bi-simplexes).

We saw that the interpretation of the 9 kinds of opposition appearing here is made easier by taking into account the two functions LMinimum and LMaximum.

A very nice property is that, like the bi-simplexes, the tri-simplexes go into infinite (tri-simplicial opposition is a new form of *n*-opposition).

What is invariant – or "transcendental" – this way (in this infinite series: tri-segment, tri-triangle, tri-tetrahedron, …): 9 elements (always 3 simplexes, three negations, three implications).

The problem: at the meta-level, the possible cases remain 2 (some kind of profound duality – or dualism – is still untouched by our generalisation).

20.
BEYOND THE LOGICAL TRI-SIMPLEXES: EXPLORING THE USEFUL LOGICAL *P*-SIMPLEXES ($4 \leq P \leq 5$)

In this chapter we want to see if the interesting result about the logical tri-simplexes can be generalised. In this view we try to explore "logical *p*-simplexes" (of dimension *m*) with $p \in \{4,5\}$. As we are going to see, the logical quinia-simplexes allow to express enough interpolated values: the "pivotal" (black) opposition as well as the "weak contrariety" and the "weak subcontrariety" (in someintuitive, qualitative respect, the further weakenings can be seen as being just more fine-grained refinements).

20.01. Logical quadri-simplexes

Having shown that the logical bi-simplexes are interesting structures (cf. ch.11-17), and that there also are logical tri-simplexes (cf. ch.19), it seems to be very natural to ask whether logical *quadri*-simplexes (or 4-simplexes) do exist. Here we explore the world of such logical quadri-simplexes, extending to them the techniques developed so far for the bi- and the tri-simplexes. Additionally to the strong (positive) suspicion towards the existence of a more general series of logical *p*-simplexes ($p \in \mathrm{N}$, $p \geq 2$), the logical bi-, tri- and quadri-simplexes suffice to lay the basis for a future study on "dynamic opposition" (i.e. the ways in which a static opposition can evolve, for instance – but not exclusively – being a *p*-simplex of changing *p*-arity).

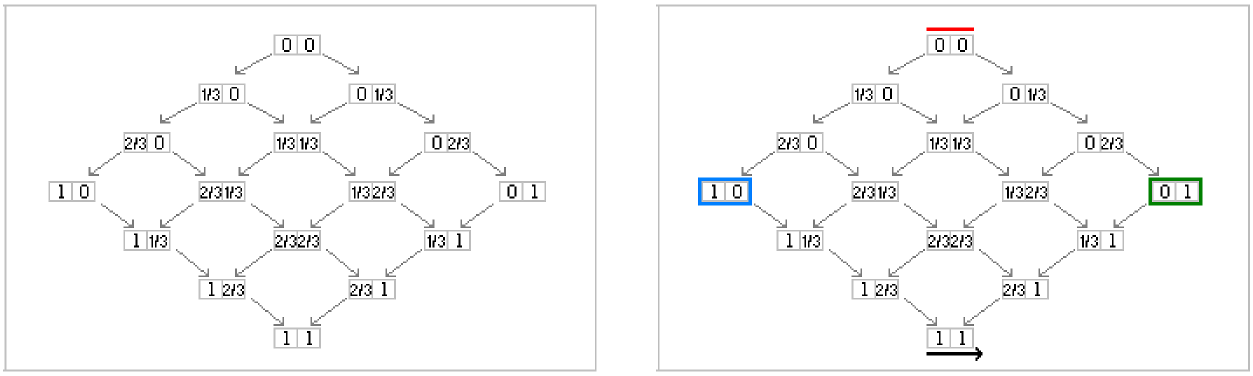
20.01.01. The Aristotelian 4^2-semantics for the logical quadri-simplexes

The first historical "Aristotelian semantics", the one leading Aristotle to his "logical square", was one with two forming questions and two possible truth values ({0, 1}) for each answer to be given to one of the two questions (cf. ch.18). This gives four possible combinations, corresponding each to one of the four kinds of possible oppositions, these four being each represented by a colour in the logical square (blue, red, green, black).

We proposed to generalise this notion of Aristotelian semantics, so to admit any number *q* of possible forming questions and any number *p* of possible truth values of the answers. From this point of view a general Aristotelian semantics is a p^q-semantics (cf.

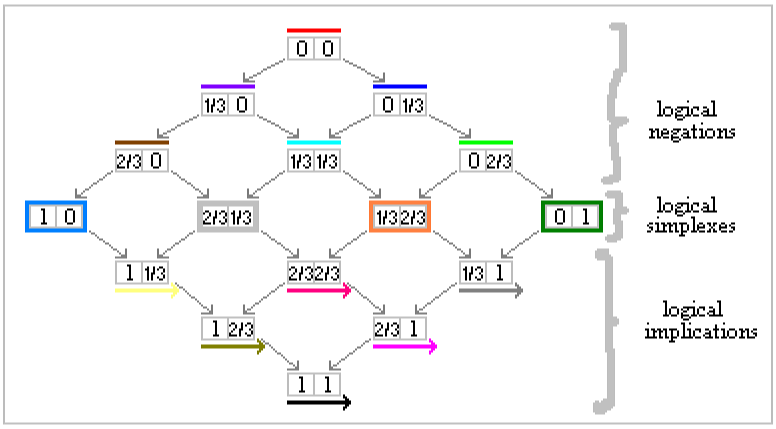
ch.18). We explained how the tri-simplexes, in order to exist, need a 3^2-semantics (neither the 2^3- nor the 3^3- do work for them).

The Aristotelian semantics we need here in order to build logical quadri-simplexes is the 4^2- one, that is, one admitting two forming questions and four truth values ({0, 1/3, 2/3, 1}) for each answer. This basis produces a combinatory of 16 possible cases (ordered couples of elements of {0, 1/3, 2/3, 1}), each one being a kind of opposition, whose lattice, the lattice of the Aristotelian 4^2-semantics, is the following (at the left side all the elements, at the right side the already known oppositions).



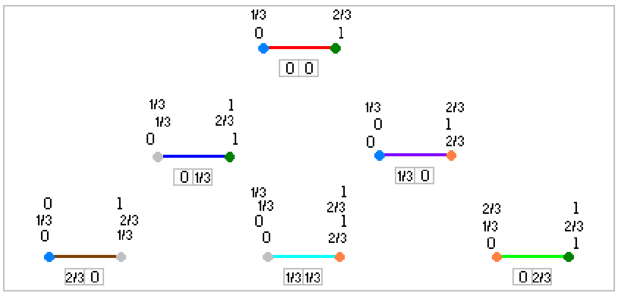
Note that the black logical simplex ([1/2|1/2]) of the logical tri-simplex has disappeared. It will appear back in the lattice of the logical *p*-simplexes where *p* is odd. So, the 2 sub-negations ([½|0], [0|½]) and the two sub-implications ([1|½], [½|1]) typical of the logical tri-simplex also disappear here. The only terms common with the logical tri-simplex are the four classical Aristotelian terms: the logical simplexes [1|0] and [0|1], the negation [0|0] and the implication [1|1] (right side of the previous figure). So there is some kind of non-conservativity of the logical quadri-simplex with respect to the logical tri-simplex.

We can give new colours to the 12 new terms. By considerations fully analogous to those made previously (cf. ch.19) about the Aristotelian lattice of the logical tri-simplex, we get a structure (a square lattice) such that it is made of 16 kinds of oppositions, of which one family is that of the logical negations (the upper triangular part of the lattice), another one is that of the logical simplexes (the horizontal middle line of the lattice) and a third one is that of the logical arrows (the lower triangular half of the lattice). In other words, besides the 16 kinds of oppositions one can say that there are, truly speaking, three “meta-kinds” of oppositions, very nicely ordered: from top to bottom, negations, simplexes and implications.

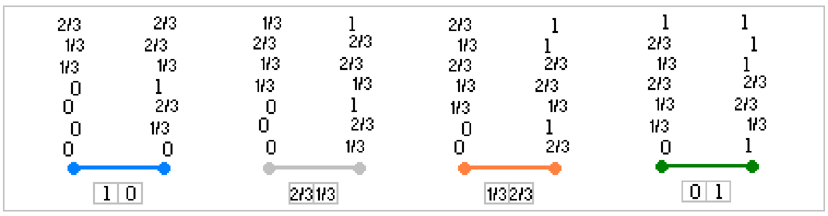


20.01.02. The four-valuedness of the logical quadri-simplexes

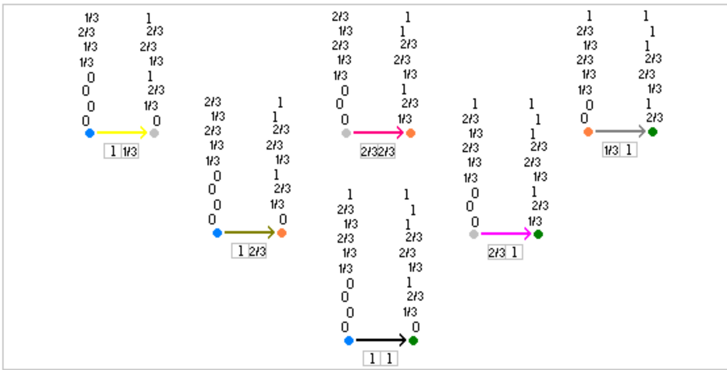
The 16 kinds of oppositions, distributed into the 3 meta-kinds of oppositions (simplexes, negations and implications) divide in turn into two kinds of relations: symmetric and asymmetric ones (the implications are asymmetric). The following three "tables" show which valuations (i.e. assignation of truth-values) are valid for each of the 16 opposition relations. For the symmetric relations (negations and logical simplexes) the numerical values put above the coloured opposition segments can and must be considered as reversible: for instance, if "0—1" is valid, "1—0" is valid as well. Not so for the asymmetric relations (the implications): if, for instance, "⅓—1" is written above the left-to-right arrow, this does not imply that "1—⅓" is valid as well. Moreover, a deeper symmetry is broken here: if, say, "1—1" is a valid valuation of an arrow, it is not granted that its "symmetric extreme", the valuation "0—0", is valid (and the other way round). Here is the table of the extensional definitions of the 5 kinds of contradiction relations of the logical quadri-simplex, showing in each of them which valid valuations are valid in it.



Here is the table of the semantical definitions of the 4 logical simplexes of the logical quadri-simplex, showing in each of them which valuations are valid in it.



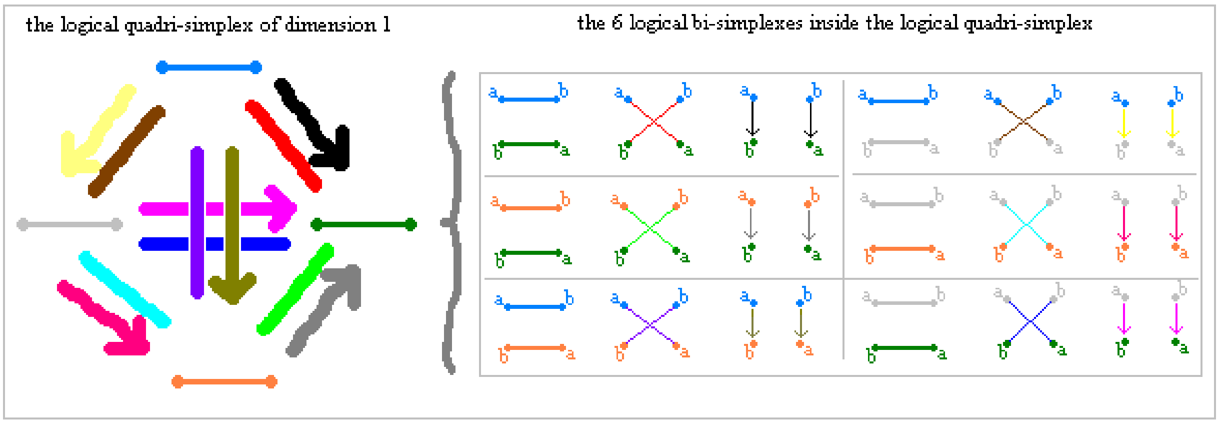
Finally, here is the tableof the extensional definitions of the 5 kinds of subalternation relations of the logical quadri-simplex, showing in each of them which valuations are valid in it.



Notice that all this is done by following the prescriptions intuitively coded by the Aristotelian 4^2-semantic.

20.01.03. The logical 4-simplex of dimension 1 (the logical 4-segment)

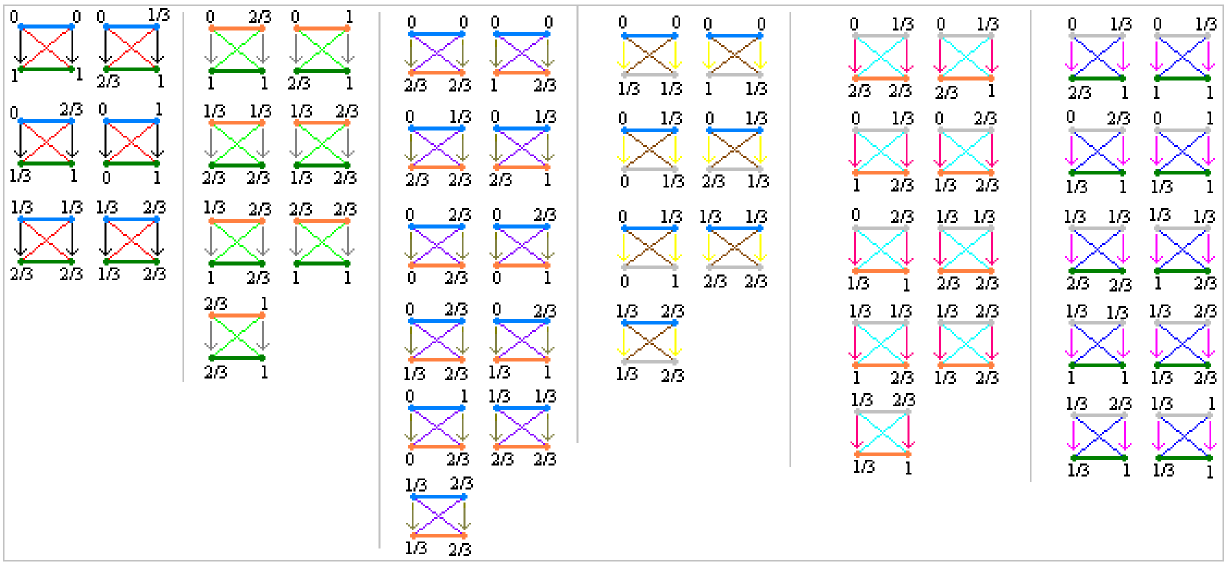
The logical quadri-simplex of dimension 1 is a logical 4-segment. The overall geometrical structure of the logical 4-segment can be summarised as follows.



Notice that the logical quadri-simplex (4-segment), clearly made of 6 logical bi-simplexes (2-segments), is also made of 4 structures isomorphic to the logical tri-simplex (3-segment).

As was the case for the logicaltri-simplexes (cf. ch. 19), if we want to find the possible valid valuations of the global logical 4-segment the easiest way seems to be to (1) calculate the valid valuations of each of the 6 logical bi-simplexes (the logical 2-segments) constituting it and (2) find by intersection the valid valuations of the whole logical 4-segment.

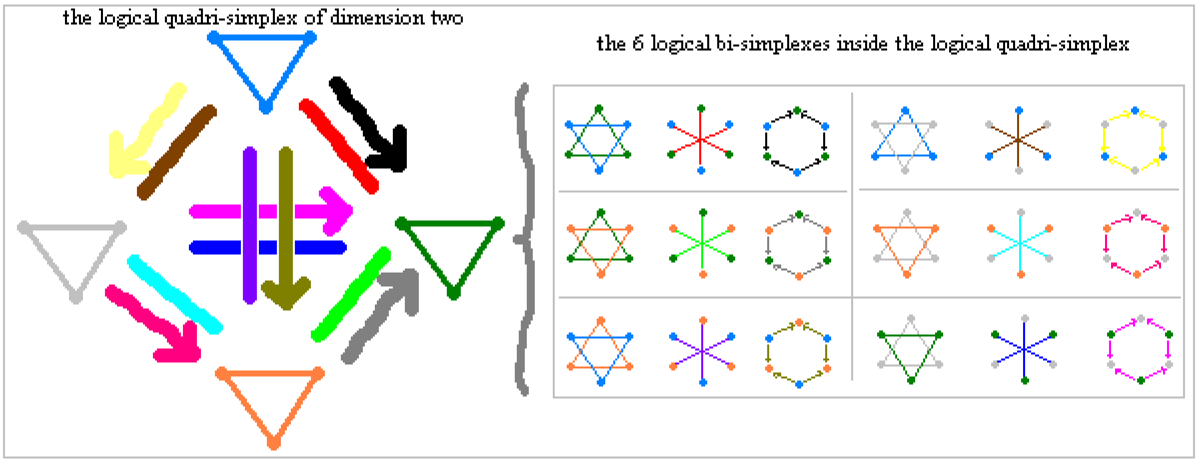
The valid valuations of the logical bi-segments constitutive of the logical 4-segment are the following (remember that, for bi-segments, blue contrariety is allowed the “0—0” valuation as valid).



As a matter of fact, a combinatorial calculation shows that there seem to be 5 valid valuations of the logical 4-segment (we will not deepen this fact here).

20.01.03. The logical 4-simplex of dimension 2 (the logical 4-triangle)

As usual, by means of a decomposition intoits constitutive NOT elements (logical simplexes, negations, implications), the overall geometrical structure of the logical 4-triangle can be summarised as follows.

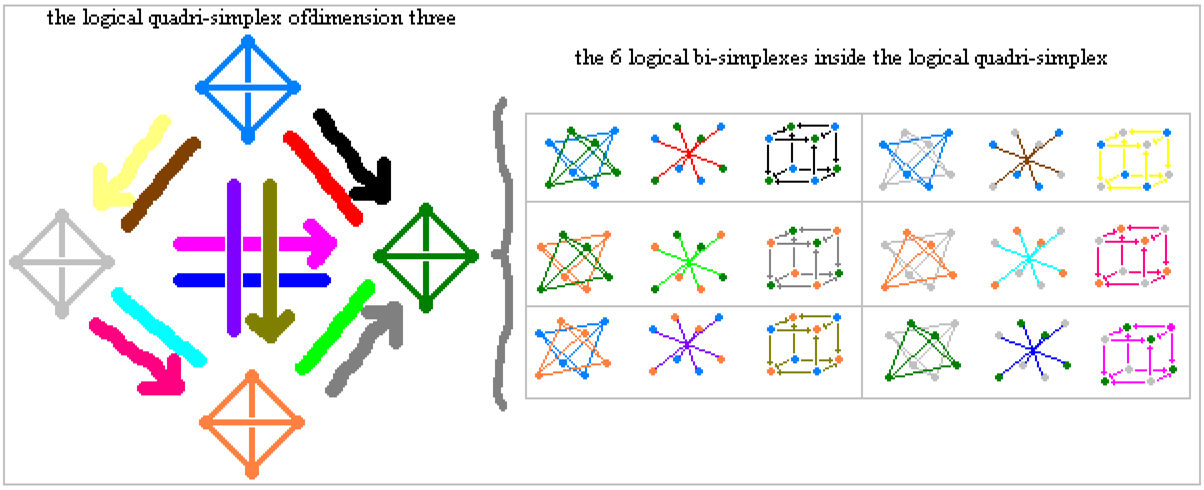


Notice that, as for the logical quadri-segment before, the logical quadri-simplex of dimension 2 (the logical 4-triangle), clearly made of 6 logical bi-simplexes (2-triangles), is made of four structures isomorphic to the logical tri-simplex of dimension 2 (the logical 3-triangle).

