Affine isometric actions of groups

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Résumé

Cette thèse a pour objet l'étude des groupes via leurs actions affines sur des espaces de Hilbert ou de Banach.

Dans la première partie, la théorie des actions affines irreductibles est développée. Un résultat analogue au Lemme de Schur pour les représentations unitaires est démontré. Plusieurs applications sont proposées parmi lesquelles une classification des actions affines irréductibles des groupes nilpotents et FC-nilpotents. La question de l'existence d'une action irreductible dont la partie linéaire est la régulière gauche d'un groupe est abordée et présente des liens avec le premier nombre de Betti L^2 du groupe. Finalement, une condition nécessaire et suffisante pour que la somme directe de deux actions soit irréductible est présentée.

La deuxième partie est consacrée à l'étude des exposants de compression des groupes. Après une brève introduction au sujet, la valeur exacte de l'exposant de compression L^p des groupes de Gal-Januszkievicz est calculée. Puis, plusieurs résultats sur la permanence des exposants de compression équivariants L^p sont présentés, dans le cas des produits libres amalgamés et dans celui des extensions HNN. Finalement, plusieurs questions et pistes de travaux à venir sont mentionnées.

Mots-clés: Théorie géométrique des groupes, théorie des représentations, représentations unitaires, actions affines isométriques, actions affines irréductibles, algèbres de von-Neumann, nombres de Betti L^2, plongements grossiers, exposants de compression, extension HNN, groupes de Gal-Januszkiewicz, exposants de compression équivariants, produits libres amalgamés.

Abstract

The purpose of this thesis is the study of groups through their affine actions on Hilbert or Banach spaces.

In the first chapter, the theory of irreducible affine actions is developed. A result similar to Schur's lemma for unitary representation is proved. Amongst several applications, a clasification of irreducible actions of nilpotent and FC-nilpotent groups is given. The question of the existence of an irreducible action with linear part the left regular representation of the group is studied and connections with the first L^2-Betti number are established. Finally, a sufficient and necessary criterion for the direct sum of two actions to be irreducible is provided.

The second chapter is devoted to the study of compression exponents of groups. After a short introduction to the matter, the exact value of the L^p-compression exponent of Gal an Januszkiewicz groups is computed. Then, several results about permanence of equivariant compression exponents are given. First in the case of amalgamated free products, then in the case of HNN extensions. Finally, several questions and ideas about further research are raised.

Keywords: Geometric group theory, representation theory, unitary representations, affine isometric actions, irreducible affine actions, von-Neumann algebras, L^2-Betti numbers, coarse embeddings, compression exponents, HNN extensions, Gal-Januszkiewicz groups, equivariant compression exponents, amalgamated free products.

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Contents

Preamble

The present text is divided into two independent chapters. The first chapter is concerned with irreducibility of affine isometric actions of groups on Hilbert spaces. The second is devoted to the computation of compression exponents of groups. The uniting idea behind this two topics is to study groups via their interactions with Hilbert spaces.

There is a well-known leitmotiv in group theory, stating that in order to understand a group you need to understand its actions. After all, what is a group if not an object designed to act upon various structures? The representation theory of groups arises completely naturally from this philosophy. As soon as a group admits an action on an object, whether it is a manifold, a graph, a measure space or a metric space, it yields a representation of the group on any function space associated with that object. This enables us to translate questions or concepts associated to the action, to the linear world.

Since this study of linear group actions has proved to be crucial in various areas of mathematics, it seems quite natural to try to extend it to the world of affine actions. But there are more motivations to this study than sole generalisation. Affine actions appear in connection with Kazhdan's property (T). Although this property was originally defined as representation-theoretic in flavour, it found a restatement in term of affine actions. The Delorme-Guichardet theorem asserts that a group has property (T) if and only if all its affine isometric actions on Hilbert spaces have a fixed point. Unsurprisingly, the Haagerup property, which strongly denies property (T), found such a reformulation too. A group has the Haagerup property if and only if it admits a metrically proper affine isometric action. Perhaps one of the most striking achievement linked to affine actions is the proof, by Higson and Kasparov, of the Novikov conjecture for groups with the Haagerup property.

Leaving the realm of groups, we can apply the same philosophy to metric spaces. Given a metric space, we can ask whether its geometry is comparable or compatible with that of a Hilbert space. Coarse embeddings into Hilbert spaces formalise that idea. They have become increasingly studied since Gromov conjectured that a space coarsely embeddable in a Hilbert space satisfy the coarse Novikov conjecture. This was confirmed by Yu who proved the coarse Baum-Connes conjecture for such spaces, which, in the case of a Cayley graph of a group, implies the Novikov conjecture.

In Chapter 1, we introduce a notion of irreducibility for affine isometric actions and undertake a systemactic study of irreducible actions. We are able to prove an analog of the classical Schur's lemma for unitary representations and this helps us classify irreducible actions for several classes of groups. The question of the existence of an irreducible affine action with linear part the regular representation reveals surprising connections with the first L^2-Betti number of the group.

Chapter 2 revolves around compression exponents. These invariants of metric spaces quantifies how well a space coarsely embeds into another, from potentially not at all to almost quasi-isometrically. After surveying the current knowledge of the topic, Section 1.2. deals with computing L^p compression exponents of a class of groups inspired by Baumslag-Solitar groups. Section 1.3. is concerned with the so-called equivariant compression exponents of groups. We study the behaviour of these invariants under free products with amalgamation and HNN extensions.

Chapter 1

Irreducible affine isometric actions of groups

This chapter consists of selected pieces of the article [BPV14] written in collaboration with B. Bekka and A. Valette.

1.1 Introduction

The theory of unitary representations of locally compact groups is by now very well understood. When one studies it, it rapidly becomes clear that irreducible representations serve as building blocks for all representations so that their understanding, and classification is crucial to the theory. Affine isometric actions are a much more recent topic. Their study is motivated by several facts. First, two properties which have proved to be valuable in several areas of mathematics admit reformulations in term of affine actions, namely Property (T) and the Haagerup Property (see [BdlHV08] and [CCJ$^+$01] for an extensive treatment of both properties). Second, the set of affine isometric actions with a prescribed linear part identifies naturally with the first cohomology group with coefficients in this representation.

In this chapter, we study a notion of irreducibility for affine isometric actions. By analogy with the case of unitary representations, we define an action to be irreducible if it has no non-trivial invariant affine subspaces. To our knowledge, this definition first appeared in [Ner98] and has not been developed since then. There are similarities between irreducible representations and irreducible actions, but to a limited extent. A striking difference between the two theories is that the direct sum of two unitary representations is never irreducible, whereas in the case of irreducible affine action it may well be.

1.1.1 Organisation of the chapter

In Section 1.2. we recall some facts and notations about affine isometric actions. We give several algebraic reformulations of irreducibility and in Proposition 1.2.11 we characterise compactly generated groups admitting an irreducible affine action.

In Section 1.3. we define the commutant of an affine isometric action and successfully establish an affine version of Schur's lemma for affine actions.

In Section 1.4. we use Schur's lemma to derive several results. We give a new proof of a theorem of Neretin on the irreducibilty of the restriction of an affine action. We characterise irreducible actions of abelian groups, groups with finite conjugacy classes, nilpotent groups, and so-called FC-nilpotent groups.

In Section 1.5. we investigate the existence of an irreducible affine isometric action with linear part the left regular representation. This turns out to have surprising connections with the first L^2-Betti number of the group.

Finally, in Section 1.6. we provide a sufficient and necessary condition for the direct sum of two affine isometric actions to be irreducible.

1.2 Characterisations of irreducible actions

1.2.1 Notations and setting

Let G be a topological group with identity element e. Throughout this chapter, α will denote an affine isometric action of G on a complex or real Hilbert space $\mathcal{H}$. That is, a homomorphism from G to the group $\mathrm{Isom}(\mathcal{H})$ of *affine* isometries of $\mathcal{H}$ which is strongly continuous in the sense that the mappings

$$g \mapsto \alpha(g)v$$

are continuous for each $v \in \mathcal{H}$. The group $\mathrm{Isom}(\mathcal{H})$ classically splits as $\mathcal{H} \rtimes \mathcal{U}(\mathcal{H})$ where $\mathcal{U}(\mathcal{H})$ denotes the group of unitary operators of $\mathcal{H}$. With respect to this splitting the affine action α can be written as

$$\alpha(g)(\cdot) = \pi(g)(\cdot) + b(g).$$

Here $\pi : G \to \mathcal{U}(\mathcal{H})$ is a unitary (or orthogonal) representation of G that we will call the *linear part* of α and $b : G \to \mathcal{H}$ will be called its *translation part* or its *cocycle.* In case π and b are given, we'll denote by $\alpha_{\pi,b}$ the associated affine isometric action of G. The use of the term cocycle is in accordance with the following easy equivalence:

Proposition 1.2.1. *Let π be a unitary or orthogonal representation of G and $b : G \to \mathcal{H}$ be a continuous map. The following statements are equivalent:*

i) b satisfies the 1-cocycle relation

$$b(gh) = \pi(g)b(h) + b(g), \quad \forall g, h \in G.$$

ii) The map $(g, v) \mapsto \alpha(g)(v) = \pi(g)v + b(g)$ defines an affine isometric action of G on $\mathcal{H}$. □

In case one of the previous statements holds, b is called a π-*cocycle.*

Example 1.2.2. If π is the trivial representation of G on $\mathcal{H}$, π-cocycles identify with homomorphisms from G to the additive group of $\mathcal{H}$.

Recall the following classical lemma (see e.g. [BdlHV08, Prop. 2.2.9]):

Lemma 1.2.3. *Let $\alpha_{\pi,b}$ be an affine isometric action of G. The following statements are equivalent:*

i) α has bounded orbits.

ii) α has a fixed point $v \in \mathcal{H}$.

iii) $b(\cdot) = v - \pi(\cdot)v$. □

Maps of the form $g \mapsto \pi(g)v - v$ will be denoted by $\partial_v(\cdot)$ and are called π-*coboundaries*. In view of the cohomological background, we'll adopt the following notations :

$$Z^1(G,\pi) = \text{set of } \pi\text{-cocycles},$$
$$B^1(G,\pi) = \text{set of } \pi\text{-coboundaries}.$$

These spaces are vector spaces, and $B_1(G,\pi) \subset Z^1(G,\pi)$. We also define

$$H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi),$$
$$\overline{H^1}(G,\pi) = Z^1(G,\pi)/\overline{B^1(G,\pi)}.$$

$H^1(G,\pi)$ is the *first cohomology group* of G with coefficients in π, $\overline{H^1}(G,\pi)$ is the corresponding *reduced* cohomology group of G and the closure is taken with respect to the topology of uniform convergence on compact subsets of G.

There is a link between affine isometric actions of groups and functions conditionally of negative type (see [BdlHV08, Appendix C.] for more details about functions on groups). Recall the definition:

Definition 1.2.4. A continuous function $\psi : G \to \mathbb{R}$ is *conditionally of negative type* (CNT) if it satisfies

1. $\psi(e) = 0$.

2. $\forall n > 0, \forall g_1, \ldots, g_n \in G$ and $\forall \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ satisfying $\sum \lambda_i = 0$,

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \psi(g_i^{-1} g_j) \leq 0.$$

Recall that a subset of $\mathcal{H}$ is *total* if it generates a dense linear subspace of $\mathcal{H}$. A GNS-type construction gives the following characterisation :

Proposition 1.2.5. *Let $\psi : G \to \mathbb{R}$ be a continuous map, the following statements are equivalent:*

1. *ψ is CNT.*

2. *There exists an affine isometric action $\alpha_{\pi,b}$ of G, on a real Hilbert space $\mathcal{H}$, such that $b(G)$ is total and*

$$\psi(g) = \|b(g)\|^2.$$

Moreover, the action α is unique in the sense that if α' satisfies the same conclusions, then α and α' are conjugated by a G-equivariant affine isometry.

The set of CNT functions forms a convex cone $\mathcal{C}_{\mathrm{CNT}}$ and its extremal rays are the CNT functions arising from affine isometric actions whose linear part is irreducible [LSV04].

1.2.2 Definition and characterisations of irreducibility

Definition 1.2.6. Let α be an affine isometric action of G on a Hilbert space $\mathcal{H}$. We say that α is irreducible if the only closed, non-empty, $\alpha(G)$-invariant, affine subspace of $\mathcal{H}$ is $\mathcal{H}$ itself.

For the rest of the text, we adopt the convention that, unless stated otherwise, all subspaces (either linear or affine) are closed and non-empty. The following two examples play an important role in the theory.

Example 1.2.7. Let $b : G \to \mathcal{H}$ be a continuous homomorphism to the additive group of $\mathcal{H}$. It gives rise to an affine action of G by translations on $\mathcal{H}$. That action is irreducible if and only if the linear span of $b(G)$ is dense in $\mathcal{H}$.

Example 1.2.8. Let π be an irreducible representation of G and let $b \in Z^1(G, \pi)$. The action $\alpha_{\pi,b}$ is irreducible if and only if b is unbounded.

Proof. Indeed, if W is an $\alpha(G)$-invariant affine subspace, it is straightforward to see that its underlying linear space, $W_0 = W - W$ is $\pi(G)$-invariant. Since π is irreducible, $W_0 = \mathcal{H}$, in which case $W = \mathcal{H}$ or $W_0 = \{0\}$. In the later case W consists of a single $\alpha(G)$-fixed point. According to Lemma 1.2.3, that happens only if b is bounded. $\square$

We observe that irreducibility is a translation-invariant notion, so that the irreducibility of an action $\alpha_{\pi,b}$ only depends on the class $[b] \in H^1(G, \pi)$. Indeed, two cocycles in $Z^1(G, \pi)$ are cohomologous if and only if they are conjugated by a translation.

For α an affine isometric action with linear part π and cocycle b, let π_0 be a sub-representation of π on a closed subspace $V_0 \subset \mathcal{H}$. Let us denote by b_0 the orthogonal projection of the map b on V_0. It is immediate to check that $g \mapsto b_0(g)$ is a cocycle with respect to π_0, so that $\alpha_0(g)v = \pi_0(g)v + b_0(g)$ defines an affine isometric action of G on V_0. We call it the *projected action* on V_0.

Proposition 1.2.9. *Let $\alpha_{\pi,b}$ be an affine isometric action and let $\psi(\cdot) = \|\cdot\|^2$ be the associated CNT function. The following properties are equivalent:*

(A1) *The affine isometric action α is irreducible.*

(A2) *For every $v \in \mathcal{H}$, the 1-cocycle $b + \partial_v$ has total image in $\mathcal{H}$.*

(A3) *For every direct sum decomposition $\pi = \pi_0 \oplus \pi_1$ with $\pi_0 \neq 0$, in the corresponding decomposition $b = b_0 \oplus b_1$, the 1-cocycle b_0 is unbounded.*

(A4) *$b(G)$ is total and, for every decomposition $\psi = \psi_0 + \psi_1$, with ψ_0, ψ_1 functions conditionally of negative type with $\psi_0 \neq 0$, the function ψ_0 is unbounded.*

(A5) *$b(G)$ is total and ψ belongs to the maximal face of $\mathcal{C}_{\mathrm{CTN}}$ containing only unbounded functions.*

(A6) *For every non-zero sub-representation π_0 of π, the projected action α_0 is irreducible.*

Proof. We follow the schemes $(A2) \Leftrightarrow (A1) \Rightarrow (A6) \Rightarrow (A3) \Rightarrow (A1)$ and $(A3) \Leftrightarrow (A4) \Leftrightarrow (A5)$

(A1)$\Leftrightarrow$ $(A2)$: Observe that $b(g) + \partial_v(g) = \alpha(g)v - v$, so that the linear span of the set $(b - \partial_v)(G)$ is the underlying linear space of the affine span of the orbit $\alpha(G)v$. It is immediate that the action is irreducible if and only if each orbit has a dense affine span[1].

$(A1) \Rightarrow (A6)$: Assume that there is a closed, $\pi(G)$-invariant subspace $V_0 \subset \mathcal{H}$ such that the projected action α_0 is reducible. So there exists a proper closed, $\alpha_0(G)$-invariant affine subspace $W \subset V_0$. Let $V_0^{\perp}$ denote the orthogonal complement of V_0. Then $W \oplus V_0^{\perp}$ is a proper closed, $\alpha(G)$-invariant affine subspace of $\mathcal{H}$, so that α is reducible.

$(A6) \Rightarrow (A3)$ is clear, as boundedness of b_0 implies reducibility of α_0.

$(A3) \Rightarrow (A1)$: Suppose α is reducible and let W be an $\alpha(G)$-invariant subspace. Let $W_0 = W - W$ be the underlying $\pi(G)$-invariant linear subspace. Observe that the unique point in $W_0^{\perp} \cap W$ must be fixed by the projection of α on to $W_0^{\perp}$. According to Lemma 1.2.3 the projection of b on that space must be bounded.

$(A3) \Leftrightarrow (A4)$: Given ψ, the uniqueness of a total coycle b satisfying $\psi(\cdot) = \|b(\cdot)\|^2$ implies that there is a unique correspondance between

[1]It is a well-known characterisation of irreducibility of a unitary representation that every vector is cyclic (i.e. with total orbit). (A2) is the analogue statement for irreducible affine actions.

decompositions of the form $b = b_0 \oplus b_1$ and decompositions of the form $\psi = \psi_0 + \psi_1$.

