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SHORT NOTE

The impact of covariance misspecification in risk-based portfolios

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Abstract The equal-risk-contribution, inverse-volatility weighted, maximum-diversification and minimum-variance portfolio weights are all direct functions of the estimated covariance matrix. We perform a Monte Carlo study to assess the impact of covariance matrix misspecification to these risk-based portfolios at the daily, weekly and monthly forecasting horizon. Our results show that the equal-risk-contribution and inverse-volatility weighted portfolio weights are relatively robust to covariance misspecification. In contrast, the minimum-variance portfolio weights are highly sensitive to errors in both the estimated variances and correlations, while errors in the estimated correlations can have a large effect on the weights of the maximum-diversification portfolio.

Keywords Covariance misspecification · Monte Carlo study · Risk-based portfolios

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1 Introduction

We study the sensitivity of daily, weekly and monthly rebalanced risk-based portfolio optimization methods to covariance matrix misspecification. We conduct this analysis for four distinct well-known risk-based portfolios, namely the minimum-variance portfolio, the inverse-volatility weighted portfolio (Leote De Carvalho et al. 2012), the equal-risk-contribution portfolio (Maillard et al. 2010), and the maximum-diversification portfolio (Choueifaty and Coignard 2008).[1] We consider risk-based portfolios that are constructed with six distinct investment universes, which include factor-based equity portfolios, industry-based equity portfolios, multi-asset portfolios and portfolios made of single equities.

Using Engle (2002)'s dynamic conditional correlation (DCC) model as the true data generating process, our Monte Carlo study reveals substantial differences in the sensitivity to covariance misspecifications between the various allocation methodologies. First, the impact of covariance matrix misspecification is substantial for the minimum-variance and the maximum-diversification portfolios. The least sensitive portfolios are the equal-risk-contribution and the inverse-volatility weighted portfolios. Second, by decomposing the impact of covariance misspecifications into variances and correlations components, we show that, with the exception of maximum-diversification, risk-based strategies are especially sensitive to misspecifications in the variance component. Third, we show that EWMA covariance matrix estimators are both statistically and economically closer to the true optimal allocations than both the Ledoit and Wolf (2003) and the sample-based estimators. Practically, our results imply that investors that are willing to invest in risk-based strategies should devote a particular attention to the methodology that is used by the investment manager to estimate the covariance matrix of the assets and, more particularly, its diagonal elements.

Risk-based allocation strategies have become extremely popular among investors during the last decade. For instance, a recent research published by JP Morgan (Kolanovic et al. 2015) indicates that the total amount managed with a risk parity approach is close to $500 BN as of August 2015. This number does not include the assets managed by investment vehicles that are pursuing minimum-variance and maximum-diversification strategies. Furthermore, most Commodity Trading Advisors (CTA funds) are also using a risk-based weighting scheme to perform their asset allocation. Despite the increasing popularity of risk-based investment strategies, there has been a shortage of scientific evidence evaluating the impact of second moment forecasting errors on the outcome of risk-based portfolio optimizations.

Our work aims at filling this gap by extending the work of Chan et al. (1999), Ledoit and Wolf (2003), Scutellà and Recchia (2013) and Kim et al. (2013) to risk-based asset allocation methodologies. Indeed, the aforementioned studies focus exclusively on the impact of the accuracy of the covariance matrix forecasts on the performance of mean-variance or benchmark-tracking portfolios. More recently, Zakamulin (2015) investigates the impact of various covariance matrix forecasting methodologies on the performance of both mean-variance and target volatility strategies but he does not pay attention to the other three very popular risk-based strategies that we investigate in this paper. Note that the objective of this study is to assess the impact of covariance matrix misspecification on the optimal weights that result from the different risk-based optimization methods. Therefore, we leave the question of the impact of covariance matrix misspecification on both portfolio performance and turnover open for further research.

[1] Throughout the paper, we use (co)variance based risk measures. E.g., see Stoyanov et al. (2013) for a sensitivity study on risk estimates based on higher order co-moments, and Ardia and Boudt (2015) for a recent review on risk-based portfolios.

The rest of this document is structured as follows. Section 2 provides a description of the various risk-based allocation methodologies. Section 3 proposes some numerical illustrations aimed at assessing the impact of second moment misspecification on the optimal allocations. The Monte Carlo study is presented in Sect. 4. Section 5 concludes.

2 Risk-based portfolios

We consider a market with N risky securities and denote a generic portfolio in this market by the $(N \times 1)$ vector $\mathbf{w} \equiv (w_1, \ldots, w_N)'$. The $(N \times N)$ covariance matrix of the $(N \times 1)$ arithmetic returns $\mathbf{r} \equiv (r_1, \ldots, r_N)'$ at the desired holding horizon is denoted by $\boldsymbol{\Sigma}$. We consider long-only portfolios in our analysis. Moreover, we define $\boldsymbol{\iota}_N$ as a $(N \times 1)$ vector of ones and $\mathbf{0}_N$ as a $(N \times 1)$ vector of zeros.

We consider four risk-based portfolios in our study.

Minimum-variance portfolio The minimum-variance portfolio is obtained as:

$$\mathbf{w}_{\min} \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \mathbf{w}' \boldsymbol{\Sigma} \mathbf{w} \right\}, \tag{1}$$

where $\mathcal{C} \equiv \left\{ \mathbf{w} \in \mathbb{R}_+^N | \mathbf{w}' \boldsymbol{\iota}_N = 1 \right\}$ is the long-only full investment constraint.