Here as well, if we want to find the possible valid valuations of the logical 4-triangle the easiest way seems to be to (1) calculate the valid valuations of the 6 constituting logical bi-simplexes (the logical 2-triangles) and (2) find by intersection the valid valuations of the whole logical 4-triangle.

20.01.04. The logical 4-simplex of dimension 3 (the logical 4-tetrahedron)

The overall structure of the logical 4-triangle can be summarised as follows.



Note that the logical quadri-simplex of dimension 3 (the 4-tetrahedron), clearly made of 6 logical bi-simplexes (2-tetrahedra), is made of 4 structures isomorphic to the logical tri-simplex (3-tetrahedron).

Here as well, if we want to find the possible valid valuations of the logical 4-tetrahedron the easiest way seems to be to (1) calculate the valid valuations of the 6 constituting logical bi-simplexes (the logical 2-tetrahedra) and (2) find by intersection the valid valuations of the whole logical 4-tetrahedron.

20.01.05. The logical 4-simplex of dimension *n*

In a near future we should be able to give the general law for the composed valid valuations (at least we must keep this aim in mind).

20.01.06. Are logical quadri-simplexes useful? How?

As we saw, the logical quadri-simplexes (of dimension *m*) are obtained from changing, with respect to the Aristotelian 2^2-semantic, the value of *p*. This means that instead of taking 2 truth values (the “false” and the “true”) we take four, which we interpreted as “false”, “almost false”, “almost true” and “true”. The question seems to be still open of knowing whether one can interpret otherwise these four values: a classical such example would be given by values like the ones of the Belnap-Nuel relevant systems (as for instance FDE), where the four are “only true”, “neither true nor false”, “both true and false”, “false only”. At this stage we do not know if an oppositional treatment of systems akin to FDE is easily possible and whether it takes the form, being four-valued, of a logical quadri-simplex.

Besides the bare fact of being 4-valued (instead of 2-valued), the logical quadri-simplexes (of dimension *m*) seem to be rather useful in so much they allow to express weakened opposition relations. In other words, by the interpolation of two new logical simplexes (the orange one and the grey one) they make it possible to express the (new) oppositional relations of “weak contrariety” and its dual, that of “weak subcontrariety”. If we want a logical *p*-simplex able to express, at the same time, weakened oppositions and the strange case of pivotal opposition (the black logical simplex emerged within the logical tri-simplexes of ch. 19), we need to go further considering the case of the logical quinia-simplexes.

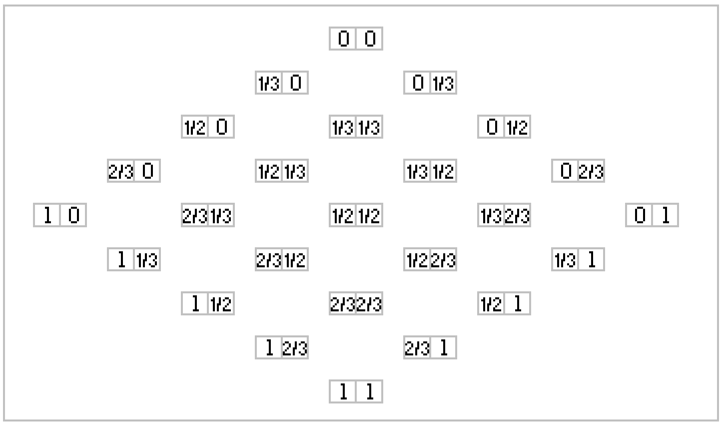
20.02. Logical quinia-simplexes (or penta-simplexes)

The logical bi-, tri- and quadri-simplexes do exist. Do the logical quinia-simplexes exist as well? The answer is not straightforward, for if the general formalism for building logical *p*-simplexes seems to be applicable for the case where p=5, semantical considerations show that it is not at all sure that such a quinia-simplex would admit valid valuations : in other words, it could be just an empty structure, unable to sustain a decoration with truth values of its empty variables. The existence of a series of the logical *p*-simplexes would be useful for building a structuralist formal theory of “dynamic opposition”. Hence the importance of the present investigation.

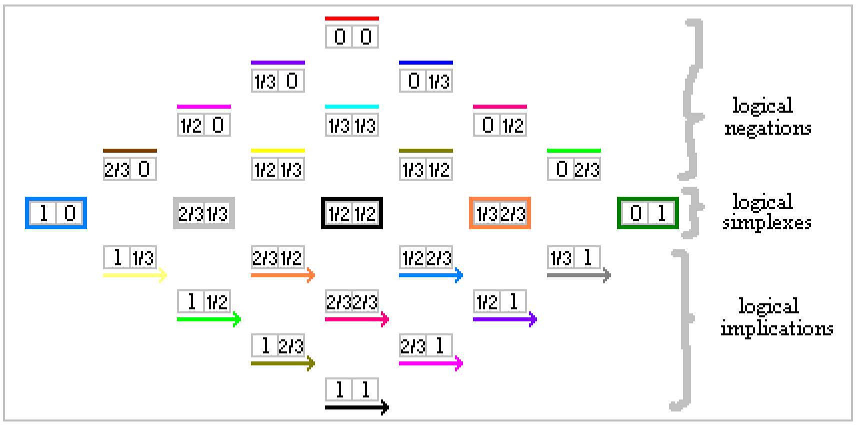
The problem is that we lack, so far, of a proof of the fact that logical *p*-simplexes in general do exist, i.e. that they always admit at least one valid valuation (the quadri-simplex, already, seems to show some kind of rarity of the valid valuations).

20.02.01. The Aristotelian 5^2-semantics for the logical quinia-simplexes

The Aristotelian 5^2-semantics is such that its outcomes are equivalent to the Cartesian product {0, 1/3, ½, 2/3, 1}×{0, 1/3, ½, 2/3, 1}. That is, it is made of 5 truth-values, with {0, 1/3, ½, 2/3, 1} meaning {false, almost false, undetermined, almost true, true}. Another possible semantic interpretation would be that of using “neither” and “both” (cf. *infra*). The lattice generated by the Aristotelian 5^2-semantics is the the following (we omit ist lattice-arrows).

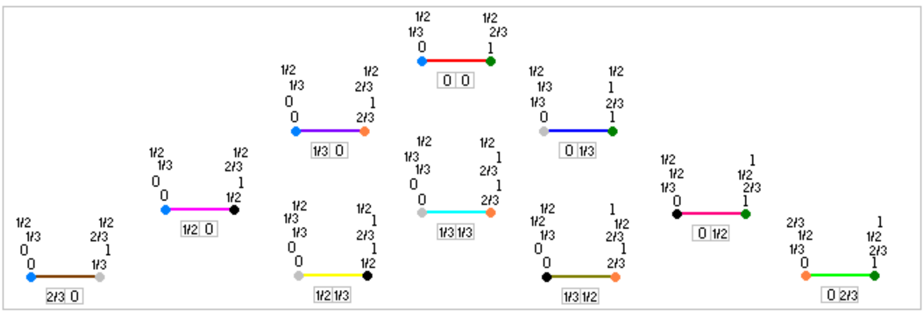


To interpret it one must recall the formation principle of the contradictions: take the minima of the two simplexes beneath it (the function LMaximum), as well as the formation principle of the subalternations: one has to take the maxima of the two simplexes above it (by the function LMinimum). By using these two functions, the opposition kinds filling its three meta-kinds of opposition (logical simplexes, contradictions, subalternations) are the following.

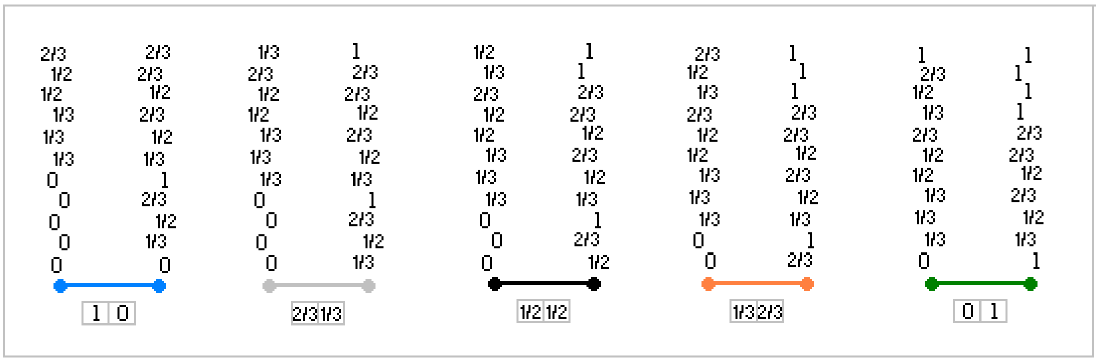


20.02.02. The five-valuedness of the logical quinia-simplexes

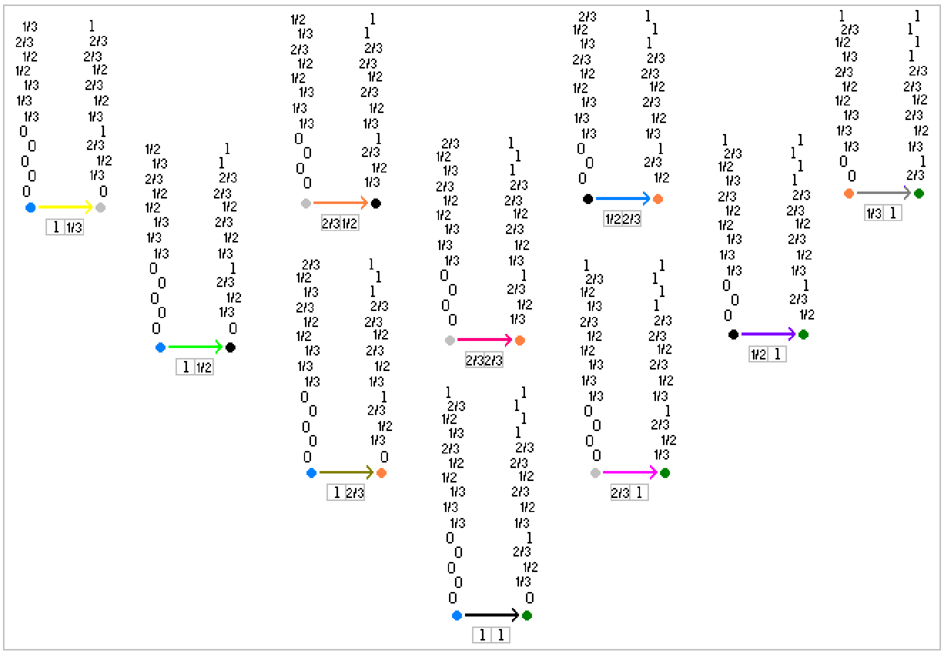
The previous figure shows the general intensional reading of the Aristotelian semantics. The "extensional definitions" of the 10 kinds of negations of the logical quinia-simplex are then given by the following table.



The extensional definitions of the 5 logical simplexes of the logical quinia-simplex are given by the following table.



Notice that the problem of interpreting the philosophical meaning of the logical simplexes, the specificity of each of them from the point of view of the concept of opposition, will be handled later. Such an interpretation could help us to find proper names for the each time new simplexes (all the simplexes interpolated between the traditional blue and green ones). The extensional definitions of the 10 kinds of implications of the quinia-simplex are the following.



Note that the previous definitions of the 10 negations of the quinia-simplex can be made more explicit (more familiar) by translating them into the truth-table notation:

0 0

¬A	A
1	0
2/3	1/3
1/2	1/2
1/3	2/3
0	1

1/3 0

⌐A	A
1	0
2/3	1/3
1/2	1/2
0	2/3
0	1

0 1/3

~A	A
1	0
1	1/3
1/2	1/2
1/3	2/3
0	1

1/2 0

~A	A
1	0
2/3	1/3
0	1/2
1/3	2/3
0	1

1/3 1/3

*A	A
1	0
1	1/3
1/2	1/2
0	2/3
0	1

0 1/2

•A	A
1	0
2/3	1/3
1	1/2
1/3	2/3
0	1

2/3 0

¤A	A
1	0
0	1/3
1/2	1/2
1/3	2/3
0	1

1/2 1/3

§A	A
1	0
1	1/3
0	1/2
1/3	2/3
0	1

1/3 1/2

$A	A
1	0
2/3	1/3
1	1/2
0	2/3
0	1

0 2/3

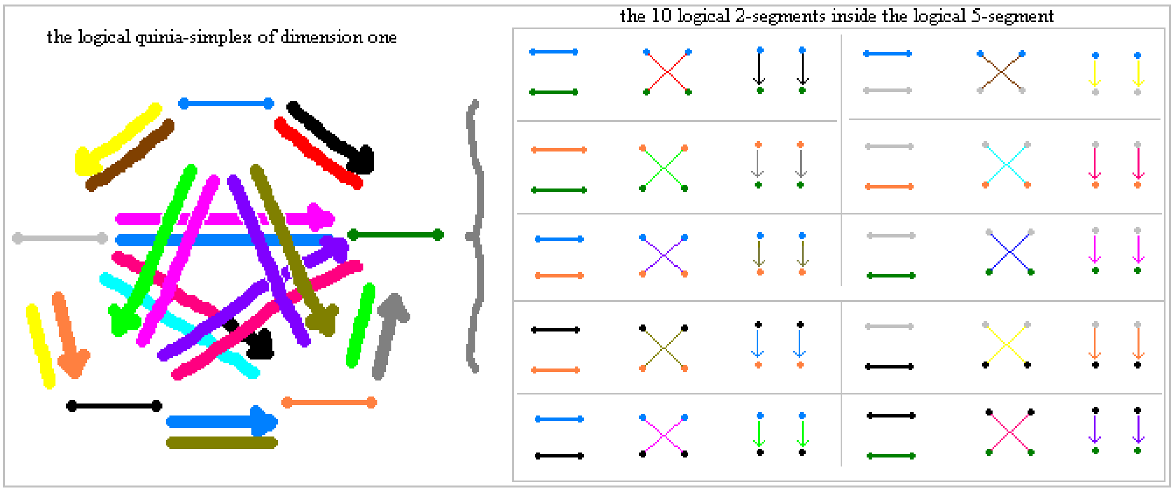
%A	A
1	0
2/3	1/3
1/2	1/2
1	2/3
0	1

(this point must be made more precise in the future).

As it seems, each layer of negations (from top to bottom) reduces uncertainty (i.e. transforms it into 0 or 1).

20.02.03. The logical 5-simplex of dimension 1 (logical 5-segment)

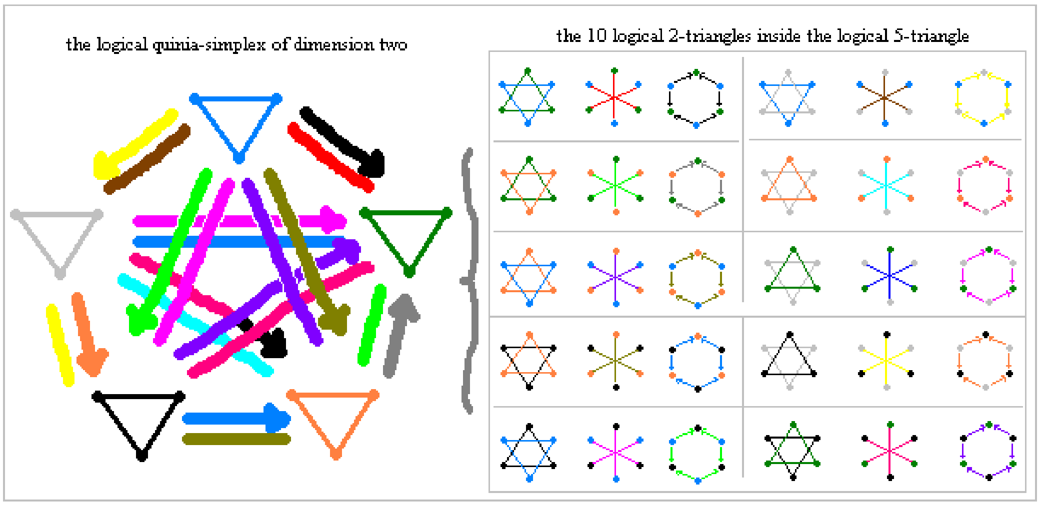
The overall structure of the logical 5-segment can be summarised as follows.



Notice that the logical quinia-simplex of dimension 1 (the logical 5-segment), clearly made of 10 logical bi-simplexes of dimension 1 (logical 2-segments), is also made of 5 structures isomorphic to the logical quadri-simplex of dimension 1 (the logical 4-segment). Here we omit giving the valid valuations of the logical quinia-simplex, reachable by the usual bottom-up valuation strategy.

20.02.04. The logical 5-simplex of dimension 2 (logical 5-triangle)

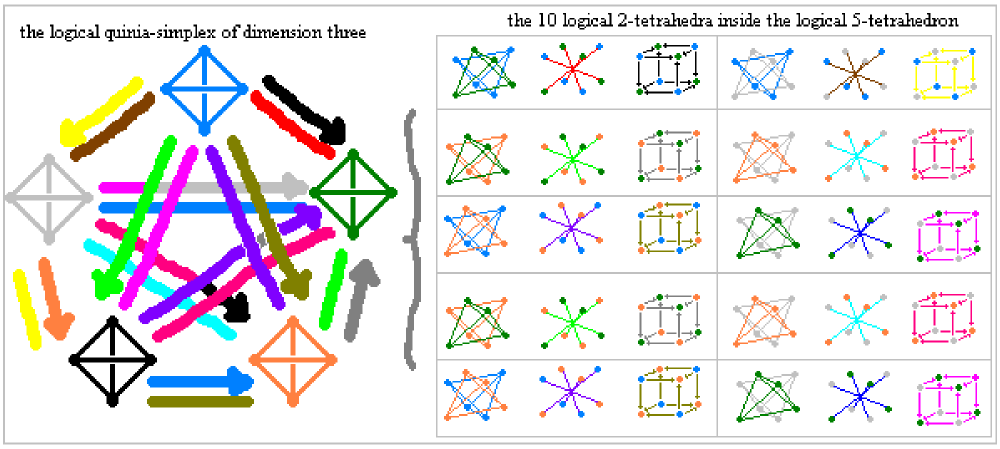
The overall structure of the logical 5-triangle can be summarised as follows.



Note that the logical quinia-simplex of dimension 2 (the logical 5-triangle), clearly made of 10 logical bi-simplexes of dimension 2 (logical 2-triangles), is also made of 5 structures isomorphic to the logical quadri-simplex of dimension 2 (the logical 4-triangle). Again,we must omit here giving the valid valuations, leaving this task for the future (by use of the bottom-up strategy seen before).

20.02.05. The logical 5-simplex of dimension 3 (logical 5-tetrahedron)

The overall structure of the logical 5-tetrahedron can be summarised as follows.



Notice that the logical quinia-simplex of dimension 3 (the logical 5-tetrahedron), clearly made of 10 logical bi-simplexes of dimension 3 (logical 2-tetrahedra), is also made of 5 structures isomorphic to the logical quadri-simplex of dimension 3 (the logical 4-tetrahedron).
The next step will be, in the future, to detail the valid valuations.

20.02.06. The logical 5-simplex of dimension *n*

In a near future an analysis of its valid valuation, isolated and/or global should be provided

20.02.07. The usefulness of the logical quinia-simplexes

Their main interest, when used with a linear order of the truth values, is to display both the extrema (totally false, totally true), the pivotal middle (totally uncertain) and the interpolated values (almost false, almost true) .

