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STOCHASTIC APPROXIMATIONS AND DIFFERENTIAL INCLUSIONS*

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Abstract. The dynamical systems approach to stochastic approximation is generalized to the case where the mean differential equation is replaced by a differential inclusion. The limit set theorem of Benaïm and Hirsch is extended to this situation. Internally chain transitive sets and attractors are studied in detail for set-valued dynamical systems. Applications to game theory are given, in particular to Blackwell's approachability theorem and the convergence of fictitious play.

Key words. stochastic approximation, differential inclusions, set-valued dynamical systems, chain recurrence, approachability, game theory, learning, fictitious play

AMS subject classifications. 62L20, 34G25, 37B25, 62P20, 91A22, 91A26, 93E35, 34F05

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1. Introduction.

1.1. Presentation. A powerful method for analyzing stochastic approximations or recursive stochastic algorithms is the so-called ODE (ordinary differential equation) method, which allows us to describe the limit behavior of the algorithm in terms of the asymptotics of a certain ODE,

$$\frac{dx}{dt} = F(x),$$

obtained by suitable averaging.

This method was introduced by Ljung [24] and extensively studied thereafter (see, e.g., the books by Kushner and Yin [23] or Duflo [14] for a comprehensive introduction and further references). However, until recently most works in this direction have assumed the simplest dynamics for F, for example, that F is linear or given by the gradient of a cost function. While this type of assumption makes perfect sense in engineering applications (where algorithms are often designed to minimize a cost function), there are several situations, including models of learning or adaptive behavior in games, for which F may have more complicated dynamics.

In a series of papers Benaïm [2, 3] and Benaïm and Hirsch [5] have demonstrated that the asymptotic behavior of stochastic approximation processes can be described with a great deal of generality beyond gradients and other simple dynamics. One of their key results is that the limit sets of the process are almost surely *compact connected attractor free* (or *internally chain transitive* in the sense of Conley [13]) for the deterministic flow induced by F.

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The purpose of this paper is to show that such a dynamical system approach easily extends to the situation where the mean ODE is replaced by a differential inclusion. This is strongly motivated by certain problems arising in economics and game theory. In particular, the results here allow us to give a simple and unified presentation of Blackwell's approachability theorem, Smale's results on the prisoner's dilemma, and convergence of fictitious play in potential games. Many other applications[1] will be considered in a forthcoming paper, by Benaïm, Hofbauer, and Sorin [7], the present one being mainly devoted to theoretical issues.

The organization of the paper is as follows. Part 1 introduces the different notions of solutions, perturbed solutions, and stochastic approximations associated with a differential inclusion. Part 2 is devoted to the presentation of two classes of examples. Part 3 is a general study of the dynamical system defined by a differential inclusion. The main result (Theorem 3.6) on the limit set of a perturbed solution being internally chain transitive is stated. Then related notions—invariant and attracting sets, attractors, and Lyapunov functions—are analyzed. Part 4 contains the proof of the limit set theorem. Finally, Part 5 applies the previous results to two adaptive processes in game theory: approachability and fictitious play.

1.2. The differential inclusion. Let F denote a set-valued function mapping each point $x \in \mathbb{R}^m$ to a set $F(x) \subset \mathbb{R}^m$. We suppose throughout that the following holds.

Hypothesis 1.1 (standing assumptions on F).

(i) F is a closed set-valued map. That is,

$$\mathrm{Graph}(F) = \{(x, y) : y \in F(x)\}$$

is a closed subset of $\mathbb{R}^m \times \mathbb{R}^m$.

(ii) $F(x)$ is a nonempty compact convex subset of $\mathbb{R}^m$ for all $x \in \mathbb{R}^m$.

(iii) There exists $c > 0$ such that for all $x \in \mathbb{R}^m$

$$\sup_{z \in F(x)} \|z\| \leq c(1 + \|x\|),$$

where $\|\cdot\|$ denotes any norm on $\mathbb{R}^m$.

Definition I. *A* solution *for the differential inclusion*

$$\frac{d\mathbf{x}}{dt} \in F(\mathbf{x}) \tag{I}$$

with initial point $x \in \mathbb{R}^m$ is an absolutely continuous mapping $\mathbf{x} : \mathbb{R} \to \mathbb{R}^m$ such that $\mathbf{x}(0) = x$ and

$$\frac{d\mathbf{x}(t)}{dt} \in F(\mathbf{x}(t))$$

for almost every $t \in \mathbb{R}$.

Under the above assumptions, it is well known (see Aubin and Cellina [1, Chapter 2.1] or Clarke et al. [12, Chapter 4.1]) that (I) admits (typically nonunique) solutions through every initial point.

[1]As pointed out to us by an anonymous referee, applications to resource sharing may be considered as in Buche and Kushner [11], where the dynamics are given by a differential inclusion. Possible applications to engineering include dry friction; see, e.g., Kunze [22].

Remark 1.2. Suppose that a differential inclusion is given on a compact convex set $C \subset \mathbb{R}^m$, of the form $F(x) = \Phi(x) - x$, such that $\Phi(x) \subset C$ for all $x \in C$ and Φ satisfies Hypothesis 1.1(i) and (ii), with $\mathbb{R}^m$ replaced by C. Then we can extend it to a differential inclusion defined on the whole space $\mathbb{R}^m$: For $x \in \mathbb{R}^m$ let $P(x) \in C$ denote the unique point in C closest to x, and define $F(x) = \Phi(P(x)) - x$. Then F satisfies Hypothesis 1.1.

1.3. Perturbed solutions. The main object of this paper is paths which are obtained as certain (deterministic or random) perturbations of solutions of (I).

DEFINITION II. *A continuous function* $\mathbf{y} : \mathbb{R}_+ = [0,\infty) \to \mathbb{R}^m$ *will be called a* perturbed solution *to* (I) *(we also say a perturbed solution to* F*) if it satisfies the following set of conditions* (II):

(i) $\mathbf{y}$ *is absolutely continuous.*

(ii) *There exists a locally integrable function* $t \mapsto U(t)$ *such that*

(a)

$$\lim_{t\to\infty} \sup_{0\le v\le T} \left\| \int_t^{t+v} U(s)\,ds \right\| = 0$$

for all $T > 0$*; and*

(b) $\frac{d\mathbf{y}(t)}{dt} - U(t) \in F^{\delta(t)}(\mathbf{y}(t))$ *for almost every* $t > 0$*, for some function* $\delta : [0,\infty) \to \mathbb{R}$ *with* $\delta(t) \to 0$ *as* $t \to \infty$*. Here* $F^\delta(x) := \{y \in \mathbb{R}^m : \exists z : \|z - x\| < \delta,\ d(y, F(z)) < \delta\}$ *and* $d(y,C) = \inf_{c\in C} \|y - c\|$.

The purpose of this paper is to investigate the long-term behavior of $\mathbf{y}$ and to describe its limit set

$$L(\mathbf{y}) = \bigcap_{t\ge 0} \overline{\{\mathbf{y}(s) : s \ge t\}}$$

in terms of the dynamics induced by F.

1.4. Stochastic approximations. As will be shown here, a natural class of perturbed solutions to F arises from certain stochastic approximation processes.

DEFINITION III. *A discrete time process* $\{x_n\}_{n\in\mathbb{N}}$ *living in* $\mathbb{R}^m$ *is a* solution for (III) *if it verifies a recursion of the form*

$$x_{n+1} - x_n - \gamma_{n+1} U_{n+1} \in \gamma_{n+1} F(x_n), \tag{III}$$

where the characteristics γ *and* U *satisfy*

- $\{\gamma_n\}_{n\ge 1}$ *is a sequence of nonnegative numbers such that*

$$\sum_n \gamma_n = \infty, \qquad \lim_{n\to\infty} \gamma_n = 0;$$

- $U_n \in \mathbb{R}^m$ *are (deterministic or random) perturbations.*

To such a process is naturally associated a continuous time process as follows.

DEFINITION IV. *Set*

$$\tau_0 = 0 \quad \textit{and} \quad \tau_n = \sum_{i=1}^n \gamma_i \quad \textit{for } n \ge 1,$$

and define the continuous time affine interpolated process $\mathbf{w} : \mathbb{R}_+ \to \mathbb{R}^m$ *by*

$$\mathbf{w}(\tau_n + s) = x_n + s\frac{x_{n+1} - x_n}{\tau_{n+1} - \tau_n}, \quad s \in [0, \gamma_{n+1}). \tag{IV}$$

1.5. From interpolated process to perturbed solutions. The next result gives sufficient conditions on the characteristics of the discrete process (III) for its interpolation (IV) to be a perturbed solution (II). If (U_i) are random variables, assumptions (i) and (ii) below have to be understood with probability one.

PROPOSITION 1.3. *Assume that the following hold:*

(i) *For all* $T > 0$

$$\lim_{n\to\infty} \sup\left\{\left\|\sum_{i=n}^{k-1}\gamma_{i+1}U_{i+1}\right\| : k = n+1,\ldots,m(\tau_n+T)\right\} = 0,$$

where

$$m(t) = \sup\{k \geq 0 : t \geq \tau_k\}; \tag{1.1}$$

(ii) $\sup_n \|x_n\| = M < \infty$.

Then the interpolated process $\mathbf{w}$ *is a perturbed solution of* F.

Proof. Let $\mathbf{U}, \gamma : \mathbb{R}_+ \to \mathbb{R}^m$ denote the continuous time processes defined by

$$\mathbf{U}(\tau_n + s) = U_{n+1}, \qquad \gamma(\tau_n + s) = \gamma_{n+1}$$

for all $n \in \mathbb{N}$, $0 \leq s < \gamma_{n+1}$.

Then, for any t,

$$\mathbf{w}(t) \in x_{m(t)} + (t - \tau_{m(t)})[\mathbf{U}(t) + F(x_{m(t)})];$$

hence

$$\dot{\mathbf{w}}(t) \in \mathbf{U}(t) + F(x_{m(t)}).$$

Let us set $\delta(t) = \|\mathbf{w}(t) - x_{m(t)}\|$. Then obviously

$$F(x_{m(t)}) \subset F^{\delta(t)}(\mathbf{w}(t)).$$

In addition,

$$\delta(t) \leq \gamma_{m(t)+1}[\|U_{m(t)+1}\| + c(1+M)]$$

hence goes to 0, using hypothesis (i) of the statement of the proposition. It remains to check condition (ii)(a) of (II), but one has

$$\left\|\int_t^{t+v}\mathbf{U}(s)ds\right\| \leq \gamma_{m(t)+1}\|U_{m(t)+1}\| + \left\|\sum_{\ell=m(t)+1}^{m(t+v)-1}\gamma_{\ell+1}U_{\ell+1}\right\| + \gamma_{m(t+v)+1}\|U_{m(t+v)+1}\|,$$

and the result follows from condition (i). ∎

Sufficient conditions. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n\}_{n\geq 0}$ a filtration of $\mathcal{F}$ (i.e., a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$). We say that a stochastic process $\{x_n\}$ given by (III) satisfies the *Robbins–Monro condition with martingale difference noise* (Kushner and Yin [23]) if its characteristics satisfy the following:

(i) $\{\gamma_n\}$ is a deterministic sequence.

(ii) $\{U_n\}$ is *adapted* to $\{\mathcal{F}_n\}$. That is, U_n is measurable with respect to $\mathcal{F}_n$ for each $n \geq 0$.

(iii) $\mathsf{E}(U_{n+1} \mid \mathcal{F}_n) = 0$.

The next proposition is a classical estimate for stochastic approximation processes. Note that F does not appear. We refer the reader to (Benaïm [3, Propositions 4.2 and 4.4]) for a proof and further references.

PROPOSITION 1.4. *Let $\{x_n\}$ given by* (III) *be a Robbins–Monro equation with martingale difference noise process. Suppose that one of the following condition holds:*

(i) *For some $q \geq 2$*

$$\sup_n \mathsf{E}(\|U_n\|^q) < \infty$$

and

$$\sum_n \gamma_n^{1+q/2} < \infty.$$

(ii) *There exists a positive number Γ such that for all $\theta \in \mathbb{R}^m$*

$$\mathsf{E}(\exp(\langle \theta, U_{n+1} \rangle) \mid \mathcal{F}_n) \leq \exp\left(\frac{\Gamma}{2}\|\theta\|^2\right)$$

and

$$\sum_n e^{-c/\gamma_n} < \infty$$

for each $c > 0$.

Then assumption (i) *of Proposition* 1.3 *holds with probability* 1.

Remark 1.5. Typical applications are

(i) U_n uniformly bounded in L^2 and $\gamma_n = \frac{1}{n}$,

(ii) U_n uniformly bounded and $\gamma_n = o(\frac{1}{\log n})$.

2. Examples.

2.1. A multistage decision making model. Let A and B be measurable spaces, respectively called the *action space* and the *states of nature*; $E \subset \mathbb{R}^m$ a convex compact set called the *outcomes space*; and $H : A \times B \to E$ a measurable function, called the *outcome* function.

At discrete times $n = 1, 2 \ldots$ a decision maker (DM) chooses an action a_n from A and observes an outcome $H(a_n, b_n)$. We suppose the following.

(A) The sequence $\{a_n, b_n\}_{n \geq 0}$ is a random process defined on some probability space $(\Omega, \mathcal{F}, P)$ and adapted to some filtration $\{\mathcal{F}_n\}$. Here $\mathcal{F}_n$ has to be understood as the history of the process until time n.

(B) Given the history $\mathcal{F}_n$, DM and nature act independently:

$$\mathsf{P}((a_{n+1}, b_{n+1}) \in da \times db \mid \mathcal{F}_n) = \mathsf{P}(a_{n+1} \in da \mid \mathcal{F}_n)\mathsf{P}(b_{n+1} \in db \mid \mathcal{F}_n)$$

for any measurable sets $da \subset A$ and $db \subset B$.

(C) DM keeps track of only the cumulative average of the past outcomes,

$$x_n = \frac{1}{n}\sum_{i=1}^{n} H(a_i, b_i), \tag{2.1}$$

and his decisions are based on this average. That is,

$$\mathsf{P}(a_{n+1} \in da \mid \mathcal{F}_n) = Q_{x_n}(da),$$

where $Q_x(\cdot)$ is a probability measure over A for each $x \in E$, and $x \in E \mapsto Q_x(da) \in [0,1]$ is measurable for each measurable set $da \subset A$. The family $Q = \{Q_x\}_{x \in E}$ is called a *strategy* for DM.

Assumption (C) can be justified by considerations of limited memory and bounded rationality. It is partially motivated by Smale's approach to the prisoner's dilemma [27] (see also Benaïm and Hirsch [4, 5]), Blackwell's approachability theory ([8]; see also Sorin [28]), as well as fictitious play (Brown [10], Robinson [26]) and stochastic fictitious play (Benaïm and Hirsch [6], Fudenberg and Levine [15], Hofbauer and Sandholm [20]) in game theory (see the examples below).

For each $x \in E$ let

$$C(x) = \left\{ \int_{A \times B} H(a,b) Q_x(da) \nu(db) : \nu \in \mathcal{P}(B) \right\},$$

where $\mathcal{P}(B)$ denotes the set of probability measures over B. Then clearly

$$\mathsf{E}(H(a_{n+1}, b_{n+1}) \mid \mathcal{F}_n) \in C(x_n) \subset \overline{C}(x_n),$$

where $\overline{C}$ denote the smallest closed set-valued extension of C with convex values. More precisely, the graph of $\overline{C}$ is the intersection of all closed subsets $G \subset E \times E$ for which the fiber $G_x = \{y \in E : (x,y) \in G\}$ is convex and contains $C(x)$.

For $x \in \mathbb{R}^m$ let $P(x)$ denote the unique point in E closest to x. Extend $\overline{C}$ as in Remark 1.2 to a set-valued map on $\mathbb{R}^m$ by setting

$$\widehat{C}(x) = \overline{C}(P(x)).$$

Then the map

$$F(x) = -x + \overline{C}(P(x)) = -x + \widehat{C}(x) \tag{2.2}$$

clearly satisfies Hypothesis 1.1, and $\{x_n\}$ verifies the recursion

$$x_{n+1} - x_n = \frac{1}{n+1}(-x_n + H(a_{n+1}, b_{n+1})),$$

which can be rewritten as (see (III))

$$x_{n+1} - x_n \in \gamma_{n+1}[F(x_n) + U_{n+1}]$$

with $\gamma_n = \frac{1}{n}$ and $U_{n+1} = H(a_{n+1}, b_{n+1}) - \int_A H(a, b_{n+1}) Q_{x_n}(da)$. Hence, the conditions of Proposition 1.4 are satisfied and one deduces the following claim.

PROPOSITION 2.1. *The affine continuous time interpolated process* (IV) *of the process* $\{x_n\}$ *given by* (2.1) *is almost surely a perturbed solution of* F *defined by* (2.2).

Example 2.2 (Blackwell's approachability theory). A set $\Lambda \subset E$ is said to be *approachable* if there exists a strategy Q such that $x_n \to \Lambda$ almost surely. Blackwell [8] gives conditions ensuring approachability. We will show in section 5.1 how Blackwell's results can be partially derived from our main results and generalized (Corollary 5.2) in certain directions.

2.2. Learning in games. The preceding formalism is well suited to analyzing certain models of learning in games.

Consider the situation where m players are playing a game over and over. Let A^i (for $i \in I = \{1, \dots, m\}$) be a finite set representing the actions (pure strategies) available to player i, and let X^i be the finite dimensional simplex of probabilities over A^i (the set of mixed strategies for player i). For $i \in I$ we let A^{-i} and X^{-i} respectively denote the actions and mixed strategies available to the opponents of i. The payoff function to player i is given by a function $U^i : A^i \times A^{-i} \to \mathbb{R}$. As usual, we extend U^i to a function (still denoted U^i) on $X^i \times X^{-i}$, by multilinearity.

Example 2.3 (fictitious and stochastic fictitious play). Consider the game from the viewpoint of player i so that the DM is player i, and "nature" is given by the other players. In fictitious or stochastic fictitious play the outcome space is the space $X^i \times X^{-i}$ of mixed strategies, and the outcome function is the "identity" function $H : A^i \times A^{-i} \to X^i \times X^{-i}$ mapping every profile of actions a to the corresponding profile of mixed strategy δ_a.

Let

$$BR^i(x^{-i}) = \operatorname*{Argmax}_{a^i \in A^i} U^i(a^i, x^{-i}) \subset A^i$$

be the set of best actions that i can play in response to x^{-i}.

Both classical fictitious play (Brown [10], Robinson [26]) and stochastic fictitious play (Benaïm and Hirsch [6], Fudenberg and Levine [15], Hofbauer and Sandholm [20]) assume that the strategy of player i, $Q^i = \{Q^i_x\}$, can be written as

$$Q^i_x(a^i) = q^i(a^i, x^{-i}),$$

where $q^i : A^i \times X^{-i} \to [0, 1]$ is such that one of the following assumptions holds:

fictitious play assumption:

$$\sum_{a^i \in BR^i(x^{-i})} q^i(a^i, x^{-i}) = 1,$$

or *stochastic fictitious play assumption*, q^i is smooth in x^{-i} and

$$\sum_{a^i \in BR^i(x^{-i})} q^i(a^i, x^{-i}) \geq 1 - \delta$$

for some $0 < \delta \ll 1$.

In this framework, if a_ℓ denotes the profile of actions at stage ℓ, one has

$$x_n = \frac{1}{n} \sum_{\ell=1}^{n} a_\ell$$

and

$$x_{n+1} - x_n = \frac{1}{n+1}(a_{n+1} - x_n).$$

Thus for each i

$$\mathsf{E}(x^i_{n+1} - x^i_n \mid \mathcal{F}_n) \in \frac{1}{n+1}(\overline{BR}^i(x^{-i}_n) - x^i_n),$$

where $\overline{BR}^i(x^{-i}) \subset X^i$ is the convex hull of $BR^i(x^{-i})$ for the standard fictitious play, and $\overline{BR}^i(x^{-i}) = \sum_{a^i \in A^i} q^i(a^i, x^{-i})\delta_{a^i}$ for the stochastic fictitious play.

Thus the set-valued map F defined in (2.2) is given as

$$F^i(x) = -x + \overline{BR}^i(x^{-i}) \times X^{-i}.$$

Observe that if a subset $J \subset I$ of players plays a fictitious (or stochastic fictitious) play strategy, then F^i has to be replaced by

$$F^J(x) = \bigcap_{i \in J} F^i(x).$$

In particular, if all players play a fictitious play strategy, the differential inclusion induced by F is the best-response differential inclusion (Gilboa and Matsui [16], Hofbauer [19], Hofbauer and Sorin [21]), while if all play a stochastic fictitious play, F is a smooth best-response vector field (Benaïm and Hirsch [6], Fudenberg and Levine [15], Hofbauer and Sandholm [20]).

Example 2.4 (Smale approach to the prisoner's dilemma). We still consider the game from the viewpoint of player i, so that the DM is player i and nature the other players, but we take for H the payoff vector function

$$\begin{gathered} H : A^i \times A^{-i} \to E, \\ a \to U(a) = (U^1(a), \ldots, U^m(a)), \end{gathered}$$

where $E \subset \mathbb{R}^m$ is the convex hull of the payoff vectors $\{U(a)\}$.

This setting fits exactly with Smale's approach to the prisoner's dilemma [27] later revisited by Benaïm and Hirsch [4]. Details will be given in section 5.2, where Smale's approach will be reinterpreted in the framework of approachability.

