Long Term Behaviour of Self-Interacting Diffusions

THÈSE

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Abstract

The purpose of this thesis is the study of the long term behaviour of *Self-Interacting Diffusions* on compact Riemannian manifolds solving formally the stochastic differential equation

$$dX_t = dB_t(X_t) - \nabla V_t(X_t)dt,$$

with $V_t(x) = \int_0^t V(x, X_s)ds$ when V is symmetric. This work is based on three published or accepted for publication papers and on one preprint.

First, in a joint work with M. Benaïm, we study the particular case

$$V(x,y) = \sum_{k=1}^{n} a_k e_k(x) e_k(y),$$

where $a_k' s$ are positive constants and the $e_k' s$ are non-constant eigenfunctions of the Laplace operator. Positiveness of the coefficients a_k implies that the particle is *repelled from its past trajectory.* We begin by introducing new variables in order to have a Markov process on an extended state space. Then, we prove that the extended process satisfies the strong Feller property and admits a unique explicit invariant probability measure. Then, we also provide rates of convergences to this invariant probability.

This work has been accepted for publication in *Probability Theory and Related Fields.*

Second, in a joint work with M. Benaïm and I. Ciotir, we restrict ourselves to the particular case of the circle $\mathbb{S}^1$ and consider the potential interaction function

$$V(x,y) = \sum_{k=1}^{\infty} a_k \big(\cos(kx)\cos(ky) + \sin(kx)\sin(ky) \big),$$

where $(a_k)_{k\geqslant 1}$ is a sufficiently regular sequence of positive values. As in the first work, we add new variables in order to obtain a true SDE, but on an *infinite dimensional manifold.* We prove that it admits a unique strong solution, which is Feller but *not* strong Feller and finally exhibit an explicit invariant probability measure.

This work is published in *Stochastic Partial Differential Equations: Analysis and Computations.*

In the third work, we consider the case of the euclidean sphere $\mathbb{S}^n$ with potential interaction function

$$V(x,y) = -\sum_{k=1}^{n+1} x_i y_i.$$

Contrary to the previous works, the particle is now *attracted by its past trajectory.* We prove that this willingness to stay where it has already been implies the almost-sure convergence of the particle. We deduce from this result the almost-sure convergence for the solution of the real valued SDE

$$dX_t = dB_t - \int_0^t \sin\big(\lambda(X_t - X_s)\big)dsdt, \quad \lambda \neq 0.$$

This work is published in *Electronic Communications in Probability.*

Finally, in the last work in collaboration with P. Monmarché, we consider on the circle $\mathbb{S}^1$ the case

$$V(x,y) = F(x)F(y).$$

Depending whether or not on the existence of a local minimum (resp. maximum) $x \in \mathbb{S}^1$ such that $F(x) > 0$ (resp. $F(x) < 0$), we show that either X_t converges almost-surely, or the positive recurrence of the process $\Big((X_t, \int_0^t F(X_s)ds)\Big)_{t\geqslant 0}$. Moreover, our arguments applies for the velocity jump process whose jump rate depends both on its present place and on its past via the process $\int_0^t F(X_s)ds$. To our knowledge, it is the first example of self-interacting Piecewise Deterministic Markov Process (PDMP).

Keywords: Self-Repelling Diffusions, Self-Attracting Diffusions, Degenerate Diffusions, Strong Feller property, Invariant probability measure, Hypocoercivity, Infinite Dimensional SDE, Asymptotic Pseudotrajectories, Almost-sure convergence, Self-interacting velocity jump process.

Résumé

Le but de cette thèse est l'étude en temps long de *Diffusions Auto-Interagissante* sur des variétés Riemannienne de la forme

$$dX_t = dB_t(X_t) - \nabla V_t(X_t)dt,$$

avec $V_t(x) = \int_0^t V(x, X_s)ds$ où V est symétrique. Ce travail s'articule autour de trois papiers publiés ou acceptés pour publication ainsi que d'un preprint.

Dans le premier travail en collaboration avec M. Benaïm, nous nous intéressons au cas particulier où

$$V(x,y) = \sum_{k=1}^{n} a_k e_k(x) e_k(y),$$

où les coefficients a_k sont strictement positifs et les e_k sont des fonctions propre non-constante du Laplacien. La positivité des coefficients a_k implique la *répulsion* de la particule par rapport à sa trajectoire passée. Nous commençons par étendre notre espace d'état par l'ajout de nouvelles variables qui rendent le processus markovien. Nous démontrons ensuite que le système étendu est fortement Feller, qu'il admet une unique probabilité invariante qui est explicite et des vitesses de convergences vers cette probabilité.

Ce travail a été accepté pour publication dans *Probability Theory and Related Fields.*

Dans le second travail en collaboration avec M. Benaïm et I. Ciotir, nous nous restreignons au cas du cercle $\mathbb{S}^1$ et étudions la situation

$$V(x,y) = \sum_{k=1}^{\infty} a_k \big(\cos(kx)\cos(ky) + \sin(kx)\sin(ky) \big),$$

où $(a_k)_{k\geqslant 1}$ est une suite de coefficients strictement positif et avec suffisamment de régularité. Comme dans le premier travail, nous introduisons de nouvelles variables dans le but d'avoir une vraie EDS, mais de *dimension infinie.* Nous montrons que ce système admet une unique solution forte, qu'elle a la propriété de Feller, mais *pas* de Feller forte, et exhibons une probabilité invariante.

Ce travail a fait l'objet d'une publication dans *Stochastic Partial Differential Equations: Analysis and Computations.*

Dans le troisième travail, nous nous plaçons sur la sphère euclidienne $\mathbb{S}^n$ et considérons le cas où

$$V(x,y) = -\sum_{k=1}^{n+1} x_i y_i.$$

Contrairement aux deux précédents travaux, la particule est à présent *attirée* par sa trajectoire. Nous prouvons que cette volonté de fixation de la particule oblige cette dernière à converger presque-sûrement. Nous déduisons de ce résultat la convergence presque-sûre pour la solution de l'EDS réelle

$$dX_t = dB_t - \int_0^t \sin\big(\lambda(X_t - X_s)\big) ds dt, \quad \lambda \neq 0.$$

Ce travail a fait l'objet d'une publication dans*Electronic Communications in Probability.*

Dans le quatrième et dernier travail en collaboration avec P. Monmarché, nous considérons sur le cercle $\mathbb{S}^1$ le cas

$$V(x,y) = F(x)F(y).$$

Selon la présence ou non d'un minimum (resp. maximum) $x \in \mathbb{S}^1$ tel que $F(x) > 0$ (resp. $F(x) < 0$), nous montrons soit la convergence de X_t, soit que le processus $\Big((X_t, \int_0^t F(X_s)ds)\Big)_{t \geqslant 0}$ est positivement récurrent. Par ailleurs, avec nos arguments, nous obtenons le même résultat limite pour le *velocity jump process* dont le taux de saut dépend de sa position sur le cercle, mais également de son passé via le processus $\int_0^t F(X_s)ds$. A notre connaissance, c'est le premier exemple de PDMP auto-interagissant.

Mots clés: Diffusions auto-repoussantes, Diffusions auto-attractives, Diffusions dégénérées, propriété de Feller forte, Probabilité invariante, Hypocoercivité , EDS infini dimensionnelle, Pseudotrajectoires asymptotiques, Convergence presque-sûre, Velocity jump process auto-interagissant.

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Contents

Chapter 1

Introduction

The purpose of this thesis is to investigate the long term behaviour of some *Self-Interacting Diffusions* on a Riemannian manifold defined by a Stochastic Differential Equation (henceforth denoted SDE) of the form

$$dX_t = dW_t(X_t) - \nabla_x f(t, X_t, X([0,t]))dt, \quad X_0 = x \tag{1.1}$$

where $X([0,t])$ is the trajectory of X during the time interval $[0,t]$ and $(W_t(.))_t$ is a Brownian vector field.

One of the main interests of such processes comes from the fact that its behaviour at time t is influenced by its history up to the time t. As well as being of mathematical interest in their own right, the general motivation of path-dependent processes comes from a variety of fields. Some examples are as follows.

In ecology, one has the relation between human activities and global warming. The random parameter might model political decisions and X_t the degree of pollution generated by human activities (industrial ones, way of life, etc.). The scenario works as follows. It has been proved over these last decades that human activities had, and still have, a deep impact in the global warming, which itself has a huge negative impact on ecosystems and finally on human populations. As a consequence, it forces members of the population to change their habits in order to limit these impacts and thus to change or improve the way they practise their different activities (e.g. the improvement of technologies in transport).

In politics, we think about the arguments developed by a politician and the opinion of the public. The randomness of this interaction is due to the uncertainty of sincere answers in surveys or to unpredictable events. Here X_t denotes the ideas defended by the politician at time t and the dynamic is the following. By his speeches, the politician will have an effect on the population, his supporters and opponents. Depending on how the opinion evolves, he will have to adapt or change his arguments and/or ideas. But in order to remain consistent, he cannot change them abruptly, and therefore he needs to recall his previous arguments and /or opinions.

In social networks, Skyrms and Pemantle [95] proposed in 2000 a path-dependent process as an attempt to explain their formation. In their model, the people are playing a repeated game by pairing and depending on the result of their games, they become friends. In a similar vein Hu, Skyrms and Tarrès [66] in 2011 for modelling the emergence of a common language.

In computer science, path-dependent processes are considered in order to model more efficient dynamic computational systems: *learning machines*. The Alphago program is one such example and it caused a mini-revolution in 2016 since it was able to beat the best players of Go in the world. Roughly speaking, the program learns an efficient strategy by playing many games and at the end of the matches, it will have gained further information to improve. For some more precise references, see [51, 94]. Such processes are also used when trying to make a machine able to recognize shapes; for instance given a picture, it should be able to say if there is a cat.

In finance, we mention the interaction between a company's market value and its reputation or the relation between the financial loan's interest rate that a state has to pay and the creditor's belief in its reimbursement's capacity. Indeed, the history of the market value or of the interest rate provides useful information to investors and therefore it influences the evolution of these economical estimators. In 1987, Arthur, Ermolieu and Kaniovski ([5]) considered path-dependent processes in order to understand creation of monopolies.

In physics, such processes were introduced in order to model the growth of polymers (see e.g [1, 45, 81, 99]). The random parameter models an exploration term whereas the drift represents physical constraints in order to minimize the loss of energy.

A last example is the modelling of an animal's position regarding to its goal. Should he keep moving to escape predators or should he localise and defend his territory?

From a mathematical point of view, there is a great challenge since the long term behaviour's analysis is very complicated due to the fact that the drift term of (1.1) is shaped by the trajectory of X. This explains the absence of general methods to deal with such processes.

Let us illustrate this complexity with the following class of Self-Interacting Diffusions which is at the core of this thesis. Let $(X_t)_{t\geqslant 0}$ be the solution of the SDE

$$dX_t = dW_t - \int_0^t F'(X_t - X_s)dsdt, \quad X_0 = 0. \tag{1.2}$$

This kind of SDEs were for instance considered in [35, 44, 79] and it will be discussed later in detail in this introduction. ***An important property*** that appears in the analysis of X_t is ***the nature of*** F ***around*** 0.

Let $T > 0$[1], I be a small interval around 0 and set $W_t^T = W_{t+T} - W_T$ as well as $X_t^T = X_{t+T} - X_T$. Then $(X_t^T)_t$ solves

$$dX_t^T = dW_t^T - g_T(X_t^T)dt - \int_0^t F'(X_t^T - X_s^T)dsdt, \quad X_0^T = 0. \tag{1.3}$$

with

$$g_T(x) = \int_0^T F'(x + X_T - X_u)du. \tag{1.4}$$

[1] T can also be a stopping time

So, if X_t^T remains in I during a large amount of time, then

$$\frac{1}{t}\int_0^t F'(X_t^T - X_s^T)ds \;\simeq\; F'(X_t^T).$$

Therefore the longer X_t^T remains in I the more dominant becomes the dynamic driven by the ordinary differential equation

$$\dot{x} = -F'(x) \tag{1.5}$$

on its behaviour. This motivates the following terminology.

Definition 1.1. *We call the solution of (1.2)* attractive *if* 0 *is an asymptotically stable equilibrium of the Ordinary Differential Equation (1.5). Otherwise, we call it* repulsive.

Now, if one supposes that there exists $\delta > 0$ such that for all $T > 0$, X_t^T leaves almost surely $[-\delta, \delta]$ and wants to obtain informations about the asymptotic of X (boundedness of X or speed of convergence to $\pm\infty$), he will have to be able to understand g_T. But this can become very tedious, even though F' has a nice shape.

A natural thing to do when dealing with non-Markovian processes is to transform them into a Markovian one by adding new variables. In the case of (1.2), observe that it can be rewritten as

$$dX_t = dW_t - (F' * L_t^X)(X_t)dt, \quad X_0 = 0, \tag{1.6}$$

where $L_t^X(x)$ stands for the local time of X on x at time t and $*$ for the convolution product between two functions.

Hence, *the pair* $(X_t, L_t^X(.))$ is a Markov process living in an possibly infinite dimensional space. However, it might be ill-adapted to obtain sufficient useful information for the long-term behaviour analysis.

The introduction is organised as follows. In sections 1.1 and 1.2, we consider functional of the form

$$f(t, x, \varphi) = \int_0^t V(x, \varphi(s))ds$$

for some function $V(.,.)$. Depending on assumptions that will be detailed later, the particle wishes either to escape or to remain to places that it has already visited. Such processes are called *reinforcement processes by the non-normalized empirical occupation measure* or *strong reinforcement processes.*

Since this thesis is mainly motivated by self-interacting diffusions solving (1.2) and that the literature about such SDEs is pretty limited, we begin each section of the introduction with a survey of existing results as well as, for most of them, a short presentation of their proofs with the underlying ideas. Then, we present the new results that have been obtained during this thesis.

For completeness we present in section 1.3 the results that have been obtained in the literature when the functional is

$$f(t, x, \varphi) = \frac{1}{t}\int_0^t V(x, \varphi(s))ds$$

for some function $V(.,.)$. Such processes are called *reinforcement processes by the normalized empirical occupation measure* or *weak reinforcement processes.*

The difference between these two models is that in the second one we consider an average of the "point to point" interaction between x and $(\varphi(s))_{s\in[0,t]}$, whereas in the first one it is an accumulation. This leads to different behaviours.

Finally, in section 1.4, the functional is

$$f(t,x,\varphi) = \big(\int_0^t F(\varphi(s))ds\big)F(x),$$

where F is a smooth function. In that case, the trajectory acts on the intensity and the sign of the potential function F.

Contents

1.1 Self-Repelling Diffusions

In this section, we review the existing results for *Self-Interacting Diffusions* solving an SDE of the form

$$dX_t = dW_t(X_t) - \nabla_x V_t(X_t)dt, \quad X_0 = x \tag{1.7}$$

with $V_t(x) = \int_0^t V(x, X_s)ds$. The existing results are mainly stated on $\mathbb{R}^d$ with $d = 1$ and $V(x,y) = F(x-y)$ for some real valued function F. In that case, equation (1.7) rewrites

$$dX_t = dW_t - \int_0^t \nabla F(X_t - X_s)dsdt, \quad X_0 = x, \tag{1.8}$$

When dealing with Self-Repelling Diffusions on $\mathbb{R}$, the main questions of interest are

Question 1.1.

- *Does $|X_t|$ converge almost-surely to infinity? If yes, can we estimate the speed?*
- *If $|X_t|$ does not converge to infinity, does it localize?*

We distinguish two families of Self-Repelling diffusions solving (1.8).

1.1.1 $\nabla F(0) \neq 0$

In 1987, J.R Norris, L.C.G Rogers and D. Williams considered in [82] SDEs of the form

$$dX_t = dW_t - g(L_t(X_t))dt, \qquad X_0 = x_0 \in \mathbb{R} \tag{1.9}$$

where $g : [0,\infty) \to \mathbb{R}$ and $L_t(x)$ is the local time of X at x after time t. One of their main result is

Theorem 1.1. *(Theorem 4 and example therein, [82]) Assume that g is a continuous, non negative, increasing function and let $(X_t)_t$ be the solution of (1.9) and. Set $f(x) = \int_0^x g(y)dy$. Then*

$$\lim_{t\to\infty} \frac{X_t}{t} = -\frac{1}{\mu},$$

where μ is the mean of the probability measure

$$\pi(dz) = C_\pi \exp\Big(- \int_0^z \frac{f(y)}{y}dy\Big)dz$$

with C_π the normalisation constant. In particular, if $g(x) = x$, then $\frac{1}{\mu} = \sqrt{\frac{\pi}{4}}$.

As a particular case, if $g(u) = u$ and $F(x) = \mathbf{1}_{[0,\infty)}(x)$ is the Heaviside function, we retrieve equation (1.8). Indeed, the derivative of F (in the weak sense) is the Dirac-delta function, ie $F'(x) = \delta_0(x)$, and Exercise 1.15 in Chapter VI of Revuz-Yor's book [92] yields

$$\int_0^t \delta_0(X_t - X_s)ds = \int_0^t \delta_{X_t}(X_s)ds = L_t(X_t).$$

Two decades later, O. Raimond and B. Schapira extended this asymptotic result to more general functions g.

Theorem 1.2. *(Theorem 1.1, [90]) Assume that g is a Borel-measurable and bounded function and let $(X_t)_t$ be the solution of (1.9). Define $f(x) = \int_0^x g(u)du$ and set*

$$C_k^{\pm} = \int_0^{\infty} x^{k-1} \exp\Big(\mp \int_0^x \frac{f(y)}{y} dy\Big) dx,\ k = 1, 2.$$

Then

1. *The process X is recurrent if and only if $C_1^+ = C_1^- = \infty$.*
2. *We have $\lim_{t\to\infty} X_t = \infty$ (resp. $-\infty$) almost surely if and only if $C_1^+ < \infty$ (resp. $C_1^- < \infty$).*
3. *We have $\lim_{t\to\infty} X_t/t = C_1^+/C_2^+$ (resp. $-C_1^-/C_2^-$) almost surely if and only if $C_2^+ < \infty$ (resp $C_2^- < \infty$).*

Their motivation differs from the one of J.R Norris, L.C.G Rogers and D. Williams since they see (1.9) as a continuous time version of a cookie random walk. Since in this thesis, we do not work with solutions of equation like (1.9) we refer interested reader to the respective papers.

The study of (1.8) saw a great leap forward in 1992 with [44] from R.T Durrett and L.C.G Rogers since they obtain bounds on the speed of escape to infinity for three families of functions whose gradient is Lipschitz continuous.

The first one is

$$\mathfrak{F}_1(M, d) = \{F : \mathbb{R}^d \to \mathbb{R}^d \mid \nabla F \text{ has compact support and } \|\nabla F\|_\infty \leqslant M\} \tag{1.10}$$

for some positive finite constant M and $d \geqslant 1$ and where $\|.\|_\infty$ stands for the uniform norm. In that case, they proved that almost surely $t \mapsto \|X_t\|$ grows at most linearly in time.

Theorem 1.3. *(Theorem 1, [44]) Let $M > 0$ and $d \in \mathbb{N}$ and let $F \in \mathfrak{F}_1(M, d)$. Then there exists a constant $\Gamma < \infty$ such that*

$$\limsup_{t\to\infty} \|X_t\|/t \leqslant \Gamma.$$

The proof of this result is based on the analysis of the sequence of stopping times $(\sigma_n)_n$ defined by

$$\sigma_n = \inf\{t > 0 \text{ s.t. } \|X_t\| = 2n\}$$

and on the observation that the result follows from

$$\liminf_{n\to\infty} \frac{\sigma_n}{n} \geqslant \gamma \tag{1.11}$$

for some constant $\gamma > 0$ sufficiently small such that $\mathbb{P}\big(\inf\{t > 0\ s.t\ Y_t = 2\} \geqslant 5\gamma\big) \geqslant 1/2$, where $(Y_t)_t$ solves

$$dY_t = dB_t + (\frac{d-1}{2Y_t + 4} + 7\gamma K)dt\ Y_0 = 0 \tag{1.12}$$

and $K = \|\nabla F\|_\infty$. In that case $\Gamma = 2/\gamma$.

To prove (1.11), they introduced the notion of *fast (resp. slow) crossing* from $2n-2$ to $2n$ when $\sigma_n - \sigma_{n-1} \leqslant 2\gamma$ (resp. $\sigma_n - \sigma_{n-1} \geqslant 5\gamma$) and showed that after a fast crossing from $2n-2$ to $2n$, the probability that the crossing from $2n$ to $2n+2$ is slow is greater than $1/2$. This is based on a comparison result between $(Y_t)_t$ and $(\|X_t\| - 2n)_t$. For the details, we refer to section 2 in [44].

The second family of functions that was investigated is

$$\mathfrak{F}_2 = \{F : \mathbb{R} \to \mathbb{R} \mid F' \leqslant 0 \text{ and } F'(0) < 0\}. \tag{1.13}$$

The key observation for this family of functions is that if X_t grows too slowly for a while, then the dominant part of $\int_0^t F'(X_t - X_s)ds$ should be of order $F'(0)t$. But then the effect of the drift is much stronger than the one of the Brownian motion and therefore, it forces the particle to accelerate until it has an "appropriate speed". This idea of minimal speed is the purpose of the following theorem.

Theorem 1.4. *(Theorem 2, [44]) If $F \in \mathfrak{F}_2$, then there exists a constant $\gamma > 0$ such that*

$$\liminf_{t\to\infty} |X_t|/t \geqslant \gamma.$$

The proof works as follows. Since $-F'$ is nonnegative, then for all $t > 0$, $X_t \geqslant W_t$, so that

$$b_t = \sup_{s\leqslant t} X_s \geqslant \sup_{s\leqslant t} W_s.$$

Since F' is negative around 0, there exists a positive constant A such that

$$1 - b_t - \inf_{s\leqslant t} W_s \geqslant X_t - W_t \geqslant \frac{At^2}{4(1 + b_t - \inf_{s\leqslant t} W_s)}.$$

This yields

$$\liminf_{t\to\infty} |X_t|/t \geqslant A^{1/2}/2.$$

Using the fact that $X_t - b_t \geqslant \inf_{s\leqslant t}(W_t - W_s)$ allows to complete the proof.