Inverse-volatility weighted portfolio The inverse-volatility weighted portfolio, called equal-risk-budget portfolio in Leote De Carvalho et al. (2012), allocates to the N stocks with volatility $\sigma_1, \ldots, \sigma_N$ the weight:

$$\mathbf{w}_{\mathrm{iv}} \equiv \left(\frac{1/\sigma_1}{\sum_{j=1}^N 1/\sigma_j}, \ldots, \frac{1/\sigma_N}{\sum_{j=1}^N 1/\sigma_j} \right)'. \tag{2}$$

Equal-risk-contribution portfolio The equal-risk-contribution portfolio is the portfolio for which all assets contribute equally to the overall portfolio volatility. Or equivalently, it is the portfolio for which the percentage volatility risk contribution of all N assets equals $1/N$, where percentage volatility risk contribution of the ith asset is given by $\%RC_i \equiv \frac{w_i [\boldsymbol{\Sigma} \mathbf{w}]_i}{\mathbf{w}' \boldsymbol{\Sigma} \mathbf{w}}$. It is computed by solving the following optimization problem:[2]

$$\mathbf{w}_{\mathrm{erc}} \equiv \underset{\mathbf{w} \in \mathcal{C}}{\operatorname{argmin}} \left\{ \sum_{i=1}^N (\%RC_i - \tfrac{1}{N})^2 \right\}. \tag{3}$$

Maximum-diversification portfolio Let $\boldsymbol{\sigma} \equiv \sqrt{\operatorname{diag}(\boldsymbol{\Sigma})}$ be the $(N \times 1)$ vector of standard deviations of arithmetic returns of the N assets in the universe. An important property of the portfolio standard deviation is its sub-additivity, $\sqrt{\mathbf{w}' \boldsymbol{\Sigma} \mathbf{w}} \leq \mathbf{w}' \boldsymbol{\sigma}$, or equivalently that the ratio between the weighted average volatility and the portfolio volatility exceeds one:

[2] If a numerical solution is not found, we follow the recommendation in Maillard et al. (2010) and slightly modify the problem and optimize over the N-dimensional vector $\mathbf{u}$ such that $\mathbf{w} \equiv \mathbf{u}/(\mathbf{u}' \boldsymbol{\iota}_N)$, under the constraint that $\mathbf{u} \geq \mathbf{0}_N$ and $\mathbf{u}' \boldsymbol{\iota}_N > 0$. This new optimization problem is easier to solve numerically as an inequality constraint is less restrictive than the full investment equality constraint.

$$DR(\mathbf{w}) \equiv \frac{\mathbf{w}'\boldsymbol{\sigma}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}} \geq 1\,. \tag{4}$$

Choueifaty and Coignard (2008) call (4) the *portfolio's diversification ratio* and define the maximum-diversification portfolio as the portfolio that has the highest diversification ratio:

$$\mathbf{w}_{\text{md}} \equiv \underset{\mathbf{w}\in\mathcal{C}}{\text{argmax}}\{DR(\mathbf{w})\}\,. \tag{5}$$

3 Numerical illustrations

Let us first consider a simplified setup with $N = 3$ risky assets in order to gain intuition on how covariance misspecification affects the weights of the risk-based portfolios. We will later see that the main conclusions from this simplified setup are confirmed in an extensive simulation study with dynamic covariances, calibrated on real data sets with dimension $N = 7$, 10 and 30. We assume the following covariance structure:

$$\boldsymbol{\Sigma} \equiv \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} \\ & 1 & \rho_{2,3} \\ & & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}, \tag{6}$$

with $(\sigma_1, \sigma_2, \sigma_3, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}) = (0.1, 0.1, 0.2, -0.1, -0.2, 0.7)$. Assets #1 and #2 have a volatility of 10% and asset #3 has a volatility of 20%. We purposely set a higher volatility for asset #3 to illustrate the case of a balanced portfolio that can contain bonds and equities. The correlations are chosen to illustrate the case of a balanced portfolio that may contain sovereign bonds, corporate bonds and equities. Sovereign bonds have been negatively correlated with both corporate bonds and equities in the past. On the other hand, the correlation between corporate bonds and equities has been positive. Table 1 reports the resulting allocations and risk contributions of each risk-based portfolio.

We can see that the minimum-variance allocation is highly concentrated both in terms of weights and risk. The weights of the maximum-diversification allocation are concentrated as well but the risk of the portfolio is better diversified than the one of the minimum-variance allocation. In order to assess the impact of misspecified volatility estimates on the optimal

Table 1 Optimal weights and risk contribution of risk-based portfolios

	Min-var		Inv-vol		ERC		Max-div	
	$\mathbf{w}_{\text{min}}$	$\%RC$	$\mathbf{w}_{\text{iv}}$	$\%RC$	$\mathbf{w}_{\text{erc}}$	$\%RC$	$\mathbf{w}_{\text{md}}$	$\%RC$
Asset #1	50.0	50.0	40.0	18.4	48.3	33.3	56.6	46.9
Asset #2	49.9	49.9	40.0	42.1	33.6	33.3	22.6	18.8
Asset #3	0.1	0.1	20.0	39.5	18.1	33.3	20.8	34.4
H^*	25.0	25.0	4.0	5.1	6.8	0.0	12.2	6.0
σ	6.7		7.8		7.4		7.5	

We consider a universe of $N = 3$ assets with covariance matrix $\boldsymbol{\Sigma}$ given in (6) with parameters $(\sigma_1, \sigma_2, \sigma_3, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}) = (0.1, 0.1, 0.2, -0.1, -0.2, 0.7)$. For the four risk-based portfolios, we report the optimal allocation $\mathbf{w}$, the percentage risk contribution $\%RC(\mathbf{w})$, the normalized Herfindahl index $H^*(\mathbf{w}) \equiv \frac{H(\mathbf{w})-1/N}{1-1/N}$ where $H(\mathbf{w}) \equiv \sum_{i=1}^{N} w_i^2$ as well as the portfolio's volatility $\sigma(\mathbf{w}) \equiv \sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}}$. The normalized Herfindahl has value in [0, 1] with a value of zero indicating perfect diversification and a value of one indicating perfect concentration. All quantities are reported in percentages