(A4)$\Leftrightarrow$(A5) : Recall that a face F in a convex cone is a subset satisfying

$$\forall \psi \in F,\ \psi = \psi_0 + \psi_1 \Rightarrow \psi_0 \in F.$$

The existence of a maximal face containing solely unbounded functions is due to Zorn's lemma. Denote by F_u such a face. Suppose F_u is not unique and let F'_u be another maximal face. Then $F_u + F'_u$ is a face containing only unbounded functions. By maximality of F_u and F'_u, we have F_u=F'_u=$F_u + F'_u$. Now observe that the smallest face containing ψ is given by $F_\psi = \{\psi_0 \mid \exists \psi_1 \text{ such that } \psi = \psi_0 + \psi_1\}$. Clearly, ψ belongs to F_u if and only if $F_\psi \subset F_u$ which happens if and only if (A4) is satisfied. □

Example 1.2.10. If α is irreducible then by $(A1) \Rightarrow (A2)$, the set $b(G)$ is total in $\mathcal{H}$. The converse is *false*: the reason is that condition $(A2)$ is translation-invariant, while $b(G)$ being total is not. Concretely, let $G = \mathbb{Z}$ act isometrically on $\mathbb{R}^2$ by

$$\alpha_n(x, y) = (x + n, (-1)^n y + 1 - (-1)^n) \quad \text{for all} \quad n \in \mathbb{Z},\ (x, y) \in \mathbb{R}^2.$$

Geometrically, this is the action by powers of the glide symmetry with axis the horizontal line $y = 1$, and translation by $+1$ to the right. Then *all* orbits are total, in particular $\alpha(G)(0) = b(G)$, but α is reducible as the axis is invariant.

Amongst compactly generated groups, groups admitting an irreducible affine isometric action are well characterised.

Proposition 1.2.11. *A locally compact compactly generated group admits an irreducible affine isometric action if and only if it does not have property (T).*

Proof. The direct implication comes from a deep result of Shalom ([Sha00, Theorem 0.2]) asserting that compactly generated groups without property (T) have an irreducible representation with non-zero cohomology. Now, according to example 1.2.8, an unbounded cocycle of an irreducible representation yields an irreducible affine isometric action.

The reverse implication is immediate since any affine isometric action of a group with property (T) fixes a point. □

1.3 The use of commutants

1.3.1 The commutant of an affine action

Let α be an affine isometric action of a group G, with linear part π. We recall that the *commutant* of π is the von Neumann algebra

$$\pi(G)' = \{T \in \mathcal{B}(\mathcal{H}) : T\pi(g) = \pi(g)T \quad \text{for all} \quad g \in G\}.$$

If b is a cocycle for π and $T \in \pi(G)'$, we observe that Tb is still a cocycle for π, so that $\pi(G)'$ acts on the space $Z^1(G, \pi)$ of 1-cocycles, and this action descends to the first cohomology space $H^1(G, \pi)$.

Definition 1.3.1. The *commutant* of α is the set of (continuous) affine transformations A on $\mathcal{H}$ such that $A \circ \alpha(g) = \alpha(g) \circ A$ for every $g \in G$.

Lemma 1.3.2. *Let A be an affine transformation written as $Av = Tv + t$ and let α be an affine isometric action with linear part π and cocycle b. The following statements are equivalent:*

i) A belongs to the commutant of α.

ii) $T \in \pi(G)'$ and $(T-1)b = \partial_t$

Proof. For $g \in G$, $v \in \mathcal{H}$, developing the commutation relation, we get:

$$\pi(g)Tv + \pi(g)t + b(g) = T\pi(g)v + Tb(g) + t.$$

Evaluating in $v = 0$, we find $(T-1)b(g) = \partial_t(g)$. Thus we get $\pi(g)Tv = T\pi(g)v$ for all g and v so that $T \in \pi(G)'$. □

We have the immediate corollary

Corollary 1.3.3. *For $T \in \pi(G)'$, the following properties are equivalent:*

i) There exists $t \in \mathcal{H}$ such that the affine transformation $Av := Tv + t$ is in the commutant of α.

ii) There exists $t \in \mathcal{H}$ such that $(T-1)b(g) = \partial_t(g)$ for every $g \in G$.

iii) $(T-1)[b] = 0$, where $[b]$ denotes the class of b in $H^1(G, \pi)$. □

Remark 1.3.4. We observe that, if $Av = Tv + t$ is in the commutant of an affine action α without fixed point, then 1 is a spectral value of T. Indeed, as the operator $(T-1)$ maps the unbounded set $b(G)$ to the bounded set $\partial_t(G)$, it cannot be invertible.

1.3.2 A Schur-type lemma

Recall that the classical Schur's Lemma for unitary representations [BdlHV08, Appendix A.2] asserts that a representation π is irreducible if and only if its commutant is reduced to scalar multiples of the identity. We get the following similar statement for irreducible affine isometric actions:

Proposition 1.3.5. *Let α be an affine isometric action on a real or complex Hilbert space $\mathcal{H}$ with linear part π. Denote by $\mathcal{H}^{\pi(G)}$ the space of $\pi(G)$-fixed vectors in $\mathcal{H}$. The following properties are equivalent.*

i) α is irreducible.

ii) The commutant of α is the set of translations along $\mathcal{H}^{\pi(G)}$.

iii) The commutant of α consists only of translations.

Proof. *(i)* ⇒ *(ii)* Let $Av = Tv + t$ be an affine transformation of $\mathcal{H}$, in the commutant of α. Then $T \in \pi(G)'$ and

$$(T-1)b(g) = \pi(g)t - t \quad \text{for every} \quad g \in G. \tag{1.1}$$

We first show that A is a translation, that is that $T = 1$. For this, consider the positive operator

$$S = T^*T - T - T^* + 2 = (T-1)^*(T-1) + 1;$$

if we show $S = 1$, then $(T-1)^*(T-1) = 0$. This implies that

$$\langle (T-1)v, (T-1)v \rangle = 0, \forall v \in \mathcal{H}$$

which readily gives $T = 1$. Observe that S is self-adjoint, so that it is enough to show that the spectrum of S is reduced to $\{1\}$. This is a direct consequence of the functional calculus for self-adjoint operators and it is valid in both the real case and the complex case. Assume by contradiction that S has some spectral value s distinct from 1. Let $[a, b]$ be a closed interval of $\mathbb{R}$ containing s in its interior, and not containing 1. Let E be the spectral projector of S associated with $[a, b]$. Since $s \in [a, b]$, $E \neq 0$. Also, $E \in \pi(G)'$ [2]. Denote by ρ the sub-representation of π on $\mathrm{Im}(E)$. Apply $(T-1)^*$ to Equation 1.1:

$$(S-1)b(g) = (\pi(g) - 1)(T^* - 1)t.$$

Then apply E and restrict to $\mathrm{Im}(E)$:

$$(S-1)Eb(g) = (\rho(g) - 1)E(T^* - 1)t.$$

But $S-1$ is invertible as a bounded operator on $\mathrm{Im}(E)$ (since $1 \notin [a, b]$); denoting by R its inverse, we obtain

$$Eb(g) = (\rho(g) - 1)RE(T^* - 1)t.$$

The projection Eb of b on $\mathrm{Im}(E)$ is therefore bounded, contradicting condition (A3) in Proposition 1.2.9.

Since $T = 1$, equality 1.1 yields $\pi(g)t - t = 0$, so that $t \in \mathcal{H}^{\pi(G)}$.

$(ii) \Rightarrow (iii)$ is trivial.

[2]It is a general fact that a spectral projector of an element of a von Neumann algebra remains in the same algebra

$(iii) \Rightarrow (i)$ Assume that α is reducible, and let W be a non-trivial closed, invariant, affine subspace of $\mathcal{H}$. Let $E : \mathcal{H} \to W$ be the projection onto W; so Ev is the point of W closest to v, for every $v \in \mathcal{H}$. Since every $\alpha(g)$ is an isometry, it follows that the affine transformation E is in the commutant of α. $\square$

We already observed that the first cohomology $H^1(G, \pi)$ is a module over the von Neumann algebra $M := \pi(G)'$; recall that a vector ξ, in a module over M, is *separating* if $S\xi = 0$ implies $S = 0$ for every $S \in M$.

Corollary 1.3.6. *Let π be a unitary representation of G. There exists an irreducible affine action α with linear part π if and only if $H^1(G, \pi)$ admits a separating vector for $\pi(G)'$.*

Proof. According to Proposition 1.3.5, the existence of α is equivalent to the existence of a 1-cocycle b such that, for every $T \in \pi(G)'$ and $t \in \mathcal{H}$ satisfying $(T - 1)b(\cdot) = \partial_t(\cdot)$, we have $T = 1$; in turn, by Lemma 1.3.3, this is equivalent to the existence of a class $[b] \in H^1(G, \pi)$ such that $(T - 1)[b] = 0$ for $T \in \pi(G)'$, implies $T = 1$; this exactly means that $[b]$ is a separating vector for $\pi(G)'$. $\square$

1.4 Applications

1.4.1 Restriction to lattices

We give a short proof of a result of Neretin (Theorem 3.6 in [Ner98]) asserting that the restriction of an irreducible affine action to a co-compact lattice, remains irreducible. Since we do not use induction of affine actions, we are able to remove the assumption of discreteness of the subgroup in [Ner98]. In order to treat non-co-compact lattices, we introduce a definition: for H a lattice in G and $b \in Z^1(G, \pi)$, we say that the cocycle b is *integrable on G/H* if there exists a measurable fundamental domain Ω for the right action of H on G, such that $\int_\Omega \|b(g)\| \, \mathrm{d}g < +\infty$, where $\mathrm{d}g$ denotes the Haar measure on G.

Theorem 1.4.1. *Let H be a closed subgroup of the locally compact group G, such that G/H carries a G-invariant probability measure μ. Let $\alpha_{\pi,b}$ be an affine isometric action of G. Assume either that H is co-compact or that H is discrete and the cocycle b is integrable on G/H. If α is irreducible, then the restriction $\alpha|_H$ is irreducible.*

Proof. Let $A(\cdot) = T(\cdot) + t$ be an affine transformation in the commutant of $\alpha|_H$. Using our Schur-type lemma, we want to show that $T = 1$, for A to be a translation. Let $\mathrm{Aff}(\mathcal{H})$ be the set of continuous affine maps from $\mathcal{H}$ to $\mathcal{H}$. Consider the map

$$G \to \mathrm{Aff}(\mathcal{H}) : g \mapsto \alpha(g)A\alpha(g)^{-1};$$

this map factors through G/H, and we wish to integrate it on G/H. For this, we compute:

$$\alpha(g)A\alpha(g)^{-1}v = \pi(g)T\pi(g)^{-1}v + \pi(g)t + [1 - \pi(g)T\pi(g)^{-1}]b(g).$$

The first two terms are bounded, and the third one is integrable on G/H under either of our assumptions. So we may define

$$B = \int_{G/H} \alpha(x)A\alpha(x)^{-1}\, d\mu(x) \tag{1.2}$$

as an element of $\mathrm{Aff}(\mathcal{H})$. By G-invariance of μ, we see that B belongs to the commutant of α. By Proposition 1.3.5, the affine transformation A is a translation. Taking linear parts in Equation (1.2), we get $\int_{G/H} \pi(x)E_0\pi(x)^{-1}\, d\mu(x) = 1$, expressing the identity 1 on $\mathcal{H}$ as an average of operators of norm ≤ 1. Since 1 is an extreme point in the unit ball of $\mathcal{B}(\mathcal{H})$ (see e.g. Proposition 1.4.7 in [Ped79]), we deduce $E_0 = 1$. □

Remark 1.4.2. The condition of integrability defined above does not hold in general. A counterexample can be found as a non-uniform lattice in the automorphism group of a regular tree. However, the condition is automatic for some classes of subgroups such as S-arithmetic groups, lattices in rank 1 simple Lie groups and twin building lattices. (See Remarks 4.2 and 4.3 in [BPV14] for a more complete treatment of the question).

Remark 1.4.3. Let Γ be a co-compact closed subgroup in the locally compact group G. Given an action α of Γ by affine isometries on a Hilbert space $\mathcal{H}$, it is possible to define an *induced* affine action $\mathrm{Ind}_\Gamma^G \alpha$ of G, as discussed in [Sha00, Section II] or [BdlHV08, p. 91].

One may ask whether $\mathrm{Ind}_\Gamma^G \alpha$ is irreducible when α is irreducible. This is not the case, even when Γ has finite index in G, as the following simple example shows. Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$ be the direct product of the cyclic group of order two and the group of integers and let $\Gamma = \mathbb{Z}$. Let α be the affine isometric action of Γ on $\mathbb{R}$ by integer translations.

The induced affine action $\mathrm{Ind}_\Gamma^G \alpha$ of G is easily seen to be defined on $\mathbb{R}^2$ by

$$(\mathrm{Ind}_\Gamma^G \alpha)(a,n)(x,y) = (x, y+n) \qquad n \in \mathbb{Z},\ (x,y) \in \mathbb{R}^2.$$

Clearly, $\mathrm{Ind}_\Gamma^G \alpha$ is not irreducible.

1.4.2 Centre and FC-centre

We denote by $Z(G)$ the centre of the topological group G.

Proposition 1.4.4. *In an irreducible affine action α of G on $\mathcal{H}$, the centre $Z(G)$ acts by translations in the direction of $\mathcal{H}^{\pi(G)}$.*

Proof. This follows immediately from Proposition 1.3.5 since the elements $\alpha(g)$ with $g \in Z(G)$ belong to the commutant of α. □

Corollary 1.4.5. *Assume that* $\mathrm{Hom}(G, \mathbb{R}) = 0$. *Then every irreducible affine action α of G factors through $G/Z(G)$.*

Proof. Let b be the cocycle defining α, and let b_0 be its projection on $\mathcal{H}^{\pi(G)}$, so that b_0 is a continuous homomorphism from G to the additive group of $\mathcal{H}^{\pi(G)}$, hence $b_0 \simeq 0$ by our assumption. This forces $\mathcal{H}^{\pi(G)} = 0$ (otherwise we would contradict condition (A3) in Proposition 1.2.9). By Proposition 1.4.4, the centre $Z(G)$ acts by the identity. □

As a consequence, we get a very short proof of a result of J.-P. Serre (see Theorem 1.7.11 in [BdlHV08]).

Corollary 1.4.6. *Let G be a compactly generated, locally compact group. Assume that the separated abelianisation $G/\overline{[G,G]}$ is compact. Let Z be a closed central subgroup of G. If G/Z has property (T), then so does G.*

Proof. Our assumption implies that $\mathrm{Hom}(G,\mathbb{R}) = 0$. Assume by contraposition that G does not have property (T), according to Proposition 1.2.11, G admits an irreducible affine action α. By Corollary 1.4.5, this action α is actually an irreducible affine action of G/Z, which therefore does not have property (T). □

The *finite conjugacy centre* (FC-centre) of G, denoted $\mathrm{FC}(G)$, is the set of elements in G whose conjugacy class is finite. Observe that the conjugacy class of an element g is finite if and only its centraliser $C_G(g)$ has finite index in G, so that we can think of the FC-centre of G as elements being *virtually* in its centre. The FC-centre is a characteristic subgroup of G.

Observe that the FC-centre of any group G is amenable. Indeed, every finitely generated subgroup $\Gamma = \langle S \rangle$ of $\mathrm{FC}(G)$ has a centre of finite index since

$$Z(\Gamma) = \mathrm{FC}(G) \cap \bigcap_{s \in S} C_G(s).$$

It follows that $\mathrm{FC}(G)$ is locally virtually abelian, hence amenable.

Proposition 1.4.7. *Let α be an irreducible affine action of the topological group G on $\mathcal{H}$. The linear part of α is trivial on the FC-centre of G; more precisely, every $g \in \mathrm{FC}(G)$ acts as a translation in the direction of $\mathcal{H}^{\pi(C_G(g))}$.*

Proof. Let $g \in \mathrm{FC}(G)$. Since $C_G(g)$ is a closed subgroup with finite index, by Theorem 1.4.1, the restriction of α to $C_G(g)$ is irreducible. Hence, by Proposition 1.4.4, $\alpha(g)$ is a translation by a vector fixed by $C_G(g)$. □

A group G is called an *FC-group* if $G = \mathrm{FC}(G)$. The following result is an immediate consequence of Proposition 1.4.7.

Proposition 1.4.8. *Let G be an FC-group. Every irreducible affine action of G on $\mathcal{H}$ is given by a homomorphism $b : G \to \mathcal{H}$ such that* $\mathrm{span}(b(G))$ *is dense.* $\square$

We now show that a result similar to Corollary 1.4.6 holds for discrete groups satisfying the following property introduced in [LZ05].

Definition 1.4.9. A discrete group Γ has property (FAb) if, for every subgroup H of finite index of Γ, we have $\mathrm{Hom}(H, \mathbb{R}) = 0$.

It is shown in [LZ05, Proposition 1.30] that Γ has property (FAb) if and only if $H^1(\Gamma, \pi) = 0$ for every complex representation π of Γ with finite image.

Corollary 1.4.10. *Let Γ be a group with property (FAb). Then every irreducible affine action α of Γ factors through $\Gamma/\,\mathrm{FC}(\Gamma)$.*

Proof. The proof is similar to the proof of Corollary 1.4.5. $\square$

We obtain from the previous result the following extension of Serre's result from Corollary 1.4.6, with a similar proof.