Finally, the third family of function considered is

$$\begin{aligned}\mathfrak{F}_3 \;=\; \{F : \mathbb{R} \to \mathbb{R} \mid\; & F' \text{ is bounded, odd, increasing for } x \geqslant q \text{ for some } q > 0 \text{ and} \\ & \text{there exists } l > 0 \text{ such that } \lim_{x\to\infty} x^\beta F'(x) = -l \text{ with } 0 < \beta < 1\}. \end{aligned} \tag{1.14}$$

Before stating the long term behaviour result, let us present their instructive heuristic.

Let $T > 0$ sufficiently large and set $x_t = T^{-\alpha} X_{tT}$ and $B_t = T^{-1/2} W_{tT}$ with $\alpha = 2/(1+\beta)$ so that $\alpha\beta = 2 - \alpha$. Then equation (1.8) rewrites

$$\begin{aligned} x_t &= T^{1/2-\alpha} B_t - T^{2-\alpha} \int_0^t \int_0^s F'\big(T^\alpha(x_s - x_u)\big) \frac{(T^\alpha(x_s - x_u))^\beta}{(T^\alpha(x_s - x_u))^\beta} duds \\ &= T^{1/2-\alpha} B_t - \int_0^t \int_0^s F'\big(T^\alpha(x_s - x_u)\big) \frac{(T^\alpha(x_s - x_u))^\beta}{((x_s - x_u))^\beta} duds \end{aligned}$$

As T tends to infinity, one may expect that if a limit exists (still denoted by x_t for simplicity), then it should satisfy

$$x_t = \int_0^t \int_0^s \frac{l}{((x_s - x_u))^\beta} duds. \tag{1.15}$$

It is easy to check that $x_t = c_0 t^\alpha$ with

$$\alpha c_0^{1+\beta} = \int_0^1 \frac{l}{(1-u^\beta)^\alpha} du \tag{1.16}$$

is a solution of (1.15).

Theorem 1.5. *(Theorem 3, [44]) Let $F \in \mathfrak{F}_3$. Then*

$$\limsup_{t\to\infty} \|X_t\|/t^\alpha \leqslant c_0,$$

where the constants α and c_0 are given by the heuristic

The proof is based on the following scaling result for a Brownian motion and on a succession of judicious choice of parameters.

Lemma 1.1. *(Lemma 4.1, [44]) Let $b > 0$ and $(B_t)_t$ a standard Brownian motion. Then for any $k < \infty$,*

$$\mathbb{P}\Big(\sup_{0\leqslant s<t\leqslant 1} \frac{|B_t - B_s|}{(t-s)^b} > x\Big) \leqslant C_k x^{-k}.$$

Because Theorem 1.5 gives an upper bound, one may wonder whether or not it is optimal. A positive answer is given by the authors.

Theorem 1.6. *(Theorem 4.2, [44]) Let $F \in \mathfrak{F}_2 \cap \mathfrak{F}_3$. Then*

$$\lim_{t\to\infty} \|X_t\|/t^\alpha = c_0$$

Theorems 1.3 and 1.4 provide respectively upper and lower bounds for the speed of convergence to infinity if the particle lives in $\mathbb{R}$ when F' is compactly supported. The next step is to obtain the *exact speed*. This has been done by M. Cranston and T. Mountford in [36].

Theorem 1.7. *(Theorem 1, [36]) Assume $d = 1$ and that F' is a non-positive, Lipschitz continuous function such that supp$(F') \subset [-k, k]$ and that $F'(x) \leqslant c < 0$ for $|x - x_0| < 5\delta$ for some $x_0 \in [-k/2, k/2]$ and some $k > 0$. Then there exists a strictly positive constant γ such that*

$$\lim_{t\to\infty} \frac{X_t}{t} = \gamma,$$

The key idea behind this result is to study the solution of the SDE

$$dX_t^T = dW_t - \int_{\max(t-T,0)}^t F'(X_t^T - X_u^T)dudt \tag{1.17}$$

for a large time T, instead of (1.8) since

$$X_t = W_t - \int_0^t \int_{\max(s-T,0)}^s F'(X_s - X_u)duds - \int_0^t \int_0^{\max(s-T,0)} F'(X_s - X_u)duds. \tag{1.18}$$

Proposition 1.1. *(Proposition 2.2, [36]) For T sufficiently large,*

$$\mathbb{P}\Big(\limsup \frac{1}{t}\int_0^t \int_0^{\max(s-T,0)} -F'(X_s - X_u)duds \leqslant \frac{1}{T}\Big) = 1.$$

A consequence of this proposition is that X_t^T/t is a good approximation for X_t/t. But, the main advantage of (1.17) compared to (1.8) is the existence of a strong law of large numbers. Indeed, the sequence of functions $(Y_n)_n \subset C([0,T])$ defined by

$$Y_n(s) = X^T_{(n-1)T+s} - X^T_{(n-1)T}$$

is a Harris-recurrent chain (see the introduction of [36]). Thus, there exists an invariant measure, denoted by μ_T, on $C([0,T])$ such that for all integrable functions,

$$\frac{1}{n}\sum_{k=1}^n g(Y_k) \to \int_{C([0,T])} g(\omega)\mu_T(d\omega)\ a.s.$$

Applying this convergence result to $g(Y) = Y(T)$ provides the existence of a positive constant c_T such that

$$\lim_{t\to\infty} \frac{X_t^T}{t} = c_T.$$

1.1.2 $\nabla F(0) = 0$, but 0 is not asymptotically stable

As we have seen in the previous subsection, R.T Durrett and L.C.G Rogers were mainly investigating the situation where $\nabla F(0) \neq 0$. However, crude estimates are obtained if $F \in \mathfrak{F}_1(M,d) \cup \mathfrak{F}_3$. As a refinement, they conjectured and explained heuristically the following behaviour, which was partially proved in [96] as we are going to explain it later.

Conjecture 1.1. *(Conjecture 3, [44]) Suppose that F' has a compact support, is odd and $xF'(x) \leqslant 0$ for all $x \in \mathbb{R}$. Then*

$$\lim_{t\to\infty} \frac{X_t}{t} = 0,$$

where $(X_t)_t$ is the solution of (1.8).

When F' is no more compactly supported, they conjectured the following behaviour which was inspired from Theorem 1.6 and later proved by P. Tarrès and T. Mountford.

Theorem 1.8. *(Theorem 1, [79]) Suppose that* $F'(x) = -\frac{x}{1+|x|^{\beta+1}}$. *Then*

$$\lim_{t\to\infty} \|X_t\|/t^{\alpha} = c_0 \ a.s,$$

where $\alpha = 2/(1+\beta)$ *and* c_0 *satisfies* $\alpha c_0^{\beta+1} = \int_0^1 \frac{du}{(1-u^{\alpha})^{\beta}}$.

Let us briefly explain the proof's guideline. For $t > 0$ and an interval I, define the functions

$$h_t^I(x) = \int_0^t F'(x-X_s)\mathbf{1}_{X_s\in I}ds,\ h_t^I(x) = \int_0^t F'(x-X_s)\mathbf{1}_{X_s\notin I}ds \text{ and } g_t(x) = \int_0^t F'(x-X_s)ds.$$

The first useful result, which would have been trivial if F' had been monotone and decreasing is the following.

Lemma 1.2. *(Lemma 2, [79]) Let* $t > 0$ *and* $a, b \in \mathbb{R}$. *Suppose there exists* $x_0 \in [a, b]$ *such that* $h_t^{[a,b]}(x_0) \geqslant 0$, *and either* $F'(b - x_0) \geqslant F'(b - a)^2$ *and* $b - x_0 \leqslant 1/16$, *or* $b - a \leqslant x_{max}$ *(the global minima of* F'*). Then, for all* $x \in [a, x_0]$, $h_t^{[a,b]}(x) \geqslant 0$.

As in [79], let denote for a stopping time S, $(B_t^S)_t$ the Brownian motion $(B_{t+S} - B_S)_t$ and $(Y_t^S)_t$ the diffusion

$$Y_t^S = X_S + B_t^S - \int_0^t h_S^I(Y_u^S)du.$$

The next result shows, in some sense, that $(Y_t^S)_t$ does not differ too much from the time-shifted process $(X_{S+t})_t$.

Lemma 1.3. *(Lemma 1, [79]) Let* S *be an almost surely finite stopping time for a filtration* $(\mathcal{F}_t)_t$, *let* I *be an interval and let* $v > 0$. *Let* $W_v(\mathbb{R})$ *be the Wiener space of continuous paths* $\omega : [0, v] \to \mathbb{R}$, *equipped with the* $\sigma-$*algebra* $\mathcal{G}$ *generated by the projection maps* $\omega \mapsto \omega(t)$.
Given $A \in \mathcal{G}$, *assume that* $\mathbb{P}(Y_{\cdot}^S \in A \mid \mathcal{F}_S) \geqslant \varepsilon$. *Then, almost surely on the event* $\{\|k_S^I\| \leqslant v\}$,

$$\mathbb{P}(X_{S+\cdot} \in A \mid \mathcal{F}_S) \geqslant Cst(\varepsilon, v).$$

Together with martingales arguments, Lemmas 1.2 and 1.3 are the ingredients of the following proposition, which is the main step of Theorem 1.8's proof.

Proposition 1.2. *(Proposition 1, [79])* $\mathbb{P}(\limsup_{t\to\infty} |X_t| = \infty) = 1$.

To prove this proposition, P. Tarrès and T. Mountford defined a suitable increasing sequence $(a_n)_n \subset [0, \infty)$ as well as the stopping times

$$S_{n,t} = \inf\{u > t \ s.t \ |X_u| > a_n\},\ n \in \mathbb{N}, t \in (0, \infty).$$

and proved the existence of a positive constant $\zeta_{n,t}$ depending only on n and t such that for all $s \geqslant t$,

$$\mathbb{P}(S_{n,t} < \infty \mid \mathcal{F}_{S_{n-1,s}}) \geqslant \zeta_{n,t} > 0 \ a.s \text{ on } \{S_{n-1,s} < \infty\}. \tag{1.19}$$

Then, a martingale convergence theorem and a recurrence argument imply the almost sure finiteness of $S_{n,t}$ for all n, t.

In 2003, S.Herrmann and B.Roynette investigated one the weakest possible kind of repulsion, ie F is constant on a neighbourhood of 0 which is a global attractor for the ordinary differential equation

$$\dot{x} = -\nabla F(x).$$

In that case, they proved the almost-sure boundedness of the trajectories.

Theorem 1.9. *(Theorem 2, [60]) Let $(X_t)_t$ be the (weak) solution of the SDE*

$$dX_t = dW_t - \int_0^t \big(\mathbf{1}_{(1,\infty)}(X_t - X_s) - \mathbf{1}_{(-\infty,-1)}(X_t - X_s)\big)dsdt,\ X_0 = 0.$$

Then, $\mathbb{P}(\sup_{t\geqslant 0} |X_t| < \infty) = 1$.

Since $(X_t)_t$ and $(-X_t)_t$ have the same law, the proof consists on showing the existence of a positive constant $\varepsilon > 0$ such that for all $n \in \mathbb{N}$,

$$\mathbb{P}\big(X \textit{ does not reach the level } 2n+2 \textit{ knowing that it has reached the level } 2n\big) \geqslant \varepsilon.$$

This is done by introducing a sequence of stopping times and between two stopping times, by using a comparison result with a simpler process.

Recently, Bálint Tóth and his co-authors (I. Horváth, P. Tarrès, B. Valkó, B. Vetö) gave a new point of view in the analysis of Self-Repelling Diffusions by studying the following generalized version of (1.8)

$$dX_t = dW_t + \xi(X_t)dt - \int_0^t \nabla F(X_t - X_s)dsdt, \quad X_0 = x. \tag{1.20}$$

Here ξ is some function with enough regularity (see [96], [97] and [64]). Because [64] and [97] are more or less generalizations to higher dimension of [96], which treat the 1−dimensional case, we mainly focus on this last paper.

To ensure that everything is well-defined, P. Tarrès, B.Tòth and B. Valkó assume that $F \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$, but their ***main and most important assumption*** is the following ***positive definiteness*** condition:

$$F \textbf{ has non-negative Fourier transform.} \tag{1.21}$$

This yields that F is even and $\sup_{x\in\mathbb{R}} |F(x)| = F(0)$. In higher dimension, F is also assumed to be spherically symmetric and sufficiently fast decaying at infinity.

As already mentioned, the process $(X_t)_t$ alone is not Markovian. However, that is the case for the pair $((X_t, \zeta_t))_t$, where ζ_t is the function

$$\zeta_t(x) = \xi(x) - \int_0^t F'(x - X_s)ds. \tag{1.22}$$

The idea is to encode the properties of $(X_t)_{t\geqslant 0}$ in a Markov process whose evolution in time does not depend explicitly on X. So, for $t \geqslant 0$, they define a function $\eta_t \in \Omega$ by

$$\eta_t(x) = \zeta_t(x + X_t), \tag{1.23}$$

where Ω is the Fréchet space

$$\Omega = \{\omega \in C^\infty(\mathbb{R}) \; s.t \; \|\omega\|_{k,l} := \sup_{x\in\mathbb{R}} (1 + |x|)^{-1/l} |\omega^{(k)}(x)| < \infty\}.$$

Therefore

$$X_t = x + W_t + \int_0^t \eta_s(0) ds, \tag{1.24}$$

and by Itô's Formula

$$d\eta_t(x) = \eta_t'(x)dW_t + \eta_t'(x)\eta_t(0)dt + \frac{\eta_t''(x)}{2} - F'(x)dt. \tag{1.25}$$

Due to the need to evaluate η_t at 0, arguments from the usual theory of Stochastic Partial Differential Equation cannot be applied.

By Minlos' Theorem, the positive definiteness and the regularity of F ensure the existence of a Gaussian probability measure π on Ω such that for all $x, y \in \mathbb{R}$,

$$\int_\Omega \omega(x)\pi(d\omega) = 0 \text{ and } \int_\Omega \omega(x)\omega(y)\pi(d\omega) = F(x - y).$$

The first main result is

Theorem 1.10. *(Theorem 1, [96] for $d = 1$, Proposition 1 in [64] for $d \geqslant 2$) The Gaussian probability measure $\pi(d\omega)$ on Ω is time invariant and ergodic for the Ω-valued process $t \mapsto \eta_t$.*

From the ergodicity of π, this almost proves conjecture 1.1.

Corollary 1.1. *(Corollary 1, [96] for $d = 1$, Corollary 1 in [64] for $d \geqslant 2$) For $\pi-$almost all initial profile $\xi(.)$, we have*

$$\lim_{t\to\infty} \frac{X_t}{t} = 0 \; a.s.$$

Because the rigorous proof of Theorem 1.10 uses the formalism of Fock and Gaussian Hilbert spaces as well as Wick product, we prefer to present the formal proof of it (see section 1.5 in [96]) which is simple to understand.

First, recall that a probability π is invariant for $(\eta_t)_t$ if and only if for any sufficiently smooth test functions u (in a dense subset of Ω), the map $t \mapsto \mathbb{E}(e^{\langle u,\eta_t\rangle_{L^2(\mathbb{R},dx)}})$ is constant[2].

The main ingredient is the following result: if X, Y, Z is jointly Gaussian with zero mean, then

$$\mathbb{E}(Ye^X) = \exp\Big(\mathbb{E}(X^2)/2\Big)\mathbb{E}(XY)$$

[2] In the sequel, we omit the subscript.

and

$$\mathbb{E}(YZe^X) = \exp\Big(\mathbb{E}(X^2)/2\Big)(\mathbb{E}(ZY) + \mathbb{E}(XY)\mathbb{E}(ZX)).$$

Since F is even, it yields from the definition of convolution product and an integration by part that

$$\langle u, F * u' \rangle = \langle u', F * u \rangle.$$

On the other hand, the integration by part also yields

$$\langle u, F * u' \rangle = -\langle u', F * u \rangle.$$

Therefore

$$\langle u, F * u' \rangle = 0.$$

Hence, if η_0 is distributed according to π, we have

$$\begin{aligned} d\mathbb{E}\Big(e^{\langle u,\eta_t\rangle}\Big) &= \mathbb{E}\Big(de^{\langle u,\eta_t\rangle}\Big) \\ &= \mathbb{E}\Big(e^{\langle u,\eta_t\rangle}(\frac{1}{2}\langle u'', \eta_t\rangle + \frac{1}{2}\langle u', \eta_t\rangle^2) - \eta_t(0)\langle u', \eta_t\rangle + \langle u', F\rangle\Big)dt \\ &= e^{\frac{1}{2}\langle u, F*u\rangle}\Big(\frac{1}{2}\langle u'', F*u\rangle + \frac{1}{2}\langle u', F*u'\rangle + \frac{1}{2}\langle u', F*u\rangle^2 - \langle u', F*u\rangle\langle u, F\rangle\Big)dt. \\ &= 0 \end{aligned}$$

In view of Theorem 1.10, a natural question is

Question 1.2. *If $\xi \in \Omega$ is distributed according to the invariant probability measure π, what is the long term behaviour of X_t?*

The answer to this question is the purpose of the second part of [96] (and [64] for $d \geqslant 3$) where the authors study

$$E(t) := \mathbb{E}(|X_t|^2)$$

and its diffusivity term

$$D(t) := \frac{E(t)}{t}.$$

In dimension 1, the result is the following.

Theorem 1.11. *(Theorem 2, [96]) We have*

$$1 \leqslant \liminf_{t\to\infty} D(t) \leqslant \limsup_{t\to\infty} D(t) \leqslant 1 + \int_{-\infty}^{\infty} p^{-2}\hat{F}(p)dp.$$

The proof is based on the property that $W_t - W_s$ and $\int_s^t \eta_u(0)du$ are uncorrelated as well as on the following lemma.

Lemma 1.4. *(Lemma 3, [96]) Let $\varphi \in L^2(\Omega,\pi)$ with $\int_\Omega \varphi d\pi = 0$. Then*

$$\limsup_{t\to\infty} t^{-1}\mathbb{E}\Big((\int_0^t \varphi(\eta_s)ds)^2\Big) \leqslant \lim_{t\to\infty} t^{-1}\mathbb{E}\Big((\int_0^t \varphi(\xi_s)ds)^2\Big),$$

where $t \mapsto \xi_t$ is the reversible Markov process on (Ω,π) whose infinitesimal generator is the opposite of the symmetric part in $L^2(\Omega,\pi)$ of the infinitesimal generator of $t \mapsto \eta_t$.

If $\hat{F}(0) > 0$, the upper bound does not provide any information. However, using a resolvent method for the Laplace transform of E:

$$\hat{E}(\lambda) = \int_0^{\infty} e^{-\lambda t} E(t)dt, \ \lambda > 0$$

and the fact that $W_t - W_s$ and $\int_s^t \eta_u(0)du$ are uncorrelated, P. Tarrès, B. Tóth and B. Valkó are still able to obtain some bounds.

Theorem 1.12. *(Theorem 3 and Remark 4, [96]) Assume that* $0 < \hat{F}(0) < \infty$. *Then there exists constants* $0 < C_1, C_2 < \infty$ *such that*

$$C_1 t^{5/4} \leqslant E(t) \leqslant C_2 t^{3/2}.$$

Generalizing the arguments to higher dimension, I. Horváth, B. Tóth, B. Valkó and B. Vetö obtained the following results.

Theorem 1.13. *(Theorem 2, [64]) For* $d \geqslant 3$, *the process* $t \mapsto D(t)$ *converges almost surely. Furthermore*

$$d \leqslant \lim_{t\to\infty} D(t) \leqslant d + \int_{\mathbb{R}^d} |p|^{-2}\hat{F}(p)dp$$

Theorem 1.14. *(Theorem 1 and Remark 1, [97]) When* $d = 2$, *there exists constants* $0 < C_1, C_2 < \infty$ *such that*

$$C_1 t \log(\log(t)) \leqslant E(t) \leqslant C_2 t \log(t).$$

Despite that Theorem 1.13 and 1.14 provide bounds, they do not give the exact behaviour of $E(t)$. Nevertheless, B. Tóth identifies by an interesting heuristic in the appendix of [97] the possible candidates.

1.1.3 Presentation of the results obtained in the thesis

The results obtained about Self-Repelling Diffusions on a compact Riemannian manifold and solving the SDE

$$dX_t = dW_t(X_t) - \nabla V_t(X_t)dt,$$

with $V_t(x) = \int_0^t V(x, X_s)ds$ lie in the framework considered in the previous subsections and they led to the redaction of two papers.

These two works have their origin in the following simple yet interesting example. Let $(X_t)_{t\geqslant 0} \subset \mathbb{R}$ be the solution of the SDE

$$dX_t = \sigma dW_t - \int_0^t F'(X_t - X_s)dsdt, \quad X_0 = 0 \tag{1.26}$$

where $(W_t)_t$ is a standard real valued Brownian motion, $\sigma > 0$ and $F(x) = \sum_{k=1} a_k \cos(kx)$ with $a_j > 0$ for all $j = 1, \cdots, n$.

Remark 1.1. *The positiveness of these coefficients corresponds to the positiveness condition (1.21). Indeed, let $b \in L^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$ be a function satisfying* (1.21) *and let*

$$\varphi_1(b)(x) = \sum_{n=-\infty}^{\infty} b(x+n)$$

be the $1-$*periodization function of* b. *It is an easy exercise of Fourier analysis to show that*

$$\varphi_1(b)(x) = \sum_{k\in\mathbb{Z}} \hat{b}(k)e^{2\pi ikx}.$$

Because b *is even, we have*

$$\varphi_1(b)(x) = \hat{b}(0) + 2\sum_{k\geqslant 1} \hat{b}(k)\cos(2\pi kx).$$

Since F is a 2π-periodic function, we can identify X_t with $(\cos(X_t), \sin(X_t))$ so that, instead of $\mathbb{R}$, we work on the *compact manifold* $\mathbb{S}^1$. Moreover, for $k \in \mathbb{N}$, the functions $x \mapsto \cos(kx)$ and $x \mapsto \sin(kx)$ are *eigenfunctions of the Laplace operator* on $[0, 2\pi]$.

Finally, we can write V as follows

$$V(x,y) = F(x-y) = \sum_{k=1} a_k\Big(\cos(kx)\cos(ky) + \sin(kx)\sin(ky)\Big).$$

All these properties can be inserted in the following general hypothesis.

