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Wreath products with the integers, proper actions and Hilbert space compression

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A la mémoire de Michel Matthey

Abstract We prove that the properties of acting metrically properly on some space with walls or some CAT(0) cube complex are closed by taking the wreath product with $\mathbb{Z}$. We also give a lower bound for the (equivariant) Hilbert space compression of $H \wr \mathbb{Z}$ in terms of the (equivariant) Hilbert space compression of H.

Keywords Wreath products · Proper (group) actions · Compression · Hilbert spaces · Trees · CAT(0) cube complexes

Mathematics Subject Classifications (2000) Primary: 20E22 · Secondary: 20E08 · 20F69

1 Introduction

A *space with walls*, as defined by Haglund and Paulin [10], is a pair $(X, \mathcal{W})$ where X is a set and $\mathcal{W}$ is a set of partitions of X (called *walls*) into two classes, submitted to the condition that any two points of X are separated by finitely many walls.

The main examples of spaces with walls are given by CAT(0) cube complexes (see [2]), i.e. metric polyhedral complexes in which each k-cell is isomorphic to the Euclidean cube $[-1/2, 1/2]^k$, and the gluing maps are isometries. Indeed, it is a result of Sageev [13] that hyperplanes in a CAT(0) cube complex endow the set of vertices

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with a structure of space with walls (see [7] and [11] for more on the relation between spaces with walls and CAT(0) cube complexes).

Our first result is the following:

Theorem 1.1 *Suppose that a group H acts metrically properly either on some space with walls, or on some $CAT(0)$ cube complex. Then so does the restricted wreath product* $H \wr \mathbb{Z} := (\bigoplus_{\mathbb{Z}} H) \rtimes \mathbb{Z}$.

Guentner–Kaminker defined the *Hilbert space compression* and the *equivariant Hilbert space compression* for any unbounded metric space (endowed with a group action in the latter case) [9]. Since we will deal with uniformly discrete[1] spaces, the following definitions are equivalent to theirs.

Let (X, d) be a uniformly discrete metric space. We define the *Hilbert space compression* of X as the supremum of the numbers $\alpha \in [0, 1]$ such that there exists a Hilbert space $\mathcal{H}$, positive constants C_1, C_2 and a map $f: X \to \mathcal{H}$ with

$$C_1 \cdot d(x,y)^{\alpha} \leqslant ||f(x) - f(y)|| \leqslant C_2 \cdot d(x,y)\ \forall x, y \in X.$$

It is denoted by $R(X, d)$ and it is a quasi-isometry invariant of (X, d). If H is a group acting on (X, d) by isometries, the *equivariant Hilbert space compression* of X is the supremum of the numbers $\alpha \in [0, 1]$ such that there exists a Hilbert space $\mathcal{H}$ endowed with an action of H by affine isometries, positive constants C_1, C_2 and a H-equivariant map $f: X \to \mathcal{H}$ with

$$C_1 \cdot d(x,y)^{\alpha} \leqslant ||f(x) - f(y)|| \leqslant C_2 \cdot d(x,y)\ \forall x, y \in X.$$

It is denoted by $R_H(X, d)$. One has trivially $R_H(X, d) \leqslant R(X, d)$.

We may view a group H as a metric space thanks to the word length associated with some (not necessarily finite) generating subset S. We denote then by $R(H, S)$ the Hilbert space compression and by $R_H(H, S)$ the equivariant Hilbert space compression. In case H is finitely generated, note that, up to bilipschitz equivalence, the word metric does not depend on the finite generating set, so that the compressions do not depend on the choice of a finite generating set. In this case, we write $R(H)$ and $R_H(H)$ for the corresponding compressions. We also use these shorter notations in the general case if there is no ambiguity about the generating set. It is a remarkable observation of Gromov (see [8, Proposition 4.4] for a proof) that $R(H) = R_H(H)$ for H finitely generated and amenable.

The first examples of finitely generated groups whose Hilbert space compression is different from 0 and 1 appeared recently in Ref. [1]: Thompson's group F and the wreath product $\mathbb{Z} \wr \mathbb{Z}$ (see Sect. 5 for more on this). Our next Theorem allows in particular to construct more examples.

Given a generating set S for H, if $\Gamma = H \wr \mathbb{Z}$, we always take $\Sigma = S \cup \{s\}$ as generating set for Γ, where s is the positive generator of $\mathbb{Z}$.

Theorem 1.2 *Let H be a group, with generating set S and let $\Gamma = H \wr \mathbb{Z}$. The non-equivariant and equivariant Hilbert space compressions satisfy:*

$$R(H,S) \geqslant R(\Gamma,\Sigma) \geqslant \frac{R(H,S)}{R(H,S)+1},$$

$$R_H(H,S) \geqslant R_{\Gamma}(\Gamma,\Sigma) \geqslant \max\left\{R_H(H,S) - \frac{1}{2}, \frac{R_H(H,S)}{2R_H(H,S)+1}\right\}.$$

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[1] That is, there exists a constant $\delta > 0$ such that $d(x, y) \geqslant \delta$ whenever $x \neq y$.

In order to select the best bound, we mention that one has $t - 1/2 \geqslant t/(2t+1)$ if and only if $t \geqslant (1+\sqrt{5})/4 \cong 0.809...$ (for $t \in [0,1]$). The above-mentioned observation by Gromov gives immediately a stronger estimate for the equivariant compression.

Corollary 1.3 *Let H be a finitely generated and amenable group and let $\Gamma = H \wr \mathbb{Z}$. The equivariant Hilbert space compression satisfies:*

$$R_H(H) \geqslant R_\Gamma(\Gamma) \geqslant \frac{R_H(H)}{R_H(H)+1}\,.$$

The proofs of Theorems 1.1 and 1.2 rest on a similar idea: we express $H \wr \mathbb{Z}$ as an HNN-extension in two different ways, which provide two different actions of $H \wr \mathbb{Z}$ on a tree. In Theorem 1.1, we use the product of these two trees, while in Theorem 1.2 we appeal to the affine actions naturally associated with each of these trees (see Sect. 7.4.1 in [4]).