Corollary 1.4.11. *Let Γ be countable discrete group with property (FAb). If $\Gamma/\,\mathrm{FC}(\Gamma)$ has property (T), then so does Γ.* $\square$

1.4.3 Abelian groups

In this section, A will denote a topological abelian group, written additively. Since A is an FC-group, we have from Proposition 1.4.4, that every irreducible affine action of A on $\mathcal{H}$ is given by a continuous homomorphism $b : A \to \mathcal{H}$ such that $\mathrm{span}(b(A))$ is dense.

Definition 1.4.12. (see [For76]) A continuous function $Q : A \to \mathbb{R}^+$ is a *non-negative quadratic form* if $Q(x+y)+Q(x-y) = 2(Q(x)+Q(y))$ for every $x, y \in A$.

Lemma 1.4.13. *A continuous, non-negative function Q on A is a quadratic form if and only if there exists a Hilbert space $\mathcal{K}$ and a continuous homomorphism $\beta : A \to \mathcal{K}$ such that $Q(x) = \|\beta(x)\|^2$ for every $x \in A$.*

Proof. If $Q(x) = \|\beta(x)\|^2$, then Q is a quadratic form due to the well-known parallelogram law in Hilbert spaces. Conversely, consider the tensor product of modules $V = A \otimes_{\mathbb{Z}} \mathbb{R}$ and observes that V has a real vector space structure given by $\mu \cdot (x \otimes \lambda) = x \otimes (\mu\lambda)$. Define a non-negative quadratic form on V by the formula

$$\tilde{Q}(\sum_i x_i \otimes \lambda_i) = \sum_i \lambda_i^2 Q(x_i) - \sum_{i<j} \lambda_i \lambda_j Q(x_i - x_j).$$

Observe that definition 1.4.12 implies $Q(x) = Q(-x)$ and $Q(nx) = n^2 Q(x)$ so that $\tilde{Q}$ is well-defined. Define $\mathcal{K}$ as the separation-completion of V with respect to the semi-norm $\|v\|^2 = \tilde{Q}(v)$. The map

$$\beta : \; A \to \mathcal{K}, \, x \mapsto x \otimes 1$$

defines a continuous homomorphism from A to the additive group of $\mathcal{K}$ as claimed. □

Recall that, for an affine isometric action α with cocycle b, we denote $\psi(.) = \|b(.)\|^2$.

Proposition 1.4.14. *Let α be an affine action of A, with $b(A)$ total in $\mathcal{H}$. The following properties are equivalent:*

i) α is irreducible;

ii) ψ is a quadratic form.

Proof. $(i) \Rightarrow (ii)$ follows immediately from Lemma 1.4.13. For $(ii) \Rightarrow (i)$, write $\psi(x) = \|\beta(x)\|^2$, with $\beta : A \to \mathcal{K}$ a continuous homomorphism, as in Lemma 1.4.13. Clearly we may assume that $\beta(A)$ is total in $\mathcal{H}$. The actions α and β (viewed as an action by translations) both have total cocycle and define the same function conditionally of negative type, so they are conjugate by an A-equivariant affine isometry according to Proposition 1.2.5. □

1.4.4 Nilpotent groups and FC-nilpotent groups

The following result generalises Corollary 5 in [Gui72], stating that for a nilpotent locally compact group, any non-trivial unitary irreducible representation has zero 1-cohomology.

Proposition 1.4.15. *Let G be a nilpotent group. Any irreducible affine action α of G on $\mathcal{H}$ is given by a continuous homomorphism $b : G \to \mathcal{H}$ such that* $\operatorname{span}(b(G))$ *is dense.*

Proof. We proceed by induction on the nilpotency rank r of G, the case $r = 1$ being Proposition 1.4.8. For the general case, let α be an irreducible affine action of G, it is enough to show that π is the trivial representation, i.e. $\mathcal{H}^{\pi(G)} = \mathcal{H}$. Assume it is not the case, and let α_0 be the projected action on the orthogonal complement of $\mathcal{H}^{\pi(G)}$. By condition (A6) in Proposition 1.2.9, the action α_0 is irreducible. Since its linear part π_0 has no non-zero fixed vector, by Proposition 1.4.4 the centre $Z(G)$ acts trivially in α_0, i.e. α_0 factors through $G/Z(G)$. By induction hypothesis α_0 is an action by translations, meaning that π_0 is the trivial representation of $G/Z(G)$. This contradiction ends the proof. □

The ascending FC-central series $(G_i)_i$ of a group G is defined inductively as follows: $G_1 = \mathrm{FC}(G)$ and G_{i+1} is the inverse image of $\mathrm{FC}(G/G_i)$ under the canonical map $G \to G/G_i$ for every $i \geq 1$. If $G_n = G$ and $G_{n-1} \neq G$, then G is said to be *FC-nilpotent* of rank n. Examples of FC-nilpotent groups include nilpotent-by-finite groups and (arbitrary) direct sums of finite groups.

Proposition 1.4.15 cannot be extended to the class of FC-nilpotent groups. The previous proof fails because, when we know that $Z(G)$ acts by translation only along $\mathcal{H}^{\pi}(G)$, $\mathrm{FC}(G)$ has no such constraint. The next proposition is the proper statement for FC-nilpotent groups

Corollary 1.4.16. *Let G be an FC-nilpotent group and let α be an irreducible affine action of G on a Hilbert space $\mathcal{H}$, with linear part π. Then π can be decomposed as a direct sum $\pi = \bigoplus_i \pi_i$, where each π_i*

is a unitary representation of G which factors through a finite quotient of G.

We start with an intermediate lemma.

Lemma 1.4.17. *Let α be an irreducible action of a group G and assume that the linear part of α does not contain any subrepresentation with finite image. Then $\mathrm{FC}(G)$ acts trivially, so that α factors through $G/\mathrm{FC}(G)$.*

Proof. Let $g \in \mathrm{FC}(G)$, from Proposition 1.4.7 we already know that $\alpha(g)$ is a translation in a $C_G(g)$-fixed direction. Let N_g be a normal subgroup of finite index contained in $C_G(g)$. The space $\mathcal{H}^{\pi(N_g)}$ is $\pi(G)$-invariant and the corresponding subrepresentation factors trough the quotient map $G \to G/N_g$. By hypothesis, we obtain that $\mathcal{H}^{\pi(N_g)} = \{0\}$ so that $\mathcal{H}^{\pi(C_G(g))} = \{0\}$. Hence $\alpha(g)$ is the identity. □

Proof of the proposition. We proceed by induction on the FC-nilpotency rank r of G. When $r = 1$, the group G is an FC-group and the claim follows from Proposition 1.4.8.

Let $r \geq 2$. Denote by $\mathcal{K}$ the closed linear space of $\mathcal{H}$ generated by all subrepresentations of π which factor through a finite quotient. It is clear by Zorn's lemma that the restriction of π to $\mathcal{K}$ can be decomposed as a direct sum $\bigoplus_i \pi_i$, where each π_i is a subrepresentation of π which factors through a finite quotient of G. The claim will be proved if we can show that $\mathcal{K} = \mathcal{H}$.

Denote by α_0 the projected action on the orthogonal complement $\mathcal{K}^\perp$ of $\mathcal{K}$. By condition (A6) in Proposition 1.2.9, the action α_0 is irreducible and its linear part π_0 does not contain any subrepresentation with finite image. By the previous lemma, α_0 factors through $G/\mathrm{FC}(G)$ which is FC-nilpotent of rank $r - 1$. By induction hypothesis, α_0 decomposes as a direct sum of finite image representations, hence $\mathcal{K}^\perp = \{0\}$ and $\mathcal{H} = \mathcal{K}$. □

Example 1.4.18. Consider the infinite dihedral group $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. D_∞ is *FC*-nilpotent of rank 2, its FC-central series being $\{1\} \lhd \mathbb{Z} \lhd D_\infty$. Consider the faithful realisation of D_∞ as the symmetry group of the lattice of integers in $\mathbb{R}$. It gives rise to an irreducibe affine action with non trivial linear part factoring through the quotient $\mathbb{Z}/2\mathbb{Z}$.

1.5 The left regular representation of a discrete group

In this section, we investigate the existence of an irreducible action of a discrete group Γ with linear part the left regular representation λ_Γ. We will also consider the question for Γ-invariant subspaces of countably many copies of $\ell^2(\Gamma)$.

Assume Γ is ICC (that is, every conjugacy class of Γ is infinite), so that the left (or right) von-Neumann algebra $L(\Gamma)$ of Γ is a factor. Every separable Hilbert module $\mathcal{H}$ over $L(\Gamma)$ is characterised by its von Neumann dimension $\dim_{L(\Gamma)}(\mathcal{H})$ and can be realised as a closed Γ-invariant subspace of a direct sum of countably many copies of $\ell^2(\Gamma)$. For $t > 0$, we denote by H_t the unique $L(\Gamma)$-module of dimension t and by λ_t the corresponding unitary representation of Γ. For the special case $t = 1$, we have $\lambda_t = \lambda_\Gamma$.

Recall that the first L^2-Betti number of Γ is

$$\beta^1_{(2)}(\Gamma) = \dim_{L(\Gamma)} H^1_{(2)}(\Gamma),$$

where $\mathcal{H}^1_{(2)}$ denotes the first L^2-cohomology group of Γ. For the purpose of this text, we are not interested in the exact definition of these groups, we just observe that in the case Γ is non-amenable, we have the following identification [BV97]:

$$H^1_{(2)}(\Gamma) \cong \overline{H^1}(\Gamma, \lambda_\Gamma) \cong H^1(\Gamma, \lambda_\Gamma).$$

In the rest of the section and to simplify notations, we will often simply write $H^1(\pi)$ for $H^1(\Gamma, \pi)$.

We get the following:

Proposition 1.5.1. *Let Γ be a non-amenable, discrete, ICC group. The following statements are equivalent:*

i) There exists an irreducible affine isometric action of Γ *with linear part* λ_t.

ii) $\beta^1_{(2)}(\Gamma) \geq t$.

Proof. Recall from Corollary 1.3.6 that there exists an irreducible affine action with linear part λ_t if and ony if the $\lambda_t(\Gamma)'$-module $H^1(\lambda_t)$ has a separating vector. In the case that a von-Neumann algebra $\mathcal{M}$ is a II_1-factor it follows from Proposition 10.2.6. in [Jon] that an $\mathcal{M}$-module $\mathcal{H}$ has a separating vector if and only if $\dim_{\mathcal{M}}(\mathcal{H}) \geq 1$. From now on, set $\mathcal{M} = L(\Gamma)$, we need to identify $\lambda_t(\Gamma)'$ in terms of $\mathcal{M}$ and to compute $\dim_{\lambda_t(\Gamma)'} H^1(\lambda_t)$.

Fix k an integer larger than t. We realise $\mathcal{H}_t$ as a subspace of $\ell^2(\Gamma) \otimes \mathbb{C}^k$ and denote by p the orthogonal projection onto $\mathcal{H}_t$. The commutant of $\lambda_\Gamma \otimes 1_k$ (this is just a direct sum of k copies of λ_Γ) is $\mathcal{M} \otimes M_k =: M_k(\mathcal{M})$, the algebra of $k \times k$ matrices over $\mathcal{M}$. The commutant of its restriction λ_t to $\mathcal{H}_t$ is then given by $pM_k(\mathcal{M})p$ and the 1-cohomology of λ_t is

$$H^1(\lambda_t) = H^1(p(\lambda_\Gamma \otimes 1_k)) = pH^1(\lambda_\Gamma \otimes 1_k).$$

Now, by Proposition 10.2.1 in [Jon]

$$\dim_{pM_k(\mathcal{M})p}(pH^1(\lambda_\Gamma \otimes 1_k)) = \tau'(p)^{-1} \dim_{M_k(\mathcal{M})} H^1(\lambda_\Gamma \otimes 1_k),$$

where τ' is the normalized trace on $\ell^2(\Gamma) \otimes \mathbb{C}^k$ (so that $\tau'(p) = \frac{t}{k}$). Moreover,

$$\dim_{M_k(\mathcal{M})} H^1(\lambda_\Gamma \otimes 1_k) = \frac{1}{k} \dim_{\mathcal{M}} H^1(\lambda_\Gamma) = \frac{\beta^1_{(2)}(\Gamma)}{k}.$$

Finally, we obtain

$$\dim_{\lambda_t(\Gamma)'} H^1(\lambda_t) = \frac{\beta^1_{(2)}(\Gamma)}{t}.$$

This concludes the proof. □

This proposition allows us to get the following definition of the first L^2-Betti number of an ICC group:

Corollary 1.5.2. *For Γ a non-amenable discrete ICC group, we have*

$$\beta^1_{(2)}(\Gamma) = \sup\{t \geq 0 \,|\, \text{ there exists an irreducible affine action with linear part } \lambda_t\}.$$

□

Example 1.5.3. The group $PSL_2(\mathbb{Z})$ is ICC and satisfies $\beta^1_{(2)}(PSL_2(\mathbb{Z})) = \frac{1}{6}$ (see Section 4 in [CG86]), so there exists no irreducible affine action with linear part the left regular representation.

For the free group $\mathbf{F}_n$ on n generators ($2 \leq n \leq +\infty$), we have $\beta^1_{(2)}(\mathbf{F}_n) = n - 1$ (see [CG86]) and it is possible to construct explicit irreducible affine isometric actions with linear part $\lambda_{\mathbf{F}_n}$. Indeed, let $(a_i)_{1 \leq i \leq n}$ be a free generating family of $\mathbf{F}_n$. Set $b(a_1) = \delta_1$ (the characteristic function of the identity of $\mathbf{F}_n$), and $b(a_i) = 0$ for $i \geq 2$. Since $\mathbf{F}_n$ is free, we may extend uniquely b to a 1-cocycle $b \in Z^1(\mathbf{F}_n, \lambda_{\mathbf{F}_n})$. It is easily seen that, for $k \geq 0$, we have $b(a_1^k) = \sum_{i=0}^{k-1} \delta_{a_1^i}$, so that b is unbounded.

Proposition 1.5.4. *For b as above, the affine isometric action of $\mathbf{F}_n$ on $\ell^2(\mathbf{F}_n)$ given by $\alpha(g)v = \lambda_{\mathbf{F}_n}(g)v + b(g)$, is irreducible.*

Proof. Let $Av = Tv + t$ be an affine transformation of $\ell^2(\mathbf{F}_n)$ in the commutant of α. Then $T \in R(\mathbf{F}_n)$ and $(T - 1)b(g) = \lambda_{\mathbf{F}_n}(g)t - t$ for every $g \in \mathbf{F}_n$. For $g = a_2$, we get $\lambda_{\mathbf{F}_n}(a_2)t = t$, hence $t = 0$ since a_2 has infinite order. So $(T - 1)b(g) = 0$ for every g. For $g = a_1$, this gives $(T - 1)\delta_1 = 0$, hence $T = 1$ since δ_1 is separating for $R(\mathbf{F}_n)$. By Proposition 1.3.5, the action α is irreducible. □

In the non-ICC case the previous statement essentially fails on a large scale. We have the following statement

Theorem 1.5.5. *Let Γ be a non-amenable, finitely generated group, and let $\mathcal{H}$ be a non-zero Hilbert $L(\Gamma)$-module with finite von Neumann dimension. Denote by $\lambda_{\mathcal{H}}$ the corresponding unitary representation of Γ in $\mathcal{H}$. The following properties are equivalent:*

i) there exists an irreducible affine isometric action of Γ with linear part $\lambda_{\mathcal{H}}$;

ii) $\mathrm{FC}(\Gamma)$ is finite, $\mathrm{FC}(\Gamma)$ acts trivially on $\mathcal{H}$, and

$$\beta^1_{(2)}(\Gamma/\,\mathrm{FC}(\Gamma)) \geq \dim_{L(\Gamma/\,\mathrm{FC}(\Gamma))}\mathcal{H}.$$

Proof. In the case where $\mathrm{FC}(\Gamma)$ is trivial, we are done by Proposition 1.5.1. So assume $\mathrm{FC}(\Gamma)$ is not trivial.
(i)$\Rightarrow$(ii): Observe that $\Gamma/\,\mathrm{FC}(\Gamma)$ is ICC and non-amenable, since $\mathrm{FC}(\Gamma)$ is amenable, but Γ is not. Suppose there exists an irreducible action α with linear part $\mathcal{H}_t$. By Proposition 1.4.7, we know that $\mathrm{FC}(\Gamma)$ acts by translations. Since $\lambda_{\mathcal{H}}$ is a subrepresentation of a multiple of the regular representation, the only possibility for the restriction of $\lambda_{\mathcal{H}}$ to $\mathrm{FC}(\Gamma)$ to be trivial is if $\mathrm{FC}(\Gamma)$ is finite[3]. Now, since the linear action of $\mathrm{FC}(\Gamma)$ is trivial, $\mathcal{H}$ can be seen as an $R(\Gamma/\,\mathrm{FC}(\Gamma))$-module and we get

$$\beta^1_{(2)}(\Gamma/\,\mathrm{FC}(\Gamma)) \geq \dim_{L(\Gamma/\,\mathrm{FC}(\Gamma))}\mathcal{H}$$

from Proposition 1.5.1.
(ii)$\Rightarrow$(i): assume that $\mathrm{FC}(\Gamma)$ is finite, that $\mathrm{FC}(\Gamma)$ acts trivially on $\mathcal{H}$, and that

$$\beta^1_{(2)}(\Gamma/\,\mathrm{FC}(\Gamma))) \geq \dim_{L(\Gamma/\,\mathrm{FC}(\Gamma))}\mathcal{H}.$$

$\mathcal{H}$ can be seen as an $R(\Gamma/\,\mathrm{FC}(\Gamma))$-module and Proposition 1.5.1 provides an action of $\Gamma/\,\mathrm{FC}(\Gamma)$ with linear part given by $\lambda_{\mathcal{H}}$. This action easily goes up to Γ which concludes the proof. □

The following corollary is immediate :

[3]In that case, $\mathcal{H}$ is contained in a multiple of the subspace of $\ell^2(\Gamma)$ of functions which are constant on the orbits of $\mathrm{FC}(\Gamma)$.

