Published in Journal of Philosophical Logic 29, issue 3, 295-313, 2000
which should be used for any reference to this work
The original publication is available at www.springerlink.com

PROPOSITIONAL LOGIC OF ESSENCE

FABRICE CORREIA

ABSTRACT. This paper presents a propositional version of Kit Fine's (quantified) logic for essentialist statements, provides it with a semantics, and proves the former adequate (i.e. sound and complete) with respect to the latter.

KEY WORDS: essence, logic, semantics.

INTRODUCTION

The present paper can be considered as a companion to Kit Fine's papers 'The Logic of Essence' and 'Semantics for the Logic of Essence'.[1] In the first paper Fine presents a logical system for quantified essentialist statements, E5.[2] In the second he presents a semantics for a variant of the system, and proves this system adequate (i.e. sound and complete) with respect to that semantics. I propose here a Kripke-style semantics for E5π, a propositional counterpart of E5, and prove the adequacy of the latter with respect to the former.

There are many, more or less natural, more or less interesting, ways to extend E5π (or one of its cousins) to a system of quantified logic of essence. E5π, together with its semantics, is intended to constitute the core of subsequent, more expressive, logics of essence. So, the study of E5π *per se*, regardless of possible quantificational extensions, is of great interest. Another interesting point about the present study lies in the fact that the completeness proof given here is much simpler than the one Fine gives for his quantificational system.

The reader is strongly urged to take a look at Fine's papers on the logic of essence, if only because no detailed comparison between Fine's material and mine will be offred.

I. THE LANGUAGE

In E5π, the basic essentialist statements have the form $\Box_X A$, A being a formula and X a delimiter. Delimiters are intended to be names for classes, and formulas of type $\Box_X A$ are to be read as 'it is true in virtue of the nature of some elements of X that A'. The language, $\mathcal{L}$, in which we formulate E5π, has the following vocabulary:

(i) a denumerable stock of basic delimiters, containing the universal delimiter V;
(ii) the union operator $+$;
(iii) the dependence closure operator c;
(iv) the constituency operator $|\ |$;
(v) the identity operator $\approx$;
(vi) a denumerable stock of sentence letters;
(vii) the truth-functional connectives $\sim$ and &;
(viii) the essentialist operator $\Box$;
(ix) the bracketing devices (and).

The *delimiters* and *formulas* of $\mathcal{L}$ are defined by the following simultaneous induction (X, Y, etc. will always be used for arbitrary delimiters and A, B, etc. for arbitrary formulas):

(i) every basic delimiter is a delimiter;
(ii) if X and Y are delimiters, so is $(X + Y)$;
(iii) if X is a delimiter, so is cX;
(iv) if A is a formula, $|A|$ is a delimiter;
(v) if X and Y are delimiters, $(X{\approx}Y)$ is a formula;
(vi) sentence letters are formulas;
(vii) if A and B are formulas, so are $(\sim A)$ and $(A \& B)$;
(viii) if X is a delimiter and A a formula, $(\Box_X A)$ is a formula.

Material implication ($\supset$) is defined in the standard way, and we use:

$$\Diamond_X A \text{ for } \sim\Box_X{\sim}A, \text{ and}$$
$$X < Y \text{ for } X + Y \approx Y.$$

Finally, a *d-atom* is a formula of type $X \approx Y$, for X and Y delimiters, and a *d-formula* is a truth-functional compound of d-atoms.

The intended meanings of symbols V, $+$, $c|\ |$ and $\approx$ remain to be explained:

- Symbol V, the universal delimiter, is intended to denote the class of all possible objects.
- Symbol $+$ is intented to express class union. So, if X and Y are delimiters, then under the present interpretation $X + Y$ denotes the class consisting of all members of X and Y (and only them).

- The identity operator $\approx$ is intended to express a form of class identity – call it identity*. Where C and D are classes, 'C is identical* to D' means 'each member of C is identical to some member of D, and each member of D is identical to some member of C'. (Accordingly, symbol $<$ is intended to express class inclusion*, where 'C is included* in D' means 'each member of C is identical to some member of D'.) So, for instance, '$\{a\}$ is identical* to $\{b\}$' means 'a is identical to b' – and not '$\{a\}$ is identical to $\{b\}$'. Of course, for any classes C and D, C is identical* to D iff C is identical to D. But the difference between identity* and identity is dramatic in the present context. Indeed, it is a theorem of E5π that $\Box_X X \approx X$. But while it is plausible that it is true in virtue of the nature of Socrates that Socrates = Socrates (i.e. that {Socrates} be identical* to {Socrates}), we may not want to say that it is true in virtue of the nature of Socrates that {Socrates} = {Socrates}.[3]
- The notion of the *constituents of a proposition*, of the objects involved in that proposition, is central to E5π. For each formula A the delimiter $|A|$ is intended to refer to the class of all the constituents of the proposition expressed by A.
- The notion of *ontological dependence* is also entral to E5π.[4] One of Fine's crucial claims about essence is this: if P is true in virtue of the nature of an object x, then x depends on each constituent of the proposition expressed by P. This thesis generalizes to the following, more suited to the form of the essentialist claims in E5π: if P is true in virtue of the nature of some elements of class C, then for every constituent y of the proposition expressed by P there is some element x of C such that x depends on y – in bastard notation: $\Box_C P \supset [\forall y \in |P|\ \exists x \in C\ dep(x, y)]$. Since there is no quantification in E5π, we must find a different means of expressing the claim. This is the role of the dependence closure operator c. Ontological dependence, as here construed, is a reflexive transitive relation. Let S be a non-empty set, $\mathcal{P}(S)$ its power set, and *dep* a binary reflexive transitive relation defined on S. Let dep^* be the function from $\mathcal{P}(S)$ onto itself defined by $dep^*(s) =_{\text{df}} \{y \in S : \exists x\ x \in s\ \&\ dep(x, y)\}$ for every $s \in \mathcal{P}(S)$. It is easy to check that for every s and t in $\mathcal{P}(S)$:

  (i) if $s \subset t$, then $dep^*(s) \subset dep^*(t)$,

  (ii) $s \subset dep^*(s)$,

  (iii) $dep^*(dep^*(s)) = dep^*(s)$, and

  (iv) $dep^*(s \cup t) = dep^*(s) \cup dep^*(t)$.

Conditions (i)–(iii) state that dep^* is a closure operator on $\mathcal{P}(S)$, and condition (iv) states that dep^* is an endomorphism of $\langle \mathcal{P}(S), \cup \rangle$. The dependence closure c is meant to refer to a function on classes similar to *dep** above, i.e. satisfying (i)–(iv).