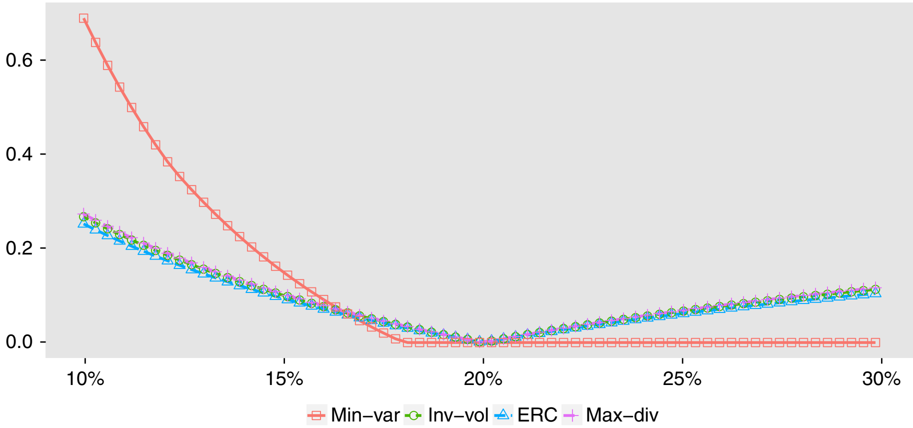

Fig. 1 Illustration of the impact of volatility misspecification. The graph reports the L^1 distance of weights (7) for the four risk-based portfolios as a function of the misspecified volatility σ_3 of asset #3. The universe consists of $N = 3$ assets with (true) covariance matrix $\mathbf{\Sigma}$ given in (6) with parameters $(\sigma_1, \sigma_2, \sigma_3, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}) = (0.1, 0.1, 0.2, -0.1, -0.2, 0.7)$

allocations, we let the volatility of asset #3 vary between 10 and 30% and compute the L^1 distance between the weights of optimal portfolios computed with the true parameters, $\mathbf{w}$, and the ones computed with the misspecified ones, $\hat{\mathbf{w}}$. The L^1 distance writes:

$$\|\mathbf{w} - \hat{\mathbf{w}}\|_1 \equiv \sum_{i=1}^{N} |w_i - \hat{w}_i| \,. \tag{7}$$

If we consider Fig. 1, which reports the distance of each portfolio as a function of the volatility estimates of asset #3, we can see significant differences between the distances of the mean-variance portfolio and those of the other 3 risk-based portfolios. The minimum-variance portfolio appears to be very sensitive to the misspecification of asset #3 volatility. Its distance from the optimal weights increases almost exponentially with the level of volatility under-estimation. The minimum-variance portfolio is less sensitive to over-estimations of asset #3 volatility as its weight stays at zero as soon as its estimated volatility is above 18%. The remaining three risk-based portfolios appear to be less sensitive to volatility misspecifications than the minimum-variance portfolio. Finally, note that even if the optimization methodology differs across the three remaining risk-based portfolios, we cannot denote significant differences between the distances of the three allocations.

Our next numerical illustration investigates the sensitivity of the risk-based portfolios to correlation misspecification. In order to do so, we shift the value of the correlation between asset #2 and asset #3, $\rho_{2,3}$, and leave all other parameters constant. As before, we use the distance measure (7) to quantify the impact of the correlation misspecification on the optimal weights. The inverse-volatility weighted portfolio is excluded from the analysis, since its weights are not influenced by the value of the estimated correlations.

Figure 2 shows the L^1 distance between the calculated weights and the true weights, due to the misspecification of $\rho_{2,3}$. We can see that both the minimum-variance and the maximum-diversification portfolios are highly sensitive to misspecification of the correlation coefficients. This is not surprising as those strategies are the ones with the most concentrated allocations. As before, the fact that asset #3 is twice as volatile as the other two assets and

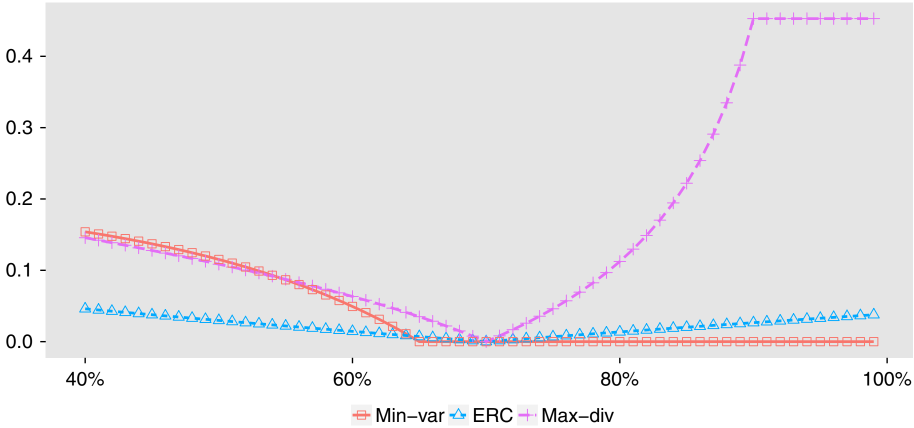

Fig. 2 Illustration of the impact of correlation misspecification. The graph reports the L^1 distance of weights (7) for the minimum-variance, the equal-risk-contribution and the maximum-diversification portfolios as a function of $\rho_{2,3}$, the correlation between asset #2 and asset #3. The (misspecified) correlation $\rho_{2,3}$ ranges from 0.4 to 0.99; for these values, the correlation matrix is positive definite. The universe consists of $N = 3$ assets with (true) covariance matrix $\mathbf{\Sigma}$ given in (6) with parameters $(\sigma_1, \sigma_2, \sigma_3, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}) = (0.1, 0.1, 0.2, -0.1, -0.2, 0.7)$

therefore very often excluded from the minimum-variance optimal portfolio, reduces the sensitivity of this allocation to over-estimation of $\rho_{2,3}$. On the other hand, the maximum-diversification portfolio is sensitive to both under- and over-estimations of the parameter $\rho_{2,3}$.

Overall, our numerical analysis reveals that the minimum-variance portfolio is sensitive to both volatility and correlation misspecifications. The maximum-diversification portfolio appears to be highly sensitive to correlation misspecification but does not show higher sensitivity to volatility misspecification than the inverse-weighted and equal-risk-contribution portfolios, which are usually less concentrated. In order to enlarge the scope of our numerical illustration, we now turn to a larger scale investigation with a Monte Carlo study.