Corollary 1.5.6. *Let Γ be a non-amenable, finitely generated group such that* $\mathrm{FC}(\Gamma)$ *is infinite. No non-zero $L(\Gamma)$-module $\mathcal{H}$ has an irreducible affine isometric action with linear part $\lambda_{\mathcal{H}}$.*

Although our methods do not apply in the amenable case, Andreas Thom was able to prove the following (see [BPV14] for a full proof):

Theorem 1.5.7. *Let Γ be a discrete, amenable group. Let α be an affine isometric action of Γ, with linear part λ_Γ. For every $\varepsilon > 0$, the action α admits a closed, affine invariant subspace $\mathcal{H}_\varepsilon$ such that the linear part $\mathcal{H}_\varepsilon^0$ satisfies $\dim_{L(\Gamma)} \mathcal{H}_\varepsilon^0 < \varepsilon$. In particular, there is no irreducible affine action of Γ with linear part λ_Γ.*

1.6 Direct sums of irreducible actions

For affine isometric actions α_1, α_2 of a group G, we may consider in an obvious way the direct sum $\alpha_1 \oplus \alpha_2$. Unlike the direct sum of unitary representations, which is always reducible, it may happen that the direct sum of two affine isometric actions is irreducible. For instance, if β_1, β_2 are linearly independent homomorphisms $G \to \mathbb{C}$, then $\beta_1 \oplus \beta_2$ defines an irreducible affine isometric action of G on $\mathbb{C}^2$. On the other hand, if α is any affine isometric action of G, then $\alpha \oplus \alpha$ is not irreducible (look at the diagonal). We shall give a sufficient and necessary condition for the direct sum of two irreducible actions to be irreducible.

In order to state the main result of this section (Theorem 1.6.2 below) we need to clarify the notion of equivalence between affine isometric actions.

Definition 1.6.1. Let α_1 and α_2 be two affine isometric actions of a group G. We say that α_1 and α_2 are equivalent if they are intertwined by an invertible continuous affine mapping. That is, if there exists an invertible continuous affine mapping $A : \mathcal{H}_{\alpha_1} \to \mathcal{H}_{\alpha_2}$ satisfying:

$$A\alpha_1(g) = \alpha_2(g)A, \quad \text{for all } g \in G.$$

If we write $A(\cdot) = T(\cdot) + t$ and $\alpha_i(g)(\cdot) = \pi_i(g)(\cdot) + b_i(g)$, the above definition boils down to $T\pi_1(g) = \pi_2(g)T$ and $Tb_1(g) = b_2(g) + \pi_2(g)t - t$ for all $g \in G$.

Since the actions are by isometries, it may seem more natural to require the intertwining in the definition of equivalence to be given by an isometric operator, in which case we would say that the actions are isometrically equivalent. To motivate our definition, one should be reminded of the similar definition for unitary representations. It is well-known that, in that case, an equivalence can always be implemented via a unitary intertwiner. This is a consequence of the fact that every invertible intertwiner can be "straightened" by replacing it with its unitary part (see e.g. [BdlHV08, Appendix A.1]). However, this fails for affine isometric actions: equivalent affine actions by isometries need not be isometrically equivalent [4].

Theorem 1.6.2. *Let α_1, α_2 be irreducible affine isometric actions of a group G. The following properties are equivalent:*

i) $\alpha_1 \oplus \alpha_2$ is reducible,

ii) α_1 and α_2 admit equivalent projected actions.

Before proving this theorem, we pinpoint two specific cases, important enough to be considered on their own.

Recall that two unitary representations π, σ of G are said to be *disjoint* if $\mathrm{Hom}_G(\mathcal{H}_\pi, \mathcal{H}_\sigma) = 0$.

Proposition 1.6.3. *Let $\alpha_1, \ldots, \alpha_k$ be irreducible affine actions of G, with linear parts $\pi_1, \ldots, \pi_k$. Assume that the π_i's are pairwise disjoint. Then the direct sum $\alpha := \alpha_1 \oplus \cdots \oplus \alpha_k$ is irreducible.*

Proof. Let $b = (b_1, \ldots, b_k)$ be the 1-cocycle defining α. Let $Av = Tv + t$ be a continuous affine mapping in the commutant of α. Write T as a

[4] As an example, consider two actions of $\mathbb{Z}$ on $\mathbb{R}$, the first one by integer translations, the second one by even translations. These actions are equivalent in our sense, but clearly they are not isometrically equivalent.

$k \times k$-matrix $(T_{ij})_{1\le i,j\le k}$ where T_{ij} is a bounded operator $\mathcal{H}_{\pi_j} \to \mathcal{H}_{\pi_i}$; similarly, write $t = (t_1, \dots, t_k)$. Since T belongs to the commutant of $\pi_1 \oplus \cdots \oplus \pi_k$, we have $T_{ij} \in \mathrm{Hom}_G(\mathcal{H}_{\pi_j}, \mathcal{H}_{\pi_i})$ and hence $T_{ij} = 0$ for $i \neq j$. The relation $(T-1)b(g) = \partial_t(g)$ then gives

$$(T_{ii} - 1)b_i(g) = \partial_{t_i}(g) \quad \text{for } 1 \le i \le k \quad \text{and } g \in G.$$

This means that the affine map $A_i v =: T_{ii}v + t_i$ is in the commutant of α_i. Since the latter is irreducible, we get $T_{ii} = 1$; hence $T = 1$ and α is irreducible. □

For π a unitary representation of G and $k \in \mathbb{N}$, we denote by $k \cdot \pi$ the representation $\pi \oplus \cdots \oplus \pi$ (k times).

Proposition 1.6.4. *Let π be an irreducible unitary representation of G on a complex Hilbert space $\mathcal{H}$. Let $b_1, \dots, b_k$ be elements in $Z^1(G, \pi)$ whose classes $[b_1], \dots, [b_k]$ are linearly independent in $H^1(G, \pi)$. Then the affine isometric action $\alpha = \bigoplus_{i=1}^k \alpha_{\pi, b_i}$ is irreducible.*

Proof. Let $Av = Tv + t$ be a continuous affine mapping in the commutant of α. In view of Proposition 1.3.5, we have to show that A is a translation, that is, $T = 1$. We know that T is in the commutant of $k \cdot \pi$ and that $(T-1)b = \partial t$, where $b = \oplus_{i=1}^k b_i$.

Write T as a $k \times k$-matrix $(T_{ij})_{1\le i,j\le k}$, where T_{ij} is a bounded operator $\mathcal{H}_\pi \to \mathcal{H}_\pi$. Then every T_{ij} interwines π with itself and hence $T_{ij} = \lambda_{ij} 1$ for some $\lambda_{ij} \in \mathbf{C}$, by Schur's lemma. On the other hand, since

$$H^1(G, k\cdot\pi) = \underbrace{H^1(G,\pi) \oplus \cdots \oplus H^1(G,\pi)}_{k \text{ times}},$$

we have

$$(T-1)\begin{pmatrix} [b_1] \\ \vdots \\ [b_k] \end{pmatrix} = 0;$$

since the $[b_i]$'s are linearly independent, we deduce that $T = 1$. □

Remark 1.6.5. This proposition is not valid anymore in the real case since the proof relies heavily on Schur's lemma for unitary representations. It is easy enough to produce a counterexample by using an irreducible orthogonal representation with non-trivial commutant.

Proof of Theorem 1.6.2. Denote by π_1, b_1 and π_2, b_2 the linear and translation parts of the actions α_1 and α_2.

$(ii) \Rightarrow (i)$ By hypothesis, there exist non zero $(\pi_1 \oplus \pi_2)(G)$-invariant closed linear subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\mathcal{H}_{\pi_i}$ such that the projected actions of α_1 and α_2 on $\mathcal{K}_1$ and $\mathcal{K}_2$ are equivalent. Let $A : \mathcal{K}_1 \to \mathcal{K}_2$ be a continuous affine, invertible map implementing the equivalence. Then the graph of A is a proper closed, invariant, affine subspace of the projected action of $\alpha_1 \oplus \alpha_2$ onto $\mathcal{K}_1 \oplus \mathcal{K}_2$. This contradicts characterisation (A6) of irreducibility from Proposition 1.2.9.

$(i) \Rightarrow (ii)$ Since $\alpha_1 \oplus \alpha_2$ is reducible, we can find, by (A3) from Proposition 1.2.9, a non-zero closed linear subspace $\mathcal{K}$ of $\mathcal{H}_{\pi_1} \oplus \mathcal{H}_{\pi_2}$ which is invariant under $(\pi_1 \oplus \pi_2)(G)$ and such that the projection of $b = b_1 \oplus b_2$ on $\mathcal{K}$ is bounded. Upon conjugating $\alpha = \alpha_1 \oplus \alpha_2$ by a translation, we may assume that the projection of b on $\mathcal{K}$ is 0.

Denote by $P_i : \mathcal{K} \to \mathcal{H}_{\pi_i}$ the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}_{\pi_i}$. We may also assume that $P_i(\mathcal{K})$ is dense in $\mathcal{H}_{\pi_i}$ for $i = 1, 2$; otherwise, we can replace α by its projected action on $\overline{P_1(\mathcal{K})} \oplus \overline{P_2(\mathcal{K})}$.

Next, observe that $\mathcal{K}$ is transverse to the $\mathcal{H}_{\pi_i}$'s. Indeed, if the intersections $\mathcal{K} \cap \mathcal{H}_{\pi_i}$ were non-zero, the projection of b_i on $\mathcal{K} \cap \mathcal{H}_{\pi_i}$ being bounded, this would contradict the irreducibility of α_i. So, P_1 and P_2 are injective. We can therefore consider the densely defined, unbounded, invertible closed operator $S = P_2 P_1^{-1}$ (for background about unbounded operators, see e.g. [Ped89, Chap. 5]). Note that $\mathcal{K}$ being $(\pi_1 \oplus \pi_2)(G)$-invariant, it is immediate that the domain $\mathcal{D}(S)$ of S is $\pi_1(G)$-invariant, that its range is $\pi_2(G)$-invariant and that S intertwines the corresponding two subrepresentations of π_1 and π_2 (on non-closed subspaces!). Now, recall that, for every $g \in G$, the vector $b(g) = b_1(g) \oplus b_2(g)$ is orthogonal to $\mathcal{K}$; hence, we have

$$\langle b_1(g), v \rangle + \langle b_2(g), Sv \rangle = 0 \quad \text{for all } v \in \mathcal{D}(S).$$

This relation implies that

$$|\langle b_2(g), Sv\rangle| = |\langle b_1(g), v\rangle| \leq \|b_1(g)\|\|v\|;$$

hence $b_2(g)$ belongs to the domain of S^* and $b_1(g) = -S^* b_2(g)$ for all $g \in G$. This shows that $-S^*$ intertwines α_2, projected on the domain of S^*, and α_1. The closed operator S^* has a polar decomposition $-S^* = UT$, where $U : \mathcal{H}_{\pi_2} \to \mathcal{H}_{\pi_1}$ is unitary and $T : \mathcal{D}(S) \to \mathcal{H}_{\pi_2}$ is a positive unbounded closed operator. Let B be a bounded Borel subset of the spectrum of T with positive measure, and denote by P_B the corresponding spectral projector. Then $-S^* P_B$ is a bounded operator and provides an equivalence between α_2 projected on $\mathrm{Im}(P_B)$ and α_1 projected on $\mathrm{Im}(S^* P_B)$. This concludes the proof. $\square$

Chapter 2

Compression exponents

2.1 Introduction

This chapter is devoted to the study of metric invariants of groups and metric spaces called *compression exponents.* These appear as quantitative measurements on coarse embeddings. Let's first recall the potential behaviours of maps between metric spaces.

Definition 2.1.1. Let $f : X \to Y$ be a map between two metric spaces and define the maps $\rho_f, \omega_f : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ by

$$\begin{aligned} \rho_f(t) &= \inf\{d\,(f(x), f(y)) \mid d(x,y) \geq t\} \\ \omega_f(t) &= \sup\{d\,(f(x), f(y)) \,|d(x,y) \geq t\} \end{aligned}$$

ρ_f is called the *compression modulus* of f and ω_f its *expansion modulus.* They are the optimal functions satisfying

$$\rho_f\,(d(x,y)) \leq d\,(f(x), f(y)) \leq \omega_f\,(d(x,y))\,.$$

Note that in this definition, we have absolutely no requirement on f. In particular, continuity is never assumed in this context. Maps are classified in terms of the possible behaviours of ρ_f and ω_f.

Definition 2.1.2.

- The map f is *coarse* if $\lim\limits_{t\to+\infty} \rho_f(t) = +\infty$ and $\omega_f(t) < \infty$.

- The map f is *large-scale Lipschitz* if there exists $C > 0$ with $\omega_f(t) \leq Ct + C$.
- The map f is *quasi-isometric* if it is large-scale Lipschitz and there exists $C > 0$ such that $\rho_f(t) \geq Ct - C$.
- The map f is *Lipschitz* if there exists $C > 0$ with $\omega_f(t) \leq Ct$. Moreover it is *1-Lipschitz* if C can be chosen equal to 1.

There are natural equivalence notions for both coarse maps and quasi-isometric maps.

Definition 2.1.3. Two maps $f_1, f_2 : X \to Y$ are *close* if there exists $C \geq 0$ such that

$$d(f_1(x), f_2(x)) \leq C, \forall x \in X.$$

- A map $f : X \to Y$ is a *coarse equivalence* if there exists a coarse map $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are close to the identity maps on X and Y.
- A map $f : X \to Y$ is a *quasi-isometry* if there exists a quasi-isometric map $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are close to the identity maps on X and Y.

The study of quasi-isometry amongst metric spaces, and especially groups, is by now very classical. Quasi-isometry is the natural equivalence relation to work with groups as metric spaces since it makes metrics arising from different generating sets equivalent. The notion of coarse equivalence seems broader but actually is not in a large class of interesting spaces.

Definition 2.1.4. A metric space (X, d) is *quasi-geodesic* if there exist $\delta > 0$, $L > 0$ such that for any $x, y \in X$ we can find a sequence of points $x = x_0, x_1, x_2, \ldots, x_n = y$ satisfying

$$d(x_i, x_{i+1}) \leq \delta \text{ and } n \leq L \cdot d(x, y).$$

To us, the most important examples of a quasi-geodesic metric will be shortest-path metrics on graphs and word metrics on groups. The following proposition is part of folklore (see e.g. [Gro93], [GK04, Prop. 2.9.])

Proposition 2.1.5. *Let X be a quasi-geodesic metric space, then any coarse map from X to any metric space Y is large-scale Lipschitz.* □

Corollary 2.1.6. *Amongst quasi-geodesic metric spaces (in particular amongst finitely generated groups), the notions of coarse equivalence and quasi-isometry coincide.* □

Although coarse equivalence is not relevant in many cases, the question of coarse-embeddability is. In 1995, M. Gromov [Gro95] suggested that metric spaces admitting a coarse embedding into a Hilbert space should always satisfy the coarse Baum-Connes conjecture. In the case of Cayley graphs of finitely generated groups, this implies the Novikov conjecture. This claim was later proved by Yu [Yu00] and generalized by Kasparov and Yu [KY06] to uniformly convex Banach spaces as target spaces. Since then, an extensive study of coarse embeddabiliy has been made. Let us mention a few important results.

Proposition 2.1.7 ([DG03])**.** *The class of groups admitting a coarse embedding into a Hilbert space is closed under*

- *Taking subgroups,*
- *Direct products,*
- *Direct limits,*
- *Free products with amalgamation,*
- *Extensions by exact groups,*
- *HNN extensions.*

Also let us note that there are very few examples of metric spaces or groups that do not admit coarse embeddings into Hilbert spaces. For a long time, the only example amongst metric spaces was provided by expander families. Recently Arzhantseva and Tessera [AT15] provided the first non-expanding example of a space not admitting such an embedding. In the case of groups, Gromov [Gro00], using random methods, showed the existence of a finitely generated group not coarsely embedded into a Hilbert space (see [AD] for a more complete treatment). More recently, Osajda [Osa] provided groups containing isometrically embedded expanders.