II. The system

The axioms and rules of system E5π are grouped under the following five headings. The axioms (2)–(3) are motivated by the above discussion, those of (4) are indeed theorems of E5, the rule (5) (iv) is a rule of E5, and finally the axioms in (5) are roughly Fine's, but adapted to our new language.

(1) *Classical*
Every $\mathcal{L}$-instance of a tautology counts as an axiom, and Modus Ponens as a rule, of E5π.

(2) *Delimiters*

(i) $X \approx X$

(ii) $X \approx X \supset Y \approx X$

(iii) $(X \approx Y \,\&\, Y \approx Z) \supset X \approx Z$

(iv) $X \approx Y \supset X + Z \approx Y + Z$

(v) $X + X \approx X$

(vi) $X + Y \approx Y + X$

(vii) $X + (Y + Z) \approx (X + Y) + Z$

(viii) $X < V$

(3) *Closure*

(i) $X < Y \supset cX < cY$

(ii) $X < cX$

(iii) $ccX \approx cX$

(iv) $c(X + Y) \approx cX + cY$

(4) *Constituency*

(i) $|X \approx Y| \approx X + Y$

(ii) $|{\sim}A| \approx |A|$

(iii) $|A\&B| \approx |A| + |B|$

(iv) $|\Box_X A| \approx X + |A|$

(5) *Essence*

(i)	$\Box_X A \supset A$	T-axiom
(ii)	$\Box_X (A \supset B) \supset (\Box_X A \supset \Box_X B)$	K-axiom
(iii)	$\Diamond_X A \supset \Box_{X+\|A\|} \Diamond_X A$	E-axiom
(iv)	$A \backslash \Box_{\|A\|} A$	Necessitation
(v)	$X < Y \supset (\Box_X A \supset \Box_Y A)$	Subsumption
(vi)	$X \approx Y \supset \Box_{X+Y} X \approx Y$	Rigidity
(vii)	$\Box_{cX} A \supset \Box_X A$	Chaining
(viii)	$\Box_X A \supset \|A\| < cX$	Localization

The part of E5π determined by (1)–(4) only constitutes what I shall call the 'logic of delimiters'. The theorems of the logic of delimiters have little intrinsic interest, and those which will be used below are easy to prove. They will be exploited without demonstration.

Some of the following propositions will be useful in later parts of this paper:

PROPOSITION II.1.

(i) *If* $\vdash A$ *then* $\vdash |A| < cX \supset \Box_X A$.

(ii) *If* $\vdash A \supset B$ *then* $\vdash |A| + |B| < cX \supset (\Box_X A \supset \Box_X B)$.

(iii) *If* $\vdash A \supset B$ *then* $\vdash |A| + |B| < cX \supset (\Diamond_X A \supset \Diamond_X B)$.

(iv) $\vdash (\Box_X A \,\&\, \Box_X B) \supset \Box_X (A \,\&\, B)$

(v) $\vdash \Box_X A \supset \Box_X \Box_X A$

(vi) *For every d-formula* A, $\vdash A \supset \Box_{|A|} A$.

Proof. (i) Suppose $\vdash A$. By Necessitation, it follows that $\vdash \Box_{|A|} A$. By Subsumption, $\vdash |A| < cX \supset (\Box_{|A|} A \supset \Box_{cX} A)$. So, $\vdash |A| < cX \supset \Box_{cX} A$. Finally, Chaining gives the result.

(ii) Assume $\vdash A \supset B$. By (i), $\vdash |A \supset B| < cX \supset \Box_X (A \supset B)$. Then by the K-axiom, $\vdash |A \supset B| < cX \supset (\Box_X A \supset \Box_X B)$. The result follows by the logic of delimiters.

(iii) Essentially from (ii).

(iv) $\vdash A \supset (B \supset A \& B)$ by truth-functional logic. By (ii) and the logic of delimiters, it follows that $\vdash |A| + |B| < cX \supset (\Box_X A \supset \Box_X (B \supset A \,\&\, B))$. Then by the K-axiom $\vdash |A| + |B| < cX \,\&\, \Box_X A \,\&\, \Box_X B \supset \Box_X (A \,\&\, B)$. But by Localization and the logic of delimiters, $\vdash \Box_X A \,\&\, \Box_X B \supset |A| + |B| < cX$. The result follows.

(v) By the T-axiom, $\vdash \Box_X A \supset \Diamond_{X+|A|} \Box_X A$. By the E-axiom (and the logic of delimiters and Subsumption), $\vdash \Diamond_{X+|A|} \Box_X A \supset \Box_{X+|A|} \Diamond_{X+|A|} \Box_X A$. By the E-axiom again, $\vdash \Diamond_{X+|A|} \Box_X A \supset \Box_X A$. These three theorems entail that $\vdash \Box_X A \supset \Box_{X+|A|}\Box_X A$. Now by Localization and the logic of delimiters, $\vdash \Box_X A \supset X + |A| < cX$. We thus have $\vdash \Box_X A \supset (X + |A| < cX \,\&\, \Box_{X+|A|}\Box_X A)$. By Subsumption, it follows that $\vdash \Box_X A \supset \Box_{cX}\Box_X A$. The result then follows by Chaining.

(vi) By induction on the length of the formulas.

(a) If A is a d-atom, we have $\vdash A \supset \Box_{|A|}A$ by Rigidity and the logic of delimiters.

(b) Suppose A is $\sim B$. By IH, $\vdash B \supset \Box_{|B|}B$. By (iii) and the logic of delimiters, it follows that $\vdash \Diamond_{|B|}B \supset \Diamond_{|B|}\Box_{|B|}B$. By the E-axiom and the T-axiom, we have then $\vdash \Diamond_{|B|}B \supset B$. Consequently, by the logic of delimiters and Subsumption, $\vdash \sim B \supset \Box_{|\sim B|}\sim B$.

(c) Suppose A is $B \& C$. By IH, $\vdash B \supset \Box_{|B|}B$ and $\vdash C \supset \Box_{|C|}C$. By Subsumption and the logic of delimiters, then, $\vdash B \supset \Box_{|B\&C|}B$ and $\vdash C \supset \Box_{|B\&C|}C$. Thus $\vdash B\&C \supset \Box_{|B\&C|}B \,\&\, \Box_{|B\&C|}c$. By (iv) it follows that $\vdash B\&C \supset \Box_{|B\&C|}(B\&C)$.

III. The semantics

Before defining the models for E5π, some notions have to be introduced.