3. Set-valued dynamical systems.

3.1. Properties of the trajectories of (I). Let $C^0(\mathbb{R}, \mathbb{R}^m)$ denote the space of continuous paths $\{\mathbf{z} : \mathbb{R} \to \mathbb{R}^m\}$ equipped with the topology of uniform convergence on compact intervals. This is a complete metric space for the distance $\mathbf{D}$ defined by

$$\mathbf{D}(\mathbf{x}, \mathbf{z}) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min(\|\mathbf{x} - \mathbf{z}\|_{[-k,k]}, 1),$$

where $\|\cdot\|_{[-k,k]}$ stands for the supremum norm on $C^0([-k,k], \mathbb{R}^m)$.

Given a set $M \subset \mathbb{R}^m$, we let $S_M \subset C^0(\mathbb{R}, \mathbb{R}^m)$ denote the set of all solutions to (I) with initial conditions $x \in M$ ($S_M = \bigcup_{x \in M} S_x$), and $S_{M,M} \subset S_M$ the subset consisting of solutions $\mathbf{x}$ that remain in M (i.e., $\mathbf{x}(\mathbb{R}) \subset M$).

LEMMA 3.1. *Assume M compact. Then S_M is a nonempty compact set and $S_{M,M}$ is a compact (possibly empty) set.*

Proof. The first assertion follows from Aubin and Cellina [1, section 2.2, Theorem 1, p. 104]. The second easily follows from the first. □

3.2. Set-valued dynamical system induced by (I). The differential inclusion (I) induces a set-valued dynamical system $\{\Phi_t\}_{t \in \mathbb{R}}$ defined by

$$\Phi_t(x) = \{\mathbf{x}(t) : \mathbf{x} \text{ is a solution to (I) with } \mathbf{x}(0) = x\}.$$

The family $\Phi = \{\Phi_t\}_{t \in \mathbb{R}}$ enjoys the following properties:

(a) $\Phi_0(x) = \{x\}$;
(b) $\Phi_t(\Phi_s(x)) = \Phi_{t+s}(x)$ for all $t, s \geq 0$;
(c) $y \in \Phi_t(x) \Rightarrow x \in \Phi_{-t}(y)$ for all $x, y \in \mathbb{R}^m$, $t \in \mathbb{R}$;
(d) $(x,t) \mapsto \Phi_t(x)$ is a closed set-valued map with compact values (i.e., $\Phi_t(x)$ is a compact set for each t and x).

Properties (a), (b), (c) are immediate to verify, and property (d) easily follows from Lemma 3.1.

For subsets $T \subset \mathbb{R}$ and $A \subset \mathbb{R}^m$ we will define

$$\Phi_T(A) = \bigcup_{t \in T} \bigcup_{x \in A} \Phi_t(x).$$

Invariant sets.

DEFINITION V. *A set $A \subset \mathbb{R}^m$ is said to be*
(i) strongly invariant *(for Φ) if $A = \Phi_t(A)$ for all $t \in \mathbb{R}$;*
(ii) quasi-invariant *if $A \subset \Phi_t(A)$ for all $t \in \mathbb{R}$;*
(iii) semi-invariant *if $\Phi_t(A) \subset A$ for all $t \in \mathbb{R}$;*
(iv) invariant *(for F) if for all $x \in A$ there exists a solution $\mathbf{x}$ to* (I) *with $\mathbf{x}(0) = x$ and such that $\mathbf{x}(\mathbb{R}) \subset A$.*

We call a set A *strongly positive invariant* if $\Phi_t(A) \subset A$ for all $t > 0$.

At first glance (at least for those used to ordinary differential equations) the good notion might seem to be the one defined by strong invariance. However, this notion is too strong for differential inclusions, as shown by the simple example below (Example 3.2), and the main notions that will really be needed here are invariance and strong positive invariance. We have included the definition of quasi invariance mainly because some of our later results may be related to a paper by Bronstein and Kopanskii [9] making use of this notion.[2] Observe, however, that by Lemma 3.3 below, quasi invariance coincides with invariance for compact sets.

Example 3.2. (a) Let F be the set-valued map defined on $\mathbb{R}$ by $F(x) = -\operatorname{sgn}(x)$ if $x \neq 0$ and $F(0) = [-1, 1]$. Then $\Phi_t(0) = \{0\}$ for $t \geq 0$, and $\Phi_t(0) = [t, -t]$ for $t < 0$. Hence $\{0\}$ is invariant and strongly positively invariant but is not strongly invariant.

(b) Let now $F(x) = x$ for $x < 0$, $F(x) = 1$ for $x > 0$, and $F(0) = [0, 1]$. Then $\Phi_t(0) = \{0\}$ for $t \leq 0$, and $\Phi_t(0) = [0, t]$ for $t \geq 0$. Hence $\{0\}$ is invariant but not strongly positively invariant.

LEMMA 3.3. *Every invariant set is quasi-invariant. Every compact quasi-invariant set is invariant.*

Proof. Suppose that A is invariant. Let $x \in A$ and $\mathbf{x}$ be a solution to (I) with $\mathbf{x}(0) = x$ and $\mathbf{x}(\mathbb{R}) \subset A$. For all $t \in \mathbb{R}$ we have $x \in \Phi_t(\mathbf{x}(-t))$. Hence A is quasi-invariant.

Conversely suppose that A is quasi-invariant and compact. Choose $x \in A$ and fix $N \in \mathbb{N}$. Then for every $p \in \mathbb{N}$ there exists, by quasi invariance and by gluing pieces of solutions together, a solution $\mathbf{x}_{p,N}$ to (I) such that $\mathbf{x}_{p,N}(0) = x$ and for all $q \in \{-2^p, \ldots, 2^p\}$, $\mathbf{x}_{p,N}(\frac{qN}{2^p}) \in A$. By Lemma 3.1, the sequence $\{\mathbf{x}_{p,N}\}_{p \in \mathbb{N}}$ is relatively compact in $C^0([-N, N], \mathbb{R}^m)$. Let $\mathbf{x}_N$ be a limit point of this sequence. Then for each dyadic point $t = \frac{qN}{2^p}$, where $q \in \{-2^p, \ldots, 2^p\}$, $\mathbf{x}_N(t) \in \overline{A}$. Continuity of $\mathbf{x}_N$ implies $\mathbf{x}_N([-N, N]) \subset \overline{A}$. Now let $\mathbf{x}$ be a limit point of the sequence $\{\mathbf{x}_N\}_{N \in \mathbb{N}}$ in $C^0(\mathbb{R}, \mathbb{R}^m)$. Then $\mathbf{x}(\mathbb{R}) \subset \overline{A}$ and $\mathbf{x}$ is a solution to (I). ∎

[2] Invariant sets in Bronstein and Kopanskii [9] coincide with what we define here as strongly invariant sets.

Remark 3.4. A invariant together with strong positive invariance implies $\Phi_t(A) = A$ for $t > 0$.

3.3. Chain-recurrence and the limit set theorem. Given a set $A \subset \mathbb{R}^m$ and $x, y \in A$, we write $x \hookrightarrow_A y$ if for every $\varepsilon > 0$ and $T > 0$ there exists an integer $n \in \mathbb{N}$, solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ to (I), and real numbers $t_1, t_2, \ldots, t_n$ greater than T such that

(a) $\mathbf{x}_i(s) \in A$ for all $0 \leq s \leq t_i$ and for all $i = 1, \ldots, n$,
(b) $\|\mathbf{x}_i(t_i) - \mathbf{x}_{i+1}(0)\| \leq \varepsilon$ for all $i = 1, \ldots, n-1$,
(c) $\|\mathbf{x}_1(0) - x\| \leq \varepsilon$ and $\|\mathbf{x}_n(t_n) - y\| \leq \varepsilon$.

The sequence $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is called an (ε, T) chain (in A from x to y) for F.

DEFINITION VI. *A set* $A \subset \mathbb{R}^m$ *is said to be* internally chain transitive*, provided that* A *is compact and* $x \hookrightarrow_A y$ *for all* $x, y \in A$.

LEMMA 3.5. *An internally chain transitive set is invariant.*

Proof. Let A be such a set and $x \in A$. Let $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be an (ε, T) chain from x to x. Set $\mathbf{y}_{\varepsilon,T}(t) = \mathbf{x}_1(t)$ for $0 \leq t \leq T$ and $\mathbf{z}_{\varepsilon,T}(t) = \mathbf{x}_n(t_n + t)$ for $-T \leq t \leq 0$. By Lemma 3.1 we can extract from $(\mathbf{y}_{1/p,T})_{p \in \mathbb{N}}$ and $(\mathbf{z}_{1/p,T})_{p \in \mathbb{N}}$ some subsequences converging, respectively, to $\mathbf{y}_T$ and $\mathbf{z}_T$, where $\mathbf{y}_T$ and $\mathbf{z}_T$ are solutions to (I), $\mathbf{y}_T(0) = x = \mathbf{z}_T(0)$, $\mathbf{y}_T([0,T]) \subset A$, and $\mathbf{z}_T([-T,0]) \subset A$. The map $\mathbf{w}_T(t) = \mathbf{y}_T(t)$ for $t \geq 0$ and $\mathbf{w}_T(t) = \mathbf{z}_T(t)$ for $t \leq 0$ is then a solution to (I) with initial condition x and such that $\mathbf{w}_T([-T,T]) \subset A$. By Lemma 3.1, again we extract from $(\mathbf{w}_T)_{T \geq 0}$ a subsequence converging to a solution $\mathbf{w}$ whose range lies in A and with initial condition x. ∎

This notion of recurrence due to Conley [13] for classical dynamical systems is well suited to the description of the asymptotic behavior of a perturbed solution to (I), as shown by the following theorem.

THEOREM 3.6. *Let* $\mathbf{y}$ *be a bounded perturbed solution to* (I). *Then, the limit set of* $\mathbf{y}$,

$$L(\mathbf{y}) = \bigcap_{t \geq 0} \overline{\{\mathbf{y}(s) : s \geq t\}},$$

is internally chain transitive.

This theorem is the set-valued version of the limit set theorem proved by Benaïm [2] for stochastic approximation and Benaïm and Hirsch [5] for asymptotic pseudotrajectories of a flow. We will deduce it from the more general results of section 4.

3.4. Limit sets. The set

$$\omega_\Phi(x) := \bigcap_{t \geq 0} \overline{\Phi_{[t,\infty)}(x)}$$

is the ω-limit set of a point $x \in \mathbb{R}^m$. Note that $\omega_\Phi(x)$ contains the limit sets $L(\mathbf{x})$ of all solutions $\mathbf{x}$ with $\mathbf{x}(0) = x$ but is in general larger than the union of these.

In contrast to the limit set of a solution, the ω-limit set of a point need not be internally chain transitive.

Example 3.7. Let F be the set-valued map defined on $\mathbb{R}$ by $F(x) = 1 - x$ for $x > 0$ and $F(0) = [0,1]$ and $F(x) = -x$ for $x < 0$. Then for every solution $\mathbf{x}$, one has $\lim_{t \to \infty} \mathbf{x}(t) = 0$ or 1. But $\omega_\Phi(0) = [0,1]$ is not internally chain transitive.

More generally one defines

$$\omega_\Phi(Y) := \bigcap_{t \geq 0} \overline{\Phi_{[t,\infty)}(Y)}.$$

Definition VII. *A set Y is* forward precompact *if $\overline{\Phi_{[t,\infty)}(Y)}$ is compact for some $t > 0$.*

Lemma 3.8. (i) *$\omega_\Phi(Y)$ is the set of points $p \in \mathbb{R}^m$ such that*

$$p = \lim_{n\to\infty} \mathbf{y}_n(t_n)$$

for some sequence $\{\mathbf{y}_n\}$ of solutions to (I) *with initial conditions $\mathbf{y}_n(0) \in Y$ and some sequence $\{t_n\} \in \mathbb{R}$ with $t_n \to \infty$.*

(ii) *$\omega_\Phi(Y)$ is a closed invariant (possibly empty) set. If Y is forward precompact, then $\omega_\Phi(Y)$ is nonempty and compact.*

Proof. Point (i) is easily seen from the definition.

(ii) Let $p = \lim_{n\to\infty} \mathbf{y}_n(t_n) \in \omega_\Phi(Y)$. Set $\mathbf{z}_n(s) = \mathbf{y}_n(t_n + s)$ for all $s \in \mathbb{R}$. By Lemma 3.1 we may extract from $(\mathbf{z}_n)_{n\geq 0}$ a subsequence converging to some solution $\mathbf{z}$ with $\mathbf{z}(0) = p$ and $\mathbf{z}(s) = \lim_{n_k\to\infty} \mathbf{y}_{n_k}(t_{n_k} + s) \in \omega_\Phi(Y)$. This proves invariance. The rest is clear. □

Note that the limit set $\omega_\Phi(Y)$ is in general not strongly positively invariant (e.g., in Example 3.7 for $x < 0$, $\omega_\Phi(x) = \{0\}$).

3.5. Attracting sets and attractors. For applications it is useful to characterize $L(\mathbf{y})$ in terms of certain compact invariant sets for Φ, namely, *the attractors,* as defined below.

Given a closed invariant set L, the induced set-valued dynamical system Φ^L is the family of (set-valued) mappings $\Phi^L = \{\Phi_t^L\}_{t\in\mathbb{R}}$ defined on L by

$$\Phi_t^L(x) = \{\mathbf{x}(t) : \mathbf{x} \text{ is a solution to (I) with } \mathbf{x}(0) = x \text{ and } \mathbf{x}(\mathbb{R}) \subset L\}.$$

Note that L is strongly invariant for Φ^L.

Definition VIII. *A compact set $A \subset L$ is called an* attracting set *for Φ^L, provided that there is a neighborhood U of A in L (i.e., for the induced topology) with the property that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that*

$$\Phi_t^L(U) \subset N^\varepsilon(A)$$

for all $t \geq t_\varepsilon$. Or, equivalently, $\Phi^L_{[t_\varepsilon,\infty)}(U) \subset N^\varepsilon(A)$. Here $N^\varepsilon(A)$ stands for the ε-neighborhood of A.

If, additionally, A is invariant, then A is called an attractor *for Φ^L.*

The set U is called a fundamental neighborhood *of A for Φ^L. If $A \neq L$ and $A \neq \emptyset$, then A is called a* proper attracting set *(or* proper attractor*) for Φ^L.*

Furthermore, an attracting set (respectively, attractor) for Φ is an attracting set (respectively, attractor) for Φ^L with $L = \mathbb{R}^m$.

Example 3.9. Let F be the set-valued map from Example 3.2(a), i.e., defined on $\mathbb{R}$ by $F(x) = -\operatorname{sgn}(x)$ if $x \neq 0$ and $F(0) = [-1, 1]$. Then $\{0\}$ is an attractor and every compact set $A \subset \mathbb{R}$ with $0 \in A$ is an attracting set.

Proposition 3.10. *Let A be a nonempty compact subset of L, and U a neighborhood of A in L. Then the following hold:*

(i) *A is an attracting set for Φ^L with fundamental neighborhood U if and only if U is forward precompact and $\omega_{\Phi^L}(U) \subset A$. In this case $\omega_{\Phi^L}(U)$ is an attractor.*

(ii) *A is an attractor for Φ^L with fundamental neighborhood U if and only if U is forward precompact and $\omega_{\Phi^L}(U) = A$.*

Proof. (i) If A is an attracting set for Φ^L with fundamental neighborhood U, then $\omega_{\Phi^L}(U) \subset \bigcap_{\varepsilon>0} N^\varepsilon(A) \subset A$. Conversely, for t large enough $V_t = \overline{\Phi^L_{[t,\infty)}(U)}$ defines a decreasing family of compact sets converging to $\omega_{\Phi^L}(U) \subset A$. Hence for any $\varepsilon > 0$ there exists t_ε with $V_{t_\varepsilon} \subset N^\varepsilon(A)$ and A is an attracting set. In particular, $\omega_{\Phi^L}(U)$ itself is an attracting set, invariant by Lemma 3.8(ii).

(ii) If $A = \omega_{\Phi^L}(U)$, then A is an attractor by (i). Conversely, if A is an attractor with fundamental neighborhood U, then $\omega_\Phi(U) \subset A$ by (i). Let $x \in A$. Since A is invariant, there exists a solution $\mathbf{y}$ to (I) with $\mathbf{y}(0) = x$ and $\mathbf{y}(\mathbb{R}) \subset A$. Set $\mathbf{y}_n(t) = \mathbf{y}(t-n)$. Then $\mathbf{y}_n(n) = x$, proving that $x \in \omega_{\Phi^L}(U)$ (by Lemma 3.8(i)). ∎

PROPOSITION 3.11. *Every attractor is strongly positively invariant. (Example* 3.2(a) *provides an attractor that is not strongly invariant.)*

Proof. By invariance, $A \subset \Phi_T^L(A)$ for all $T > 0$. Hence, given $t > 0$,

$$\Phi_t^L(A) \subset \Phi_{t+T}^L(A) \subset \Phi_{t+T}^L(U) \subset \Phi_{[t+T,\infty)}^L(U)$$

for all $T > 0$. Thus $\Phi_t^L(A) \subset N^\varepsilon(A)$ for all $\varepsilon > 0$, and hence $\Phi_t^L(A) \subset A$ for all $t > 0$. ∎

Remark 3.12. In the family of attracting sets A with a given fundamental neighborhood U, there exists a minimal one, which is in addition invariant, strongly positively invariant, and independent of the set U used to define the family. It is also the largest positively quasi-invariant set included in U.

Any attractor $A \subset L$ can be written as $A = \omega_{\Phi^L}(U)$ for some U. Hence any fundamental neighborhood uniquely determines the attractor A. This implies, as in Conley [13], that Φ^L can have at most countably many attractors.

3.6. Attractors and stability.

DEFINITION IX. *A set $A \subset L$ is* asymptotically stable *for Φ^L if it satisfies the following three conditions:*

(i) *A is invariant.*

(ii) *A is* Lyapunov stable; *i.e., for every neighborhood U of A there exists a neighborhood V of A such that $\Phi_{[0,\infty)}(V) \subset U$.*

(iii) *A is* attractive; *i.e., there is a neighborhood U of A such that for every $x \in U : \omega_\Phi(x) \subset A$.*

Alternatively, instead of (iii) *one could ask for the following weaker requirement:*

(iii′) *There is a neighborhood U of A such that for every solution $\mathbf{x}$ with $\mathbf{x}(0) \in U$ one has $L(\mathbf{x}) \subset A$.*

We show now that for compact sets the concepts of attractor and asymptotic stability are equivalent. The proof of Corollary 3.18 below shows that it makes no difference whether one uses (iii) or (iii′) in the definition of asymptotic stability.

We start with an upper bound for entry times.

LEMMA 3.13. *Let V be an open set and K compact such that for all solutions $\mathbf{x}$ with $\mathbf{x}(0) \in K$ there is $t > 0$ with $\mathbf{x}(t) \in V$. Then there exists $T > 0$ such that for every solution $\mathbf{x}$ with $\mathbf{x}(0) \in K$ there is $t \in [0, T]$ with $\mathbf{x}(t) \in V$.*

Proof. Suppose that there is no such upper bound T for the entry times into V. Then for each $n \in \mathbb{N}$ there is $\mathbf{x}_n(0) = x_n \in K$ and a solution $\mathbf{x}_n$ such that $\mathbf{x}_n(t) \notin V$ for $0 \le t \le n$. Since K is compact, we can assume that $x_n \to x \in K$. And by Lemma 3.1 a subsequence of $\mathbf{x}_n$ converges to a solution $\mathbf{x}$ with $\mathbf{x}(0) = x$ and $\mathbf{x}(t) \notin V$ for all $t > 0$. ∎

LEMMA 3.14. *If a closed set A is Lyapunov stable, then it is strongly positively invariant.*

Proof. A is the intersection of a family of strongly positively invariant neighborhoods. ∎

Lemma 3.15. *If a compact set A satisfies* (ii) *and* (iii′), *it is attracting.*

Proof. Let B be a compact neighborhood of A, included in the fundamental neighborhood U, and let W be a neighborhood of A. A being Lyapunov stable, there exists an open neighborhood V of A with $\Phi^L_{[0,\infty)}(V) \subset W$. For any $x \in B$ and any solution $\mathbf{x}$ with $\mathbf{x}(0) = x$, there exists $t > 0$ with $\mathbf{x}(t) \in V$. Applying Lemma 3.13 implies $\Phi^L_T(B) \subset \Phi^L_{[0,T]}(V)$; hence $\Phi^L_{[T,\infty)}(B) \subset W$ and A is attracting. □

Lemma 3.16. *If the set A is attracting and strongly positively invariant, then it is Lyapunov stable.*

Proof. Let A be attracting with fundamental neighborhood U, and V be any other (open) neighborhood of A. Then by definition there is $T > 0$ such that $\Phi^L_{[T,\infty)}(U) \subset V$. A being strongly positively invariant, $\Phi^L_{[0,T]}(A) \subset A$. Upper semicontinuity gives an $\varepsilon > 0$ such that $\Phi^L_{[0,T]}(N^\varepsilon(A)) \subset V$ and $N^\varepsilon(A) \subset U$. Hence $\Phi^L_{[0,\infty)}(N^\varepsilon(A)) \subset V$, which shows Lyapunov stability. □

Corollary 3.17. *For a compact set A, properties* (ii) *and* (iii′) *of Definition* IX, *together, are equivalent to attracting and strong positive invariance.*

Corollary 3.18. *A compact set A is an attractor if and only if it is asymptotically stable.*

We conclude with a simple useful condition ensuring that an open set contains an attractor.