4 Monte Carlo study

This section performs a Monte Carlo study to assess the impact of covariance misspecification to risk-based portfolios. We first present the various covariance estimators used in our study, then describe the data and the Monte Carlo setup, discuss the results for various forecasting horizons, and perform a robustness check with respect to the presence of fat tails in the distribution of the innovations in the underlying data generating process.

4.1 Covariance matrix estimators

Suppose we have a time series of T past returns, $\mathbf{r}_1, \ldots, \mathbf{r}_T$, to estimate the covariance matrix $\mathbf{\Sigma}$ of $\mathbf{r}_{T+1}$. We consider the following estimators:

Sample-based (SMPL) The most well-known and simple estimator of covariance, which sets $\mathbf{\Sigma}$ to the sample covariance of the T historical returns.

Ledoit–Wolf (LW) Weighted average of the sample covariance matrix and a *prior*. The prior is given by a one-factor model and the factor is equal to the cross-sectional average of all returns. The shrinkage intensity is the plug-in estimate of the mean square optimal one. See Ledoit and Wolf (2003). Like the sample-based covariance estimator, the LW estimate of covariance ignores the potential time-variation in the conditional covariance matrix.

Exponentially weighed moving average (EWMA) Simple dynamic model in which recent returns have more weights than past returns in the estimation. The decay parameter is set to 0.94 as advocated by RiskMetrics Group (1996) for daily returns.

Dynamic conditional correlation (DCC) Sophisticated model by Engle (2002) in which the variances and correlations are of GARCH(1,1) type, allowing persistence in the variance and the correlation dynamics. The model is estimated via the composite likelihood technique as suggested by Engle and Sheppard (2001).[3]

4.2 Setup

Previous research shows that the impact of comovements on optimal weights depends on the general level of correlation between the assets of the investment universe (e.g., see Schumann 2013) and the specific composition of the investment universe (e.g., see Bertrand and Lapointe 2015). Therefore, we conduct our study on six investment universes (datasets) of various variance/correlation structure and dimensions; see Table 2. For five out of six datasets, we use value-weighted portfolios obtained from the data library of Kenneth French.[4] This includes the well-known size, book-to-market and past performance (momentum) sorted portfolios, as well as industry portfolios based on the firms' four-digit SIC code. The Fama-French research portfolios contain US equities that are listed on NYSE, AMEX or NASDAQ. Our fifth universe covers seven major asset classes whose prices are obtained from the Thomson Reuters Datastream database. The sixth dataset is formed by the 30 constituents of the Dow Jones Industrial Average index as of June 2014. For each dataset, we use the daily adjusted closing prices from December 2008 to November 2014.

For each universe, we set up our Monte Carlo experiment by first fitting a DCC model by the composite likelihood technique on the $T = 1{,}500$ daily log-returns (about 6 years of data).[5] For each dataset, our estimated parameters indicate persistence in the conditional variances and persistence in correlations; as expected for daily data.

For each Monte Carlo replication, we generate DCC multivariate normal scenarios of length $T + h$, where h is the forecasting (rebalancing) horizon. The main analysis is for the daily forecasting horizon ($h = 1$), but, in the robustness section, we also provide results for the weekly ($h = 5$) and monthly ($h = 20$) horizons. To deal with the difference between the daily frequency of the log-return used for the DCC estimation and the various forecasting horizons, we proceed as follows. First, because of the time-aggregation property of log-returns and since the simulation setup has no serial correlation in the log-returns,

[3] Estimations are performed with an adapted version of the `R` package **rmgarch** (Ghalanos 2012).

[4] Data are available at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

[5] The use of the DCC model to describe the daily returns in our six data setups is common in the academic literature and in practice. See in particular the real-time conditional variance and correlation estimates available on Robert Engle's V-Lab available at https://vlab.stern.nyu.edu/. We use a parametric setup instead of a block bootstrap approach, since, for our analysis on the effect of covariance missspecification on the risk-based portfolios, we need the true covariance matrix of the data generating process.

Table 2 List of datasets/universes considered

#	Dataset/universe	N	Volatility			Correlation		
			Min	Med	Max	Min	Med	Max
1	Portfolios formed on size	10	17.2	21.6	27.3	87.9	96.2	98.8
2	Portfolios formed on book-to-market	10	17.2	20.4	28.2	83.0	92.6	95.8
3	Portfolios formed on momentum	10	18.5	21.7	37.2	58.0	87.9	96.0
4	Industry portfolios	10	13.7	18.9	28.5	68.2	79.0	92.1
5	Core portfolio	7	13.0	16.3	27.5	−50.2	38.7	78.2
6	DJIA portfolio	30	14.6	25.0	44.1	27.1	48.3	84.5

N denotes the number of "asset" in each universe. Min, med, max report the minimum, median and maximum values of the unconditional volatilities (annualized, in percent) and the unconditional pairwise correlations (in percent) in the universe. For the core portfolio, we consider the following assets from the Thomson Reuters Datastream database: US Benchmark 30 year Govt. Bond Index (BMUS30Y), MSCI EUROPE Stock Index (MSEROP), MSCI Emerging Markets Stock Index (MSEMKF), MSCI PACIFIC Stock Index (MSPACF), MSCI US Stock Index (MSUSAML), S&P US REIT Index (SBBRUSL), S&P Goldman Sachs Commodity Index (GSCITOT). For the DJIA portfolio, we consider the 30 constituents belonging to the Dow Jones Industrial Average universe as of July 2014

the covariance matrix of the h-period ahead return is obtained as the sum of the iterated 1-step ahead covariance predictions; see Ghysels et al. (2009) and Zakamulin (2015). Second, the covariance matrix of the h-period log-return is converted into the covariance matrix of the h-period arithmetic return using the approach by Meucci (2001). This leads to the *true* covariance matrix used to benchmark the various estimators. For the four tested estimation methods, we proceed in a similar fashion. We first estimate the 1-day ahead covariance matrices from the first T simulated returns, then project them at horizon h, and convert them into h-day ahead covariance forecasts of arithmetic returns using Meucci (2001)'s methodology.