• [Semilattice] A *semilattice* is a couple $\langle S, \sqcup\rangle$, where S is a non-empty set and $\sqcup$ a binary operation on S, such that for every x, y and z in S:

(i) $x \sqcup x = x$,

(ii) $x \sqcup y = y \sqcup x$, and

(iii) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$.

Given a semilattice $\langle S, \sqcup\rangle$, we define the partial order $\sqsubset$ on S by $x \sqsubset y$ iff $x \sqcup y = y$.

An object 1 is said to be a *greatest element* of a semilattice $\langle S, \sqcup\rangle$ iff $1 \in S$ and for every $x \in S$, $x \sqcup 1 = 1$ (i.e. $x \sqsubset 1$). A semilattice has at most one greatest element.

• [Dependence structure] A *closure operator* on a semilattice $\langle S, \sqcup\rangle$ is a function γ from S onto S such that for every x and y in S:

(i) if $x \sqsubset y$, then $\gamma(x) \sqsubset \gamma(y)$,

(ii) $x \sqsubset \gamma(x)$, and

(iii) $\gamma(\gamma(x)) = \gamma(x)$.

An element x of a semilattice $\langle S, \sqcup\rangle$ is said to be *closed* (with respect to the closure operator γ defined on $\langle S, \sqcup\rangle$) iff there is some y in S such that

$x = \gamma(y)$, i.e. iff $\gamma(x) = x$. It should be noticed that if $\langle S, \sqcup \rangle$ has a greatest element 1, then 1 is closed for any closure operator defined on $\langle S, \sqcup \rangle$.

A closure operator γ defined on a semilattice $\langle S, \sqcup \rangle$ is said to be a *dependence closure operator* iff γ is an endomorphism of $\langle S, \sqcup \rangle$, i.e. iff for every x and y in S, $\gamma(x \sqcup y) = \gamma(x) \sqcup \gamma(y)$.

A *dependence structure* is any quadruple $\langle S, 1, \sqcup, \gamma \rangle$, where $\langle S, \sqcup \rangle$ is a semilattice with greatest element 1 and γ is a dependence closure operator on $\langle S, \sqcup \rangle$.

• [Constituency model] Where $\Delta = \langle S, 1, \sqcup, \gamma \rangle$ is a dependence structure, we define a *constituency function* for $\mathcal{L}$ on Δ as a function i which assigns to each nonlogical primitive expression of $\mathcal{L}$ (basic delimiter or sentence letter) a member of S, such that $i(V) = 1$. A *constituency model* for language $\mathcal{L}$ is a pair $\langle \Delta, i \rangle$, where Δ is a dependence structure and i is a constituency function for $\mathcal{L}$ on Δ.

Given any constituency model $\langle \Delta, i \rangle$, with $\Delta = \langle S, 1, \sqcup, \gamma \rangle$, function i is extended to every delimiter and every formula by the following conditions:[5]

(1) $(X \approx Y)^i = X^i \sqcup Y^i$,

(2) $(\sim A)^i = A^i$,

(3) $(A \& B)^i = A^i \sqcup B^i$,

(4) $(\Box_X A)^i = X^i \sqcup A^i$,

(5) $(X + Y)^i = X^i \sqcup Y^i$,

(6) $(cX)^i = \gamma(X^i)$, and

(7) $|A^i| = A^i$.

The notion of *satisfaction of a d-formula* by a constituency model $\langle \Delta, i \rangle$ is defined recursively in the obvious way:

(i) a d-atom $X \approx Y$ is satisfied by $\langle \Delta, i \rangle$ iff $X^i = Y^i$
(ii) $\sim A$ is satisfied by $\langle \Delta, i \rangle$ iff A is not satisfied by $\langle \Delta, i \rangle$
(iii) $A \& B$ is satisfied by $\langle \Delta, i \rangle$ iff A and B are both satisfied by $\langle \Delta, i \rangle$.

• [E5π-model] An *E5π-model* for $\mathcal{L}$ is a 6-tuple $\langle W, @, \Delta, \text{dom}, i, [\cdot] \rangle$, where:

(i) W (worlds) is a set,
(ii) @ (the actual world) is in W,
(iii) $\Delta = \langle S, 1, \sqcup, \gamma \rangle$ is a dependence structure,
(iv) dom is a surjective function from W onto $\gamma(S)$ such that dom(@) $= 1$ ($\gamma(S)$ is the set of all closed elements of S),
(v) i is a constituency function for $\mathcal{L}$ on Δ, and

(vi) $[\cdot]$ is a function (a 'truth-set function') taking each sentence letter A of $\mathcal{L}$ into some subset W' of W, with the condition that for every w in W', $A^i \sqsubset \text{dom}(w)$.

For every element x of S, we shall say that $w \in W$ is an x-*world* iff $x \sqsubset \text{dom}(w)$, and we define W_x as the set of all x-worlds. Since for every $x \in S, x \sqsubset 1 = \text{dom}(@)$, each W_x contains @ and thus is not empty. The condition expressed in (vi) can then be rephrased as requiring that for every sentence letter A, $[A]$ be a part of W_{A^i}. The proof of the following proposition is straightforward:

PROPOSITION III.1. *For every elements x and y of S:*

(i) $x \sqsubset \gamma(y)$ *iff* $W_y \subset W_x$.

(ii) $\gamma(x) = \gamma(y)$ *iff* $W_x = W_y$.

(iii) $W_{x \sqcup y} = W_x \cap W_y$.

(iv) $W_{\gamma(x)} = W_x$.

(v) $@ \in W_1 \subset W_x$.

• [Truth] For every E5π-model $\langle W, @, \Delta, \text{dom}, i, [\cdot]\rangle$, function $[\cdot]$ is extended to the set of all formulas by the following conditions:

(1) $[X \approx Y] =$ (i) W_{A^i} if $X^i = Y^i$,
(ii) $\emptyset$ otherwise.

(2) $[\sim A] = W_{A^i} - [A]$.

(3) $[A\&B] = [A] \cap [B]$.

(4) $[\Box_X A] =$ (i) W_{X^i} if $W_{X^i} \subset [A]$,
(ii) $\emptyset$ otherwise.

Note that for every d-formula A, $[A] = W_{A^i}$ iff A is satisfied by constituency model $\langle \Delta, i\rangle$.

One can easily verify that for every formula A:

(i) $[A] \subset W_{A^i}$,

(ii) $[A] \cup [\sim A] = W_{A^i}$, and

(iii) $[A] \cap [\sim A] = \emptyset$.