2 Preliminaries: wreath products and trees

Let Λ be a group, H a subgroup and $\vartheta\colon H \to \Lambda$ an injective homomorphism. The *HNN-extension* with *basis* Λ and *stable letter* t relatively to H and ϑ is defined by $HNN(\Lambda, H, \vartheta) = \langle \Lambda, t \mid t^{-1}ht = \vartheta(h)\ \forall h \in H\rangle$.

Our definition of graphs and trees are those of [14]. Given an HNN-extension $\Gamma = HNN(\Lambda, H, \vartheta)$, the associated *Bass–Serre tree* is defined by

$$V(T) = \Gamma/\Lambda, \quad E(T) = \Gamma/H \sqcup \Gamma/\vartheta(H), \quad \overline{\gamma H} = \gamma t\vartheta(H);\ \overline{\gamma\vartheta(H)} = \gamma t^{-1}H,$$

$$(\gamma H)^- = \gamma\Lambda, \quad (\gamma H)^+ = \gamma t\Lambda, \quad (\gamma\vartheta(H))^- = \gamma\Lambda, \quad (\gamma\vartheta(H))^+ = \gamma t^{-1}\Lambda,$$

where, given an edge e, its origin is denoted by e^- and its terminal vertex by e^+. It is a tree [14, Theorem 12]. We turn T to an oriented tree by setting $Ar_+(T) = \Gamma/H$, $(\gamma H)^- = \gamma\Lambda$, $(\gamma H)^+ = \gamma t\Lambda$ and the Γ-action on T preserves this orientation. Moreover we remark that the oriented tree is bi-regular: for each vertex of T the outgoing edges are in bijection with Λ/H and the incoming edges are in bijection with $\Lambda/\vartheta(H)$.

We turn to wreath products. Let G, H be groups. We set

$$\Lambda = H^{(G)} = \bigoplus_{g\in G} H = \{\lambda\colon G \to H \text{ with finite support}\}.$$

The group G acts on Λ by automorphisms: $(g\cdot\lambda)(x) = \lambda(g^{-1}x)$. The *restricted wreath product $H \wr G$* is the semi-direct product $\Lambda \rtimes G$, with respect to the above action. The group H embeds in $H \wr G$ and will be identified with its copy indexed by $1_G \in G$. It is easy to see that, given generating sets of G and H, their union generates $H \wr G$.

In case $G = \mathbb{Z}$, one may express $H \wr \mathbb{Z}$ as an HNN-extension in two ways (we denote by s the positive generator of $\mathbb{Z}$ in $H \wr \mathbb{Z}$ and by t_+, t_- the stable letters of the HNN-extensions):[2]

1. Set $\Lambda_+ = \bigoplus_{n\geqslant 0} H$ and $\vartheta_+ : \Lambda_+ \to \Lambda_+$ given by $\vartheta_+(\lambda)_0 = 1_H$ and $\vartheta_+(\lambda)_n = \lambda_{n-1}$ for $n \geqslant 1$. One has $HNN(\Lambda_+, \Lambda_+, \vartheta_+) = H \wr \mathbb{Z}$ and the isomorphism is given by $\lambda \mapsto \lambda$ and $t_+ \mapsto s^{-1}$.

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[2] There is a third way of expressing $H \wr \mathbb{Z}$ as an HNN-extension: set $\vartheta\colon \Lambda \to \Lambda; \vartheta(\lambda)_n = \lambda_{n-1}$. One has $HNN(\Lambda, \Lambda, \vartheta) = H \wr \mathbb{Z} = \Lambda \rtimes_\vartheta \mathbb{Z}$ and the isomorphism is given by $\lambda \mapsto \lambda$ and $t \mapsto s^{-1}$. Nevertheless, this expression will be useless in this article.

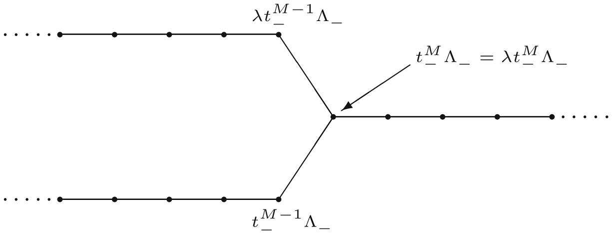

Fig. 1 The geodesics c and $\lambda \cdot c$

2. Set $\Lambda_- = \bigoplus_{n\leqslant 0} H$ and $\vartheta_- : \Lambda_- \to \Lambda_-$ given by $\vartheta_-(\lambda)_0 = 1_H$ and $\vartheta_-(\lambda)_n = \lambda_{n+1}$ for $n \leqslant -1$. One has $HNN(\Lambda_-, \Lambda_-, \vartheta_-) = H \wr \mathbb{Z}$ and the isomorphism is given by $\lambda \mapsto \lambda$ and $t_- \mapsto s$.

Given a wreath product $H \wr \mathbb{Z}$, we will denote by T_+, respectively, T_-, the Bass–Serre tree associated to the first, respectively, second, HNN-extension above. We take as base points (when necessary) the vertices Λ_+ and Λ_-.

We collect now some observations about the $H \wr \mathbb{Z}$-actions on T_+ and T_- which will be relevant in the next sections. Set $\Gamma = H \wr \mathbb{Z}$ and $\gamma = \lambda s^n \in \Gamma$. If λ is non-trivial, we set $m = \min\{k \in \mathbb{Z} : \lambda_k \neq 1_H\}$ and $M = \max\{k \in \mathbb{Z} : \lambda_k \neq 1_H\}$. The following lemma is straightforward.

Lemma 2.1 *If* $\lambda = 1$, *one has* $d_{T_+}(\Lambda_+, \gamma\Lambda_+) = |n| = d_{T_-}(\Lambda_-, \gamma\Lambda_-)$.