(H1). (**The manifold**) M is a smooth, finite dimensional, compact, oriented, connected and without boundary Riemannian manifold.

(H2). (**Interaction potential**) $V(x,y) = \sum_{i=1}^n a_i e_i(x)e_i(y)$, with $a_i > 0$ and $e_1, \cdots, e_n$ are eigenfunctions for the Laplace operator Δ_M and their respective eigenvalues are $\lambda_1, \cdots, \lambda_n < 0$[3].

Letting $\Psi_n(x) = (\sqrt{a_1}e_1(x), \cdots, \sqrt{a_n}e_n(x))$ yields

$$V(x,y) = \langle \Psi_n(x), \Psi_n(y)\rangle,$$

so that V captures the idea of angular distance between two points on M.

At first glance, the finiteness of the sum might be quite restrictive. However, that is not the case since P. Bérard (Theorem 13 in [25]) and more recently J.W Portegies ([86]) show that for n sufficiently large and a suitable choice of (a_i), Ψ_n is a quasi-isometric embedding of M in $\mathbb{R}^n$ in the sense that

$$\Phi_\lambda : (M, dx) \to \mathbb{R}^{N(\lambda)+1} : x \mapsto \Big(\sum_{j=0}^{N(\lambda)} \phi_j^2(x)\Big)^{-1/2}\Big(\phi_0(x), \phi_1(x), \ldots, \phi_{N(\lambda)}(x)\Big)$$

is an isometric embedding for $\lambda > 0$ large enough. Here dx is the standard Riemannian metric, $0 = \mu_0 > \mu_1 \geq \mu_2 \geq \ldots$ is the spectrum of the Laplacian operator on M, the sequence $(\phi_j)_{j\geq 0}$ is an orthonormal basis of $L^2(M, dx)$ such that $\Delta_M\varphi_j = \mu_j\varphi_j$ and for any $\lambda > 0$, $N(\lambda) = Card\{j \geq 1 \ : \ |\mu_j| \leq \lambda\}$.

[3] In particular, they are non-constant functions

Self-Repelling diffusions on a Riemannian manifold

We present the result obtained in [16] in collaboration with M. Benaïm. They have been accepted for publication in *Probability Theory and Related Fields*.

From Proposition 2.5 in [19], we can choose as Brownian vector field

$$dW_t(x) = \sum_{j=1}^{N} F_j(x) \circ dW_t^{(j)}$$

for some $N > 0$, where $(W^{(1)}, \cdots, W^{(N)})$ is a standard Brownian motion on $\mathbb{R}^N$, $\circ$ denotes the Stratonovitch integral and $\{F_i\}$ is a family of smooth vectors fields on M such that

$$\sum_{i=1}^{N} F_i(F_i f) = \Delta_M f,\ f \in \mathcal{C}^\infty(M).$$

Since Hypothesis $(H2)$ implies

$$V_t(x) = \sum_{k=1}^{n} a_k e_k(x) \int_0^t e_k(X_s) ds,$$

then, by letting $U_{k,t}$ denotes the variable

$$U_{k,t} = \int_0^t e_k(X_s) ds,$$

we obtain the following system on the extended manifold $\mathbb{M} := M \times \mathbb{R}^n$

$$\begin{cases} dX_t &= \sigma \sum_{j=1}^N F_j(X_t) \circ dW_t^{(j)} - \sum_{j=1}^n a_j \nabla e_j(X_t) U_{j,t} dt \\ dU_{k,t} &= e_k(X_t) dt, \qquad\qquad k = 1, \cdots, n \end{cases} \tag{1.27}$$

with initial condition $(x, 0, \cdots, 0)$. A first important result is

Proposition 1.3. *(Proposition 2.6, Chapter 2) For all $y = (x, u) \in \mathbb{M}$ there exists a unique global strong solution $(Y_t^y)_{t \geqslant 0}$ to (1.27) with initial condition $Y_0^y = y = (x, u)$. Moreover, we have*

$$Y_t^y = (X_t^y, U_t^y) \in M \times \bar{B}(u, Kt), \tag{1.28}$$

where $K = (\max_{y \in M} \sum_{j=1}^n e_j(y)^2)^{1/2}$ and $\bar{B}(u, R) = \{v \in \mathbb{R}^n \ :\ \|v - u\| \leqslant R\}$.

From now on, we let $(P_t)_{t \geqslant 0}$ denote the semigroup induced by (1.27), i.e

$$(P_t f)(y) = \mathbb{E}(f(Y_t) \mid Y_0 = y) \tag{1.29}$$

for any bounded and measurable function f and $y \in \mathbb{M}$. Finally, let $\mathcal{L}$ denotes its infinitesimal generator.

On the set of $\mathcal{C}^2$ bounded function having first and second bounded derivatives, the infinitesimal generator coincides with the Kolmogorov operator, defined on the space of twice differentially bounded functions having first and second bounded derivatives by

$$L = \frac{\sigma^2}{2}\Delta_M - \sum_{k=1}^{n} a_k u_k (\nabla e_k(x), \nabla_x .)_{TM} + \sum_{k=1}^{n} e_k(x)\partial_{u_k},$$

where $(\Delta_M f)(x,u) = (\Delta_M f(.,u))(x)$ and $(.,.)_{TM}$ stands for the inner product on the tangent bundle of M. Our first main result is

Theorem 1.15. *(Theorem 2.9, Chapter 2) Let $(P_t)_{t\geqslant 0}$ be the semi-group associated to the system (1.27) and $P_t(y_0, dy)$ its transition probability. Then*

1) *The semi-group $(P_t)_{t\geqslant 0}$ is strongly Feller (meaning that $P_t f$ is a bounded continuous function for whatever bounded measurable function f) and there exists a $\mathcal{C}^\infty((0,\infty),\mathbb{M},\mathbb{M})$ function $p_t(y_0,y)$ such that $P_t(y_0,dy) = p_t(y_0,y)dy$ for all $y_0 \in \mathbb{M}$ and $(L_z^* - \partial_t)p_t(y,.) = 0$. Here L^* denotes the adjoint operator of L in $L^2(\mathbb{M}, dy)$.*

2) *The unique invariant probability measure is*

$$\mu(dy) = \nu(dx) \otimes \frac{e^{-\Phi(u)}}{C} du =: \varphi(y)dy,$$

with $y = (x,u)$, $\Phi(u) = \frac{1}{2}\sum_{k=1}^n a_k|\lambda_k|u_k^2$, C is a normalization constant and $\nu(dx)$ is the uniform probability measure on M.

Moreover for all $y \in \mathbb{M}$ and for all bounded measurable function f, we have

$$\lim_{t\to\infty} P_t f(y) = \int_{\mathbb{M}} f(z)\mu(dz).$$

3) *The process $(Y_t)_t$ is positive Harris recurrent, i.e for any Borelian set R such that $\mu(R) > 0$,*

$$\int_0^\infty \mathbf{1}_R(Y_t^y)dt = \infty \ a.s$$

for all $y \in \mathbb{M}$.

4) $\lim_{t\to\infty}\int_{\mathbb{M}} |p_t(z,y) - \varphi(y)|dy = 0$ *for all $z \in \mathbb{M}$.*

The proof mainly consists on proving that the *Hörmander condition* holds. This notion is recalled in section 2.1.1.

Let us briefly explain the strategy used to achieve it. Due to the structure of (1.27), we begin to define a condition (E') which will imply the Hörmander condition (see subsection 2.1.1) and which involves *only* the variable of the manifold M via the eigenfunctions and their successive derivatives.

First we prove that condition (E') holds if the eigenfunctions e_i have the *same non zero eigenvalue* λ. To do this, we use that every linear combination of these eigenfunctions is again an eigenfunction associated to the eigenvalue λ. Then, the fact that every non-zero eigenfunction of the Laplacian on a C^∞ manifold with C^∞ metric never vanishes to infinite order will allow us to conclude.

Finally, if there are different eigenvalues, we consider a partition $(E_\lambda)_{\lambda\in\Lambda}$ of the set $\{e_1, \cdots, e_n\}$, where $\Lambda = \{\lambda \in (-\infty, 0),\ s.t\ \exists j \in \{1, \cdots, n\} \text{ with } \Delta_M e_j = \lambda e_j\}$ and $e_j \in E_\lambda$ if and only if $\Delta_M e_j = \lambda e_j$.

By the previous step, the condition (E') holds on E_λ for all $\lambda \in \Lambda$. We complete the proof by putting everything together via the family of operator

$$P^\lambda(\Delta_M) = \prod_{\alpha\in\Lambda\alpha\neq\lambda} (\Delta_M - \alpha I),$$

where I is the identity operator.

Once the *existence of a unique* probability measure and the convergence of $Y_t' s$ distribution to it in total variation is known, the next step consists to find out, if possible, some rate of convergence. This is the purpose of the next Theorems.

Theorem 1.16. *(Theorem 2.10, Chapter 2) For every $\eta > 0$ and $g \in L^2(\mu)$*

$$\|P_t g - \int_{\mathbb{M}} g(y)\mu(dy)\|_{L^2(\mu)} \leqslant \sqrt{1+2\eta}\|g - \int_{\mathbb{M}} g(y)\mu(dy)\|_{L^2(\mu)} e^{-\lambda t},$$

where

$$\lambda = \frac{\eta}{1+\eta}\frac{K_1\sigma^2}{1+K_2\sigma^2+K_3\sigma^4},$$

with explicit constants K_1, K_2 and K_3.

Theorem 1.17. *(Theorem 2.11, Chapter 2) For all $z_0 \in \mathbb{M}$ and $t \geqslant 1$,*

$$\|P_t(z_0, dz) - \mu(dz)\|_{TV} \leqslant \sqrt{1+2\eta}\|h(1, z_0, z) - 1\|_{L^2(\mu)} e^{-\lambda(t-1)},$$

where $h(1, z_0, z) = \frac{p_1(z_0,z)}{\varphi(z)}$ and λ is as in Theorem 1.16.

Theorem 1.17 turns out to be a consequence of Theorem 1.16, which itself follows from Theorem 1.18.

The idea behind the proof is the use of a Grönwall like inequality. However, it is hopeless to try to obtain such an inequality from classical functional inequalities like the Poincaré inequality because for sufficiently smooth function ξ, we have

$$(L\xi, \xi)_{L^2(\mu)} = -\int |\partial_x \xi(y)|^2 \mu(dy), \tag{1.30}$$

where L is the Kolmogorov operator induced by (1.27). In particular, if ξ does not depend on the variable x, the left hand side vanishes.

Remark 1.2. *Equation 1.30 explains why the coefficient λ from Theorem 1.16 is bad as σ goes to infinity. Indeed, on one hand, if σ is large then the effect of the Brownian motion on the particle (the X-variable) is important so that the particle should cover uniformly the manifold M rapidly. Therefore, for any $g \in L^2(\mu)$, $P_t g$ becomes less and less dependent on the x-variable.*

On the other hand, since $\frac{d}{dt}P_t g = Lg$, we have

$$\frac{d}{dt}\|P_t g - \int_{\mathbb{M}} g(y)\mu(dy)\|^2_{L^2(\mu)} = -2\int_0^t |\partial_x(P_t g)(y)|^2\mu(dy).$$

To overcome this problem we use the hypocoercivity method introduced by J.Dolbeault, C. Mouhot and C. Schmeiser in [43] and thoroughly analysed by M.Grothaus and P.Stilgenbauer in [56]. As an application, they studied the case $M = \mathbb{S}^d$, $d \geqslant 1$ and $e_j(x) = x_j$ for $j = 1, \cdots, d+1$.

Set $S = \frac{1}{2}\Delta_M$ and $A = S - L$. A first important observation, is that for all $\phi, \psi \in D$, a dense set in $L^2(\mathbb{M}, \mu)$ of sufficiently smooth function, we have

$$\begin{aligned} (S\psi, \phi)_{L^2(\mu)} &= (S\phi, \psi)_{L^2(\mu)} \\ (A\psi, \phi)_{L^2(\mu)} &= -(A\phi, \psi)_{L^2(\mu)}. \end{aligned}$$

Let P be an orthogonal projection such that

$$P(D) \subset D(A) \text{ and } PAP\phi = 0, \quad \forall \phi \in D \tag{1.31}$$

and define on $D((AP)^*AP)$ the operator

$$B_0 = (I + (AP)^*AP)^{-1}(AP)^*.$$

One can prove that there exists a constant $c > 0$ such that

$$\|B_0 f\|_{L^2(\mu)} \leqslant c\|(I - P)f\|_{L^2(\mu)}$$

for all $f \in L^2(\mathbb{M}, \mu)$. So, instead of working with the square of the $L^2(\mu)$-norm, the idea is to consider the following equivalent object

Definition 1.2. *For $0 \leqslant \varepsilon < 1$, the modified entropy functional is*

$$H_\varepsilon[f] = \frac{1}{2}\|f\|^2_{L^2(\mu)} + \varepsilon(B_0 f, f)_{L^2(\mu)}.$$

In order to apply a Grönwall like argument, they introduced the following tool.

Definition 1.3. *For $0 \leqslant \varepsilon < 1$, The dissipative entropy functional is*

$$D_\varepsilon(t, f) = -\frac{d}{dt}H_\varepsilon(P_t f).$$

Under a quite long list of assumptions and hypothesis that can be found in [56], they proved the existence of a constant $\kappa > 0$ such that

$$D_\varepsilon(t, f) \geqslant \kappa\|P_t f\|_{L^2(\mu)}.$$

Thus, everything boils down to

Theorem 1.18 (Theorem 2 in [43], Theorem 1 in [42], Theorem 2.18 in [56])**.** *Under the list of assumptions and hypothesis from [56], there exist constants* $\kappa_1, \kappa_2 \in (0, \infty)$ *explicitly computable such that for all* $g \in L^2(\mu)$ *and* $t \geqslant 0$*,*

$$\|P_t g\|_{L^2(\mu)} \leqslant \kappa_1 e^{-\kappa_2 t} \|g\|_{L^2(\mu)} \tag{1.32}$$

Self-Repelling diffusions via an infinite dimensional approach

We present the results obtained in [14] in collaboration with M. Benaïm and I. Ciotir. They are published in *Stochastic Partial Differential Equations: Analysis and Computations*.

The purpose of this work is to generalize as much as possible the results obtained in [16] for the circle to more general profile functions. More precisely, we analyse the long term behaviour of

$$X_t = x + \int_0^t g(X_s)\, ds - \int_0^t \int_0^s F'(X_s - X_r)\, dr ds + W_t \tag{1.33}$$

where $x \in \mathbb{R}$ and W_t is a standard real Brownian motion. We assume that g, F are sufficiently regular 2π-periodic functions and F is even. In terms of Fourier series' representation, it yields

$$\begin{aligned} F(x) &\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \\ g(x) &\sim \sum_{n=1}^{\infty} a_n^{1/2} n \left(u_0^{(n)} \sin(nx) + v_0^{(n)} \cos(nx) \right) \end{aligned}$$

where it is assumed that

$$(a_n)_{n\geqslant 1} \in \left\{ (a_n)_n ; \sum_{n=1}^{\infty} \left(1+n^2\right)^5 a_n^2 < \infty \right\} \bigcap \left\{ (b_n)_n ; b_n > 0\ \forall n \right\}$$

and $\left(u_0^{(n)}\right)_n, \left(v_0^{(n)}\right)_n \in l^2$.

Introducing the variables

$$u_t^{(n)} = u_0^{(n)} + + a_n^{1/2} \int_0^t \cos(nX_s)\, ds$$

and

$$v_t^{(n)} = v_0^{(n)} - a_n^{1/2} \int_0^t \sin(nX_s)\, ds,$$

we obtain the following system in the Hilbert space $H = \mathbb{R} \times l^2 \times l^2$

$$\begin{cases} X_t = x + \int_0^t \sum_n n \left(a_n^{1/2} \sin(nX_s)\, u_s^{(n)} + a_n^{1/2} \cos(nX_s)\, v_s^{(n)} \right) ds + W_t, \\ u_t^{(n)} = u_0^{(n)} + a_n^{1/2} \int_0^t \cos(nX_s)\, ds, \quad n \geq 1, \\ v_t^{(n)} = v_0^{(n)} - a_n^{1/2} \int_0^t \sin(nX_s)\, ds, \quad n \geq 1. \end{cases}$$

Since H is an infinite dimensional state space, a natural formalism is the one of Stochastic Partial Differential Equations. The usual questions are

Question 1.3.

- *Do we have existence of a unique strong solution?*
- *Does the system have the Feller property? the Strong Feller property?*
- *Does an invariant probability measure exists?*

To shorten the notation, we write $Y_t = (X_t, (u_t^n)_n, (v_t^n)_n)$. The answer to the first question is given by the following proposition

Proposition 1.4. *(Proposition 3.1, Chapter 3) For each $y \in H$, there is a unique analytically strong solution*

$$Y \in C\left([0,\infty);H\right) \cap L^{\infty}_{loc}\left([0,\infty);\ H\right)$$

to equation (1.1.3).

Moreover, for $T < \infty$, we have that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|Y_t\|_H^2\right) < \infty.$$

When investigating the existence of an invariant probability measure, the first step consists in establishing the Feller property, which is often expected to hold. However, that is rarely the case for the Strong Feller property, even if everything seems to be as regular as wanted to ensure that all the finite dimensional system that approximates the infinite dimensional one have the Strong Feller property (see e.g example 3.1.5 in [58]).

Proposition 1.5. *(Proposition 3.2, Chapter 3) The solution of (1.1.3) satisfies the Feller property, but not the Strong Feller property.*

The proof of the Feller property is based on the continuity with respect to the initial condition whereas the lack of Strong Feller property follows from the existence of non-empty invariant subsets of H. This phenomena does not occur in the finite dimensional setting.

In view of Theorem 1 in [96] and Theorem 1.15, it is not surprising to have the following Theorem

Theorem 1.19. *(Proposition 3.3, Chapter 3) The probability measure*

$$\mu\left(dy\right) = \frac{dx}{2\pi} \otimes \prod_{n\geq 1} N\left(0, \frac{1}{n^2}\right) du_n \otimes \prod_{n\geq 1} N\left(0, \frac{1}{n^2}\right) dv_n$$

is invariant for the semigroup induced by (1.1.3).

The Theorem's proof is based on the Galerkin approximation method and on Theorem 1.15.

For $N \in \mathbb{N}$, set

$$H_N = \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N.$$

Since $H = H_N \times l^2 \times l^2$, we define the operator $\Pi_N : H \to H_N \times \{0\}^\infty \times \{0\}^\infty$ by

$$\Pi_N \left(x, (u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}\right) = \left(x, (u_n)_{n=1}^N \times \{0\}^\infty, (u_n)_{n=1}^N \times \{0\}^\infty\right).$$

Then, define the process $(Y_t^{(N)})_{t\geqslant 0}$ as the solution of the stochastic differential equation on H_N

$$\begin{cases} dY_t^{(N)} = \Pi_N \left(F\left(Y_t^{(N)}\right)\right) dt + \sigma dW_t \\ Y_0^{(N)} = \Pi_N y \end{cases} \tag{1.34}$$

and the measure

$$\begin{aligned} \mu_\infty^N (dy) &= \frac{dx}{2\pi} \otimes \prod_{n=1}^N N\left(0, \frac{1}{n^2}\right) du_n \otimes \prod_{n>N} \delta_0 (du_n) \\ &\quad \otimes \prod_{n=1}^N N\left(0, \frac{1}{n^2}\right) dv_n \otimes \prod_{n>N} \delta_0 (dv_n) \\ &\overset{Denote}{=} \frac{dx}{2\pi} \otimes \mu^N \left(d\left(u_n\right)_{n=1}^N\right) \otimes \mu^{N+} \left(d\left(u_n\right)_n\right) \\ &\qquad \otimes \mu^N \left(d\left(v_n\right)_{n=1}^N\right) \otimes \mu^{N+} \left(d\left(v_n\right)_n\right), \end{aligned}$$

where δ_0 is the Dirac measure on $\mathbb{R}$.

By a similar argumentation as for Theorem 1.15, μ_∞^N is invariant for the semigroup $(P_t^N)_t$ induced by (1.34).

In order to complete the proof of Theorem 1.19, we show, on one hand, that μ_∞^N converges to μ for the topology of weak convergence; and on the other hand, that

$$\lim_{N\to\infty} \int (P_t^N f)(y)\mu(dy) = \int (P_t f)(y)\mu(dy)$$

for any bounded continuous function $f : H \to \mathbb{R}$.

1.2 Self-Attracting Diffusions

Contrary to Section 1.1, we are now interested in the case where the particle is attracted by its past. Therefore, the question is now

Question 1.4.

- *Does the particle converge almost-surely? If yes, can we estimate the rate of convergence?*

1.2.1 Convergence results

The first results about *Self-Attracting Diffusions* were obtained in 1995 by M. Cranston and Y. Le Jan ([35]). They studied the real valued SDE

$$dX_t = dW_t - \int_0^t f(X_t - X_s)dsdt,\ X_0 = 0 \tag{1.35}$$

when $f(x) = ax$ (linear case) and $f(x) = a \times sgn(x)$ (constant case), with $a > 0$. In both cases, they proved the *almost-sure convergence* of X_t.

For the linear case, the solution of (1.35) is

$$X_t = \int_0^t h(t,s)dW_s, \quad \text{with } h(t,s) = 1 - ase^{as^2/2}\int_s^t e^{-au^2/2}du. \tag{1.36}$$

Therefore, the natural limit candidate as t goes to infinity is

$$X_\infty = \int_0^\infty h(s)dW_s, \quad \text{with } h(s) = 1 - ase^{as^2/2}\int_s^\infty e^{-au^2/2}du. \tag{1.37}$$

Using that

$$h(t,s) - h(s) = ase^{as^2/2}\int_t^\infty e^{-au^2/2}du \leqslant \frac{s}{t}e^{\frac{a(s^2-t^2)}{2}},$$

they proved

Theorem 1.20. *(Theorem 1, [35]) If $f(x) = ax, a > 0$, then X_t converges almost-surely and in L^2 to X_∞.*

When the Brownian motion $(W_t)_t$ is replaced by a *fractional Brownian motion*, Litan Yan, Yu Sun and Yunscheng Lu still obtain the almost-sure convergence of the particle.