Let us define truth-at-a-world and falsity-at-a-world by the following natural conditions:

A is true at w iff $w \in [A]$, and
A is false at w iff $w \in [\sim A]$.

(ii) above implies that formula A has a truth value at world w iff w is an A^i-world, that is, iff the constituents of the proposition expressed by A belong to the domain of w. Notice that consequently, every formula has a truth value at the actual world.

Here are some consequences of the conditions on $[\cdot]$ in terms of truth values:

(1) A sentence letter A has a truth value at world w iff w is an A^i-world.
(2) Let A be d-atom $X \approx Y$.

    (i) A has a truth value at w iff w is an A^i-world (i.e. iff w is both an X^i-world and a Y^i-world).
    (ii) A is true at w iff A has a truth value at w and $X^i = Y^i$.
    (iii) A is false at w iff A has a truth value at w and $X^i \neq Y^i$.

(3) (i) $\sim A$ has a truth value at w iff w is a $(\sim A)^i$-world (i.e. iff w is an A^i-world).
    (ii) $\sim A$ is true at w iff A is false at w.
    (iii) $\sim A$ is false at w iff A is true at w.
(4) (i) $A\&B$ has a truth value at w iff w is an $(A\&B)^i$-world (i.e. iff w is both an A^i-world and a B^i-world).
    (ii) $A\&B$ is true at w iff both A and B are true at w.
    (iii) $A\&B$ is false at w iff both A and B have a truth value at w and at least one of them is false at w.
(5) (i) $\Box_X A$ has a truth value at w iff w is a $(\Box_X A)^i$-world (i.e. iff w is both an X^i-world and an A^i-world).
    (ii) $\Box_X A$ is true at w iff $X^i \sqsubset \text{dom}(w)$ and for every w' such that $X^i \sqsubset \text{dom}(w')$, A is true at w'.
    (iii) $\Box_X A$ is false at w iff $X^i \sqcup A^i \sqsubset \text{dom}(w)$ and for some w' such that $X^i \sqsubset \text{dom}(w')$, either it is not the case that $A^i \sqsubset \text{dom}(w')$ or A is false at w'.

The conditions at the actual world are more standard:

(1) every sentence letter has a truth value.
(2) d-atom $X \approx Y$ is true iff $X^i = Y^i$ (and false iff $X^i \neq Y^i$).
(3) $\sim A$ is true iff A is false (and $\sim A$ is false iff A is true).
(4) $A\&B$ is true iff both A and B are true (and $A\&B$ is false iff A or B is false).
(5) $\Box_X A$ is true iff A is true at every X^i-world (and $\Box_X A$ is false iff either A lacks a truth value at some X^i-world or A is false at some X^i-world).

Notice the analogy with the modal (S5) truth-conditions for the necessity operator $\Box$. The analogy amounts to identity in the case of the universal essentialist operator $\Box_V$.

• [Validity] A formula A is said to be *valid on E5π-model* $\langle W, @, \Delta, \text{dom}, i, [\cdot]\rangle$ iff $[A] = W_{A^i}$. That is, A is valid on a model iff it is true at every world of that model where it has a truth value. A formula A is said to be *valid* iff A is valid on every E5π-model. We already saw that given any model $\langle W, @, \Delta, \text{dom}, i, [\cdot]\rangle$, with $\Delta = \langle S, 1, \sqcup, \gamma, i\rangle$, for every d-formula A, $[A] = W_{A^i}$ iff A is satisfied by constituency model $\langle S, 1, \sqcup, \gamma\rangle$. Consequently, a d-formula is valid on an E5π-model iff it is satisfied by the corresponding constituency model, and it is valid iff it is satisfied by every constituency model.

Before turning to the adequacy theorems, let me briefly explain the intended interpretation of the semantics. Let D be the class of all possible objects, that is, the class of everything that can be referred to or quantified over. Under a standard version of possible world semantics for quantified modal languages, each world w is assigned a subclass D_w of D, its 'domain', which is the class of all objects *existing* in w, and even if the domain of a world w is strictly included in D, every member of D can be referred to 'from' w (e.g. for object a not in D_w, '$a = a$' is still true at w). Now, in the semantics presented above, the domains are not to be thought of in the same way. The domain of world w is intended to be the class of all *possible objects* of w, that is, of all the objects than can be referred to or quantified over from w.

The requirement that the domains be closed (under the dependence relation) may be argued for as follows. Assume that object x belongs to the domain of world w, and that consequently x can be referred to from w. Suppose, moreover, that x depends on y, i.e. that x's essence contains y as a constituent, or equivalently, that y belongs to some proposition that is true in virtue of x's nature. Then such a proposition is true at w, and thus y belongs to the domain of w. Conversely, arguably the only reason why a part C of D could not be a domain is that C contains an object x whose nature contains an object y not in C – i.e. that C is not closed under dependency. Hence the surjectivity requirement for function dom.

Under the intended interpretation of our semantics, the actual world is assigned the biggest possible domain, namely D, the class of all possible objects.

Let us turn now to the truth conditions. As we saw, our semantic rules have the consequence that a sentence may be neither true nor false at some world w. Indeed, A has a truth value at w iff w is an A^i-world, that is, iff the constituents of the proposition expressed by A belong to the domain of w. This point is in accordance with the previous discussion, provided that a certain claim is accepted. The claim is that one can only refer to (or quantify over) possible entities. I shall not discuss this, but, if it is accepted, then on the view that a sentence containing denotationless terms has no truth value, the semantical point under discussion is surely correct.

The truth conditions for negation, conjunction and essence all seem quite natural. And similarly for the truth conditions for d-formulas. The only substantial point about them is this: at a given world w, a d-formula A is either deprived of a truth value (this is the case where some constituent of the proposition expressed by A is not in w), or essentially true, or essentially false – to be more precise: true/false in virtue of the constituents of the proposition it expresses. But, I think, the point is harmless, for under the intended interpretation of $\mathcal{L}$, d-formulas basically express identities.

IV. SOUNDNESS

PROPOSITION IV.1. *E5π is sound* (*with respect to the class of E5π-models*).

Proof. The proof does not present any special difficulty. However, I shall go through it in some details.

(1) As we saw, a d-formula is valid on an E5π-model iff it is satisfied by the underlying constituency model. And obviously, the axioms for delimiters, closure and constituency are satisfied by every constituency model.