Proposition 3.19. *Let U be an open set with compact closure. Suppose that $\Phi_T(\overline{U}) \subset U$ for some $T > 0$. Then U is a fundamental neighborhood of some attractor A.*

Proof. Since Φ has a closed graph, $\Phi_T(\overline{U})$ is compact. Therefore $\Phi_T(\overline{U}) \subset V \subset \overline{V} \subset U$ for some open set V. By upper semicontinuity of Φ_T (which follows from property (d) of a set-valued dynamical system) there exists $\varepsilon > 0$ such that $\Phi_t(\overline{U}) \subset V$ for $T - \varepsilon \le t \le T + \varepsilon$. Let $t_0 = T(T+1)/\varepsilon$. For all $t \ge t_0$ write $t = kT + r$ with $k \in \mathbb{N}$ and $r < T$. Hence $t = k(T + r/k)$ with $0 \le r/k < \varepsilon$. Thus

$$\Phi_t(\overline{U}) = \Phi_{T+r/k} \circ \cdots \circ \Phi_{T+r/k}(\overline{U}) \subset V.$$

Hence $\omega_\Phi(U) = \bigcap_{t \ge t_0} \overline{\Phi_{[t,\infty)}(U)} \subset \overline{V} \subset U$ is an attractor with fundamental neighborhood U. □

3.7. Chain transitivity and attractors.

Proposition 3.20. *Let L be internally chain transitive. Then L has no proper attracting set for Φ^L.*

Proof. Let $A \subset L$ be an attracting set. By definition, there exists a neighborhood U of A, and for all $\varepsilon > 0$ a number t_ε such that $\Phi^L_t(U) \subset N^\varepsilon(A)$ for all $t > t_\varepsilon$. Assume $A \ne L$ and choose ε small enough so that $N^{2\varepsilon}(A) \subset U$ and there exists $y \in L \setminus N^{2\varepsilon}(A)$. Then, for $T \ge t_\varepsilon$ and $x \in A$, there is no (ε, T) chain from x to y. In fact, $\mathbf{x}_1(0) \in N^{2\varepsilon}(A)$, and hence $\mathbf{x}_1(t_1) \in N^\varepsilon(A)$; by induction, $\mathbf{x}_i(t_i) \in N^\varepsilon(A)$ so that $\mathbf{x}_{i+1}(0) \in N^{2\varepsilon}(A)$ as well. Thus we arrive at a contradiction. □

Remark 3.21. This last proposition can also be deduced from Bronstein and Kopanskii [9, Theorem 1] combined with Lemma 3.1. Also the converse is true.

Recall that an attracting set (respectively, attractor) for Φ is an attracting set (respectively, attractor) for Φ^L with $L = \mathbb{R}^m$.

Lemma 3.22. *Let A be an attracting set for Φ and L a closed invariant set. Assume $A \cap L \ne \emptyset$. Then $A \cap L$ is an attracting set for Φ^L.*

Proof. The proof follows from the definitions. □

If A is a set, then

$$B(A) = \{x \in \mathbb{R}^m : \omega_\Phi(x) \subset A\}$$

denotes its *basin of attraction.*

THEOREM 3.23. *Let A be an attracting set for Φ and L an internally chain transitive set. Assume $L \cap B(A) \neq \emptyset$. Then $L \subset A$.*

Proof. Suppose $L \cap B(A) \neq \emptyset$. Then there exists a solution $\mathbf{x}$ to (I) with $\mathbf{x}(0) = x \in B(A)$ and $\mathbf{x}(\mathbb{R}) \subset L$. Hence $d(\mathbf{x}(t), A) \to 0$ when $t \to \infty$, proving that L meets A. Proposition 3.20 and Lemma 3.22 imply that $L \subset A$. ∎

A *global attractor* for Φ is an attractor whose basin of attraction consists of all $\mathbb{R}^m$. If a global attractor exists, then it is unique and coincides with the maximal compact invariant set of Φ. The following corollary is an immediate consequence of Theorem 3.23 or even more easily of Lemma 3.5.

COROLLARY 3.24. *Suppose Φ has a global attractor A. Then every internally chain transitive set lies in A.*

3.8. Lyapunov functions.

PROPOSITION 3.25. *Let Λ be a compact set, $U \subset \mathbb{R}^m$ be a bounded open neighborhood of Λ, and $V : \overline{U} \to [0, \infty[$. Let the following hold:*

(i) *For all $t \geq 0$, $\Phi_t(U) \subset U$ (i.e., U is strongly positively invariant);*

(ii) *$V^{-1}(0) = \Lambda$;*

(iii) *V is continuous and for all $x \in U \setminus \Lambda$, $y \in \Phi_t(x)$ and $t > 0$, $V(y) < V(x)$;*

(iv) *V is upper semicontinuous, and for all $x \in \overline{U} \setminus \Lambda$, $y \in \Phi_t(x)$, and $t > 0$, $V(y) < V(x)$.*

(A) *Under* (i), (ii), *and* (iii), *Λ is a Lyapunov stable attracting set, and there exists an attractor contained in Λ whose basin contains U, and with $V^{-1}([0, r))$ as fundamental neighborhoods for small $r > 0$.*

(B) *Under* (i), (ii), *and* (iv), *there exists an attractor contained in Λ whose basin contains U.*

Proof. For the proof of (A), let $r > 0$ and $U_r = \{x \in U : V(x) < r\}$. Then $\{\overline{U}_r\}_{r>0}$ is a nested family of compact neighborhoods of Λ with $\bigcap_{r>0} \overline{U}_r = \Lambda$. Thus for $r > 0$ small enough, $\overline{U_r} \subset U$. Moreover, $\Phi_t(\overline{U}_r) \subset U_r$ for $t > 0$ by our hypotheses on U and V. Proposition 3.19 then implies the result.

For (B), let $A = \omega_\Phi(U)$, which is closed and invariant (by Lemma 3.8) and hence compact, since it is included in $\overline{U}$. Let $\alpha = \max_{y \in A} V(y)$ be reached at x, since V is upper semicontinuous. By invariance there exists a solution $\mathbf{x}$ and $t > 0$ with $z = \mathbf{x}(0) \in A$ and $\mathbf{x}(t) = x$. This contradicts (iv) unless $\alpha = 0$ and $A \subset \Lambda$. Thus U is a neighborhood of A, which is an attractor included in Λ. ∎

Remark 3.26. Given any attractor A, there exists a function V such that Proposition 3.25(iv) holds for $\Lambda = A$. Take $V(x) = \max\{d(y, A)g(t),\ y \in \Phi_t(x),\ t \geq 0\}$, where $d > g(t) > c > 0$ is any continuous strictly increasing function.

Let Λ be any subset of $\mathbb{R}^m$. A continuous function $V : \mathbb{R}^m \to \mathbb{R}$ is called a *Lyapunov function* for Λ if $V(y) < V(x)$ for all $x \in \mathbb{R}^m \setminus \Lambda$, $y \in \Phi_t(x)$, $t > 0$, and $V(y) \leq V(x)$ for all $x \in \Lambda$, $y \in \Phi_t(x)$, and $t \geq 0$. Note that for each solution $\mathbf{x}$, V is constant along its limit set $L(\mathbf{x})$.

The following result is similar to Benaïm [3, Proposition 6.4].

PROPOSITION 3.27. *Suppose that V is a Lyapunov function for Λ. Assume that $V(\Lambda)$ has empty interior. Then every internally chain transitive set L is contained in Λ and $V \mid L$ is constant.*

Proof. Let

$$v = \inf\{V(y) : y \in L\}.$$

Since L is compact and V is continuous, $v = V(x)$ for some point $x \in L$. Since L is invariant, there exists a solution $\mathbf{x}$ with $\mathbf{x}(t) \in L$ and $\mathbf{x}(0) = x$. Then $v = V(x) > V(\mathbf{x}(t))$, and thus is impossible for $t > 0$. Since $\mathbf{x}(t) \in \Phi_t(x)$, we conclude $x \in \Lambda$.

Thus v belongs to the range $V(\Lambda)$. Since $V(\Lambda)$ contains no interval, there is a sequence $v_n \notin V(\Lambda)$ decreasing to v. The sets $L_n = \{x \in L : V(x) < v_n\}$ satisfy $\Phi_t(\overline{L}_n) \subset L_n$ for $t > 0$. In fact, either $x \in \Lambda \cap \overline{L}_n$ and $V(y) \leq V(x) < v_n$ or $V(y) < V(x) \leq v_n$, for any $y \in \Phi_t(x)$, $t > 0$.

Thus, using Propositions 3.19 and 3.20, one obtains $L = \bigcap_n \overline{L}_n = \{x \in L : V(x) = v\}$. Hence V is constant on L. L being invariant, this implies, as above, $L \subset \Lambda$. □

COROLLARY 3.28. *Let V and Λ be as in Proposition* 3.27. *Suppose furthermore that V is C^m and Λ is contained in the critical points set of V. Then every internally chain transitive set lies in Λ and $V \mid L$ is constant.*

Proof. By Sard's theorem (Hirsch [18, p. 69]), $V(\Lambda)$ has empty interior and Proposition 3.27 applies. □

4. The limit set theorem.

4.1. Asymptotic pseudotrajectories for set-valued dynamics. The translation flow $\Theta : C^0(\mathbb{R}, \mathbb{R}^m) \times \mathbb{R} \to C^0(\mathbb{R}, \mathbb{R}^m)$ is the flow defined by

$$\Theta^t(\mathbf{x})(s) = \mathbf{x}(s+t).$$

A continuous function $\mathbf{z} : \mathbb{R}^+ \to \mathbb{R}^m$ is an asymptotic pseudotrajectory (APT) for Φ if

$$\lim_{t\to\infty} \mathbf{D}(\Theta^t(\mathbf{z}), S_{\mathbf{z}(t)}) = 0 \tag{4.1}$$

(or $\lim_{t\to\infty} \mathbf{D}(\Theta^t(\mathbf{z}), S) = 0$, where $S = \bigcup_{x\in\mathbb{R}^m} S_x$ denotes the set of all solutions of (I)).

Alternatively, for all T

$$\lim_{t\to\infty} \inf_{\mathbf{x}\in S_{\mathbf{z}}(t)} \sup_{0\leq s\leq T} \|\mathbf{z}(t+s) - \mathbf{x}(s)\| = 0.$$

In other words, for each fixed T, the curve

$$[0, T] \to \mathbb{R}^m : s \to \mathbf{z}(t+s)$$

shadows some Φ trajectory of the point $\mathbf{z}(t)$ over the interval $[0, T]$ with arbitrary accuracy for sufficiently large t. Hence $\mathbf{z}$ has a forward trajectory under Θ attracted by S. As usual, one extends $\mathbf{z}$ to $\mathbb{R}$ by letting $\mathbf{z}(t) = \mathbf{z}(0)$ for $t < 0$.

The next result is a natural extension of Benaïm and Hirsch [4], [5, Theorem 7.2].

THEOREM 4.1 (characterization of APT). *Assume* $\mathbf{z}$ *is bounded. Then there is equivalence between the following statements:*

(i) $\mathbf{z}$ *is an APT for* Φ.

(ii) $\mathbf{z}$ *is uniformly continuous, and any limit point of* $\{\Theta^t(\mathbf{z})\}$ *is in* S.

In both cases the set $\{\Theta^t(\mathbf{z});\ t \geq 0\}$ *is relatively compact.*

Proof. By hypothesis, $K = \overline{\{\mathbf{z}(t);\ t \geq 0\}}$ is compact.

For any $\varepsilon > 0$, there exists $\eta > 0$ such that $\|z - x\| < \varepsilon/2$, for any $x \in K$, any $z \in \Phi_s(x)$, and any $|s| < \eta$, using property (d) of the dynamical system.

$\mathbf{z}$ being an APT, there exists T such that $t > T$ implies

$$d(\mathbf{z}(t+s), \Phi_s(\mathbf{z}(t))) < \frac{\varepsilon}{2} \quad \forall |s| < \eta;$$

hence

$$\|\mathbf{z}(t+s) - \mathbf{z}(t)\| \leq \varepsilon$$

and $\mathbf{z}$ is uniformly continuous. Clearly any limit point belongs to S by the condition (4.1) above.

Conversely, if $\mathbf{z}$ is uniformly continuous, then the family of functions $\{\Theta^t(\mathbf{z});\ t \geq T\}$ is equicontinuous and hence (K being compact) relatively compact by Ascoli's theorem. Since any limit point belongs to S, property (4.1) follows. ∎

4.2. Perturbed solutions are APTs.

THEOREM 4.2. *Any bounded solution* $\mathbf{y}$ *of* (II) *is an APT of* (I).

Proof. Let us prove that $\mathbf{y}$ satisfies Theorem 4.1(ii). Set $v(t) = \dot{\mathbf{y}}(t) - U(t) \in F^{\delta(t)}(\mathbf{y}(t))$. Then,

$$\mathbf{y}(t+s) - \mathbf{y}(t) = \int_0^s v(t+\tau)d\tau + \int_t^{t+s} U(\tau)d\tau. \tag{4.2}$$

By assumption (iii) of (II), the second integral goes to 0 as $t \to \infty$. The boundedness of $\mathbf{y}$, $\mathbf{y}(\mathbb{R}) \subset M$, M compact (combined with the fact that F has linear growth) implies boundedness of v and shows that $\mathbf{y}$ is uniformly continuous. Thus the family $\Theta^t(\mathbf{y})$ is equicontinuous, and hence relatively compact. Let $\mathbf{z} = \lim_{t_n \to \infty} \Theta^{t_n}(\mathbf{y})$ be a limit point. Set $t = t_n$ in (4.2) and define $v_n(s) = v(t_n + s)$. Then, using the assumption (iii) on U, the second term in the right-hand side of this equality goes to zero uniformly on compact intervals when $n \to \infty$. Hence

$$\mathbf{z}(s) - \mathbf{z}(0) = \lim_{n \to \infty} \int_0^s v_n(\tau)d\tau.$$

Since (v_n) is uniformly bounded, it is bounded in $L^2[0,s]$, and by the Banach–Alaoglu theorem, a subsequence of v_n will converge weakly in $L^2[0,s]$ (or weak* in $L^\infty[0,s]$) to some function v with $v(t) \in F(\mathbf{z}(t))$, for almost every t, since $v_n(t) \in F^{\delta(t+t_n)}(\mathbf{y}(t+t_n))$ for every t. Here we use (ii) and that F is upper semicontinuous with convex values. In fact, by Mazur's theorem, a convex combination of $\{v_m, m \geq n\}$ converges almost surely to v and $\lim_{m \to \infty} \text{Co}(\bigcup_{n \geq m} F^{\delta(t+t_n)}(\mathbf{y}(t+t_n))) \subset F(\mathbf{z}(t))$. Hence $\mathbf{z}(s) - \mathbf{z}(0) = \int_0^s v(\tau)d\tau$, proving that $\mathbf{z}$ is a solution of (I) and hence $\mathbf{z} \in S_{M,M}$. ∎

4.3. APTs are internally chain transitive.

THEOREM 4.3. *Let* $\mathbf{z}$ *be a bounded APT of* (I). *Then* $L(\mathbf{z})$ *is internally chain transitive.*

Proof. The set $\{\Theta^t(\mathbf{z}) : t \geq 0\}$ is relatively compact, and hence the ω-limit set of $\mathbf{z}$ for the flow Θ,

$$\omega_\Theta(\mathbf{z}) = \bigcap_{t \geq 0} \overline{\{\Theta^s(\mathbf{z}) : s \geq t\}},$$

is internally chain transitive. (By standard properties of ω-limit sets of bounded semiorbits, $\omega_\Theta(\mathbf{z})$ is a nonempty, compact, internally chain transitive set invariant under Θ; see Conley [13]; a short proof is also in Benaïm [3, Corollary 5.6].) By property (4.1), $\omega_\Theta(\mathbf{z}) \subset S$, the set of all solutions of (I).

Let $\Pi : (C^0(\mathbb{R}, \mathbb{R}^m), \mathbf{D}) \to (\mathbb{R}^m, \|\cdot\|)$ be the projection map defined by $\Pi(\mathbf{z}) = \mathbf{z}(0)$. One has $\Pi(\omega_\Theta(\mathbf{z})) = L(\mathbf{z})$. In fact if $p = \lim_{n\to\infty} \mathbf{z}(t_n)$, let $\mathbf{w}$ be a limit point of $\Theta^{t_n}(\mathbf{z})$. Then $\mathbf{w} \in \omega_\Theta(\mathbf{z})$ and $\Pi(\mathbf{w}) = p$.

It then easily follows that $L(\mathbf{z})$ is nonempty compact and invariant under Φ since $\omega_\Theta(\mathbf{z}) \subset S$. Since Π has Lipschitz constant 1, Π maps every (ε, T) chain for Θ to an (ε, T) chain for Φ. This proves that $L(\mathbf{z})$ is internally chain transitive for Φ. ∎

5. Applications.

5.1. Approachability. An application of Proposition 3.25 is the following result, which can be seen as a continuous asymptotic deterministic version of Blackwell's approachability theorem [8]. Note that one has no property on uniform speed of convergence.

Given a compact set $\Lambda \in \mathbb{R}^m$ and $x \in \mathbb{R}^m$, we let $\Pi_\Lambda(x) = \{y \in \Lambda : d^2(x, \Lambda) = \|x - y\|^2 = \langle x - y, x - y\rangle\}$.

COROLLARY 5.1. *Let $\Lambda \subset \mathbb{R}^m$ be a compact set, $r > 0$, and $U = \{x \in \mathbb{R}^m : d(x, \Lambda) < r\}$. Suppose that for all $x \in U \setminus \Lambda$ there exists $y \in \Pi_\Lambda(x)$ such that the affine hyperplane orthogonal to $[x, y]$ at y separates x from $x + F(x)$. That is,*

$$\langle x - y, x - y + v\rangle \leq 0 \tag{5.1}$$

for all $v \in F(x)$. Then Λ contains an attractor for (I) *with fundamental neighborhood U.*

Proof. Set $V(x) = d(x, \Lambda)$. To apply Proposition 3.25 it suffices to verify condition (iii) of Proposition 3.25. Condition (i) will follow, and condition (ii) is clearly true.

Let $\mathbf{x}$ be a solution to (I) with initial condition $x \in U \setminus \Lambda$. Set $\tau = \inf\{t > 0 : \mathbf{x}(t) \in \Lambda\} \leq \infty$, $g(t) = V(\mathbf{x}(t))$, and let $I \subset [0, \tau[$ be the set of $0 \leq t < \tau$ such that $g'(t)$ and $\dot{\mathbf{x}}(t)$ exist and $\dot{\mathbf{x}}(t) \in F(\mathbf{x}(t))$. For all $t \in I$ and $y \in \Pi_\Lambda(\mathbf{x}(t))$

$$\begin{aligned} g(t+h) - g(t) &\leq \|\mathbf{x}(t+h) - y\| - \|\mathbf{x}(t) - y\| \\ &= \|\mathbf{x}(t) + \dot{\mathbf{x}}(t)h - y\| - \|\mathbf{x}(t) - y\| + |h|\varepsilon(h), \end{aligned}$$

where $\lim_{h\to 0} \varepsilon(h) = 0$. Hence

$$\begin{aligned} g'(t) &\leq \frac{1}{\|\mathbf{x}(t) - y\|}\langle \mathbf{x}(t) - y, \dot{\mathbf{x}}(t)\rangle \\ &= -g(t) + \frac{1}{\|\mathbf{x}(t) - y\|}\langle \mathbf{x}(t) - y, \mathbf{x}(t) - y + \dot{\mathbf{x}}(t)\rangle. \end{aligned}$$

Thus, $\dot{x} \in F(x)$ and (5.1) imply $g'(t) \leq -g(t)$ for all $t \in I$. Since g and $\mathbf{x}$ are absolutely continuous, I has full measure in $[0, \tau[$. Hence $g(t) \leq e^{-t}g(0)$ for all $t < \tau$. Therefore $V(\mathbf{x}(t)) < V(x)$ for all $0 < t < \tau$, which shows (iii). Finally, $V(\mathbf{x}(t)) \leq e^{-t}V(x)$ shows that the sets $V^{-1}[0, r')$ (with $0 < r' \leq r$) are fundamental neighborhoods of the attractor in Λ. ∎

In particular, if any point of E has a unique projection on Λ (for example, Λ convex), then $\overline{C} = C$, and one recovers exactly Blackwell's sufficient condition for approachability.