Several tools and properties have been developed to study coarse embeddability. Let us mention one of the most important : Yu's property A :

Definition 2.1.8 ([Yu00])**.** Let X be a uniformly discrete metric space. We say that X has *Property A* if for every $\varepsilon > 0$ and $R > 0$, there exists a collection $(A_x)_{x\in X}$ of finite subsets of $X \times \mathbb{N}$ and $S > 0$ such that

(a) $\dfrac{\# A_x \triangle A_y}{\# A_x \cap A_y} \leq \varepsilon$ whenever $d(x, y) \leq R$, and

(b) $A_x \subset B(x, S) \times \mathbb{N}$.

Property A appears as a non-equivariant version of amenability and time has proved that it is *the* good definition of amenability in metric spaces. It turned out to be equivalent to several reformulations [CTWY08, BNŠ+13, Sak14, RW14] , and in case the metric space is the Cayley graph of a finitely generated group, to the exactness of the group reduced C*-algebra [GK04]. The definition was shaped to satisfy the following result

Proposition 2.1.9 ([Yu00])**.** *Every metric space with property A admits a coarse embedding into a Hilbert space.*

The reciprocal statement was an open question for a while before Nowak [Now07] showed that the disjoint union of infinitely many cubes of growing dimension provides a counter-example. Later, Arzhantseva, Guentner and Spakula [AGŠ12] came up with a counter-example of bounded geometry, namely infinite families of quotients of the free group on two generators. Finally, the existence of a group admitting a coarse embedding into a Hilbert space but not enjoying property A was proved by Osajda [Osa].

In their study of the equivalence between Property A for groups and exactness, Guentner and Kaminker introduced compression exponents to prove a partial converse to the previous proposition.

Definition 2.1.10.

- Let $f : X \to Y$ be a coarse map, the *compression exponent* $\text{comp}(f)$ of f is the supremum of all $\alpha \geq 0$ for which there exists $C > 0$ with
$$d(f(x), f(y))) \geq \frac{1}{C} d(x,y)^{\alpha} - C, \ \forall x, y \in X.$$
Equivalently, $\text{comp}(f) = \liminf_{t\to\infty} \frac{\log \rho_f(t)}{\log t}$.

- Let X be a metric space and $\mathcal{Y}$ be a family of metric spaces, the $\mathcal{Y}$*-compression exponent* of X is defined by
$$\alpha^*_{\mathcal{Y}}(X) = \sup_f \{\text{comp}(f)\},$$
where the supremum is taken over all large-scale Lipschitz maps from X to an element of $\mathcal{Y}$.

- Let G be a finitely generated group equipped with the word metric associated to some finite generating set and let $\mathcal{B}$ be a family of Banach spaces. The $\mathcal{B}$*-equivariant compression exponent* of G is defined by
$$\alpha^{\#}_{\mathcal{B}}(G) = \sup_f \{\text{comp}(f)\},$$
where the supremum is taken over all equivariant maps from G to an element B of $\mathcal{B}$, namely orbits of an affine isometric action of G on B.

- When $\mathcal{B}$ will be the class of L^p spaces, we will always write $\alpha^*_{\mathcal{B}}(X) = \alpha^*_p(X)$ and $\alpha^{\#}_{\mathcal{B}}(G) = \alpha^{\#}_p(G)$.

Observe from the definition that the compression exponent of a group is comprised between 0 and 1, 0 meaning that the compression modulus of any coarse lipschitz map is asymptotically bounded by any power functions $t \mapsto t^\alpha$. 1 meaning that the group is arbitrarily close to being quasi-isometrically embeddable (but may not be). Guentner and Kaminker get the following:

Theorem 2.1.11 ([GK04]).

- *Let X be a uniformly discrete metric space satisfying $\alpha^*_2(X) > \frac{1}{2}$, then X has property A.*
- *Let G be a finitely generated group satisfying $\alpha^{\#}_2(G) > \frac{1}{2}$, then G is amenable.*

We collect a few results about compression exponents :

- Simpicial trees with uniformly bounded degree have L^p-compression exponent equal to 1 for all $p \geq 1$ [GK04]. Compare this to the fact that infinite trees do not embed quasi-isometrically into uniformly convex spaces [Bou86] to see that the supremum in the definition of the compression exponent need not be attained.
- Every finitely generated hyperbolic group embeds into a product of simplicial trees [BDS07] so that they also have L^p-compression exponents equal to 1.
- Every finite dimensional CAT(0) cube complex has Hilbert compression exponent equal to 1 [CN05].
- For any $0 < \alpha < 1$, there exists a non-amenable group G_α satisfying $\alpha^*_B(G_\alpha) = \alpha$ for any uniformly convex Banach space B [ADS09].

- Every connected Lie group has L^p compression exponent equal to 1 [Tes11].

On the side of equivariant compression, we have

- If G is an amenable finitely generated group, then $\alpha_p^*(G) = \alpha_p^{\#}(G)$ for all $p \geq 1$ [NP11]. For p=2, this is sometimes reffered to as Gromov's trick (see [dCTV07] for the original proof).
- The free group on 2 generators F_2 satisfies $\alpha_2^{\#}(F_2) = \frac{1}{2}$ [GK04]. Furthermore $\alpha_p^{\#}(F_2) = \max(\frac{1}{2}, \frac{1}{p})$ for $p \geq 1$ [NP08].
- Polycyclic groups, amenable connected Lie groups and wreath products $F \wr \mathbb{Z}$ with F finite all have L^p-equivariant compression exponent 1 for all $p \geq 1$ [Tes11].
- Yu proves that finitely generated hyperbolic groups admit proper affine isometric actions on L^p for p large enough [Yu05]. The proof also provides $\alpha_p^{\#}(G) \geq \frac{1}{p}$ in this case.
- There exist amenable groups satisfying $\alpha_p^{\#}(G) = 0, \forall p \geq 1$ [Aus11].
- The wreath product $G_2 = \mathbb{Z} \wr \mathbb{Z}$ satisfies $\alpha_2^{\#}(G) = \frac{2}{3}$ [ANP09]. Furthermore, if $G_k = G_{k-1} \wr \mathbb{Z}$ then $\alpha_2^{\#}(G_k) = \frac{1}{2-2^{1-k}}$ [NP08].

2.2 Compression exponents of $\mathfrak{N}$-BS groups

This section is devoted to the study of coarse embeddings of a large class of groups similar to Baumslag-Solitar groups. It is a joint work with P.-N. Jolissaint and has been published in the Journal of Group Theory [JP13].

2.2.1 $\mathfrak{N}$-BS groups

In [GJ03] the authors introduced the notion of an $\mathfrak{N}$-BS group. These are groups arising as HNN extensions satisfying properties similar to Baumslag-Solitar groups. In order to prove the Haagerup property for such groups, they developed a framework that we shall heavily rely on and that we now recall.

Let $\mathfrak{N}$ be a locally compact compactly generated group and let G be a closed subgroup of $\mathfrak{N}$. Let $i_1, i_2 : H \to G$ be two inclusions of a group H onto open subgroups of finite index of G. Assume i_1 and i_2 are conjugated by an automorphism φ of $\mathfrak{N}$ so that $\varphi \circ i_1 = i_2 \circ \varphi$. The $\mathfrak{N}$-BS group Γ is then the HNN extension $\mathrm{HNN}(G, i_1(H), i_2(H))$ whose presentation is given by $\langle S, t | R, ti_1(h)t^{-1} = i_2(h)\ \forall h \in H\rangle$, where $G = \langle S|R\rangle$.

For later use, we recall the main technical result of [GJ03]. Let $\tilde{\mathfrak{N}} = \mathfrak{N} \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on $\mathfrak{N}$ by iterations of φ and let $j_{\mathfrak{N}} : \Gamma \to \tilde{\mathfrak{N}}$ be the homomorphism defined by $g \mapsto (g, 0)$ for $g \in G$ and $t \mapsto (1, 1)$. This is indeed a homomorphism, since the defining relations $ti_1(h)t^{-1} = i_2(h)$ of Γ are satisfied in $\tilde{\mathfrak{N}}$. Then consider T, the Bass-Serre tree associated with the HNN extension Γ and denote by $j_T : \Gamma \to \mathrm{Aut}(T)$ the inclusion induced by the action of Γ on T.

Theorem 2.2.1 ([GJ03])**.** *The homomorphism $j : \Gamma \to \tilde{\mathfrak{N}} \times \mathrm{Aut}(T)$, $g \mapsto (j_{\mathfrak{N}}(g), j_T(g))$ is injective and has closed image. In particular, it is proper.*

Elaborating on the quantitative aspect of this theorem, we prove the following

Theorem 2.2.2. *Let $\mathfrak{N}$ be a connected Lie group and G a closed co-compact subgroup of $\mathfrak{N}$ and let Γ be an $\mathfrak{N}$-BS group as defined above. Then, for all $p > 1$, $\alpha_p^*(\Gamma) = 1$.*

The strategy to prove this theorem is to construct a metric space M on which Γ acts continuously, properly, cocompactly and by isometries, so that M and Γ are quasi-isometric by Schwarz lemma: this is done in subsection 2.2.2, where we also give a quantitative comparison between two natural metrics on M. In the last subsection, we prove Theorem 2.2.2 and treat some concrete examples.

2.2.2 Fibred product

Following Proposition 2.1 in [CdCMT12], we will define a metric space Y on which $\tilde{\mathfrak{N}}$ acts continuously, properly, cocompactly and by isometries. Endow $\tilde{\mathfrak{N}}$ with a left-invariant Riemannian metric. For each coset $L_i = \mathfrak{N} \times \{i\}$ of $\mathfrak{N}$ in $\tilde{\mathfrak{N}}$ we consider a strip $L_i \times [0,1]$, equipped with the product Riemannian metric, and attach it to $\tilde{\mathfrak{N}}$ by identifying $(l,0)$ to l and $(l,1)$ to $l \cdot (1,1)$. Denote by Y the space obtained in this way. Y has a natural shortest-path metric induced by the riemannian metric on each of the strips. Furthermore, Y is naturally homeomorphic (but not necessarily isometric!) to $\mathfrak{N} \times \mathbb{R}$. Using this obvious parametrization, $\tilde{\mathfrak{N}}$ acts on Y by $(n,k) \cdot (y,s) = (n\varphi^k(y), k+s)$, for $(n,k) \in \tilde{\mathfrak{N}}$ and $(y,s) \in Y$. As in Proposition 2.1 in [CdCMT12], Y is a locally compact, geodesic metric space on which $\tilde{\mathfrak{N}}$ acts continuously, properly, cocompactly and by isometries. We denote by b the projection map $(y,s) \mapsto s$.
Let us recall briefly the construction of the Bass-Serre tree T of Γ. It is an oriented graph whose vertices are the left-cosets Γ/G and the edges correspond to the left cosets $\Gamma/i_1(H)$. The edge $\gamma/i_1(H)$ is directed from $\gamma t^{-1}G$ to γG. As the $i_k(H)$ are of finite index in G, T is locally finite. Then, by construction, Γ acts naturally on T by left multiplication.
Now, let $p : \Gamma \to \mathbb{Z}$ be the homomorphism defined on the generators by $p(t) = 1$ and $p(g) = 0$, for every $g \in G$. Since the vertices of T

correspond to the left cosets of G in Γ, we can define a map c on the vertices of T by $c(xG) = p(x)$ and extend it to the metric tree T by affine interpolation. This allows us to define the fibred product M :

$$M = \{(x, y) \in T \times Y \;:\; c(x) = b(y)\}.$$

The subspace M is Γ-invariant for the diagonal action of Γ on $T \times Y$. Indeed, for all $x \in T$, $c(g \cdot x) = c(x)$ if $g \in G$ and $c(t \cdot x) = c(x) + 1$. In a similar fashion, for all $y \in Y$, $b(g \cdot y) = b(y)$ if $g \in G$ and $b(t \cdot y) = b(y) + 1$. Hence, if $c(x) = b(y)$, it implies that $c(\gamma \cdot x) = b(\gamma \cdot y)$ for any $\gamma \in \Gamma$.
We endow $T \times Y$ with the product metric, namely, $d((x, y), (x', y')) = d_T(x, x') + d_Y(y, y')$.

Lemma 2.2.3. *M is path-connected. Furthermore, denoting by d_M the shortest-path metric induced by d on M, the metrics d and d_M are bilipschitz equivalent.*

Proof : First, observe that, for any point $y = (n, s) \in Y$, the path

$$\alpha_y : \mathbb{R} \to Y : \; u \mapsto (n, u + s)$$

is a geodesic such that $\alpha_y(0) = y$ and $b(\alpha_y(u)) = b(y) + u$, $\forall u \in \mathbb{R}$. Similarly, for any point $x \in T$ one can chose a geodesic path $\beta_x : \mathbb{R} \to T$ such that $\beta_x(0) = x$ and $c(\beta_x(u)) = c(x) + u$. Let $(x_0, y_0), (x_1, y_1) \in M$. We will build a path linking those points in two steps. For the first one, let $\sigma : [0, d_T(x_0, x_1)] \to T$ be the geodesic from x_0 to x_1. Let θ_1 be the path defined by

$$\theta_1(u) = (\sigma(u), \alpha_{y_0}(c(\sigma(u)) - b(y_0)).$$

The left component links x_0 to x_1, while the right component starts from y_0 and ends at a certain point y_2. Moreover, the path θ_1 is contained in M. Indeed, for all $u \in [0, d_T(x_0, x_1)]$, we have:

$$\begin{aligned} b(\alpha_{y_0}(c(\sigma(u)) - b(y_0))) &= b(y_0) + c(\sigma(u)) - b(y_0) \\ &= c(\sigma(u)). \end{aligned}$$

So, θ_1 connects (x_0, y_0) to a point $(x_1, y_2) \in M$ satisfying $b(y_2) = c(x_1) = b(y_1)$. For the second step, we will find a path in M between (x_1, y_2) and (x_1, y_1). In a similar way, let $\tilde{\sigma} : [0, d_Y(y_2, y_1)] \to Y$ be a geodesic path linking y_2 to y_1 in Y. Then, it is easy to check that the path

$$\theta_2 : [0, d_Y(y_2, y_1)] \to M : \theta_2(u) = (\beta_{x_1}(b(\tilde{\sigma}(u)) - c(x_1)), \tilde{\sigma}(u))$$

does the job. This shows that M is path-connected. Now, the inequality $d \leq d_M$ being immediate, we need to analyze the length of the path we just considered in order to finish the proof. Denoting by $L(\theta_j)$ the length of the path θ_j, we get the following estimates:

$$L(\theta_1) \leq 2d_T(x_0, x_1)$$

and

$$L(\theta_2) \leq 2d_Y(y_2, y_1) \leq 2(d_Y(y_0, y_1) + d_Y(y_1, y_2))$$

By construction, $d_Y(y_0, y_2) \leq d_T(x_0, x_1)$. We can conclude:

$$\begin{aligned} d_M((x_0, y_0), (x_1, y_1)) &\leq L(\theta_1) + L(\theta_2) \\ &\leq 2d_T(x_0, x_1) + 2d_Y(y_0, y_1) + 2d_Y(y_1, y_2) \\ &\leq 4d_T(x_0, x_1) + 2d_Y(y_0, y_1) \\ &\leq 4 \cdot d((x_0, x_1), (y_0, y_1)). \end{aligned}$$

□

2.2.3 Proof of Theorem 2.2.2 and Applications

As before, let Γ be an $\mathfrak{N}$-BS group with G a closed cocompact subgroup of $\mathfrak{N}$. In order to apply the Schwarz Lemma, we prove that the action of Γ is proper and cocompact.

Lemma 2.2.4. *The action of Γ on $T \times Y$ is proper. That is, for all $(x, y) \in T \times Y$, there exists $r > 0$ so that the set of all γ's such that*

$$\gamma \cdot B((x, y), r) \cap B((x, y), r) \neq \emptyset$$

is relatively compact in Γ.

In particular, as M is a closed subset of $T \times Y$, we get immediately the following Corollary.

Corollary 2.2.5. *The action of* Γ *on the fibred product* M *is proper.*

Proof of Lemma 2.2.4 : The action of $Aut(T) \times \tilde{\mathfrak{N}}$ on $T \times Y$ is proper, therefore, by Theorem 2.2.1, it is also the case for the action of Γ. As M is closed and Γ-invariant, we can conclude. $\square$

Lemma 2.2.6. *The action of* Γ *on* M *is cocompact.*

Proof : It is enough to see that, for any sequence $(x_k, y_k)_k \subset M$, we can find a sequence $(\gamma_k)_k \subset \Gamma$ so that the sequence $(\gamma_k \cdot (x_k, y_k))_k$ converges. Since Γ acts transitively on the edges of T, we can assume that the sequence $(x_k)_k$ belongs to the edge $[G, tG]$. This implies that $0 \leq c(x_k) = b(y_k) \leq 1$, for all but possibly finitely many k, so that the sequence $(y_k)_k$ is contained in the strip of Y corresponding to the coset $\mathfrak{N}$ in $\tilde{\mathfrak{N}}/\mathfrak{N}$. Using the fact that the action of G on $\mathfrak{N}$ is cocompact, we can multiply by elements of G in such a way that the sequence $(y_k)_k$ converges. But since G stabilizes the vertex G in T, this process maintains the sequence $(x_k)_k$ inside the edges adjacent to G. Since there are only finitely many on these, the sequence $(x_k, y_k)_k$ converges up to extracting a subsequence. This concludes the proof. $\square$

We are now able to prove Theorem 2.2.2.