(2) It is also clear that every PC-tautology is valid on every E5π-model. Modus Ponens is validity-preserving, as the following argument shows. Let A and B be two formulas, with A and $A \supset B$ valid. Let $M = \langle W, @, \Delta, \mathrm{dom}, i, [\cdot] \rangle$ be an arbitrary E5π-model, with w in W a B^i-world. Now we must prove that B is true at w in M. Let $N = \langle W, @, \Delta, \mathrm{dom}, k, \{\cdot\} \rangle$ be the E5π-model defined by the following conditions (X is any basic delimiter, and C is any sentence letter):

$$X^k = \text{(i) dom}(w) \text{ if } X \text{ is in } A \text{ but not in } B,$$
$$\text{(ii) } X^i \text{ otherwise.}$$
$$C^k = \text{(i) dom}(w) \text{ if } C \text{ is in } A \text{ but not in } B,$$
$$\text{(ii) } X^i \text{ otherwise.}$$
$$\{C\} = \text{(i) } W_{\mathrm{dom}(w)} \text{ if } C \text{ is in } A \text{ but not in } B,$$
$$\text{(ii) } [C] \text{ otherwise.}$$

By construction, w is both an A^k-world and an $(A \supset B)^k$-world. So, since A and $A \supset B$ are valid, A and $A \supset B$ are both true at w in N, and consequently, B is true at w in N as well. But now, one can prove that B is true at w in N iff B is true at w in M. So, B is true at w in M.

For the remaining axioms and rule, let $\langle W, @, \Delta, \text{dom}, i, [\cdot]\rangle$ be an arbitrary E5π-model, with $\Delta = \langle S, 1, \sqcup, \gamma\rangle$.

(3) *T-axiom*. Let w be in $W_{X^i} \cap W_{A^i}$, and suppose that $w \in [\Box_X A]$. Then $[\Box_X A] \neq \emptyset$, which implies that $W_{X^i} \subset [A]$. But then since $w \in W_{X^i}$, $w \in [A]$.

(4) *K-axiom*. Let w be in $W_{X^i} \cap W_{A^i} \cap W_{B^i}$, and suppose that $w \in [\Box_X (A \supset B)]$ and $w \in [\Box_X A]$. This implies: $W_{X^i} \subset [A \supset B]$ and $W_{X^i} \subset [A]$. Now, $[A \supset B] = W_{A^i} \cap W_{B^i} \cap (-[A] \cup [B])$. But since $W_{X^i} \subset [A]$, it is not the case that $W_{X^i} \subset -[A]$, and thus since $W_{X^i} \subset [A \supset B]$, we have: $W_{X^i} \subset [B]$. Therefore, $[\Box_x B] = W_{X^i}$. Now since $w \in W_{X^i}$, $w \in [\Box_X B]$.

(5) *E-axiom*. Let w be in $W_{X^i} \cap W_{A^i}$, and suppose that $w \in [\sim\Box_X A]$. By definition:

(i) $[\Box_{X+|A|}\sim\Box_X A] = W_{X^i} \cap W_{A^i}$ if $W_{X^i} \cap W_{A^i} \subset [\sim\Box_X A]$, and $[\Box_{X+|A|}\sim\Box_X A] = \emptyset$ otherwise, and

(ii) $[\sim\Box_X A] = \emptyset$ if $W_{X^i} \subset [A]$, and $[\sim\Box_X A] = W_{X^i} \cap W_{A^i}$ otherwise. Now, since $w \in [\sim\Box_X A]$, $[\sim\Box_X A] \neq \emptyset$, and thus by (ii) $[\sim\Box_X A] = W_{X^i} \cap W_{A^i}$. By (i) it follows that $[\Box_{X+|A|}\sim\Box_X A] = W_{X^i} \cap W_{A^i}$. But since $w \in W_{X^i} \cap W_{A^i}$, we have $w \in [\Box_{X+|A|}\sim\Box_X A]$.

(6) *Necessitation*. Suppose that A is valid in M, i.e. that $[A] = W_{A^i}$. By definition: $[\Box_{|A|} A] = W_{A^i}$ if $W_{A^i} \subset [A]$, and $[\Box_{|A|} A] = \emptyset$ otherwise. Therefore $[\Box_{|A|} A] = W_{A^i}$.

(7) *Subsumption*. Let w be in $W_{X^i} \cap W_{Y^i} \cap W_{A^i}$, and suppose that (a) $w \in [X < Y]$ and (b) $w \in [\Box_X A]$. We have then: (a) $X^i \sqsubset Y^i$ and consequently $W_{Y^i} \subset W_{X^i}$, and (b) $W_{X^i} \subset [A]$. So, $W_{Y^i} \subset [A]$, and then $W_{Y^i} = [\Box_X A]$. Now since $w \in W_{X^i}$, $w \in [\Box_X A]$.

(8) *Rigidity*. Let w be in $W_{X^i} \cap W_{Y^i}$. Suppose $w \in [X \approx Y]$. We then have: $[X \approx Y] \neq \emptyset$, and thus $[X \approx Y] = W_{(X+Y)^i}$. But this implies that $[\Box_{X+Y} X \approx Y] = W_{(X+Y)^i}$. Since $W_{(X+Y)^i} = W_{X^i} \cap W_{Y^i}$, we have: $w \in [\Box_{X+Y} X \approx Y]$.

(9) *Chaining*. Let w be in $W_{X^i} \cap W_{A^i}$. Suppose $w \in [\Box_{cX} A]$. We then have: $[\Box_{cX} A] \neq \emptyset$, and thus $W_{(cX)^i} \subset [A]$. But since $W_{(cX)^i} = W_{X^i}$, we have $W_{X^i} \subset [A]$, and consequently $[\Box_X A] = W_{X^i}$. Thus $w \in [\Box_X A]$.

(10) *Localization*. Let w be in $W_{X^i} \cap W_{A^i}$. Suppose $w \in [\Box_X A]$. We then have: $[\Box_X A] \neq \emptyset$, and thus $W_{X^i} \subset [A]$. But since $[A] \subset W_{A^i}$, we have $W_{X^i} \subset W_{A^i}$. Hence, $A^i \sqsubset \gamma(X^i) = (cX)^i$. Thus $\langle S, 1, \sqcup, \gamma, i\rangle$ satisfies $|A| < cX$, and then $[|A| < cX] = W_{X^i} \cap W_{A^i}$. Consequently $w \in [|A| < cX]$.

V. Completeness

Let α be a non-theorem of E5π, i.e. be such that $\nvdash \alpha$, and let @ be an $\mathcal{L}$-maximal E5π-consistent extension of $\{\sim\alpha\}$ (we may use a Lindenbaum-type construction to prove the existence of @).

