

Unconventional treatment of focal shift

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Abstract

We present an unconventional approach for the explanation of focal shift behind a lens. It is based on the fact that, within the approximation of Fresnel diffraction, the intensity distributions in the conjugate planes of a lens are equal to their geometrical images. We show that the focus (position of highest intensity) is always shifted towards the lens. The results for a Gaussian beam and a uniform converging spherical wave are presented.

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1. Introduction

Focal shift is a known phenomena. It occurs when a converging wave is diffracted at an aperture. Then, the point of highest intensity is not at the geometrical focus but somewhat closer to the aperture. In the case of conventional optical systems, which have high Fresnel numbers, the focal shift is negligible. However, the focal shift becomes significant if the Fresnel number is small. With decreasing beam diameter the maximum intensity on the optical axis moves towards the aperture, causing an asymmetric intensity distribution about the geometrical focal plane of the lens [1]. The publication of Erkkila and Rogers [2] in 1981, describing the asymmetric intensity distribution about the geometrical focal plane, has initiated investigations of this phenomena. Li and Wolf [3], Parker Givens [4], and Stammes and Spjelkavik [5] showed the focal shift for a uniform converging wave. Carter [6] discussed the focal shift of a converging Gaussian beam and its implication to optical design. The focal shift of a truncated Gaussian beam was investigated by Li and Wolf

[7]. In a succeeding publication [8] they presented the modification of the three-dimensional intensity distribution near the focus when the Fresnel number is gradually decreased. Li and Platzler demonstrated experimentally the existence of the focal shift and showed the good agreement between theory and measurements [1]. In a more recent paper Wang et al. [9] has investigated the focused field in systems with large Fresnel numbers.

So far, publications have treated the focal shift by analysing the converging wave. In the present paper, we present a different approach for the explanation of focal shift. First, we show that, within the approximation of Fresnel diffraction, the intensity distribution of a converging wave behind a lens is equal to the geometrical image of the intensity distribution in the conjugate plane of the incoming wave. Based on this fact, we will develop a procedure to investigate the focal shift.

In Section 2, we shall derive the theory of diffraction for this paper based on the Fresnel approximation. We can use this approximation, since the focal shift becomes significant when the Fresnel approximation becomes valid. Section 3 will use the equivalence shown in Section 2 to develop a procedure to find the focal shift of a converging wave with arbitrary intensity distribution. In Section 4 we

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apply this procedure to evaluate the focal shift of a Gaussian beam and a uniform converging spherical wave.

2. Theory

The following derivation is based on Figs. 1 and 2. Fig. 1 shows a plane wavefront $U_0(\mathbf{r}_0)$ at distance $z_0 = 0$ focused by a thin lens with focal length f placed at $z_1 = +0$. The vector \mathbf{r}_0 has the components x and y and is therefore perpendicular to the axis ζ_1 . The ρ_0 stands for the radius of the wave in the transverse plane at $z_0 = 0$ (e.g., the radius of an aperture or the waist of a Gaussian beam). The observation plane is placed at the distance $z_1 > 0$, where the wave function $U_1(\mathbf{r}_1, z_1)$ is observed. We will use the expression ‘converging’ in order to refer to the propagating wave behind the lens in Fig. 1.

Fig. 2 shows how the plane wavefront $U_0(\mathbf{r}_0)$ of Fig. 1 would propagate freely into the half space $z_2 > 0$. We assume that the angular spectrum of the wave falls within a small solid angle. Such a wave is known as a (optical) beam [10] and we will use this expression in the following. An observation plane is placed at the distance z_2 ($z_2 > 0$), where the wave function $U_2(\mathbf{r}_2, z_2)$ is observed. We would like to point out that for the following we will keep strictly to the introduced symbols and indices. The index ‘o’ refers always to the location $\zeta = 0$ in both of Figs. 1 and 2. The indices 1 and 2 refer to the waves U_1 and U_2 , respectively. The vector \mathbf{r} has the two components x and y and therefore is perpendicular to the axis ζ .