Lemma 2.2 *If* $\lambda \neq 1$, *the distances* $d_{T_\pm}(\Lambda_\pm, \gamma\Lambda_\pm)$ *are given by formulas:*

$$d_{T_+}(\Lambda_+, \gamma\Lambda_+) = \begin{cases} |n| & \text{if } m \geqslant n \text{ or } m \geqslant 0, \\ n - 2m & \text{if } m < n \text{ and } m < 0, \end{cases}$$

$$d_{T_-}(\Lambda_-, \gamma\Lambda_-) = \begin{cases} |n| & \text{if } M \leqslant n \text{ or } M \leqslant 0, \\ 2M - n & \text{if } M > n \text{ and } M > 0. \end{cases}$$

In particular, the inequalities $d_{T_+}(\Lambda_+, \gamma\Lambda_+) \geqslant -m$, $d_{T_-}(\Lambda_-, \gamma\Lambda_-) \geqslant M$, $d_{T_+}(\Lambda_+, \gamma\Lambda_+) \geqslant |n|$ *and* $d_{T_-}(\Lambda_-, \gamma\Lambda_-) \geqslant |n|$ *hold.*

Proof We prove the second equality, leaving the first one, which is very similar, to the reader. We remark that $\gamma = \lambda s^n = \lambda t_-^n$ and that, for any $k \in \mathbb{Z}$, the stabilizer of the vertex $t_-^k\Lambda_-$ satisfies

$$\mathrm{Stab}(t_-^k\Lambda_-) = t_-^k\Lambda_- t_-^{-k} = s^k\Lambda_- s^{-k} = \bigoplus_{i\leqslant k} H$$

so that we obtain

$$\lambda \in \mathrm{Stab}(t_-^k\Lambda_-) \iff k \geqslant M. \tag{2.3}$$

Consider now the infinite geodesic c, whose set of vertices is $\{t_-^k\Lambda_- : k \in \mathbb{Z}\}$. By Eq. 2.3, half of it is fixed by λ, while the remainder is not (see Fig. 1). To draw points Λ_-, $t_-^n\Lambda_-$ and $\gamma\Lambda_- = \lambda t_-^n\Lambda_-$ in Fig. 1 (distinguish cases $M > \max(0, n)$, $M \leqslant 0$ and $M \leqslant n$) will now convince the reader that the result holds. □

Let us now state a formula computing the length of an element of $H \wr \mathbb{Z}$, which is a direct consequence of [12, Theorem 1.2]. Note that, even if the theorem was stated for finitely generated groups, it also applies in our case.

Proposition 2.4 *Keep the above notations. Let* $\gamma = \lambda s^n \in \Gamma = H \wr \mathbb{Z}$. *In case* $\lambda = 1$, *one has* $|\gamma| = |n|$, *while in case* $\lambda \neq 1$, *the length of* γ *satisfies:*

$$|\gamma| = L_{\mathbb{Z}}(\gamma) + \sum_{i \in \mathbb{Z}} |\lambda_i| \,,$$

where $L_{\mathbb{Z}}(\gamma)$ *denotes the length of the shortest path starting from* 0, *ending at* n *and passing through* m *and* M *in the (canonical) Cayley graph of* $\mathbb{Z}$.

The length $L_{\mathbb{Z}}(\gamma)$ appearing in Proposition 2.4 can be estimated as follows:

Proposition 2.5 *Let* $\gamma \in \Gamma = H \wr \mathbb{Z}$. *The following inequalities hold:*

$$d_{T_\pm}(\Lambda_\pm, \gamma\Lambda_\pm) \leqslant L_{\mathbb{Z}}(\gamma) \leqslant d_{T_+}(\Lambda_+, \gamma\Lambda_+) + d_{T_-}(\Lambda_-, \gamma\Lambda_-).$$

Proof If $\gamma = s^n$, the result is obvious. We suppose now $\gamma = \lambda s^n$ with $\lambda \neq 1$. The proof is then a consideration of eight cases which are listed in the following table:

n	m	M	d_{T_+}	d_{T_-}	$L_{\mathbb{Z}}$	$d_{T_+} + d_{T_-}$
$\geqslant 0$	$\geqslant 0$	$> n$	n	$2M - n$	$2M - n$	$2M$
$\geqslant 0$	$\geqslant 0$	$\leqslant n$	n	n	n	$2n$
$\geqslant 0$	< 0	$> n$	$n - 2m$	$2M - n$	$2M - 2m - n$	$2M - 2m$
$\geqslant 0$	< 0	$\leqslant n$	$n - 2m$	n	$n - 2m$	$2n - 2m$
< 0	$< n$	$\leqslant 0$	$n - 2m$	$-n$	$n - 2m$	$-2m$
< 0	$\geqslant n$	$\leqslant 0$	$-n$	$-n$	$-n$	$-2n$
< 0	$< n$	> 0	$n - 2m$	$2M - n$	$2M - 2m + n$	$2M - 2m$
< 0	$\geqslant n$	> 0	$-n$	$2M - n$	$2M - n$	$2M - 2n$

The values of $d_{T_\pm}(\Lambda_\pm, \gamma\Lambda_\pm)$ come from Lemma 2.2; those of $L_{\mathbb{Z}}(\gamma)$ are easy to compute. We now observe that the result is true in the eight cases. □

Combining Propositions 2.4 and 2.5, one obtains immediately:

Corollary 2.6 *Let* $\gamma = \lambda s^n \in \Gamma = H \wr \mathbb{Z}$. *The following inequalities hold:*

$$d_{T_\pm}(\Lambda_\pm, \gamma\Lambda_\pm) + \sum_{i \in \mathbb{Z}} |\lambda_i| \leqslant |\gamma| \leqslant d_{T_+}(\Lambda_+, \gamma\Lambda_+) + d_{T_-}(\Lambda_-, \gamma\Lambda_-) + \sum_{i \in \mathbb{Z}} |\lambda_i|.$$

3 Metrically proper actions

Let us consider a group G, acting by isometries on a metric space X.

Definition 3.1 The action is *metrically proper* if, whenever B is a bounded subset of X, the set $\{g \in G : g \cdot B \cap B \neq \varnothing\}$ is finite.