PROPOSITION V.1. *Every theorem of E5π is contained in @, and if $A \in @$ and $A \supset B \in @$ then $B \in @$.*

Proof. Standard.

PROPOSITION V.2. *There exists a constituency model which satisfies every d-formula of @ and only them.*

Proof. Let D be the set of all delimiters of $\mathcal{L}$. For every delimiter X, let $\rho(X)$ be $\{Y : Y \in D$ and $X \approx Y \in @\}$. And finally, let 1 be $\rho(V)$, and let S be $\{\rho(X) : X \in D\}$.

From Proposition V.1 and the logic of delimiters, one can prove that:

(i) for every X and Y in D, $\rho(X) = \rho(Y)$ iff $X \approx Y \in @$.
(ii) for every X and Y in D, $\rho(X) \sqsubset \rho(Y)$ iff $X < Y \in @$.

Let us define two operations $\sqcup$ and γ on S by:

> $x \sqcup y = \rho(X + Y)$ for some delimiters X and Y such that $\rho(X) = x$ and $\rho(Y) = y$, and
> $\gamma(x) = \rho(cX)$ for some delimiter X such that $\rho(X) = x$.

From (i) above and the logic of delimiters, it follows that for every X, X', Y and Y' in D, if $\rho(X) = \rho(X')$ and $\rho(Y) = \rho(Y')$, then $\rho(X + Y) = \rho(X' + Y')$, and $\rho(cX) = \rho(cX')$. As a consequence, our two operations do not depend on the particular delimiters chosen. It is easy to check that $\Delta = \langle S, 1, \sqcup, \gamma \rangle$ is a dependence structure (use (i), (ii) and the logic of delimiters).

Now, let i be the function defined by the following two conditions:

If X is a basic delimiter, $i(X) = \rho(X)$, and
if A is a sentence letter, $i(A) = \rho(|A|)$.

Since $i(V) = \rho(V) = 1$, i is a constituency function for $\mathcal{L}$ on Δ. One can then verify that for every delimiter X, $X^i = \rho(X)$. From (i) above, it then follows that for every d-atom, A, $\langle \Delta, i \rangle$ satisfies A iff $A \in @$. By an easy induction, one can then prove that for every d-formula A, $\langle \Delta, i \rangle$ satisfies A iff $A \in @$.

If X is a delimiter, let $@[X]$ be the set of all formulas A such that $\Box_X A \in @$.

PROPOSITION V.3. *For all delimiters X and Y:*

(i) *if* $X^i \sqsubset Y^i$ *then* $@[X] \subset @[Y]$
(ii) *if* $X^i = Y^i$ *then* $@[X] = @[Y]$
(iii) $@[X] \cup @[Y] \subset @[X+Y]$
(iv) $@[cX] = @[X]$

Proof. (i) (a) Assume $X^i \sqsubset Y^i$. Then by Proposition V.2(ii), $X < Y \in @$. Thus by Subsumption and Proposition V.1, for every formula A, if $\Box_X A \in @$ then $\Box_Y A \in @$. That is: for every formula A, if $A \in @[X]$ then $A \in @[Y]$.

(ii) From (i) and the fact that in any semilattice $\langle S, \sqcup \rangle$, $x = y$ iff ($x \sqsubset y$ and $y \sqsubset x$).

(iii) $X^i \sqsubset (X+Y)^i$. Thus by (i), $@[X] \subset @[X+Y]$. Similarly, $@[Y] \subset @[X+Y]$. Hence $@[X] \cup @[Y] \subset @[X+Y]$.

(iv) $X^i \sqsubset (cX)^i$. Thus by (i) $@[X] \subset @[cX]$. Now suppose $\Box_{cX} A \in @$. By Chaining and Proposition V.1, $\Box_X A \in @$. So, $@[cX] \subset @[X]$.

PROPOSITION V.4. *For every delimiter X, $@[X]$ is consistent and for every A in $@[X]$, $A^i \sqsubset \gamma(X^i)$.*

Proof. (1) If $A \in @[X]$, then $\Box_X A \in @$, and so $A \in @$ by the T-axiom and Proposition V.1. Thus, $@[X]$ is a part of $@$, and is thereby consistent.

(2) If $A \in @[X]$, then $\Box_X A \in @$, and so $|A| < cX \in @$ by Localization and Proposition V.1. By Proposition V.2(ii), $|A|^i \sqsubset cX^i$. So, $A^i \sqsubset \gamma(X^i)$.

Let us call a set Σ of sentences *X-maximal* provided that (i) for every formula $A \in \Sigma$, $A^i \sqsubset \gamma(X^i)$, and (ii) for every formula A such that $A^i \sqsubset \gamma(X^i)$, either $A \in \Sigma$ or $\sim A \in \Sigma$. Clearly, every consistent set of formulas satisfying (i) (in particular every $@[X]$) has some X-maximal consistent extension (adapt the usual Lindenbaum-type construction). Let us call $[X]$ the set of all X-maximal consistent extensions of $@[X]$. We have:

PROPOSITION V.5. *For all delimiters X and Y, the three following conditions are equivalent:*

(i) $[X] = [Y]$
(ii) $[X] \cap [Y] \neq \emptyset$
(iii) $\gamma(X^i) = \gamma(Y^i)$.

Proof. (i) $\Rightarrow$ (ii) For every delimiter X, $\Box_X X \approx X \in @$ by Proposition II.1(vi), the logic of delimiters and Proposition V.1. Thus $@[X]$ is never empty. Consequently, for every delimiter X, $[X] \neq \emptyset$. It follows that for all delimiters X and Y, if $[X] = [Y]$ then $[X] \cap [Y] \neq \emptyset$.

(ii) $\Rightarrow$ (iii) Suppose that $[X] \cap [Y] \neq \emptyset$, and let w be in $[X] \cap [Y]$. Since $X \approx X \in @[X]$, then $X \approx X \in w$. But since $w \in [Y]$, we have $X^i = (X \approx X)^i \sqsubset \gamma(Y^i)$. By a similar argument, $Y^i \sqsubset \gamma(X^i)$. Consequently, $\gamma(X^i) = \gamma(Y^i)$.

(iii) $\Rightarrow$ (i) Suppose that $\gamma(X^i) = \gamma(Y^i)$, that is: $(cX)^i = (cY)^i$. Then by Proposition V.2(i), $cX \approx cY \in @$. By Subsumption, Chaining and Proposition V.1, it follows that for every formula A, $\Box_X A \in @$ iff $\Box_Y A \in @$. That is: $@[X] = @[Y]$. So, $[X] = [Y]$.