In order to decompose the sensitivity of the allocation methodologies into their variance and correlation components, we proceed as follows. For the variance component, we fit each variance estimator from the T simulated log-returns, which are then projected/converted at horizon h together with the supposedly *true* DCC correlations in order to get the covariance matrix. For the correlation component, we fit correlations for each estimator from the T simulated log-returns and we project/convert them together with the *true* DCC volatilities (i.e., GARCH(1,1)) at horizon h to get the covariance matrix.

For each risk-based portfolio, we assess the performance of the various covariance methods by using two distinct metrics. First, for each Monte Carlo replication, the L^1 distance in (7) is computed. Second, we compute objective-based distances to assess the economic/financial impact of misspecification. For the minimum-variance portfolio, we compute the ratio $\sigma(\hat{\mathbf{w}})/\sigma(\mathbf{w})$ which is larger than one when misspecification is observed. For the inverse-volatility portfolio, we compute $H^*(\%RC(\hat{\mathbf{w}})) - H^*(\%RC(\mathbf{w}))$, the difference between the normalized Herfindahl index on risk contributions for the misspecified and true portfolio. This difference will be larger than zero if the estimated portfolio weights are misspecified. For the equal-risk-contribution portfolio we simply compute $H^*(\%RC(\hat{\mathbf{w}}))$, as the normalized Herfindahl index is zero for the true portfolio. Finally, for the maximum-diversification portfolio, we compute the ratio $DR(\hat{\mathbf{w}})/DR(\mathbf{w})$ which is lower than one when misspecification is observed.

4.3 Main results

Table 3 reports the main results of our Monte Carlo study when the forecasting horizon equals one day. For each universe and covariance estimator, the average (over one hundred Monte Carlo replications) for the L^1 and objective-based distances are reported. Overall, the results are in line with the conclusions of the numerical illustrations performed in Sect. 3. Indeed, we clearly notice that the minimum-variance portfolio is the most sensitive one to covariance misspecification. It displays a substantially higher average distance. On top, the objective-based distance is substantially higher than one for all universes and covariance matrix estimators. The maximum-diversification allocation methodology is the second most

Table 3 Monte Carlo results for misspecification on the covariance matrix $\boldsymbol{\Sigma}$ in case of a daily forecasting horizon ($h = 1$)

#	Cov type	L^1 distance				Objective-based distance			
		Min-var	Inv-vol	ERC	Max-div	Min-var	Inv-vol	ERC	Max-div
1	DCC	16.690	1.388	1.409	9.455	1.002	0.004	0.004	1.000
1	EWMA	39.569	2.552	2.618	20.884	1.007	0.013	0.014	0.999
1	LW	96.528	8.798	8.920	58.615	1.064	0.146	0.156	0.995
1	SMPL	96.534	8.798	8.920	58.613	1.064	0.146	0.156	0.995
2	DCC	27.014	1.690	1.757	12.584	1.003	0.006	0.006	1.000
2	EWMA	46.879	3.423	3.534	25.023	1.012	0.023	0.027	0.999
2	LW	135.950	11.841	12.102	79.611	1.117	0.305	0.327	0.988
2	SMPL	135.963	11.841	12.102	79.603	1.117	0.305	0.327	0.988
3	DCC	23.214	1.919	2.021	11.951	1.003	0.008	0.010	1.000
3	EWMA	44.002	4.046	4.194	25.003	1.012	0.039	0.041	0.998
3	LW	159.031	17.415	17.811	88.420	1.238	0.664	0.708	0.978
3	SMPL	159.031	17.415	17.811	88.434	1.237	0.664	0.708	0.978
4	DCC	19.789	2.285	2.495	12.194	1.004	0.010	0.013	0.999
4	EWMA	35.595	4.804	5.672	25.227	1.012	0.034	0.070	0.997
4	LW	94.820	15.114	16.212	78.989	1.119	0.529	0.620	0.970
4	SMPL	94.844	15.114	16.212	79.032	1.119	0.529	0.620	0.970
5	DCC	10.596	2.542	3.008	9.510	1.005	0.033	0.105	0.997
5	EWMA	28.519	5.309	8.150	29.325	1.028	0.095	0.577	0.975
5	LW	51.294	18.170	19.077	31.090	1.123	2.053	3.451	0.958
5	SMPL	51.311	18.170	19.076	31.114	1.123	2.053	3.458	0.958
6	DCC	23.470	2.942	3.661	21.876	1.007	0.006	0.009	0.997
6	EWMA	59.801	6.122	12.598	69.876	1.048	0.018	0.124	0.965
6	LW	85.685	16.939	18.064	57.908	1.134	0.238	0.262	0.972
6	SMPL	85.998	16.939	18.058	58.226	1.134	0.238	0.262	0.972

Left part: average L^1 distances (multiplied by 100 for convenience) of weights (7) obtained with the various universes and covariance matrix estimators, for the four risk-based portfolios (minimum-variance, inverse-volatility weighted, equal-risk contribution and maximum-diversification). Right: average objective-based distances. For the Min-vol portfolio: $\sigma(\hat{\mathbf{w}})/\sigma(\mathbf{w})$, for the Inv-vol portfolio: $H^*(\%RC(\hat{\mathbf{w}})) - H^*(\%RC(\mathbf{w}))$, for the ERC portfolio: $H^*(\%RC(\hat{\mathbf{w}}))$, and for the Max-div portfolio: $DR(\hat{\mathbf{w}})/DR(\mathbf{w})$. Averages are computed over one hundred Monte Carlo replications. See Table 1 for details sensitive one to covariance misspecifications. This confirms the fact that higher portfolio concentration increases the sensitivity to covariance misspecifications. The inverse-volatility weighted and equal-risk-contribution allocations under misspecified covariance are closer to the true DCC allocations. Interestingly, in five cases out of six, the inverse-volatility weighted method does not display a much lower sensitivity to covariance misspecification than the equal-risk-contribution method even if this latter requires $N(N-1)$ additional parameter estimates. The maximum-diversification portfolio lies between the second best performer and the minimum-variance portfolio.