21.
THE GEOMETRY OF *P*-VALUED OPPOSITION: THE "LOGICAL *P*-SIMPLEXES OF DIMENSION *M*" ($P \in$ N, $P \geq 2$)

After the tri-simplexes, as we saw, it seems possible to construct "logical quadri-simplexes". Moreover, it seems to be possible to construct "logical 5-simplexes", and so on (logical *p*-simplexes, for any finite $p \in$ N, $p \geq 2$). If the analogy is to be kept (as we believe), that is, if the general theory of opposition is, among other, a theory of the *p*-simplexes, this seems to show that there are numerical constraints on the possible kinds of systems of opposition (so far, the only conceivable systems of opposition would be ones having as kinds of different oppositions sets of cardinality 2^2, 3^2, 4^2, …, n^2, etc.). In other words, it seems, so far, impossible to have a system of, say, 5 (or 23, or 74, etc.) different kinds of oppositions (because 5, 23, 74, etc, are not square powers of any integer). But remember: we can build systems of opposition for any number *n* of terms (n∈N, n≥2) (*via* the bi-simplicial *n*-opposition theory taken as a basis)

22.
A FIRST APPLICATION OF THESE NEW IDEAS (WITH F. SCHANG): THE ASSERTORIC LOGIC "AR_4". THE RE-DISCOVERY OF PIAGET'S "I.N.R.C. GROUP"[235]

In a joint work with the French philosopher and logician Fabien Schang, starting from an adaptation (by him) of my Aristotelian p^q-semantics to "pragmatics", we developed a new kind of pragmatic logic for "assertion" and "denial". This logic, besides its intrinsic philosophical interest (it tackles in a new way some classical problems inherent to the logical formalisation of pragmatic notions – it formalises "assertion", "denial", "conjecture" and "doubt"), bears a special interest from the point of view of NOT. As a matter of fact, one can show that it offers two kinds of oppositions: a static one and a (more) dynamic one. At the static level of its main four modalities (assertion, denial, conjecture and doubt), AR_4 generates a logical tetraicosahedron. At the dynamic level of its iterated modalities (assertions of assertions, denials of assertions, doubts of denials, etc…), it generates a strange "opposition square", different from Aristotle's one, made of only three out of the four classical opposition relations. A closer examination shows, surprisingly, that this square is isomorphic with Piaget's INRC group (cf. ch. 06 *supra*), which seems to mean at least two things: (1) that we have, thanks to Piaget, simple rules for the expression of iterated modalities (like the dynamic ones of AR_4) and (2) that we can have a look "from the inside" into the strange square geometry of the INRC group. This second consequence is important, for it offers us an understanding of Piaget's strange oppositions that was unavailable to us until now. In other words, it opens NOT to the case of "molecular oppositions" (a totally new subject, NOT deals with the oppositions between mono-valued atoms). As a final remark, the enquiry over AR_4 bears *a posteriori* many similarities with some results, over NOT, of Luzeaux, Sallantin and Dartnell (cf. ch. 17 *supra*).

22.01. Introduction: pragmatics and logic

The starting problem at the origin of this joint work of ours with F. Schang (which originated in some of his intuitions about pragmatics) was the current irreducibility of

[235] I wish here to thank Fabien for having offered me, by his friendly intellectual pressure, this (to me) very interesting field of application of NOT, and for having allowed me to mention here to some extent some of our common, still unpublished results. We presented these results at the LNAT Conference in Brussels, 5-7 November 2008 (proceedings to be published).

"pragmatics" to formal semantics. This situation is stated very clearly by the reference work on speech act theory by J. Searle and D. Vanderveken[236]. In particular, Vanderveken more recently said in this respect that "The success and satisfaction conditions of the illocutionary acts are not reducible to the truth-conditions of their propositional content"[237]. It is in reaction to this (closed) perspective that Schang wanted to adopt, in order to counter Vanderveken's (and Searle's) claim, some kind of non-Fregean logic[238].

22.02. Getting inspired by the Aristotelian p^q-semantics

In developing further his intuition in terms of a semantic game suitable for pragmatics (pragmatics by "*n*-valuation"), Schang was inspired by three sources: (1) H. Smessaert's "ordered values for oppositions" (cf. ch. 12 *supra*)[239], (2) the "Aristotelian p^q-semantics" I developed in this very study (cf. ch.18 *supra*) and which Schang happened to read (around February 2008), and (3) Łukasiewicz's and Prior's "products of classical matrices".

22.03. The four starting modalities of the logic AR_4

In a way similar to that of the Aristotelian p^q-semantics (the inspiring model), we adopt two questions:

- (1) "is φ held to be true?";
- (2) "is ¬φ held to be true" (i.e. "is φ held to be false?").

The possible (mutually exclusive) answers are: "no" and "yes".

So, the possible [x,y] outcomes of this game for modelling the possible basic pragmatics attitudes towards some proposition "φ" amount to four:

- (1) "[1,0](φ)" = "assertion of φ" (φ is held to be true but ¬φ isn't)
- (2) "[1,1](φ)" = "conjecture of φ " (both φ and ¬φ are held to be true)
- (3) "[0,1](φ)" = "negassertion of φ " (φ is not held to be true but ¬φ is)
- (4) "[0,0](φ)" = "doubt of φ " (neither φ nor ¬φ are held to be true)

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236 Cf. J. Searle and D. Vanderveken, *Foundation of Illocutionary Logic*, Cambridge, CUP, 1985.

237 "Les conditions de succès et de satisfaction des actes illocutoires ne sont pas réductibles aux conditions de vérité de leur contenu propositionnel", quotation from "Logique du discours et fondements de la pragmatique" (unpublished manuscript).

238 The notion of "non-Fregean logics" is originally due to R. Suszko, "Non-Fregean Logic and Theories", *Analele Universitatii Bucurestii*, Acta Logica 11, 1968, pp.105-125.

239 Cf. H. Smessaert, "On the 3D visualisation of logical relations", *Logica Universalis*, 3, 2, 2009.

Remark that there are therefore 3 distinct levels of “valuation” (i.e. assignment of some kind of “value”): (i) the meta-level of the answers (“0” or “1”, i.e. “no” or “yes”) of the constituting semantic game; (ii) the level of the constituted pragmatic attitudes ([0,0], [0,1], [1,0] and [1,1]), which therefore belong to an assertionally 4-valued pragmatic space; and (iii) the level of the (classical) truth values used in order to judge ontically the truth or falsity of the 4 pragmatic attitudes, for a [x,y](φ) act may obtain (v([x,y](φ))=1) or not (v([x,y](φ))=0).

These four modalities can be parted into 2 classes:

- [1,-], which summarises [1,1] and [1,0], is the “acceptance” and is symbolised by “A”;
- [0,-], which summarises [0,0] and [0,1], is the “rejection” (i.e. the “denial”) and is symbolised by “R”.

So, considering these last two notions to be possibly instantiated in a strong or in a weak way, and considering the valuation function “v” mapping the set “S” of sentences into the set “PV” = {[0,0], [0,1], [1,0], [1,1]} of the pragmatic values (with $\varphi \in S$), we have the following possible notations of the four modalities of AR_4 (the symbol “:=:” stands for “definition”):

- $A^{+}\varphi$:=: v(φ) = [1,0];
- $A^{-}\varphi$:=: v(φ) = [1,1];
- $R^{+}\varphi$:=: v(φ) = [0,0];
- $R^{-}\varphi$:=: v(φ) = [0,1];

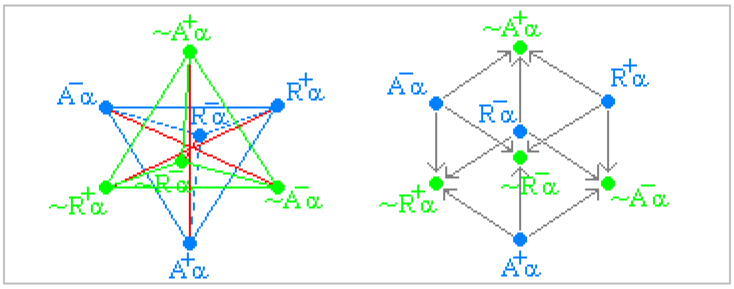
With this respect, the logic AR_4 will be a device allowing to formalise pragmatic situations previously thought to be unformalisable.

22.04. The oppositional properties of AR_4

The logics of the AR_4 system still being under examination (i.e. we have not yet proven any metalogical properties about it), we decided to at least study it from the point of view of the geometry of its “modalities”, so to get some philosophical (conceptual) clarification by that means. As it turned out, there are two different plans in which one can study the geometry of the oppositions of AR_4: one static and one dynamic. The latter is quite new and interesting.

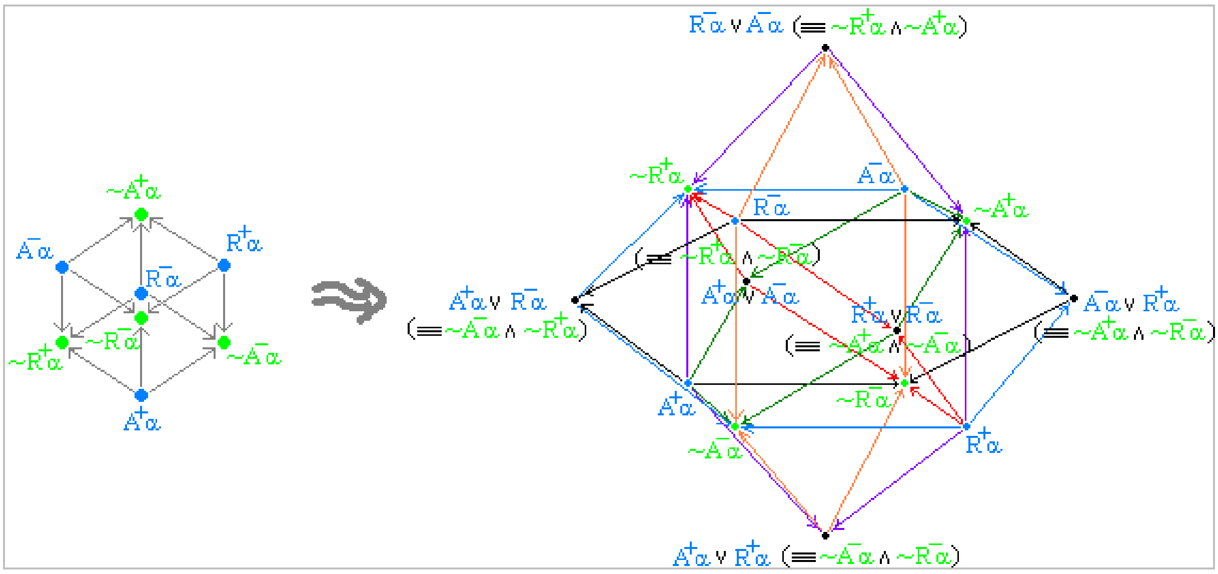
22.04.01. The static oppositions of AR_4 form a logical tetraicosahedron

Studying the geometry of the static modalities of AR_4 forces us to face a first problem: we do not know the “basic modalities” of this system. Of it we only know the first four starting modalities, which are the A^+, A^-, R^+ and R^-. Of these four, we can (as Schang did) prove the mutual contrariety (two by two). So we have, from the point of view of NOT, (at least) a blue tetrahedron of opposition, whence we can (and must) construct its overall logical cube (AR_4 embodies at least a 4-opposition).



(we see here that a first clarification brought by NOT to AR_4 is that it shows the necessity of taking into account 4 further modalities, that is: the contradictory negations of the starting four)

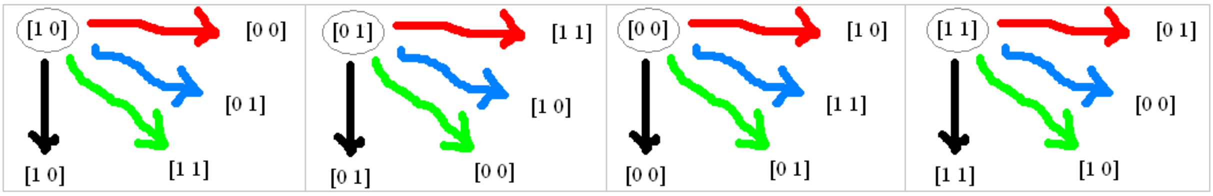
Now, around this pragmatic cube for AR_4, there is room for taking into account its natural closure, a logical (pragmatic) tetraicosahedron. As usual (cf. ch. 12-15 *supra*), this is done by taking into account the possible “logical Egyptian pyramids” or “spikes” over each of the six faces of the previous logical cube.



This expansion of the cube only shows that some composed modalities (i.e. composed by means of propositional connectives) may embody some previously unseen modal notions forming opposition nets.

22.04.02. The "one-iteration" modalities of AR_4 form a strange square

Having at least partly settled the question of the static modalities of AR_4, we wanted to consider the possible ways in which the four pragmatic *acts* embodied by the 4 starting modalities of AR_4 could be mutually put into relation. Remark that this is a totally new problem for us: it consisted in understanding (if possible geometrically) how each of the 4 acts can be transformed into another one. We examined this point combinatorially, by taking isolately into account each of the 4 pragmatic acts and by putting it in relation with each of the four pragmatic acts (including itself). The possible pragmatic transformations of the "assertion" [1,0], of the "negassertion" [0,1], of the "doubt" [0,0] and of the "conjecture" [1,1] are the following.

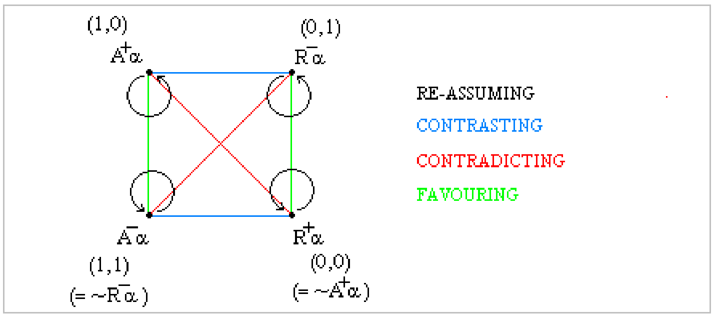


Whence our colours? In fact, we chose to interpret as *operational counterparts* of the standard opposition non-operational relations (i.e. of the classical four colours of Aristotle's theory of opposition) the following *pragmatic dynamic* features:

- "contradicting" (in red) changes the first value (i.e. the "x");
- "contrasting", i.e. "contrarying" (in blue), changes both values (i.e. both the "x" and the "y");
- "favouring", i.e. "subcontrarying" (in green), changes the second value (i.e. the "y");
- "re-assuming", i.e. applying "identity" (in black), changes no value (i.e. neither the "x" nor the "y").

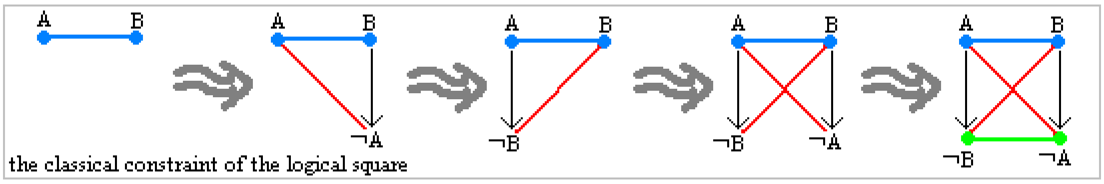
Now, once we combined these four elements (these four "tetra-chromatic" bundles of relations), we got a tetra-chromatic square[240].

[240] Remark that Schang and myself hesitated (and still hesitate) when choosing which operations (among the 4 possible ones) to attribute, respectively, to "contradicting" and to "contrarying" (it could have been the other way round). However, it is easy to show and to realise that even with the other choice (i.e. "contradicting" in blue, and "contrasting" in red), the final result would have been the same "strange square".



This square is, in some sense, a “square of opposition”, for it expresses opposition relations between any 2 of its 4 terms. But, clearly, it is not Aristotle’s square of opposition: it has no subalternation relation, and has instead two segments (instead of one) for expressing contrariety and two segments (instead of one) for expressing subcontrariety.

And, as a matter of fact, a possible objection with respect to this “assertoric square” is that the normal behaviour of opposition (with respect to a square of opposition), starting from any contrariety, is the following (we already recalled this in ch. 4 and 6 *supra*):

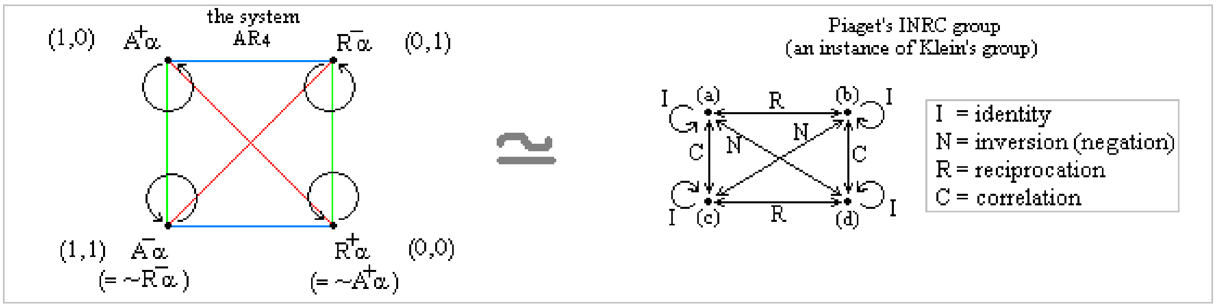


So, it is strange to have, as above (in our strange “assertoric square”) an upper contrariety blue segment generating, by the contradictions (red diagonals) of its two vertexes, a lower blue contrariety segment. This structure is not, *stricto sensu*, a logical square!

However, one must recall how we constructed it: the oppositions were based on the possible relations between elements each admitting two valued atoms ([x|y], with x and y ∈{0,1}). So, this “square of opposition” is (very) different from the traditional “square of opposition” (i.e. the logical bi-segment of NOT) because here the oppositions are, in fact, not relations but operations: for instance, it is not of “contradiction” that we are speaking about, but of “contradicting”; the difference may seem subtle, but is in fact crucial.

22.05. An unexpected surprise: back to Piaget’s INRC group

Now, this “strange structure”, in fact, is an old friend of ours, for if one examines it carefully, geometrically speaking it is none other than Piaget’s “INRC group” (cf. ch. 06 *supra*).



As we saw in ch. 6 *supra*, this Piagetian structure of INRC group is an instance of Kleinian group, which means it has the following simple rules of composition between its 4 elements (the three operations N, R, C and the identity I), each of which represents a mental operation (cf. ch.06 *supra*):

I = NRC
N = RC
R = NC
C = NR

Piaget

	I	N	R	C
I	I	N	R	C
N	N	I	C	R
R	R	C	I	N
C	C	R	N	I

But then, for AR_4, this means that we have, so to say "gratis", a general law for any finite number of iterations of its modalities!

I = NRC
N = RC
R = NC
C = NR

Piaget

	I	N	R	C
I	I	N	R	C
N	N	I	C	R
R	R	C	I	N
C	C	R	N	I

⇒

X = Y Z W
Y = Z W
Z = Y W
W = Y Z

us

	X	Y	Z	W
X	X	Y	Z	W
Y	Y	X	W	Z
Z	Z	W	X	Y
W	W	Z	Y	X

In other words, we started by studying the simple "one-shot" applications of AR_4 modalities to AR_4 modalities, and we got, thanks to "Piaget's gift", a general, stable law of composition for the pragmatic modalities of AR_4.

Remark, as already said in ch. 6, that this structure is also isomorphic to Gottschalk's quaternarity group (cf. ch. 5 *supra*).

22.06. Remark: there are parallels with Luzeaux *et alii* (2008).

One must mention that in the already quoted paper by D. Luzeaux, J. Sallantin and C. Dartnell, one can find elements of thought partly similar to the ones F. Schang and myself developed for working out the logic AR_4[241]. Our discovery of such convergencies (with

[241] D. Luzeaux, J. Sallantin, C. Dartnell, "Logical Extensions of Aristotle's Square", *Logica Universalis*, 2, N.1 (2008), cf. particularly the pages 177-178.