Let W be $\bigcup\{[X] : X \text{ a delimiter}\}$, and let dom be the function from W onto $\gamma(S)$ such that for every w in W, $\text{dom}(w) = \gamma(X^i)$ for some delimiter X such that $w \in [X]$ (by Proposition V.5, this condition determines only one function). Note that (i) $\text{dom}(@) = \gamma(V^i) = \gamma(1) = 1$, and that (ii) for every w in W, $\text{dom}(w)$ is closed. Also notice that dom is surjective. In effect, let x be in $\gamma(S)$ with, say, $x = \gamma(X^i)$, and let w be in $[X]$. Suppose that $\text{dom}(w) = \gamma(Y^i)$. We then have $w \in [Y]$. So, $[X] \cap [Y] \neq \emptyset$, and then by Proposition V.5, $\gamma(X^i) = \gamma(Y^i)$. Thus, $x = \text{dom}(w)$. Finally let us put $\|A\| =_{\text{df}} \{w \in W : A \in w\}$ for every formula A. By the definition of the worlds, for every formula A, $\|A\| \subset W_{A^i}$. All this entails that $\langle W, @, \Delta, \text{dom}, i, [\cdot]\rangle$, where $[\cdot]$ is the truth-set function satisfying $[A] = \|A\|$ for every sentence letter A, is an E5π-model. Our aim is now to prove that for every formula A, $[A] = \|A\|$.

PROPOSITION V.6. *Every theorem A of E5π is contained in every A^i-world, and for every world, w, if $A \in w$ and $A \supset B \in w$ then $B \in w$.*

Proof. (1) By maximality, for every formula A and every A^i-world w, either $A \in w$ or $\sim A \in w$. It follows by consistency that for every theorem A and every A^i-world w, $A \in w$.

(2) Let $w \in W$ be such that $S \in w$ and $A \supset B \in w$. Since $A \supset B \in w$, then w is an $(A \supset B)^i$-world. But $(A \supset B)^i = A^i \sqcup B^i$. Thus w is a B^i-world. Therefore, either $B \in w$ or $\sim B \in w$ by maximality. Now by truth-functional logic, it is a theorem of E5π that $\sim(A \& A \supset B \& \sim B)$. Therefore since w is consistent, $\sim B \notin w$. So, $B \in w$.

PROPOSITION V.7. *For every formula A and delimiter X, if $A \in @[X]$ then $W_{X^i} \subset \|A\|$.*

Proof. Let A be a formula, X a delimiter and w a world with, say, $\text{dom}(w) = \gamma(Y^i)$. Suppose that $w \in W_{X^i}$. This means that $X^i \sqsubset \gamma(Y^i) = (cY)^i$. Thus by Propositions V.3 (i) and (iv), $@[X] \subset @[Y]$. Since $w \in [Y]$, $@[Y] \subset w$. So, $@[X] \subset w$.

PROPOSITION V.8. *Let A be a d-atom. The three following conditions are equivalent:*

(i) $\|A\| \neq \emptyset$

(ii) $\|A\| = W_{A^i}$

(iii) $@ \in \|A\|$

Proof. (i) $\Rightarrow$ (iii) Suppose that $@ \notin \|A\|$ – i.e. $A \notin @$. By maximality, $\sim A \in @$. By Proposition II.1(vi), $\vdash \sim A \supset \Box_{|A|} \sim A$. So by Proposition V.1, $\Box_{|A|} \sim A \in @$, i.e. $\sim A \in @[|A|]$. Now suppose that $\|A\| \neq \emptyset$ – i.e. that for some w in W, $A \in w$ with, say, $\mathrm{dom}(w) = \gamma(Y^i)$. We then have $A^i \sqsubset \gamma(Y^i)$, and thus $@[|A|] \subset @[Y]$ by Propositions V.3(i) and (iv). We consequently have $\sim A \in @[Y]$, and thus (since $@[Y] \subset w$) $\sim A \in w$. But this is impossible since w is consistent. So, if $@ \notin \|A\|$ then $\|A\| = \emptyset$.

(iii) $\Rightarrow$ (i) Trivial.

(ii) $\Rightarrow$ (iii) Since $@ \in W_{X^i}$ for every delimiter X.

(iii) $\Rightarrow$ (ii) As we have seen, for every formula A, $\|A\| \subset W_{A^i}$. So, we have to prove that for every d-atom A, if $@ \in \|A\|$, then $W_{A^i} \subset \|A\|$. Let A be a d-atom and suppose $A \in @$. By Proposition II.1(vi), $\vdash A \supset \Box_{|A|} A$. Hence by Proposition V.1, $\Box_{|A|} A \in @$, i.e. $A \in @[|A|]$. By Proposition V.7, we have the result.

PROPOSITION V.9. *Let A be a formula and X a delimiter. Then $W_{X^i} \subset \|A\|$ iff $\Box_X A \in @$.*

Proof. (i) Suppose $\Box_X A \in @$, i.e. $A \in @[X]$. By Proposition V.7, $W_{X^i} \subset \|A\|$.

(ii) Either $A^i \sqsubset \gamma(X^i)$ is the case or it is not.

(a) Suppose it is not the case. Let w be a world such that $\gamma(X^i) = \mathrm{dom}(w)$ (such a world exists since dom is surjective). Thus, $w \in W_{X^i}$. Since A^i is not part of $\mathrm{dom}(w)$, then *a fortiori* $A \notin w$, i.e. $w \notin \|A\|$. So, it is not the case that $W_{X^i} \subset \|A\|$.

(b) Now suppose $A^i \sqsubset \gamma(X^i)$ and $\Box_X A \notin @$. I shall first prove that $@[X] \cup \{\sim A\}$ is consistent. Suppose it is not. Then there are $B_1, \ldots, B_n$ in $@[X]$ such that $\vdash (B_1 \& \cdots \& B_n) \supset A$. By Necessitation (and the logic of delimiters), then, $\vdash \Box_{|B_1|+\cdots+|B_n|+|A|}[(B_1 \& \cdots \& B_n) \supset A]$, and hence by Proposition V.1 $(\Box_{|B_1|+\cdots+|B_n|+|A|}[(B_1 \& \cdots \& B_n) \supset A]) \in @$. Since $A^i \sqsubset \gamma(X^i) = (cX)^i$, we have $|A| < cX \in @$ by Proposition V.2(ii). For the same reasons, we have also $|B_k| < cX \in @$ for all k between 1 and n. So by the logic of delimiters, Subsumption and Proposition V.1, $(\Box_{cX}[(B_1 \& \cdots \& B_n) \supset A]) \in @$. Finally by Chaining and Proposition V.1, $(\Box_X[(B_1 \& \cdots \& B_n) \supset A]) \in @$. By the K-axiom and Proposition V.1, then, if $\Box_X(B_1 \& \cdots \& B_n) \in @$ then $\Box_X A \in @$. But now, since $B_1, \ldots, B_n$