COROLLARY 5.2 (Blackwell's approachability theorem). *Consider the decision making process described in section* 2.1*, Example* 2.2*. Let $\Lambda \subset E$ be a compact set. Assume that there exists a strategy Q such that for all $x \in E\setminus\Lambda$ there exists $y \in \Pi_\Lambda(x)$ such that the hyperplane orthogonal to $[x, y]$ through y separates x from $\overline{C}(x)$. Then Λ is approachable.*

Proof. Let $L(x_n)$ denote the limit set of $\{x_n\}$. By Corollary 5.1, Λ is an attractor with fundamental neighborhood E, hence a global attractor. Thus Theorem 3.6 with Proposition 2.1 and Corollary 3.24 imply that $L(x_n)$ is almost surely contained in Λ. ∎

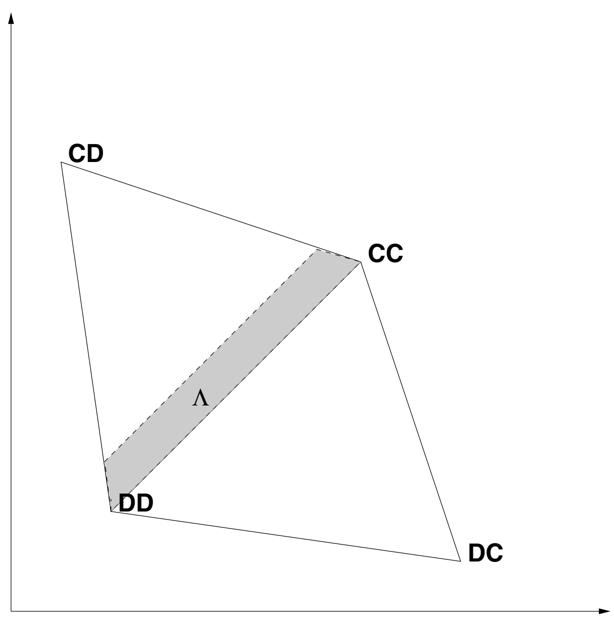
5.2. Smale's approach to the prisoner's dilemma. We develop here Example 2.4. Consider a 2×2 prisoner's dilemma game. Each player has two possible actions: cooperate (play C) or defect (play D). If both cooperate, each receives α; if both defect, each receives λ; if one cooperates and the other defects, the cooperator receives β and the defector γ. We suppose that $\gamma > \alpha > \lambda > \beta$, as is usual with a prisoner's dilemma game. We furthermore assume that

$$\gamma - \alpha < \alpha - \beta,$$

so that the outcome space E is the convex quadrilateral whose vertices are the payoff vectors

$$\mathbf{CD} = (\beta, \gamma), \qquad \mathbf{CC} = (\alpha, \alpha), \qquad \mathbf{DC} = (\gamma, \beta), \qquad \mathbf{DD} = (\lambda, \lambda);$$

see the figure below.



The outcome space E

Let δ be a nonnegative parameter. Adapting Smale [27] and Benaïm and Hirsch [4, 5], a *δ-good* strategy for player 1 is a strategy $Q^1 = \{Q^1_x\}$ (as defined in section 2.1) enjoying the following features:

$$Q^1_x(\text{play C}) = 1 \quad \text{if } x^1 > x^2$$

and

$$Q^1_x(\text{play C}) = 0 \quad \text{if } x^1 < x^2 - \delta.$$

The following result reinterprets the results of Smale [27] and Benaïm and Hirsch [4, 5] in the framework of approachability. It also provides some generalization (see Remark 5.4 below).

THEOREM 5.3. (i) *Suppose that player* 1 *plays a* δ*-good strategy. Then the set*

$$\Lambda = \{x \in E : x^2 - \delta \leq x^1 \leq x^2\}$$

is approachable.

(ii) *Suppose that both players play a* δ*-good strategy and that at least one of them is continuous (meaning that the corresponding function* $x \to Q^i_x$*(play* C*) is continuous). Then*

$$\lim_{n\to\infty} x_n = \mathbf{CC}$$

almost surely.

Proof. (i) Let $x \in E \setminus \Lambda$. If $x^1 > x^2$, then

$$C(x) = \overline{C}(x) = [\mathbf{CC}, \mathbf{CD}],$$

and the line $\{u \in \mathbb{R}^2 : u^1 = u^2\}$ separates x from $\overline{C}(x)$. Similarly if $x^1 < x^2 - \delta$, then

$$C(x) = \overline{C}(x) = [\mathbf{DD}, \mathbf{DC}],$$

which is separated from x by the line $\{u \in \mathbb{R}^2 : u^1 = u^2 - \delta\}$. Assertion (i) then follows from Corollary 5.2.

(ii) If both play a δ-good strategy, then (i) and its analogue for player 2 imply that the diagonal

$$\Delta = \{x \in E : x^1 = x^2\}$$

is approachable. Thus $L(x_n) \subset \Delta$. Also (by Proposition 2.1, Theorem 3.6, and Lemma 3.5) $L(x_n)$ is invariant under the differential inclusion induced by

$$F(x) = -x + \overline{C}(x),$$

where $C(x) = C^1(x) \cap C^2(x)$ and $C^i(x)$ is the convex set associated with Q^i (the strategy of player i). Suppose that one player, say 1, plays a continuous strategy. Then $\overline{C}(x) \subset \overline{C^1}(x) = C^1(x)$ and for all $x \in \Delta$, $C^1(x) = [\mathbf{CD}, \mathbf{CC}]$. Now, there is only one subset of Δ which is invariant under $\dot{x} \in -x + [\mathbf{CD}, \mathbf{CC}]$; this is the point $\mathbf{CC}$. This proves that $L(x_n) = \mathbf{CC}$. ∎

Remark 5.4. (i) In contrast to Smale [27] and Benaïm and Hirsch [4, 5], observe that assertion (i) makes no hypothesis on player 2's behavior. In particular, it is unnecessary to assume that player 2 has a strategy of the form defined by section 2.1.

(ii) The regularity assumptions (on strategies) are much weaker than in Benaïm and Hirsch [4, 5].

(iii) A 0-good strategy makes the diagonal Δ approachable. However, if both players play a 0-good strategy, then $\overline{C}(x) = E$ for all $x \in \Delta$, and we are unable to predict the long-term behavior of $\{x_n\}$ on Δ.

5.3. Fictitious play in potential games. Here we generalize the result of Monderer and Shapley [25]. They prove convergence of the classical discrete fictitious play process, as defined in Example 2.3, for n-linear payoff functions. Harris [17] studies the best-response dynamics in this case but does not derive convergence of fictitious play from it. Our limit set theorem provides the right tool for doing this, even in the following, more general setting.

Let X^i, $i = 1, \dots, n$, be compact convex subsets of Euclidean spaces and $U\colon X^1 \times \cdots \times X^n \to \mathbb{R}$ be a C^1 function which is concave in each variable. U is interpreted as the common payoff function for the n players. We write $x = (x^i, x^{-i})$ and define $BR^i(x^{-i}) := \mathrm{Argmax}_{x^i \in X^i} U(x)$ the set of maximizers. Then $x \mapsto BR(x) = (BR^1(x^{-1}), \dots, BR^n(x^{-n}))$ is upper semicontinuous (by Berge's maximum theorem, since U is continuous) with nonempty compact convex values. Consider the best response dynamics

$$\dot{\mathbf{x}} \in BR(\mathbf{x}) - \mathbf{x}. \tag{5.2}$$

Its constant solutions $\mathbf{x}(t) \equiv \hat{x}$ are precisely the Nash equilibria $\hat{x} \in BR(\hat{x})$; i.e., $U(\hat{x}) \geq U(x^i, \hat{x}^{-i})$ for all i and $x^i \in X^i$. Along a solution $\mathbf{x}(t)$ of (5.2), let $u(t) = U(\mathbf{x}(t))$. Then for almost all $t > 0$,

$$\dot{u}(t) = \sum_{i=1}^{n} \frac{\partial U}{\partial x^i}(\mathbf{x}(t))\dot{\mathbf{x}}^i(t) \tag{5.3}$$

$$\geq \sum_{i=1}^{n} [U(\mathbf{x}^i(t) + \dot{\mathbf{x}}^i(t), \mathbf{x}^{-i}(t)) - U(\mathbf{x}(t))] \tag{5.4}$$

$$= \sum_{i=1}^{n} \left[\max_{y^i \in X^i} U(y^i, \mathbf{x}^{-i}(t)) - U(\mathbf{x}(t))\right] \geq 0, \tag{5.5}$$

where from (5.3) to (5.4) we use the concavity of U in x^i, and (5.5) follows from (5.2) and the definition of BR^i. Since the function $t \mapsto u(t)$ is locally Lipschitz, this shows that it is weakly increasing. It is constant in a time interval T, if and only if $\mathbf{x}^i(t) \in BR^i(\mathbf{x}^{-i}(t))$ for all $t \in T$ and $i = 1, \dots, n$, i.e., if and only if $\mathbf{x}(t)$ is a Nash equilibrium for $t \in T$ (but $\mathbf{x}(t)$ may move in a component of the set of Nash equilibria (NE) with constant U).

THEOREM 5.5. *The limit set of every solution of* (5.2) *is a connected subset of NE, along which U is constant. If, furthermore, the set $U(NE)$ contains no interval in* $\mathbb{R}$, *then the limit set of every fictitious play path is a connected subset of NE along which U is constant.*

Proof. The first statement follows from the above. The second statement follows from Theorem 3.6 together with Proposition 3.27 with $V = -U$ and $\Lambda = NE$. ∎

Remark 5.6. The assumption that the set $U(NE)$ contains no interval in $\mathbb{R}$ follows via Corollary 3.28 if U is smooth enough (e.g., in the n-linear case) and if each X^i has at most countably many faces, by applying Sard's lemma to the interior of each face.

Example 5.7 (2×2 coordination game). The global attractor of (5.2) consists of three equilibria and two line segments connecting them. The internally chain transitive sets are the three equilibria. Hence every fictitious play process converges to one of these equilibria.

The case of (continuous concave-convex) two-person zero-sum games was treated in Hofbauer and Sorin [21], where it is shown that the global attractor of (5.2) equals the set of equilibria. In this case the full strength of Theorem 3.6 and the notion of chain transitivity are not needed; the invariance of the limit set of a fictitious play path implies that it is contained in the global attractor; compare Corollary 3.24.

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Stochastic Approximations and Differential Inclusions, Part II: Applications

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We apply the theoretical results on "stochastic approximations and differential inclusions" developed in Benaïm et al. [M. Benaïm, J. Hofbauer, S. Sorin. 2005. Stochastic approximations and differential inclusions. *SIAM J. Control Optim.* **44** 328–348] to several adaptive processes used in game theory, including classical and generalized approachability, no-regret potential procedures (Hart and Mas-Colell [S. Hart, A. Mas-Colell. 2003. Regret-based continuous time dynamics. *Games Econom. Behav.* **45** 375–394]), and smooth fictitious play [D. Fudenberg, D. K. Levine. 1995. Consistency and cautious fictitious play. *J. Econom. Dynam. Control* **19** 1065–1089].

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1. Introduction. The first paper of this series (Benaïm et al. [10]), henceforth referred to as BHS, was devoted to the analysis of the long-term behavior of a class of continuous paths called *perturbed solutions* that are obtained as certain perturbations of trajectories solutions to a *differential inclusion* in $\mathbb{R}^m$

$$\dot{\mathbf{x}} \in M(\mathbf{x}). \tag{1}$$

A fundamental and motivating example is given by (continuous time-linear interpolation of) discrete stochastic approximations of the form

$$X_{n+1} - X_n = a_{n+1} Y_{n+1} \tag{2}$$

with

$$\mathsf{E}(Y_{n+1} \mid \mathcal{F}_n) \in M(X_n),$$

where $n \in \mathbb{N}$, $a_n \geq 0$, $\sum_n a_n = +\infty$, and $\mathcal{F}_n$ is the σ-algebra generated by $(X_0, \ldots, X_n)$, under conditions on the increments $\{Y_n\}$ and the coefficients $\{a_n\}$. For example, if:

(i) $\sup_n \|Y_{n+1} - \mathsf{E}(Y_{n+1} \mid \mathcal{F}_n)\| < \infty$ and
(ii) $a_n = o(1/\log(n))$,

the interpolation of a process $\{X_n\}$ satisfying Equation (2) is almost surely a perturbed solution of Equation (1).

Following the dynamical system approach to stochastic approximations initiated by Benaïm and Hirsch (Benaïm [5], [6], Benaïm and Hirsch [8], [9]), it was shown in BHS that the set of limit points of a perturbed solution is a *compact invariant attractor free set* for the set-valued dynamical system induced by Equation (1).

From a mathematical viewpoint, this type of property is a natural generalization of Benaïm and Hirsch's previous results.[1] In view of applications, it is strongly motivated by a large class of problems, especially in game theory, where the use of differential inclusions is unavoidable since one deals with unilateral dynamics where the strategies chosen by a player's opponents (or nature) are unknown to this player.

In BHS, a few applications were given: (1) in the framework of approachability theory (where one player aims at controlling the asymptotic behavior of the Cesaro mean of a sequence of vector payoffs corresponding to the outcomes of a repeated game) and (2) for the study of fictitious play (where each player uses, at each stage of a repeated game, a move that is a best reply to the past frequencies of moves of the opponent).

[1] Benaïm and Hirsch's analysis was restricted to asymptotic pseudotrajectories (perturbed solutions) of differential equations and flows.

The purpose of the current paper is to explore much further the range of possible applications of the theory and to convince the reader that it provides a unified and powerful approach to several questions such as approachability or consistency (no regret). The price to pay is a bit of theory, but as a reward we obtain neat and simpler (sometimes much simpler) proofs of numerous results arising in different contexts.

The general structure for the analysis of such discrete time dynamics relies on the identification of a state variable for which the increments satisfy an equation like (2). This requires in particular vanishing step size (for example, the state variable will be a time average–of payoffs or moves–) and a Markov property for the conditional law of the increments (the behavioral strategy will be a function of the state variable).

The organization of the paper is as follows. Section 2 summarizes the results of BHS that will be needed here. In §3, we first consider generalized approachability where the parameters are a correspondence N and a potential function Q adapted to a set C, and we extend some results obtained by Hart and Mas-Colell [25]. In §4 we deal with (external) consistency (or no regret): The previous set C is now the negative orthant, and an approachability strategy is constructed explicitly through a potential function P, following Hart and Mas-Colell [25]. A similar approach (§5) also allows us to recover conditional (or internal) consistency properties via generalized approachability. Section 6 shows analogous results for an alternative dynamics: smooth fictitious play. This allows us to retrieve and extend certain properties obtained by Fudenberg and Levine [19], [21] on consistency and conditional consistency. Section 7 deals with several extensions of the previous results to the case where the information available to a player is reduced, and §8 applies to results recently obtained by Benaïm and Ben Arous [7].

2. General framework and previous results. Consider the differential inclusion (Equation 1). All the analysis will be done under the following condition, which corresponds to Hypothesis 1.1 in BHS:

Hypothesis 2.1 (Standing Assumptions). *M is an upper semicontinuous correspondence from $\mathbb{R}^m$ to itself, with compact convex nonempty values and which satisfies the following growth condition. There exists $c>0$ such that for all $x\in\mathbb{R}^m$,*

$$\sup_{z\in M(x)} \|z\| \le c(1+\|x\|).$$

Here $\|\cdot\|$ denotes any norm on $\mathbb{R}^m$.

Remark. These conditions are quite standard and such correspondences are sometimes called Marchaud maps (see Aubin [1, p. 62]). Note also that in most of our applications, one has $M(x)\subset K_0$, where K_0 is a given compact set, so that the growth condition is automatically satisfied.

In order to state the main results of BHS that will be used here, we first recall some definitions and notation.

The *set-valued dynamical system* $\{\Phi_t\}_{t\in\mathbb{R}}$ induced by Equation (1) is defined by

$$\Phi_t(x) = \{\mathbf{x}(t)\colon \mathbf{x} \text{ is a solution to Equation (1) with } \mathbf{x}(0)=x\},$$

where a solution to the differential inclusion (Equation 1) is an absolutely continuous mapping $\mathbf{x}\colon \mathbb{R}\to\mathbb{R}^m$, satisfying

$$\frac{d\mathbf{x}(t)}{dt}\in M(\mathbf{x}(t))$$

for almost every $t\in\mathbb{R}$.

Given a set of times $T\subset\mathbb{R}$ and a set of positions $V\subset\mathbb{R}^m$,

$$\Phi_T(V)=\bigcup_{t\in T}\bigcup_{v\in V}\Phi_t(v)$$

denotes the set of possible values, at some time in T, of trajectories being in V at time 0. Given a point $x\in\mathbb{R}^m$, let

$$\omega_\Phi(x)=\bigcap_{t\ge 0}\overline{\Phi_{[t,\infty)}(x)}$$

denote its *ω-limit set* (where as usual the bar stands for the closure operator). The corresponding notion for a set Y, denoted as $\omega_\Phi(Y)$, is defined similarly with $\Phi_{[t,\infty)}(Y)$ instead of $\Phi_{[t,\infty)}(x)$.

A set A is *invariant* if, for all $x\in A$ there exists a solution $\mathbf{x}$ with $\mathbf{x}(0)=x$ such that $\mathbf{x}(\mathbb{R})\subset A$ and is *strongly positively invariant* if $\Phi_t(A)\subset A$ for all $t>0$. A nonempty compact set A is an *attracting* set if there exists a neighborhood U of A and a function $\mathbf{t}$ from $(0,\varepsilon_0)$ to $\mathbb{R}^+$ with $\varepsilon_0>0$ such that

$$\Phi_t(U)\subset A^\varepsilon$$

for all $\varepsilon < \varepsilon_0$ and $t \geq \mathbf{t}(\varepsilon)$, where A^ε stands for the ε-neighborhood of A. This corresponds to a strong notion of attraction, uniform with respect to the initial conditions and the feasible trajectories. If additionally A is invariant, then A is an *attractor*.

Given an attracting set (respectively attractor) A, its *basin of attraction* is the set

$$B(A) = \{x \in \mathbb{R}^m\colon \omega_\Phi(x) \subset A\}.$$

When $B(A) = \mathbb{R}^m$, A is a *globally* attracting set (resp. a global attractor).

REMARK. The following terminology is sometimes used in the literature. A set A is *asymptotically stable* if it is

(i) invariant,

(ii) *Lyapounov stable*, i.e., for every neighborhood U of A there exists a neighborhood V of A such that its forward image $\Phi_{[0,\infty)}(V)$ satisfies $\Phi_{[0,\infty)}(V) \subset U$, and

(iii) *attractive*, i.e., its basin of attraction $B(A)$ is a neighborhood of A.

However, as shown in (BHS, Corollary 3.18) attractors and compact asymptotically stable sets coincide.

Given a closed invariant set L, the induced dynamical system Φ^L is defined on L by

$$\Phi_t^L(x) = \{\mathbf{x}(t)\colon \mathbf{x} \text{ is a solution to Equation (1) with } \mathbf{x}(0) = x \text{ and } \mathbf{x}(\mathbb{R}) \subset L\}.$$

An invariant set L is *attractor free* if there exists no proper subset A of L that is an attractor for Φ^L.

We now turn to the discrete random perturbations of Equation (1) and consider, on a probability space $(\Omega, \mathcal{F}, P)$, random variables X_n, $n \in \mathbb{N}$, with values in $\mathbb{R}^m$, satisfying the difference inclusion

$$X_{n+1} - X_n \in a_{n+1}[M(X_n) + U_{n+1}], \tag{3}$$

where the coefficients a_n are nonnegative numbers with

$$\sum_n a_n = +\infty.$$

Such a process $\{X_n\}$ is a *discrete stochastic approximation* (DSA) of the differential inclusion (Equation 1) if the following conditions on the perturbations $\{U_n\}$ and the coefficients $\{a_n\}$ hold:

(i) $\mathsf{E}(U_{n+1} \mid \mathcal{F}_n) = 0$ where $\mathcal{F}_n$ is the σ-algebra generated by $(X_1, \ldots, X_n)$,

(ii) (a) $\sup_n \mathsf{E}(\|U_{n+1}\|^2) < \infty$ and $\sum_n a_n^2 < +\infty$ or

(b) $\sup_n \|U_{n+1}\| < K$ and $a_n = o(1/\log(n))$.

REMARK. More general conditions on the characteristics (a_n, U_n) can be found in (BHS, Proposition 1.4).

A typical example is given by equations of the form Equation (2) by letting

$$U_{n+1} = Y_{n+1} - \mathsf{E}(Y_{n+1} \mid \mathcal{F}_n).$$

Given a trajectory $\{X_n(\omega)\}_{n\geq 1}$, its set of accumulation points is denoted by $L(\omega) = L(\{X_n(\omega)\})$. The *limit set* of the process $\{X_n\}$ is the random set $L = L(\{X_n\})$.

The principal properties established in BHS express relations between limit sets of DSA and attracting sets through the following results involving *internally chain transitive* (ICT) sets. (We do not define ICT sets here since we only use the fact that they satisfy Properties 2 and 4 below; see BHS §3.3.)

PROPERTY 1. The limit set L of a bounded DSA is almost surely an ICT set.

This result is, in fact, stated in BHS for the limit set of the continuous time interpolated process, but under our conditions both sets coincide.

Properties of the limit set L will then be obtained through the next result (BHS, Lemma 3.5, Proposition 3.20, and Theorem 3.23):

PROPERTY 2.

(i) ICT sets are nonempty, compact, invariant, and attractor free.

(ii) If A is an attracting set with $B(A) \cap L \neq \varnothing$ and L is ICT, then $L \subset A$.

Some useful properties of attracting sets or attractors are the two following (BHS, Propositions 3.25 and 3.27).

PROPERTY 3 (STRONG LYAPOUNOV). Let $\Lambda \subset \mathbb{R}^m$ be compact with a bounded open neighborhood U and $V\colon \bar{U} \to [0, \infty[$. Assume the following conditions:

(i) U is strongly positively invariant,

(ii) $V^{-1}(0) = \Lambda$,

(iii) V is continuous and for all $x \in U \setminus \Lambda$, $y \in \Phi_t(x)$ and $t > 0$, $V(y) < V(x)$.

Then, Λ contains an attractor whose basin contains U. The map V is called a strong Lyapounov function associated to Λ.