Proof of Theorem 2.2.2 : Firstly, we show that $\alpha_p^*(\Gamma) \geq \alpha_p^*(\tilde{\mathfrak{N}})$. Indeed, by Schwarz lemma, Γ is quasi-isometric to (M, d_M), which is quasi-isometric to (M, d) by Lemma 2.2.3. Moreover, Y is quasi-isometric to $\tilde{\mathfrak{N}}$. Hence, $\alpha_p^*(\Gamma) = \alpha_p^*(M) \geq \alpha_p^*(T \times Y)$ and $\alpha_p^*(Y) = \alpha_p^*(\tilde{\mathfrak{N}})$. Then, the lower bound follows from the propositions:

- For a tree T, $\alpha_p^*(T) = 1$, for all $p > 1$. (See Theorem 2.6 in [BS08])
- For two metric spaces X and X', the compression of $X \times X'$ is the minimum of the compressions of the factors. (See [GK04])

Finally, we conclude the proof by noting that $\alpha_p^*(\tilde{\mathfrak{N}}) = 1$, which follows from the following propositions:

- Any semi-direct product of a connected Lie group with $\mathbb{Z}$ is quasi-isometric to a connected Lie group. (By an unpublished result of Y. Cornulier)
- Let $\mathcal{K}$ be a connected Lie group. Then, $\alpha_p^*(\mathcal{K}) = 1$, for all $p > 1$. (See [Tes11])

□

We remark that, if $\mathfrak{N}$ is a soluble connected Lie group, then Cornulier's result is a simple consequence of a lemma of Mostow. Here is a short proof that we owe to Alain Valette. In this case, $\tilde{\mathfrak{N}}$ is soluble and Noetherian (i.e. every closed subgroup is compactly generated). Mostow's lemma (see Lemma 5.2 in [Mos71]) asserts that there exists a compact normal subgroup K of $\tilde{\mathfrak{N}}$ and a soluble almost connected Lie group $\mathcal{M}$ such that the quotient $\tilde{\mathfrak{N}}/K$ is isomorphic to $\mathcal{L}$, where $\mathcal{L}$ is a cocompact, closed subgroup of $\mathcal{M}$. Then, the connected component of unity $\mathcal{M}^0$ is quasi-isometric to $\tilde{\mathfrak{N}}$. Indeed, on one hand, by compactness, $\tilde{\mathfrak{N}}$ is quasi-isometric to $\tilde{\mathfrak{N}}/K$ and by Mostow we deduce that $\tilde{\mathfrak{N}}$ is quasi-isometric to $\mathcal{L}$. On the other hand, by cocompactness, $\mathcal{L}$ is quasi-isometric to $\mathcal{M}$ and, since $\mathcal{M}$ has only finitely many connected components, it is quasi-isometric to $\mathcal{M}^0$.

In particular, Theorem 2.2.2 allows us to cover all the examples appearing in [GJ03].

Corollary 2.2.7. *The following groups have* L^p*-compression exponent* 1.

1. *The Baumslag-Solitar groups* $BS_q^p = \langle x, t \mid x^p = tx^q t^{-1}\rangle = HNN(\mathbb{Z}, \mathbb{Z}, p\cdot, q\cdot)$*, with parameters* $p, q \in \mathbb{Z}_+$*. (For a different proof, see also [CV15].)*
2. *Torsion free, finitely presented abelian-by-cyclic groups.*

3. *Let* $\mathfrak{N}$ *be a homogeneous nilpotent Lie group. So, it admits a dilating automorphism* φ*. Suppose that* $\mathfrak{N}$ *contains a discrete, cocompact subgroup* G *which is invariant by* φ*. Then, for any finite index subgroup* H *in* G*, the extension* $HNN(G, H, i_1, \varphi|_H)$*, where* i_1 *is the canonical injection, has compression* 1*.*

2.3 Equivariant compression under group constructions

The question of how compression exponents behave under group constructions is natural and an extensive study has been made. The case of HNN extensions and amalgamated free products over finite subgroups has been studied by Dreesen [Dre11] and Hume [Hum]. For non-equivariant compression exponents Hume completely solved the question in a more general setting :

Theorem 2.3.1 ([Hum]). *Let G be a finitely generated group hyperbolic relative to a family of subgroups $\{H_i\}$ and fix $p \geq 1$. Then*

$$\alpha_p^*(G) = \min_i \alpha_p^*(H_i).$$

In the equivariant case, the picture is not as clear. Dreesen [Dre11] has the following partial results

Theorem 2.3.2 ([Dre11]).

- *Let A, B be finitely generated groups with a common finite subgroup F, then*

$$\alpha_2^\#(A *_F B) = \min\left(\alpha_2^\#(A), \alpha_2^\#(B), \frac{1}{2}\right).$$

- *Let A be a finitely generated group, F a finite group and i_1, i_2 be injective homomorphisms $F \to A$, then*

$$\alpha_2^\#(A *_F) = \min\left(\alpha_2^\#(A), \frac{1}{2}\right).$$

If the considered free product or HNN extension is non-virtually cyclic.

The picture in L^p spaces is less complete. The above construction for HNN extensions over finite subgroups extends to L^p but gives a non-sharp lower bound on the compression exponent. The case of free products with amalgamation doesn't extend to L^p, however free products without amalgamation are covered by Dreesen and Antolin's result on graph products [APD]. We get the following picture

Proposition 2.3.3. *Under the same assumptions as in Theorem 2.3.2, we get*

1. $$\min\left(\alpha_p^{\#}(A), \alpha_p^{\#}(B), \tfrac{1}{p}\right) \leq \alpha_p^{\#}(A * B)$$
$$\leq \min\left(\alpha_p^{\#}(A), \alpha_p^{\#}(B), \max(\tfrac{1}{2}, \tfrac{1}{p})\right).$$

2. $$\min\left(\alpha_p^{\#}(A), \tfrac{1}{p}\right) \leq \alpha_p^{\#}(A*_F) \leq \min\left(\alpha_p^{\#}(A), \max(\tfrac{1}{2}, \tfrac{1}{p})\right).$$

The factor $\max(\frac{1}{2}, \frac{1}{p})$ in the upper bound of both cases comes from the non-amenability of the considered group. However, up to now, any known attempt to produce explicit constructions of proper affine isometric actions gives compression exponents bounded above by $\frac{1}{p}$. This usually comes from the use of a convexity inequality involving p-norms. In the case $p \leq 2$ this is not a problem, since $\frac{1}{p} \geq \frac{1}{2}$ and the above theorem gives an exact value for the equivariant compression exponent of a free product or an HNN extension. However, when $p > 2$ the lower bound and the upper bound do not agree.

Here, we complete the picture to include free products with amalgamation over a finite subgroup. We also provide a proof in the case of HNN extensions, but we claim no originality on the result. Later we show how to slightly improve the lower bound in both cases and understanding that proof will be necessary. Finally, we address the question of how to fill the $1/p$-$1/2$ gap. A possible strategy involving embedding some Banach spaces into L^p spaces is proposed, but the question remains open.

2.3.1 Free products with amalgamation over finite subgroups

Proposition 2.3.4. *Let A and B be finitely generated groups and F be a finite subgroup of both A and B. Moreover assume that F is of index at least 3 in A, so that $A *_F B$ is not virtually cyclic. Fix $p \geq 1$, then*

$$\min\left(\alpha_p^{\#}(A), \alpha_p^{\#}(B), \frac{1}{p}\right) \leq \alpha_p^{\#}(A *_F B) \leq \min\left(\alpha_p^{\#}(A), \alpha_p^{\#}(B), \max(\frac{1}{2}, \frac{1}{p})\right).$$

We start with an easy lemma combining two known facts : first, for all purposes, it is always possible to assume that a cocycle vanishes on a finite subgroup. Second, we can get rid of additive constants in the compression modulus of a cocycle.

Lemma 2.3.5. *Let G be a finitely generated group and F be a finite subgroup of G. Let π be an isometric representation of G on some L^p space and let b be a π-cocycle satisfying for some constants $C, D, r > 0$:*

$$\frac{1}{C}|x|^r - D \leq \|b(x)\|_p \leq C|x|, \quad \forall x \in G.$$

Then, there exists an isometric representation $\tilde{\pi}$ of G on an L^p space and a $\tilde{\pi}$-cocycle $\tilde{b}$ satisfying for some $\tilde{C} > 0$:

$$\tilde{b}(x) = 0, \quad \forall x \in F$$
$$\textit{and} \quad \frac{1}{\tilde{C}}|x|^r \leq \|b(x)\|_p \leq \tilde{C}|x|, \quad \forall x \in G \setminus F.$$

Proof. First, since F is finite, it fixes a point in L^p. Hence, up to translation, we can assume that $\forall g \in F$, $b(g) = 0$. Let λ be the left quasi-regular representation of G on $\ell^p(G/F)$. Set $\tilde{\pi} = (\pi \oplus \lambda)_p$ and

$\tilde{b}(g) = b(g) \oplus (\lambda(g)\delta_F - \delta_F)$, with δ_F being the Dirac mass at the point F in $\ell^p(G/F)$. Note that $\tilde{b}$ still vanishes on F.

Now, let $g \in G \setminus F$ and set $M = (2CD)^{\frac{1}{r}}$.
If $|g| \geq M$ then $\frac{1}{2C}|g|^r \geq D$ and

$$\|\tilde{b}(g)\| \geq \|b(g)\|_p \geq \frac{1}{C}|g|^r - D \geq \frac{1}{2C}|g|^r.$$

If $|g| \leq M$, we have $\frac{1}{2CD}|g|^r \leq 1$ and we obtain

$$\|\tilde{b}(g)\| \geq \|\lambda(g)\delta_F - \delta_F\|_p = 2^{1/p} \geq \frac{2^{1/p}}{2CD}|g|^r.$$

By setting $\tilde{C} = \max\left(2C, \frac{2CD}{2^{1/p}}\right)$, we see that $\tilde{b}$ satisfy the claim. □

Proof of Prop. 2.3.4. The upper bound is immediate. Due to [NP08, Theorem 1.1.], $\alpha_p^{\#}(A *_F B) \leq \max(\frac{1}{2}, \frac{1}{p})$ since $G = A *_F B$ is non-amenable. Also, it is well known that the embeddings $A \hookrightarrow A *_F B$ and $B \hookrightarrow A *_F B$ are bi-lipschitz, so that $\alpha_p^{\#}(A *_F B) \leq \min(\alpha_p^{\#}(A), \alpha_p^{\#}(A))$.

Fix $r < \min(\alpha_p^{\#}(A), \alpha_p^{\#}(B), \frac{1}{p})$, we're going to produce an explicit action of G with compression exponent at least r. Pick L^p isometric actions (π_A, b_A) of A and (π_B, b_B) of B satisfying

$$\begin{aligned}
&\frac{1}{C}|x|^r \leq \|b_A(x)\|, \ \forall x \in A \setminus F, \\
&\frac{1}{C}|x|^r \leq \|b_B(x)\|, \ \forall x \in B \setminus F, \\
&\text{and } b_A(x) = b_B(x) = 0, \ \forall x \in F,
\end{aligned}$$

according to Lemma 2.3.5. Without loss of generality, we will assume that both A and B act on the same space $\mathrm{L}^p(\Omega)$.

Denote by ρ_A (respectively ρ_B) the representation induced from A (resp. B) to G by π_A (resp. π_B). Recall that the underlying space of the linear representation ρ_A (e.g.) is given by the set of functions

$f : G \to \mathrm{L}^p(\Omega)$ satisfying

$$\begin{aligned} \pi_A(\alpha^{-1})f(x) &= f(x\alpha), \forall x \in G, \alpha \in A \\ \text{and } \|f\| &:= \left(\sum_{x \in G/A} \|f(x)\|_p^p \right)^{1/p} < \infty. \end{aligned}$$

The action of G is then given by $\rho_A(g)f(x) = f(g^{-1}x)$

Let $\rho = (\rho_A \oplus \rho_B)_p$, our goal is to build a well-defined ρ-cocycle b. We first define the value of b on elements of A and B. For $\alpha \in A$, let $b(\alpha) = \tilde{b}_A(\alpha) \oplus 0$, where $\tilde{b}_A(\alpha)$ denotes the function $G \to \mathrm{L}^p(\Omega)$ defined by

$$\tilde{b}_A(\alpha)(x) = \begin{cases} \pi_A(x^{-1})b_A(\alpha) & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

For $\beta \in B$ let $b(\beta) = 0 \oplus \tilde{b}_B(\beta)$ where $\tilde{b}_B(\beta)$ is defined similarly as above.

By construction, b belongs to the space of π, but we need to check that b is indeed a cocycle when restricted to A (resp. B). Let $\alpha_1, \alpha_2, x \in A$, we have

$$\begin{aligned} \left(\tilde{b}_A(\alpha_1) + \rho_A(\alpha_1)\tilde{b}_A(\alpha_2)\right)(x) &= \tilde{b}_A(\alpha_1)(x) + \tilde{b}_A(\alpha_2)(\alpha_1^{-1}x) \\ &= \pi_1(x^{-1})b_A(\alpha_1) + \pi_1(x^{-1}\alpha_1)b_A(\alpha_2) \\ &= \pi_1(x^{-1})b_A(\alpha_1\alpha_2) \\ &= \tilde{b}_A(\alpha_1\alpha_2)(x). \end{aligned}$$

Now, we extend b to composite elements of G (i.e. words in elements of A and B) by using the cocycle relation $b(\alpha\beta) = b(\alpha) + \rho(\alpha)b(\beta)$. Namely if $g = \alpha_1\beta_1\alpha_2\beta_2 \dots \alpha_n\beta_n \in G$ with the α's in A and the β's in B, we set

$$b(g) = b(\alpha_1) + \rho(\alpha_1)b(\beta_1) + \rho(\alpha_1\beta_1)b(\alpha_2) + \dots + \rho(\alpha_1\beta_1 \dots \beta_{n-1}\alpha_n)b(\beta_n).$$

To see that this definition is consistent, we need to check that the cocycle vanishes on the relations defining G, namely those of the form

$f \cdot f^{-1}$ where $f \in F$ is first seen as an element of A and then as an element of B. But this is indeed the case since both b_A and b_B vanish on F.

It remains to check that this cocycle has the desired compression exponent. Let $g = \alpha_1\beta_1\alpha_2\beta_2 \ldots \alpha_n\beta_n \in G$ with the α's in A and the β's in B. One has

$$b(g) = b(\alpha_1) + \rho(\alpha_1)b(\beta_1) + \rho(\alpha_1\beta_1)b(\alpha_2) + \cdots + \rho(\alpha_1\beta_1 \ldots \beta_{n-1}\alpha_n)b(\beta_n)$$

Note that ρ acts by permuting the left cosets of A and B therefore every term of the above sum is supported, as a function $G \to \mathrm{L}^p(\Omega)$, on a unique and different coset of either A or B. We conclude that :

$$\begin{aligned}
\|b(g)\|^p &= \|b(\alpha_1)\|^p + \|\rho(\alpha_1)b(\beta_1)\|^p + \cdots + \|\rho(\alpha_1\beta_1 \ldots \alpha_n)b(\beta_n)\|^p \\
&= \|b(\alpha_1)\|^p + \|b(\beta_1)\|^p + \|b(\alpha_2)\|^p + \cdots + \|b(\beta_n)\|^p \\
&\geq \frac{1}{C}|\alpha_1|_A^{pr} + \frac{1}{C}|\beta_1|^{pr} + \cdots + \frac{1}{C}|\beta_n|^{pr} \\
&\geq \frac{1}{C}(|\alpha_1| + |\beta_1| + \cdots + |\beta_n|)^{pr} \\
&= \frac{1}{C}|g|^{pr}
\end{aligned}$$

Where the last inequality holds because $pr < 1$. Finally, we obtain

$$\frac{1}{C}|g|^r \leq \|b(g)\|.$$

□

To get a better lower bound we can try to use the action of G on its Bass-Serre tree. This action is well suited to the study of the geometry of the group equipped with another metric, the block-length metric. The simple idea to get a result for the group equipped with its word metric is to split into two cases : a long word in a free product either has a large block length or contains a large element of one of the factor. We get the following :

Proposition 2.3.6. *Let A, B and F be as in the setting of Proposition 2.3.4 and set $s = \min\left(\alpha_p^{\#}(A), \alpha_p^{\#}(B)\right)$. Then*

$$\alpha_p^{\#}(A *_F B) \geq \frac{s}{1+2s}.$$

Remark 2.3.7. Whether or not this is an improvement of Proposition 2.3.4 depends on the specific values of p and s. For example, if $\alpha_p^{\#}(A) = \alpha_p^{\#}(B) = 1$, we get that $\alpha_p^{\#}(A *_F B) \geq 1/3$ which is an improvement as soon as $p > 3$.

Recall that in the amalgamated free product G, every element g can be written as $g = \alpha_1\beta_1 \cdots \alpha_n\beta_n f$ with $\alpha_i \in A \setminus F$, $\beta_i \in B \setminus F$, $f \in F$, and possibly $\alpha_1 = 1$ and/or $\beta_n = 1$. The number of non trivial α's and β's (here $2n$, $2n-1$, or $2n-2$) is called the *block length* of g. We'll denote it by $|g|_{\mathrm{bl}}$. We start by studying the equivariant L^p-compression exponent of G equipped with the pseudo-metric $|\cdot|_{\mathrm{bl}}$. This metric is intimately linked with the action of the group on its Bass-Serre tree and we use a classical cocycle associated to any group action on a tree. We refer to [Ser77] for proper definitions and results related to Bass-Serre theory.