From now on, we shall write “proper” instead of “metrically proper”. Let us recall that the action is proper if and only if the following property holds, for some $z \in X$:

$$\text{for any } R > 0, \quad \text{the set } \{g \in G : d(z, g \cdot z) \leqslant R\} \text{ is finite.} \qquad (Prop_z)$$

Let now $\mathcal{F} = (X_i, d_i, b_i)_{i\in I}$ be a family of pointed metric spaces (where d_i is the metric on X_i and $b_i \in X_i$ is a base-point), and let $p \geqslant 1$. We call ℓ^p*-product* of the family the space

$$\ell^p(\mathcal{F}) = \left\{ x \in \prod_{i\in I} X_i : \sum_{i\in I} d_i(b_i, x_i)^p < +\infty \right\}.$$

It is a metric space with metric $\delta(x,y) = \left(\sum_{i\in I} d_i(x_i,y_i)^p\right)^{1/p}$. We set $(b_i)_{i\in I}$ as base point. Consider now the case $(X_i, b_i) = (X, b)$ for all $i \in I$. One has:

$$\ell^p(I; X, b) := \ell^p(\mathcal{F}) = \left\{ \phi\colon I \to X : \sum_{i\in I} d(b, \phi_i)^p < +\infty \right\}.$$

If a group H acts by isometries on X, the group $H \wr G$ acts by isometries on $\ell^p(G; X, b)$ in the following way:

$$\begin{aligned} (\lambda\cdot\phi)_g &= \lambda_g\cdot\phi_g && \text{for } \lambda \in \textstyle\bigoplus_{g\in G} H, \\ (g\cdot\phi)_{g'} &= \phi_{g^{-1}g'} && \text{for } g \in G. \end{aligned} \tag{3.2}$$

Given G infinite, observe that, even if the action of H is proper, the action of $H \wr G$ on $\ell^p(G; X, b)$ is *not*. Indeed, G has a global fixed point in $\ell^p(G; X, b)$.

Theorem 1.1 will follow from the following statement.

Proposition 3.3 *Let H be a group acting properly on a metric space X, $b \in X$ and $p \geqslant 1$. Then, the action of $\Gamma = H \wr \mathbb{Z}$ on $T_+ \times T_- \times \ell^p(\mathbb{Z}; X, b)$, where the product is endowed with the ℓ^p metric, is proper.*

Proof We are going to prove property ($Prop_z$) for $z = (\Lambda_+, \Lambda_-, (b)_{i\in\mathbb{Z}})$. Thus, let $R > 0$ and $A = \{\gamma \in \Gamma : d(z, \gamma\cdot z) \leqslant R\}$. Take $\gamma = \lambda s^n \in A$. We have $d_{T_+}(\Lambda_+, \gamma\Lambda_+) \leqslant R$, $d_{T_-}(\Lambda_-, \gamma\Lambda_-) \leqslant R$ and $\sum_{i\in\mathbb{Z}} d(b, \lambda_i\cdot b)^p \leqslant R^p$.

By Lemmas 2.1 and 2.2, one has $M \leqslant R$, $m \geqslant -R$ (if M and m are defined) and $|n| \leqslant R$. Set $B = \{h \in H : d(b, h\cdot b) \leqslant R\}$. It is a finite set since the H-action is proper.

Hence, one has $|n| \leqslant R$, $\lambda_i = 1_H$ for $|i| > R$ and $\lambda_i \in B$ for $|i| \leqslant R$. This leaves finitely many choices for γ, and proves thus that A is finite. □

Remark 3.4 The space $T_+ \times T_- \times \ell^p(\mathbb{Z}; X, b)$ is canonically isometric to the product $\ell^p(\mathcal{F})$ with $I = \{+,-\} \cup \mathbb{Z}$ and $\mathcal{F}$ given by $\mathcal{F}(+) = (T_+, \Lambda_+)$, $\mathcal{F}(-) = (T_-, \Lambda_-)$ and $\mathcal{F}(i) = (X, b)$ for $i \in \mathbb{Z}$.

Proof of Theorem 1.1 We recall first that a tree is a CAT(0) cube complex, hence a space with walls.

It is shown in [5, Sect. 5] that a ℓ^1-product of spaces with (measured) walls carries the same structure. Hence the conclusion for spaces with walls by Proposition 3.3.

Given a CAT(0) cube complex Y, we denote by $Y^{(k)}$ the set of k-cells in Y. Take now a family $\mathcal{F} = (X_i, b_i)_{i\in I}$ of CAT(0) cube complexes with $b_i \in X_i^{(0)}$ and set $\mathcal{F}^{(0)} = (X_i^{(0)}, b_i)_{i\in I}$. We are going to construct a subspace X of $\ell^2(\mathcal{F})$ which is a CAT(0) cube complex.

We define first $X^{(0)} = \ell^2(\mathcal{F}^{(0)})$. Since the distance between two distinct vertices is at least 1, one has

$$X^{(0)} = \bigoplus_{i\in I}(X_i^{(0)}, b_i) := \left\{ v \in \prod_{i\in I} X_i^{(0)} : \{i \in I : v_i \neq b_i\} \text{ is finite} \right\}.$$

For $k \geqslant 1$, we define then the set of k-cells as

$$X^{(k)} = \left\{ \begin{array}{c} c \in \prod_{i\in I}(X_i^{(0)} \cup \ldots \cup X_i^{(k)}) : \\ \sum_{i\in I} \dim(c_i) = k \text{ and } \{i \in I : c_i \neq b_i\} \text{ is finite} \end{array} \right\}.$$

It is clear that every k-cell, as a subset of $\ell^2(\mathcal{F})$, is isometric to $[-1/2, 1/2]^k$. If $c \in X^{(k)}$, the faces of c are the $(k-1)$-cells c' such that c'_j is a face of c_j for some j and $c'_i = c_i$ for $i \neq j$. The gluing maps are isometric. Finally, the space $\ell^2(\mathcal{F})$ inherits the CAT(0) property, so that X is a CAT(0) cube complex.