Luzeaux, Sallantin and Dartnell) was however posterior to the discovery/invention by us of AR_4 itself.

22.07. Conclusion: seemingly a comforting symptom

There is ground to think that this casual discovery of a convergence between the oppositional geometry generated by our and Schang's logic AR_4 (for assertion and rejection) and Piaget's classical INRC group (for modelling the logical, mental capabilities of children) is a rather important fact. First, Piaget's strange instance of "logical square" (isomorphic to Gottschalk's quaternality group) has a sense (a meaning which can be regained by a question-answer semantics akin to the Aristotelian p^q-semantics we developed for opposition theory (cf. ch.18 *supra*). Second, this could mean that it is possible to extend the notion of opposition, so to obtain a "second geometry" (in some sense parallel to the one we devised here in term of logical *p*-simplexes) for opposition phenomena. All this will have to be studied more specifically in the future, but we can already say that the relation between semantic games (in the sense of Game-Theoretical Semantics) like the Aristotelian p^q-semantics and NOT is confirmed, after the logical *p*-simplexes, by the rediscovery (with a notable gain in understanding of its hidden inner structure) of Piaget's square.

23.
LOGICAL *P*-SIMPLEXES AND MANY-VALUEDNESS: BEYOND SUSZKO, WITH MALINOWSKI AND SHRAMKO-WANSING[242]

In this chapter we face a possible radical objection to the logical *p*-simplexes: as they are *p*-valued logical structures, they could fall under the same severe criticisms made against the very idea of many-valued logic by the Polish logician Roman Suszko (1975, 1979). Studying some very interesting responses given to that famous, very polemic and dramatic challenge (that is: mainly those of Malinowski, Shramko and Wansing), we put forward arguments suggesting that there could be strong underlying links relating the "non-Suszkian many-valued logics" of these three authors to the logical *p*-simplexes. From this point of view, freed of Suszko's threat, and considering that many-valued logics bear a strong relation to substractural logics (a very important family of logics, subsuming for instance relevance logics and linear logic), it could turn out that the logical *p*-simplexes bear a considerable significance for logic even outside the field of NOT.

23.01. The logical *p*-simplexes are *p*-valued logics

In the previous pages of this Part III, we wanted to test the limitations of NOT. In particular, we wanted to see if the Aristotelian limitation to four and only four kinds of opposition (contradiction, contrariety, subcontrariety and subalternation) was a truly transcendental (i.e. inescapable) one. We saw that the oppositional quaternality (four colours) of NOT rested on the "bi-" prefix of its key structure: the "logical *bi*-simplexes" (of dimension *m*). So we wanted to test the logical bi-simplexes, to see if their structure is a transcendental one. In other words, we looked for "logical tri-simplexes", ignoring if they really existed (for: if logical tri-simplexes do exist, then the logical bi-simplexes are relativised). We devised (basing ourselves on Aristotle's original combinatorial intuition underlying the logical square) a so-called Aristotelian 2^2-semantics for explaining the logical bi-simplexes. Considering it as being more generally a so-called Aristotelian p^q-semantics (with the particular values p=2 and q=2), we arrived at the conclusion that, in order to try to

[242] For writing this chapter we relied mainly (but not only), because of its particular clarity on this subject, on Y. Shramko and H. Wansing, "Suszko's Thesis, Inferential Many-Valuedness, and the Notion of a Logical System", *Studia Logica*, 88 (2008), pp. 405-429. We partly treat the same subject in "The Critics to Paraconsistency and to Many-Valuedness and the Geometry of Oppositions" (submitted for the *Proceedings* of the 4th World Congress on Paraconsistency, Melbourne, July 2008. We thank the University of Melbourne for a travel grant).

go beyond (the bi-simplexes), we had to let the values of the parameters *p* and *q* change. In this study we limited ourselves to allowing *p* variations (the study of the variations of *q* leads to many-dimensional "p^q-lattices", whereas the p^q-lattices of the p^2-semantics, that is the p^2-lattices, remain bi-dimensional), and it worked: the logical tri-simplexes turned out to be generated by the 3^2-semantics (the logical quadri-simplexes are generated by the 4^2-semantics, and so on, cf. ch. 19 and 20 *supra*). But changing the value of *p* turns out (by definition of the p^q-semantics) to change the number of the possible truth values. Hence the general result: the logical *p*-simplexes are possible and in fact incorporate the expression of *p*-valued opposition (i.e. opposition admitting *p* kinds of truth-values). We entered, as citizens, the realm of many-valued logics.

23.02. Suszko's attack against the idea of many-valued logics

One big problem with this many-valuedness of the logical *p*-simplexes is that many-valued logics have been conceptually (and rather heavily) attacked by the venerable Polish logician Roman Suszko. If his reasoning, where he claims that many-valued logic are a theoretical deceit, is correct, aren't we to feel worried for being, ourselves (NOT-people), part of a deceitful enterprise? Before judging, let us recall the debate.

23.02.01. The so-called "Suszko's Thesis"

Suszko claims that the so-called generalised truth-values are in fact always composed of two distinct components: two among their elements are logical truth-values: these are "0" ("false") and "1" ("true"); the other elements (like, to take arbitrary examples, the "half-true" "½", the "almost false" "1/3", the "almost true" "2/3", etc.) are only "algebraic values". The reason of this distinction is that algebraic values are just admissible referents of formulas; their only use is to call things "almost false", "undecided", "almost true", etc. On the contrary "0" and "1" are *logical* values, because they play a special extra role (one essential to logic): they are needed in order to define "valid semantic consequence", a very important concept of logic. Classically (i.e. when there are only 2 truth-values) "valid semantic consequence" is one where, if all the members of the antecedent are true, then the consequent is also true. Suszko remarks that the creators of many-valued logic do preserve this binary property, just by introducing, with respect to the set of all the admitted truth-values, a sub-set of it, called "designated set", the truth-values of which are, so to say, "positive", whereas the elements of the rest of the set are, so to say, “negative”: in many-valued logic “valid semantic consequence” is thus defined as an inference such that if the elements of the antecedent all belong to the designated (i.e. positive) set, then the consequent belongs in turn to the designated set (of truth-values). A reverse consequence (contraposition) may be defined by saying that if the consequent of such a relation has a truth-value which does not belong to the designated set (i.e. it belongs to the “negative” one), then at least one of the elements of the antecedent must have a truth-value not belonging to the designated set. So, if logic is the science of inferences, and if inferences, in order to have a notion of valid consequence relation, only need the truth-values “false” and “true”, then the only “logical truth values” (seriously speaking) are the “false” and the “true”. This (strong) claim – to which Suszko also adds the very heavy comment that, for this reason, “Łukasiewicz is the chief perpetrator of a magnificent conceptual deceit lasting out in mathematical logic to the present day”[243] – is known as “Suszko’s Thesis”.

23.02.02. The so-called “Suszko’s Reduction”

The strength of Suszko’s reasoning becomes even more evident when he adds to his so to say philosophical analysis of many-valuedness (“there are only two *logical* truth-values”) a powerful and very technical theorem showing that in all interesting logical cases (i.e. when we speak of logical systems in which the consequence operator is “Tarskian”) we can reduce (i.e. translate) many-valued systems into two-valued ones (Suszko uses, to build his own, the so-called “Wójcicki’s theorem”). This impressive result is known as “Suszko’s Reduction”[244]. Together with Suszko’s Thesis, it seems to show, logically, that many-valued logics are indeed a deceit, i.e. a deceitful appearance deprived of serious logical meaning. So, coming back to us (who are investigating the foundations of NOT), are the logical *p*-simplexes also a deceit (i.e. a false novelty, a false logical discovery)? Let us see some answers which have been given to Suszko’s severe and killjoy argument.

[243] R. Suszko, “The Fregean Axiom and Polish Mathematical Logic in the 1920^{s}”, *Studia Logica*, XXXVI, 4 (1977), p. 377. Cf. also R. Suszko, “Remarks on Łukasiewicz’s Three-Valued Logic”, *Bulletin of the Section logic*, Polish Academy of Sciences, vol. 4 (1975), no. 3, pp. 87-90.
[244] It says that every structural Tarskian consequence relation, and hence every structural Tarskian many-valued propositional logic as well, is characterised by a bivalent semantics.

23.03. Malinowski's answer to Suszko: the *q*-logics (1990)

A frontal answer to Suszko has been given by another Polish logician, Grzegorz Malinowski. He showed that, contrary to the implicit opinion in Suszko's line of thought, it is possible to build a "non-Suszkian" formal framework for reasoning, one admitting rules of inference that lead from non-rejected assumption to accepted conclusions. Let us see sketchily how[245].

23.03.01. Malinowski's first remark

As we saw, one important piece of Suszko's reasoning was the existence, in many-valued logics, of the partition of the set of all "truth-values" (or, more precisely, the set of all algebraic values) into two sub-sets: the set D^+ of "designated values" (the values counting as "true" with respect to the validity of the consequence relation), and the set $D\backslash D^+$ (i.e. "D" minus "D^+") constituted by the set of "undesignated values", i.e. the set-theoretical complement of the first. But, and this is Malinowski's opening remark, one can also conceive a set D^- of "anti-designated" algebraic values (this abstract idea was already present years before in Gottwald, Malinowski himself and Rescher). And this leaves room for (sets of) values which are neither designated nor anti-designated, as well as for (sets of) values which are both designated and anti-designated. This implies, at the level of the subsets of D (the set of all algebraic values), the replacement of logical 2-valuedness by logical 4-valuedness, whereas in Suszko's view, at the level of such subsets one could only find 2-valuedness (designated / undesignated). The 2-valuedness can be replaced by logical 3-valuedness when one "sacrifices" one of the 2 new subsets: if it is postulated that $D^+ \cap D^- = \varnothing$ or if it is postulated that $D^+ \cup D^-$, the set of all algebraic values is available (Malinowski and Gottwald chose the first, the second may be used for expressing in a non-Suszkian way paraconsistent systems).

23.03.02. Building a counterexample to Suszko's Thesis: *q*-matrices

Wanting to build a counter-example, Malinowski defined a "*quasi*-consequence" relation and its semantic counterpart, "*q*-entailment". The latter is defined by requiring that if

[245] G. Malinowski, "On Many-Valuedness, Sentential Identity, Inference and Łukasiewicz Modalities", *Logica Trianguli*, 1, 1997, pp. 59-71; "Beyond three inferential values", handout of the 2008 Workshop on Truth Values, Dresden, 30 May – 1 June, 2008 (internet).

every premise is not anti-designated, then the conclusion is designated. Malinowski demonstrated that for every structural *q*-consequence relation, there exists a characterising class of so-called *q*-matrices, i.e. matrices which, besides the subset D^+, also have a disjoint subset D^- of anti-designated values.

23.03.03. Consequences of this: not everything in logic can be turned into bivalence

The important fact, then, with respect to Suszko, is that not every *q*-consequence relation has a bivalent semantics. In fact, another result of Malinowski's introduction of the *q*-matrices, to be taken in account in order to see if one is ready to pay the price for Malinowski's solution, is that the *q*-consequence relations need not to be reflexive (which can be shown to be strange in some sense).

23.04. Da Costa, Béziau, Bueno and Tsuji's answers to Suszko

The answers to Suszko can be seen from at least two points of view: they have given rise to discussions over Suszko's use of Wojcicki's theorem (where logicians hope to find some weak point in Suszko's strategy) and to discussions over Malinowski's way of trying to escape from Suszko.

23.04.01. da Costa-Béziau-Bueno and Caleiro-Carnielli-Coniglio-Marcos

Several authors have all remarked that Suszko's result can in fact be even strengthened. C. Caleiro, W.A. Carnielli, M. Coniglio and J. Marcos (2008) show that Suszko's reduction can be carried out for any Tarskian consequence relation (even without structurality)[246]. N.C.A. da Costa, J.-Y. Béziau and O. Bueno made a partly similar remark (1996)[247]. And J.-Y. Béziau showed (1998) that a reduction to a bivalent semantics can even

[246] C. Caleiro, W. Carnielli, M. E. Coniglio and J. Marcos, "Suszko's Thesis and dyadic semantics", (preprint); "Dyadic semantics for many-valued logics", (preprint); "Two's Company: "The Humbug of Many Logical Values"", in J.-Y. Béziau (ed), *Logica Universalis*, Basel – Boston –Berlin, Birkhäuser, 2005.
[247] N.C.A. da Costa, J.-Y. Béziau and O. Bueno, "Malinowski on Many-Valued Logics: On the Reduction of Many-Valuedness to Two-Valuedness", *Modern Logic*, Vol. 6, no. 3 (July 1996), pp. 272-299.

be carried out by systems logically much weaker. As for the rest, Béziau looked for strategies escaping Suszko (2004), by making remarks on valuation theory[248].

23.04.02. Tsuji using Béziau's definition of logic against Malinowski

M. Tsuji makes an analysis of Malinowski using Béziau's concept of Universal Logic (of which we spoke in ch.10 *supra*)[249]. According to Tsuji, Malinowski's analysis is not wrong, but misses the main point revealed by Suszko's intervention. Tsuji invokes a notion of "abstract logical structure" <L, ⊢ >inspired by Béziau's framework (a definition very ambitious in its abstractness) for showing that, because all logics in Béziau's style have reflexive consequence relations, and because Malinowski's *q*-consequence relation does not need to be reflexive, Malinowski's result is not very interesting (Béziau's very abstract notion of logic seeming a non-negotiable standard to Tsuji). Starting from this concept of abstract logical structure he then derives a theorem and the consequence of this theorem is, according to Tsuji, that the fundamental constraints (which are of a Suszkian flavour) resting on logic (this one being Béziauian) are due, *ultima ratio*, to the *geometry* of the consequence operator (i.e. to the antecedent-consequent structure of the turnstile "⊢").

23.05. Shramko and Wansing's answer to Suszko (2005)

Y. Shramko and H. Wansing have produced a series of important studies where, relying on Malinowski's result, they develop considerations (and technicalities) radicalising it in a promising way. In their opinion Suszko's reduction is doubly important: not only for the thesis it develops (a dramatic examination of the conditions of possibility of many-valued logic); but also because in discussing the conditions of the Tarskian consequence operators it touches some of the most profound issues of logic in general (as we have partly seen with Tsuji's remarks). Shramko and Wansing disagree with Tsuji, for they consider that Malinowski's analysis captures the central aspect of the explanation of the feasibility of Suszko's reductions and the distinction introduced by him between algebraic and logical values. They reproach Tsuji the fact of begging the question when, for demonstrating (by his theorem) that Suszko's reduction to 2-valuedness applies to almost any (serious) logic, he

[248] J.-Y. Béziau, "Non Truth-Functional Many-Valuedness", in: J.-Y. Béziau, A. Costa-Leite and A. Facchini (eds), *Aspects of Universal Logic*, special issue of *Travaux de Logique*, Neuchâtel, Université de Neuchâtel, 2004.
[249] M. Tsuji, "Many-Valued Logics and Suszko's Thesis Revisited", *Studia Logica*, **60**, pp. 299-309 (1998).

assumes a notion of “logical structure” <L, ⊢ > (from Béziau) which admits at most logical two-valuedness if the single assumed consequence relation is reflexive. As a matter of fact, Shramko and Wansing show that with a (tenable) concept of logical system – other than Béziau-Tsuji’s one – , “every such system is logically *k*-valued (for some $k \in N$, $k \geq 2$) and, moreover, the definition of a logical system is such that every entailment relation in a logically *k*-valued logic is reflexive” (p. 407). So Tsuji’s objection to Malinowski’s strategy is settled here. In sum, the “correction” brought to Malinowski (considered as the discoverer of the right solution to Suszko’s challenge) is that Shramko and Wansing, beside the Malinowskian augmentation of the number of logical values (*via* the augmentation of the number of the separate subsets of the set of algebraic values) also restore (what was lost with Malinowski) the preservation of logical value from premises to conclusion (or *vice versa*).

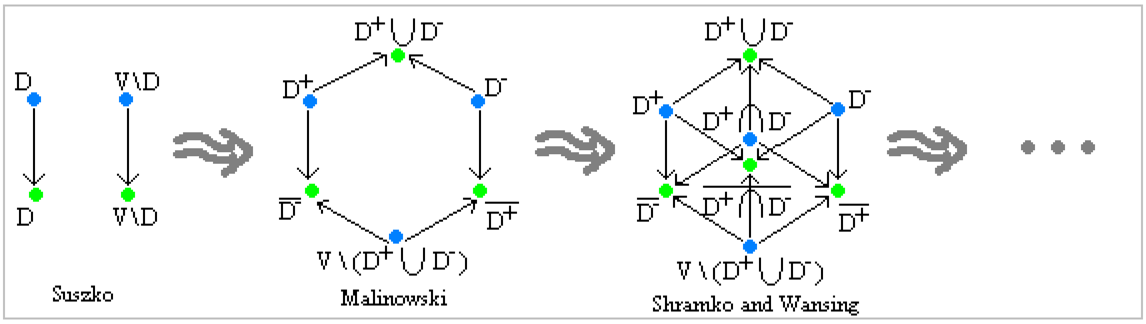
Shramko and Wansing propose, therefore, an increase not only of the number of logical values but also of the number of entailment relations. They show that there are many more than Malinowski thought. To the “*q*-entailment”of the latter, another Polish logician, Frankowski, had already shown the possibility of adding an alternative “*p*-entailment”. Shramko and Wansing show thus that in a general case, based on considerations touching “bilattice theory”, there can in fact be at least 4 such *a priori* distinct entailments (in the sense of consequence-relations based on the subsets of the set V of algebraic truth-values): *t*-entailment (“t” is for truth), *f*-entailment (“f” is for falsity), *q*-entailment and *p*-entailment. Hence, also, a new definition of the notion of “(logical) truth-value”: a logical value is a value that gives rise to an entailment relation in a canonical way (p.426).

The consequence of this on Béziau is an argument relativising his proposal of the notion of an abstract logical structure <L, ⊢ >. This point is important in order to better develop the elaboration program of a “true to the facts” universal logic.

Starting from these elements they also construct a series of strong (so to say “trans-Suszkian”) *n*-valued logics. This is an important fact for logic: Malinowski, Frankowski and Shramko and Wansing seemingly freed many-valued logic from Suszko’s impressive threat. And this point is important for NOT, in so far as its central notion of logical bi-simplex is just a particular case of an essentially many-valued notion (the notion of logical *p*-simplex).

23.06. Is this not a little familiar with the logical *p*-simplexes?

Actually, the subsets discussed by Suszko, Malinowski, Shramko and Wansing can be displayed so as to show which *n*-opposition they are playing.



As one sees, from this point of view Suszko corresponds to a logical square, Malinowski to a logical hexagon and Shramko and Wansing to a logical cube. As inside NOT there is no finite limit to such a progression of growing opposition, the question arises naturally as to whether looking for higher logical bi-simplexes decorated in such a way can make sense.

Moreover, NOT has provided us with an unexpected demultiplication of subalternations (i.e. implications), of contradictions (i.e. classical and sub-classical negations) and of logical simplexes. It is therefore tempting to look for more profound interrelations (supposing there are some) between generalised NOT and generalised many-valued logic. As many-valued logic bears many relations with substructural logics, this part of logic as well seems to deserve to be investigated with NOT-eye in the future[250].

[250] The deep link between many-valued logics and substructural logics is partly discussed in Y. Shramko and H. Wansing, "Entailment Relations and/as Truth Values", *Bulletin of the Section Logic*, Vol. 36:3 / 4 (2007), p. 131.