We shall show, that the geometrical imaging relation between the conjugate planes of the lens holds for the intensity distribution also within the approximation of Fresnel diffraction. This result has been previously published by Kogelnik [11] and Yariv [12], but we shall follow here a different approach. For convenience, we develop first the expression for the beam $U(\mathbf{r}_2, z_2)$ in Fig. 2 (bold

lines). A beam propagating along the axis ζ_2 can be described by the Fresnel approximation [13]

$$U_2(\mathbf{r}_2, z_2) = \frac{\exp(ikz_2)}{i\lambda z_2} \iint_{-\infty}^{+\infty} U_0(x_0, y_0) \times \exp\left(\frac{ik}{2z_2}((x_0 - x_2)^2 + (y_0 - y_2)^2)\right) dx_0 dy_0. \quad (1)$$

If we introduce the Fourier transform

$$G(x_2, y_2) = \text{FT}\{g(x_0, y_0)\} = \iint_{-\infty}^{+\infty} g(x_0, y_0) \times \exp\left(-\frac{ik}{2z_2}(x_0 x_2 + y_0 y_2)\right) dx_0 dy_0 \quad (2)$$

and use the vector notation

$$\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \quad (3)$$

the Eq. (1) can be written as

$$U_2(\mathbf{r}_2, z_2) = \frac{1}{i\lambda z_2} \exp\left(ikz_2 + \frac{ik}{2z_2} \mathbf{r}_2^2\right) \times \text{FT}\left\{U_0(\mathbf{r}_0) \exp\left(\frac{ik}{2z_2} \mathbf{r}_0^2\right)\right\}. \quad (4)$$

In Fig. 1 a thin aberration free lens with focal length f is introduced. Therefore, the wave function $U_0(\mathbf{r}_0)$ has to

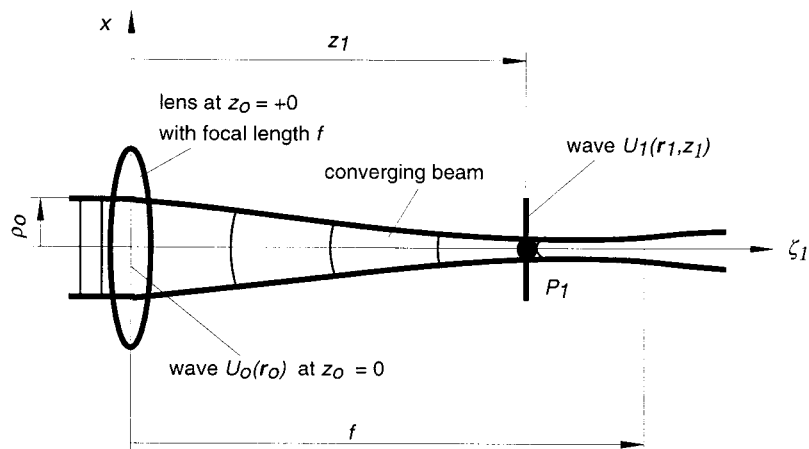


Fig. 1. Propagation of a converging wave with $U_0(\mathbf{r}_0)$ at $z_0 = 0$.

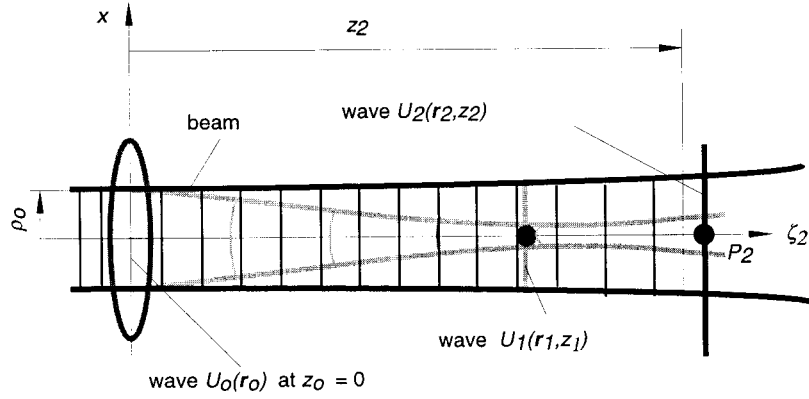


Fig. 2. Propagation of a wave with $U_0(\mathbf{r}_0)$ at $z_0 = 0$ without the lens (solid line) and with the lens (dashed line).

be multiplied by the phase function ϕ of the lens, which is

$$\phi(\mathbf{r}_0) = -\frac{ik}{2f}r_0^2. \quad (5)$$

The converging beam $U_1(\mathbf{r}_1, z_1)$ in Fig. 1 is then found to be given by

$$U_1(\mathbf{r}_1, z_1) = \frac{1}{i\lambda z_1} \exp\left(ikz_1 + \frac{ik}{2z_1}r_1^2\right) \times \text{FT}\left\{U_0(\mathbf{r}_0) \exp\left(\frac{ik}{2}r_0^2\left(\frac{1}{z_1} - \frac{1}{f}\right)\right)\right\}. \quad (6)$$