Suppose now that H acts on a CAT(0) cube complex Y and take v_0 a vertex of Y. We consider the family $\mathcal{F}$ given by $I = \{+, -\} \cup \mathbb{Z}$, $\mathcal{F}(+) = (T_+, \Lambda_+)$, $\mathcal{F}(-) = (T_-, \Lambda_-)$ and $\mathcal{F}(i) = (Y, v_0)$ for $i \in \mathbb{Z}$. The action of $H \wr \mathbb{Z}$ on $\ell^2(\mathcal{F})$ is proper by Proposition 3.3 and the CAT(0) cube complex X constructed as above is an invariant subset, so that it is endowed with a proper action of $H \wr \mathbb{Z}$ too. □

Remark 3.5 The same techniques can be used, if H acts properly on some Hilbert space $\mathcal{H}$, to prove that $H \wr \mathbb{Z}$ acts properly on the Hilbert direct sum $\ell^2(E(T_+)) \oplus \ell^2(E(T_-)) \oplus \bigoplus_{i\in\mathbb{Z}} \mathcal{H}$. Hence, we recover the known fact that Haagerup property is preserved by taking wreath products with $\mathbb{Z}$ [4, Proposition 6.1.1 and Example 6.1.6]. The interest of our technique is that we obtain an *explicit* proper action of $H \wr \mathbb{Z}$, knowing a proper action of H.

Remark 3.6 It is known [5, Theorem 1], that a discrete group satisfies the Haagerup property if and only if it acts properly on some space with measured walls. According to Remark 3.5, whenever H acts properly on a space with measured walls, the same holds for $H \wr \mathbb{Z}$. Again, our techniques give an explicit action, as Theorem 1.1 is also valid for spaces with measured walls.

4 Hilbert space compression: Theorem 1.2

We recall that a map $f\colon X \to Y$ between metric spaces is *Lipschitz* if there exists $M > 0$ such that $d_Y(f(x), f(y)) \leqslant M \cdot d_X(x, y)$ for all $x, y \in X$. Given a Lipschitz map $f\colon X \to \mathcal{H}$ of a metric space X into a Hilbert space $\mathcal{H}$, we set[3] R_f to be the supremum of the numbers $\alpha \in [0, 1]$ such that there exists $D > 0$ with $D \cdot d_X(x, y)^\alpha \leqslant ||f(x) - f(y)||$ for all $x, y \in X$.

Given a generating set S of a group H, we recall our convention to take $\Sigma = S \cup \{s\}$ as generating set for $H \wr \mathbb{Z}$, where s is the positive generator of $\mathbb{Z}$. In order to simplify notations, we do not mention explicitly S and Σ, which we fix throughout this section.

The key result to prove Theorem 1.2 is the following:

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[3] It does not coincide with the *asymptotic compression* of f defined in Ref. [9].

Proposition 4.1 *Let H be a group (with a generating set S) and $\Gamma = H \wr \mathbb{Z}$. Suppose that maps $f\colon H \to \mathcal{H}$ and $f_\pm\colon V(T_\pm) \to \mathcal{H}_\pm$ are Lipschitz with $R_{f_+} = R_{f_-} > 0$ and $R_f > 0$. Consider the map*

$$\sigma\colon \Gamma \to \mathcal{H}' := \mathcal{H}_+ \oplus \mathcal{H}_- \oplus \bigoplus_{i\in\mathbb{Z}} \mathcal{H}\,,$$

where, given $\gamma = \lambda s^n \in H \wr \mathbb{Z}$, we set $\sigma(\gamma)_\pm = f_\pm(\gamma\Lambda_\pm)$ and $\sigma(\gamma)_i = f(\lambda_i)$ for $i \in \mathbb{Z}$. Then $R_\sigma \geqslant R_f \cdot R_{f_\pm}/(R_f + R_{f_\pm})$ and $R_\sigma \geqslant \min\{R_{f_\pm}, R_f - \frac{1}{2}\}$.

Moreover, if f is H-equivariant and $f_\pm$ are Γ-equivariant (with respect to some actions by affine isometries), σ is Γ-equivariant with respect to some action of Γ on $\mathcal{H}'$ by affine isometries.

Proof We show first that σ is Lipschitz (the reader could remark that it is trivial if H is a finitely generated group; however, this case is also covered by the proof below). Let us take $K, K_+, K_- > 0$ such that

$$\begin{aligned} \|f(h_1) - f(h_2)\| &\leqslant K \cdot |h_1^{-1}h_2| \quad \forall h_1, h_2 \in H,\\ \|f_\pm(u) - f_\pm(v)\| &\leqslant K_\pm \cdot d_{T_\pm}(u,v) \quad \forall u, v \in V(T_\pm).\end{aligned}$$

Let $x, y \in \Gamma$. We set $\gamma = x^{-1}y$ and write $x = \xi s^p$, $y = \eta s^q$, $\gamma = \lambda s^n$ in $H \wr \mathbb{Z} = \Lambda \rtimes \mathbb{Z}$ (so that $n = q - p$ and $\lambda_i = \xi_{i-p}^{-1}\eta_{i-p}$). One has then

$$\left(\sum_{i\in\mathbb{Z}} \|\sigma(x)_i - \sigma(y)_i\|^2\right)^{1/2} \leqslant \sum_{i\in\mathbb{Z}} \|f(\xi_i) - f(\eta_i)\| \leqslant \sum_{j\in\mathbb{Z}} K \cdot |\lambda_j| \leqslant K \cdot |\gamma|.$$

Corollary 2.6 implies:

$$\|\sigma(x)_\pm - \sigma(y)_\pm\| \leqslant K_\pm \cdot d_{T_\pm}(x\Lambda_\pm, y\Lambda_\pm) = K_\pm \cdot d_{T_\pm}(\Lambda_\pm, \gamma\Lambda_\pm) \leqslant K_\pm \cdot |\gamma|.$$

Thus, we get finally $\|\sigma(x) - \sigma(y)\| \leqslant (K_+ + K_- + K) \cdot |x^{-1}y|$, which proves that σ is Lipschitz, as desired.