Lemma 2.3.8. *Fix $p \geq 2$, there exists a cocycle $c\colon G \to \mathrm{L}^p$ satisfying*

$$\|c(g)\| \geq \frac{1}{C}|g|_{bl}^{1/2}, \forall g \in G.$$

Proof. Denote by T the Bass-Serre tree of $G = A *_F B$ and denote by σ the associated unitary representation of G on $\ell^2(E(T))$. Recall that the vertices of T are given by left-cosets of A and B in G and denote by x_0 the coset associated to A. Define $\delta : V(T) \times V(T) \to \ell^2(E(T))$ by

$$\delta(x,y)(e) = \begin{cases} 1 & \text{if } e \in [x,y] \\ -1 & \text{if } e \in [y,x] \\ 0 & \text{otherwise} \end{cases},$$

where $[x,y]$ denotes the *o*riented geodesic from x to y. Now define a σ-cocycle $\tilde{c}\colon G \to \ell^2(E(T))$ by $\tilde{c}(g) = \delta(x_0, g \cdot x_0)$.

It is classical that $\tilde{c}$ is indeed a σ-cocycle and that

$$\|\tilde{c}(g)\| = \delta(x_0, g \cdot x_0)^{\frac{1}{2}}.$$

In our case, the distance between x_0 and $g \cdot x_0$ is given by $|g|_{\mathrm{bl}} \pm 1$ (the ± 1 factor depending on whether g, written as a word in A and B, starts and/or ends with an element of A). Thus we obtain

$$\|\tilde{c}(g)\| \geq |g|_{\mathrm{bl}}^{\frac{1}{2}} - 1$$

so that the equivariant compression exponent of this cocycle is exactly $\frac{1}{2}$. Now, according to [NP08], we can embed $\ell^2(V(T))$ isometrically into some L^p space equivariantly with respect to the affine action given by σ and $\tilde{c}$. Getting rid of the additive constant in a similar fashion as Lemma 2.3.5, we get the claim. □

Proof of Proposition 2.3.6. Consider the representation of G given by $(\rho \oplus \sigma)_p$ and its cocycle $b \oplus c$ as in the proofs of prop. 2.3.4 and Lemma 2.3.8. Recall that b was constructed from cocycles b_A and b_B whose compression exponents can be chosen arbitrarily close to $s = \min\left(\alpha_p^{\#}(A), \alpha_p^{\#}(B)\right)$. For simplicity, we suppose that indeed there exists C so that $b_{A/B}(g) \geq \frac{1}{C}|g|^s$.

Now fix $g = \alpha_1\beta_1 \cdots \alpha_n\beta_n$ in G, set $u = \frac{2s}{1+2s}$ and consider two cases

1. If $|g|_{\mathrm{bl}} \geq |g|^u$, then

$$\|b(g) \oplus c(g)\| \geq \|c(g)\| \geq \frac{1}{C}|g|_{\mathrm{bl}}^{1/2} \geq \frac{1}{C}|g|^{u/2}.$$

2. If $|g|_{\mathrm{bl}} \leq |g|^u$, then by the pigeon hole principle, there is at least one element α_i or β_i of length at least $|g|^{1-u}$. Say that it is α_1 for example, we get

$$\|b(g) \oplus c(g)\| \geq \|b(g)\| \geq \|b_A(\alpha_1)\| \geq \frac{1}{C}|\alpha_1|^s \geq \frac{1}{C}|g|^{(1-u)s}.$$

Now, u was chosen acurately so that $\frac{u}{2} = (1-u)s = \frac{s}{1+2s}$ and the claim is proved.

□

2.3.2 HNN extensions over finite subgroups

We can apply the strategy of Proposition 2.3.6 to obtain a similar result for HNN extensions. But first, for the sake of completeness, we give a full proof of Dreesen's result for HNN extensions in the L^p case.

Proof of Proposition 2.3.3 2. Recall that $G = A*_F$ is the group generated by A and a free letter t subject to the relation $t^{-1}i_1(f)t = i_2(f)$ for each $f \in F$. Any element $g \in G$ can be written as $g = t^{k_1}\alpha_1 t^{k_2}\alpha_2 \dots t^{k_n}\alpha_n t^{k_{n+1}}$, with the α's in A and $k_1, \dots, k_{n+1}$ non-zero integers (except maybe k_1 and k_{n+1}).

Fix $r < \min(\alpha_p^{\#}(A), \frac{1}{p})$, let π be linear representation of G and let b be a π-cocycle satisfying

$$\|b(g)\| > \frac{1}{C}|g|^r - D.$$

According to Lemma 2.3.5, we can drop the additive constant and assume that $b(i_1(f)) = 0$ for all $f \in F$. Also, since $i_2(F)$ is finite, the affine action defined by π and b has an $i_2(F)$-fixed point, let's denote it by ξ_0, so that

$$b(i_2(f)) = \xi_0 - \pi(i_2(f))\xi_0, \ \forall f \in F.$$

Now set $\rho = \mathrm{Ind}_A^G \pi$ and define a ρ-cocycle c by

$$c(\alpha)(x) = \begin{cases} \pi(x^{-1})b(\alpha) & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

for $\alpha \in A$, and by

$$c(t)(tx) = \begin{cases} -\pi(x^{-1})\xi_0 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

When restricted to A, this is indeed a well defined cocycle. It will extend to a cocycle on G, by using the 1-cocycle relation, if the defining relations of the group hold, i.e. if $c(i_1(f)t) = c(t \cdot i_2(f))$, $\forall f \in F$. We evaluate in tx for convenience, on the one hand we have :

$$\begin{aligned} c\left(i_1(f)\cdot t\right)(tx) &= c\left(i_1(f)\right)(tx) + \rho(i_1(f))c(t)(tx) \\ &= 0 + c(t)\left(i_1(f)^{-1}\cdot tx\right) \\ &= c(t)\left(t\cdot i_2(f)^{-1}x\right) \\ &= \begin{cases} -\pi\left(x^{-1}i_2(f)\right)\xi_0 & \text{if } x\in A, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand we have :

$$\begin{aligned} c\left(t\cdot i_2(f)\right)(tx) &= c(t)(tx) + \rho(t)c\left(i_2(f)\right)(tx) \\ &= c(t)(tx) + c\left(i_2(f)\right)(x) \\ &= -\pi(x^{-1})\xi_0 + \pi(x^{-1})b\left(i_2(f)\right) \\ &= -\pi(x^{-1})\xi_0 + \pi(x^{-1})\left(\xi_0 - \pi(i_2(f))\xi_0\right) \\ &= \begin{cases} -\pi\left(x^{-1}i_2(f)\right)\xi_0, & \text{if } x\in A. \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We start estimating the compression exponent of c. Fix $g \in G$ and write $g = t^{k_1}\alpha_1 t^{k_2}\alpha_2 \ldots t^{k_n}\alpha_n t^{k_{n+1}}$.

We have

$$c(g) = c(t^{k_1}) + \rho(t^{k_1})c(\alpha_1) + \ldots + \rho(t^{k_1}\alpha_1 \ldots t^{k_n}\alpha_n)c(t^{k_{n+1}}).$$

Now each term of this sum is a function from G to E_π, but their support are not disjoint as in the case of amalgamated free products. We need to carefully analyse each element. First note that

$$c(t^k)(t^i x) = \begin{cases} -\pi(x^{-1})\xi_0 & \text{if } 1\le i\le k \text{ and } x\in A, \\ 0 & \text{otherwise,} \end{cases}$$

if $k > 0$ and

$$c(t^k)(t^i x) = \begin{cases} \pi(x^{-1})\xi_0 & \text{if } k+1\le i\le 0 \text{ and } x\in A, \\ 0 & \text{otherwise,} \end{cases}$$

if $k < 0$.

We see that elements of the form $\rho(\gamma)c(t^k)$ are supported on the cosets $\gamma t A$, $\gamma t^2 A$, ... , $\gamma t^k A$ if k is positive, and on the cosets $\gamma t^k A$, $\gamma t^{k+1} A, \dots, \gamma t^{-1} A$ if k is negative. But elements of the form $\rho(\gamma t^k)c(\alpha)$ are also supported on $\gamma t^k A$. So the value $c(g)(\gamma t^{k_i})$ differs from $c(\alpha_i)$ depending on whether k_i is positive or negative, and also on whether the power of t succeeding to α is negative or positive. In any case, we get that $c(g)(\gamma t^{k_i}) = c(\alpha_i) + \eta_i$ with the error term η_i satisfying $\|\eta_i\| \leq 2\|\xi_0\|$. We obtain for the norm of $c(g)$:

$$\begin{aligned}
\|c(g)\|^p &= \sum_{i=1}^{n+1}\sum_{j=1}^{k_i} \|c(g)(t^{k_1}\alpha_1 \dots \alpha_{i-1}t^j)\|^p \\
&\geq \sum_{i=1}^{n} \|b(\alpha_i) + \eta_i\|^p + \sum_{i=1}^{n+1}(|k_i| - 2)\|\xi_0\|^p \\
&\geq \sum_{i=1}^{n} \left(\frac{1}{2^{p-1}}\|b(\alpha_i)\|^p - \|\eta_i\|^p\right) + \|\xi_0\|^p \sum_{i=1}^{n+1} |k_i| - 2(n+1)\|\xi_0\|^p \\
&\geq \frac{1}{2^{p-1}} \sum_{i=1}^{n} \|b(\alpha_i)\|^p - n2^p\|\xi_0\|^p + \|\xi_0\|^p \sum_{i=1}^{n+1} |k_i| - 2(n+1)\|\xi_0\|^p \\
&= \frac{1}{2^{p-1}} \sum_{i=1}^{n} \|b(\alpha_i)\|^p + \|\xi_0\|^p \sum_{i=1}^{n+1} |k_i| - n(2^p - 2)\|\xi_0\|^p - 2\|\xi_0\|^p
\end{aligned}$$

Now, the second term of this estimate is problematic because it can represent a large proportion of the length of G. To compensate it, we use the action of G on its Bass-Serre tree. Let d be the cocycle associated to this action, which satisfies

$$\|d(g)\|^p = d(A, g \cdot A),$$

where the distance is measured between the vertices A and gA in the tree. In the case of amalgamated free products, this distance was connected to the block length of G, here it is connected to the number of occurences of the free letter t in g. Precisely, if g is written as before, we have

$$d(A, g \cdot A) = \sum_{i=1}^{n+1} |k_i|.$$

Observing that $\sum |k_i| \geq n-1$, we may take a direct sum $\left(c \oplus \left((2^p-2)^{1/p}\|\xi_0\| \cdot d\right)\right)_p$ to get :

$$\begin{aligned}
\|c(g) \oplus \left((2^p-2)^{1/p}\|\xi_0\| \cdot d(g)\right)\|^p &\geq \frac{1}{2^{p-1}} \sum_{i=1}^{n} \|b(\alpha_i)\|^p + \sum_{i=1}^{n+1} |k_i| \|\xi_0\|^p + \text{Cst} \\
&\geq \frac{1}{2^{p-1}C} \sum |\alpha_i|^{pr} + \sum_{i=1}^{n+1} |k_i|^{pr} \|\xi_0\|^p + \text{Cst} \\
&\geq \frac{1}{\text{Cst}} \left(\sum |\alpha_i| + \sum_{i=1}^{n+1} |k_i| \right)^{pr} + \text{Cst} \\
&= \frac{1}{\text{Cst}} |g|^{pr} + \text{Cst}.
\end{aligned}$$

□

Now we are ready to give an analogue statement to Proposition 2.3.6 for HNN extensions.

Proposition 2.3.9. *Under the same assumptions as before,*

$$\alpha_p^{\#}(A *_F) \geq \frac{\alpha_p^{\#}(A)}{1 + 2\alpha_p^{\#}(A)}.$$

Proof. We use the cocycle we just produced. Since it will not lead to confusion, we just denote it by c. Getting rid of some constants, we have

$$\|c(g)\| \geq \left(\sum_{i=1}^{n} \|b(\alpha_i)\|^p \right)^{\frac{1}{p}},$$

where $g = t^{k_1}\alpha_1 t^{k_2}\alpha_2 \dots t^{k_n}\alpha_n t^{k_{n+1}}$ as before. As was done in Lemma 2.3.8, we can distort the cocycle d induced by the action on the Bass-Serre tree so that

$$\|d(g)\| = \left(\sum_{i=1}^{n+1} |k_i|\right)^{\frac{1}{2}}.$$

Now we consider $c \oplus d$ and split into two cases. Set $s = \alpha_p^{\#}(A)$ and $u = \frac{s}{1+2s}$.

1. If $\sum |k_i| \geq |g|^u$, then

$$\|c(g) \oplus d(g)\| \geq \|d(g)\| = \left(\sum |k_i|\right)^{\frac{1}{2}} \geq |g|^{\frac{u}{2}}$$

2. If $\sum |k_i| \leq |g|^u$, then we get that $n - 1 \leq \sum |k_i| \leq |g|^u$ and $\sum |\alpha_i| = |g| - \sum |k_i| \geq |g| - |g|^u$. So that at least one of the α's must satisfy

$$\begin{aligned} |\alpha| &\geq \frac{1}{n}\sum |\alpha_i| \\ &\geq \frac{|g| - |g|^u}{|g|^u + 1} \\ &\geq |g|^{1-u} - 1 \end{aligned}$$

   Thus we get

$$\|c(g) \oplus d(g)\| \geq \|c(g)\| \geq \|b(\alpha)\| \geq \frac{1}{C}|\alpha|^s \geq \frac{1}{C}|g|^{(1-u)s}$$

Again, u was chosen so that $\frac{u}{2} = (1-u)s = \frac{s}{1+2s}$, so that the claim is proved.

□

2.3.3 The $1/p$-$1/2$ gap

As mentioned earlier, when $p > 2$, in all of the previous results, the lower and upper bounds do not agree. The ‘ $\frac{1}{2}$ ’ term in upper bounds comes from the non-amenability of the considered group. Indeed, Naor and Peres show that for any finitely generated G and any Banach space X of modulus of smoothness of power type p the following inequality holds

$$\alpha_X^{\#}(G)\beta^*(G) \leq \frac{1}{p}.$$

β^* is a numerical invariant of the group measuring the escape rate of a simple random walk on G. More precisely, $\beta^*(G)$ is the largest β so that there exists a finite generating set S satisfying $\mathbb{E}[d(W_t, 1)] \geq \mathrm{Cst} \cdot t^\beta$, where W_t is the canonical random walk associated to S. It is a classical result of Kesten [Kes59] that if G is non-amenable then $\beta^*(G) = 1$, and since L^p spaces have modulus of smoothness of power type $\min(2, p)$, we get that

$$\alpha_p^{\#}(G) \leq \max(\frac{1}{2}, \frac{1}{p}).$$

Now, at first glance, and knowing that classical L^p-cocycles usually have compression exponents $\frac{1}{p}$, one may think that the above method is not sharp when $p > 2$ and that the natural upper bound should be $\frac{1}{p}$. But this is of course not the case because we've already seen that the $\frac{1}{p}$ threshold can be beaten. Also, in many cases, groups exhibit L^p equivariant compression behaviours containing expressions of the type $\max(1/p, 1/2)$, e.g. free solvable groups [Sal], a-T-menable graphs of $\mathbb{Z}^n$'s [CV15] and, of course, free groups. In the last case, the previous upper bound actually is exact value :

$$\alpha_p^{\#}(F_2) = \max(\frac{1}{2}, \frac{1}{p})$$

This is due to Naor and Peres [NP08, Remark 2.2.], who show that for a finitely generated G, $\alpha_2^{\#}(G) \leq \alpha_p^{\#}(G)$ for all $p \geq 1$. The core of the proof of this result has a more general flavour and may be stated as follow.

Proposition 2.3.10 ([NP08] Lemma 2.3.)**.** *Let G be a finitely generated group and let $(\pi, \mathcal{H})$ be a unitary representation of G. Then there* *exists a standard measured space (Ω, μ) equipped with a μ-preserving G-action such that $\mathcal{H}$ embeds linearly, G-equivariantly into $\bigcap_{p\geq 1} \mathrm{L}^p(\Omega, \mu)$. Moreover, there exists a constant C, only depending on p such that*

$$\|v\|_{\mathcal{H}} = C\|v\|_{\mathrm{L}^p(\Omega)}.$$

Now using this result, we get that any affine isometric action of G on a Hilbert space can be isometrically and G-equivariantly embedded into some L_p space which yields the aforementionned bound on $\alpha_p^{\#}(G)$.

Let us try to apply the same line of thought to our problem. Fix $p > 2$ for the rest of this section, since these are the values of p for which we need a deeper study. In both the proofs of Proposition 2.3.4 and Dreesen's result on HNN extensions we used L^p versions of induced representations. Now the space of such a representation can be seen in different ways.