are in $@[X]$, $\Box_X B_1, \ldots, \Box_X B_n$ are in $@$, and thus by Proposition II.1(iv) and Proposition V.1, $\Box_X(B_1 \& \cdots \& B_n) \in @$. So $\Box_X A \in @$. Contradiction. Hence, $@[X] \cup \{\sim A\}$ is consistent. Let w be an X-maximal consistent extension of $@[X] \cup \{\sim A\}$. w is in W_{X^i} since w is an X-maximal consistent extension of $@[X]$. Now, we have $\sim A \in w$, and thus by consistency $A \notin w$. Therefore, it is not the case that $W_{X^i} \subset \|A\|$.

PROPOSITION V.10. *Let A be a formula and X a delimiter. The three following conditions are equivalent:*

(i) $\|\Box_X A\| \neq \emptyset$

(ii) $\|\Box_X A\| = W_{X^i}$

(iii) $@ \in \|\Box_X A\|$.

Proof. (i) $\Rightarrow$ (iii) Suppose that $\|\Box_X A\| \neq \emptyset$, i.e. that for some world w, $\Box_X A \in w$. By Localization and Proposition V.6, $|A| < cX \in w$. By Proposition V.8, $|A| < cX \in @$. But since $\vdash |A| < cX \supset X + |A| < cX$, we have $X + |A| < cX \in @$ by Proposition V.1. Now suppose $@ \notin \|\Box_X A\|$, i.e. $\Box_X A \notin @$. By consistency, $\sim\Box_X A \in @$. By the E-axiom and Proposition V.1, $\Box_{X+|A|}\sim\Box_X A \in @$. Since $X + |A| < cX \in @$, we have $\Box_{cX}\sim\Box_X A \in @$ by Subsumption and Proposition V.1, and thus by Chaining and Proposition V.1, $\Box_X\sim\Box_X A \in @$, i.e. $\sim\Box_X A \in @[X]$. So, $W_{X^i} \subset \|\sim\Box_X A\|$ by Proposition V.7. Now, since $\Box_X A \in w$, we have $(\Box_X A)^i \sqsubset \mathrm{dom}(w)$, and so $X^i \sqsubset \mathrm{dom}(w)$, i.e. $w \in W_{X^i}$. Therefore, $w \in \|\sim\Box_X A\|$, that is: $\sim\Box_X A \in w$. Contradiction. So, if $\|\Box_X A\| \neq \emptyset$, then $@ \in \|\Box_X A\|$.

(iii) $\Rightarrow$ (i) Trivial.

(ii) $\Rightarrow$ (iii) Follows from the fact that $@ \in W_{X_i}$ for every delimiter X.

(iii) $\Rightarrow$ (ii) In any case, $\|\Box_X A\| \subset W_{X^i}$. For (a) in any case if $w \in \|\Box_X A\|$ then w is a $(\Box_X A)^i$-world and (b) every $(\Box_X A)^i$-world is an X^i-world. Suppose now that $@ \in \|\Box_X A\|$, i.e. that $\Box_X A \in @$. By Proposition II.1(v) and Proposition V.1, $\Box_X\Box_X A \in @$, i.e. $\Box_X A \in @[X]$. By Proposition V.7, we get: $W_{X^i} \subset \|\Box_X A\|$.

PROPOSITION V.11. *For all delimiters X and Y and for all formulas A and B:*

(1) $\|X \approx Y\| =$ (i) W_{A^i} *if* $X^i = Y^i$
(ii) $\emptyset$ *otherwise*

(2) $\|\sim A\| = W_{A^i} - \|A\|$

(3) $\|A \& B\| = \|A\| \cap \|B\|$

(4) $\|\Box_X A\| =$ (i) W_{X^i} *if* $W_{X^i} \subset \|A\|$
(ii) $\emptyset$ *otherwise*

Proof. (1) Let X and Y be delimiters. By Proposition V.2(i), $X^i = Y^i$ iff $X \approx Y \in @$, i.e. iff $@ \in \|X \approx Y\|$. The result then follows from Proposition V.8.

(2) and (3) One can easily verify (using maximality and consistency) that (i) $\sim A \in w$ iff $w \in W_{A^i}$ and $A \notin w$, and that (ii) $A\&B \in w$ iff $A \in w$ and $B \in w$.

(4) By Proposition V.9 and V.10.

PROPOSITION V.12. *For every formula* A, $[A] = \|A\|$.

Proof. From Proposition V.11.

PROPOSITION V.13. *E5π is complete* (*with respect to the class of E5π-models*).

Proof. Since $\sim\alpha \in @$ and $@$ is maximal, $\alpha \notin @$. (Recall that α is the non-theorem introduced at the beginning of the present section). But since for every delimiter X, $@ \in W_{X^i}$, we have $\|\alpha\| \neq W_{\alpha^i}$. Hence by Proposition V.12 $[\alpha] \neq W_{\alpha^i}$. I.e. α is not valid in $\langle W, @, \Delta, \text{dom}, i, [\cdot]\rangle$, and thus is not valid *tout court*.

ACKNOWLEDGMENT

This work was supported by the Fond National pour la Recherche Scientifique Suisse, grant number 11-55845.98. I would like to thank an anonymous referee of the JPL for helpful comments.

NOTES

[1] At the time I worked on the present paper only the first of Fine's papers was available to me. I became acquainted with 'Semantics for the Logic of Essence' after obtaining the results presented here.

[2] Fine aims at developing such a system in response to his own objections to the standard modal contruals of essence. See his 'Essence and Modality'.

[3] The problem of interpreting $\approx$ as class identity was suggested to me by an anonymous referee of the JPL.

[4] According to Fine, x ontologically depends on y iff (roughly) y is a constituent of some proposition which is true in virtue of the nature of x. See his 'Ontological Dependence' for a detailed discussion, in particular for the distinction between ontological dependence and existential dependence. It is not within the scope of the present paper to say very much about these notions.

[5] I use $\ldots^i$ or $(\ldots)^i$ for $i(\ldots)$, and in order to simplify notation, I omit the object language brackets.

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Université de Genève,
Faculté des Lettres,
Département de Philosophie,
Uni Bastions
CH-1211 Genève 4,
Switzerland