If the condition

$$\frac{1}{z_2} = \frac{1}{z_1} - \frac{1}{f} \quad (7)$$

is satisfied, the Fourier transforms (FT) of Eqs. (4) and (6) are equal. Eq. (7) is recognised as the condition for geometric imaging, i.e. the Gaussian lens formula and z_1 and z_2 are known as the positions of the conjugate planes of the lens. Furthermore, if we compare the Fourier kernels in Eqs. (4) and (6), using the definition of the Fourier transform given in Eq. (1), we get for the conjugate planes

$$\frac{ik}{z_2}(\mathbf{r}_2 \cdot \mathbf{r}_0) = \frac{ik}{z_1}(\mathbf{r}_1 \cdot \mathbf{r}_0), \quad (8)$$

which leads to the magnification

$$|\beta| = \frac{|\mathbf{r}_1|}{|\mathbf{r}_2|} = \frac{|z_1|}{|z_2|}, \quad (9)$$

well known from geometrical optics.

The intensity distribution is given by the $I = |U|^2$. Consequently, the intensity distributions in the planes are related by

$$I_2(\mathbf{r}_2, z_2) = \beta^2 I_1(\mathbf{r}_1, z_1). \quad (10)$$

3. Evaluating the focal shift of a converging wave

Let us use the findings of Section 2 in order to evaluate the focal shift or the location of the maximum intensity on the axis ζ_1 . A plane wavefront at $z_0 = 0$ with an arbitrary intensity distribution, focused by an aberration free lens at $z_0 = +0$ of a focal length $f > 0$ (see definition in Eq. (5)), propagates along the axis ζ_1 into the half space $z_1 > 0$. As shown above, $I_1(z_1)$ on the axis ζ_1 is imaged on the axis ζ_2 , yielding the intensity $I_2(z_2)$. The relation between these two intensities is given by Eq. (10). Expressing the magnification β using Eqs. (7) and (9) we find

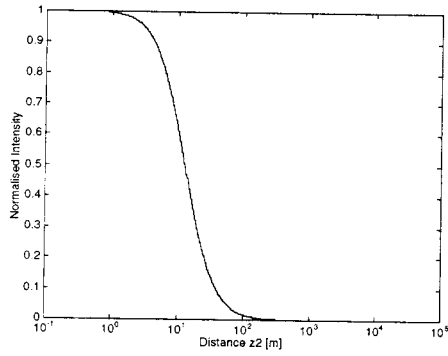
$$I_2(z_2) = \left(\frac{f - z_1}{f_1}\right)^2 I_1(z_1). \quad (11)$$

The point of maximum intensity $I_1(z_1)$ on ζ_1 is then given by the condition

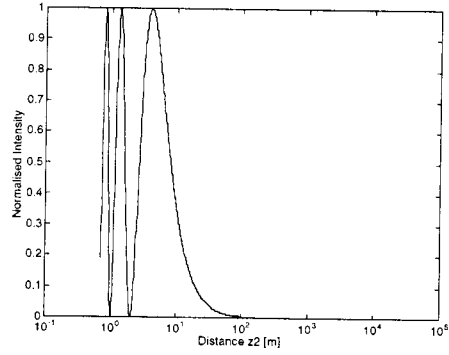
$$\frac{dI_1}{dz_1} = \frac{(z_2 + f)^3}{f^4} \left(2I_2 + (f + z_2) \frac{dI_2}{dz_2}\right) = 0. \quad (12)$$

To get further information about the behaviour of $I_1(z_1)$ in the vicinity of the focus of the lens, we shall investigate the farfield for I_2 , which corresponds, according to Eq. (7), to the focal region on ζ_1 . At far distance ($z_2 \gg \rho_0^2/\lambda$) the Fresnel integral of Eq. (1) becomes

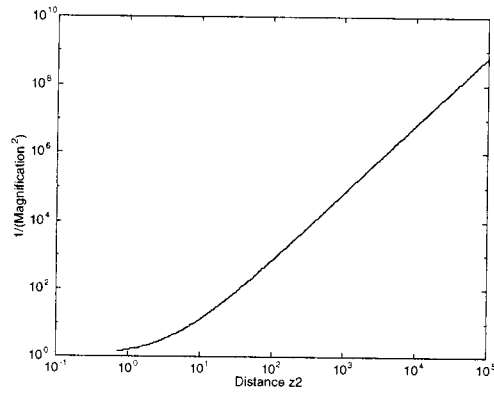
$$U_2(\mathbf{r}_2, z_2) = \frac{\exp\left(ik\left(z_2 + \frac{r_2^2}{2z_2}\right)\right)}{i\lambda z_2} \text{FT}\{U_0(\mathbf{r}_0)\}, \quad (13)$$