If we denote by π a linear representation of a subgroup $H < G$ on a space $\mathrm{L}^p(\Omega)$ and set $\rho = \mathrm{Ind}_H^G \pi$ then the underlying space X_ρ of ρ can be identified with $\ell^p(G/H, \mathrm{L}^p(\Omega))$. Namely

$$X_\rho \ = \ \{f : G/H \to \mathrm{L}^p(\Omega) \mid \sum_{xH\in G/H} \|f(xH)\|_p^p \leq \infty\}.$$

We may also think of it as $\ell^p(G/H) \otimes \mathrm{L}^p(\Omega)$ equipped (and completed) with its natural p-norm, or as a p-direct sum $\left(\bigoplus_{G/H} \mathrm{L}^p(\Omega)\right)_p$.

Now, observe that both our constructions would work equally fine using $\ell^2(G/H) \otimes \mathrm{L}^p(\Omega)$ or equivalently $\left(\bigoplus_{G/H} \mathrm{L}^p(\Omega)\right)_2$, and would produce a cocycle with compression exponent $\min\left(\alpha_p^{\#}(A), \alpha_p^{\#}(B), \frac{1}{2}\right)$ for $A *_F B$ and $\min\left(\alpha_p^{\#}(A), \frac{1}{2}\right)$ for $A * F$. The draw back is that these spaces are not L^p spaces anymore, but we may still ask the question :

Question 2.3.11. *Let X be a Banach space of the form $\mathrm{L}^2(\Lambda) \otimes \mathrm{L}^p(\Omega)$ equipped with the natural norm where elements of the tensor product are seen as square summable functions $\Lambda \to \mathrm{L}^p(\Omega)$. Let G be a group and equip X with a linear isometric G-action. Does the space X embeds linearly, isometrically and G-equivariantly into some L^p space ?*

One may try to use Naor and Peres' result to produce an embedding of $\mathrm{L}^2(\Lambda) \otimes \mathrm{L}^p(\Omega)$ into $\mathrm{L}^p(\tilde{\Lambda}) \otimes \mathrm{L}^p(\Omega)$, but this approach fails because tensor norms behave badly when taking subspaces. To the extent of our knowledge, this question is still open even if we drop the G-action and the equivariance requirement in the statement. Should the answer be positive, it would produce sharp lower bounds on equivariant compression behaviours for free products with amalgamation and HNN extensions over a finite subgroups. But it would also help in the case of graph products treated by Antolin and Dreesen [APD] and would sharpen Yu's result on existence of proper affine isometric actions of hyperbolic groups on L^p spaces [Yu05], answering positively a question of Naor and Peres [NP08, Question 7.7.]. Indeed, all these constructions are using similar spaces as we do and produce cocycles with $\frac{1}{p}$ compression for the sole reason of using convexity inequalities of the form

$$|x|^{pr} + |y|^{pr} \geq (|x| + |y|)^{pr}, \text{ if } r \leq \frac{1}{p}.$$

Also, it gives a way of solving $1/p$-$1/2$ issues generically instead of cases by cases.

Appendix. Perspectives

We gather here a few thoughts and questions about compression exponents.

Although it is known that compression exponents of groups can take any value between 0 and 1, the picture is much less satisfactory for equivariant compression exponents. Only a handful of values are known:

- $\alpha_2^{\#}(G) = 1$ is achieved by abelian groups, polycyclic groups and amenable connected Lie groups [Tes11].
- $\alpha_2^{\#}(G) = 1/2$ is achieved by virtually free groups, Thompson's group F [AGS06], Baumslag solitar groups [CV15].
- $\alpha_2^{\#}(G) = \frac{1}{2-2^{1-k}}$ is achieved by an iterated wreath product of $\mathbb{Z}$ with itself.[NP08]
- $\alpha_2^{\#}(G) = 0$ is achieved by solvable groups constructed by Austin [Aus11]. It is also a consequence of an embedding result of Osin an Olshanskii [OO13] that amenable groups can have arbitrarily bad constraints on the compression modulus of their equivariant embeddings. Also, Bartholdi and Erschler [BE14] managed to prove the same statement for groups of intermediate growth.

Question A.1. *Fix $0 \leq \alpha \leq 1$. Does there exist an amenable group G with $\alpha_2^{\#}(G) = \alpha$?*

Due to Gromov's trick, it is enough to consider non-equivariant maps to attack this question. This allows some flexibility in the construction of embeddings. The case of non-amenable groups may be much more difficult to deal with.

Question A.2. *Fix* $0 \leq \alpha \leq 1$. *Does there exist a non-amenable group* G *with* $\alpha_2^{\#}(G) = \alpha$*?*

Let $F = F_r$ be the free group of rank r, let $F^{(1)} = [F, F]$ be its derived subgroup and let $F^{(d)} = [F^{(d-1)}, F^{(d-1)}]$ be its d-th derived subgroup. Recall that the free solvable group of rank r and derived length d is the group $S_{r,d} = F_r/F^{(d)}$. It is universal in the sense that every r-generated solvable group of derived length d can be realised as a quotient of $S_{r,d}$. Sale ([Sal]) proves the following:

$$\alpha_p^{\#}(S_{r,d}) \geq \frac{1}{d}\max(\frac{1}{2}, \frac{1}{p}).$$

These bounds provide new possible values for equivariant compression exponents.

Question A.3. *Is it true that*

$$\alpha_p^{\#}(S_{r,d}) = \frac{1}{d}\max(\frac{1}{2}, \frac{1}{p})?$$

A class of groups where nothing is known about compression exponents is the class of groups with intermediate growth. The first, and the most famous, of these groups is Grigorchuk's group, but it has proved to be reluctant to computation. In particular, its exact growth behaviour is still unknown. Bartholdi and Erschler [BE12] have provided the first groups with growth behaviour exactly e^{r^ν} for some ν. Later, Brieussel [Bri14] proved that for any ν in the range $[\nu_0 \approx 0.78, 1]$ there exists a group with growth behaviour ν. We ask the following :

Question A.4.

- *Let G_ν be a group with growth behaviour of type e^{r^ν}. Is there any good lower (resp. upper) bound on $\alpha_2^\#(G_\nu)$ in terms of ν?*
- *What are the exact compression exponents of Brieussel's groups?*

Both Olshanskii-Osin and Bartholdi-Erschler groups with arbitrarily poor compression modulus relies on the same idea. Given a family M_n of finite groups whose Cayley graph form a family of expander, they embed the countable (infinitely generated) group $\prod_n M_n$ into a finitely generated amenable (resp. of intermediate growth) group G. This is done in such a way that the group M_n is quasi-isometrically embedded into G with good control on the quasi-isometry constants. It naturally provides an upper bound on the possible compression modulus of any embedding of the group into a Banach space. However, to keep some lower control on the compression exponents seems difficult.

Question A.5. *Fix $0 \leq \alpha \leq 1$. Is it possible to embed the group $\prod_n M_n$ in an amenable finitely generated group G so that*

$$\alpha_p^\#(G) = \alpha?$$

Finally, we would like to mention the following questions, which we believe are of interest.

Question A.6. *Set*

$$\mathcal{P}(G) = \{p \geq 1 : G \text{ admits a proper affine isometric action on } \mathrm{L}^p\}.$$

What is the topology of the set $\mathcal{P}(G)$?

Question A.7. *Consider the function $p \mapsto \alpha_p^\#(G)$. Is it continuous on $\mathcal{P}(G)$?*

Note that it is possible to prove continuity of $p \mapsto \alpha_p^\#(G)$ under the assumption that the constants in the compression inequality are controlled. Precisely, fix a sequence p_n converging to some p and, for each n, fix b_n an L^p_n-cocycle of G. If we have

$$\|b_n(g)\| \geq \frac{1}{C_n}|g|^{\alpha}_n$$

and if the C_n's are uniformly bounded above, then we can prove that

$$\alpha^{\#}_{\lim p_n}(G) \geq \lim \alpha_n.$$

But the control of these constants is definitely a non trivial condition.

Bibliography

[AD] Goulnara Arzhantseva and Thomas Delzant. Examples of random groups. available at www.unige.ch/math/folks/arjantse/Abs/random.pdf.

[ADS09] Goulnara Arzhantseva, Cornelia Druţu, and Mark Sapir. Compression functions of uniform embeddings of groups into Hilbert and Banach spaces. *J. Reine Angew. Math.*, 633:213–235, 2009.

[AGS06] G. N. Arzhantseva, V. S. Guba, and M. V. Sapir. Metrics on diagram groups and uniform embeddings in a Hilbert space. *Comment. Math. Helv.*, 81(4):911–929, 2006.

[AGŠ12] Goulnara Arzhantseva, Erik Guentner, and Ján Špakula. Coarse non-amenability and coarse embeddings. *Geom. Funct. Anal.*, 22(1):22–36, 2012.

[ANP09] Tim Austin, Assaf Naor, and Yuval Peres. The wreath product of $\mathbb{Z}$ with $\mathbb{Z}$ has Hilbert compression exponent $\frac{2}{3}$. *Proc. Amer. Math. Soc.*, 137(1):85–90, 2009.

[APD] Yago Antolín Pichel and Dennis Dreesen. The Haagerup property is stable under graph products. arxiv:1305.6748.

[AT15] Goulnara Arzhantseva and Romain Tessera. Relative expanders. *Geom. Funct. Anal.*, 25(2):317–341, 2015.

[Aus11] Tim Austin. Amenable groups with very poor compression into Lebesgue spaces. *Duke Math. J.*, 159(2):187–222, 2011.

[BdlHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette. *Kazhdan's property (T)*, volume 11 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2008.

[BDS07] Sergei Buyalo, Alexander Dranishnikov, and Viktor Schroeder. Embedding of hyperbolic groups into products of binary trees. *Invent. Math.*, 169(1):153–192, 2007.

[BE12] Laurent Bartholdi and Anna Erschler. Growth of permutational extensions. *Invent. Math.*, 189(2):431–455, 2012.

[BE14] Laurent Bartholdi and Anna Erschler. Imbeddings into groups of intermediate growth. *Groups Geom. Dyn.*, 8(3):605–620, 2014.

[BNŠ+13] Jacek Brodzki, Graham A. Niblo, Ján Špakula, Rufus Willett, and Nick Wright. Uniform local amenability. *J. Noncommut. Geom.*, 7(2):583–603, 2013.

[Bou86] J. Bourgain. The metrical interpretation of superreflexivity in Banach spaces. *Israel J. Math.*, 56(2):222–230, 1986.

[BPV14] Bachir Bekka, Thibault Pillon, and Alain Valette. Irreducible affine isometric actions. arXiv:1411.2267, 2014.

[Bri14] Jérémie Brieussel. Growth behaviors in the range e^{r^α}. *Afr. Mat.*, 25(4):1143–1163, 2014.

[BS08] N. Brodsky and D. Sonkin. Compression of uniform embeddings into Hilbert space. *Topology Appl.*, 155(7):725–732, 2008.

[BV97] Mohammed E. B. Bekka and Alain Valette. Group cohomology, harmonic functions and the first L^2-Betti number. *Potential Anal.*, 6(4):313–326, 1997.

[CCJ+01] Pierre-Alain Cherix, Michael Cowling, Paul Jolissaint, Pierre Julg, and Alain Valette. *Groups with the Haagerup property*, volume 197 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001. Gromov's a-T-menability.

[CdCMT12] P.-E. Caprace, Y. de Cornulier, N. Monod, and R. Tessera. Amenable hyperbolic groups. *preprint, arXiv:1202.3585v1.*, 2012.

[CG86] Jeff Cheeger and Mikhael Gromov. L_2-cohomology and group cohomology. *Topology*, 25(2):189–215, 1986.

[CN05] Sarah Campbell and Graham A. Niblo. Hilbert space compression and exactness of discrete groups. *J. Funct. Anal.*, 222(2):292–305, 2005.

[CTWY08] Xiaoman Chen, Romain Tessera, Xianjin Wang, and Guoliang Yu. Metric sparsification and operator norm localization. *Adv. Math.*, 218(5):1496–1511, 2008.

[CV15] Yves Cornulier and Alain Valette. On equivariant embeddings of generalized Baumslag–Solitar groups. *Geom. Dedicata*, 175:385–401, 2015.

[dCTV07] Yves de Cornulier, Romain Tessera, and Alain Valette. Isometric group actions on Hilbert spaces: growth of cocycles. *Geom. Funct. Anal.*, 17(3):770–792, 2007.

[DG03] Marius Dadarlat and Erik Guentner. Constructions preserving Hilbert space uniform embeddability of discrete groups. *Trans. Amer. Math. Soc.*, 355(8):3253–3275 (electronic), 2003.

[Dre11] Dennis Dreesen. Hilbert space compression for free products and HNN-extensions. *J. Funct. Anal.*, 261(12):3585–3611, 2011.

[For76] Gunnar Forst. The Lévy-Hinčin representation of negative definite functions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 34(4):313–318, 1976.

[GJ03] Światosław R. Gal and Tadeusz Januszkiewicz. New a-T-menable HNN-extensions. *J. Lie Theory*, 13(2):383–385, 2003.

[GK04] Erik Guentner and Jerome Kaminker. Exactness and uniform embeddability of discrete groups. *J. London Math. Soc. (2)*, 70(3):703–718, 2004.

[Gro93] M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, volume 182 of *London Math. Soc. Lecture Note Ser.*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.

[Gro95] Mikhael Gromov. Geometric reflections on the Novikov conjecture. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, volume 226 of *London Math. Soc. Lecture Note Ser.*, pages 164–173. Cambridge Univ. Press, Cambridge, 1995.

[Gro00] Misha Gromov. Spaces and questions. *Geom. Funct. Anal.*, (Special Volume, Part I):118–161, 2000. GAFA 2000 (Tel Aviv, 1999).

[Gui72] Alain Guichardet. Sur la cohomologie des groupes topologiques. II. *Bull. Sci. Math. (2)*, 96:305–332, 1972.

[Hum] David Hume. Direct embeddings of relatively hyperbolic groups with optimal p compression exponent. to appear in J. Reine Angew. Math., arxiv:1111.6013.

[Jon] Vaughan F. R. Jones. Von Neumann algebras. available at http://www.math.ucdavis.edu/ kaminker/VonNeumann.pdf.

[JP13] Pierre-Nicolas Jolissaint and Thibault Pillon. L^p compression of some HNN extensions. *J. Group Theory*, 16(6):907–913, 2013.

[Kes59] Harry Kesten. Full Banach mean values on countable groups. *Math. Scand.*, 7:146–156, 1959.

[KY06] Gennadi Kasparov and Guoliang Yu. The coarse geometric Novikov conjecture and uniform convexity. *Adv. Math.*, 206(1):1–56, 2006.

[LSV04] Nicolas Louvet, Yves Stalder, and Alain Valette. Fonctions conditionnellement de type négatif, représentations irréductibles et propriété (T). *Enseign. Math. (2)*, 50(3-4):239–266, 2004.

[LZ05] Alexander Lubotzky and Andrzej Żuk. On property (τ). in preparation, available at http://www.ma.huji.ac.il/ alexlub/BOOKS/On2005.

[Mos71] G. D. Mostow. Some applications of representative functions to solvmanifolds. *Amer. J. Math.*, 93:11–32, 1971.

[Ner98] Yurii A. Neretin. Notes on affine isometric actions of discrete groups. In *Analysis on infinite-dimensional Lie groups and algebras (Marseille, 1997)*, pages 274–320. World Sci. Publ., River Edge, NJ, 1998.

[Now07] Piotr W. Nowak. Coarsely embeddable metric spaces without Property A. *J. Funct. Anal.*, 252(1):126–136, 2007.

[NP08] Assaf Naor and Yuval Peres. Embeddings of discrete groups and the speed of random walks. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 076, 34, 2008.

[NP11] Assaf Naor and Yuval Peres. L_p compression, traveling salesmen, and stable walks. *Duke Math. J.*, 157(1):53–108, 2011.

[OO13] Alexander Yu. Olshanskii and Denis V. Osin. A quasi-isometric embedding theorem for groups. *Duke Math. J.*, 162(9):1621–1648, 2013.

[Osa] Damian Osajda. Small cancellation labellings of some infinite graphs and applications. arxiv:1406.5015.

[Ped79] Gert K. Pedersen. *C^*-algebras and their automorphism groups*, volume 14 of *London Mathematical Society Monographs*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979.

[Ped89] Gert K. Pedersen. *Analysis now*, volume 118 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989.

[RW14] John Roe and Rufus Willett. Ghostbusting and property A. *J. Funct. Anal.*, 266(3):1674–1684, 2014.

[Sak14] Hiroki Sako. Property A and the operator norm localization property for discrete metric spaces. *J. Reine Angew. Math.*, 690:207–216, 2014.

[Sal] Andrew Sale. The geometry of the conjugacy problem in wreath products and free solvable groups. arxiv:1307.6812.

[Ser77] Jean-Pierre Serre. *Arbres, amalgames,* SL_2. Société Mathématique de France, Paris, 1977. Avec un sommaire anglais, Rédigé avec la collaboration de Hyman Bass, Astérisque, No. 46.

[Sha00] Yehuda Shalom. Rigidity of commensurators and irreducible lattices. *Invent. Math.*, 141(1):1–54, 2000.

[Tes11] Romain Tessera. Asymptotic isoperimetry on groups and uniform embeddings into Banach spaces. *Comment. Math. Helv.*, 86(3):499–535, 2011.

[Yu00] Guoliang Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139(1):201–240, 2000.

[Yu05] Guoliang Yu. Hyperbolic groups admit proper affine isometric actions on l^p-spaces. *Geom. Funct. Anal.*, 15(5):1144–1151, 2005.