We now turn to the estimation of R_σ, Fix any α, β such that $0 < \alpha < R_f$ and $0 < \beta < R_{f_\pm}$. There exists constants $C, C_+, C_- > 0$ such that:

$$\begin{aligned} \|f(h_1) - f(h_2)\| &\geqslant C \cdot |h_1^{-1}h_2|^\alpha, \quad \forall h_1, h_2 \in H\,,\\ \|f_\pm(u) - f_\pm(v)\| &\geqslant C_\pm \cdot d_{T_\pm}(u,v)^\beta, \quad \forall u, v \in V(T_\pm).\end{aligned}$$

We notice first that σ is injective. More precisely, for any $x, y \in \Gamma$, one has

$$x \neq y \implies \|\sigma(x) - \sigma(y)\| \geqslant \min\{C, C_+, C_-\}\,. \tag{4.2}$$

Indeed, we express $x = \xi s^p$ and $y = \eta s^q$ as above. If $p \neq q$, we obtain $\|\sigma(x)_\pm - \sigma(y)_\pm\| \geqslant C_\pm \cdot d_{T_\pm}(x\Lambda_\pm, y\Lambda_\pm)^\beta \geqslant C_\pm$ and if $\xi_i \neq \eta_i$ for some i, we obtain $\|\sigma(x)_i - \sigma(y)_i\| \geqslant C \cdot |\xi_i^{-1}\eta_i|^\alpha \geqslant C$.

Let us take x, y and γ as above. According to Corollary 2.6, one (at least) of the following cases occurs. We treat them separately. As the case $x = y$ is trivial, we assume $x \neq y$, that is $|\gamma| \geqslant 1$, in what follows.

(a) Case $d_{T_+}(\Lambda_+, \gamma\Lambda_+) \geqslant \frac{1}{3}|\gamma|$: We obtain

$$\begin{aligned} ||\sigma(x) - \sigma(y)|| &\geqslant ||\sigma(x)_+ - \sigma(y)_+|| \geqslant C_+ \cdot d_{T_+}(x\Lambda_+, y\Lambda_+)^\beta \\ &= C_+ \cdot d_{T_+}(\Lambda_+, \gamma\Lambda_+)^\beta \geqslant \frac{C_+}{3^\beta}|x^{-1}y|^\beta \, . \end{aligned}$$

(b) Case $d_{T_-}(\Lambda_-, \gamma\Lambda_-) \geqslant \frac{1}{3}|\gamma|$: We obtain the same way

$$||\sigma(x) - \sigma(y)|| \geqslant ||\sigma(x)_- - \sigma(y)_-|| \geqslant \frac{C_-}{3^\beta}|x^{-1}y|^\beta \, .$$

(c) Case $\sum_{i\in\mathbb{Z}} |\lambda_i| \geqslant \frac{1}{3}|\gamma|$: We establish two independent estimates.
First, for all $i \in \mathbb{Z}$, one has $||\sigma(x)_i - \sigma(y)_i|| = ||f(\xi_i) - f(\eta_i)|| \geqslant C|\xi_i^{-1}\eta_i|^\alpha = C|\lambda_{i+p}|^\alpha$. Recall that $m = \min\{k \in \mathbb{Z} : \lambda_k \neq 1_H\}$ and $M = \max\{k \in \mathbb{Z} : \lambda_k \neq 1_H\}$. Using the Cauchy–Schwarz inequality for the third step below and $\alpha \leqslant 1$ for the fourth one, we obtain

$$\begin{aligned} ||\sigma(x) - \sigma(y)|| &\geqslant \left(\sum_{i\in\mathbb{Z}} ||\sigma(x)_i - \sigma(y)_i||^2\right)^{1/2} \geqslant C\left(\sum_{j\in\mathbb{Z}} |\lambda_j|^{2\alpha}\right)^{1/2} \\ &\geqslant \frac{C}{\sqrt{M-m+1}} \sum_{j=m}^{M} |\lambda_j|^\alpha \geqslant \frac{C}{\sqrt{M-m+1}} \left(\sum_{j=m}^{M} |\lambda_j|\right)^\alpha \end{aligned}$$

By Proposition 2.4, one has $|\gamma| \geqslant M - m + 1$, so that we obtain

$$||\sigma(x) - \sigma(y)|| \geqslant \frac{C}{\sqrt{|\gamma|}}\left(\frac{1}{3} \cdot |\gamma|\right)^\alpha \geqslant \frac{C}{3^\alpha} \cdot |\gamma|^{\alpha - 1/2} \, . \qquad (*)$$

This is our first estimate for case (c).
Second, we fix any $\zeta \in \,]0,1[$. Then, either there exists $k \in \mathbb{Z}$ such that $|\lambda_k| \geqslant (\frac{1}{3} \cdot |\gamma|)^\zeta$, or one has $M - m + 1 \geqslant (\frac{1}{3} \cdot |\gamma|)^{1-\zeta}$. We distinguish the two subcases:

- if there exists $k \in \mathbb{Z}$ such that $|\lambda_k| \geqslant (\frac{1}{3} \cdot |\gamma|)^\zeta$, we have

  $$\begin{aligned} ||\sigma(x) - \sigma(y)|| &\geqslant ||\sigma(x)_{k-p} - \sigma(y)_{k-p}|| \geqslant C \cdot |\xi_{k-p}^{-1}\eta_{k-p}|^\alpha \\ &= C \cdot |\lambda_k|^\alpha \geqslant \frac{C}{3^{\alpha\zeta}}|\gamma|^{\alpha\zeta} \, ; \end{aligned}$$

- in case $M - m + 1 \geqslant (\frac{1}{3} \cdot |\gamma|)^{1-\zeta}$, having $L_\mathbb{Z}(\gamma) \geqslant M - m$ by definition, Proposition 2.5 gives

  $$d_{T_+}(\Lambda_+, \gamma\Lambda_+) + d_{T_-}(\Lambda_-, \gamma\Lambda_-) \geqslant L_\mathbb{Z}(\gamma) \geqslant \left(\frac{1}{3} \cdot |\gamma|\right)^{1-\zeta} - 1 \, .$$