Three very interesting facts emerge from the examination of the results for each different asset universe. First, the core portfolio (universe #5) appears to be the most challenging one for the equal-risk-contribution allocation. Indeed, as shown in Table 3, such a balanced universe is characterized by both a low average level of correlations between the assets and a high level of dispersion between the correlation coefficients. In that context, the equal-risk-contribution allocation underperforms all allocations but the minimum-variance one. The objective-based distance of the equal-risk-contribution allocation is substantially higher than zero for the core portfolio universe. Note also that the objective-based metric of the maximum diversification allocation deteriorates when the investment universe is characterized by lowly-correlated asset classes. The least sensitive allocation is the inverse-volatility weighted allocation, which does not make use of correlation coefficients. This result has practical implications as most equal-risk-contribution investment funds (risk-parity funds) invest in a mix of asset classes made of sovereign bonds, corporate bonds, real estate, equities and commodities.

Second, there does not seem to be any relationship between the number of assets in the universe and the portfolios sensitivity to covariance misspecification. Indeed, the distance of the portfolios in the DJIA universe (universe #6), which is made of 30 assets is not higher than the ones of the other universes that is made of fewer assets.

Third, the statistical and economic distances of the portfolios that are constructed with the EWMA covariance matrix are much smaller than those that are constructed with the Ledoit and Wolf (2003) and the sample-based estimates.

The Monte Carlo results for the decomposition of sensitivities into the variance and correlation components are reported in Panels A and B of Table 4, respectively. To save space, we henceforth focus on the comparison between universes #4, #5 and #6.[6] By comparing the results for variance and correlation misspecification, we see that, for universe #4, where the correlations are high and relatively homogeneous across the stocks, the minimum-variance and equal-risk-contribution methodologies display a much higher sensitivity (both statistically and economically) to variance misspecifications than to correlation misspecifications. In contrast, for universes #5 and #6, the observed range in correlation statistics (between −50.2 and 78.2% for #5, and between 27.1 and 84.5% for #6) is much higher than for #4 (between 68.2 and 92.1%), and hence correct estimation of the correlation turns out to be an important driver of misspecification in the weights. Indeed, we note that both the balanced (portfolio #5) and DJIA stocks (portfolio #6) portfolios constructed with the EWMA estimators display more sensitivity to correlation than to variance misspecifications. In fact, for the two aforementioned portfolios, the statistical and economic distances of the EWMA allocations are higher than those of both the Ledoit and Wolf (2003) and sample-based estimators. Finally, the maximum-diversification methodology, seems to be much more sensitive to correlation misspecifications than to misspecification in the variance. This is true for all considered portfolio universes and all estimators.

[6] Results for universes #1, #2 and #3 are available from the corresponding author. They are similar to those of universe #4, because, as can be seen in Table 2, the distributions of the variance, and especially the correlation statistic, is similar for the first four universes.

Table 4 Monte Carlo results for misspecification on the variances (Panel A) and the correlation matrix (Panel B) in case of a daily forecasting horizon ($h = 1$)

#	Cov type	L^1 distance				Objective-based distance			
		Min-var	Inv-vol	ERC	Max-div	Min-var	Inv-vol	ERC	Max-div
Panel A: misspecification on the variances									
4	DCC	19.355	2.285	2.305	2.313	1.004	0.010	0.011	1.000
4	EWMA	33.821	4.804	4.880	4.827	1.011	0.034	0.047	1.000
4	LW	95.679	15.114	15.178	15.158	1.108	0.529	0.559	0.996
4	SMPL	95.679	15.114	15.178	15.158	1.108	0.529	0.559	0.996
5	DCC	7.467	2.542	2.568	2.396	1.003	0.033	0.080	0.999
5	EWMA	15.885	5.309	5.270	4.744	1.010	0.095	0.254	0.998
5	LW	48.296	18.170	17.910	15.018	1.106	2.053	3.242	0.975
5	SMPL	48.296	18.170	17.910	15.018	1.106	2.053	3.242	0.975
6	DCC	19.837	2.942	2.954	2.946	1.005	0.006	0.006	1.000
6	EWMA	41.308	6.122	6.186	6.290	1.022	0.018	0.024	0.999
6	LW	83.464	16.939	16.638	14.205	1.118	0.238	0.227	0.996
6	SMPL	83.464	16.939	16.638	14.205	1.118	0.238	0.227	0.996
Panel B: misspecification on the correlations									
4	DCC	5.638	0.000	0.679	11.530	1.000	0.000	0.001	0.999
4	EWMA	14.167	0.000	2.009	24.187	1.003	0.000	0.011	0.998
4	LW	24.288	0.000	4.006	77.397	1.008	0.000	0.035	0.973
4	SMPL	24.317	0.000	4.007	77.437	1.008	0.000	0.035	0.973
5	DCC	7.013	0.000	1.552	9.019	1.002	0.000	0.032	0.998
5	EWMA	27.410	0.000	6.262	28.856	1.025	0.000	0.414	0.976
5	LW	23.168	0.000	5.176	26.520	1.020	0.000	0.299	0.979
5	SMPL	23.186	0.000	5.178	26.552	1.020	0.000	0.301	0.979
6	DCC	10.357	0.000	1.685	21.350	1.001	0.000	0.002	0.997
6	EWMA	50.359	0.000	10.007	68.954	1.036	0.000	0.085	0.967
6	LW	24.075	0.000	4.844	55.628	1.008	0.000	0.014	0.976
6	SMPL	24.429	0.000	4.860	56.108	1.009	0.000	0.014	0.976

See Table 3 for details

4.4 Results for the weekly and monthly forecasting horizons

The choice of covariance estimate can thus have a substantial impact on the accuracy of the composition of the daily rebalanced risk-based portfolio. In order to avoid turnover, investors often use a lower frequency of rebalancing. We now investigate the effect of covariance misspecification when the rebalancing frequency is weekly ($h = 5$) and monthly ($h = 20$). This thus implies setting risk-based weights using the covariance estimates of the weekly and monthly returns, respectively.