Let $\Lambda \subset \mathbb{R}^m$ be a set and $U \subset \mathbb{R}^m$ an open neighborhood of Λ. A continuous function $V\colon U \to \mathbb{R}$ is called a *Lyapounov function* for $\Lambda \subset \mathbb{R}^m$ if $V(y) < V(x)$ for all $x \in U \setminus \Lambda$, $y \in \Phi_t(x)$, $t > 0$; and $V(y) \le V(x)$ for all $x \in \Lambda$, $y \in \Phi_t(x)$ and $t \ge 0$.

Property 4 (Lyapounov). Suppose V is a Lyapounov function for Λ. Assume that $V(\Lambda)$ has an empty interior. Then, every internally chain transitive set $L \subset U$ is contained in Λ and $V \mid L$ is constant.

3. Generalized approachability: A potential approach. We follow here the approach of Hart and Mas-Colell [25], [27]. Throughout this section, C is a closed subset of $\mathbb{R}^m$ and Q is a "potential function" that attains its minimum on C. Given a correspondence N, we consider a dynamical system defined by

$$\dot{\mathbf{w}} \in N(\mathbf{w}) - \mathbf{w}. \tag{4}$$

We provide two sets of conditions on N and Q that imply convergence of the solutions of Equation (4) and of the corresponding DSA to the set C. When applied in the approachability framework (Blackwell [11]), this will extend Blackwell's property.

Hypothesis 3.1. *Q is a $\mathscr{C}^1$ function from $\mathbb{R}^m$ to $\mathbb{R}$ such that*

$$Q \ge 0 \qquad \text{and} \qquad C = \{Q = 0\}$$

and N is a correspondence satisfying the standard Hypothesis 2.1.

3.1. Exponential convergence.

Hypothesis 3.2. *There exists some positive constant B such that for $w \in \mathbb{R}^m \setminus C$*

$$\langle \nabla Q(w), N(w) - w \rangle \le -BQ(w),$$

meaning $\langle \nabla Q(w), w' - w \rangle \le -BQ(w)$, for all $w' \in N(w)$.

Theorem 3.3. *Let $\mathbf{w}(t)$ be a solution of Equation (4). Under Hypotheses 3.1 and 3.2, $Q(\mathbf{w}(t))$ goes to zero at exponential rate and the set C is a globally attracting set.*

Proof. If $\mathbf{w}(t) \notin C$

$$\frac{d}{dt} Q(\mathbf{w}(t)) = \langle \nabla Q(\mathbf{w}(t)), \dot{\mathbf{w}}(t) \rangle$$

hence,

$$\frac{d}{dt} Q(\mathbf{w}(t)) \le -BQ(\mathbf{w}(t))$$

so that

$$Q(\mathbf{w}(t)) \le Q(\mathbf{w}(0)) e^{-Bt}.$$

This implies that, for any $\varepsilon > 0$, any bounded neighborhood V of C satisfies $\Phi_t(V) \subset C^\varepsilon$, for t large enough.

Alternatively, Property 3 applies to the forward image $W = \Phi_{[0,\infty)}(V)$. □

Corollary 3.4. *Any bounded DSA of Equation (4) converges a.s. to C.*

Proof. Being a DSA implies Property 1. C is a global attracting set, thus Property 2 applies. Hence, the limit set of any DSA is a.s. included in C. □

3.2. Application: Approachability. Following again Hart and Mas-Colell [25], [27] and assuming Hypothesis 3.2, we show here that the above property extends Blackwell's approachability theory (Blackwell [11], Sorin [33]) in the convex case. (A first approach can be found in BHS, §5.)

Let I and L be two finite sets of moves. Consider a two-person game with vector payoffs described by an $I \times L$ matrix A with entries in $\mathbb{R}^m$. At each stage $n+1$, knowing the previous sequence of moves $h_n = (i_1, l_1, \ldots, i_n, l_n)$, player 1 (resp. 2) chooses i_{n+1} in I (resp. l_{n+1} in L). The corresponding stage payoff is $g_{n+1} = A_{i_{n+1}, l_{n+1}}$ and $\bar{g}_n = (1/n) \sum_{m=1}^n g_m$ denotes the average of the payoffs until stage n. Let $X = \Delta(I)$ denote the simplex of mixed moves (probabilities on I) and similarly $Y = \Delta(L)$. $\mathscr{H}_n = (I \times L)^n$ denotes the space of all possible sequences of moves up to time n. A *strategy* for player 1 is a map

$$\sigma\colon \bigcup_n \mathscr{H}_n \to X, \qquad h_n \in \mathscr{H}_n \to \sigma(h_n) = (\sigma_i(h_n))_{i \in I}$$

and similarly $\tau\colon \bigcup_n \mathcal{H}_n \to Y$ for player 2. A pair of strategies (σ, τ) for the players specifies at each stage $n+1$ the distribution of the current moves given the past according to the formulae:

$$P(i_{n+1}=i,\, l_{n+1}=l \mid \mathcal{F}_n)(h_n) = \sigma_i(h_n)\tau_l(h_n),$$

where $\mathcal{F}_n$ is the σ-algebra generated by h_n. It then induces a probability on the space of sequences of moves $(I\times L)^{\mathbb{N}}$ denoted $\mathsf{P}_{\sigma,\tau}$.

For x in X we let xA denote the convex hull of the family $\{xA_l = \sum_{i\in I} x_i A_{il};\ l\in L\}$. Finally $d(.,C)$ stands for the distance to the closed set C: $d(x,C)=\inf_{y\in C} d(x,y)$.

Definition 3.5. Let N be a correspondence from $\mathbb{R}^m$ to itself. A function $\tilde{x}$ from $\mathbb{R}^m$ to X is *N-adapted* if

$$\tilde{x}(w)A \subset N(w), \quad \forall\, w \notin C.$$

Theorem 3.6. *Assume Hypotheses 3.1 and 3.2 and that $\tilde{x}$ is N-adapted. Then, any strategy σ of player 1 that satisfies $\sigma(h_n)=\tilde{x}(\bar{g}_n)$ at each stage n, whenever $\bar{g}_n\notin C$, approaches C: explicitly, for any strategy τ of player 2,*

$$d(\bar{g}_n, C)\to 0 \quad \mathsf{P}_{\sigma,\tau} \text{ a.s.}$$

Proof. The proof proceeds in two steps.

First, we show that the discrete dynamics associated to the approachability process is a DSA of Equation (4), as in BHS, §2 and §5. Then, we apply Corollary 3.4. Explicitly, the sequence of outcomes satisfies:

$$\bar{g}_{n+1}-\bar{g}_n = \frac{1}{n+1}(g_{n+1}-\bar{g}_n).$$

By the choice of player 1's strategy, $E_{\sigma,\tau}(g_{n+1}\mid \mathcal{F}_n)=\gamma_n$ belongs to $\tilde{x}(\bar{g}_n)A\subset N(\bar{g}_n)$, for any strategy τ of player 2. Hence, one writes

$$\bar{g}_{n+1}-\bar{g}_n = \frac{1}{n+1}(\gamma_n-\bar{g}_n+(g_{n+1}-\gamma_n)),$$

which shows that $\{\bar{g}_n\}$ is a DSA of Equation (4) (with $a_n=1/n$ and $Y_{n+1}=g_{n+1}-\bar{g}_n$, so that $E(Y_{n+1}\mid\mathcal{F}_n)\in N(\bar{g}_n)-\bar{g}_n$). Then, Corollary 3.4 applies. □

Remark. The fact that $\tilde{x}$ is N-adapted implies that the trajectories of the deterministic continuous time process when player 1 follows $\tilde{x}$ are always feasible under N, while N might be much more regular and easier to study.

Convex Case. Assume C convex. Let us show that the above analysis covers the original framework of Blackwell [11]. Recall that Blackwell's sufficient condition for approachability states that for any $w\notin C$, there exists $x(w)\in X$ with:

$$\langle w-\Pi_C(w),\, x(w)A-\Pi_C(w)\rangle \le 0, \tag{5}$$

where $\Pi_C(w)$ denotes the projection of w on C.

Convexity of C implies the following property:

Lemma 3.7. *Let $Q(w)=\|w-\Pi_C(w)\|_2^2$, then Q is C^1 with $\nabla Q(w)=2(w-\Pi_C(w))$.*

Proof. We simply write $\|w\|^2$ for the square of the L^2 norm:

$$\begin{aligned} Q(w+w')-Q(w) &= \|w+w'-\Pi_C(w+w')\|^2-\|w-\Pi_C(w)\|^2\\ &\le \|w+w'-\Pi_C(w)\|^2-\|w-\Pi_C(w)\|^2\\ &= 2\langle w', w-\Pi_C(w)\rangle+\|w'\|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} Q(w+w')-Q(w) &\ge \|w+w'-\Pi_C(w+w')\|^2-\|w-\Pi_C(w+w')\|^2\\ &= 2\langle w', w-\Pi_C(w+w')\rangle+\|w'\|^2. \end{aligned}$$

C being convex, Π_C is continuous (1 Lipschitz); hence, there exist two constants c_1 and c_2 such that

$$c_1\|w'\|^2 \le Q(w+w')-Q(w)-2\langle w', w-\Pi_C(w)\rangle \le c_2\|w'\|^2.$$

Thus, Q is C^1 and $\nabla Q(w)=2(w-\Pi_C(w))$. □

PROPOSITION 3.8. *If player 1 uses a strategy σ which, at each position $\bar{g}_n = w$, induces a mixed move $x(w)$ satisfying Blackwell's condition (Equation 5), then approachability holds: for any strategy τ of player 2,*

$$d(\bar{g}_n, C) \to 0 \quad \mathsf{P}_{\sigma,\tau} \text{ a.s.}$$

PROOF. Let $N(w)$ be the intersection of $\mathbf{A}$, the convex hull of the family $\{A_{il};\ i \in I,\ l \in L\}$, with the closed half space $\{\theta \in \mathbb{R}^m;\ \langle w - \Pi_C(w), \theta - \Pi_C(w)\rangle \leq 0\}$. Then, N is u.s.c. by continuity of Π_C and Equation (5) makes x N-adapted. Furthermore, the condition

$$\langle w - \Pi_C(w), N(w) - \Pi_C(w)\rangle \leq 0$$

can be rewritten as

$$\langle w - \Pi_C(w), N(w) - w\rangle \leq -\|w - \Pi_C(w)\|^2,$$

which is

$$\left\langle \frac{1}{2}\nabla Q(w), N(w) - w\right\rangle \leq -Q(w)$$

with $Q(w) = \|w - \Pi_C(w)\|^2$ by the previous Lemma 3.7. Hence, Hypotheses 3.1 and 3.2 hold and Theorem 3.6 applies. □

REMARK. (i) The convexity of C was used to get the property of Π_C, hence of Q ($\mathscr{C}^1$) and of N (u.s.c.). Define the support function of C on $\mathbb{R}^m$ by:

$$w_C(u) = \sup_{c \in C}\langle u, c\rangle.$$

The previous condition of Hypothesis 3.2 holds in particular if Q satisfies

$$\langle \nabla Q(w), w\rangle - w_C(\nabla Q(w)) \geq B.Q(w) \tag{6}$$

and N fulfills the following inequality:

$$\langle \nabla Q(w), N(w)\rangle \leq w_C(\nabla Q(w)) \quad \forall\, w \in \mathbb{R}^m \backslash C, \tag{7}$$

which are the original conditions of Hart and Mas-Colell [25, p. 34].

(ii) Blackwell [11] obtains also a speed of convergence of $n^{-1/2}$ for the expectation of the distance: $\rho_n = \mathsf{E}(d(\bar{g}_n, C))$. This corresponds to the exponential decrease $\rho_t^2 = Q(\mathbf{x}(t)) \leq Le^{-t}$ since in the DSA, stage n ends at time $t_n = \sum_{m \leq n}(1/m) \sim \log(n)$.

(iii) BHS proves results very similar to Proposition 3.8 (Corollaries 5.1 and 5.2 in BHS) for arbitrary (i.e., not necessarily convex) compact sets C but under a stronger separability assumption.

3.3. Slow convergence. We follow again Hart and Mas-Colell [25] in considering a hypothesis weaker than Hypothesis 3.2.

HYPOTHESIS 3.9. *Q and N satisfy, for $w \in \mathbb{R}^m \backslash C$:*

$$\langle \nabla Q(w), N(w) - w\rangle < 0.$$

REMARK. This is in particular the case if C is convex, inequality (7) holds, and whenever $w \notin C$:

$$\langle \nabla Q(w), w\rangle > w_C(\nabla Q(w)). \tag{8}$$

(A closed half space with exterior normal vector $\nabla Q(w)$ contains C and $N(w)$ but not w (see Hart and Mas-Colell [25, p. 31])).

THEOREM 3.10. *Under Hypotheses 3.1 and 3.9, Q is a strong Lyapounov function for Equation (4).*

PROOF. Using Hypothesis 3.9, one obtains if $\mathbf{w}(t) \notin C$:

$$\frac{d}{dt}Q(\mathbf{w}(t)) = \langle \nabla Q(\mathbf{w}(t)), \dot{\mathbf{w}}(t)\rangle = \langle \nabla Q(\mathbf{w}(t)), N(\mathbf{w}(t)) - \mathbf{w}(t)\rangle < 0. \quad \square$$

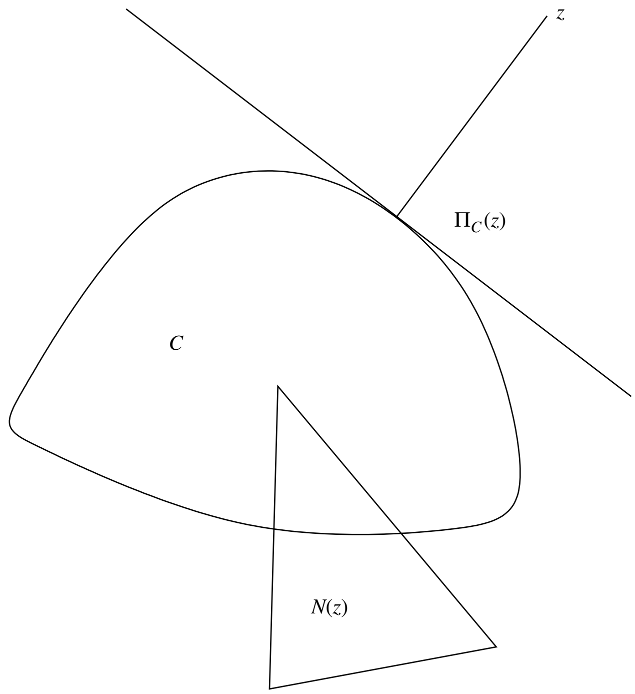

FIGURE 1. Condition (5).

COROLLARY 3.11. *Assume Hypotheses 3.1 and 3.9. Then, any bounded DSA of Equation (4) converges a.s. to C. Furthermore, Theorem 3.6 applies when Hypothesis 3.2 is replaced by Hypothesis 3.9.*

PROOF. The proof follows from Properties 1, 2, and 3. The set C contains a global attractor; hence, the limit set of a bounded DSA is included in C. □

We summarize the different geometrical conditions as in Figures 1, 2, and 3.

The hyperplane through $\Pi_C(z)$ orthogonal to $z - \Pi_C(z)$ separates z and $N(z)$ (Blackwell [11]) as in condition (5) (see Figure 1).

The supporting hyperplane to C with orthogonal direction $\nabla Q(z)$ separates $N(z)$ from z (Hart and Mas-Colell [24]) as in Conditions (7) and (8) (see Figure 2).

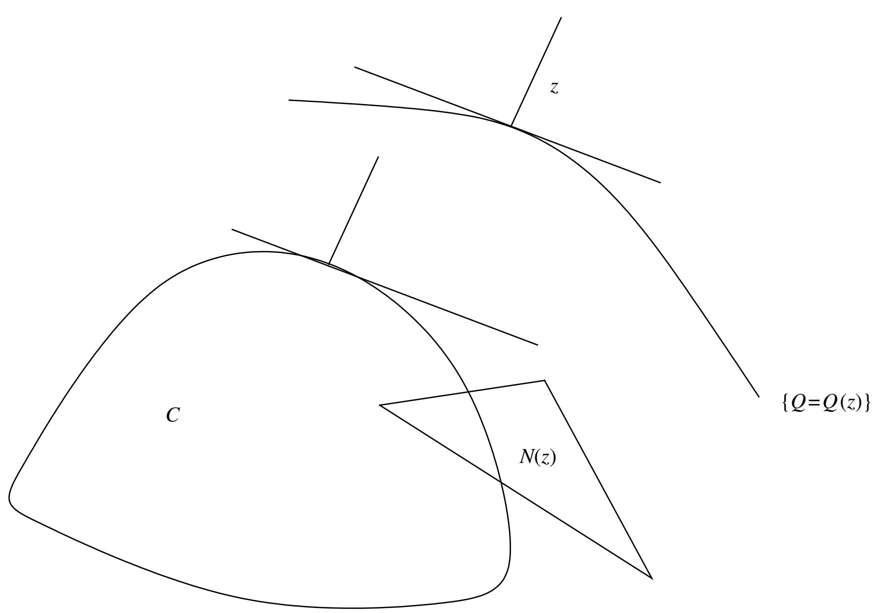

FIGURE 2. Conditions (7) and (8).

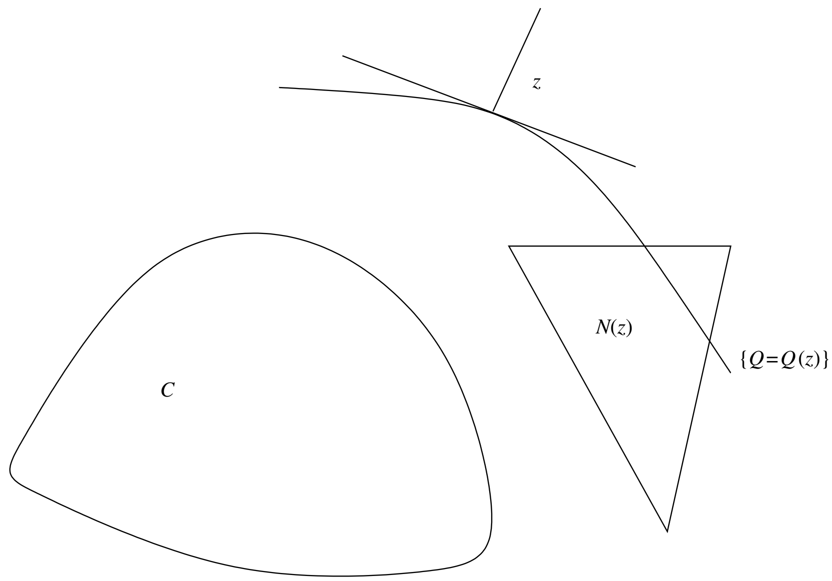

FIGURE 3. Condition of Hypothesis 3.9.

$N(z)$ belongs to the interior of the half space defined by the exterior normal vector $\nabla Q(z)$ at z as in Figure 3.

4. Approachability and consistency. We consider here a framework where the previous set C is the negative orthant and the vector of payoffs describes the vector of regrets in a strategic game (see Hart and Mas-Colell [25], [27]). The consistency condition amounts to the convergence of the average regrets to C. The interest of the approach is that the same function P will be used to play the role of the function Q on the one hand and to define the strategy and, hence, the correspondence N on the other. Also, the procedure can be defined on the payoff space as well as on the set of correlated moves.

4.1. No regret and correlated moves. Consider a finite game in strategic form. There are finitely many players labeled $a = 1, 2, \ldots, A$. We let S^a denote the finite moves set of player a, $S = \prod_a S^a$, and $Z = \Delta(S)$ the set of probabilities on S (correlated moves). Since we will consider everything from the view point of player 1, it is convenient to set $S^1 = I$, $X = \Delta(I)$ (mixed moves of player 1), $L = \prod_{a\neq 1} S^a$, and $Y = \Delta(L)$ (correlated mixed moves of player 1's opponents), hence $Z = \Delta(I \times L)$. Throughout, $X \times Y$ is identified with a subset of Z through the natural embedding $(x, y) \to x \times y$, where $x \times y$ stands for the product probability of x and y. As usual, I (L, S) is also identified with a subset of X (Y, Z) through the embedding $k \to \delta_k$. We let U: $S \to \mathbb{R}$ denote the payoff function of player 1, and we still denote by U its linear extension to Z and its bilinear extension to $X \times Y$. Let m be the cardinality of I and $R(z)$ denote the m-dimensional vector of regrets for player 1 at z in Z, defined by

$$R(z) = \{U(i, z^{-1}) - U(z)\}_{i\in I},$$

where z^{-1} stands for the marginal of z on L. (Player 1 compares his payoff using a given move i to his actual payoff, assuming the other players' behavior, z^{-1}, given.)

Let $D = \mathbb{R}^m_-$ be the closed negative orthant associated to the set of moves of player 1.

DEFINITION 4.1. H (for Hannan's set; see Hannan [22]) is the set of probabilities in Z satisfying the no-regret condition for player 1. Formally:

$$H = \{z \in Z\colon U(i, z^{-1}) \le U(z),\ \forall\, i \in I\} = \{z \in Z\colon R(z) \in D\}.$$

DEFINITION 4.2. P is a *potential function* for D if it satisfies the following set of conditions:

(i) P is a $\mathscr{C}^1$ nonnegative function from $\mathbb{R}^m$ to $\mathbb{R}$,
(ii) $P(w) = 0$ iff $w \in D$,
(iii) $\nabla P(w) \ge 0$, and
(iv) $\langle \nabla P(w), w\rangle > 0$, $\forall\, w \notin D$.