  Thus, $\exists\, j \in \{+,-\}$ such that $d_{T_j}(\Lambda_j, \gamma\Lambda_j) \geqslant \frac{1}{2}(\frac{1}{3} \cdot |\gamma|)^{1-\zeta} - \frac{1}{2}$. For $|\gamma| \geqslant 4$, there exists $K > 0$ such that $d_{T_j}(\Lambda_j, \gamma\Lambda_j) \geqslant K \cdot |\gamma|^{1-\zeta}$, so that $||\sigma(x) - \sigma(y)|| \geqslant C_j K^\beta \cdot |x^{-1}y|^{\beta(1-\zeta)}$ as in cases (a)–(b).
  Otherwise, for $|\gamma| \leqslant 3$, Eq. 4.2 gives

  $$||\sigma(x) - \sigma(y)|| \geqslant (\min\{C, C_+, C_-\}) \cdot 3^{-\beta(1-\zeta)} \cdot |\gamma|^{\beta(1-\zeta)} \, .$$

  Hence, there exists $C'_\zeta > 0$ with $||\sigma(x) - \sigma(y)|| \geqslant C'_\zeta \cdot |x^{-1}y|^{\beta(1-\zeta)}$.

Consequently, setting $m_\zeta = \min\{\alpha\zeta, \beta(1-\zeta)\}$, it comes

$$||\sigma(x) - \sigma(y)|| \geqslant \min\left\{\frac{C}{3^{\alpha\zeta}}, C'_\zeta\right\} \cdot |\gamma|^{m_\zeta} \ . \qquad (**_\zeta)$$

The largest value for m_ζ is obtained for $\alpha\zeta = \beta(1-\zeta)$, that is $\zeta = \frac{\beta}{\alpha+\beta}$. It gives $m_\zeta = \frac{\alpha\beta}{\alpha+\beta}$. This is our second estimate for case (c).

As one has $\beta > \alpha\beta/(\alpha+\beta)$, combination of cases (a)–(c) gives

$$||\sigma(x) - \sigma(y)|| \geqslant C'' \cdot |x^{-1}y|^{\frac{\alpha\beta}{\alpha+\beta}}, \qquad \forall x, y \in \Gamma,$$

$$||\sigma(x) - \sigma(y)|| \geqslant C'' \cdot |x^{-1}y|^{\min\{\beta, \alpha - \frac{1}{2}\}}, \quad \forall x, y \in \Gamma$$

for some $C'' > 0$. Hence, we get $R_\sigma \geqslant \alpha\beta/(\alpha+\beta)$ and $R_\sigma \geqslant \min\{\beta, \alpha - \frac{1}{2}\}$ for all α, β satisfying $0 < \alpha < R_f$ and $0 < \beta < R_{f_\pm}$. This implies immediately $R_\sigma \geqslant R_f \cdot R_{f_\pm}/(R_f + R_{f_\pm})$ and $R_\sigma \geqslant \min\{R_{f_\pm}, R_f - \frac{1}{2}\}$.

To conclude the proof of Proposition 4.1, we pass now to the last statement. We thus suppose that f is H-equivariant and $f_\pm$ are Γ-equivariant (with respect to some actions by affine isometries). To establish the Γ-equivariance of σ, we only have to define a Γ-action (by affine isometries) on $\oplus_{i\in\mathbb{Z}} \mathcal{H}$ and check the Γ-equivariance with respect to it.

The Γ-action on $\bigoplus_{i\in\mathbb{Z}} \mathcal{H} = \ell^2(\mathbb{Z}, \mathcal{H}, 0)$ is defined by Eq. 3.2. To check the equivariance, we set $\gamma = \lambda s^n$ and $g = \mu s^p$ with $\lambda, \mu \in \Lambda$ and $n, p \in \mathbb{Z}$. We have $(\gamma \cdot \sigma(g))_i = \lambda_i \cdot f(\mu_{i-n})$ and $\sigma(\gamma g)_i = f(\lambda_i \mu_{i-n})$ and we get $(\gamma \cdot \sigma(g))_i = \sigma(\gamma g)_i$ for all i by H-equivariance of f. □

Theorem 1.2 will be obtained by applying Proposition 4.1 with good embeddings of the trees $T_\pm$. We explain now how to embed a tree in a Hilbert space with high values of the constant "R_f". First, the following result can be obtained by a straightforward adaptation of [9, Proposition 4.2].

Proposition 4.3 *Let $T = (V, E)$ be a tree. Then $R(V) = 1$.*

More precisely, if we denote by E_G the set of geometric (or unoriented) edges of T and if we fix a base vertex v_0, then for any $\varepsilon \in]0, 1/2[$ we may consider the map

$$f_\varepsilon\colon\ V \longrightarrow \ell^2(E_G)\ , \quad x \longmapsto \sum_{k=1}^{d(v_0,x)} k^\varepsilon \delta_{e_k(x)},$$

where the $e_k(x)$'s are the consecutive edges on the unique geodesic from x to v_0 and δ_e is the Dirac mass at e. It is a Lipschitz map with $R_{f_\varepsilon} \geqslant 1/2 + \varepsilon$. We refer to the proof of [9, Proposition 4.2] for this fact.

To prove the "equivariant" part of Theorem 1.2, we need some explicit equivariant embeddings into Hilbert spaces. Let $T = (V, E)$ be a tree. We recall from Sect. 7.4.1 in [4] how to embed equivariantly T in a Hilbert space. We recall that we denote by $e \mapsto \bar{e}$ the "orientation-reversing" involution on E, and we endow $\ell^2(E)$ with the scalar product:

$$\langle \xi | \eta \rangle = \frac{1}{2} \sum_{e\in E} \xi(e)\overline{\eta(e)}.$$

Define a map c: $V \times V \to \ell^2(E)$: $(x, y) \mapsto c(x, y)$ with

$$c(x,y) = \sum_{e \in (x \to y)} \delta_e - \delta_{\overline{e}},$$

where δ_e is the Dirac mass at e and the summation is taken over coherently oriented edges in the oriented geodesic from x to y. The map c satisfies, for every $x, y, z \in V$:

$$c(x,y) + c(y,z) = c(x,z), \tag{4.4}$$

$$\|c(x,y)\|^2 = d(x,y). \tag{4.5}$$

Moreover if a group G acts on T, then for every $g \in G$:

$$c(gx, gy) = \pi(g)c(x,y), \tag{4.6}$$

where π is the permutation representation of G on $\ell^2(E)$.