Theorem 1.21. *(Theorem 3.3, [100]) Let $(Z_t)_t$ be the solution of*

$$Z_t = B_t^H - a\int_0^t\int_0^s (Z_s - Z_u)duds,$$

where $(B_t^H)_t$ is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. Then X_t converges almost surely to $\int_0^\infty \left(1 - ase^{as^2/2}\int_s^\infty e^{-au^2/2}du\right)dB_s^H$.

An important difference between the linear and the constant case is about the intensity of the interaction function. Indeed, in the linear case the further X_t is far X_s the stronger is the restoring force whereas in the constant case it remains unchanged. However, if X_t is close to X_s, then the force is stronger than in the linear case. So, one may expect the convergence of the particle for the constant interaction.

Theorem 1.22. *(Theorem 2, [35]) If $f(x) = a \times sgn(x), a > 0$, then X_t converges almost-surely.*

The constant case required a more sophisticated proof and mainly relies on the following lemma which implies the almost-sure boundedness of X.

Lemma 1.5. *(Lemma 1, [35]) Let $\mathbb{P}^{(V)}$ be the law of the solution of*

$$dY_t = dW_t - \big(V(Y_t) + a\int_0^t sgn(Y_t - Y_s)dsdt\big),\ Y_0 = 0$$

where V is a measurable function, assumed to be nonnegative on $\mathbb{R}_+$. Then

1. $\sup_{t\geqslant 0} Y_t < \infty$, $\mathbb{P}^{(V)}$ *almost-surely.*
2. *For every $\varepsilon > 0$, there exists $M(\varepsilon) > 0$ such that if $V(x) \geqslant M(\varepsilon)$ for all $x > 0$, then $\mathbb{P}^{(V)}(\sup_{t\geqslant 0} Y_t > \varepsilon) < \varepsilon$.*

The proof is built upon an adequate sequence of stopping times $(\tau_n)_{n\geqslant 0}$ and, knowing $(Y_t)_{0\leqslant t\leqslant \tau_n}$, on the comparison between $(Y_t)_{t\geqslant \tau_n}$ with a simpler process[4].

Note that despite $(Y_t)_t$ is not Markovian, the pair $(Y_t, \mu_t = \int_0^t \delta_{Y_s} ds)_t$ is strongly Markovian.

In order to move from a conditional distribution to a non conditional one, the authors need the following lemma.

Lemma 1.6. *(Lemma 2, [35]) For any stopping time τ, the conditional law with respect to $\mathbb{P}^{(V)}$ of $Y_{t+\tau} - Y_\tau$ given (Y_τ, μ_τ) coincides with $\mathbb{P}^{(V_\tau)}$, where $V_\tau(x) = V(x+Y_\tau) + a\int sgn(x + Y_\tau - y)\mu_\tau(dy)$.*

The price paid by applying the lemma is the loss of knowledge of the potential function's shape. But it is not dramatic if good estimates are available. These have been obtained by the authors from the explicit expression of f.

We refer to [35] for the details.

In 2003, S. Herrmann and B. Roynette were able to extend the proof to a larger class of functions.

Theorem 1.23. *(Theorem 1, [60]) Assume that f is an odd increasing bounded and continuous function such that in a neighbourhood of the origin there exists $C > 0, \rho > 0$ and $k \in \mathbb{Z}^+$ such that*

$$|f(x)| \geqslant Ce^{-\rho|x|^{-k}}.$$

Then the solution of (1.35) converges almost-surely.

So far, all the previous results are stated in $\mathbb{R}$. Hence, a possible extension is to generalise them in $\mathbb{R}^d$ for $d \geqslant 2$. The linear case is obvious since it reduces to d times the one dimensional linear case. A more interesting challenge is the extension of the constant case. This was done by O. Raimond in 1997.

[4]Nonetheless, this simpler process depends on $(Y_u)_{0\leqslant u\leqslant \tau_n}$

Theorem 1.24. *(Theorem 1, [88]) Let $a > 0$, $(B_t)_t$ a $\mathbb{R}^d$-Brownian motion and $(X_t)_t$ the solution of the SDE*

$$dX_t = dW_t - a\int_0^t \frac{X_t - X_s}{\|X_t - X_s\|} dsdt,\ X_0 = 0. \tag{1.38}$$

Then X_t converges almost-surely.

The proof follows more or less the same structure as in the one dimensional case where Lemma 1.5's non-negativeness property is replaced by $\langle V(x), x\rangle \geqslant 0$. Contrary to the one dimensional case, we cannot infer at that level the convergence of X_t since Cranston-Le Jan's argument is purely one dimensional. Furthermore, an other phenomenon that is possible only in $\mathbb{R}^d$ with $d \geqslant 2$, is that X_t could turn around a point.

In order to prove that is not possible, O. Raimond shows that $\liminf_t \|X_t - C_t\| = 0$ almost surely, where C_t is the *unique* minimum of the convex function V_t defined by

$$V_t(x) = a\int_0^t \|x - X_s\| ds$$

whose gradient is nothing else as the drift of (1.38). Then, he uses the boundedness of X_t to complete the proof.

The intuition behind the proof is the following. When X_t localizes for a while, then it digs a hole whose bottom is C_t and the slope of V_t becomes progressively steeper. Hence the force felt by X_t becomes progressively more and more important, so that X_t get increasingly closer to C_t. Therefore, at infinity, the slope is "vertical" and it kills the Brownian motion's effect. Therefore X_t converges.

1.2.2 Rate of convergence

The first paper providing rates of convergence is [61] from S. Herrmann and M. Scheutzow and it was published in 2004. The first main result is

Theorem 1.25. *(Theorem 1, [61]) Suppose that f is an odd increasing function of class C^1 and there exists $\eta > 0, \gamma \geqslant 1$ and $C_\gamma > 0$ such that*

$$|f(x) - f(y)| \geqslant C_\gamma |x - y|^\gamma \ \textit{if}\ |x - y| \leqslant \eta.$$

Then, for all $\mu < 1/(1 + \gamma)$,

$$\lim_{t\to\infty} \Big((\frac{t}{\ln(t)})^\mu \sup_{s\geqslant t} |X_s - X_t| \Big) = 0\ a.s.$$

As for [35] and [60], the idea is to compare locally $(X_t)_{t\geqslant 0}$ with simpler processes. The proof is divided in three steps.

In the first one, they first consider the SDE

$$dY_t = dW_t - t\Phi(Y_t)dt \tag{1.39}$$

for some function Φ and prove the following long term behaviour.

Proposition 1.6. *(Proposition 1, [61])*

1. *Assume that* Φ *is non decreasing,* $x\Phi(x) > 0$ *for all* $x \neq 0$. *Then* Y_t *converges almost surely to* 0.

2. *Moreover, if there exists* $\eta, \gamma, C_\gamma > 0$ *such that* $|\Phi(x)| \geqslant C_\gamma |x|^\gamma$ *for* $|x| \leqslant \eta$, *then there exists* $K_\gamma > 0$ *such that*

$$\lim_{t\to\infty} \left((\frac{t}{\ln(t)})^{1/(1+\gamma)} \sup_{s\geqslant t} |Y_s| \right) \leqslant K_\gamma.$$

A reason to study (1.39) is the following. Since X_t converges to X_∞ (by Theorem 1.23), then for t sufficiently large

$$\int_0^t f(X_t - X_s)ds \sim \int_0^t f(X_t - X_\infty + o(1))ds \sim tf(X_t - X_\infty)$$

Hence, if $Y_t := X_t - X_\infty$ solves some true SDE[5], then it must be

$$dY_t = dW_t - tf(Y_t)dt. \tag{1.40}$$

The proof of the proposition being technical, we choose to not say anything more about it except that comparison results are used several times.

In the second step, we recover the idea developed by O. Raimond in [88]; namely the study of the distance between X_t and the unique zero, denoted by C_t, of the function

$$D(t,.) : \mathbb{R} \to \mathbb{R} : \ x \mapsto \int_0^t f(x - X_s)ds.$$

The uniqueness of C_t is ensured by the monotonicity of f. S. Herrmann and M. Scheutzow call the process $(C_t)_t$, with initial condition $C_0 = X_0$, *the mean-process.*

Proposition 1.7. *(Proposition 2, [61])*

1. *If there exists* $c > 0$ *and* $R > 0$, *such that* $|f'(x)| \geqslant c$ *for* $|x| \geqslant R$, *then*

$$\lim_{t\to\infty} (X_t - C_t) = 0$$

almost-surely.

2. *If* f *satisfies the assumption of Theorem 1.25, then there exists* $K_\gamma > 0$ *such that*

$$\lim_{t\to\infty} \left(\left(\frac{t}{\ln(t)}\right)^{1/(1+\gamma)} \sup_{s\geqslant t} |X_s - C_s| \right) \leqslant K_\gamma.$$

[5]By true SDE, we mean a SDE of the form $dY_t = \sigma(t, Y_t)dW_t + h(t, Y_t)dt$.

The proof works as follows. Define $Z_t = X_t - C_t$ so that $(Z_t)_t$ solves the SDE

$$dZ_t = \Big(- \int_0^t f(X_t - X_s)ds + \frac{f(-Z_t)}{\int_{\mathbb{R}} f'(C_t - y)\mu_t(dy)}\Big)dt + dW_t,\ Z_0 = 0$$

Then, by the assumptions on the function f, they are able to define a function Φ such that

$$-sgn(Z_t)f(Z_t + C_t - y)\mu_t(dy) \leqslant -t sgn(Z_t)\Phi(Z_t).$$

Therefore, they obtain from a comparison result that

$$\mathbb{P}(Z_t^2 \leqslant U_t,\ \forall\, t \geqslant 0) = 1,$$

for some process $(U_t)_{t\geqslant 0}$ having the same distribution as $(Y_t^2)_{t\geqslant 0}$.

Finally, they complete Theorem 1.25's proof in the third and last part. They define first a sequence of stopping times $(\tau_n)_n$ by $\tau_0 = 0$ and

$$\tau_{n+1} = \inf\{t \geqslant \tau_n + n \mid Z_t = 0\}$$

where Z_t is defined in the second step. Then for any $\delta > 0$, Propositions 1.6 and 1.7 provide the existence of a random variable N such that for $n \geqslant N$, $\tau_{n+1} \leqslant \tau_n + 2n$ and for $t \geqslant \tau_N$,

$$\sup_{s\geqslant t} |Z_s| \leqslant (K_\gamma + \delta)\Big(\frac{\ln(t)}{t}\Big)^{\frac{1}{1+\gamma}}. \tag{1.41}$$

Given $n \geqslant N$, they define from τ_n and $(C_t)_{t\geqslant\tau_n}$ a new sequence of stopping time $(S_k)_k \subset [\tau_m, \infty)$ such that

$$\sup_{s\geqslant\tau_n} |C_s - C_{\tau_n}| \leqslant \sum_{k\geqslant 0} \Big(\frac{\ln(S_k)}{S_k}\Big)^{\frac{1}{1+\gamma}}$$

and prove that $\lim_{k\to\infty} S_k = \infty$ and

$$(K_\gamma + \gamma)\sum_{k\geqslant 0} \Big(\frac{\ln(S_k)}{S_k}\Big)^{\frac{1}{1+\gamma}} \leqslant K\Big(\frac{\ln(\tau_n)}{\tau_n}\Big)^r$$

for $r < \frac{1}{1+\gamma}$ and some constant K. For details, we refer the interested reader to the original paper.

The conclusion now follows from the following inequalities

$$\begin{aligned}\limsup_{t\to\infty} \Big(\frac{t}{\ln(t)}\Big)^\mu \sup_{s\geqslant t}|X_s - X_t| &\leqslant \limsup_{t\to\infty} \Big(\frac{t}{\ln(t)}\Big)^\mu (\sup_{s\geqslant t}|Z_s| + \sup_{s\geqslant t}|C_s - C_t|) \\ &\leqslant 2(K_\gamma + \gamma)\limsup_{m\to\infty} \Big(\frac{\tau_m}{\ln(\tau_m)}\Big)^\mu \Big(\frac{\ln(S_k)}{S_k}\Big)^{\frac{1}{1+\gamma}}\end{aligned}$$

Beside this general result, S. Herrmann and M. Scheutzow also give the exact rate of convergence for the linear case.

Theorem 1.26. *(Proposition 4, [61]) Assume that $f(x) = ax$ with $a > 0$. Then*

$$\limsup_{t\to\infty} \sqrt{\frac{t}{\ln(t)}}(X_t - X_\infty) = -\liminf_{t\to\infty} \sqrt{\frac{t}{\ln(t)}}(X_t - X_\infty) = \sqrt{\frac{2}{a}},$$

where X_∞ is defined by (1.37).

The proof is based on the explicit expression of the solution, the Dubins-Schwartz Theorem and the Law of Iterated Logarithm.

1.2.3 Presentation of the results obtained in the thesis

The results presented in this subsection are the purpose of the article [53] and are published in *Electronic Communication in Probability.*

Let $(X_t)_{t\geqslant 0} \subset \mathbb{S}^n$ be the solution of the SDE

$$dX_t = P(X_t, \circ dW_t) - a\nabla V_t(X_t)dt, \quad X_0 = x \tag{1.42}$$

where $P(x,u) = u - \langle x, u\rangle x$ denotes the orthogonal projection of $u \in \mathbb{R}^{n+1}$ on the tangent space of $\mathbb{S}^n$ at x, $(W_t)_{t\geqslant 0}$ is a standard Brownian motion on $\mathbb{R}^{n+1}$, $a < 0$ and

$$V_t(x) = \int_0^t \langle x, X_s\rangle ds = \sum_{i=1}^{n+1} x^{(i)} \int_0^t X_s^{(i)} ds.$$

The interest of this potential function is its connection with the notion of distance on $\mathbb{S}^n$ because for $x, y \in \mathbb{S}^n$, one has

$$\|x - y\|^2 = 2 - 2\langle x, y\rangle = 2 - 2\cos(D(x,y)), \tag{1.43}$$

where $D(.,.)$ is the geodesic distance on $\mathbb{S}^n$ and $\|.\|$ is the standard Euclidean norm on $\mathbb{R}^{n+1}$.

When $a < 0$, the drift term of equation (1.42) shows that X_t tends to minimize its distance with its pasts positions.Therefore, one expects that X_t localizes on a part of the sphere. But this implies that the potential function V_t becomes progressively steeper around some mean point (the bottom of the hole). On the other hand, the deeper the hole is, the more difficult it becomes for the particle to move away from this mean point.

Hence, in view of the result in the linear case (see Theorem 1.20), one would expect almost-sure convergence. This is the purpose of the next theorem, which is our main result.

Theorem 1.27. *(Theorem 4.5, Chapter 4) If $a < 0$, there exists a random variable $X_\infty \in \mathbb{S}^n$ such that*

$$\|X_t - X_\infty\| = \begin{cases} O\big(t^{-1/2}\sqrt{\ln(t)}\big) & \text{if } n = 1 \\ O\big((\frac{\ln(t)}{t})^{1/4}\big) & \text{otherwise} \end{cases}.$$

Let us briefly explain how to prove this result. First of all, as in subsection 1.1.3, we add new variables in order to get the following true SDE on $\mathbb{S}^n \times \mathbb{R}^{n+1}$

$$\begin{cases} dX_t = P(X_t, \circ dW_t) + P(X_t, U_t)dt \\ dU_t = X_t dt \end{cases} \tag{1.44}$$

Remark 1.3. *We have $P(X_t, \circ dB_t) = P(X_t, dB_t) - \frac{n}{2} X_t dt$.*

Let us now take the problem in the reverse sense and assume that X_t converges to $X_\infty \in \mathbb{S}^n$.

Then, from the definition of U_t, we have

$$\lim_{t\to\infty} \frac{U_t}{t} = X_\infty.$$

This yields two important informations:

$$\lim_{t\to\infty} \frac{\|U_t\|}{t} = 1$$

and

$$\lim_{t\to\infty} \frac{\langle U_t, X_t \rangle}{\|U_t\|} = 1.$$

This motivates the following strategy.

(S1) Show that $R_t := \|U_t\|$ converges almost-surely to infinity sufficiently fast (see Lemma 4.2).

(S2) Prove that it implies that $C_t := \frac{\langle U_t, X_t \rangle}{\|U_t\|}$ converges almost-surely to 1.

(S3) Use it in order to deduce that $\lim_{t\to\infty} \frac{R_t}{t} = 1$.

(S4) Obtain a sufficiently fast rate of convergence for C_t.

(S5) Conclude.

A first important result in this strategy is

Lemma 1.7. *(Lemma 4.1, Chapter 4) There exists a real valued Brownian motion $(B_t)_{t\geqslant 0}$ such that $((C_t, R_t))_{t\geqslant}$ is solution to*

$$\begin{cases} dC_t = \sqrt{1 - C_t^2} dB_t + [(R_t + \frac{1}{R_t})(1 - C_t^2) - \frac{n}{2} C_t] dt \\ dR_t = C_t dt \end{cases} \tag{1.45}$$

whenever $R_t > 0$.

This proposition shows that $(S2)$ implies $(S3)$. Since $|C_t| \leqslant 1$, then one may expect that if R_t converges sufficiently fast to infinity, then it kills the noise term, so that the long term behaviour of C_t is determined by the ordinary differential equation

$$\dot{C} = (1 - C^2)$$

This intuition is formalised by the *Asymptotic Pseudotrajectories' theory*, which is recalled in Section 4.1.1. This is used to prove that $(S1)$ implies $(S2)$.

In order to conclude, we set $V_t = \frac{U_t}{R_t}$. Then, from the definition of U_t and R_t, it yields

$$dV_t = \frac{1}{R_t}(X_t - C_t V_t)dt.$$

Since

$$\|X_t - C_t V_t\| = \sqrt{1 - C_t^2},$$

$(S3)$ and $(S4)$ implies that $\frac{1}{R_t}\|X_t - C_t V_t\|$ is an integrable quantity. So V_t, and thus X_t, converges almost-surely.

1.3 Self-Interacting diffusions with a reinforcement by the normalized empirical occupation measure

The aim of this section is to present the results obtained when the interaction is normalised, ie $(X_t)_{t\geqslant 0}$ solves

$$dX_t = dW_t(X_t) - \frac{1}{t}\int_0^t \nabla_x V(X_t, X_s)ds dt, \quad X_0 = x. \tag{1.46}$$

As mentioned by R. Pemantle in his survey (section 6.1, page 58 in [84]), these processes are a more natural extension to a continuous time process on non-discrete state space of *Reinforced Random Walk* whose reinforcement is given via the definition of the transition probability measure (some references written after 2007 are for instance [77], [24], [46], [23] and [8]; and for those written before, we refer to [84]). For instance, the transition probability for Edge Reinforced Random Walk at time n is given by

$$\mathbb{P}\Big(X_{n+1} = y | \sigma(X_0, X_1, \cdots, X_n)\Big) = \mathbb{1}_{\{y \sim X_n\}} \frac{w(Z_n(\{X_n, y\})}{\sum_{i \sim X_n} w(Z_n(\{X_n, i\}))},$$

where $Z_n(\{x, y\})$ is the number of crossing of the non-oriented edge $\{x, y\}$ of the walker at time n and w is some positive increasing function.

As explained in subsection 1.3.1, equation (1.46) can be related to a Markov process by considering the normalized occupation measure probability $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$ as an additional variable. This is the core of the investigations which have a dynamical system flavour.

This section is based on the papers [19] and [21] for the case of a compact, connected, smooth and without boundary Riemannian manifold, and on the papers [73] and [76] for the non-compact setting.

1.3.1 The compact case

In this subsection, we assume that the Riemannian manifold M and the potential function V enjoy the following properties:

- M is a smooth, finite dimensional, compact, oriented, connected and without boundary Riemannian manifold.
- For all $y \in M$, the map $V_y : M \to \mathbb{R} : x \mapsto V(x, y)$ is a C^2 function whose first and second derivatives are continuous in both variables.
- The mapping $x \mapsto \int_M V_y(x)\lambda(dy)$ is constant, where λ stands for the Riemannian probability on M.
- V is symmetric: $V(x, y) = V(y, x)$ for all $x, y \in M$.

These hypothesis ensure existence and uniqueness of the solution (see Proposition 2.5 in [19]).

Let $\mathcal{M}(M)$ denote the set of Borel bounded measure on M endowed with the weak topology and $\mathcal{P}(M)$ the subset of Borel probabilities with the induced topology. Note that the compactness of M implies the relatively compactness of $\mathcal{P}(M)$ by Prohorov's Theorem.

Given a probability measure μ, let V_μ be the function on M defined by

$$V_\mu(x) = \int_M V(x, y)\mu(dy). \tag{1.47}$$

The analysis of the long term behaviour of (1.46) was initiated by M. Benaïm, M. Ledoux and O. Raimond in 2002 ([19]) and later further investigated by M. Benaïm and O. Raimond ([20], [21] and [22]) via the normalized occupation measure

$$\mu_t(dy) = \frac{1}{t}\int_0^t \delta_{X_s}(dy)ds.$$

When considering the pair $\Big((X_t, \mu_t)\Big)_{t\geqslant 0}$, we have a Markov process that solves the following SDE on $M \times \mathcal{P}(M)$:

$$\begin{cases} dX_t = dW_t(X_t) - \nabla V_{\mu_t}(X_t)dt \\ d\mu_t = \frac{1}{t}(\delta_{X_t} - \mu_t)dt. \end{cases} \tag{1.48}$$

Since the variables are evolving in two different time scales, one may expect the following behaviour[6]. Because μ_t changes with a time scale of order $1/t$, the evolution of μ_t becomes progressively slower compared to X_t's one. Thus, for all $t_0 > 1$ sufficiently large, any

[6]The following heuristic was first explain explicitly by V.Kleptsyn and A. Kurtzmann in [73]

$T > 0$ and any $t \in [t_0, t_0 + T]$, V_{μ_t} should be close to $V_{\mu_{t_0}}$. Hence, one might expect that $(X_{t_0+t})_{t\geqslant 0}$ should not differ too much from the solution of the SDE

$$dX_t^{\mu_{t_0}} = dW_t(X_t^{\mu_{t_0}}) - \nabla V_{\mu_{t_0}}(X_t^{\mu_{t_0}})dt, \; X_0^{\mu_{t_0}} = X_{t_0}. \tag{1.49}$$

On one hand, the invariant probability measure of (1.49) is

$$\Pi(\mu_{t_0})(dx) = \frac{e^{-2V_{\mu_{t_0}}(x)}}{Z_{\mu_{t_0}}}\lambda(dx), \tag{1.50}$$

where $Z_{\mu_{t_0}}$ is the normalization constant. On the other hand, by Birkhoff's Ergodic Theorem, $\frac{1}{T}\int_{t_0}^{t_0+T} \delta_{X_s^{\mu_{t_0}}} ds$ converges to $\Pi(\mu_{t_0})$ as T goes to infinity. Therefore

$$\begin{aligned} \mu_{t_0+T} &= \frac{t_0}{t_0+T}\mu_{t_0} + \frac{T}{t_0+T}\frac{1}{T}\int_{t_0}^{t_0+T} \delta_{X_s} ds \\ &\approx \frac{t_0}{t_0+T}\mu_{t_0} + \frac{T}{t_0+T}\Pi(\mu_{t_0}) \\ &= \mu_{t_0} + \frac{T}{t_0+T}(-\mu_{t_0} + \Pi(\mu_{t_0})). \end{aligned}$$

Thus

$$\frac{\mu_{t_0+T} - \mu_{t_0}}{T} \approx \frac{1}{t_0+T}(-\mu_{t_0} + \Pi(\mu_{t_0})) \overset{t_0 \gg T}{\approx} \frac{1}{t_0}(-\mu_{t_0} + \Pi(\mu_{t_0})).$$

Consequently, it is reasonable to hope to be able to describe μ_t's long term behaviour from the ODE (on $\mathcal{P}(M)$)

$$\dot{\mu} = -\mu + \Pi(\mu). \tag{1.51}$$

Denoting by $L((\mu_t)_t)$ the set of accumulation of $(\mu_t)_t$ and Ψ the flow induced by this ODE, M. Benaïm, M. Ledoux and O. Raimond proved that it is the case.