24.
FROM STATIC TO DYNAMIC OPPOSITION? FIRST ELEMENTS THROUGH FIVE STRATEGIES 'OPPOSITION DYNAMICS' AND 'OPPOSITION FIELDS'

In this chapter, relying on what we have seen so far, we face some possible challenges to a general theory of opposition: they concern mainly the "static" character of the present theory. In particular, using some of the tools freshly discovered by us, we try to show that scientific models of opposition enable to think somehow dynamic (and maybe even intensive) opposition phenomena.

24.01. Being dynamic VS being static

As we already said (cf. ch 1), one can seemingly conceive of an important distinction between at least three kinds of general oppositions: static (like the black-white one), dynamic (like in active fight) and intensive (like in bad feelings). In our study so far we motivated the choice of the static kind of opposition (the classic one in Western thought), as being the simplest one to formalise logically and geometrically. Now that we have some robust elements of formalisation, we can face, at least sketchily (very sketchily in fact), the question of somehow resuming the other two sides of general opposition.

A few words on the possible interest of this kind of enquiry. If the static approach has a relation to the notion of 'negation' (formal sciences), the dynamic one seems to have a deep relation to the notion of 'system' (we are thinking of the theory of the closed as well as the open systems). Historical examples of concern for 'dynamics', beside science (kinematics, ballistics, physics, etc.) can be found, paradigmatically in sociology (Durkheim's: 'social dynamics') and psychology (Lewin's: 'group dynamics'), to which more recently one should probably add, between science and humanities, cybernetics of first and second order (*inter alia*: E. Morin, G. Spencer-Brown and N. Luhmann).

In what follows, aiming at effectively widening our scope (particularly in §24.02), we are going to try to put out natural-looking challenges to ourselves (i.e. challenges against the static aspect of NOT). Starting from this basis for possible criticisms, we will try to explore (in §24.04) some possible answers to them, and eventually to restate more finely the starting challenges.

24.02. Five natural challenges to the actual NOT

We see five possible criticisms (of course, there must be many more) of the actual shape of NOT. If the first is, truly speaking, already resolved (we state it here a bit rhetorically in order to give a more complete view on the subject), the second and third one are still in need of an answer. The fourth and fifth ones so far seem very difficult to us (but heuristically stimulating). Let us see.

24.02.01. The challenge of having more than 2 opposed terms

As already mentioned (cf. ch.4), the classical theory of opposition deals with 2 terms. Opposition traditionally was conceived of as, if I dare say, a "couple affair". But if it is undeniable that the 2-opposition case is very interesting because very simple (apparently the simplest possible opposition case), in "real life" things are definitely more complex. For oppositions can concern more that 2 people (or things) at the same time. As we are going to see, this, in turn, opens out on to many other interesting problems. Of course, we already know (no scoop) that *n*-opposition theory allows to have *n* opposed terms (with $n \geq 2$, $n \in \mathbb{N}$). But, as we are going to see, it does not yet say clearly how to face the possible problems (the possible interesting new configurations) originating in this plurality bigger than 2. In other words, NOT has not yet been explored from a philosophical point of view.

24.02.02. The challenge of seeing oppositions change

The most natural challenge to issue to static opposition is to be able to change. To this, there seems to be a single abstract answer, which can take different forms (a backbone for future research). If we consider oppositional geometrical objects (the square, the hexagon, etc.) to be "pictures" of opposition situations, it seems natural to conceive some kind of general "movie solution": one can study (or represent) opposition changes by passing from one such picture to another. This general solution (let us call it "opposition movies") can already take different forms: if we knew the logical square only (as many people still do nowadays), we could watch a movie where, over the unchanging (decorated) square, the valuations (the truth-values) of the four objects decorating the square do change. This movie would be simple, of course, but it would be a movie nevertheless. But now that we know infinitely many logical bi-simplexes (of dimension *m*), we can see movies where a given bi-simplex of dimension *a* (a (*a*+1)-opposition) suddenly gives way to a related bi-simplex of dimension *b* (a (*b*+1)-opposition): this will modelise the fact that, inside a given opposition, one (or more) opposed term(s) is (are) added or eliminated (according to the fact that $a<b$ or $a>b$). Of course, now that we know, beside the logical bi-simplexes, of the existence of the general logical *p*-simplexes, we can also go beyond and see still richer movies, where a given logical *c*-simplex of dimension *m* suddenly gives way to a related logical *d*-simplex of dimension *m*. And we can combine both, having fantastic opposition movies where the picture of a given *p*-simplex of dimension *m* suddenly gives place to the picture of a related other one, where *p* and/or *m* are changed.

One last remark here: this "movie solution" is very general; it really keeps the analogy with the difference between picture and movie. Which means that if, in the future, other refinements of the possible oppositions are found, these new refinements will possibly be adopted by "opposition Hollywood". The movie technique is generalisable *ad libitum*. But this also means that the problem of having changing oppositions has to be studied inside the problem of having different qualities of opposition. We need actors and plots!

24.02.03. The challenge of having different degrees of a same opposition

As we said at the beginning of this chapter, being allowed (as we are) to have oppositions between more than 2 terms implies a series of new (interesting) problems. For instance: if in a given situation there are more than 2 opposed terms, must they be "all in a same plan", or can there be "alliances"? Which can be restated thus: can there be differences, inside one same kind of opposition? (can 2 terms be mutually "more contrary" than they both are with respect to a third contrary term? This point seems interesting in several respects, for it has to do, for instance, with the question of being able to represent "power relations" (the French "rapports de force"), "domination relations" (resuscitating Hegel: "who is who's master? Who is who's slave?"). Below in this same chapter (§24.04.02), we are going to examine whether something can be done in this respect with the help of the freshly discovered logical *p*-simplexes.

One last word here: were we able, somehow, to answer positively to this challenge; this could be a way to cope with the question of intensive oppositions (the baby's notion of "crying her/his dislike" for instance), perhaps an oppositional approach (some kind of model, of course very simplistic) to *feelings* (at least some of them)[251].

24.02.04 The challenge of combining mutually independent oppositions

A further challenge we can mention here is a very natural one, but also one seemingly very difficult to face. It concerns the possibility (or impossibility) of combining several independent oppositions at the same time. This challenge is very general, and the way of formulating it probably leads to different kinds of answers. So let us try to give some alternative formulations for it.

Example 1: from linguistic oppositions to a linguistic network

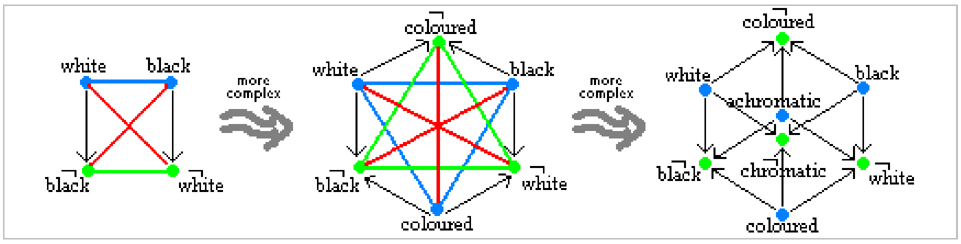
The problem emerges quickly when dealing with semantic networks (by oppositions).

As we recalled it (cf. ch.02 *supra*), the "post-tractarian" Wittgenstein has taught us that it is generally erroneous to look for the "platonic essence" of a word (or concept), for, most of the time human languages (and conceptual networks) are such that a word is made of several distinct meanings (related to uses), having thus no unique common element. Meaning is not an intersection, so to say, but a union. Put in another way, words are poly-semantical. Wittgenstein's classical example is that of the "family relations" (of similarity): two by two, many elements of the "family" share common "family features" but the global intersection of them is nevertheless empty. Consequently, Wittgenstein's solution for understanding language in a better way amounts to studying patiently, for every word (object of investigation) of a given language (for instance: English) its ways of common use (this gave birth to the movement called "philosophy of ordinary language"). Thus, like Peirce (the author of the Pragmatist Maxim, cf. ch. 2 *supra*), Wittgenstein thinks that "meaning is in the use". But then, with the help of the "spirit of NOT", we can add to this: "and hence, meaning lays somehow in the available oppositions related to the scrutinised word (or concept)". This means that it can be interesting to join more opposition schemes (more αn-structures) all relative to a same word.

So let us take a simple example: the familiar English word "white". This word, seemingly univocal (we all know what "white" is), quickly shows, when investigated linguistically, the richness of its constitutive oppositional semantic network. We can easily

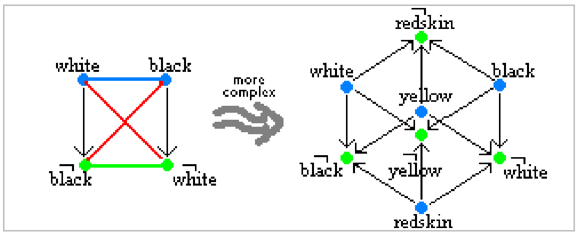
[251] In classical philosophy (see for instance Descartes' *Les passions de l'âme* or Spinoza's *Ethic*) emotions are often studied from the point of view of their mutual (usually binary or ternary) oppositions.

exhibit at least three different common contexts of use of “white”. Firstly, of course, this word belongs to the world of colours, where it is opposed to other kinds of colours.



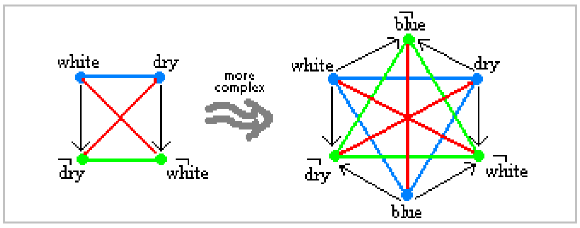
(by “coloured” we mean here “admitting a colour other than black or white”; by “achromatic” we mean here “admitting no colour whatsoever”)

Secondly, “white” belongs, in common if not in scientifically grounded language, to the world of popular “races” knowledge (phenotype as seemingly expressed by human skin “colours”).



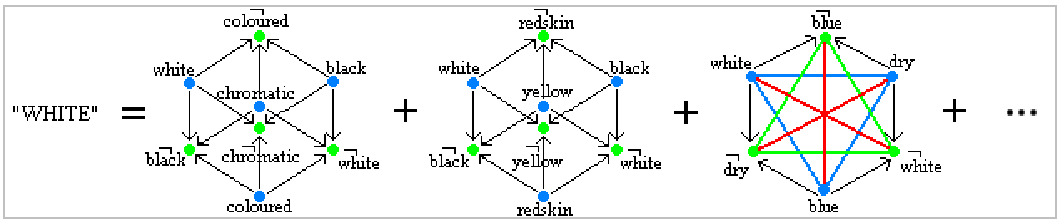
(by “yellow” we mean far-Asian people, whereas by “redskins” we mean American Indians)

Thirdly, the same word can belong (in French, as “blanc”) to the world of cheese types.

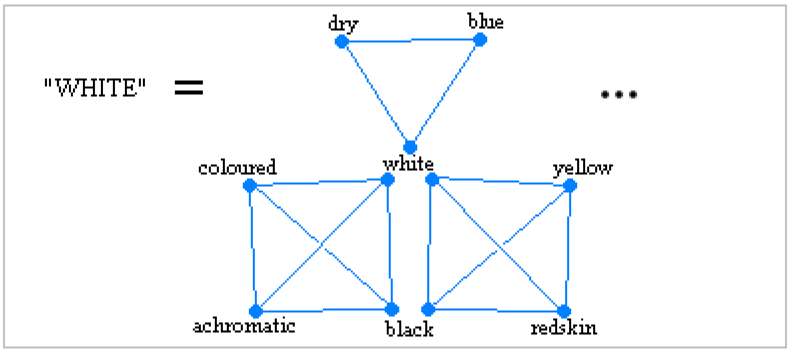


(“white” means fresh, liquid cheese, “dry” means solid, “blue” means cheese with blue moisture)

So “white” seems to need, from an easy-to-accept Wittgensteinian point of view over the negation of the existence of a Platonic “essence” of words, the faculty of joining distinct semantically constitutive opposition schemes (αn-structures) into some kind of global conceptual network.



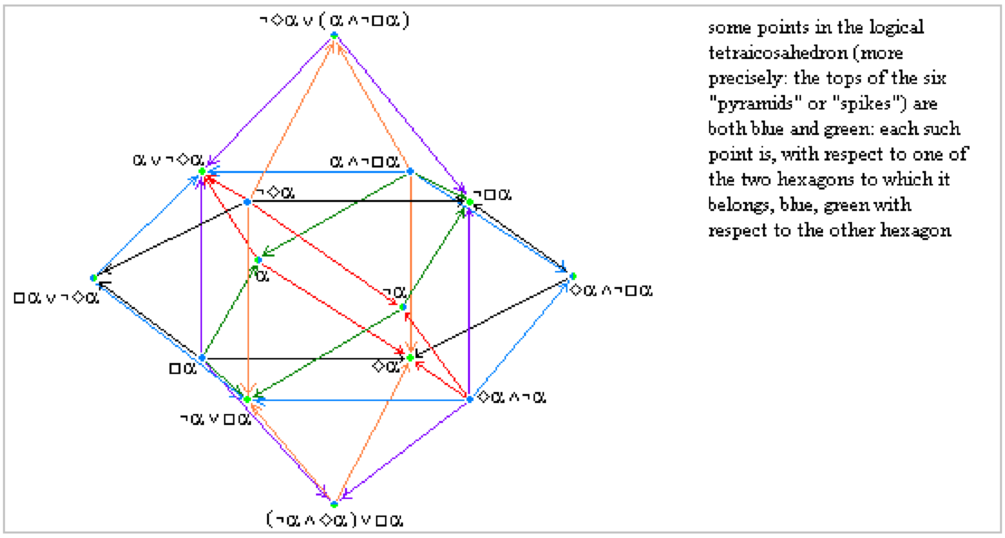
Is such a junction of distinct opposition schemas (here in the example: three of them) a possible object of representation? Can it lead to a new geometry of oppositions?



Remark that this begins to look like Gärdenfors' conceptual spaces (cf. ch.02 *supra*). All these considerations lead to the question of the general feasibility of a composition of oppositional solids (cf. example 3 below). Is everything straightforward for this, or are there technical pitfalls to be avoided? Are there regularities to be discovered, explained and used? Again: a positive answer could open to us the gates of Gärdenfors' theory of the conceptual spaces (to which NOT could add something useful).

Example 2: being contrary in one place, subcontrary in another one …

A second example is related to the possibility, for a given object A (abstract or concrete), of belonging to two (and perhaps more?) distinct opposition relations (each relative to a different set of mutually opposed objects – say, $\{B_1, B_2, B_3, \ldots, B_n\}$ and $\{C_1, C_2, C_3, \ldots, C_m\}$). Classically, i.e. for the logical bi-simplexes, this seems to concern mainly contrariety and subcontrariety. Remark that this is a different problem from to the previous one.



Remark also that some of the already known oppositional structures of NOT, namely the β*n*-structures, already embody this phenomenon by exhibiting the feature of having terms (among all the possible ones) that do belong both to the blue contrariety simplexes and to the green subcontrariety simplexes. This appears clearly in Pellissier's logical tetraicosahedron (the β3-structure, cf.previous figure).

As a matter of fact, one sees here that while the heart of the logical tetraicosahedron, the logical cube, is composed of 4 blue points (for contrariety) and of 4 green points (for subcontrariety) without chromatic ambiguity (blue is blue and green is green), the remaining 6 points (i.e. those outside the cube, at the top of each of the 6 "Egyptian pyramids" or "spikes") are each both blue and green (blue with respect to one logical hexagon, green with respect to the other). So this behaviour is logically possible: a same element, inside NOT, can at the same time belong to a contrariety network and to a subcontrariety network.

So the question now is: can we have this outside the β*n*-structures? But in this case, what is the outcome of this? (which geometrical laws, if any?) We lack, so far, a geometry of the free (arbitrary) compositions of oppositions.

Considerations similar to those made in the previous example about linguistic oppositions (and those of §24.02.05 *infra*) tend to suggest that it would be interesting to have such compositions of qualitatively different opposition relations for a same given object.

Example 3: joining oppositional solids together (for what use?)

So the question now is: is it possible to have this behaviour in more general configurations? Put in another way: is it possible to have constructions made of α*n*- or β*n*-structures joined freely and meaningfully to other ones without turning globally into an already known β*m*-structure?

24.02.05. The challenge of having asymmetric opposition relations

A fifth (and last) challenge to the actual static character of NOT could be that of being able to express *asymmetric* opposition relations. We mean by that the fact that, of two objects A and B (abstract or concrete), one (say A) could have with respect to the other (i.e. B) a certain "opposition" relation while the latter (i.e. B) could have towards the former (i.e. A) another kind of "opposition" relation. In a similar vein, two objects A and B (abstract or concrete) could be such that they are related by a same "opposition" relation, but in such a way that while A has towards B the degree X of the considered opposition relation, B has towards A the degree Y of the considered opposition relation (X≠Y). If feasible, all this could be useful for modelling "human psychological opposition relations" (such as friendship, adversity, hatred, attraction, etc.). For, A can think to be friend of B, while B does not think the symmetric. Or A can hate B at a degree X, while B hates A at a different degree, say Y. In other words, we want to be able to express the fact that the points of view (and the perceptions) of A and B might diverge. All this would require us to give away the actual opposition relations of NOT (which, with the exception of the implication arrows – i.e. subalternation –, are *symmetric* relations) in order to get at their place asymmetric opposition relations. But we actually do not know any such asymmetric opposition relations.

Remark also that if it seems rather natural to interpret contrariety relations as "opposition" in the common sense (i.e. hostility, incompatibility, adversity) it seems there is a need to philosophically clarify the possible interpretations of subcontrariety. As a matter of fact, while subcontrariety is classically understood as some kind of "compatibility" ("they can be true at the same time"), this opposition relation seems to have, possibly, a stronger philosophical meaning (useful for our present purpose), something like friendship-collaboration: two subcontrary things are such that always one of the two will be there: this is more than compatibility, this is collaboration and cooperation, some kind of reliability ("of us two, there will always be at least one [for you]").

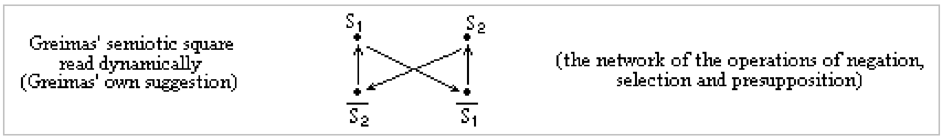
So, having mentioned some possible challenges to the global static aspect of NOT, it is time now to move to the possible solutions to these challenges.

24.03. Intermezzo: two existing dynamic readings of opposition

Before going to the dynamic solutions that we will propose in the rest of this chapter, let us have a quick look at two mysterious (but interesting) already existing reinterpretations of, respectively, the logical square and the logical hexagon. They are interesting for us because they (already) involve a *dynamical* reading of their static structure.

24.03.01. Wandering inside oppositional graphs as Greimas does

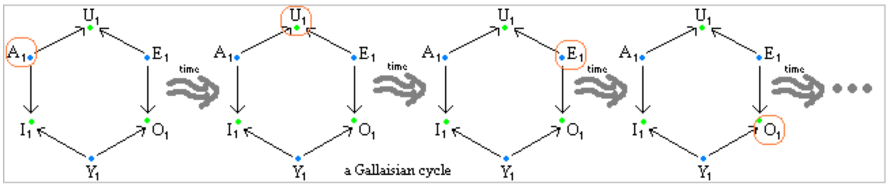
We already reminded something of Greimas' structuralist theory of "narratology", the formal heart of which is the so-called "semiotic square" (cf. ch. 06 *supra*). Now, the semiotic square has in fact two uses: a static one and a dynamic one[252].