DEFINITION 4.3. Given a potential P for D, the *P-regret-based dynamics* for player 1 is defined on Z by

$$\dot{\mathbf{z}} \in N(\mathbf{z}) - \mathbf{z} \tag{9}$$

where

(i) $N(z) = \varphi(R(z)) \times Y \subset Z$, with
(ii) $\varphi(w) = \nabla P(w)/|\nabla P(w)| \in X$ whenever $w \notin D$ and $\varphi(w) = X$ otherwise.
Here $|\nabla P(w)|$ stands for the L^1 norm of $\nabla P(w)$.

REMARK. This corresponds to a process where only the behavior of player 1, outside of H, is specified. Note that even the dynamics is truly independent among the players ("uncoupled" according to Hart and Mas-Colell; see Hart [23]) the natural state space is the set of correlated moves (and not the product of the sets of mixed moves) since the criteria involves the actual payoffs and not only the marginal empirical frequencies.

The associated discrete process is as follows. Let $s_n \in S$ be the random variable of profile of actions at stage n and $\mathcal{F}_n$ the σ-algebra generated by the history $h_n = (s_1, \ldots, s_n)$. The average $\bar{z}_n = (1/n)\sum_{m=1}^n s_m$ satisfies:

$$\bar{z}_{n+1} - \bar{z}_n = \frac{1}{n+1}[s_{n+1} - \bar{z}_n]. \tag{10}$$

DEFINITION 4.4. A *P-regret-based strategy* for player 1 is specified by the conditions:
(i) For all $(i, l) \in I \times L$

$$\mathsf{P}(i_{n+1} = i,\, l_{n+1} = l \mid \mathcal{F}_n) = \mathsf{P}(i_{n+1} = i \mid \mathcal{F}_n)\mathsf{P}(l_{n+1} = l \mid \mathcal{F}_n), \quad \text{and}$$

(ii) $\mathsf{P}(i_{n+1} = i \mid \mathcal{F}_n) = \varphi_i(R(\bar{z}_n))$ whenever $R(\bar{z}_n) \notin D$, where $\varphi(\cdot) = \{\varphi_i(\cdot)\}_{i\in I}$ is like in Definition 4.3.
The corresponding discrete time process (Equation 10) is called a *P-regret-based discrete dynamics*.

Clearly, one has the following property:

PROPOSITION 4.5. *The P-regret-based discrete dynamics Equation* (10) *is a DSA of Equation* (9).

The next result is obvious but crucial.

LEMMA 4.6. *Let $z = x \times y \in X \times Y \subset Z$, then*

$$\langle x, R(z)\rangle = 0.$$

PROOF. One has

$$\sum_{i\in I} x_i[U(i, y) - U(x \times y)] = 0. \quad \square$$

4.2. Blackwell's framework. Given $w \in \mathbb{R}^m$, let w^+ be the vector with components $w_k^+ = \max(w_k, 0)$. Define $Q(w) = \sum_k (w_k^+)^2$. Note that $\nabla Q(w) = 2w^+$; hence, Q satisfies the conditions (i)–(iv) of Definition 4.2. If Π denotes the projection on D, one has $w - \Pi(w) = w^+$ and $\langle w^+, \Pi(w)\rangle = 0$.

In the game with vector payoff given by the regret of player 1, the set of feasible expected payoffs corresponding to xA (cf. §3.2), when player 1 uses θ, is $\{R(z);\, z = \theta \times z^{-1}\}$. Assume that player 1 uses a Q-regret-based strategy. Since at $w = \bar{g}_n$, $\theta(w)$ is proportional to $\nabla Q(w)$, hence to w^+, Lemma 4.6 implies that condition (5): $\langle w - \Pi w, xA - \Pi w\rangle \le 0$ is satisfied; in fact, this quantity reduces to: $\langle w^+, R(y) - \Pi w\rangle$, which equals 0. Hence, a Q-regret-based strategy approaches the orthant D.

4.3. Convergence of P-regret-based dynamics. The previous dynamics in §3 were defined on the payoff space. Here, we take the image by R (which is linear) of the dynamical system (Equation 9) and obtain the following differential inclusion in $\mathbb{R}^m$:

$$\dot{\mathbf{w}} \in \widehat{N}(\mathbf{w}) - \mathbf{w} \tag{11}$$

where

$$\widehat{N}(w) = R(\varphi(w) \times Y).$$

The associated discrete dynamics to Equation (10) is given as

$$\bar{w}_{n+1} - \bar{w}_n = \frac{1}{n+1}(w_{n+1} - \bar{w}_n) \tag{12}$$

with $w_n = R(z_n)$.

THEOREM 4.7. *The potential P is a strong Lyapounov function associated to the set D for Equation* (11) *and, similarly, $P \circ R$ to the set H for Equation* (9). *Hence, D contains an attractor for Equation* (11) *and H contains an attractor for Equation* (9).

PROOF. Remark that $\langle \nabla P(w), \widehat{N}(w)\rangle = 0$; in fact, $\nabla P(w) = 0$ for $w \in D$, and for $w \notin D$ use Lemma 4.6. Hence, for any $\mathbf{w}(t)$ solution to Equation (11),

$$\frac{d}{dt}P(\mathbf{w}(t)) = \langle \nabla P(\mathbf{w}(t)), \dot{\mathbf{w}}(t)\rangle = -\langle \nabla P(\mathbf{w}(t)), \mathbf{w}(t)\rangle < 0,$$

and P is a strong Lyapounov function associated to D in view of conditions (i)–(iv) of Definition 4.2. The last assertion follows from Property 3. □

COROLLARY 4.8. *Any P-regret-based discrete dynamics (Equation 10) approaches D in the payoff space; hence, H is in the action space.*

PROOF. D (resp. H) contains an attractor for Equation (11) whose basin of attraction contains $R(Z)$ (resp. Z) and the process Equation (12) (resp. Equation (10)) is a bounded DSA, hence Properties 1, 2, and 3 apply. □

REMARK. A direct proof is available as follows:

Let $\mathbf{R}$ be the range of R and define for $w \notin D$

$$N(w) = \{w' \in \mathbb{R}^m;\ \langle w', \nabla P(w)\rangle = 0\} \cap \mathbf{R}.$$

Hypotheses 3.1 and 3.9 are satisfied and Corollary 3.11 applies.

5. Approachability and conditional consistency. We keep the framework of §4 and the notation introduced in §4.1 and follow Hart and Mas-Colell [24], [25], [26] in studying conditional (or internal) regrets. One constructs again an approachability strategy from an associate potential function P. As in §4, the dynamics can be defined either in the payoff space or in the space of correlated moves.

We still consider only player 1 and denote by U his payoff.

Given $z = (z_s)_{s\in S} \in Z$, introduce the family of m comparison vectors of dimension m (testing k against j with $(j, k) \in I^2$) defined by

$$C(j,k)(z) = \sum_{l\in L} [U(k,l) - U(j,l)] z_{(j,l)}.$$

(This corresponds to the change in the expected gain of player 1 at z when replacing move j by k.) Remark that if one let $(z \mid j)$ denote the conditional probability on L induced by z given $j \in I$ and z^1 the marginal on I, then

$$\{C(j,k)(z)\}_{k\in I} = z^1_j R((z \mid j)),$$

where we recall that $R((z \mid j))$ is the vector of regrets for player 1 at $(z \mid j)$.

DEFINITION 5.1. The set of no conditional regret (for player 1) is

$$C^1 = \{z;\ C(j,k)(z) \le 0,\ \forall\, j, k \in I\}.$$

It is obviously a subset of H since

$$\sum_j \{C(j,k)(z)\}_{k\in I} = R(z).$$

PROPERTY. The intersection over all players a of the sets C^a is the set of correlated equilibria of the game.

5.1. Discrete standard case. Here we will use approachability theory to retrieve the well-known fact (see Hart and Mas-Colell [24]) that player 1 has a strategy such that the vector $C(\bar{z}_n)$ converges to the negative orthant of $\mathbb{R}^{m^2}$, where $\bar{z}_n \in Z$ is the average (correlated) distribution on S.

Given $s \in S$, define the auxiliary "vector payoff" $B(s)$ to be the $m \times m$ real valued matrix, where if $s = (j, l) \in I \times L$, hence j is the move of player 1 and the only nonzero line is line j with entry on column k being $U(k, l) - U(j, l)$. The average payoff at stage n is thus a matrix B_n with coefficient

$$B_n(j,k) = \frac{1}{n} \sum_{r,\, i_r = j} (U(k, l_r) - U(j, l_r)) = C(j,k)(\bar{z}_n),$$

which is the test of k versus j on the dates up to stage n where j was played.

Consider the Markov chain on I with transition matrix

$$M_n(j,k) = \frac{B_n(j,k)^+}{b_n}$$

for $j \ne k$ where $b_n > \max_j \sum_k B_n(j,k)^+$. By standard results on finite Markov chains, M_n admits (at least) one invariant probability measure. Let $\mu_n = \mu(B_n)$ be such a measure. Then (dropping the subscript n),

$$\mu_j = \sum_k \mu_k M(k,j) = \sum_{k\ne j} \mu_k \frac{B(k,j)^+}{b} + \mu_j \left(1 - \sum_{k\ne j} \frac{B(j,k)^+}{b}\right).$$

Thus, b disappears and the condition writes

$$\sum_{k\ne j} \mu_k B(k,j)^+ = \mu_j \sum_{k\ne j} B(j,k)^+.$$

Theorem 5.2. *Any strategy of player* 1 *satisfying* $\sigma(h_n) = \mu_n$ *is an approachability strategy for the negative orthant of* $\mathbb{R}^{m^2}$. *Namely,*

$$\forall\, j, k \quad \lim_{n\to\infty} B_n(j, k)^+ = 0 \quad a.s.$$

Equivalently, $(\bar{z}_n)$ *approaches the set of no conditional regret for player* 1:

$$\lim_{n\to\infty} d(\bar{z}_n, C^1) = 0.$$

Proof. Let Ω denote the closed negative orthant of $\mathbb{R}^{m^2}$. In view of Proposition 3.8, it is enough to prove that inequality (5)

$$\langle b - \Pi_\Omega(b), b' - \Pi_\Omega(b)\rangle \le 0, \quad \forall\, b \notin \Omega$$

holds for every regret matrix b', feasible under $\mu = \mu(b)$.

As usual, since the projection is on the negative orthant Ω, $b - \Pi_\Omega(b) = b^+$ and $\langle b - \Pi_\Omega(b), \Pi_\Omega(b)\rangle = 0$. Hence, it remains to evaluate

$$\sum_{j,k} B(j, k)^+ \mu_j [U(k, l) - U(j, l)],$$

but the coefficient of $U(j, l)$ is precisely

$$\sum_k B^+(k, j)\mu_k - \mu_j \sum_k B^+(j, k) = 0$$

by the choice of $\mu = \mu(b)$. □

5.2. Continuous general case. We first state a general property (compare Lemma 4.6):

Lemma 5.3. *Given* $a \in \mathbb{R}^{m^2}$, *let* $\mu \in X$ *satisfy*:

$$\sum_{k:k\neq j} \mu_k a(k, j) = \mu_j \sum_{k:k\neq j} a(j, k), \quad \forall\, j \in I,$$

then

$$\langle a, C(\mu \times y)\rangle = 0, \quad \forall\, y \in Y.$$

Proof. As above, one computes:

$$\sum_j \sum_k a(j, k)\mu_j [U(k, y) - U(j, y)],$$

but the coefficient of $U(j, y)$ is precisely

$$\sum_k a(k, j)\mu_k - \mu_j \sum_k a(j, k) = 0. \quad \square$$

Let P be a potential function for Ω the negative orthant of $\mathbb{R}^{m^2}$; for example, $P(w) = \sum_{ij} (w_{ij}^+)^2$, as in the standard case above.

Definition 5.4. The *P-conditional regret dynamics* in continuous time is defined on Z by:

$$\dot{\mathbf{z}} \in \mu(\mathbf{z}) \times Y - \mathbf{z}, \tag{13}$$

where $\mu(z)$ is the set of $\mu \in X$ that are solution to:

$$\sum_k \mu_k \nabla P_{kj}(C(z)) = \mu_j \sum_k \nabla P_{jk}(C(z))$$

whenever $C(z) \notin \Omega$ (∇P_{jk} denotes the jk component of the gradient of P). In particular, $\mu(z) = X$ whenever $C(z) \in \Omega$.

The associated process in $\mathbb{R}^{m^2}$ is the image under C:

$$\dot{\mathbf{w}} \in C(\nu(\mathbf{w}) \times Y) - \mathbf{w}, \tag{14}$$

where $\nu(w)$ is the set of $\nu \in X$ with

$$\sum_k \nu_k \nabla P_{kj}(w) = \nu_j \sum_k \nabla P_{jk}(w).$$

THEOREM 5.5. *The processes* (13) *and* (14) *satisfy*:

$$C^+(j,k)(\mathbf{z}(t)) = \mathbf{w}^+_{jk}(t) \to_{t\to\infty} 0.$$

PROOF. Apply Theorem 3.10 with:

$$N(w) = \{w' \in (\mathbb{R}^m)^2 \colon \langle \nabla P(w), w'\rangle = 0\} \cap \mathbf{C},$$

where $\mathbf{C}$ is the range $C(Z)$ of C. Since $\mathbf{w}(t) = C(\mathbf{z}(t))$, Lemma 5.3 implies that $\dot{\mathbf{w}}(t) \in N(\mathbf{w}(t)) - \mathbf{w}(t)$. □

The discrete processes corresponding to Equations (13) and (14) are, respectively, in Z

$$\bar{z}_{n+1} - \bar{z}_n = \frac{1}{n+1}[\mu_{n+1} \times z^{-1}_{n+1} - \bar{z}_n + (z_{n+1} - \mu_{n+1} \times z^{-1}_{n+1})], \tag{15}$$

where μ_{n+1} satisfies:

$$\sum_{k\in S} \mu^k_{n+1} \nabla P_{kj}(C(\bar{z}_n)) = \mu^j_{n+1} \sum_k \nabla P_{jk}(C(\bar{z}_n))$$

and in $\mathbb{R}^{m^2}$

$$\bar{w}_{n+1} - \bar{w}_n = \frac{1}{n+1}[C(\mu_{n+1} \times z^{-1}_{n+1}) - \bar{w}_n + (w_{n+1} - C(\mu_{n+1} \times z^{-1}_{n+1}))]. \tag{16}$$

COROLLARY 5.6. *The discrete processes* (15) *and* (16) *satisfy*:

$$C^+(j,k)(\bar{z}_n) = \bar{w}^{jk,+}_n \to_{t\to\infty} 0 \qquad a.s.$$

PROOF. Equations (15) and (16) are bounded DSA of Equations (13) and (14), and Properties 1, 2, and 3 apply. □

COROLLARY 5.7. *If all players follow the above procedure, the empirical distribution of moves converges a.s. to the set of correlated equilibria.*

6. Smooth fictitious play (SFP) and consistency. We follow the approach of Fudenberg and Levine [19], [21] concerning consistency and conditional consistency, and deduce some of their main results (see Theorems 6.6 and 6.12) as corollaries of dynamical properties. Basically, the criteria are similar to the ones studied in §§4 and 5, but the procedure is different and based only on the previous behavior of the opponents. As in §§4 and 5, we continue to adopt the point of view of player 1.

6.1. Consistency. Let

$$V(y) = \max_{x\in X} U(x,y).$$

The *average regret evaluation* along $h_n \in \mathcal{H}_n$ is

$$e(h_n) = e_n = V(\bar{y}_n) - \frac{1}{n}\sum_{m=1}^{n} U(i_m, l_m),$$

where as usual $\bar{y}_n$ stands for the time average of (l_m) up to time n. (This corresponds to the maximal component of the regret vector $R(\bar{z}_n)$.)

DEFINITION 6.1 (FUDENBERG AND LEVINE [19]). Let $\eta > 0$. A strategy σ for player 1 is said η-consistent if for any opponent's strategy τ

$$\limsup_{n\to\infty} e_n \le \eta \quad \mathsf{P}_{\sigma,\tau} \text{ a.s.}$$

6.2. Smooth fictitious play. A *smooth perturbation* of the payoff U is a map

$$U^\varepsilon(x,y) = U(x,y) + \varepsilon\rho(x), \quad 0 < \varepsilon < \varepsilon_0$$

such that:

(i) $\rho\colon X \to \mathbb{R}$ is a $\mathscr{C}^1$ function with $\|\rho\| \le 1$,

(ii) $\arg\max_{x\in X} U^\varepsilon(.,y)$ reduces to one point and defines a continuous map

$$\mathbf{br}^\varepsilon\colon Y \to X$$

called a *smooth best reply function*, and

(iii) $D_1 U^\varepsilon(\mathbf{br}^\varepsilon(y), y).D\mathbf{br}^\varepsilon(y) = 0$ (for example, $D_1 U^\varepsilon(.,y)$ is zero at $\mathbf{br}^\varepsilon(y)$). (This occurs in particular if $\mathbf{br}^\varepsilon(y)$ belongs to the interior of X.)

REMARK. A typical example is

$$\rho(x) = -\sum_k x_k \log x_k, \tag{17}$$

which leads to

$$\mathbf{br}_i^\varepsilon(y) = \frac{\exp(U(i, y)/\varepsilon)}{\sum_{k\in I} \exp(U(k, y)/\varepsilon)} \tag{18}$$

as shown by Fudenberg and Levine [19], [21].

Let

$$V^\varepsilon(y) = \max_x U^\varepsilon(x, y) = U^\varepsilon(\mathbf{br}^\varepsilon(y), y).$$

LEMMA 6.2 (FUDENBERG AND LEVINE [21]).

$$DV^\varepsilon(y)(h) = U(\mathbf{br}^\varepsilon(y), h).$$

PROOF. One has

$$DV^\varepsilon(y) = D_1U^\varepsilon(\mathbf{br}^\varepsilon(y), y).D\mathbf{br}^\varepsilon(y) + D_2U^\varepsilon(\mathbf{br}^\varepsilon(y), y)$$

The first term is zero by condition (iii) above. For the second term, one has

$$D_2U^\varepsilon(\mathbf{br}^\varepsilon(y), y) = D_2U(\mathbf{br}^\varepsilon(y), y),$$

which by linearity of $U(x, .)$ gives the result. □

DEFINITION 6.3. A smooth fictitious play strategy for player 1 associated to the smooth best response function $\mathbf{br}^\varepsilon$ (in short a $\mathsf{SFP}(\varepsilon)$ strategy) is a strategy σ^ε such that

$$\mathsf{E}_{\sigma^\varepsilon,\tau}(i_{n+1} \mid \mathcal{F}_n) = \mathbf{br}^\varepsilon(\bar{y}_n)$$

for any τ.

There are two classical interpretations of $\mathsf{SFP}(\varepsilon)$ strategies. One is that player 1 chooses to randomize his moves. Another one called *stochastic fictitious play* (Fudenberg and Levine [20], Benaïm and Hirsch [9]) is that payoffs are perturbed in each period by random shocks and that player 1 plays the best reply to the empirical mixed strategy of its opponents. Under mild assumptions on the distribution of the shocks, it was shown by Hofbauer and Sandholm [28] (Theorem 2.1) that this can always be seen as an $\mathsf{SFP}(\varepsilon)$ strategy for a suitable ρ.

6.3. SFP and consistency. Fictitious play was initially used as a global dynamics (i.e., the behavior of each player is specified) to prove convergence of the empirical strategies to optimal strategies (see Brown [12] and Robinson [32]; for recent results, see BHS, §5.3 and Hofbauer and Sorin [29]).

Here we deal with unilateral dynamics and consider the consistency property. Hence, the state space can not be reduced to the product of the sets of mixed moves but has to incorporate the payoffs.

Explicitly, the discrete dynamics of averaged moves is

$$\bar{x}_{n+1} - \bar{x}_n = \frac{1}{n+1}[i_{n+1} - \bar{x}_n], \qquad \bar{y}_{n+1} - \bar{y}_n = \frac{1}{n+1}[l_{n+1} - \bar{y}_n]. \tag{19}$$

Let $u_n = U(i_n, l_n)$ be the payoff at stage n and $\bar{u}_n$ be the average payoff up to stage n so that

$$\bar{u}_{n+1} - \bar{u}_n = \frac{1}{n+1}[u_{n+1} - \bar{u}_n]. \tag{20}$$

LEMMA 6.4. *Assume that player* 1 *plays a* $\mathsf{SFP}(\varepsilon)$ *strategy. Then, the process* $(\bar{x}_n, \bar{y}_n, \bar{u}_n)$ *is a DSA of the differential inclusion*

$$\dot{\boldsymbol{\omega}} \in N(\boldsymbol{\omega}) - \boldsymbol{\omega}, \tag{21}$$

where $\omega = (x, y, u) \in X \times Y \times \mathbb{R}$ *and*

$$N(x, y, u) = \{(\mathbf{br}^\varepsilon(y), \beta, U(\mathbf{br}^\varepsilon(y), \beta))\colon \beta \in Y\}.$$

PROOF. To shorten notation, we write $\mathsf{E}(. \mid \mathcal{F}_n)$ for $\mathsf{E}_{\sigma^\varepsilon,\tau}(. \mid \mathcal{F}_n)$, where τ is any opponent's strategy. By assumption, $\mathsf{E}(i_{n+1} \mid \mathcal{F}_n) = \mathbf{br}^\varepsilon(\bar{y}_n)$. Set $\mathsf{E}(l_{n+1} \mid \mathcal{F}_n) = \beta_n \in Y$. Then, by conditional independence of i_{n+1} and l_{n+1}, one gets that $\mathsf{E}(u_{n+1} \mid \mathcal{F}_n) = U(\mathbf{br}^\varepsilon(\bar{y}_n), \beta_n)$. Hence, $\mathsf{E}((i_{n+1}, l_{n+1}, u_{n+1}) \mid \mathcal{F}_n) \in N(x_n, y_n, u_n)$. □

Theorem 6.5. *The set* $\{(x, y, u) \in X \times Y \times \mathbb{R}: V^\varepsilon(y) - u \leq \varepsilon\}$ *is a global attracting set for Equation* (21). *In particular, for any* $\eta > 0$, *there exists* $\bar{\varepsilon}$ *such that for* $\varepsilon \leq \bar{\varepsilon}$, $\limsup_{t\to\infty} V^\varepsilon(\mathbf{y}(t)) - \mathbf{u}(t) \leq \eta$ (*i.e., continuous* $\mathsf{SFP}(\varepsilon)$ *satisfies* η*-consistency*).