Theorem 1.28. *(Theorem 2.4, [21]) With probability one, $L((\mu_t)_t)$ is a non empty compact connected subset of*

$$Fix(\Pi) = \{\mu \in \mathcal{P}(M) \mid \Pi(\mu) = \mu\}.$$

Remark 1.4. *The result is in general no longer valid when V is not symmetric, whereas the statements of Theorems* (1.29) *and* (1.30) *remain valid.*

The proof is based on the heuristic described above. For $t, s > 0$, let $\varepsilon_t(s)$ be the measure defined by

$$\varepsilon_t(s) = \int_t^{t+s} (\delta_{X_{e^u}} - \Pi(\mu_{e^u}))du = \int_{e^t}^{e^{t+s}} \frac{\delta_{X_u} - \Pi(\mu_u)}{u}du. \tag{1.52}$$

This measure corresponds to the accumulating error between the evolution of μ_{e^t} driven by (1.48) and the one it would have if it were solving (1.51) during a period of time s. Indeed

$$\begin{aligned} d\mu_{e^t} &= (\delta_{X_{e^t}} - \mu_{e^t})dt \\ &= \Big(((\delta_{X_{e^t}} - \Pi(\mu_{e^t})) - \mu_{e^t} + \Pi(\mu_{e^t})\Big)dt. \end{aligned}$$

As stated in the next theorem, the error term $\varepsilon_t(s)$ is small as t converges to infinity.

Theorem 1.29. *(Theorem 3.6, [19])*

1. *For all $f \in C^\infty(M)$ and every $T > 0$:*
   (a) *There exists a positive constant K such that for all $\delta > 0$,*
   $$\mathbb{P}(\sup_{0 \leqslant s \leqslant T} |\varepsilon_t(s) f| \geqslant \delta \mid \mathcal{F}_{e^t}) \leqslant \frac{K}{\delta^2} \|f\|_\infty e^{-t}.$$
   (b) *Almost surely, $\limsup_{t\to\infty} \frac{1}{t} \log(\sup_{0\leqslant s\leqslant T} |\varepsilon_t(s) f|) \leqslant -\frac{1}{2}$.*
2. *The function $t \mapsto \mu_{e^t}$ is an asymptotic pseudotrajectory for Ψ.*

As a consequence, we have

Theorem 1.30. *(Theorem 3.8, [19]) The limit set $L((\mu_t)_t)$ is almost surely an attractor free set of Ψ.*

The remaining part consists to show that all attractor free sets of Ψ are subsets of $Fix(\Pi)$. This basically follows from the fact that Ψ is gradient like whenever V is symmetric. We refer to section 4 of [21] for the details[7].

If $Fix(\Pi)$ is a set of isolated points, Theorem 1.28 ensures that μ_t converges almost-surely to one of these points.

Question 1.5.
- *Under which conditions, $Fix(\Pi)$ reduces to one point?*
- *If $Fix(\Pi)$ does not reduce to one point, is every point of $Fix(\Pi)$ the limit of $(\mu_t)_t$ with positive probability?*

In order to answer these questions, we assume for the remainder of this subsection that
$$V = V_+ - V_-,$$
where V_+, V_- are Mercer kernels. Recall that a map $K : M \times M \to \mathbb{R}$ is a Mercer kernel if it is a continuous and symmetric function such that
$$\int_M \int_M K(x,y) f(y) \lambda(dy) f(x) \lambda(dx) \geqslant 0$$
for all function $f \in L^2(\lambda(dx))$. Some examples are given in section 2 of [21], but one of them is
$$K(x,y) = \langle x, y \rangle = \sum_{j=1}^{n+1} x_j y_j, \; x, y \in \mathbb{S}^n.$$
An answer to the first question is given by the following theorem.

[7] If the reader is not familiar with the notions involved in the statements of these two theorems, we invite him either to go to section 4.1.1 or to look at [11].

Theorem 1.31. *(Theorem 2.13 and Theorem 2.22, [21])*

1. *If* $V_- = 0$, *then* $Fix(\Pi) = \{\lambda(dx)\}$ *and* $\lim_{t\to\infty} \mu_t = \lambda(dx)$ *almost-surely.*

2. *If* $\sup_{x\in M} V_-(x,x) < 1$ *or* $\sup_{x,y\in M}\left(\frac{V_-(x,x)+V_-(y,y)}{2} - V_-(x,y)\right) < 1$, *then* $Fix(\Pi)$ *reduces to one point.*

Before answering the second question, let us recall some informal definitions[8]. We say that $\mu_* \in Fix(\Pi)$ is non degenerate if the "derivative of $\mu \mapsto -\mu + \Pi(\mu)$" does not vanish at μ_*. In that case, we say that μ_* is a sink, if there exists a neighbourhood $\mathcal{V}$ of μ_* such that for all initial point in $\mathcal{V}$, the solution of (1.51) converges to μ_*; otherwise, we say that μ_* is a saddle.

Theorem 1.32. *(Theorem 2.24 and Theorem 2.26, [21]) Let* $\mu_* \in Fix(\Pi)$ *be non degenerated. Then*

1. *If* μ_* *is a sink, then* $\mathbb{P}(\lim_{t\to\infty} \mu_t = \mu_*) > 0$.

2. *If* μ_* *is a saddle, then* $\mathbb{P}(\lim_{t\to\infty} \mu_t = \mu_*) = 0$.

In view of Theorem 1.31, we need some conditions on V_- to ensure that the limit point is *maybe not* the Riemannian probability measure λ. So, from Theorem 1.32, we obtain

Theorem 1.33. *(Theorem 2.28, [21])*
Let $\rho(V) = \inf\{\langle Vu, u\rangle_\lambda \ s.t\, u \in L^2(\lambda), \langle u, 1\rangle_\lambda = 0, \|u\|_\lambda = 1\}$, *where* $Vu(x) = \int_M V(x,y)u(y)\lambda(dy)$.

1. *If* $\rho(V) > -1$, *then* $\mathbb{P}(\lim_{t\to\infty} \mu_t = \lambda) > 0$.

2. *If* $\rho(V) < -1$, *then* $\mathbb{P}(\lim_{t\to\infty} \mu_t = \lambda) = 0$.

Before completing this subsection, let us enumerate the corresponding results for the special case of the sphere (and the circle) since it has been at the origin of this thesis and of the series of paper [19], [20],[21] and [22].

Theorem 1.34. *(Theorem 4.5, [19]) For* $a \neq 0$, *let* $(X_t)_{t\geqslant 0}$ *be the solution of the SDE*

$$dX_t = dW_t(X_t) - \frac{a}{t}\int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X_0 = x \in \mathbb{S}^n,$$

where $V_u(x) = \langle x, u\rangle$. *Set* $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$.

1. *If* $a \geqslant -(n+1)/2$, *then* $\{\mu_t\}$ *converges almost surely (for the topology of weak* convergence) toward the Riemannian probability measure on* $\mathbb{S}^n$.

[8]The rigorous corresponding definitions are given at page 1723 of [21].

2. *If $a < -(n+1)/2$, then there exists a random variable $\varsigma \in \mathbb{S}^n$ such that $\{\mu_t\}$ converges almost surely toward the measure*

$$\mu_{c,\varsigma}(dx) = \frac{\exp(\beta(a)\langle x, \varsigma\rangle)}{Z_a},$$

where Z_a is the normalization constant, $\beta(a)$ is the unique positive solution to the implicit equation

$$2a\Lambda_n'(\beta) + \beta = 0,$$

$\Lambda_n(\beta) = \log(\int_0^\pi \exp(-\beta\cos(x))\lambda_n(dx))$ *and* $\lambda_n(dx) = \frac{(\sin(x))^{n-1}}{\int_0^\pi (\sin(x))^{n-1}dx}dx.$

For the circle, we have the following picture.

Theorem 1.35. *(Theorem 1.2, [19]) Let $(\theta_t)_t$ be the solution of the real valued SDE*

$$d\theta_t = dW_t - \frac{1}{t}\int_0^t V'(\theta_t - \theta_s)dsdt, \qquad \theta_0 = 0,$$

where $V(x) = 2\sum_{k=1}^n \big(a_k\cos(kx) + b_k\sin(kx)\big)$. Set $\mu_t = \frac{1}{t}\int_0^t \delta_{e^{i\theta_s}}ds$.

i. Suppose there exists $1 \leqslant k \leqslant n$ such that $a_k < -1/2$. Then μ_t almost surely does not converge toward $\lambda(dx)$, the normalized Lebesgue measure on $\mathbb{S}^1 \sim [0, 2\pi]$.

ii. Suppose that for all $1 \leqslant k \leqslant n$, $a_k > -1/2$. Then μ_t converges toward λ with positive probability.

iii. Suppose that one of the two following conditions holds

(a) For all $1 \leqslant k \leqslant n$, $b_k = 0$ and $a_k \geqslant 0$,

(b) For all $1 \leqslant k \leqslant n$, $b_k = 0$, $a_k \leqslant 0$ and $\sum_{k=1}^n a_k > -1/2$.

Then μ_t converges almost surely toward λ.

1.3.2 The non-compact case

The extension to the non-compact case was initiated by A. Kurtzmann during her Phd-thesis ([75]) and it has led to the papers [76] and [73]; the last one being in collaboration with V. Klepstyn.

As in the compact case, the authors investigate the long term of $\mu_t(dx) = \frac{1}{t}\int_0^t \delta_{X_s}(dx)ds$ where $(X_t)_t \subset \mathbb{R}^n$ is the solution of the SDE

$$dX_t = dW_t - \Big(\nabla F(X_t) + \frac{1}{t}\int_0^t \nabla_x V(X_t, X_s)ds\Big)dt, \quad X_0 = 0, \tag{1.53}$$

where $(W_t)_t$ stands for a standard Brownian motion on $\mathbb{R}^n$ and V is a nonnegative C^2 function. Given a finite measure μ, let $\Pi(\mu)$ be the probability measure

$$\Pi(\mu) = \frac{e^{-2(F(x)+V_\mu(x))}}{Z_\mu}dx.$$

where Z_μ is the normalization constant. The strategy used in [76] is similar to the compact case. However the main technical difference is the possibility for the limit set $L((\mu_t)_t)$ to be empty. To avoid this problem, A. Kurtzmann needed to add an extra-potential function F with the following assumptions

(i) (regularity and positivity): F is a C^2 function such that $F \geqslant 1$.

(ii) (growth): There exists $C > 0$ such that for all $x, y \in \mathbb{R}^n$,

$$|\nabla F(x) - \nabla F(y)| \leqslant C\big(\inf(1, |x-y|)\big)\big(V(x) + V(y)\big).$$

(iii) (domination): there exists $K \geqslant 1$ such for all $x, y \in \mathbb{R}^n$,

$$V(x,y) \leqslant K(F(x) + F(y)), \quad |\nabla^2_{xx} V(x,y)| + |\nabla_x V(x,y)| K(F(x) + V(x,y))$$

and

$$\lim_{|x|\to\infty} \sup_{y\in\mathbb{R}^d} \frac{|\nabla F(x)|^2 + 2\langle \nabla F(x), \nabla_x V(x,y)\rangle}{F(x) + V(x,y)} = \infty$$

(iv) (curvature): There exist $\alpha, a > 0$, $\delta > 1$ and $M \in \mathbb{R}$ such that for all $x, y, z \in \mathbb{R}^n$,

$$\langle x, \nabla F(x) + \nabla_x V(x,y)\rangle \geqslant a|x|^{2\delta} - \alpha \text{ and } \langle (\nabla^2 F(x) + \nabla^2_{xx} V(x,y))z, z\rangle \geqslant M|z|^2.$$

The domination condition allows A. Kurtzmann to prove the following tightness result

Theorem 1.36. *(Theorem 3.2, [76]) Almost surely,*

$$\sup_{t\geqslant 0} \int F(x)\mu_t(dx) < \infty.$$

Since tightness is a weak form of compactness by Prohorov's Theorem, one can hope to be able to adapt the machinery developed in the compact case and prove the following theorem.

Theorem 1.37. *(Theorem 3.6, [76]) The map is $t \mapsto \mu_{e^t}$ almost-surely an asymptotic pseudotrajectory for the semi-flow Φ, generated by*

$$\Phi_t(\mu) = e^{-t}\mu + e^{-t} \int_0^t e^u \Pi(\Phi_u(\mu)) du. \tag{1.54}$$

Due to the non compactness of $\mathbb{R}^n$, one has to be more careful in estimating the error measure $\varepsilon_t(s)$ defined in (1.52). To overcome this difficulty, Kurtzmann had to use the curvature condition to show an uniform ultra-contractivity property.

As for the compact case, A. Kurtzmann was able to derive from Theorem 1.37 the following long term behaviour of μ_t.

Theorem 1.38. *(Theorem 1.2, [76]) Under the above assumptions on F and V, we have*

1. *$L((\mu_t)_t)$ is a non empty, weakly compact, invariant by the flow Φ defined in (1.54) and admits no other attractor than itself.*

2. *If V is symmetric, then $L((\mu_t)_t)$ is a subset of $Fix(\Pi)$, the set of fixed point of $\mu \mapsto \Pi(\mu)$.*

If the function V is assumed to have more structure, the function F is useless.

Theorem 1.39. *(Theorem 1.2, [73]) Assume that there exist a function $W \in C^2(\mathbb{R}^n)$ such that $V(x,y) = W(x-y)$, $F = 0$ and*

(i) (spherical symmetry): $W(x) = W(|x|)$.

(ii) (uniform convexity): There exists $C > 0$ such that for all $x \in \mathbb{R}^n$ and $v \in \mathbb{S}^{n-1}$,

$$\frac{\partial^2 W}{\partial v^2} |_x \geqslant C.$$

(iii) (polynomial growth): There exists some polynomial P such for all $x \in \mathbb{R}^n$,

$$|W(x)| + |\nabla W(x)| + \|\nabla^2 W(x)\| \leqslant P(|x|).$$

Then, there exists a unique symmetry density $\rho_\infty : \mathbb{R}^n \to [0,\infty)$ so that, almost surely, there exists a random c_∞ such that $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$ in the $*$-weak topology to

$$\mu_\infty(dx) = \rho_\infty(x - c_\infty)dx.$$

Moreover, there exists $a > 0$ such that the speed of convergence of μ_t toward μ_∞ for the Wasserstein distance is at least $\exp(-a \sqrt[k+1]{\log(t)})$, where k is the degree of P.

This last theorem has two interests. The first one is the absence of the additional potential function F. However, the price to pay is the strict convexity assumption (so that uniqueness of the invariant probability is expected). The second one, and maybe the most interesting, is the rate of convergence.

The philosophy of the proof differs from Theorem 1.37 and the compact case, and it is much more closer to the spirit of [61] with the use, as reference point, of the unique minimum of the potential V_{μ_t}[9]. Interested reader are encouraged to have a look at [73].

1.3.3 A link between the reinforcement by the normalized and the the non-normalized empirical occupation measure

As we have seen, the study of *Self-Interacting Diffusions with a reinforcement by the normalized empirical occupation measure* provided a numerous of interesting results via the analysis of an ordinary differential equation on the space of probability measure $\mathcal{P}(M)$, where M is either $\mathbb{R}^n$ or a compact manifold.

[9] The uniqueness comes from the convexity assumption

So a natural extension is to consider Self-Interacting Diffusions of the form

$$dX_t = dW_t(X_t) - \frac{w(t)}{t}\nabla_x V(X_t, X_s)dsdt, \quad X_0 = x \tag{1.55}$$

where w is a positive non-decreasing function. If $w(t) = t$, we recover the *reinforcement processes by the non-normalized empirical occupation measure*.

The gap between the weight function $w(t) = 1$ and the weight function $w(t) = t$ is studied by O. Raimond in [89] where w converges to infinity, but not too fast, ie there exists positives constants c, t_0 such that for $t \geqslant t_0$, $w(t) \leqslant c\log(t)$ and $|w'(t)| = O(t^{-\gamma})$, with $\gamma \in]0, 1]$.

As a particular case, he considers on the unit sphere $\mathbb{S}^n$ the interaction function

$$V(x, y) = \sum_{i=1}^{n+1} x_i y_i$$

and proves the following Theorem.

Theorem 1.40. *(Theorem 3.1, [89]) Under the above assumptions, there exists a random variable X_∞ in $\mathbb{S}^n$ such that almost surely, $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$ converges weakly towards δ_{X_∞}.*

This Theorem lies between Theorem 1.34 and Theorem 1.27.

1.4 Some other path-dependent processes

In the previous sections, we presented Self Interacting Diffusions where we had a kind of “point by point” interaction between the present position and all its past positions. In a different spirit, solution of Stochastic Differential Equation of the form

$$dX_t = g(t, X([0, t]))dW_t - f(t, X([0, t]))dt$$

have received some attention these last two decades. They are known as *Stochastic Functional Equations* and time delays systems falls into this framework (see [50] and references therein).

The question of interest are therefore more functional like; and in this sense, there are close to SPDEs. Indeed, the main problems are about existence/uniqueness of the solutions and with functional stochastic calculus.

However, in [7], Y. Bakhtin and J.C Mattingly were interested in the existence/uniqueness of an invariant probability and on its properties.

An intermediate model between the stochastic functional equations and the self interacting diffusions presented in the previous is the solution of the "Stochastic Differo-Functional Equation"[10]

$$dX_t = dW_t - f(t, X([0, t]))F'(X_t)dt + g(t, X_t)dL_t, \tag{1.56}$$

[10] I used this term since it is a mixture between classical SDEs and Stochastic Functional Equations, but it is not known in the literature.

where L_t is a non-decreasing adapted process.

Such path-dependent processes are considered by K. Burdzy and co-authors in [9] and [31]. In [9], the first object of analysis is the evolution of a Brownian motion in a bounded, connected domain that accumulates drift when it hits the boundary. Using the notation of (1.56), it yields $F(x) = x$ and

$$f(t, \gamma([0,t])) = \int_0^t n(\gamma(s)) dL_s \text{ and } g(t,x) = n(x) \mathbb{1}_{x \in \partial D},$$

where L is the process' local time on ∂D and $n(x)$ is the unit inward normal vector of D at $x \in \partial D$. In that case, the authors prove that the extended process $\left((X_t, \int_0^t n(X_s) dL_s)\right)_{t \geqslant 0}$ on $D \times \mathbb{R}^n$ admits as invariant measure the product of the uniform distribution over D and a Gaussian measure over $\mathbb{R}^n$.

In [31], they investigate the case

$$f(t, \gamma([0,t])) = -\int_0^t F''(\gamma(s)) ds \text{ and } g(t,x) = 0,$$

for some 2π-periodic real valued C^5 function F and W is a stable Lévy process. Again, they are able to prove that the extended process $\left((e^{iX_t}, -\int_0^t F''(X_s) ds)\right)$ admits an invariant measure given as the product of the uniform distribution on $\mathbb{S}^1$ and a Gaussian measure over $\mathbb{R}^n$.

Finally, let us mention that two other path-dependent processes were considered by K. Burdzy ([30], [32]). In [30], the authors consider a stable process evolving on $\mathbb{S}^1$ until a clock rings. At this time, it jumps to an other point on the circle. The jump time depends on the current position and the target one, but also on all its past via the process $S_t = \int_0^t F(X_s) ds$. As before, it turns out that an invariant measure for (X, S) is the product of the uniform distribution on $\mathbb{S}^1$ and a Gaussian measure over $\mathbb{R}$.

In [32], the state space on which lives the particle X is finite and it jumps from a point to an other one according to some rule which depends on the time spent on each point or only on some particular states. Hence, it has a memory. K. Burdzy and D. White give some conditions on the jump rule to ensure that the extended process particle-memory has an invariant measure such that the marginal distribution of the memory is Gaussian over $\mathbb{R}^d$.

1.4.1 Presentation of the results obtained in the thesis

In this thesis, we investigate the long term behaviour of the solution of the Stochastic Differo-Functional Equation (1.56) when the functional f, g are given by

$$f(t, \gamma([0,t])) = \int_0^t F(\gamma(s)) ds \text{ and } g(t,x) = 0,$$

where F is a 2π-periodic real valued function that does not vanish at its local extrema. This is a joint work with Pierre Monmarché.

Introducing the variable $U_t = \int_0^t F(X_s)ds$, we obtain the following SDE on $\mathbb{S}^1 \times \mathbb{R}$

$$\left\{ \begin{array}{rcl} dX_t & = & dW_t - U_t F'(X_t)dt \\ dU_t & = & F(X_t)dt. \end{array} \right. \tag{1.57}$$

This equation has to be compared with the motivating example of section 1.1.3. The periodicity of the function F allows us to identify X_t with e^{iX_t}.