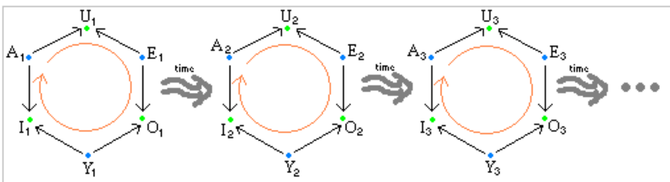
We already asked the question (ch. 17 *supra*): are there "semiotic bi-simplexes"? Another question can be now: can we do inside NOT the same as Greimas did? (that is, a dynamic use) I would incline here to defend a small conjecture: there is a "Greimasian" dynamic reading of the logical bi-simplexes (but this will need a future separate study).

24.03.02. Wandering "à la Gallais" inside (and between nested) hexagons

As we (briefly) saw in ch. 9, Gallais proposed to make a very dynamical use of the notion of logical hexagon. In his theory (very different from Greimas'), the (narratologic) meaning of a standard medieval novel is structured by a game of tensions and oppositions obeying the logical hexagon (reinterpreted). The hexagon's structure displays its action through a succession of positions, all mutually ordered. Concretely, this gives rise to some kind of "opposition movie" (in the sense in which we proposed to understand this expression). Gallais' "movies" have the peculiarity of being circular: after each cycle, a new one starts. So, truly speaking, there are two dimensions of the movie. The first dimension is circular (it consists in circulating clockwise in a logical hexagon; this is the basis of Gallais' theory).



The second dimension (also a part of Gallais' theory) consists in having a succession of "tours": and this makes an oppositional movie (Gallais would represent this with growing spirals, cf. ch. 09 *supra*).



[252] Cf. Groupe d'Entrevernes, *Analyse sémiotique des textes*, Lyon, Presses Universitaires de Lyon, 1979 (despite the general title, this book is an introduction to Greimas' thought, and in fact one of the best ones).

(here we do not present everything exhaustively, but one must be aware of the fact that in Gallais' theory the "paths" are not all "positive", there can be involutions, seahorse-like spirals, etc.)

We have not yet been able to propose an achieved extension of Gallais' theory *via* NOT (that is: using the fact that instead of a logical hexagon we can take higher logical bi-simplexes, like for instance logical cubes). But this, if verified, could offer us a rather complex and rather interesting "Gallaisian oppositional dynamic".

In these two uses (Greimas, Gallais) it is not the oppositional geometry that we change. What the two tell us, is that a simple oppositional structure (the square for Greimas, the hexagon for Gallais) is the hidden spring (or engine) of meaning (the semiotic-narrative sense for Greimas, the meaning of the standard medieval narration for Gallais). If I am to make a conjecture, I must say that I suspect something like this (*mutatis mutandis*!) to be (hiddenly) at stake with René Thomas' bio-mathematics (cf. ch. 09 *supra*).

All this shows that one can do rather unexpected dynamical things with oppositional static structures. But in order to give answers to our five challenges, we need something different (something more technical).

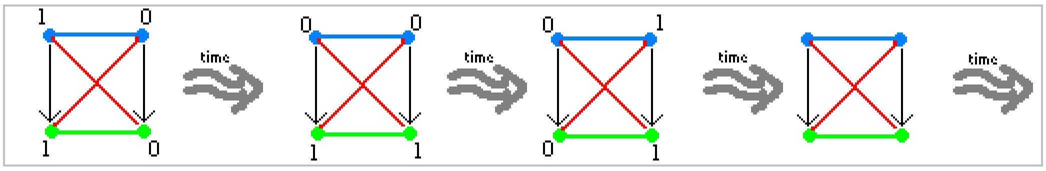
24.04. Strategies suggested by the very logic of opposition

The most natural reaction to the aforementioned challenges consists in using in a new way the formal possibilities (i.e. NOT) developed so far. The backbone of all our coming approaches here will be what we already hinted at as being the "movie strategy": we have "pictures" of opposition situations (*via* NOT), which can evolve in time. The simple juxtaposition of different pictures may simulate time changes in opposition, that is "opposition dynamics". Because the two main parameters of opposition are (so far) n and p (the n of n-opposition and the p of the p-simplexes), we will first examine some changes related to the modulation of these two parameters.

24.04.00. Changing the valuation of a given oppositional solid

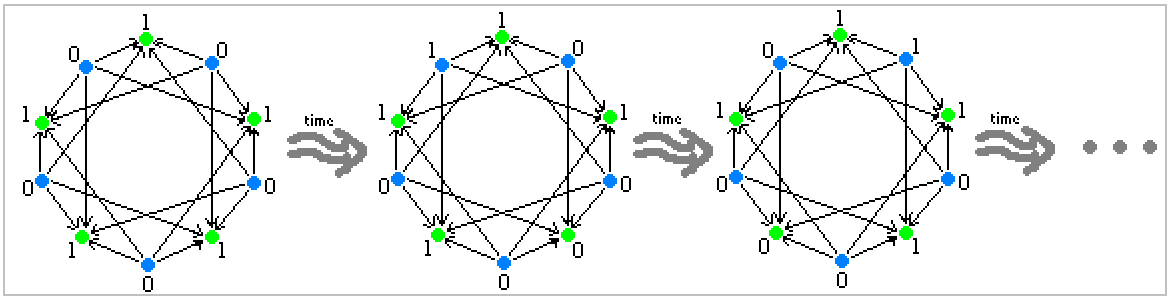
But, before that, let us mention some kind of important "prehistoric ancestor" of the coming changes: for, any oppositional solid (αn- or βn-structure) may be decorated with truth-values by a "valuation" (we treated this topic in the ch. 19-20 *supra*). So, a very first version of the "movie strategy" just sketched may concern the variations of valuation for a given oppositional solid. Let us start with Aristotle's square ("0" and "1" being the possible valuations).

(in fact, there are only these three possible ways of classically valuating the logical square,



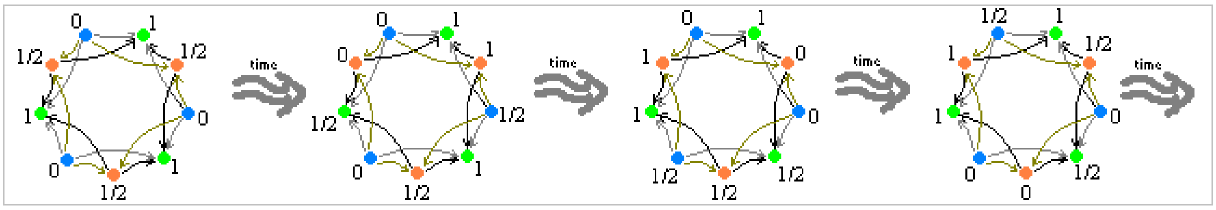
but their transition order is of course open to infinite variation)

But this valuation change, of course, can in fact concern any given logical bi-simplex, for instance the α5-structure.



(in fact there are only 6 possible ways of classically valuating the α5-structure, but, again, their possible alternations are infinite)

And further, this valuation change can concern any given logical *p*-simplex (of dimension *m*), as for instance the logical tri-triangle (i.e. the logical tri-simplex of dimension 2). Here we will have three possible truth-values, as we saw in ch. 19: the values 0, ½, and 1.



(in fact, there are only 7 possible ways of valuating classically – i.e. tri-valently – the logical tri-triangle, but infinitely many ways of alternating them)

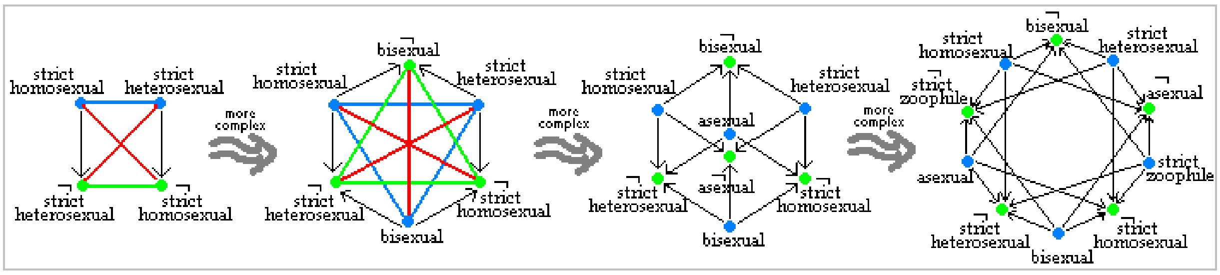
So, generally we can express the fact that a given oppositional structure (unchanged over time as for its shape) sees the truth-values of its decorating elements change through time (this is an answer to the second challenge).

24.04.01. Changing the number *n* of opposed terms

But we can build more complex "movies", of course. Changing the number of opposed terms is one of the main features offered by NOT since 2004 (with respect to Aristotle's opposition theory). Such a "gift" can be used for different purposes. Here we give two examples.

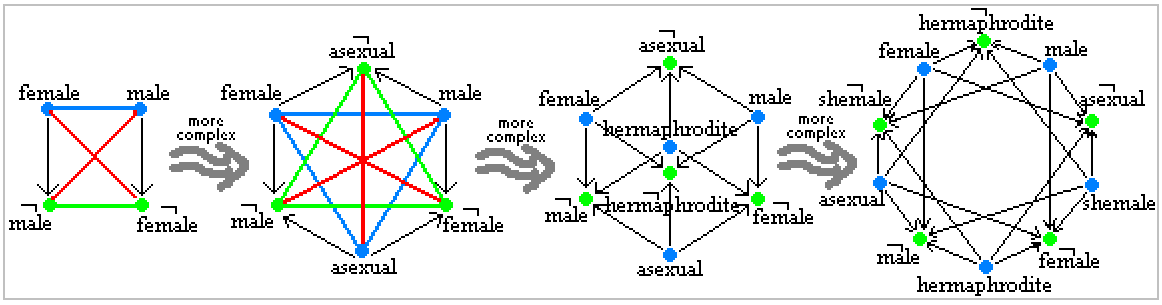
Example 1: refining the oppositions (the cases of gender and sexuality)

In a classificatory context one first purpose may be that of allowing the change of the number of opposed qualities. "Sexual choice" offers a nice and clear example (*una tantum*, sexier and funnier than Socrates' mortality or than the crows' feathers being black or not). Here we can shift from a down-to-earth vision of sexual possibilities (homosexual-heterosexual) to more fine-grained ones (there seems to be no conceptual limit).



One sees that the list could seemingly be expanded almost into infinity, by adding increasing precisions over the possible sexual choices (*cum grano salis*: the kind of animal, the animal's gender, and whatever). Remark also that a same term (say, "strict heterosexual") may belong to different (nested) kinds of opposition schemes (i.e. to different terms of the previous series, i.e. to different decorated αn-structures), different degrees of descriptive precision.

In a similar way, "gender" offers another nice and simple (and not too boring) example of qualitative shifts of the categorisation, from down-to-earth (female-male) to more fine-grained ("she-males" are women with a penis – a possible outcome of transsexuality).



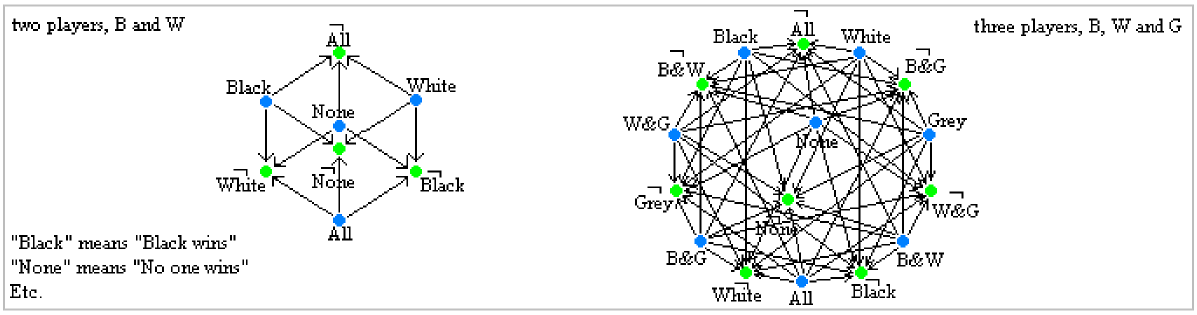
Here again, incredible as it may seem, there probably are ways to extend the series beyond actual prevision (there could be more sexes in the future!)[253]. As previously, a same

[253] Maybe ones coming from outer space …

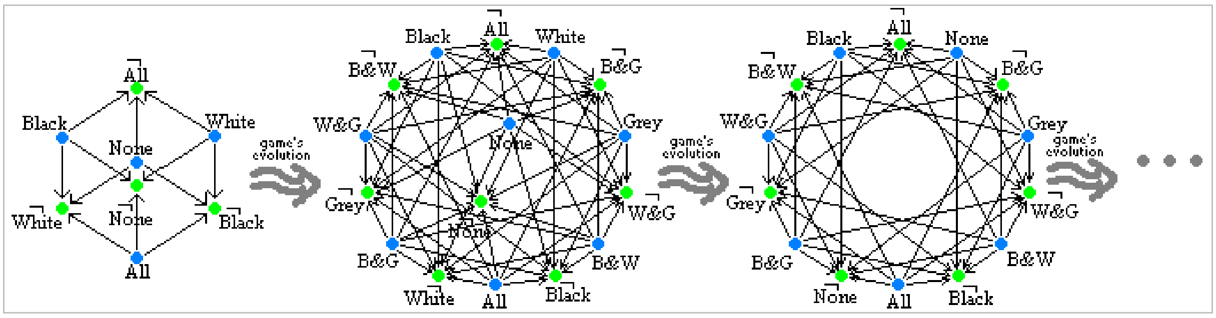
nominal term (say “hermaphrodite”) may belong here as well to different (nested) terms of the series. More impressively, remark also that a same term (say, “asexual”) may belong to different oppositional series (here “asexual” belongs both to the previously seen “sexual choice series” and to the presently examined “gender series”), which brings us back to the issue described in the Example 1 of § 24.02.04 *supra*.

Example 2: changing the number of fighters (and the “winning distributions”)

Another possible example is given by changing the number of the players during a game or conflict. As a matter of fact, for any number of players the complete set of possible issues – in fact the power set of the set of all players – can be expressed by means of a logical bi-simplex. For instance, if we have two players (a “2-conflict”) the possible issues of their fight seem to be 4 (the blue points are the possible issues). With three players (a “3-conflict”) the possible issues (blue points) seem to be 8 (cf. figure). With four players (a “4-conflict”) the possible issues seem to be 16. And so on (same power-set algorithm).



So, knowing this, it is easy to see that with our “movie strategy” we can represent changes in the evolution of a competition game (or fight).



(here we see that the game starts with two players, Black and White; then a new player, Grey, appears; then we see that player White will not win alone; and so on)

This may be interesting for “game theory”, a fertile mathematised discipline that has deep relations with logic (as shown by Lorenzen and Lorenz, Hintikka, van Benthem and many others). Remark also that it would be interesting to be able to express “partial gain” (or "partial defeat"), which could be possible by using our many-valued logical *p*-simplexes, cf. *infra*.

So here we showed how to answer the first challenge (changing the number of terms) as well as the second one (seeing opposition change).

24.04.02. Changing the number *p* of logical simplexes in a log. *p*-simplex

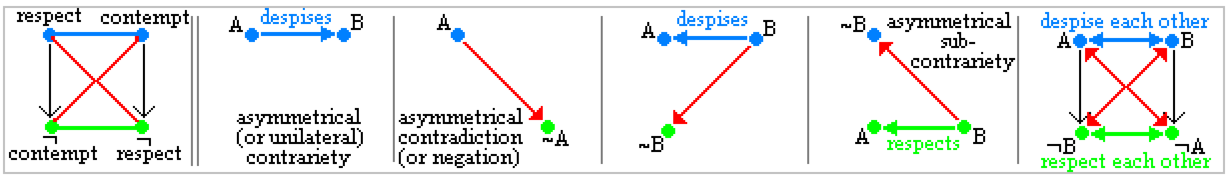
We want to express degrees of opposition (with the aim of being able to enrich our "movies", so to make them more dynamic). Now, it seems that the logical *p*-simplexes do allow that. As a matter of fact, they permit to express, beyond the possibility of having more fundamental qualities of opposition (which was the *p*-simplexes' founding aim), the interpolation of weakened degrees of contrariety and of subcontrariety between the two pure poles of contrariety and of subcontrariety (cf. ch.19-21 *supra*). So, for instance, two objects linked by a [1|0] opposition are "contrary", whereas two objects linked by a [2/3|1/3] opposition are "almost contrary". Two terms linked by a [0|1] opposition are subcontrary, whereas two objects linked by a [1/3|2/3] opposition are "almost subcontrary" (and so on). So this seems to be a way of answering the third challenge (seeing different degrees of a same opposition).

24.04.03. Combining separated oppositional structures

We already discussed before some of the main issues concerning the combination of separated oppositional structures (for instance when we discussed the English word "white"). We mentioned that each logical tetraicosahedron comprises 6 points that have this feature (each of these 6 points belongs to one logical hexagon's "triangle of contrariety" and to another logical hexagon's triangle of subcontrariety). However, at this stage, it is not quite clear how this could be done. Maybe a possible solution could come from the theory of the Aristotelian 2^q-semantics and 2^q-lattices (and more generally from the Aristotelian p^q-semantics and lattices). So, the fourth challenge (combining mutually independent oppositions) remains an open issue.

24.04.04. Having asymmetric opposition relations (for feelings etc.)

A possible answer to the challenge of having asymmetric opposition relations (other than the subalternations arrows) could consist in diffracting the notion of "oppositional



picture" (that is, a logical poly-simplex or a gathering of logical poly-symplexes, or at least a gathering of logical bi-simplexes). Getting inspired by the historical invention (discovery) of the complex numbers (based on the *ad hoc* axiom "$-1 = i^2$"), we could (try to) establish *ad hoc* the notion of "oppositional field", meaning by that some oppositional solid (or set or gathering of such solids) centred on a single object: by definition this would express all the ordinary opposition relations (extended by the notion of logical *p*-simplex) but *truncated*: such opposition relations (positive or negative, and whatever their degree) would express the, so to say, "emotional field" of the concerned "object" (abstract or concrete), with no "return backwards", i.e. no mention of the kind or degree of intensity of the "feelings" of the other objects for this one. So symmetries (of opposition quality) would be only a possible (not granted) coincidence (the general case being the asymmetry). As an example, we give here an illustration of the "asymmetrical treatment" of the opposition of "respect" and "contempt".

One sees on the previous scheme the difference between the abstract case (the impersonal concepts of "respect" and "contempt", related by a classical symmetric contrariety) and the concrete case (the embodied situation of "A despises B", which needs not be symmetrical *a priori*).

The next step could consist in multiplying such oppositional fields, attributing one to every object taken into account by the modelising process. So, the global reality (at least inside the model) would be given by a large series (or a large sedimentation) of "oppositional fields". At this stage it is not clear if such an evolution of NOT would still be an opposition-technology: it could be the case that this evolution would simply boil down to abolishing NOT (i.e. entering classical first order logic, with suitable predicates or relations for expressing emotions or feelings). Again, if this could work (this is not yet clear – it deserves a specific future investigation), it could yield some kind of new model of emotions (again: emotions *do* have in any case oppositional features – except for the "embodied" or corporal side of emotions, clearly out of reach). So, the fifth and last challenge (having asymmetric opposition relations) remains open, providing however an interesting (far-fetched?) conjecture (the notion of "opposition field") as to a possible way of answering it.

24.04.05. In general: studying transitions of changing opposition states

Again, the study of opposition dynamics would, in general, mainly rely on the notion of "oppositional movie". The principle of this seems rather simple, the major refinements coming from the fine-grained analysis made possible by the notions of logical bi-simplex and logical poly-simplex of opposition. The exploration of the q parameter (in the Aristotelian p^q-semantics) leaves room for the possible future emergence of new answers.