Proof. Let $\mathbf{w}^\varepsilon(t) = V^\varepsilon(\mathbf{y}(t)) - \mathbf{u}(t)$. Taking time derivative, one obtains, using Lemma 6.2 and Equation (21):

$$\begin{aligned}\dot{\mathbf{w}}^\varepsilon(t) &= DV^\varepsilon(\mathbf{y}(t)).\dot{\mathbf{y}}(t) - \dot{\mathbf{u}}(t)\\ &= U(\mathbf{br}^\varepsilon(\mathbf{y}(t)), \beta(t)) - U(\mathbf{br}^\varepsilon(\mathbf{y}(t)), \mathbf{y}(t)) - U(\mathbf{br}^\varepsilon(\mathbf{y}(t)), \beta(t)) + \mathbf{u}(t)\\ &= \mathbf{u}(t) - U(\mathbf{br}^\varepsilon(\mathbf{y}(t)), \mathbf{y}(t))\\ &= -\mathbf{w}^\varepsilon(t) + \varepsilon\rho(\sigma^\varepsilon(\mathbf{y}(t))).\end{aligned}$$

Hence,

$$\dot{w}^\varepsilon(t) + w^\varepsilon(t) \leq \varepsilon$$

so that $w^\varepsilon(t) \leq \varepsilon + Ke^{-t}$ for some constant K and the result follows. □

Theorem 6.6. *For any* $\eta > 0$, *there exists* $\bar{\varepsilon}$ *such that for* $\varepsilon \leq \bar{\varepsilon}$, $\mathsf{SFP}(\varepsilon)$ *is* η*-consistent.*

Proof. The assertion follows from Lemma 6.4, Property 1, Property 2(ii), and Theorem 6.5. □

6.4. Remarks and generalizations. The definition given here of an $\mathsf{SFP}(\varepsilon)$ strategy can be extended in some interesting directions. Rather than developing a general theory, we focus on two particular examples.

1. *Strategies Based on Pairwise Comparison of Payoffs.* Suppose that ρ is given by Equation (17). Then, playing an $\mathsf{SFP}(\varepsilon)$ strategy requires for player 1 the computation of $\mathbf{br}^\varepsilon(\bar{y}_n)$ given by Equation (18) at each stage. In a case where the cardinality of S^1 is very large (say, 2^N with $N \geq 10$), this computation is not feasible! An alternative feasible strategy is the following: Assume that I is the set of vertices set of a connected symmetric graph. Write $i \sim j$ when i and j are neighbours in this graph, and let $N(i) = \{j \in I\backslash\{i\}: i \sim j\}$. The strategy is as follows: Let i be the action chosen at time n (i.e., $i_n = i$). At time $n+1$, player 1 picks an action j at random in $N(i)$. He then switches to j (i.e., $i_{n+1} = j$) with probability

$$R(i, j, \bar{y}_n) = \min\left[1, \frac{|N(i)|}{|N(j)|}\exp\left(\frac{1}{\varepsilon}(U(j, \bar{y}_n) - U(i, \bar{y}_n))\right)\right]$$

and keeps i (i.e., $i_{n+1} = i$) with the complementary probability $1 - R(i, j, \bar{y}_n)$. Here $|N(i)|$ stands for the cardinal of $N(i)$. Note that this strategy only involves at each step the computation of the payoff's difference $(U(j, \bar{y}_n) - U(i, \bar{y}_n))$. While this strategy is not an $\mathsf{SFP}(\varepsilon)$ strategy, one still has:

Theorem 6.7. *For any* $\eta > 0$, *there exists* $\bar{\varepsilon}$ *such that, for* $\varepsilon \leq \bar{\epsilon}$, *the strategy described above is* η*-consistent.*

Proof. For fixed $y \in Y$, let $Q(y)$ be the Markov transition matrix given by $Q(i, j, y) = (1/|N(i)|)R(i, j, y)$ for $j \in N(i)$, $Q(i, j, y) = 0$ for $j \notin N(i) \cup \{i\}$, and $Q(i, i, y) = 1 - \sum_{j\neq i} Q(i, j, y)$. Then, $Q(y)$ is an irreducible Markov matrix having $\mathbf{br}^\varepsilon(y)$ as unique invariant probability; this is easily seen by checking that $Q(y)$ is reversible with respect to $\mathbf{br}^\varepsilon(y)$. That is, $\mathbf{br}_i^\varepsilon(y)Q(i, j, y) = \mathbf{br}_j^\varepsilon(y)Q(j, i, y)$.

The discrete time process (19) and (20) is not a DSA (as defined here) to Equation (21) because $\mathsf{E}(i_{n+1} \mid \mathcal{F}_n) \neq \mathbf{br}^\varepsilon(\bar{y}_n)$. However, the conditional law of i_{n+1} given $\mathcal{F}_n$ is $Q(x_n, \cdot, \bar{y}_n)$ and using the techniques introduced by Métivier and Priouret [31] to deal with Markovian perturbations (see, e.g., Duflo [14, Chapter 3.IV]), it can still be proved that the assumptions of Proposition 1.3 in BHS are fulfilled, from which it follows that the interpolated affine process associated to Equations (19) and (20) is a perturbed solution (see BHS for a precise definition) to Equation (21). Hence, Property 1 applies and the end of the proof is similar to that for the proof of Theorem 6.6. □

2. *Convex Sets of Actions.* Suppose that X and Y are two convex compact subsets of finite dimensional Euclidean spaces. U is a bounded function with $U(x, .)$ linear on Y. The discrete dynamics of averaged moves is

$$\bar{x}_{n+1} - \bar{x}_n = \frac{1}{n+1}[x_{n+1} - \bar{x}_n], \qquad \bar{y}_{n+1} - \bar{y}_n = \frac{1}{n+1}[y_{n+1} - \bar{y}_n], \tag{22}$$

with $x_{n+1} = \mathbf{br}^\varepsilon(\bar{y}_n)$. Let $u_n = U(x_n, y_n)$ be the payoff at stage n and $\bar{u}_n$ be the average payoff up to stage n so that

$$\bar{u}_{n+1} - \bar{u}_n = \frac{1}{n+1}[u_{n+1} - \bar{u}_n]. \tag{23}$$

Then, the results of §6.3 still hold.

6.5. SFP and conditional consistency. We keep here the framework of §4 but extend the analysis from consistency to conditional consistency (which is like studying external regrets (§4) and then internal regrets (§5)). Given $z \in Z$, recall that we let $z^1 \in X$ denote the marginal of z on I. That is,

$$z^1 = (z_i^1)_{i\in I} \quad \text{with } z_i^1 = \sum_{l\in L} z_{il}.$$

Let $z[i] \in \mathbb{R}^L$ be the vector with components $z[i]_l = z_{il}$. Note that $z[i]$ belongs to tY for some $0 \le t \le 1$. A conditional probability on L induced by z given $i \in I$ satisfies

$$z \mid i = (z \mid i)_{l\in L} \quad \text{with } (z \mid i)_l z_i^1 = z_{il} = z[i]_l.$$

Let $[0,1].Y = \{ty\colon 0 \le t \le 1,\ y \in Y\}$. Extend U to $X \times ([0,1] \times Y)$ by $U(x, ty) = tU(x, y)$ and similarly for V. The conditional evaluation function at $z \in Z$ is

$$ce(z) = \sum_{i\in I} V(z[i]) - U(i, z[i]) = \sum_{i\in I} z_i^1 [V(z \mid i) - U(i, z \mid i)] = \sum_{i\in I} z_i^1 V(z \mid i) - U(z),$$

with the convention that $z_i^1 V(z \mid i) = z_i^1 U(i, z \mid i) = 0$ when $z_i^1 = 0$.

As in §5, conditional consistency means consistency with respect to the conditional distribution given each event of the form "i was played." In a discrete framework, the conditional evaluation is thus

$$ce_n = ce(\bar{z}_n),$$

where as usual $\bar{z}_n$ stands for the empirical correlated distribution of moves up to stage n. Conditional consistency is defined like consistency but with respect to (ce_n). More precisely:

Definition 6.8. A strategy σ for player 1 is said to be η-conditionally consistent if for any opponent's strategy τ

$$\limsup_{n\to\infty} ce_n \le \eta \quad \mathsf{P}_{\sigma,\tau} \text{ a.s.}$$

Given a smooth best reply function $\mathbf{br}^\varepsilon\colon Y \to X$, let us introduce a correspondence $\mathbf{Br}^\varepsilon$ defined on $[0,1] \times Y$ by $\mathbf{Br}^\varepsilon(ty) = \mathbf{br}^\varepsilon(y)$ for $0 < t \le 1$ and $\mathbf{Br}^\varepsilon(0) = X$. For $z \in Z$, let $\mu^\varepsilon(z) \subset X$ denote the set of all $\mu \in X$ that are solutions to the equation

$$\sum_{i\in I} \mu_i b^i = \mu \tag{24}$$

for some vectors family $\{b^i\}_{i\in I}$ such that $b^i \in \mathbf{Br}^\varepsilon(z[i])$.

Lemma 6.9. *μ^ε is an u.s.c. correspondence with compact convex nonempty values.*

Proof. For any vector's family $\{b^i\}_{i\in I}$ with $b^i \in X$, the function $\mu \to \sum_{i\in I} \mu_i b^i$ maps continuously X into itself. It then has fixed points by Brouwer's fixed point theorem, showing that $\mu^\varepsilon(z) \ne \varnothing$. Let $\mu, \nu \in \mu^\varepsilon(z)$. That is, $\mu = \sum_i \mu_i b^i$ and $\nu = \sum_i \nu^i c^i$ with $b^i, c^i \in \mathbf{Br}^\varepsilon(z[i])$. Then, for any $0 \le t \le 1$ $t\mu + (1-t)\nu = \sum_i (t\mu_i + (1-t)\nu_i) d^i$ with $d^i = (t\mu_i b^i + (1-t)\nu_i c^i)/(t\mu_i + (1-t)\nu_i)$. By convexity of $\mathbf{Br}^\varepsilon(z[i])$, $d^i \in \mathbf{Br}^\varepsilon(z[i])$. Thus, $t\mu + (1-t)\nu \in \mu^\varepsilon(z)$, proving convexity of $\mu^\varepsilon(z)$. Using the fact that $\mathbf{Br}^\varepsilon$ has a closed graph, it is easy to show that μ^ε has a closed graph, from which it will follow that it is u.s.c. with compact values. Details are left to the reader. □

Definition 6.10. A conditional smooth fictitious play (CSFP) strategy for player 1 associated to the smooth best response function $\mathbf{br}^\varepsilon$ (in short a $\mathsf{CSFP}(\varepsilon)$ strategy) is a strategy σ^ε such that $\sigma^\varepsilon(h_n) \in \mu^\varepsilon(\bar{z}_n)$.

The random discrete process associated to $\mathsf{CSFP}(\varepsilon)$ is thus defined by:

$$\bar{z}_{n+1} - \bar{z}_n = \frac{1}{n+1}[z_{n+1} - \bar{z}_n], \tag{25}$$

where the conditional law of $z_{n+1} = (i_{n+1}, l_{n+1})$ given the past up to time n is a product law $\sigma^\varepsilon(h_n) \times \tau(h_n)$. The associated differential inclusion is

$$\dot{\mathbf{z}} \in \mu^\varepsilon(\mathbf{z}) \times Y - \mathbf{z}. \tag{26}$$

Extend $\mathbf{br}^\varepsilon$ to a map, still denoted $\mathbf{br}^\varepsilon$, on $[0,1] \times Y$ by choosing a nonempty selection of $\mathbf{Br}^\varepsilon$ and define

$$V^\varepsilon(z[i]) = U(\mathbf{br}^\varepsilon(z[i]), z[i]) - \varepsilon z_i^1 \rho(\mathbf{br}^\varepsilon(z[i]))$$

(so that if $z_i^1 > 0$ $V^\varepsilon(z[i]) = z_i^1 V^\varepsilon(z \mid i)$ and $V^\varepsilon(0) = 0$). Let

$$ce^\varepsilon(z) = \sum_i (V^\varepsilon(z[i]) - U(z[i])) = \sum_i V^\varepsilon(z[i]) - U(z).$$

The evaluation along a solution $t \to z(t)$ to (26) is

$$\mathbf{W}^\varepsilon(t) = ce^\varepsilon(\mathbf{z}(t)).$$

The next proof is in spirit similar to §6.3 but technically heavier. Since we are dealing with smooth best reply to conditional events, there is a discontinuity at the boundary and the analysis has to take care of this aspect.

Theorem 6.11. *The set* $\{z \in Z\colon ce^\varepsilon(z) \le \varepsilon\}$ *is an attracting set for Equation* (26), *whose basin is* Z. *In particular, conditional consistency holds for continuous* $\mathsf{CSFP}(\varepsilon)$.

Proof. We shall compute

$$\dot{\mathbf{W}}^\varepsilon(t) = \frac{d}{dt}\sum\nolimits_i V^\varepsilon(\mathbf{z}[i](t)) - \frac{d}{dt}U(\mathbf{z}(t)).$$

The last term is

$$\frac{d}{dt}U(\mathbf{z}(t)) = U(\mu^\varepsilon(t), \beta(t)) - U(\mathbf{z}(t))$$

by linearity, with $\beta(t) \in Y$ and $\mu^\varepsilon(t) \in \mu^\varepsilon(\mathbf{z}(t))$. We now pass to the first term. First, observe that

$$\frac{d}{dt}\mathbf{z}_i^1 \in \mu_i^\varepsilon(\mathbf{z}) - \mathbf{z}_i^1 \ge -\mathbf{z}_i^1.$$

Hence, $\mathbf{z}_i^1(t) > 0$ implies $\mathbf{z}_i^1(s) > 0$ for all $s \ge t$. It then exists $\tau_i \in [0, \infty]$ such that $\mathbf{z}_i^1(s) = 0$ for $s \le \tau_i$ and $\mathbf{z}_i^1(s) > 0$ for $s > \tau_i$. Consequently, the map $t \to V^\varepsilon(\mathbf{z}[i](t))$ is differentiable everywhere but possibly at $t = \tau_i$ and is zero for $t \le \tau_i$. If $t > \tau_i$, then

$$\begin{aligned}\frac{d}{dt}V^\varepsilon(\mathbf{z}[i](t)) &= \frac{d}{dt}U^\varepsilon(\mathbf{br}^\varepsilon(\mathbf{z}[i](t)), \mathbf{z}[i](t)) - \varepsilon \mathbf{z}_i^1(t)\rho(\mathbf{br}^\varepsilon(\mathbf{z}[i](t)))\\ &= U^\varepsilon(\mathbf{br}^\varepsilon(\mathbf{z}[i](t)), \dot{\mathbf{z}}[i](t)) - \dot{\mathbf{z}}_i^1(t)\varepsilon\rho(\mathbf{br}^\varepsilon(\mathbf{z}[i](t)))\end{aligned} \tag{27}$$

by Lemma 6.2. If now $t < \tau_i$, both $\dot{\mathbf{z}}[i](t)$ and $(d/dt)V^\varepsilon(\mathbf{z}[i](t))$ are zero, so that equality (27) is still valid.

Finally, using $(d/dt)\mathbf{z}_{ij}(t) = \mu^\varepsilon{}_i(t)\beta_j(t) - \mathbf{z}_{ij}(t)$, we get that

$$\dot{\mathbf{W}}^\varepsilon(t) = \sum_i U^\varepsilon(\mathbf{br}^\varepsilon(\mathbf{z}[i](t)), \mu_i^\varepsilon(t)\beta(t) - \mathbf{z}[i](t)) + \sum_i (\mu_i^\varepsilon(t) - \mathbf{z}_i^1(t))\varepsilon\rho(\mathbf{br}^\varepsilon(\mathbf{z}[i](t))) - U(\mu^\varepsilon(t), \beta(t)) + U(\mathbf{z}(t))$$

for all (but possibly finitely many) $t \ge 0$. Replacing gives

$$\dot{\mathbf{W}}^\varepsilon(t) = -\mathbf{W}^\varepsilon(t) + \mathbf{A}(t),$$

where

$$\mathbf{A}(t) = -U(\mu^\varepsilon(t), \beta(t)) + \sum_i U^\varepsilon(\mathbf{br}^\varepsilon(\mathbf{z}[i](t)), \mu_i^\varepsilon(t)\beta(t)) + \sum_i \mu_i^\varepsilon(t)\varepsilon\rho(\mathbf{br}^\varepsilon(\mathbf{z}[i](t))).$$

Thus, one obtains:

$$\mathbf{A}(t) = -U(\mu^\varepsilon(t), \beta(t)) + \sum_i \mu_i^\varepsilon(t)[U(\mathbf{br}^\varepsilon(\mathbf{z}[i](t)), \beta(t)) + \varepsilon\rho(\mathbf{br}^\varepsilon(\mathbf{z}[i](t)))].$$

Now Equation (24) and linearity of $U(., y)$ implies

$$U(\mu^\varepsilon(t), \beta(t)) = \sum_i \mu_i^\varepsilon(t) U(\mathbf{br}^\varepsilon(\mathbf{z}[i](t)), \beta(t))).$$

Hence,

$$\mathbf{A}(t) = \varepsilon \sum_i \mu_i^\varepsilon(t)\rho(\mathbf{br}^\varepsilon(\mathbf{z}[i](t)))$$

so that

$$\dot{\mathbf{W}}^\varepsilon(t) \le -\mathbf{W}^\varepsilon(t) + \varepsilon$$

for all (but possibly finitely many) $t \ge 0$. Hence,

$$\mathbf{W}^\varepsilon(t) \le e^{-t}(\mathbf{W}^\varepsilon(0) - \varepsilon) + \varepsilon$$

for all $t \ge 0$. □

Theorem 6.12. *For any $\eta > 0$, there exists $\bar{\varepsilon} > 0$ such that for $\varepsilon \le \bar{\varepsilon}$ a* CSFP(ε) *strategy is η-consistent.*

Proof. Let $\mathcal{L} = \mathcal{L}(\bar{z}_n)$ be the limit set of $(\bar{z}_n)$ defined by Equation (25). Since $(\bar{z}_n)$ is a DSA to Equation (26) and $\{z \in Z\colon ce^{\varepsilon}(z) \le \varepsilon\}$ is an attracting set for Equation (26), whose basin is Z (Theorem 6.11), it suffices to apply Property 2(ii). ☐

7. Extensions. We study in this section extensions of the previous dynamics in the case where the information of player 1 is reduced: Either he does not recall his past moves or he does not know the other players' moves sets, or he is not told their moves.

7.1. Procedure in law. We consider here procedures where player 1 is uninformed of his previous sequences of moves but knows only its law (team problem).

The general framework is as follows. A discrete time process $\{w_n\}$ is defined through a recursive equation by:

$$w_{n+1} - w_n = a_{n+1} V(w_n, i_{n+1}, l_{n+1}), \tag{28}$$

where $(i_{n+1}, l_{n+1}) \in I \times L$ are the moves[2] of the players at stage $n+1$ and $V\colon \mathbb{R}^m \times I \times L \to \mathbb{R}^m$ is some bounded measurable map.

A typical example is given in the framework of approachability (see §3.2) by

$$V(w, i, l) = -w + A_{il}, \tag{29}$$

where A_{il} is the vector valued payoff corresponding to (i, l) and $a_n = 1/n$. In such case $w_n = \bar{g}_n$ is the average payoff.

Assume that player 1 uses a strategy (as defined in §3.2) of the form

$$\sigma(h_n) = \psi(w_n),$$

where for each w, $\psi(w)$ is some probability over I. Hence, w plays the role of a state variable for player 1, and we call such σ a ψ-strategy. Let $V_{\psi}(w)$ be the range of V under σ at w, namely, the convex hull of

$$\left\{ \int_I V(w, i, l)\psi(w)\,(di);\ l \in L \right\}.$$

Then, the associated continuous time process associated to Equation (28) is

$$\dot{\mathbf{w}} \in V_{\psi}(\mathbf{w}). \tag{30}$$

We consider now another discrete time process, where, after each stage n, player 1 is not informed upon his realized move i_n but only upon l_n. Define by induction the new input at stage $n+1$:

$$w^*_{n+1} - w^*_n = a_{n+1} \int_I V(w^*_n, i, l_{n+1})\psi(w^*_n)\,(di). \tag{31}$$

Remark that the range of V under $\psi(w^*)$ at w^* is $V_{\psi}(w^*)$ so that the continuous time process associated to Equation (31) is again Equation (30). Explicitly Equations (28) and (31) are DSA of the same differential inclusion (Equation 30).