Let us explain what is going on and assume that $U_0 > 0$. Since the equation satisfied by X_t is gradient like, then X_t is pushed to the minima of F, say for instance x_0, and to remain on a small neighbourhood of x_0 for a while. Depending on the sign of $F(x_0)$, we will have two possible behaviour.

If $F(x_0) > 0$, then U_t increases as long as X_t remains on the small neighbourhood. The consequence is that the slope of the environment around 0 becomes progressively steeper. Hence it is more difficult for the particle to leave that neighbourhood. If it remains stuck forever, the expected behaviour is the convergence of X_t to x_0.

If $F(x_0) < 0$, then U_t decreases as long as X_t remains on the small neighbourhood. So, after a finite times, U_t becomes negative. Thus, there is an inversion of the slope of the environment. Consequently, a local minima is now repelling whereas a local maxima is attracting.

This heuristic explains the following results

Theorem 1.41. *(Theorem 5.1, Chapter 5)*

1. *If there exists a local minima x of F such that $F(x) > 0$ or a local maxima y of F such that $F(y) < 0$, then X_t converges almost surely.*

2. *If for all local minima x of F, one has $F(x) < 0$ and for all local maxima y of F one has $F(y) > 0$, then (1.57) admits a unique invariant probability measure.*

In a second phase, we consider the velocity jump process counterpart, which lies in the family of *Piecewise Deterministic Markov Processes*. Let us give a brief explanation on how this process works.

The process X moves at constant speed on the circle until a clock rings. At that time, the process goes to the opposite direction until the next clock rings; and so on. We let Y denotes the speed-direction component and assume that it takes its values in $\{-1, 1\}$ and U is the process described in (1.57).

Given i.i.d standard exponential random variables E_0, F_0, the next jump occurs after a time $\theta_1 \wedge \theta_2$ where

$$\begin{array}{rcl} \theta_1 & = & \inf \left\{ s > 0, \displaystyle\int_0^s (U_v Y_v F'(X_v))_+ dv > E_0 \right\} \\ \theta_2 & = & \dfrac{1}{\lambda} F_0 \end{array}$$

and $\lambda > 0$ is some deterministic constant.

So from the heuristic described above, one may expect to have an equivalent statement as for Theorem (1.41) for velocity jump process... and that is the case.

1.5 Structure of the thesis

This thesis has six chapters including the Introduction. Each of the Chapters 2, 3 and 4 led to accepted for publication or published papers. However, in order to make them as accessible as possible, we begin chapters 2 and 4 with a reminder of the main tools or concepts involved in those articles.

The purpose of Chapter 5 is the content of a recently submitted paper.

The chapters 2-5 can be read independently from each other, but it worth pointing out that they are interconnected.

Finally, we address in chapter 6 some open problems that this thesis has arisen with some partial results.

Chapter 2

Self-Repelling diffusions on a Riemannian manifold

This chapter is based on the paper [16] accepted for publication in *Probability Theory and related fields*. It is a joint work with M. Benaïm.

The chapter is organised as follows. In the first section, we recall the concepts lying behind the main results. In the second one, we present some results obtained for the solution of the Self-Repelling Stochastic Differential equation

$$dX_t = \sigma dB_t + \Big(u_0 \sin(X_t) - v_0 \cos(X_t) + \int_0^t \sin(X_t - X_s) ds\Big) dt, \qquad (2.1)$$

where $\sigma > 0$. This equation is the toy model at the origin of this thesis. Then, we consider the zero noise case (ie $\sigma = 0$) since apart of its own interest, it shows how the presence of the Brownian motion destroys a rigidity structure. It constitutes the appendix of [16] as well as a part of the paper's introduction.

Finally, we reproduce the core of the article in the next sections. Therefore some redundancies with the Introduction chapter are possible.

Keywords: Self-interacting diffusions, strong Feller property, degenerate diffusions, hypocoercivity, invariant probability measure

MSC primary: 58J65, 60K35, 60H10, 60J60
MSC secondary: 37A25, 37A30

Contents

2.1 Basic tools

2.1.1 The Hörmander condition

The goal of this subsection is to recall what the *Hörmander condition* is since it plays a primordial role in the paper [16].

Let M be a smooth connected d-dimensional manifold (e.g $M = \mathbb{R}^d$ or $M = \mathbb{S}^d$) and let $G_0, G_1, \cdots, G_n$ be smooth vector fields. Let X_t^x be the solution of the SDE

$$dX_t^x = G_0(X_t^x)dt + \sum_{k=1}^{n} G_k(X_t^x) \circ dB_t^{(k)} \tag{2.2}$$

with initial condition $X_0^x = x$. Here $\circ$ stands for the Stratonovich integral and $(B^{(1)}, \cdots, B^{(n)})$ is a standard Brownian motion on $\mathbb{R}^n$. Then the infinitesimal operator is

$$L = \frac{1}{2}\sum_{k=1}^{n} G_k^2 + G_0.$$

We denote by $P_t(x, dy)$ the law of X_t^x and by P_t the semigroup induced by L. Note that for a bounded and measurable function,

$$P_t f(x) = \mathbb{E}(f(X_t^x)).$$

Since the Hörmander condition is connected to the notion of Lie-bracket, let us first recall its definition.

Definition 2.1. *Given two smooth vector fields A and B on M, the Lie-bracket of A and B is the vector field on M characterized by*

$$[A,B](f) = A(B(f)) - B(A(f))$$

for all $f \in \mathcal{C}^\infty(M)$. In case $M = \mathbb{R}^d$, then for all $x \in \mathbb{R}^d$

$$[A,B](x) = DB(x)A(x) - DA(x)B(x)$$

where $DA(x)$ (resp. $DB(x)$) stands for the derivative of A (resp. B) at x.

With this definition in hands, we are now ready to introduce the *Hörmander condition.* Let $\mathcal{G}_0 = \{G_1, \cdots, G_n\}$ and define recursively $\mathcal{G}_k, k \geqslant 1$, by

$$\mathcal{G}_k = \mathcal{G}_{k-1} \cup \{[B, G_j],\ B \in \mathcal{G}_{k-1} \text{ and } j = 0, \cdots, N.\}$$

Let then $\mathcal{G}_\infty = \bigcup_{k\geqslant 0} \mathcal{G}_k$ and for all $x \in M$

$$\mathcal{G}_\infty(x) = \{V(x)\ : V \in \mathcal{G}_\infty\}.$$

Definition 2.2. *The dynamics (2.2) satisfies the Hörmander condition if for all $x \in M$, $\mathcal{G}_\infty(x)$ spans T_xM.*

The intuition behind this definition is that the noise term is sufficiently well "distributed" to the different coefficient of X in order to allow the particle to go at every direction at every point. The strength of the Hörmander condition is its smoothing property.

Theorem 2.1. *(Theorem 3 and Lemma 5.1, [67]) Assume that the Hörmander condition holds. Then there exists a $\mathcal{C}^\infty((0,\infty) \times M \times M)$ function $p_t(x,y)$ such that*

1. *$P_t(x,dy) = p_t(x,y)dy$*
2. *$Lp_t(.,y) = \partial_t p_t(.,y)$*
3. *$L^* p_t(x,.) = \partial_t p_t(x,.)$, where L^* is the adjoint operator of L in $L^2(M,dx)$.*
4. *The semigroup P_t is Strong Feller, ie for all bounded and measurable function f, $P_t f$ is a bounded and continuous function.*

This result was first proven in 1967 by Lars Hörmander ([63]) on an open subset of $\mathbb{R}^d$ where he used analytical tools and arguments. In 1986, James R. Norris gave a probabilistic proof by using Malliavin Calculus ([80]) and it was revisited in 2011 by Martin Hairer in [57].

We emphasize that this result is true only if the state space has *finite* dimension. A particular case in an infinite dimensional setting, with only a finite vector fields modelling the noise term, was studied by M.Hairer and J.C Mattingly in [59]. Their proof uses Malliavin calculus and follows the idea from [80]. However, the semi-group is not Strong Feller, but *asymptotically Strong Feller*; which is a notion that they introduced in [58]. This notion is weaker than the Strong Feller, but stronger than the Feller property. We refer to this paper for the definition (Definition 3.8), its smoothing property (Theorem 3.16) and examples/applications.

2.1.2 Hypocoercivity and functional inequalities

Even if functional inequalities are not playing a key role in the article, they are not very far for two reasons. The first one lies in the fact that they are closely related to rate of convergence to an invariant probability measure. The second one is that we cannot dissociate them from the notion of *Hypocoercivity* since the underlying ideas are similar.

Functional inequalities

The purpose of this subsection is not to give a complete overview of the world of functional inequalities since it is very large and the literature on that subject is important. Here, we present only the Poincaré inequality in order to fix the idea about the power of functional inequalities. This subsection is based on the book [2], written in French.

Let L be the infinitesimal generator of a diffusion process $(X_t)_t$ in a state space M and let P_t be the associated semigroup. We assume that L admits a unique invariant probability measure μ. Let $\mathcal{A}$ be a set of sufficiently smooth function and such that it is dense in $L^2(\mu)$ (see Definition 2.4.2 in [2] for a precise description). For $f \in \mathcal{A}$, we denote by $\mu(f)$ the mean of f with respect to μ

$$\mu(f) = \int_M f(x)\mu(dx).$$

Definition 2.3. *(Definition 2.5.4, [2]) We say that μ satisfies a Poincaré inequality for L with constant $c > 0$ if for all $f \in \mathcal{A}$,*

$$\|f - \mu(f)\|^2_{L^2(\mu)} \leqslant c\||\nabla f|\|^2_{L^2(\mu)}.$$

The Poincaré inequality is useful if one wants to estimate the speed of convergence of the law of X_t to μ.

Theorem 2.2. *(Theorem 2.5.5, [2]) The following assertions are equivalent:*

1. *There exists $\lambda > 0$ such that for all $f \in \mathcal{A}$,*

$$\|P_t f - \mu(f)\|_{L^2(\mu)} \leqslant e^{-\lambda t}\|f\|_{L^2(\mu)}$$

2. *μ satisfies a Poincaré inequality for L for some constant c.*

Proving such an inequality might not be an easy task, especially when the state space is of high dimension. Fortunately, if $L = L_1 + L_2$ and $\mu = \mu_1 \otimes \mu_2$, then one can apply the following theorem.

Theorem 2.3. *(Theorem 3.2.1, [2]) Assume that for $j = 1, 2$, μ_j satisfies a Poincaré inequality for L_j with constant c_j. Then μ satisfies Poincaré inequality for L with constant* $\max(c_1, c_2)$.

If $M = \mathbb{R}^d$ and $L = \Delta - \langle U'(.), \nabla\rangle$, where U is some potential of class $\mathcal{C}^2$ such that $\int_{\mathbb{R}^d} e^{-U(x)}dx < \infty$, then one has the following theorem for the invariant probability measure $\mu_U(dx) = \frac{1}{C(U)}e^{-U(x)}dx$.

Theorem 2.4. *(Corollary 5.5.2 and discussion in section 1.2.6, [2]) If $U = \Phi + G$, where Φ is a strictly convex function and G is bounded, then μ_U satisfies a Poincaré inequality.*

If $M = \mathbb{R}$, we also have

Theorem 2.5. *(Theorem 6.4.3, [2]) Assume there exists $\kappa, K > 0$ such that $|U'(x)| > \kappa$ for $|x| > K$ and $\lim_{|x|\to\infty} U''(x)/(U'(x))^2 = 0$. Then μ_U satisfies a Poincaré inequality.*

Hypocoercitivity

When the Brownian motion term does *not* act directly on all coordinates of a SDE, it is hopeless to prove such functional inequalities. A classical example is the generalised kinetic Langevin equation

$$\begin{aligned} dX_t &= Y_t dt \\ dY_t &= -Y_t dt - U'(X_t)dt + \sqrt{2}dB_t \end{aligned}$$

where $(B_t)_t$ is a real valued Brownian motion and U is a $\mathcal{C}^2$ function such that $x \mapsto e^{-U(x)}$ is integrable for the Lebesgue measure on $\mathbb{R}$. Then, its infinitesimal generator is given by

$$Lf(x,y) = \partial_{yy} f(x,y) - U'(x)\partial_y f(x,y) - y\partial_y f(x,y) + y\partial_x f(x,y)$$

and its invariant probability measure is

$$\mu(dx,dy) = \frac{1}{C} e^{-U(x)-y^2/2} dxdy,$$

where C is a normalisation constant. It is not difficult to prove the following equality: for all $f \in \mathcal{C}^\infty$ converging to 0 as $|x|$ goes to infinity,

$$\int_{\mathbb{R}^2} f(x,y)(Lf)(x,y)\mu(dx,dy) = -\int_{\mathbb{R}^2} (\partial_y f)^2(x,y)\mu(dx,dy). \tag{2.3}$$

Since

$$\frac{d}{dt}\|P_t f - \mu(f)\|^2_{L^2(\mu)} = 2\int_{\mathbb{R}^2} P_t f(x,y)(Lf)(x,y)\mu(dx,dy),$$

it is not possible to obtain a rate of convergence without any modification.

Since the missing terms of the left hand side of (2.3) are partial derivatives, a natural idea is to "add" them. To do this, we consider the Sobolev H^1 norm or an other equivalent norm. This is the method that was first introduced and studied by C. Villani in 2006 in his book [98].

Applying this method to the generalised kinetic Langevin equation, it allowed C. Villani to obtain a rate of convergence that is exponential, like in Theorem 2.2, but for the H^1 norm (Theorem 35 in [98]).

Behind the term "hypocoercitivity", the idea is to change our measurement tool in order to have an inequality that looks like the Poincaré inequality. In [43], J. Dolbeault, C. Mouhot and C. Schmeiser had a different approach from . Villani by assuming that the infinitesimal generator can be decomposed in a symmetric operator in $L^2(\mu)$ and an antisymmetric one. Then they introduced a new object that they called *modified entropy*, which is equivalent to the $L^2(\mu)$ norm. This entropy is built upon some projection operator and the antisymmetric part of L. For more details, we refer to [43], [42] and [56].

We will use this last method in order to obtain our rate of convergence.

2.2 A motivating toy model: case of the circle

In this warm-up section, we consider the following real-valued self-interacting differential equation which is at the origin of this thesis

$$dX_t = \sigma dB_t + \big(u_0 \sin(X_t) - v_0 \cos(X_t)\big)dt + \int_0^t \sin(X_t - X_s)ds dt, \tag{2.4}$$

where $(B_t)_{t\leqslant 0}$ is a standard real valued Brownian motion and $\sigma \geqslant 0$. Since the drift part of the equation is 2π-periodic, we can interpret X_t as an angle. Indeed, if $\Xi_t = \big(\cos(X_t), \sin(X_t)\big)$, it follows from (2.4) and the trigonometric rules that Ξ_t solves

$$d\begin{pmatrix}\Xi_t^{(1)}\\ \Xi_t^{(2)}\end{pmatrix} = \sigma \begin{pmatrix}-\Xi_t^{(2)}\\ \Xi_t^{(1)}\end{pmatrix} \circ dB_t + \Big[\Big\langle \begin{pmatrix}-\Xi_t^{(2)}\\ \Xi_t^{(1)}\end{pmatrix}, \begin{pmatrix}u_0\\ v_0\end{pmatrix}\Big\rangle + \int_0^t \Big\langle \begin{pmatrix}-\Xi_t^{(2)}\\ \Xi_t^{(1)}\end{pmatrix}, \begin{pmatrix}\Xi_s^{(1)}\\ \Xi_s^{(2)}\end{pmatrix}\Big\rangle ds\Big] \begin{pmatrix}-\Xi_t^{(2)}\\ \Xi_t^{(1)}\end{pmatrix} dt \tag{2.5}$$

Setting $U_t = u_0 + \int_0^t \Xi_s^{(1)} ds$ and $V_t = v_0 + \int_0^t \Xi_s^{(2)} ds$ we get the following SDE on $\mathbb{S}^1 \times \mathbb{R}^2$:

$$\begin{cases} d\,\Xi_t &= \sigma \begin{pmatrix}-\Xi_t^{(2)}\\ \Xi_t^{(1)}\end{pmatrix} \circ dB_t + (\Xi_t^{(2)} U_t - \Xi_s^{(1)} V_t) \begin{pmatrix}-\Xi_t^{(2)}\\ \Xi_t^{(1)}\end{pmatrix} dt \\ dU_t &= \Xi_t^{(1)} dt \\ dV_t &= \Xi_t^{(2)} dt \end{cases} \tag{2.6}$$

with initial condition $((\cos(X_0), \sin(X_0)), u_0, v_0)$. This system enjoys the following properties, summarized by the next Theorem and whose proof follows from general results stated in Theorems 2.9,2.10,2.11 and Proposition 2.6.

Theorem 2.6. *Given any initial condition (x_0, u_0, v_0), the Markov process $\Big((\Xi_t, U_t, V_t)\Big)_{t\geqslant 0}$ is positive Harris and admits a unique invariant probability given as*

$$\mu(dxdudv) = \nu(d\xi) \otimes \frac{\exp(-u^2/2)}{\sqrt{2\pi}} du \otimes \frac{\exp(-v^2/2)}{\sqrt{2\pi}} dv,$$

where $\nu(d\xi)$ is the uniform distribution on $\mathbb{S}^1$. Furthermore, the law of $\big(\Xi_t, U_t, V_t\big)$ converges exponentially fast to μ in $L^2(\mu)$ and in total variation.

Remark 2.1. *A similar result holds for the decoupled SDE when $V(x,y) = \sum_{j=1}^n a_j \cos(j(y-x))$ and $a_j > 0$ for all $j = 1, \cdots, n$, by setting $U_j(t) = \int_0^t \cos(jX_s)ds$ and $V_j(t) = \int_0^t \sin(jX_s)ds$.*

Theorem 2.7. *Almost surely, the solution of (2.6) with initial condition $(X_0, U_0, V_0) = ((1,0),0,0)$ does not converge on $\mathbb{S}^1$ and a fortiori on $\mathbb{R}$. However, on $\mathbb{R}$,*

$$\frac{X_t}{t} \to 0 \ a.s.\ as\ t \to \infty.$$

Proof. Let $\varepsilon > 0$ and set $R_j^\varepsilon = \bigcup_{k\in\mathbb{Z}}((2k+j)\pi - \varepsilon, (2k+j)\pi + \varepsilon) \times \mathbb{R}^2$, $j = 0, 1$. Then by positive Harris recurrence of $(\Xi_t, U_t, V_t)_t$ and definition of Ξ_t, we have

$$X_t \in \bigcup_{k\in\mathbb{Z}}((2k+j)\pi - \varepsilon, (2k+j)\pi + \varepsilon),$$

infinitely often for $j = 0, 1$. This proves the first assertion.
Applying now Corollary 2.2 in section 2.5 to the function $f(\xi, u, v) = \xi_2 u - \xi_1 v$ gives us

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(\Xi_s, U_s, V_s)ds = \int_{\mathbb{S}^1\times\mathbb{R}^2} f(\xi, u, v)\mu(dx, du, dv) = 0 \quad \mathbb{P}_{(0,0,0)} a.s.$$

Consequently,

$$\frac{X_t}{t} = \sigma\frac{B_t}{t} + \frac{1}{t}\int_0^t f(\Xi_s, U_s, V_s)ds$$

converges $\mathbb{P}_{(0,0,0)}$ almost surely to 0. □

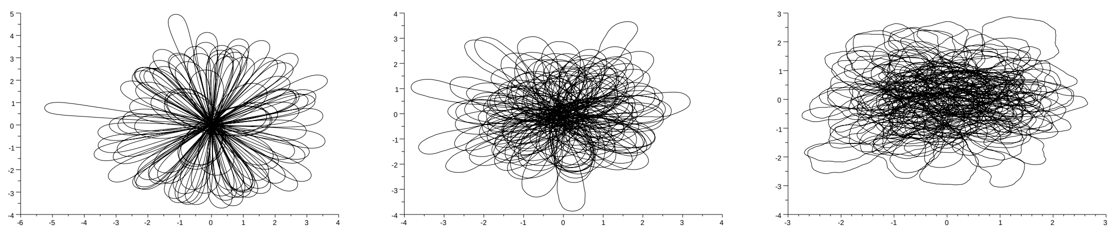

Figure 2.1: Evolution of the coordinate (U_t, V_t) after time $T = 750$, where σ is respectively $0.1, 1$ and 4

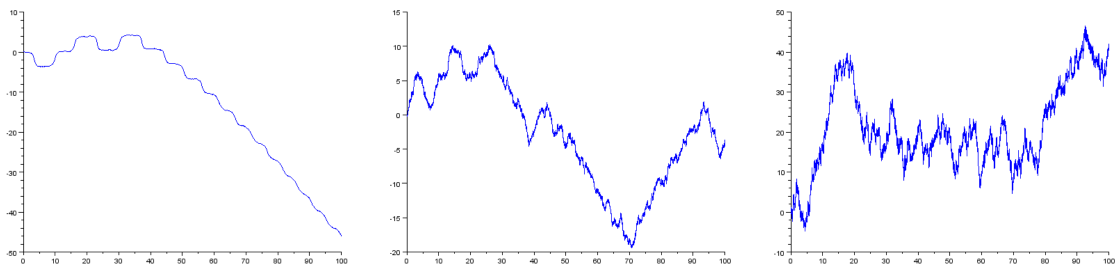

Figure 2.2: Evolution of the angle X_t after time $T = 100$, where σ is respectively $0.1, 1$ and 4.

The zero noise limit

We point out that (2.6) is -for $\sigma \ll 1$- a random perturbation of the following ordinary differential equation (ODE)

$$\begin{cases} \dot{X}(t) &= \sin(X(t))U(t) - \cos(X(t))V(t) \\ \dot{U}(t) &= \cos(X(t)) \\ \dot{V}(t) &= \sin(X(t)) \end{cases} \tag{2.7}$$

Thanks ot the periodicity of the sin(.) and cos(.) functions, we may identify $X(t)$ with $\big(\cos(X(t)), \sin(X(t))\big)$ for simplicity.