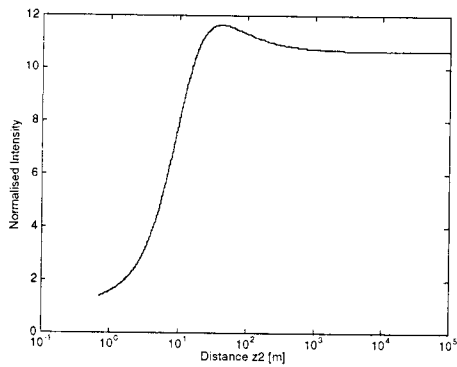
(a)



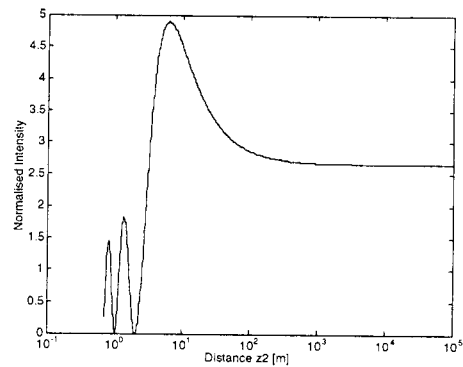
(b)



(c)

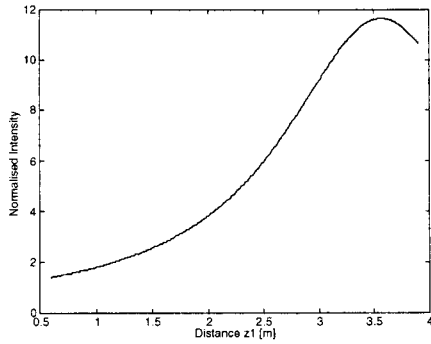


(d)

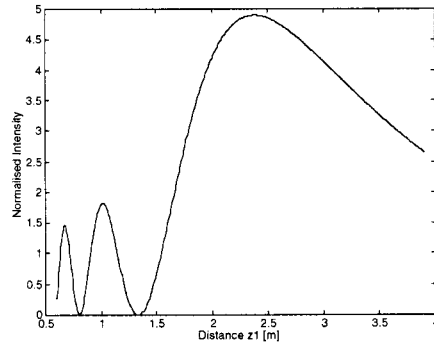


(e)

Fig. 4. (a) intensity along the axis ζ_2 of a Gaussian beam, (b) intensity along the axis ζ_2 of a uniform beam; (c) inverse of the squared magnification, i.e., $1/\beta^2$; (d) Gaussian beam multiplied with $1/\beta^2$; (e) uniform beam multiplied with $1/\beta^2$; (f) intensity along the axis ζ_1 of the converging Gaussian wave; (g) intensity along the axis ζ_1 of the converging uniform wave.



(f)



(g)

Fig. 4 (continued).

lens as expected. Substituting Eq. (21) into Eq. (7) and using the relation of Eq. (17) yields for the focal shift

$$\delta = -\frac{f}{1 + \frac{a}{bf^2}} = -\frac{f}{1 + \left(\frac{\pi w_0^2}{\lambda f}\right)^2}, \quad (23)$$

which is in accordance with equation (65) of Ref. [14] for $d_1 = 0$.

4.2. Focal shift of a converging uniform spherical wave

As next we will evaluate the focal shift of a converging spherical wave, having a uniform intensity distribution at the diffracting circular aperture. But first, let us derive the intensity distribution along the ζ_2 axis of the plane wave diffracted at the aperture (uniform beam) using the Rayleigh–Sommerfeld integral.

In general, the scalar field at the point $P(\mathbf{r}_2, z_2)$ is given by [15]

$$U_2(\mathbf{r}_2, z_2) = \frac{1}{i\lambda} \int_{\Sigma} A_o(\mathbf{r}_o) \frac{\exp(ikR)}{R} \cos(\theta) dS, \quad (24)$$

where Σ stands for the surface limited by the aperture in $z_0 = 0$ (Fig. 3). For a plane wave diffracted at the circular aperture and propagating along the axis ζ_2 we have $A_o(\mathbf{r}_o) = A_o$ and for $R = \sqrt{(z_2^2 + r_o^2)}$ with $r_o = |\mathbf{r}_o|$. Furthermore, for the paraxial approximation we can assume that $\cos(\theta) \approx 1$. Substituting these relations into Eq. (24) and introducing polar co-ordinates yields