24.05. Concluding remarks so far on "Opposition Dynamics"

It seems that opposition dynamics could be developed (there are directions of thought to be developed). So far, people are even unaware of the basic of (static) NOT, but such a theory could find many interesting applications. Up to now, some of the challenges proposed by us are still not clearly answered. And, of course, there probably are more, waiting to be recognised and formulated.

Conclusive Part

THINKING OPPOSITION
AND
THINKING THROUGH OPPOSITION

25.
LAST REMARKS

Here we want to focus our attention on both some of the main gains of our approach (not already – or not enough – mentioned) and some last points that constitute a possible bridge between what we did see in this work and what could be studied in future works relying on it.

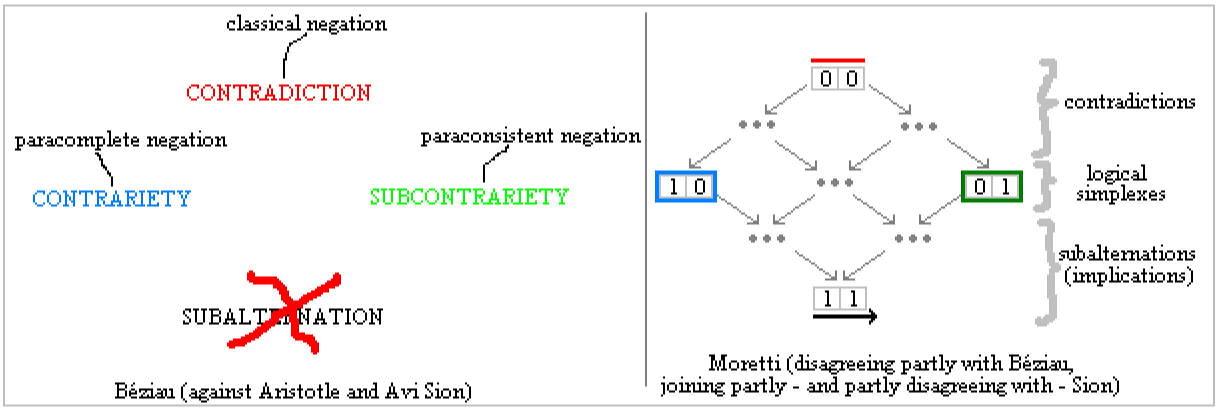
25.01. Arrows seem to be opposition relations as well

There is one important philosophical issue concerning which relations, among those generated, ruled and explained by the formalism of NOT, must be accounted for as "oppositions".

Aristotle's own opinion seems to have been that contradiction and contrariety are "oppositions", whereas subcontrariety and subalternation (both neither theorised, nor named by Aristotle) are not (cf. ch. 04 *supra*).

Béziau (2003) claims that subcontrariety must be taken to be an opposition as well (as contradiction and contrariety). In this respect he invokes the duality of contrariety and subcontrariety. This is motivated by the fact that Béziau wants to defend paraconsistency (and paraconsistency relies on subcontrariety). But he thinks that subalternation (i.e. implication) is not an opposition, and therefore – erasing the arrows from the hexagons – he calls "stars" the logical hexagons (cf. ch. 10 *supra*). The fact of having seen, following Béziau, subcontrariety as a kind of opposition seems to be, *a posteriori*, from the point of view of NOT, a significant progress.

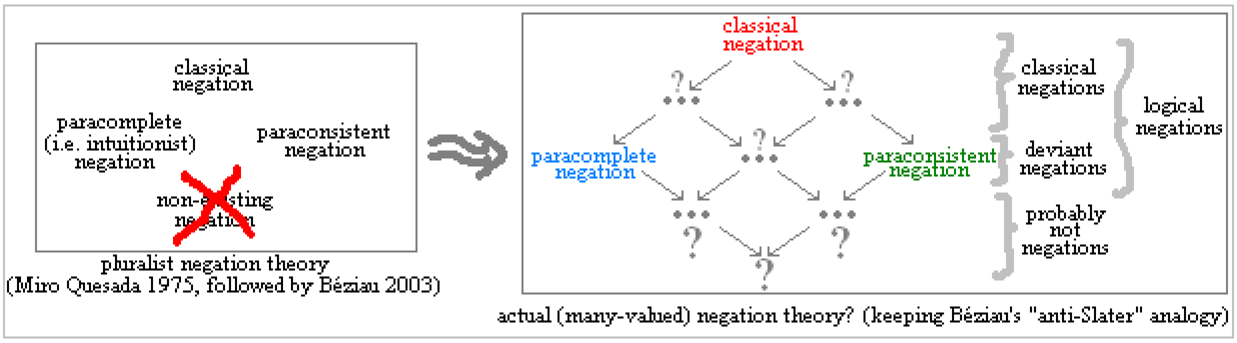
However, my claim is that without taking into account the arrows (i.e. the subalternations), the coherence of the whole is not clear. As a matter of fact, it was the refusal to see the arrows in the opposition hexagons that prevented Béziau from analysing with full correctness his "solid of oppositions". On the contrary, it was by concentrating on the arrows that I managed to correctly identify the geometrical solution to the problem of finding a solid of opposition (relatively to the 12 vertices found by Béziau). Therefore, because of the impressiveness of NOT (in terms of results and of inner coherence), the subalternations should be taken as oppositions. And in fact, even philosophically, this position seems very plausible. For subalternations *oppose* a dominant to a subordinate (which is a potentially violent opposition!). Remark that, formally speaking, this is not shocking: between the two terms of a binary order relation there is indeed a difference (one of … order!): therefore an opposition.



This view that I propose seems to be the most convincing so far. However, it does not take into account the oppositional structures obtained by making the q parameter vary in the Aristotelian p^q-semantics (cf. ch. 18 *supra*). So, the future will perhaps tell us more on these matters, and surprises cannot be excluded *a priori*.

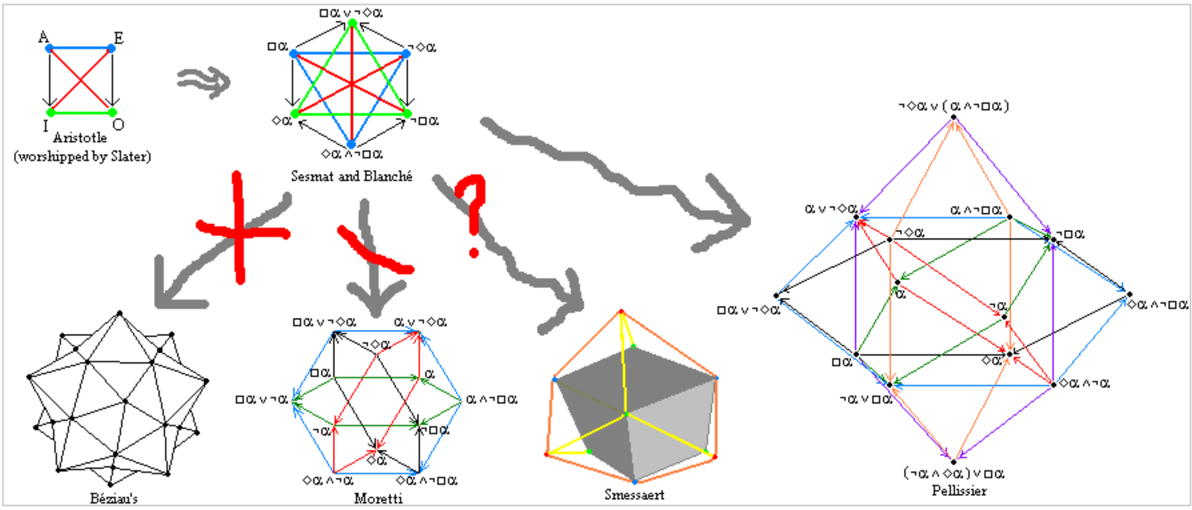
25.02. Back to Béziau's answer to Slater

A direct consequence of what we just saw pertains to Béziau's philosophical theory of negation, as conceived in response to Slater, by relying on opposition theory (cf. ch. 10 *supra*). As we saw, it consisted in claiming that paraconsistent and paracomplete negations *are* indeed negations. Now, NOT, through the theory of the logical p-simplexes, seems to make Béziau's position more precise.



The big change apparently consists in admitting a plurality of contradiction relations and in conceiving of the general notion of logical negation as being composed both by the contradiction kinds (upper half of the lattice) and by the logical simplexes (the lattice's diagonal), the latter generalising the notions of contrariety and subcontrariety.

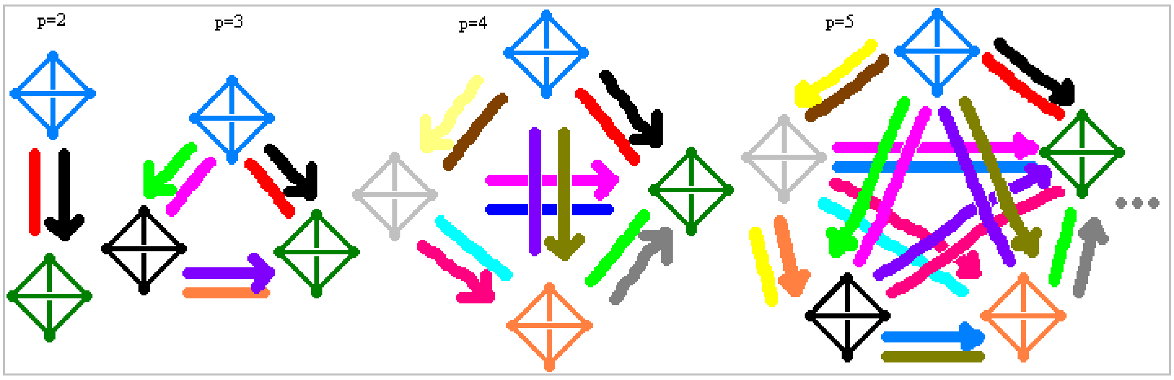
In general, Béziau's move consisted in proposing some kind of transcendental alternative to Slater's own transcendental reading of the logical square (cf. ch. 10). I showed, by discovering the logical cuboctahedron, that his solution was not yet correct (cf. ch. 10 and 11), and Smessaert and Pellissier, by discovering respectively the rhombic dodecahedron and logical tetraicosahedron, showed that my own was incomplete (cf. ch. 12). I would propend to see the logical tetraicosahedron as the only perfect solution: it is the only *complete figure*.



By his "topological hexagon", Pellissier also showed (this time in his second paper on NOT) that Béziau's general idea of reading paraconsistency in the geometry of the oppositions was correct. Pellissier nevertheless got to this "confirmation" by a way essentially different (and more powerful) than Béziau's one: Béziau's arguments are semantic, Pellissier's ones are syntactic (cf. ch. 12)[254].

25.03. Conceiving of the logical *p*-simplexes more radically?

We saw that the logical *p*-simplexes apparently do exist, and this is important in so far as it allows to claim some kind of strong deepening of the concept of opposition (and hence of bivalence): we thus demonstrated that there can be more than 4 kinds of opposition (cf. ch. 18-21).



However, the way we used to decorate the logical *p*-simplexes (with logical truth-values) seems not to be the only possible one. In particular, we always relied on a vision about

[254] Cf. R. Pellissier, "2-opposition and the topological hexagon", (forthcoming).

truth-values which is, so to say, linear, whereas the set of the truth values needs not to be a total order (it can be a partial order). It is thus tempting to explore a non-linear decoration of the logical *p*-simplexes. There seem to be different ways of having non-linear interpolations of extra truth-values[255]. One is the classical quaternary one of, say, the relevant FDE system ("First Degree Entailment"), where we have as truth-values: "only true", "true and false", "only false", "neither true nor false"[256]. Another one could be one generating truth values of a radically new kind (a "multi-polarity" still hard to imagine). But we leave the exploration of this to a further study.

25.04. Exploring the *q* dimension of opposition

Of this exploration of the *q* parameter we only saw the very premises (cf. ch. 18 *supra*). The path to follow in order to do this is, up to now, not clearly straightforward. We still need to understand in which sense it can be seen as an extension (conservative?) of the logics generated by the Aristotelian p^2-semantics. It seems to be a possibly very interesting field of geometrical-logical investigation – related, as we hinted at in ch. 18, to another series of logical simplexes –, and it could possibly generate rather useful families of new structures (useful at least for the elaboration of what we proposed as a future "opposition dynamics").

25.05. Are there dimensions of opposition other than the *n*, *p*, *q*?

The answer to this seemingly far-fetched question seems to be "yes" (but this is not sure). Fabien Schang (cf. ch. 22 *supra*) in some sense realised a meaningful modification of the notion of Aristotelian p^q-semantics, which he used in order to think some kind of "pragmatic values". This goes outside of NOT, but shows that there can be meaningful changes in the way of playing the question-answer game we proposed. So, an interesting question still ahead of us is that of understanding both how to characterise the notion of oppositional game and how to relate it to the already existing field of game-theoretical semantics.

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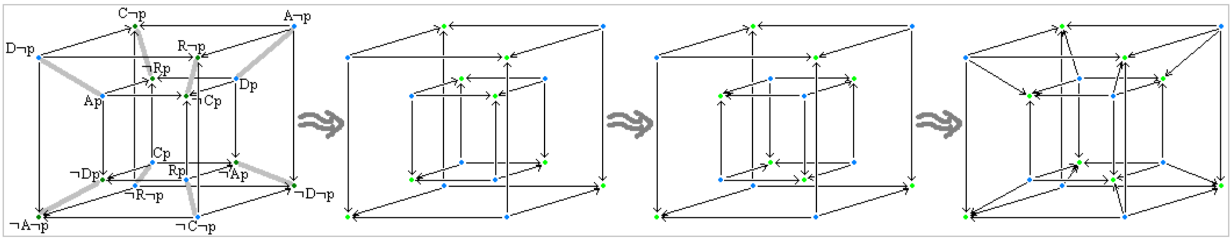
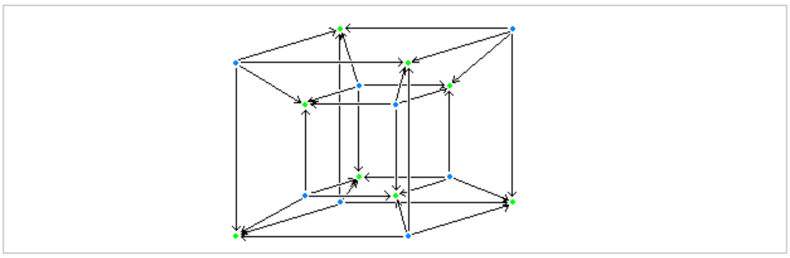
[255] On this subject, still to be explored in NOT style, cf. the interesting seminal work by A. Varzi and M. Warglien, "The Geometry of Negation", *Journal of Applied Non-Classical Logics*, 13:1 (2003), pp.9-19.

25.06. A new series of oppositional solids? The δ*n*-structures?

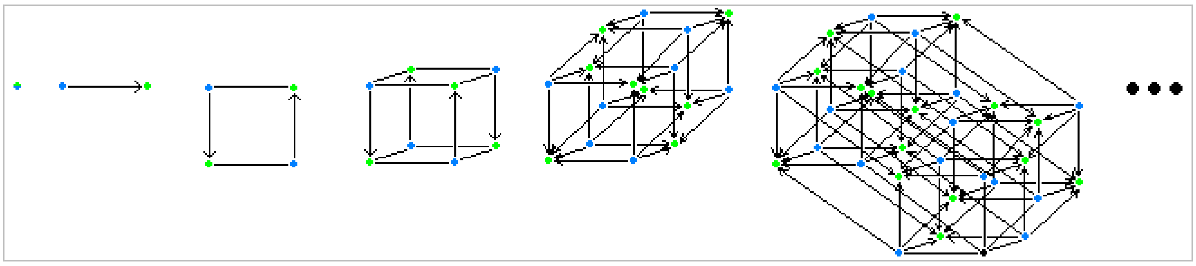
We would like to mention one more case, strange but interesting and possibly (if confirmed as oppositionally meaningful) important for the future development of NOT. We saw in § 17.02.05 *supra*, as a model of pragmatic assertion/denial acts, an object made of two nested logical cubes. These cubes were so to say “parallel” (with respect to the directions of their arrows in space). Now, if we reverse in one of the two squares (let’s say; the inner one) the direction of all its arrows (and if we correspondingly exchange blue vertices with green *et vice versa*), the segments linking each of the 8 vertices of one cube to one and only one of the eight vertices (its “parallel”, not its symmetric) of the other cube can in fact be read plainly as logical arrows (in the next figure we show how).



This gives an elegant (four-dimensional) hyper-cube made of NOT-coherent arrows and vertices: in this hyper-cube, each of the 8 constituent cubes is a logical cube (in the sense of NOT, cf. ch. 11 *supra*). So one would be tempted to consider it globally as being a new oppositional solid, a “logical hyper-cube”.



Now, at least two crucial remarks must be made here: (1) it is not at all clear if this object can



be considered as a fully-fledged oppositional solid: for some of its opposition relations could be failing to fulfil the NOT-prerequisites (but this is not yet clear); (2) this solid, as such (i.e.

[256] Cf. G. Priest, *An Introduction to Non-Classical Logics*, Cambridge, CUP, 2001.

as a geometrical structure made of arrows and of blue or green vertices) clearly belongs to an infinite series: and this opens the suspicion that it could be a fourth NOT-group of opposition-structures (maybe the "δ*n*-structures"?), beside the α*n*-, β*n*- and γ-structures.
The main objection or remark (from the point of view of NOT) is that this "logical hypercube" would be such that each of its terms would have four distinct (and equally important) symmetry centres and therefore four distinct contradictory negations. This seems strange, but it could be just the "strangeness" inherent to the fresh discovery of a new world. So, if confirmed after a proper examination (which we cannot afford here), this would open up a possible new field of NOT. Remark that this (hypothetically logical) series has one term, the logical cube, that also belongs to another series: that of the α*n*-structures (if we called "series of the δ*n*-structure" this new candidate for the NOT-citizenship, we would say that the logical cube is both the α4-structure and, say, the δ3-structure). Remark that we already know of a similar behaviour inside NOT (this one solidly established mathematically speaking): the logical hexagon is common to both the α*n*-structures (it is the α3-structure) and the β*n*-structures (it is the β2-structure). It is fascinating to think that this intersection behaviour could be just one small fragment of a big, coherent new field. Remark also that, if confirmed, this new series could be related with (i) the mysterious "*q*" parameter (cf. 25.04 *supra*) and/or with (ii) Luzeaux's notion (by now only verbal, never pictured in his joint papers) of "logical hypercube" (cf. § 12.03 and § 17.03.07 *supra*).

25.07. Is there a future for opposition dynamics?

The static NOT seems to be very useful, at least in so much as it touches virtually all situations. As for the dynamics built on top of NOT, the same considerations seem to hold. A special mention could be deserved by the (until now) hypothetical theory of the asymmetrical opposition relations, for they seem to possibly somehow open the door of the realm of emotion. However, it is not clear whether the notion of "cluster of opposition fields", on which this treatment of emotions *via* opposition theory could be built, can really have a meaning. If it does, it can bring valuable light on the notion of emotion (at least it can offer some models of it).

25.08. What is the place of NOT inside logic?

NOT seems to be important for discussing some very hard questions of the philosophy of logic. This is evident with respect to the issues on paraconsistency, for NOT originated from that (inside the already mentioned Priest-Slater-Béziau debate). But then we saw (Part III of this work) that there seem to be unsuspected deep relations of NOT with many-valued logics (*via* the theory of the Aristotelian p^q-semantics and lattices and the notion of logical poly-simplexes). Additionally, NOT reveals its considerable relevance for the discussion of logic when it is observed relatively to Béziau's research line of UL, for NOT seems to confirm Béziau's intuition that logic has something specific with respect to the classical Bourbakian so-called mother-structures. But NOT also goes in the direction of the works of Shramko and Wansing, which at least partly contradict some of Béziau's theses on the general structure and meaning of logic (in particular they contradict Béziau's definition of an abstract logical structure <L, ⊢> as being one made of a set and of *one* consequence relation, cf. ch. 23 *supra*).