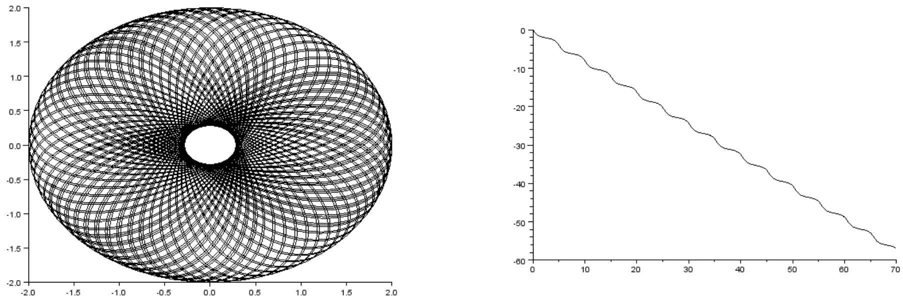

Figure 2.3: Evolution of $(U(t), V(t)$ after time $T = 1000$ (left) and evolution of $X(t)$ until time $T = 70$ (right). Both simulations started with initial condition $(x, u, v) = (0, 0, 2)$.

Since the vectorial field F defined by

$$F(X, U, V) = \begin{pmatrix} (\sin(X)U - \cos(X)V) \\ \cos(X) \\ \sin(X) \end{pmatrix} \tag{2.8}$$

is smooth and sub-linear,it induces a smooth flow $\psi : \mathbb{R} \times (\mathbb{S}^1 \times \mathbb{R}^2) \to \mathbb{S}^1 \times \mathbb{R}^2$. A first and important observation is

Proposition 2.1. *If the initial condition for the ODE (2.7) is*

$$(X_0, U_0, V_0) = (X_0, \cos(X_0), \sin(X_0)),$$

then

$$\psi_t(X_0, U_0, V_0) = (X_0, \cos(X_0)(t+1), \sin(X_0)(t+1)) \; \forall t \in \mathbb{R}.$$

In particular, the line

$$\{(X, Y, Z) \in \mathbb{S}^1 \times \mathbb{R}^2 \;:\; X = X_0, \; \exists t \in \mathbb{R} \;\text{ such that } (Y, Z) = (\cos(X_0)t, \sin(X_0)t)\}$$

is invariant under ψ.

Proof. By the hypothesis, we have $\dot{X}(0) = 0$. Hence $X(t) = X_0$ for all $t \in \mathbb{R}$. Therefore, $U(t) = \cos(X_0)(t+1)$ and $V(t) = \sin(X_0)(t+1)$ □

An immediate consequence is

Corollary 2.1. *If $\dot{X}(0) > 0$ (respectively $\dot{X}(0) < 0$), then $\dot{X}(t) > 0$ (respectively $\dot{X}(t) < 0$) for all t.*

Proof. We proceed by contradiction. Hence, by continuity of $\dot{X}$, there exists t_0 such that $\dot{X}(t_0) = 0$. Then the two last Propositions imply that $\dot{X}(t) = 0$ for all t. In particular $\dot{X}(0) = 0$, which is a contradiction. □

Let

$$\begin{pmatrix} x \\ u \\ v \end{pmatrix} = \Xi(\begin{pmatrix} X \\ U \\ V \end{pmatrix}) = \begin{pmatrix} X \\ \cos(X)U + \sin(X)V \\ -\sin(X)U + \cos(X)V \end{pmatrix}. \tag{2.9}$$

Note that (u, v) is obtained from (U, V) by a rotation of angle $-X$. Then, in the new variable, the ODE (2.7) becomes the ODE

$$\dot{x}(t) = -v(t) \tag{2.10}$$

$$\left\{ \begin{array}{rcl} \dot{u}(t) & = & 1 - v(t)^2 \\ \dot{v}(t) & = & u(t)v(t) \end{array} \right. \tag{2.11}$$

Let

$$H(u, v) = \left\{ \begin{array}{l} \frac{1}{2}(u^2 + v^2 - \log(v^2)), \text{ if } v \neq 0, \\ \infty, \text{ if } v = 0. \end{array} \right. \tag{2.12}$$

Proposition 2.2. *The function H is a first integral for the ODE (2.11).*

Proof. Let $v_0 \neq 0$. Deriving H with respect to t and applying the chain rule, we obtain

$$\begin{aligned} \frac{d}{dt}H(u, v) &= (u\dot{u} + v\dot{v}) - \frac{\dot{v}}{v} \\ &= (u - uv^2 - vuv) - u \\ &= 0 \end{aligned}$$

□

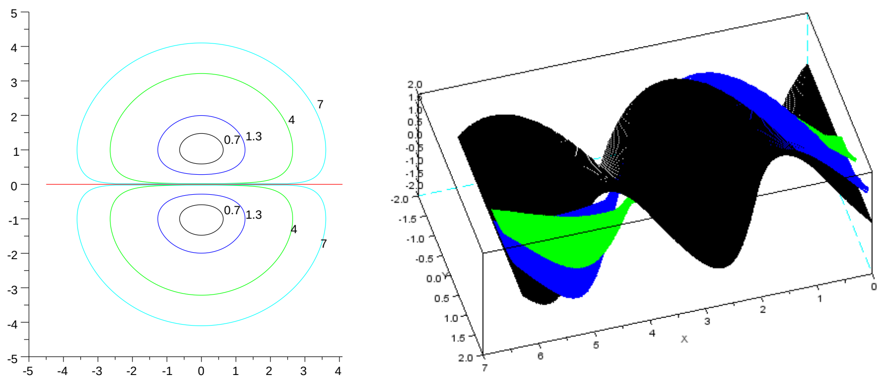

Figure 2.4: The left picture shows level sets of the function H whereas the right picture shows the full twisted strip (in black) and two torus T_c^+, with $c = \sqrt{2}$ (in green) and $c = 2$ (in blue).

Note that H is convex, reaches its global minimum in $(0, \pm 1)$ and takes the value $1/2$ at these points.
For $c \in [1/2, \infty[$, let

$$H_c^+ = H^{-1}(c) \cap \{v > 0\}, \; H_c^- = H^{-1}(c) \cap \{v < 0\}$$

and set $H_\infty = \{v = 0\}$. Then, we define $\mathbb{T}_c^\alpha = \mathbb{S}^1 \times H_c^\alpha$ for $\alpha \in \{+, -\}$ and $T_\infty = \mathbb{S}^1 \times H_\infty$.

Since the function H is strictly convex on $\{v > 0\}$ and $\{v < 0\}$, we observe that $T_{1/2}^\alpha$ is a closed curve, T_c^α a torus and T_∞ a cylinder. A first result is

Proposition 2.3. *Let $(x(t), u(t), v(t))$ be a solution of the ODE defined by (2.10) and (2.11).*

(i) $\mathbb{T}^{\alpha}_{1/2}$ is a periodic orbit with period 2π, $\alpha \in \{+,-\}$

(ii) On T_∞, the dynamic takes the form $(x(t), u(t), v(t)) = (x(0), u(0)+t, 0)$.

For $c > 1/2$, let T_c be the period of (2.11) on H^{α}_c

(iii) If $\frac{x(T_c)}{2\pi} \in \mathbb{Q}$, then every trajectory on T^{α}_c is periodic with period qT_c if the irreducible fraction of $\frac{x(T_c)}{2\pi}$ writes $\frac{p}{q}$.

(iv) If $\frac{x(T_c)}{2\pi} \notin \mathbb{Q}$, then every trajectory on $\mathbb{S}^1 \times H^{-1}(c)$ is dense either on T^+_c or T^-_c.

Proof. Points (i) and (ii) follow immediately from (2.10), (2.11) and the function H.

Without loss of generality, we assume that $x(0) = 0$. Let $c > 1/2$. Because for $m \in \mathbb{N}^*$, we have

$$\begin{aligned} x(mT_c) &= \int_0^{mT_c} \dot{x}(t)dt = -\int_0^{mT_c} v(t)dt \\ &= -m\int_0^{T_c} v(t)dt \\ &= m\int_0^{T_c} \dot{x}(t)dt, \\ &= mx(T_c) \end{aligned} \tag{2.13}$$

we obtain that when $(u(t), v(t))$ is back to its initial condition, then $x(t)$ does a rotation of angle $x(T_c)$. Hence if $\frac{x(T_c)}{2\pi} = \frac{p}{q}$, with $q \in \mathbb{N}^*$, $p \in \mathbb{Z}$ and such that the fraction is irreducible, then

$$\begin{aligned} 2p\pi &= qx(T_c) \\ &= x(qT_c). \end{aligned}$$

This proves (iii).

If $\frac{x(T_c)}{2\pi} \notin \mathbb{Q}$, then $(x(qT_c))_{q\in\mathbb{N}}$ is dense on $\mathbb{S}^1$. Now, assume without lost of generality that $v(0) < 0$ and denote by t_n the time of the n^th passage of $x(t)$ on $x(0)$. By (2.10) and (2.11), t_n does not depend on $x(0)$. Since the flow induced by (2.10) and (2.11) is smooth, then one can defined a smooth function $F : H^-_c \to H^-_c$ such that

$$\big(u(t_{n+1}), v(t_{n+1})\big) = F\big(u(t_n), v(t_n)\big).$$

Since H^-_c is a closed simple curve and F preserves the orientation, then $\Big(\big(u(t_n), v(t_n)\big)\Big)_{n\geq 1}$ is either dense or periodic.

If it is periodic with period t_n, then there exists $q \in \mathbb{N}$ such that $t_n = qT_c$. Therefore, by (2.13), we have $2n\pi = x(qT_c) = qx(T_c)$; so that $\frac{x(T_c)}{2\pi} = \frac{n}{q}$. This is a contradiction.

Since for all $(q, n) \in \mathbb{N} \times \mathbb{N}^*$ we have

$$\Big(x(qT_c + t_n), u(qT_c + t_n), v(qT_c + t_n)\Big) = \Big(x(qT_c), u(t_n), v(t_n)\Big),$$

then the density of $(x(qT_c))_{q\in\mathbb{N}}$ on $\mathbb{S}^1$ and the one of $(u(nT), v(nT))_{n\in\mathbb{N}}$ on H_c^- implies the density of $((x(t), u(t), v(t)))_{t\geqslant 0}$ on T_c^-. This proves (iv). □

From now, we assume without lost of generality that $v(0) < 0$ (the case $v(0) > 0$ being symmetric). In order to derive properties of $c \mapsto T_c$ (see Proposition (2.3)), we change the time scale by use of $t \mapsto x(t)$. This is possible because it is strictly increasing. We denote by y the inverse function of x. Since we have assumed that $x(0) = 0$, it follows that $y(0) = 0$.
Set $u_2(t) = u(y(t))$ and $v_2(t) = v(y(t))$. Therefore (u_2, v_2) is solution to the ODE

$$\begin{cases} \dot{u}_2(t) &= (v_2(t) - \frac{1}{v_2(t)}) \\ \dot{v}_2(t) &= -u_2(t) \end{cases} \tag{2.14}$$

with initial condition $(u(0), v(0))$. Observe that H is still a first integral for this system.

Proposition 2.4. *Let $(x(t), u(t), v(t))$ be a solution to the ODE defined by equation (2.10) with initial condition $(0, u_0, v_0)$ and let $(t, u_2(t), v_2(t))$ where $(u_2(t), v_2(t))$ is the solution to the ODE defined by equation (2.14) with initial condition (u_0, v_0).*
Then $(x(t), u(t), v(t))$ is periodic in $\mathbb{S}^1 \times \mathbb{R}^2$ iff $(t, u_2(t), v_2(t))$ is periodic in $\mathbb{S}^1 \times \mathbb{R}^2$. Further, if T is the period of $(x(t), u(t), v(t))$, then $x(T)$ is the period of $(t, u_2(t), v_2(t))$.

Proof. Straightforward. □

Denote by $T_{c,2}$ the period of $(u_2(t), v_2(t))$, where $c = H(u_2(0), v_2(0)) > 1/2$. Then

$$T_{c,2} = x(T_c). \tag{2.15}$$

An immediate consequence of Propositions 2.3 and 2.4 is that $(t, u_2(t), v_2(t))$ is periodic if and only if

$$\frac{T_{c,2}}{2\pi} \in \mathbb{Q}. \tag{2.16}$$

We complete the zero noise limit's analysis with the study of the *"period-function"*

$$f : (1/2, +\infty) \to \mathbb{R}_+ : c \mapsto T_{c,2}. \tag{2.17}$$

First notice that $(0, 1)$ and $(0, -1)$ are stationary points for the ODE (2.14).
Let $(u_0, v_0) \in \mathbb{R} \times (0, \infty)$. By symmetry of H along the line $v_2 = 0$, what follow remains true for $v_0 < 0$.

Set $c = H(u_0, v_0)$. Since H is a first integral, then $H(u_2(t), v_2(t)) = c$ for all t.
Using the fact that $\dot{v}_2 = -u_2$, we have that

$$\frac{1}{2}\dot{v}_2^2 + (\frac{v_2^2}{2} - \log(v_2)) = c. \tag{2.18}$$

Set $\phi(v) = (\frac{v^2}{2} - \log(v))$.

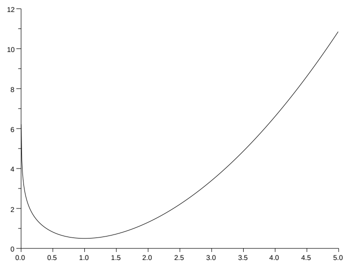

Figure 2.5: Graph of the function ϕ.

Since the curve $H^{-1}(c)$ is symmetric along the line $u_2 = 0$, we have that

$$\frac{T_{c,2}}{2} = \int_{c_1}^{c_2} \frac{dv}{\sqrt{2(c-\phi(v))}}, \tag{2.19}$$

ie

$$T_{c,2} = \sqrt{2}\int_{c_1}^{c_2} \frac{dv}{\sqrt{(c-\phi(v))}}, \tag{2.20}$$

where $0 < c_1 < 1 < c_2 < \infty$ are the roots of the function $v \mapsto \phi(v) - c$.
Denote by h the inverse function of ϕ restricted to $[1, \infty)$ and by g the inverse function of ϕ restricted to $(0, 1)$. By a change of variable, we then obtain

$$\int_{1}^{c_2} \frac{dv}{\sqrt{(c-\phi(v))}} = \int_{\frac{1}{2}}^{c} \frac{h'(v)dv}{\sqrt{(c-v)}} \tag{2.21}$$

and

$$\int_{c_1}^{1} \frac{dv}{\sqrt{(c-\phi(v))}} = -\int_{\frac{1}{2}}^{c} \frac{g'(v)dv}{\sqrt{(c-v)}}. \tag{2.22}$$

Therefore

$$f(c) = T_{c,2} = \sqrt{2}\int_{\frac{1}{2}}^{c} \frac{(h'-g')(v)}{\sqrt{(c-v)}}dv = \int_{\mathbb{R}} \Lambda(v)A(c-v)dv = (\Lambda * A)(c), \tag{2.23}$$

where $*$ stands for the convolution product, $\Lambda(v) = \sqrt{2}(h'-g')(v)\mathbf{1}_{v>1/2}$ and $A(v) = \frac{1}{\sqrt{v}}\mathbf{1}_{v>0}$.
Hence

$$f'(c) = (\Lambda * A')(c). \tag{2.24}$$

Since $g(v) \in (0,1)$ and $h(v) > 1$ for $v \in (1/2, c)$, then $g'(v) = \frac{1}{\phi'(g(v))} < 0$ and $h'(v) = \frac{1}{\phi'(h(v))} > 0$. Using the fact that $A'(v) = -\frac{1}{2}\mathbf{1}_{v>0}\frac{1}{\sqrt{v^3}}$, we have

$$f'(c) < 0 \text{ for all } 1/2 < c < \infty. \tag{2.25}$$

Our next goal is now to study the limiting behaviour $c \to 1/2$ and $c \to \infty$

Lemma 2.1. *Let $c > 1/2$ and let c_1 and c_2 the two roots of the function $v \mapsto \phi(v) - c$. Then*

$$T_{c,2} \geqslant 2\sqrt{2}[\sqrt{\frac{c_1}{1+c_1}} + \sqrt{\frac{c_2}{1+c_2}}].$$

Proof. By convexity of ϕ, we have $\frac{\phi(v)-\phi(c_1)}{v-c_1} \geqslant \phi'(c_1)$. Hence

$$\sqrt{c-\phi(v)} \leqslant \sqrt{-\phi'(c_1)}\sqrt{v-c_1}.$$

Therefore

$$\int_{c_1}^{1} \frac{dv}{\sqrt{c-\phi(v)}} \geqslant \frac{1}{\sqrt{-\phi'(c_1)}} \int_{c_1}^{1} \frac{dv}{\sqrt{v-c_1}} = 2\frac{\sqrt{1-c_1}}{\sqrt{-\phi'(c_1)}}.$$

Since $-\phi'(v) = \frac{1}{v} - v$, $-\phi'(c_1) = (1-c_1^2)/c_1$ and thus

$$\int_{c_1}^{1} \frac{dv}{\sqrt{c-\phi(v)}} \geqslant 2\sqrt{\frac{c_1}{(1+c_1)}}.$$

Once again convexity of ϕ implies $\frac{\phi(c_2)-\phi(v)}{c_2-v} \leqslant \phi'(c_2)$, so that $c-\phi(v) \leqslant \phi'(c_2)(c_2-v)$. By proceeding as above, we obtain

$$\int_{1}^{c_2} \frac{dv}{\sqrt{c-\phi(v)}} \geqslant 2\sqrt{\frac{c_2}{(1+c_2)}}.$$

Hence

$$f(c) = T_{c,2} = \sqrt{2}[\int_{c_1}^{1} \frac{dv}{\sqrt{c-\phi(v)}} + \int_{1}^{c_2} \frac{dv}{\sqrt{c-\phi(v)}}] \geqslant 2\sqrt{2}[\sqrt{\frac{c_1}{1+c_1}} + \sqrt{\frac{c_2}{1+c_2}}].$$

□

Lemma 2.2. $\lim_{c\to 1/2} f(c) = \sqrt{2}\pi$.

Proof. We have $c_1, c_2 \to 1$ as $c \to 1/2$. Thus, it implies that $\log(v) \approx (v-1) - \frac{1}{2}(v-1)^2$ for $v \in (c_1, c_2)$ and therefore

$$\phi(v) = \frac{1}{2}(v-1+1)^2 - \log(v) \approx \frac{1}{2} + (v-1)^2.$$

But

$$\begin{aligned}
\int_{c_1}^{c_2} \frac{dv}{\sqrt{c-\frac{1}{2}-(v-1)^2}} &= \frac{1}{\sqrt{c-\frac{1}{2}}} \int_{c_1-1}^{c_2-1} \frac{dv}{\sqrt{1-(v/\sqrt{c-\frac{1}{2}})^2}} \\
&= \int_{\frac{c_1-1}{\sqrt{c-1/2}}}^{\frac{c_2-1}{\sqrt{c-1/2}}} \frac{du}{\sqrt{1-u^2}} \\
&= \arcsin(\frac{c_2-1}{\sqrt{c-1/2}}) + \arcsin(\frac{1-c_2}{\sqrt{c-1/2}})
\end{aligned}$$

Since for c sufficiently close to $1/2$, $c = \phi(1+c_j-1) \approx \frac{1}{2}+(c_j-1)^2$, then $\lim_{c\to 1/2} \frac{|c_j-1|}{\sqrt{c-\frac{1}{2}}} = 1, j = 1, 2$.
Thus, $\lim_{c\to 1/2} \int_{c_1}^{c_2} \frac{dv}{\sqrt{c-(v-1)^2}} = \pi$ and therefore

$$\lim_{c\to 1/2} f(c) = \lim_{c\to 1/2} T_{c,2} = \lim_{c\to 1/2} \sqrt{2}\int_{c_1}^{c_2} \frac{dv}{\sqrt{c-\frac{1}{2}-(v-1)^2}} = \sqrt{2}\pi. \tag{2.26}$$

□

Remark 2.2. *One can prove that $\sqrt{2}\pi$ is the period of the orbits from the linear ODE*

$$\begin{cases} \dot{u}(t) &= 2v(t) \\ \dot{v}(t) &= -u(t). \end{cases} \tag{2.27}$$

But this is nothing else than the linearized system at $(0, 1)$ from the ODE (2.14).

Summarizing all these information concerning $T_{c,2}$, we obtain

Proposition 2.5. *The "period-function" $f : (1/2, \infty) \to \mathbb{R}_+ : c \mapsto T_{c,2}$ is continuous, decreasing, bounded from below by $2\sqrt{2}$ and converge to $\sqrt{2}\pi$ when c tends to $1/2$.*

Proof. The decreasing property comes from (2.25) whereas the continuity follows from (2.23). While c_1 converges to 0 and $\frac{c_2}{1+c_2}$ converges to 1 when c tends to ∞, then Lemma 2.1 combined with the decreasing property implies that $f(c) \geqslant 2\sqrt{2}$ for all $c > 1/2$.

Since f is decreasing, then $\sup_{c>1/2} f(c) = \lim_{c\to 1/2} f(c) = \sqrt{2}\pi$. □

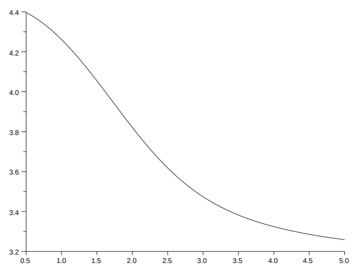

Figure 2.6: Graph of the function $c \mapsto T_{c,2}$.

2.3 Introduction to the general case

Let M be a smooth (i.e $\mathcal{C}^\infty$) Riemannian manifold, $V : M \times M \to \mathbb{R}$ a smooth function and $w : [0, \infty[\to [0, \infty[$ a continuous function. Adopting the terminology now coined in the literature we define a *Self Interacting Diffusion with potential V and weight function w* to be a continuous time stochastic process $(X_t)_{t\geq 0}$ living on M defined by the stochastic differential equation

$$dX_t = \sigma dB_t(X_t) - \nabla V_t(X_t)dt, \tag{2.28}$$

where $\sigma > 0$, $\{B_t\}$ is a Brownian vector field[1] on M and

$$V_t(x) = w_t \int_0^t V(X_s, x) ds, \tag{2.29}$$

The case M compact and $w_t = t^{-1}$ has been thoroughly analyzed in a series of papers by the first named author in collaboration with Raimond ([19], [20], [21]) and Ledoux [19]. In particular, it was shown that long term behavior of the normalized occupation measure $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$ can be precisely related to the long term behavior of a deterministic semi-flow defined on the space of probability measures over M. Pemantle's survey paper ([84]) contains a comprehensive discussion of these results among others and further references. Some extensions to noncompact spaces have been considered by Kurtzmann in [73], [76] and other weight functions decreasing to zero by Raimond in [89].