To do this analysis, we use the same setup for the Monte Carlo simulations, namely the stationary DCC model. Due to the stationarity in the DCC process, the conditional covariance forecasts are getting closer to their unconditional values as we increase the forecast horizon. It is therefore expected that the impact of the choice of covariance forecasting method (i.e.,

Table 5 Monte Carlo results for misspecification on the covariance matrix $\boldsymbol{\Sigma}$ when the forecasting horizon is 1 week (Panel A) and one month (Panel B)

#	Cov type	L^1 distance				Objective-based distance			
		Min-var	Inv-vol	ERC	Max-div	Min-var	Inv-vol	ERC	Max-div
Panel A: weekly forecasting horizon ($h = 5$)									
4	DCC	26.755	3.293	3.520	19.103	1.008	0.020	0.026	0.998
4	EWMA	40.733	5.779	6.576	31.079	1.018	0.045	0.093	0.996
4	LW	103.958	14.461	15.731	74.070	1.137	0.493	0.553	0.973
4	SMPL	104.010	14.461	15.730	74.086	1.137	0.493	0.553	0.973
5	DCC	15.045	3.919	4.274	8.937	1.009	0.126	0.210	0.997
5	EWMA	29.500	6.428	9.367	27.304	1.031	0.215	0.794	0.977
5	LW	52.444	17.528	18.816	31.235	1.121	1.532	3.546	0.956
5	SMPL	52.448	17.528	18.813	31.223	1.121	1.532	3.553	0.956
6	DCC	31.549	4.086	4.791	24.161	1.014	0.011	0.015	0.996
6	EWMA	62.888	6.843	12.825	71.798	1.053	0.022	0.126	0.963
6	LW	83.025	16.042	17.317	57.311	1.117	0.232	0.258	0.971
6	SMPL	83.373	16.042	17.313	57.637	1.118	0.232	0.258	0.970
Panel B: monthly forecasting horizon ($h = 20$)									
4	DCC	62.920	7.270	7.712	34.048	1.042	0.106	0.122	0.994
4	EWMA	71.631	8.877	9.978	44.059	1.054	0.154	0.210	0.990
4	LW	98.069	14.098	15.160	75.923	1.128	0.417	0.488	0.975
4	SMPL	98.081	14.098	15.159	75.923	1.128	0.417	0.488	0.975
5	DCC	25.286	8.025	8.677	13.002	1.031	0.402	0.820	0.991
5	EWMA	37.132	9.801	12.838	30.355	1.058	0.486	1.490	0.968
5	LW	51.702	17.944	19.315	30.386	1.122	1.788	3.499	0.956
5	SMPL	51.694	17.944	19.312	30.386	1.122	1.788	3.507	0.956
6	DCC	56.176	8.154	9.285	36.144	1.048	0.054	0.060	0.989
6	EWMA	80.572	10.477	15.857	79.485	1.095	0.074	0.181	0.953
6	LW	86.407	15.941	17.053	55.550	1.122	0.213	0.239	0.974
6	SMPL	86.878	15.941	17.048	55.684	1.123	0.213	0.239	0.974

See Table 3 for details

the conditional DCC covariance forecast versus the unconditional sample or Ledoit and Wolf (2003) based covariance estimates) diminishes when the forecasting horizon becomes longer.

The results of our investigation for the covariance matrix are reported in Table 5 for weekly and monthly horizons, respectively. We find that the statistical and economic distances of the portfolios that are constructed with the EWMA covariance matrix are still smaller than those of Ledoit and Wolf (2003) and sample-based estimates. However, as expected, the differences between the distances decrease as the forecasting horizon increases. If we consider the various investment universes, we can see that the impact of the forecasting horizon extension is the smallest for the core (multi-asset) universe. Finally, consistent with our previous results, both minimum-variance and maximum-diversification portfolios display the highest sensitivities to the extension of the forecasting horizon.

We also performed the analysis of disentangling the sensitivity of the weekly and monthly rebalanced risk-based portfolio weights to misspecification in the variance and

Table 6 Monte Carlo results for misspecification on the variances (Panel A) and the correlation matrix (Panel B) when the forecasting horizon is one month ($h = 20$)

#	Cov type	L^1 distance				Objective-based distance			
		Min-var	Inv-vol	ERC	Max-div	Min-var	Inv-vol	ERC	Max-div
Panel A: misspecification on the variances									
4	DCC	62.388	7.270	7.345	7.876	1.039	0.106	0.110	0.999
4	EWMA	71.078	8.877	8.969	9.274	1.052	0.154	0.169	0.999
4	LW	91.377	14.098	14.167	13.741	1.106	0.417	0.445	0.997
4	SMPL	91.377	14.098	14.167	13.741	1.106	0.417	0.445	0.997
5	DCC	23.958	8.025	7.980	7.050	1.027	0.402	0.640	0.994
5	EWMA	29.159	9.801	9.700	8.652	1.037	0.486	0.936	0.991
5	LW	47.432	17.944	18.127	15.693	1.104	1.788	3.051	0.973
5	SMPL	47.432	17.944	18.127	15.693	1.104	1.788	3.051	0.973
6	DCC	51.657	8.154	8.143	7.768	1.040	0.054	0.048	0.999
6	EWMA	66.429	10.477	10.519	10.552	1.064	0.074	0.074	0.998
6	LW	80.627	15.941	15.665	12.696	1.104	0.213	0.206	0.997
6	SMPL	80.627	15.941	15.666	12.696	1.104	0.213	0.206	0.997
Panel B: misspecification on the correlations									
4	DCC	14.496	0.000	1.895	32.086	1.002	0.000	0.008	0.995
4	EWMA	22.179	0.000	2.905	41.959	1.006	0.000	0.021	0.992
4	LW	26.473	0.000	3.837	73.771	1.008	0.000	0.030	0.977
4	SMPL	26.523	0.000	3.838	73.784	1.008	0.000	0.030	0.977
5	DCC	8.989	0.000	2.406	9.945	1.003	0.000	0.083	0.997
5	EWMA	27.340	0.000	7.375	28.757	1.026	0.000	0.562	0.975
5	LW	23.008	0.000	5.757	25.517	1.020	0.000	0.393	0.980
5	SMPL	22.993	0.000	5.771	25.534	1.020	0.000	0.398	0.980
6	DCC	16.127	0.000	2.860	34.294	1.004	0.000	0.005	0.991
6	EWMA	54.162	0.000	10.050	78.735	1.045	0.000	0.081	0.956
6	LW	24.589	0.000	4.496	52.484	1.009	0.000	0.013	0.978
6	SMPL	24.941	0.000	4.504	52.673	1.009	0.000	0.013	0.978