$$U_2(\mathbf{r}_2 = 0, z_2) = \frac{A_o}{i\lambda} \int_{\phi=0}^{2\pi} \int_{r_o=0}^{\rho_o} \frac{\exp\left(ik\sqrt{z_2^2 + r_o^2}\right)}{\sqrt{z_2^2 + r_o^2}} r_o dr_o d\phi, \quad (25)$$

where ρ_o is the radius of the limiting aperture at $z_0 = 0$. Integrating Eq. (25) and approximating the square root in the phase by the first two terms of the Taylor series yields

$$I_2(\mathbf{r}_2 = 0, z_2) = \left(2A_o \sin\left(\frac{k\rho_o^2}{4z_2}\right)\right)^2, \quad (26)$$

which is the function describing the intensity distribution along the axis ζ_2 , in accordance with equation (3.2) of Ref. [3] for $f \rightarrow \infty$. Introducing Eq. (26) into Eq. (12) and using Eqs. (7) and (17) we get after some algebraic transformations

$$\tan\left(\frac{k\rho_o^2\delta}{4f(f+\delta)}\right) = \frac{k\rho_o^2\delta}{4(f+\delta)^2}. \quad (27)$$

An approximative value for the focal shift δ is found by expanding the tangent into a Taylor series about $\delta = 0$ and by taking the first two terms. Finally we get for the focal shift

$$\delta = -\frac{f}{1 + \frac{1}{12} \left(\frac{\pi\rho_o^2}{\lambda f}\right)^2}, \quad (28)$$

which is in accordance with equation (3.15) of Ref. [3].

4.3. Finding focal shifts numerically

A numerical method of finding the focal shift, or the point of maximum intensity, is shown in Fig. 4. We take the examples of the Gaussian and the uniform beam. We chose a wavelength of $\lambda = 633$ nm, a focal length of lens of $f = 3.9$ m and beam diameters of $2w_0 = 2\rho_o = 3.2$ mm. The focal shift for these parameters has been experimentally determined for the converging uniform spherical wave by Li and Platzer [1].

The steps followed in order to get the maximum intensity of the converging wave are: first, calculate and plot the intensity function $I_2(z_2)$ along the axis ζ_2 of the beams (Fig. 4a and 4b). Second, evaluate the magnification factor $\beta(z_2)$ using Eqs. (9) and (7) (Fig. 4c). Third, multiply the abscissa of the graph (intensity) by the inverse of the squared magnification factor (Fig. 4d and 4e). Finally, multiply the ordinate of the graph (distance) by the magnification factor in order to get the correct scaling for the distance z_1 (Fig. 4f and 4g). The maximum intensity is then found in the plot for the converging Gaussian beam at $z_1 = 3.6$ m (Fig. 4f) and in the plot for the converging uniform wave at $z_1 = 2.4$ m (Fig. 4g) yielding a focal shift of $\delta = -0.3$ m and $\delta = -1.5$ m (Eq. (17)), respectively. Calculating the focal shifts by the Eqs. (23) and (27), we get $\delta = -0.340$ m and $\delta = -1.52$ m, respectively. Hence, the values found in the plots agree well with the calculated values of Eqs. (23) and (27). However, with Eq. (28), which is an approximation of Eq. (27), we find a focal shift of $\delta = -2.07$ m. This is about a 30% difference of the value obtained by Eq. (27). This means, that for the chosen parameters the approximation made in Eq. (28) is not good enough. From the expansion of Eq. (27) we get as condition for an error smaller than 1%

$$\frac{\rho_0^2 \delta}{\lambda f^2} > -0.1. \quad (29)$$

For the given parameters above, we get -0.55 for the left side hand side of Eq. (29), and therefore the inequality is not satisfied.

5. Conclusions

We have shown that the effect of focal shift can be explained by the imaging laws of geometrical optics and the diffraction of the incident light wave. We have proven, in a different way than Kogelnik [11] and Yariv [12], that within the approximation of Fresnel diffraction the intensity distribution of a converging wave behind a lens is equal to the geometrical image of the intensity distribution in the conjugate plane of the incoming wave. The position of highest intensity behind the lens is then determined by the combination of the increasing cross-section of the

incident wave due to diffraction and the variable magnification of the lens along the axes, governed by geometrical optics.

Using this approach, we have shown that the focus (position of highest intensity) is always shifted towards the lens, independent of the intensity distribution of the incident light wave. As examples, we have calculated the focal shift for a Gaussian beam and a uniform converging spherical wave. The results agree with those published in the open literature. We believe, that the different view of focal shift presented in this paper, gives a good physical understanding of the origin of the focal shift. In addition, the separation of the effect in two parts, diffraction of the incident wave and optical imaging, simplifies the numerical calculations comparing the focal shift for different geometries.

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