Fix now a base-vertex $v_0 \in V$. Define a map

$$\iota_{v_0}: V \to \ell^2(E): v \mapsto c(v_0, v)$$

and, for $g \in G$, an affine isometry $\alpha_{v_0}(g)$ of $\ell^2(E)$:

$$\alpha_{v_0}(g)\xi = \pi(g)\xi + c(v_0, gv_0).$$

Using Eqs. 4.4–4.6 above, the following lemma is immediate.

Lemma 4.7

1. *For all $g, h \in G$: $\alpha_{v_0}(gh) = \alpha_{v_0}(g)\alpha_{v_0}(h)$, so that α_{v_0} defines an affine isometric action of G on $\ell^2(E)$;*
2. *the map ι_{v_0} is G-equivariant with respect to the action α_{v_0} on $\ell^2(E)$;*
3. *one has $||\iota_{v_0}(x) - \iota_{v_0}(y)|| = \sqrt{d(x,y)}$ for all $x, y \in V$, so that $R_{\iota_{v_0}} = 1/2$.*

It is an immediate consequence that $R_G(V) \geqslant 1/2$.

Proof of Theorem 1.2 The inequalities $R(H) \geqslant R(\Gamma)$ and $R_H(H) \geqslant R_\Gamma(\Gamma)$ are trivial.

One has $R(V(T_\pm)) = 1$ by Proposition 4.3, so that Proposition 4.1 gives $R(\Gamma) \geqslant R(V(T_\pm)) \cdot R(H)/(R(V(T_\pm)) + R(H)) = R(H)/(R(H) + 1)$.

Finally, one has $R_\Gamma(V(T_\pm)) \geqslant 1/2$ by Lemma 4.7, so that we obtain

$$R_\Gamma(\Gamma) \geqslant \frac{R_\Gamma(V(T_\pm)) \cdot R_H(H)}{R_\Gamma(V(T_\pm)) + R_H(H)} \geqslant \frac{R_H(H)}{2R_H(H) + 1},$$

$$R_\Gamma(\Gamma) \geqslant \min\left\{R_\Gamma(V(T_\pm)), R_H(H) - \frac{1}{2}\right\} = R_H(H) - \frac{1}{2}$$

by Proposition 4.1. □

5 Hilbert space compression: examples

We begin this section with known results about the compression of groups of the form $H \wr \mathbb{Z}$. Let us first state a generalization of [1, Theorem 3.9] which gives upper bounds for many of them.

Proposition 5.1 *Let G be a finitely generated group with growth function satisfying $\kappa(n) \succcurlyeq n^k$ for some $k > 0$ and let H be a group. We assume the generating set of H chosen such that the word metric is unbounded. Then, the Hilbert space compression of $\Gamma = H \wr G$ satisfies*

$$R(\Gamma, \Sigma) \leqslant \frac{1 + k/2}{1 + k},$$

where Σ is the union of the generating sets of G and H. In particular, with $G = \mathbb{Z}$, we get $R(H \wr \mathbb{Z}) \leqslant 3/4$.

The proof is a straightforward adaptation of [1, Theorem 3.9].[4]

Remark 5.2 If H is finitely generated, the hypothesis "the word metric is unbounded" means exactly that H is infinite.

Lower bounds on compression were found by Tessera [15, Corollary 14]. In particular:

Proposition 5.3 *Let H be a finitely generated group. If H has polynomial growth, one has $R(H \wr \mathbb{Z}) \geqslant 2/3$.*

Together, Propositions 5.1 and 5.3 give immediately:

Corollary 5.4 *If H is an infinite group with polynomial growth, then one has $R(H \wr \mathbb{Z}) \in [2/3, 3/4]$.*

In a similar spirit, Proposition 5.1 and our Theorem 1.2 imply immediately:

Corollary 5.5 *Let H be an infinite, finitely generated group.*

(a) *If $R(H) = 1$, then $R(H \wr \mathbb{Z}) \in [1/2, 3/4]$.*
(b) *If $R(H) = R_H(H) = 1/2$, then $R(H \wr \mathbb{Z}) \in [1/3, 1/2]$ and $R_{H \wr \mathbb{Z}}(H \wr \mathbb{Z}) \in [1/4, 1/2]$ (in particular, if $R_{H \wr \mathbb{Z}}(H \wr \mathbb{Z}) < 1/3$, then H is non-amenable).* □

The interest of part (a) in Corollary 5.5 stems from the fact that numerous groups satisfy $R(H) = 1$: among amenable groups, we mention polycyclic groups and lamplighter groups $F \wr \mathbb{Z}$ with F finite [15, Theorem 1]; among (usually) non-amenable groups, we cite hyperbolic groups [3, Theorem 4.2], groups acting properly co-compactly on finite-dimensional CAT(0) cube complexes [6], co-compact lattices in connected Lie groups, irreducible lattices in higher rank semi-simple Lie groups [15, Theorem 2].

Our excuse for isolating (b) in Corollary 5.5 is a remarkable result by Arzhantseva et al. [1, Theorem 1.8]: for Thompson's group F, one has $R(F) = R_F(F) = 1/2$.

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[4] The only point to notice is that, since the word length is unbounded on H, for every b in the ball of radius n in G, we can find an element $h_b \in H$ such that $|h_b| = 2n + 1 - |b|$. In Ref. [1], the hypothesis $H = \mathbb{Z}$ is used only to have an explicit formula for h_b.

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