25.09. Some missing but necessary comparisons

In the perspective of building a systematic exploration of the relations of NOT to logic in general, some important exploration can already be perceived as missing: we think of the useful work that could result from comparing systematically NOT and the so-called "IF (modal) logics" (system inspired by Hintikka's work on Game-Theoretical Semantics and internalising into logic the game-theoretical notion of "informational independence"). Another field that could turn out very interesting, could be that of the systematic comparisons of NOT with "linear logic" (at least in its modal flavour), a study that would open up to a still more general one, that of the relations of NOT to "substructural logics" (these comprising, among others, linear logic and relevant logic). A third easily perceivable *desideratum* is the study of the possible relations between NOT and "hybrid logics", a non-standard approach to modal logic (and in particular to tense logic) which enables the expression of modal properties (especially indexical ones) previously out of reach. In all these cases (and we probably forgot many others), one aspect of the problem would be that of adapting NOT to the study of new (non-standard) families of quantifiers.

25.10. Is NOT related to Gärdenfors or Matte Blanco?

This last question is very important from the point of view of the possible applications of NOT to the theory of concepts and to that of mind. The answer to the first part of the question, so far, seems to be "not" (or at least not yet "yes"), for the geometry of NOT remains very regular and rigid (it makes one think of crystals), whereas Gärdenfors' "conceptual spaces" are intended to be geometrically very malleable. However, this point is not yet clear, we still do not know the whole story about NOT (we have not yet investigated the *q* parameter, for instance) and what we saw in §24.02.04 suggests that there could be possible inspiring intersections of NOT and Gärdenfors' theory: the composition, in a network, of different α*n*-structures (cf. § 24.02.04) could lead to some particular kinds of conceptual spaces. It could be the case that, at worst, NOT embodies (just) one aspect of the reductive geometry of concepts, if not the whole of it. In this respect, it seems likely that a future task will be that of comparing (and possibly joining) NOT with some other existing formal theories of concepts (like Wille *et alii*'s "formal concept analysis" or Sowa *et alii*'s "conceptual graphs"). This question clearly needs a specific enquiry. As for the relation to Matte Blanco's theory, the NOT owes much to him, for it is there that, as long as I'm concerned, I found inspiration for thinking the notion of a logical simplex. Again, it is still too early to make a definite commentary, a specific examination is still needed in order to get clear ideas on this subject; but there seems to be a community of inspiration, for both approaches use – so to say against some of Kant's misleading dogmata – *n*-dimensional spaces.

26.
CONCLUSION

We give here the very final remarks of our present work. We divide them into three headings: the results, the present consequences and the future possible developments.

26.01. What has been done

The results of our enquiry can roughly be classified along four main sections, depending on the field concerned (mainly: philosophy or logic) and on the two particular points where some quite precise gains were obtained (the comprehension of Aristotle's implicit intuition of opposition and the general concept of modal logic).

26.01.01. Philosophical clarification on the concept of "opposition"

At the beginning of our study, we proposed a typology of oppositions classifying them in three main classes: the static, the dynamic and the intensive ones. We proposed to limit ourselves, at least in the beginning, to the first class, this one being the easiest to formalise. We suggested that, despite the early relations of the notion of opposition with spatial thinking and geometry, some kind of problem with space occurred in our Western philosophical tradition, which involved a privilege granted to logic as a fundamentally non-spatial discipline. We analysed some possible historical reasons for that and suggested that some major trends in this direction could have been created by some dramatic and badly-known events (an undesired geometrical discovery, similar to the famous dramatic discovery of "$\sqrt{2}$" in the Pythagorean school) which occurred inside the Platonic Academy at the time of Plato and Aristotle: according to the rigorous philological and historical enquiries of the so-called Tübingen-Milano school (and in particular to those of Ch. Mugler, I. Toth and V. Hösle), this seems to have been the shocking (pro-Sophistic) discovery of the possibility of having non-Euclidean geometries. Aristotle's clear abandon of mathematics as a guideline for ontology and philosophy in general shifted philosophy toward the progressive adoption of "logic" as its main tool. And we underlined how Kant has seemingly played a rather bad part in that respect (deepening Aristotle's move), and how the so-called analytical philosophy has remained, despite its epistemophile appearances, in the same anti-geometrical framework. We went on to recall the evolutions of the theory of (static) opposition in the Western thought, since Aristotle to Sesmat, Blanché and, in recent times, Béziau (who gave it a remarkable impulsion).

26.01.02. Strong technical (formal) results: general *n*-opposition theory

Starting from Aristotle, Sesmat, Blanché and Béziau's theories on opposition and based on some new results by Pellissier, Smessaert and ourselves, we proposed a renewed theory of (static) opposition ("NOT"). First, we showed that it implies the strict distinction of three kinds of graphs, which we called α-structures, β-structures and γ-structures (respectively the oppositional solids, the gatherings of oppositional solids and the modal graphs). Second, we showed that the α-structures (i.e. the family of the logical square and hexagon) are based on the notion of logical bi-simplex of dimension *m* (their genetic algorithm, opening to an infinite series of growing finite oppositions). Two important points, at least, depend on this: (i) this theory allows to express the opposition relations among any finite number of terms (whereas the classical theory until 2004 – reaching the logical hexagon – was limited to three); and (ii) this theory allows to translate almost all systems of modal logic (and not only) into precise, exhaustive logical-geometrical solids (belonging to a single infinite linear series, that of the gatherings), by way of suitable translation rules. For short: NOT is able to translate (through a mapping) the γ-structures (i.e. modal logic, and in fact even more than this) into the series of the βn-structures, and each βn-structure easily gives (by a suitable partition technique, due to Pellissier) *all* the αn-structures (i.e. all the oppositional solids, all the logical bi-simplexes). Furthermore, facing the hard challenge of escaping the until then inescapable constraint of the transcendental quaternarity inherent to the kinds of opposition (which persisted since Aristotle's theory, *via* Gottschalk's and Piaget quaternarity groups, to Béziau and even to our notion of logical bi-simplex), we succeeded in showing that this whole theory (i.e. NOT) could be strongly generalised by accessing the (logically tenable) new notion of logical *p*-simplex (instead of logical bi-simplex) of dimension *m*. This move – allowing to have p^q kinds of oppositions instead of just 4 (= 2^2) –, which turns out to open up opposition theory to many-valued oppositions and therefore to many-valued logics (and thus to substructural logics, a very important and powerful family of logical systems), is done with the help of a suitable powerful new tool, the so-called Aristotelian p^q-semantics, inspired to us by Aristotle's original combinatorial definitions of opposition, and generating correlated Aristotelian p^q-lattices. Using all these new ingredients we proposed the first elements of a possible future theory of the *dynamical* oppositions (or "opposition dynamics"), thus leaving the sole ground of static opposition and (apparently) accessing formally the two other possible classes of opposition (the dynamic and the intensive ones). Dynamic oppositions can be approached through "movies" made of "NOT-pictures" (the formal specific requirements and invariants of such dynamic theory still need to be investigated). As for formally approaching the still more distant idea of intensive oppositions, they are conjectured by us as being possibly modelised by means of “opposition fields”, an opposition field being made of asymmetric oppositions (an *ad hoc* weakening of the notion of opposition, possibly comparable for its strangeness, if successful, to the invention/discovery of $\sqrt{(-1)}$), centred not on sets of simplectically displayed ontological contrary objects anymore (the logical simplexes constituting the logical p-simplexes of NOT), but rather on singular “sentient” (or emotional) “subjects” (an opposition field would be some kind of simplified emotional viewpoint – hence the stress over asymmetry – over a subject’s world). The superposition of many subject-centred opposition fields could open up on to some kind of theory of “oppositional intersubjectivity”.

26.01.03. A clarification of Aristotle’s (implicit) notion of opposition

These discoveries, which let impressive formal regularities emerge, confirmed us in holding a particular hermeneutic position with respect to Aristotle’s line of thought. In other words, NOT shows that the notion of opposition as discovered, at least partly, by Aristotle is a whole “package” made of 4 (and not 2 or 3) sides, 4 kinds of opposition. This result is rather new – it shows that “subalternation” (i.e. logical implication) can (and must) be seen as a kind of opposition – and brings much clarity. Moreover, the part of NOT dealing with Aristotelian p^q-semantics (and lattices) shows – and this result is philosophically very new and very strong – that there can be more than 4 kinds of opposition, and gives the complete algorithm for generating and handling them. This result is totally new, the constraint of having four and only four kinds of opposition being a logically (and thus philosophically) very strong one.

26.01.04. A general consequence over the understanding of modal logic

The meaning of NOT is not limited to the expression of the opposition relations among any (finite) number n of terms, or to the expression of p-valued oppositions: another important side of it is to have shown that the compositions of “basic modalities” (by means of unary and binary connectives) are also full modalities of their own. This means that when speaking the language of modal logic, many modalities are systematically forgotten: the only way to have a clear and complete mastery of it is to express (or recognise) the underlying oppositional backbone. In other words, for any modal system the so-called “βn-structures” (the gatherings) are, at least conceptually speaking, the fundamental underlying form of modal logic (and this result even holds for larger conceptions of modal logic, i.e. ones possibly taking into account either multi-modal operators or null modalities. In this respect, even if it has not caused much of a stir so far, NOT seems to be some kind of (maybe not bloody!) revolution inside logic, the β*n*-structures being apparently fundamental to logic, but totally new.

26.02. Other consequences of our enquiry

The consequences of our study seem to pertain to at least three main fields: logic, philosophy of logic and general philosophy.

26.02.01. Relevance of NOT for the logical research

By its simple existence, NOT showed the presence of a new logical tool: the geometry of the oppositions. It seems clear to us that this tool can (and will) be useful to pure logical research. As it stands, because of its abstract character (not to be hidden by the presence of a visual, geometrical aspect), NOT could be implemented in computer science. Moreover (and consequently), there could be applications in *pure* (theoretical) computer science. By now we seem to witness the emergence of two new fields: general opposition theory (oppositional geometry, already furnished with its own regularities, laws and vocabulary) and opposition dynamics (this discipline being much more conjectural, we only prospected some possibly seminal ideas in the present work; time will judge). If a further conjecture of ours is to be verified, there could be room for a third new domain: the theory of the opposition fields.

26.02.02. Relevance of NOT for the philosophy of logic

As far as the philosophy of logic is concerned – a discipline of the highest importance for philosophy in our eyes –, it seems that there is a strong relevance of NOT for it. This is not astonishing, NOT admitting some of its deepest origins in Béziau pioneering and fundamental work on the philosophy of logic. We saw clear examples of the strong links between NOT and the philosophy of logic (at its best) both with the so-called Slater debate on paraconsistent logic (cf. ch. 10) and with the so-called Suszko debate on many-valued logic (cf. ch. 23). Both debates are raging, very actual and situated at the very heart of the logically possible (or impossible). From this respect NOT has deep links with the understanding of the so-called "universal logic" project, which, again, is no mystery if one remembers the role of Béziau at the origin of our investigation, Béziau being additionally the proponent of the compelling (and very open and technically powerful) research line thus called (UL). Of course, NOT already told us several new things – from the vantage viewpoint of general opposition – about the foundations of logic (namely the notion, still to be analysed and better understood philosophically, of "logical simplex"). In this respect it could be the case that new important elements will be brought to light, in the near future, about the aforementioned Slater- and Suszko- debates.

26.02.03. Relevance of NOT for philosophy in general

The relevance of NOT for philosophy, if less apparent, is not minor in my opinion. A starting school case is the study of McNamara's logical systems, which showed that even the best scholars and philosophers (McNamara's philosophical analysis of "supererogation" is recognised as the current world standard, and his consequent formal systems embodying it, like DWE, have been proven logically complete and sound) do need a method for enquiring the geometry of their oppositions (his geometrical expressions of his logic are – NOT proves it – both logically redundant and logically incomplete): random geometry – and without NOT any inquiry over the geometry of the logical oppositions is destined to remain random – can betray any sound and complete logical systems, as it did with McNamara's. In a similar vein, our joint work with Schang (cf. ch. 22) can claim the usefulness of NOT for clarifying philosophical concepts (in this case those of Searle and Vanderweken's standard theory of "formal pragmatics") and, as a by-product (if we dare say), it happened to offer a totally new understanding (and legitimisation) of Piaget's famous "INRC group", his until then strange variation on Aristotle's square. As for conceptual clarification in general, it will be interesting in the future to try to join other approaches (as those of Wille's "formal concept analysis", of Sowa's "conceptual graphs" and, most of all, of Gärdenfors' theory of the "conceptual spaces") with the aim of powerfully complementing them. Again, this powerful clarifying role would be no surprise, if one recalls that one of the discoverers of the logical hexagon (clearly one of the starting points of the NOT adventure), Blanché, developed with it a robust theory of the systematic organisation of concepts. In this respect remark that, as Blanché was an (isolated) "structuralist" (Piaget and Greimas where famous non-isolated ones), the NOT seems to suggest that this old research line (structuralism) has probably been buried too quickly (probably because of theoretical developments of it that lost the link to the search of concrete structures, as in Michel Foucault's "structuralism without structures", or with Jacques Derrida who diluted structuralism into a generic and controversial language-disrupting inquiry over abstract "difference", the famous "différance"). Béziau's project of investigating a pluralistic but interconnected "universal logic" seems to be clearly structuralist (like the Bourbakian school of mathematics that inspired him in the search of formal "mother structures"). And our reflections over the embarrassing neglect of space (and more generally of "structures") by most analytical philosophers (who keep targeting instead the banner-concept of "symbolic calculus") could be related to the fact that analytical philosophy *de facto* took (or stole) the place of structuralism as a formal general world research program including philosophy and the cognitive sciences (notably against the very idea of psychoanalysis), but forgot the constitutive aim of looking for concrete "structures". Analytical philosophy – in that inspired, paradoxically, by a general mathematical anti-geometrical trend (embodied even by the Bourbakian movement) – overstressed the importance of "logic" (against space and visualisation in general), thus putting to death the more general (and more promising, more powerful) notion of "structure". If taken seriously, NOT could help accessing to a more balanced and fruitful general research program, one dribbling the blinding fetishism of logic (solely conceived of as a symbolic calculus) and breathing new life into structuralist thinking (and hopefully integrating all the gains of analytical philosophy at ist best)[257].

26.04. Other applications of NOT

Other applications of NOT are not yet known, but we have elements suggesting that it could, among others, underlie the bio-mathematical theory of the Belgian biologist René Thomas (with whom we try to work thereupon, cf. ch. 9). It would not be totally astonishing (but for this we have no elements so far) to see, sometime or other, the emergence of relations with quantum mechanics, this discipline admitting at its heart symmetry groups and similar structures (so far, this is just a conjecture of mine). We mentioned already the plausibility of seeing one day applications in computer science, both applied and theoretical (of this, as well, we have no concrete elements as of now). We can anyway make the easy guess that there will be the emergence of some kind of "*general* oppositional technology" – wider than the one developed here –, possibly matching partly some of the guesses we made when speaking of the idea of a future discipline to be called "opposition dynamics" (or "opposition fields").

[257] As for instance with, but not limited to, huge thinkers (philosophers and/or logicians) such as Quine, Strawson, Davidson, David Lewis…

26.04. Open future

The conceptual horizon of the research line developed in this study is, hopefully, quite open and promising. Three points must be mentioned here: the questions left open, the current community of researchers working on the theory and, of course, the unpredictability of the future, which will be – I swear it, patient reader! – our very last word.

26.04.01. Points still to be dealt with

The list of things deserving to be done seems to be (at least partly) rather clear. There are some quite precise tasks ahead of us, “opposition people”. One is the determination of an algorithm serving as general translation rule for switching from *any* finite modal graph to its corresponding β*n*-structure. The current problem is that of giving a general algorithm for applying (or adapting) Pellissier’s general “setting” technique. Another important task, still a bit vague at present, is that of exploring more accurately the logical *p*-simplexes, of which we gave here only the first ideas and characterisations, but which clearly lack a clearer treatment. Then one more huge task could consist in exploring what we called the *q* parameter (of the Aristotelian p^q-semantics). Of this we gave here almost no element, despite the fact that we already explored this issue privately (but our materials were not ready, neither in quantity nor in quality, to be exposed here). A further task will then be that of looking systematically for the next possible generalisations of the theory, that is, looking for further fundamental parameters of opposition theory, beyond the known *n*, *p* and *q* (cf. ch. 18). Another task will be that of exploring the opposition relations between logical molecules admitting more than one valuation (as we partly did with Schang, re-discovering then Piaget). Another task, as suggested by Luzeaux, Sallantin and Dartnell, will be that of studying geometrically (i.e. in NOT style) the so-called modal graphs (or γ-structures) of the logical modal systems ruled, at a metalevel, by a metalanguage other than classical. One last task which we must mention here is that of systematising the exploration of what we called opposition dynamics. Of course there can be more tasks to be faced – at least we hope so –, many of which exceed what we can imagine as of now.

26.04.02. The emergence (so far) of a NOT-working community

As for the NOT-working community (if I dare say – for actually, on the contrary, I really hope to find a job!), it seems that at the very moment of this writing, it is a very small (but productive) one, consisting, as far as I know, of four people: Régis Pellissier, Hans Smessaert, Dominique Luzeaux and myself. Of course, it must be understood here that I speak only about the abstract theory closely related to what we called NOT in this work. Outside this precise (but constraining) framework, much valuable work is being made by several people, including for instance (of course) *in primis* Jean-Yves Béziau, but also Alexandre Costa-Leite and Gillman Payette (who are working on the "square of imagination"), Ferdinando Cavaliere (who rediscovered – ignoring Sesmat and Blanché – the logical hexagon and now develops a very original geometrical approach to "fuzzy syllogistics") and many others (for instance most of the many people who took part to the First World Congress on the Square of Opposition in Montreux, 2007, as well as many known or unknown others). As for the applications of NOT, we have to mention here the Montpellier group (LIRMM) on "debate theory" and Fabien Schang (with ourselves) on pragmatics. Of course, it will be helpful (and wise) to unite in order to fortify and promote a research group on opposition that could be a world leader in developing this powerful and potentially useful and universal technology.

But the future is unpredictable, and therefore we cannot assume for sure that our enterprise ("our" to be taken in a plural and open sense, of course) will grow, and grow fast, so to become a new classic (of formal thought). Resistances can clearly be felt, as I have felt them up to now ("What the hell is this stuff?", "What's the use of it? None!", "Where is Possible World Semantics in all this? Nowhere, therefore this is not science", "Who do these guys think they are?", "Do you really need geometry to do this? This is not geometry!", "This is disguised Pythagorism", …): to use an euphemism, at the present day it is not always easy to be heard (or read) when speaking about these topics (opposition theory meets … opposition), their concepts and terminology being new (sorry for that!) and therefore (seemingly) offensive (or outrageous) to some. The reference to the structuralist element of NOT may explain the instinctive hostility against it shown by analytical philosophy (and its reigning, symbolic-calculatory fetishist paradigm). It seems however plausible to think that opposition phenomena being as fundamental as they clearly are, the interest for them will grow, and, NOT being possibly the right approach to them (we firmly believe and claim it is), it could be the case that the generalised *n*-opposition theory we tried to serve here as best as we could (for some years, with effort, solitude and unsponsored sacrifices, but also with great intellectual pleasure and fun) is destined to deserve some particular and more friendly attention and discussion in the coming times. May it be so!

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