Definition 7.1. A ψ-procedure in law is a strategy σ of the form $\sigma(h_n) = \psi(w^*_n)$, where for each w, $\psi(w)$ is some probability over I and $\{w^*_n\}$ is given by Equation (31).

The key observation is that a procedure in law for player 1 is independent on the moves of player 1 and only requires the knowledge of the map V and the observation of the opponents' moves. The interesting result is that such a procedure will in fact induce, under certain assumptions (see Hypothesis 7.2 below), the same asymptotic behavior in the original discrete process.

Suppose that player 1 uses a ψ-procedure in law. Then, the coupled system (Equations 28 and 31) is a DSA to the differential inclusion

$$(\dot{w}, \dot{w}^*) \in V^2_{\psi}(w, w^*), \tag{32}$$

where $V^2_{\psi}(w, w^*)$ is the convex hull of

$$\left\{ \left(\int_I V(w, i, l)\psi(w^*)\,(di),\ \int_I V(w^*, i, l)\psi(w^*)\,(di) \right);\ l \in L \right\}.$$

[2] For convenience, we keep the notation used for finite games, but it is unnecessary to assume here that the move spaces are finite.

We shall assume, from now on, that Equation (32) meets the standing Hypothesis 2.1. We furthermore assume the following:

HYPOTHESIS 7.2. *The map V satisfies one of the two following conditions*:

(i) *There exists a norm $\|\cdot\|$ such that $w \to w + V(w, i, l)$ is contracting uniformly in $s = (i, l)$. That is*

$$\|w + V(w, s) - (u + V(u, s))\| \leq \rho \|w - u\|$$

for some $\rho < 1$.

(ii) *V is C^1 in w and there exists $\alpha > 0$ such that all eigenvalues of the symmetric matrix*

$$\frac{\partial V}{\partial w}(w, s) + \frac{{}^t\partial V}{\partial w}(w, s)$$

are bounded by $-\alpha$.

^{t} stands for the transpose. Remark that Hypothesis 7.2 holds trivially for Equation (29). Under this later hypothesis, one has the following result.

THEOREM 7.3. *Assume that $\{w_n, w_n^*\}$ is a bounded sequence. Under a ψ-procedure in law the limit sets of $\{w_n\}$ and $\{w_n^*\}$ coincide, and this limit set is an ICT set of the differential inclusion (Equation 30). Under a ψ-strategy the limit set of $\{w_n\}$ is also an ICT set of the same differential inclusion.*

PROOF. Let $\mathscr{L}$ be the limit set of $\{w_n, w_n^*\}$. By Properties 1 and 2, $\mathscr{L}$ is compact and invariant. Choose $(w, w^*) \in \mathscr{L}$ and let $t \to (\mathbf{w}(t), \mathbf{w}^*(t))$ denote a solution to Equation (32) that lies in $\mathscr{L}$ (by invariance) with initial condition (w, w^*). Let $\mathbf{u}(t) = \mathbf{w}(t) - \mathbf{w}^*(t)$.

Assume condition (i) in Hypothesis 7.2. Let $Q(t) = \|\mathbf{u}(t)\|$. Then, for all $0 \leq s \leq 1$,

$$Q(t+s) = \|\mathbf{u}(t) + \dot{\mathbf{u}}(t)s + o(s)\| = \|(1-s)\mathbf{u}(t) + (\dot{\mathbf{u}}(t) + \mathbf{u}(t))s + o(s)\| \leq (1-s)Q(t) + s\|\dot{\mathbf{u}}(t) + \mathbf{u}(t)\| + o(s).$$

Now $\dot{\mathbf{u}}(t) + \mathbf{u}(t)$ can be written as

$$\mathbf{w}(t) - \mathbf{w}^*(t) + \int_{I \times L} [V(\mathbf{w}(t), i, l) - V(\mathbf{w}^*(t), i, l)]\psi(\mathbf{w}^*(t))\,(di)\,d\nu(l)$$

for some probability measure ν over L. Thus, by condition (i),

$$Q(t+s) \leq (1-s)Q(t) + s\rho Q(t) + o(s),$$

from which it follows that

$$\dot{Q}(t) \leq (\rho - 1)Q(t)$$

for almost every t. Hence, for all $t \geq 0$:

$$Q(0) \leq e^{(\rho-1)t} Q(-t) \leq e^{(\rho-1)t} K$$

for some constant K. Letting $t \to +\infty$ shows that $Q(0) = 0$. That is, $w = w^*$.

Assume now condition (ii). Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^m$ and $\langle \cdot, \cdot \rangle$ the associated scalar product. Then,

$$\begin{aligned}\langle V(w, s) - V(w^*, s), w - w^* \rangle &= \int_0^1 \langle \partial_w V(w^* + u(w - w^*), s).(w - w^*), w - w^* \rangle\, du \\ &\leq -\frac{\alpha}{2}\|w - w^*\|^2.\end{aligned}$$

Therefore,

$$\frac{d}{dt}Q^2(t) = 2\langle \mathbf{w}(t) - \mathbf{w}^*(t), \dot{\mathbf{w}}(t) - \dot{\mathbf{w}}^*(t) \rangle \leq -\alpha Q^2(t),$$

from which it follows (like previously) that $Q(0) = 0$.

We then have proved that given Hypothesis 7.2, $\{w_n\}$ and $\{w_n^*\}$ have the same limit set under a ψ-procedure in law. Since $\{w_n^*\}$ is a DSA to Equation (30), this limit set is ICT for Equation (30) by Property 1. The same property holds for $\{w_n\}$ under a ψ-strategy. □

REMARK. Let $\mathscr{R}$ denote the set of chain-recurrent points for Equation (28). Hypothesis 7.2 can be weakened to the assumption that conditions (i) or (ii) are satisfied for V restricted to $\mathscr{R} \times I \times L$.

The previous result applies to the framework of §§4 and 5 and show that the discrete regret dynamics will have the same properties when based on the (conditional) expected stage regret $E_x R(s)$ or $E_x C(s)$.

7.2. Best prediction algorithm. Consider a situation where at each stage n an unknown vector U_n ($\in [-1,+1]^I$) is selected and a player chooses a component $i_n \in I$. Let $\omega_n = U_n^{i_n}$. Assume that U_n is announced after stage n. Consistency is defined through the evaluation vector V_n with $V_n^i = \bar{U}_n^i - \bar{\omega}_n$, $i \in I$, where, as usual, $\bar{U}_n$ is the average vector and $\bar{\omega}_n$ the average realization. Conditional consistency is defined through the evaluation matrix W_n with $W_n^{jk} = (1/n)(\sum_{m,\, i_m=j} U_m^k - \omega_m)$. This formulation is related to online algorithms; see Foster and Vohra [17] or Freund and Schapire [18] for a general presentation. In the previous framework, the vector U_n is $U(.,l_n)$, where l_n is the choice of players other than 1 at stage n. The claim is that all previous results go through (V_n or W_n converges to the negative orthant) when dealing with the dynamics expressed on the payoffs space. This means that player 1 does not need to know the payoff matrix or the set of moves of the other players; only a compact range for the payoffs is requested. A sketch of proofs is as follows.

7.2.1. Approachability: Consistency. We consider the dynamics of §4. The regret vector R^* if i is played is $R^*(i) = \{U^j - U^i\}_{j\in I}$. Lemma 4.6 is now, for $\theta \in \Delta(I)$,

$$\langle \theta, R^*(\theta)\rangle = 0$$

and since $R^*(\theta)$, the expectation of R^* under θ is

$$R^*(\theta) = \sum_{i\in I} \theta(i) R^*(i) = \{U^j - \langle \theta, U\rangle\}_j;$$

hence, the properties of the P-regret-based dynamics on the payoff space $\mathbb{R}^m$ still hold (Theorem 4.7 and Corollary 4.8).

7.2.2. Approachability: Conditional consistency. The content of §5 extends as well. The $I \times I$ regret matrix is defined at stage n, given the move i_n, by all lines being zero except line i_n, which is the vector $\{U_n^j - U_n^i\}_{j\in J}$. Then, the analysis is identical, and the convergence of the regret to the negative orthant holds for P-conditional regret dynamics as in Theorem 5.5 and Corollary 5.6.

7.2.3. SFP: Consistency. In the framework of §6, the only hypothesis used on the set Y was that it was convex compact; hence, one can take $L = [-1,+1]^I$ and $U(x,l) = \langle x, l\rangle$. Then, all computations go through.

7.2.4. SFP: Conditional consistency. For the analog of §6.5, let us define the $I \times I$ evaluation matrix M_n at stage n and, given the move i_n, by all lines equal to zero, except line i_n being the vector U_n. Its average at stage n is $\bar{M}_n$. μ_n is an invariant measure for the Markov matrix defined by the family $\mathrm{BR}^\epsilon(\bar{M}_n^i)$, where $(\bar{M}_n^i)$ denotes the i-line of $(\bar{M}_n)$.

7.3. Partial information. We consider here the framework of §7.2 but where only ω_n is observed by player 1, not the vector U_n. In a game theoretical framework, this means that the move of the opponent at stage n is not observed by player 1 but only the corresponding payoff $U(i_n, l_n)$ is known.

This problem has been studied in Auer et al. [2, 3], Foster and Vohra [15], Fudenberg and Levine [21], Hart and Mas-Colell [26], and, in a game theoretical framework, by Banos [4] and Megiddo [30]. (Note that working in the framework of §7.2 is more demanding than finding an optimal strategy in a game, since the payoffs can actually vary stage after stage.)

The basic idea is to generate, from the actual history of payoffs and moves $\{\omega_n, i_n\}$ and the knowledge of the strategy σ, a sequence of pseudovectors $\tilde{U}_n \in \mathbb{R}^S$ to which the previous procedures apply.

7.3.1. Consistency. We follow Auer et al. [2] and define $\tilde{U}_n$ by

$$\tilde{U}_n^i = \frac{\omega_n}{\sigma_n^i} \mathbf{1}_{\{i=i_n\}},$$

where as usual i_n is the component chosen at stage n and σ_n^i stands for $\sigma(h_{n-1})(i)$. The associated pseudoregret vector is $\{\tilde{R}_n^i = \tilde{U}_n^i - \omega_n\}_{i\in I}$. Notice that

$$E(\tilde{R}_n^i \mid h_{n-1}) = U_n^i - \langle \sigma_n, U_n\rangle;$$

hence, in particular

$$\langle \sigma_n, E(\tilde{R}_n \mid h_{n-1})\rangle = 0.$$

To keep $\widetilde{U}_n$ bounded, one defines first τ_n adapted to the vector $\widetilde{U}_n$ as in §7.2, namely, proportional to $\nabla P((1/(n-1)) \cdot \sum_{m=1}^{n-1} \widetilde{R}_m)$ (see §4), then σ is specified by

$$\sigma_n^i = (1-\delta)\tau_n^i + \delta/K$$

for $\delta > 0$ small enough, K being the cardinality of the set I.

The discrete dynamics is thus

$$\bar{\bar{R}}_n - \bar{\bar{R}}_{n+1} = \frac{1}{n}(\widetilde{R}_{n+1} - \bar{\bar{R}}_n).$$

The corresponding dynamics in continuous time satisfies:

$$\dot{\mathbf{w}}(t) = \alpha(t) - \mathbf{w}(t),$$

with $\alpha(t) = U_t - \langle p(t), U_t \rangle$ for some measurable process U_t with values in $[-1, 1]$ and $p(t) = (1-\delta)q(t) + \delta/K$ with

$$\nabla P(w(t)) = \|\nabla P(w(t))\| q(t).$$

Define the condition

$$\langle \nabla P(w), w \rangle \geq B\|\nabla P(w)\| \|w^+\| \tag{33}$$

on $\mathbb{R}^S \setminus D$ for some positive constant B (satisfied, for example, by $P(w) = \sum_s (w_s^+)^2$).

Proposition 7.4. *Assume that the potential satisfies in addition Equation (33). Then, consistency holds for the continuous process $\widetilde{R}_t$ and both discrete processes $\widetilde{R}_n$ and R_n.*

Proof. One has

$$\begin{aligned}\frac{d}{dt}P(w(t)) &= \langle \nabla P(w(t)), \dot{w}(t) \rangle \\ &= \langle \nabla P(w(t)), \alpha(t) - w(t) \rangle.\end{aligned}$$

Now,

$$\begin{aligned}\langle \nabla P(w(t)), \alpha(t) \rangle &= \|\nabla P(w(t))\| \langle q(t), \alpha(t) \rangle \\ &= \|\nabla P(w(t))\| \left\langle \frac{1}{1-\delta} p_t - \frac{\delta}{(1-\delta)K}, \alpha(t) \right\rangle \\ &\leq \|\nabla P(w(t))\| \frac{\delta}{(1-\delta)K} R\end{aligned}$$

for some constant R since $\langle p(t), \alpha(t) \rangle = 0$ and the range of α is bounded. It follows, using Equation (33), that given $\varepsilon > 0$, $\delta > 0$ small enough and $\|w^+(t)\| \geq \varepsilon$ implies

$$\begin{aligned}\frac{d}{dt}P(w(t)) &\leq \|\nabla P(w(t))\| \left(\frac{\delta}{(1-\delta)K} R - B\|w^+(t)\| \right) \\ &\leq -\|\nabla P(w(t))\| B\varepsilon/2.\end{aligned}$$

Now, $\langle \nabla P(w), w \rangle > 0$ for $w \notin D$ implies $\|\nabla P(w)\| \geq a > 0$ on $\|w^+\| \geq \varepsilon$. Let $\beta > 0$, $A = \{P \leq \beta\}$, and choose $\varepsilon > 0$ such that $\|w^+\| \leq \varepsilon$ is included in A. Then, the complement of A is an attracting set, and consistency holds for the process $\bar{\bar{R}}_t$ hence, as in §4, for the discrete time process $\widetilde{R}_n$. The result concerning the actual process R_n with $R_n^k = U_n^k - \omega_n$ finally follows from another application of Theorem 7.3, since both processes have the same conditional expectation. □

7.3.2. Conditional consistency. A similar analysis holds in this framework. The pseudoregret matrix is now defined by

$$\widetilde{C}_n(i, j) = \frac{\sigma_n^i}{\sigma_n^j} U_n^j \mathbf{1}_{\{j=i_n\}} - U_n^i \mathbf{1}_{\{i=i_n\}},$$

hence

$$E(\widetilde{C}_n(i, j) \mid h_{n-1}) = \sigma_n^i (U_n^j - U_n^i),$$

and this relation allows us to invoke ultimately Theorem 7.3, hence to work with the pseudoprocess. The construction is similar to that in §5.2, in particular Equation (A6). The measure $\mu(w)$ is a solution of

$$\sum_k \mu^k(w)\nabla_{kj}P(w) = \mu^j(w)\sum_k \nabla_{jk}P(w)$$

and player 1 uses a perturbation $\nu(t) = (1-\delta)\mu(w(t)) + \delta u$ where u is uniform. Then, the analysis is as above and leads to the following proposition:

Proposition 7.5. *Assume that the potential satisfies, in addition, Equation* (33). *Then, consistency holds for the continuous process $\tilde{C}_t$ and both discrete processes $\tilde{C}_n$ and C_n.*

8. A learning example. We consider here a process analyzed by Benaïm and Ben Arous [7]. Let $S = \{0, \ldots, K\}$,

$$X = \Delta(S) = \left\{x \in \mathbb{R}^{K+1}\colon x_k \geq 0, \sum_{k=0}^{K} x_k = 1\right\}$$

be the K dimensional simplex and $f = \{f_k\}$, $k \in S$ a family of bounded real valued functions on X. Suppose that a "player" has to choose an infinite sequence $x_1, x_2, \ldots \in S$ (identified with the extreme points of X) and is rewarded at time $n+1$ by

$$y_{n+1} = f_{x_{n+1}}(\bar{x}_n),$$

where

$$\bar{x}_n = \frac{1}{n}\sum_{1\leq m\leq n} x_m.$$

Let

$$\bar{y}_n = \frac{1}{n}\sum_{1\leq m\leq n} y_m$$

denote the average payoff at time n. The goal of the player is thus to maximize its long-term average payoff $\liminf \bar{y}_n$. In order to analyze this system, note that the average discrete process satisfies

$$\bar{x}_{n+1} - \bar{x}_n = \frac{1}{n}(x_{n+1} - \bar{x}_n),$$

$$\bar{y}_{n+1} - \bar{y}_n = \frac{1}{n}(f_{x_{n+1}}(\bar{x}_n) - \bar{y}_n).$$

Therefore, it is easily seen to be a DSA of the following differential inclusion

$$(\dot{\mathbf{x}}, \dot{\mathbf{y}}) \in -(\mathbf{x}, \mathbf{y}) + N(\mathbf{x}, \mathbf{y}), \tag{34}$$

where $(x, y) \in X \times [\alpha_-, \alpha_+]$, $\alpha_- = \inf_{S,X} f_k(x)$, $\alpha_+ = \sup_{S,X} f_k(x)$, and N is defined as

$$N(x, y) = \{(\theta, \langle\theta, f(x)\rangle)\colon \theta \in X\}.$$

Definition 8.1. The function f has a gradient structure if, letting

$$g_k(x_1, \ldots, x_K) = f_0\left(1 - \sum_{k=1}^{K} x_k, x_1, \ldots, x_K\right) - f_k\left(1 - \sum_{k=1}^{K} x_k, x_1, \ldots, x_K\right),$$

there exists a C^1 function V, defined in a neighborhood of

$$Z = \{z \in \mathbb{R}^K,\ z = \{z_k\},\ k = 1, \ldots, K, \text{ with } (x_0, z) \in X \text{ for some } x_0 \in [0, 1]\},$$

satisfying

$$\nabla V(z) = g(z).$$

Theorem 8.2. *Assume that f has a gradient structure. Then, every compact invariant set of Equation* (34) *meets the graph*

$$S = \{(x, y) \in X \times [\alpha_-, \alpha_+]\colon y = \langle f(x), x\rangle\}.$$

Proof. We follow the computation in Benaïm and Ben Arous [7]. Note that Equation (34) can be rewritten as

$$\dot{x} + x \in X$$
$$\dot{y} = \langle x + \dot{x}, f(x)\rangle - y.$$

Hence,

$$\begin{aligned}\frac{y(s+t) - y(s)}{t} &= \frac{1}{t}\int_s^{s+t} \dot{y}(u)\,du \\ &= \frac{1}{t}\left[\int_s^{s+t} \langle f(x(u)), x(u)\rangle - y(u)\,du + \int_s^{s+t} \langle f(x(u)), \dot{x}(u)\rangle\,du\right],\end{aligned}$$

but $x(u) \in X$ implies

$$\begin{aligned}\langle f(x(u)), \dot{x}(u)\rangle &= \sum_{k=0}^{K} f_k(x(u))\dot{x}_k(u) \\ &= \sum_{k=1}^{K}[-f_0(x(u)) + f_k(x(u))]\dot{x}_k(u) \\ &= -\sum_{k=1}^{K} g_k(z(u))\dot{z}_k(u) \\ &= -\frac{d}{dt}V(z(u)),\end{aligned}$$

where $z(u) \in \mathbb{R}^m$ is defined by $z_k(u) = x_k(u)$. So that

$$\frac{1}{t}\int_s^{s+t} (\langle f(x(u)), x(u)\rangle - y(u))\,du = \frac{(y(s+t) + V(z(s+t)) - (y(s) + V(z(s))}{t}$$

and the right-hand term goes to zero uniformly (in s, y, z) as $t \to \infty$. Let now $\mathscr{L}$ be a compact invariant set. Replacing $\mathscr{L}$ by one of its connected components we can always assume that $\mathscr{L}$ is connected. Suppose that $\mathscr{L} \cap S = \varnothing$. Then, $(\langle f(x), x\rangle - y)$ has constant sign on $\mathscr{L}$ (say, >0) and, by compactness, is bounded below by a positive number δ. Thus, for any trajectory $t \to (x(t), y(t))$ contained in $\mathscr{L}$

$$\frac{1}{t}\int_s^{s+t} (\langle f(x(u)), x(u)\rangle - y(u))\,du \geq \delta,$$

a contradiction. □

Corollary 8.3. *The limit set of* $\{(\bar{x}_n, \bar{y}_n)_n\}$ *meets* S. *In particular,*

$$\liminf \bar{y}_n \leq \sup_{x \in X} \langle x, f(x)\rangle.$$

If, furthermore, (x_n) *is such that* $\lim_{n\to\infty} \bar{x}_n = x^*$, *then*

$$\lim_{n\to\infty} \bar{y}_n = \sup_{x \in X} \langle x^*, f(x^*)\rangle.$$

Proof. One uses the fact that the discrete process is a DSA, hence the limit set is invariant, being ICT by Property 2. The second part of the corollary follows from the proof part (a) of Theorem 4 in Benaïm and Ben Arous [7]. □

9. Concluding remarks. The main purpose of the paper was to show that stochastic approximation tools are extremely effective for analyzing several game dynamics and that the use of differential inclusions is needed. Note that certain discrete dynamics do not enter this framework: One example is the procedure of Hart and Mas-Colell [25], which depends both on the average regret and on the last move. The corresponding continuous process generates in fact a differential equation of order two. Moreover, as shown in Hart and Mas-Colell [27] (see also Cahn [13]), this continuous process has regularity properties not shared by the discrete counterpart.

Among the open problems not touched upon in the present work are the questions related to the speed of convergence and to the convergence to a subset of the approachable set.

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