See Table 3 for details

correlation components. The results in case of the monthly forecasting horizon are shown in Table 6. They confirm our main findings, namely that the largest effects of misspecifying the variance are observed for the minimum-variance portfolio weights, and that misspecification of the correlations leads to a large misspecification in the weights of the minimum-variance and maximum-diversification portfolios. Comparing Table 4 with Table 6, we also see that, for a longer forecasting horizon, the choice of covariance methods has less impact. The performance of the risk-based portfolios using the estimated DCC process, the EWMA, the shrinkage covariance estimate and the sample variances and correlations are more similar at the monthly forecasting horizon than at the daily forecasting horizon.

4.5 Results in case of a DCC model with fat-tailed innovations

Part of the covariance misspecification is driven by the sampling uncertainty. Until now, we have assumed a Gaussian distribution for the innovations in the DCC model. We perform now a Monte Carlo study with innovations that are Student-t distributed with six degrees of freedom (i.e., fat-tailed). Changing the distribution of the innovations affects the estimates in two ways. First, the approach of maximizing the Gaussian (pseudo) log-likelihood is no longer the maximum likelihood approach. The estimator is still consistent, but is no more efficient. Also for the estimation of the unconditional covariance, there are more efficient methods available than the standard sample covariance. A second channel through which the distribution of the innovations affects our results is that the DCC and EWMA covariance estimates use the past innovations to forecast the variance and correlation dynamics. When the innovations are fat-tailed, misspecification in the dynamic structure is expected to have larger effects in terms of errors in the covariance estimates, and hence also in the estimated risk-based portfolio weights.

The Monte Carlo results for the misspecification in the covariance estimates are reported in Table 7. We find that, in general, and as predicted in the previous paragraph, all statistical and economic distances are higher than under the Gaussian assumption. The largest effect is observed for the DCC and EWMA estimation methods in terms of a substantially higher L^1 distance, compared with the sample-based estimators. Finally, in terms of differences in sensitivity across the four risk-based portfolios, we find that our previous conclusions remain the same: The inverse-volatility weighted and the equal-risk-contribution portfolios are by design more robust to covariance misspecification than the minimum-variance and the maximum-diversification portfolio. Comparing the results of universe #4 with those of universes #5 and #6 we also find confirmation of the finding that the equal-risk-contribution portfolio weights tend to be more affected by covariance misspecification when the correlations in the universe are more dispersed.

Table 7 Monte Carlo results for misspecification on the covariance matrix $\mathbf{\Sigma}$ at the daily forecasting horizon ($h = 1$) when the innovations are Student-t distributed with six degrees of freedom

#	Cov type	L^1 distance				Objective-based distance			
		Min-var	Inv-vol	ERC	Max-div	Min-var	Inv-vol	ERC	Max-div
4	DCC	25.504	3.048	3.267	18.862	1.007	0.019	0.023	0.999
4	EWMA	48.636	6.047	7.410	41.000	1.022	0.063	0.117	0.994
4	LW	96.539	16.412	16.949	71.606	1.136	0.656	0.741	0.978
4	SMPL	96.663	16.412	16.944	71.799	1.136	0.656	0.740	0.978
5	DCC	15.017	3.624	4.440	13.576	1.010	0.069	0.218	0.993
5	EWMA	29.762	6.375	10.080	30.591	1.035	0.125	1.089	0.970
5	LW	57.969	21.258	23.446	41.164	1.209	2.811	5.205	0.913
5	SMPL	57.968	21.258	23.452	41.212	1.208	2.811	5.227	0.913
6	DCC	34.399	4.058	5.618	31.355	1.015	0.012	0.022	0.991
6	EWMA	62.824	7.806	16.158	64.911	1.052	0.025	0.277	0.964
6	LW	98.201	19.615	23.513	90.784	1.206	0.351	0.448	0.911
6	SMPL	98.634	19.615	23.487	91.769	1.207	0.351	0.448	0.910

See Table 3 for details

5 Conclusions

Risk-based allocation methodologies have become increasingly popular among investors. These strategies have two main advantages compared to other widespread quantitative asset allocation strategies such as mean-variance. First, they do not require the estimation of expected returns. Second, they place risk management at the center of the allocation process. As a consequence, the superiority of risk-based investment methods is highly dependent on the quality of the risk parameter estimates, which are used as input to construct the portfolios.

In this paper, we first document differences between the sensitivities of the various methodologies to estimation risk. Minimum-variance and the maximum-diversification portfolios are much more sensitive to misspecifications of the covariance matrix than the other risk-based allocation methodologies. Their vulnerability to a misspecified covariance matrix is due to the higher concentration of their portfolio weights. Second, by decomposing the sensitivity of the allocation methodologies into their variance and correlation components, we show that, for both the minimum-variance and the equal-risk-contribution methodologies, sampling errors in variances have a higher impact on the portfolio allocation than correlation misspecifications. The reverse is true for maximum-diversification strategies. Third, we show that the composition of the opportunity set impacts the degree of sensitivity of some allocation strategies to covariance misspecifications. In particular, equal-risk-contribution strategies are very sensitive to covariance misspecifications when the opportunity set is characterized by low and dispersed correlations between its comprising assets. Finally, we document that the aforementioned effects exist not only at the daily, but also at the weekly and monthly forecasting horizons, and that similar findings are obtained when the return innovations are driven by a fat-tailed Student-t distribution.

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