When w doesn't converge to zero, say $w_t = 1$, the literature on the subject mainly consists of case studies under the assumption that $M = \mathbb{R}$ (or $\mathbb{R}^d$) and $V(x, y) = v(y - x)$. Self attracting processes, that is $xv'(x) \geq 0$ (or $\langle x, v'(x)\rangle \geq 0$ in $\mathbb{R}^d$), have been considered by Cranston and Le Jan [35], Raimond [88], Herrmann and Roynette [60], Herrmann and Scheutzow [61] and typically converge almost surely. For self repelling processes, that is $xv'(x) \leq 0$, the process tends to be "transient" and strong law of large numbers and rate of escapes have been obtained under various assumptions by Cranston and Mountford [36], Durrett and Rogers [44], Mountford and Tarrès [79]. In [96], Tarrès, Tóth and Valkó consider the situation when v is a sufficiently smooth function having a nonnegative Fourier transform. Under this condition and other technical assumptions, they show that the *environment seen from* X_t, that is the mapping $x \mapsto \int_0^t v'(x + X_t - X_s) ds$, admits an ergodic invariant Gaussian measure.

In this paper we will pursue this line of research and investigate the long term behaviour of (2.28) under the assumptions that:

(i) (**Strong interaction**) $w_t = 1$.

(ii) (**Compactness**) M is smooth, finite dimensional, compact, oriented, connected and without boundary.

(iii) (**Self repulsion**) V is a *Mercer kernel*. That is, $V(x, y) = V(y, x)$ and

$$\int_M \int_M V(x, y) f(x) f(y) dx dy \geqslant 0$$

for all $f \in L^2(dx)$, where dx stands for the Riemannian measure.

By Mercer Theorem, V can be written as

$$V(x, y) = \sum_i a_i e_i(x) e_i(y) \tag{2.30}$$

where $a_i \geq 0$ and $\{e_i\}$ is an orthonormal (in $L^2(dx)$) family of eigenfunctions of the operator $f \mapsto T_V f$, where $T_V f(x) = \int V(x, y) f(y) dy$.

[1] See next subsection for a precise formulation

Thus, if one interpret the sequence

$$\Psi(x) = (\sqrt{a_i} e_i(x))_i$$

as a *feature vector* representing x in l^2,

$$V(x, y) = \langle \Psi(x), \Psi(y) \rangle_{l^2}$$

can be thought of as a similarity between the feature vectors $\Psi(x)$ and $\Psi(y)$. The process is therefore *self-repelling* in the sense that the drift term $-\nabla V_t(X_t)$ in equation (2.28) tends to minimize the similarity between the current feature vector $\Psi(X_t)$ and the cumulative feature $\int_0^t \Psi(X_s) ds$.

Here we will focus on the particular situation where the operator T_V commute the Laplacian on M, so that we assume the following additional assumption.

(iii') (**Diagonal decomposition**) The sum in (2.30) is finite and the $\{e_i\}$ are eigenfunctions of the Laplace operator.

Our motivation for such a restriction is twofold. First, for a suitable choice of n and (a_i), the feature map

$$\Psi : M \mapsto \mathbb{R}^n,$$

$$x \mapsto (\sqrt{a_1} e_1(x), \ldots, \sqrt{a_n} e_n(x))$$

is a quasi-isometric embedding of M in $\mathbb{R}^n$ in the sense that

$$\Phi_\lambda : (M, dx) \to \mathbb{R}^{N(\lambda)+1} : x \mapsto \Big(\sum_{j=0}^{N(\lambda)} \phi_j^2(x) \Big)^{-1/2} \Big(\phi_0(x), \phi_1(x), \ldots, \phi_{N(\lambda)}(x) \Big)$$

is an isometric embedding for $\lambda > 0$ large enough. Here dx is the standard Riemannian metric, $0 = \mu_0 > \mu_1 \geq \mu_2 \geq \ldots$ is the spectrum of the Laplacian operator on M, the sequence $(\phi_j)_{j \geq 0}$ is an orthonormal basis of $L^2(M, dx)$ such that $\Delta_M \varphi_j = \mu_j \varphi_j$ and for any $\lambda > 0$, $N(\lambda) = Card\{j \geq 1 \ : \ |\mu_j| \leq \lambda\}$.

We refer the reader to the recent paper (Portegies 2015 [86]) for a precise statement (Theorem 5.1), and further interesting discussions and references on embedding by eigenfunctions. In particular, for some $\varepsilon > 0$

$$-V(x, y) \leq \frac{1}{2} \|\Psi(x) - \Psi(y)\|^2 \leq (1 + \varepsilon) \frac{d(x, y)^2}{2},$$

where d stands for the Riemannian distance on M. Hence, with this choice of (a_i), the smaller is $V_t(X_t)$ the larger is the cumulative quadratic distance $\int_0^t d^2(X_t, X_s) ds$.

Secondly, under hypothesis $(iii)'$, an invariant probability measure of the process $(X_t, V_t(x))$ can be explicitly computed. It turns out that this will be of fundamental importance for our analysis.

2.4 Description of the model

Let us start by fixing some notation. Throughout all the paper, we let ∇ denote the gradient on M, Δ_M the Laplacian on M and for some vector field $\mathcal{X}$ on a manifold $\mathcal{N}$, we denote by $\mathcal{X}(f)$ the Lie derivative of f along $\mathcal{X}$; f being a smooth function.

For a smooth function $V : M \times M \to \mathbb{R}$ and for a Borel measure μ, we let $V\mu : M \to \mathbb{R}$ denotes the function defined by

$$V\mu(x) = \int_M V(u, x)\mu(du).$$

We then consider the model

$$dX_t = \sigma \sum_{j=1}^{N} F_j(X_t) \circ dB_t^{(j)} - \nabla V\mu_t(X_t)dt, \qquad X_0 = x, \tag{2.31}$$

where $\sigma > 0$, $(B^{(1)}, \cdots, B^{(N)})$ is a standard Brownian motion on $\mathbb{R}^N$, $\circ$ denotes the Stratonovitch integral, $\{F_i\}$ is a family of smooth vectors fields on M such that

$$\sum_{i=1}^{N} F_i(F_i f) = \Delta_M f, \; f \in \mathcal{C}^\infty$$

and μ_t is the random occupation measure defined by

$$\mu_t = \int_0^t \delta_{X_s} ds.$$

Note that there exists at least one such family $\{F_i\}$ since by Nash's embedding Theorem, there exists $N \in \mathbb{N}$ large enough such that M is isometrically embedded in $\mathbb{R}^N$ with the standard metric (see Theorem 3.1.4 in [65] or Proposition 2.5 in [19]).
In this paper, we suppose that the function V has the following form

$$V(x, y) = \sum_{j=1}^{n} a_j e_j(x) e_j(y), \tag{2.32}$$

where $(e_j)_{j=1,\cdots,n}$ are eigenfunctions for the Laplacian associated to non zero eigenvalues $\lambda_1, \cdots, \lambda_n < 0$ such that

$$\int_M e_j(x) e_k(x) dx = \delta_{k,j},$$

where $\delta_{k,j}$ is the Kronecker symbol and dx stands for the Riemannian measure on M. We also assume that $a_j > 0$ for all $j = 1, \cdots, n$.
Due to the particular form for V, we can obtain a "true" stochastic differential equation by introducing the new variables $U_{k,t} = \int_0^t e_k(X_s)ds$. Therefore we get the following system on $\mathbb{M} := M \times \mathbb{R}^n$

$$\begin{cases} dX_t &= \sigma \sum_{j=1}^{N} F_j(X_t) \circ dB_t^{(j)} - \sum_{j=1}^{n} a_j \nabla e_j(X_t) U_{j,t} dt \\ dU_{k,t} &= e_k(X_t)dt, \qquad\qquad k = 1, \cdots, n \end{cases} \tag{2.33}$$

with initial condition $(x, 0, \cdots, 0)$. In the rest of the paper, we will work with the system (2.33) and prove that:

1. There exists a unique global strong solution for the system (2.33);
2. Strong Feller property holds;
3. The system admits a unique invariant measure which is given explicitly as the product of the uniform probability on M and a Gaussian probability on $\mathbb{R}^n$;
4. The law of the solution converges to μ exponentially fast.

The paper is organized as follows. In the next section, we present the main results and the proof of point 1.

In section 2.6, we provide the proofs of points 2 and 3. To this end, we introduce a property, called *condition* (E') and prove that it implies the Strong Feller property.

In section 2.7 is given the proof of an exponential decay in $L^2(\mathbb{M}, \mu)$, where μ is the unique invariant probability whereas a proof for an exponential decay in the Total Variation norm is presented in Section 2.8.

2.5 Presentation of the results

Recall that $\mathbb{M} = M \times \mathbb{R}^n$. Throughout, we denote by $\mathcal{C}_0(\mathbb{M})$ the set of function $f : \mathbb{M} \to \mathbb{R} : (x, u) \mapsto f(x, u)$ which are continuous and such that $f(x, u) \to 0$ when $\|u\| \to \infty$, and by $\mathcal{C}_c^k(\mathbb{M})$ the set of function which are k times continuously differentiable with compact support.
We equip $\mathcal{C}_0(\mathbb{M})$ with the supremum norm

$$\|f\|_\infty := sup_{y\in\mathbb{M}}|f(y)|.$$

Let G_0, $G_1, \cdots$, G_N be the vector fields on $\mathbb{M}$ defined by

$$G_0(x,u) = \begin{bmatrix} -\sum_{j=1}^n a_j \nabla e_j(x) u_j \\ e_1(x) \\ \vdots \\ e_n(x) \end{bmatrix},$$

and for $j = 1, \cdots , N$,

$$G_j(x,u) = \begin{bmatrix} \sigma F_j(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

with $x \in M$ and $u \in \mathbb{R}^n$. So (2.33) can be rewritten as:

$$dY_t = \sum_{j=1}^N G_j(Y_t) \circ dB_t^j + G_0(Y_t)dt. \tag{2.34}$$

Proposition 2.6. *For all $y = (x, u) \in \mathbb{M}$ there exists a unique global strong solution $(Y_t^y)_{t \geqslant 0}$ to (2.34) with initial condition $Y_0^y = y = (x, u)$. Moreover, we have*

$$Y_t^y = (X_t^y, U_t^y) \in M \times \bar{B}(u, Kt), \tag{2.35}$$

where $K = (\max_{y \in M} \sum_{j=1}^n e_j(y)^2)^{1/2}$ and $\bar{B}(u, R) = \{v \in \mathbb{R}^n \ : \ \|v - u\| \leqslant R\}$.

Proof. Existence and uniqueness is standard since G_0 is locally Lipschitz and sub-linear (see for example [92], page 383). Concerning (2.35), note that we have

$$\begin{aligned} \sum_{j=1}^n (U_{j,t} - u_j)^2 &\leqslant t \int_0^t \sum_{j=1}^n e_j(X_s)^2 ds \\ &\leqslant t^2 \max_{y \in M} \sum_{j=1}^n e_j(y)^2 < \infty; \end{aligned}$$

which proves (2.35). □

Throughout, we let $(P_t)_{t \geqslant 0}$ denote the semi-group induced by (2.34). Recall that for any bounded or nonnegative measurable function $f : \mathbb{M} \to \mathbb{R}$, $P_t f$ is the function defined by

$$P_t f(y) = \mathbb{E}(f(Y_t^y)) \text{ for all } y \in \mathbb{M}. \tag{2.36}$$

Lemma 2.3. *The semi-group $(P_t)_{t \geqslant 0}$ is Feller, meaning that*

1. *For all $t \geqslant 0$, $P_t(\mathcal{C}_0(\mathbb{M})) \subset \mathcal{C}_0(\mathbb{M})$.*
2. *For all $f \in \mathcal{C}_0(\mathbb{M})$, $\lim_{t \to 0} \|P_t f - f\|_\infty = 0$.*

Proof. By Proposition 2.6, for all $T > 0$, $(Y_t^y)_{t \in [0,T]}$ lies on a deterministic compact set depending only y and T. Hence, by standard results (see eg Theorem IX.2.4 in [92]), $y \mapsto Y_t^y$ is continuous. Thus, by dominated convergence, $y \mapsto P_t f(y)$ lies in $\mathcal{C}_0(\mathbb{M})$ for all $f \in \mathcal{C}_0(\mathbb{M})$.
In order to prove the second point, it suffices to show that $\lim_{t \downarrow 0} P_t f(y) = f(y)$ (see Proposition III.2.4 in [92]). This follows again from continuity of $t \mapsto Y_t^y$ and dominated convergence. □

The next result gives further informations on the semi-group.

Proposition 2.7. *The set $\mathcal{C}_c^2(\mathbb{M})$ is stable for P_t, $t \geqslant 0$, ie for all $t \geqslant 0$, $P_t(\mathcal{C}_c^2(\mathbb{M})) \subset \mathcal{C}_c^2(\mathbb{M})$.*

Proof. Let $f \in \mathcal{C}_c^2(\mathbb{M})$. The fact that $P_t f$ has a compact support is a consequence of Equation (2.35). Let us now prove that $P_t f$ is twice continuously differentiable.
Let $y = (x_0, u) \in \mathbb{M}$ and $R > 0$. For $\tilde{y} \in M \times B(u, R)$, we have, by Proposition 2.6,

$$(Y_s^{\tilde{y}})_{0 \leqslant s \leqslant t} \subset M \times \bar{B}(u, Kt + R). \tag{2.37}$$

Pick a smooth function $\psi : \mathbb{R}^n \to \mathbb{R}_+$ which is 1 on the ball $B(u, Kt+R)$, 0 outside the ball $\bar{B}(u, Kt+R+1)$ and $\psi(v) \leqslant 1$ for all v.
Consider now the SDE defined by

$$d\tilde{Y}_t = \sum_{j=1}^{N} G_j(\tilde{Y}_t) \circ dB_t^j + \tilde{G}_0(\tilde{Y}_t)dt, \tag{2.38}$$

where $\tilde{G}_0(x,v) = G_0(x, u+\psi(v)(v-u))$. Let us denote by $\tilde{P}_t$ its associated semi-group. The fact that G_0 is smooth and locally Lipschitz implies that $\tilde{G}_0$ is smooth and Lipschitz. By Nash's embedding Theorem and proceeding in the same way as in Proposition 2.5 in [19], we can extend (2.38) to a SDE on $\mathbb{R}^N \times \mathbb{R}^n$ and f to a function in $\mathcal{C}^2(\mathbb{R}^N \times \mathbb{R}^n)$. Therefore, in view of subsection 3.2.1 in [37] and of Proposition 2.5 in [38], it follows that $\tilde{P}_s f$ is a function of class $\mathcal{C}^2$ for all $s \geqslant 0$. Since

$$P_s f(\tilde{y}) = \tilde{P}_s f(\tilde{y}) \text{ for all } 0 \leqslant s \leqslant t \text{ and all } \tilde{y} \in M \times B(u,R), \tag{2.39}$$

it follows that $P_t f$ is of class $\mathcal{C}^2$ on $M \times B(u,R)$.
Consequently, $P_t f \in \mathcal{C}_c^2(\mathbb{M})$. □

The infinitesimal generator of $(P_t)_{t \geqslant 0}$ is the operator

$$\mathcal{L} : D(\mathcal{L}) \to \mathcal{C}_0(\mathbb{M}) : f \mapsto \lim_{t \downarrow 0} \frac{P_t f - f}{t}, \tag{2.40}$$

where $D(\mathcal{L}) := \{f \in \mathcal{C}_0(\mathbb{M}) : \frac{P_t f - f}{t}$ converges in $\mathcal{C}_0(E)$ when $t \downarrow 0\}$.Then (see for example Theorem 17.6 in [69]) for all $f \in D(\mathcal{L})$,

$$P_t f - f = \int_0^t \mathcal{L}(P_s f)ds = \int_0^t P_s(\mathcal{L}f)ds \tag{2.41}$$

We briefly recall the following result which characterize the elements of $D(\mathcal{L})$:

Theorem 2.8. *(Propositions VII.1.6 and VII.1.7 in [92])*
For $g, h \in \mathcal{C}_0(\mathbb{M})$, the following assertions are equivalent:

1. *$h \in D(\mathcal{L})$ and $\mathcal{L}h = g$.*

2. *For all $y \in E$, the process*
$$h(Y_t^y) - \int_0^t g(Y_s^y)ds$$
is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(Y_s^y : 0 \leqslant s \leqslant t)$.

Since the definition of the infinitesimal generator is implicit, it is convenient to introduce a more tractable operator: the *Kolmogorov operator.*

Definition 2.4. *The Kolmogorov operator associated to (2.33) is the operator defined on $\mathcal{C}^2$ bounded functions having first and second bounded derivatives by*

$$L = \frac{\sigma^2}{2}\Delta_M - \sum_{k=1}^{n} a_k u_k (\nabla e_k(x), \nabla_x \cdot)_{TM} + \sum_{k=1}^{n} e_k(x)\partial_{u_k},$$

with the convention $(\Delta_M f)(x,u) = (\Delta_M f(.,u))(x)$ and $(.,.)_{TM}$ stands for the inner product on the tangent bundle of M.

The link between the infinitesimal and the Kolmogorov operator is given by the next proposition.

Proposition 2.8. *Let f be a $\mathcal{C}^2$ bounded function having first and second bounded derivatives, then $f \in D(\mathcal{L})$ and*

$$Lf = \mathcal{L}f.$$

Proof. It follows from Itô's formula and Theorem 2.8. □

Definition 2.5. *Let $\Phi : \mathbb{R}^n \to \mathbb{R} : u = (u_1 \cdots, u_n) \mapsto \ln(C(\Phi)) + \frac{1}{2}\sum_{k=1}^{n} a_k |\lambda_k| u_k^2$ with*

$$C(\Phi) = \int_{\mathbb{R}^n} \exp(-\frac{1}{2}\sum_{k=1}^{n} a_k |\lambda_k| u_k^2) du = \prod_{i=1}^{n} \sqrt{\frac{2\pi}{|\lambda_i| a_i}} < \infty.$$

Recall that $\lambda_i < 0$ is the eigenvalue associated to the eigenfunction e_i of Δ_M. On $\mathbb{M}$, we define the probability measure

$$\mu(dx \otimes du) = \nu(dx) \otimes e^{-\Phi(u)} du =: \varphi(y) dy, \tag{2.42}$$

with $y = (x,u)$ and $\nu(dx) = \frac{dx}{\int_M dz}$ is the uniform probability measure on M.

Remark 2.3. *Note that $\mu(dy)$ does not depend on the noise term σ.*

We can now state our first main result.

Theorem 2.9. *Let $(P_t)_{t \geqslant 0}$ be the semi-group associated to the system (2.33) and $P_t(y_0, dy)$ its transition probability. Then*

1) *The semi-group $(P_t)_{t\geqslant 0}$ is strongly Feller (meaning that $P_t f$ is a bounded continuous function for whatever bounded measurable function f) and there exists a $\mathcal{C}^\infty((0,\infty), \mathbb{M}, \mathbb{M})$ function $p_t(y_0, y)$ such that $P_t(y_0, dy) = p_t(y_0, y)dy$ for all $y_0 \in \mathbb{M}$ and $(L_z^* - \partial_t) p_t(y,z) = 0$,*

2) *The probability $\mu(dy) = \varphi(y)dy$, where φ is given in Definition 2.5, is the unique invariant probability. Moreover for all $y \in \mathbb{M}$ and for all bounded measurable function f, we have*

$$\lim_{t\to\infty} P_t f(y) = \int_{\mathbb{M}} f(z)\mu(dz).$$

Furthermore, the process $(Y_t)_t$ is positive Harris recurrent, ie for all Borelian set R such that $\mu(R) > 0$, then

$$\int_0^\infty \mathbf{1}_R(Y_t^y)dt = \infty \ a.s$$

for all $y \in \mathbb{M}$.

3) $\lim_{t\to\infty} \int_{\mathbb{M}} |p_t(z,y) - \varphi(y)|dy = 0$ *for all $z \in \mathbb{M}$.*

Remark 2.4. *The fact that μ is independent of the parameter σ implies that it is also an invariant probability of the deterministic system obtained with $\sigma = 0$. However, in that case it is not necessarily unique (compare with Theorem 2.6, where there exists infinitely many compact disjoint invariant sets, thus infinitely many ergodic probabilities.)*

As an immediate consequence of the Harris positive recurrence property, we have

Corollary 2.2. *For all $f \in L^1(\mu)$,*

$$\frac{1}{t}\int_0^t f(Y_s^y)ds \to \int_{\mathbb{M}} f(y)\mu(dy)$$

almost surely for any $y \in \mathbb{M}$.

Proof. Apply Theorem 3.1 in [6] to the positive and negative part of f. □

The next results establish exponential rate of convergence of $(P_t)_{t\geqslant 0}$ to μ.

Theorem 2.10. *For every $\eta > 0$ and $g \in L^2(\mu)$*

$$\|P_t g - \int_{\mathbb{M}} g(y)\mu(dy)\|_{L^2(\mu)} \leqslant \sqrt{1+2\eta}\|g - \int_{\mathbb{M}} g(y)\mu(dy)\|_{L^2(\mu)} e^{-\lambda t},$$

where

$$\lambda = \frac{\eta}{1+\eta}\frac{K_1\sigma^2}{1+K_2\sigma^2+K_3\sigma^4},$$

with

$$K_1 = \frac{1}{4(2+(1+N_2)^2)}(\frac{\Lambda}{1+\Lambda})^2,$$

$$K_2 = \frac{(1+N_2)\sum_{j=1}^n|\lambda_j|}{2+(1+N_2)^2},$$

$$K_3 = \frac{(\sum_{j=1}^n|\lambda_j|)^2}{4(2+(1+N_2)^2)},$$

$$\Lambda = \min_{i=1,\dots,n}|\lambda_i|a_i$$

and

$$N_2 = 2\frac{n}{\min\{|\lambda_j|, j=1,\cdots,n\}}\sup_{i=1,\dots,n}\|\nabla e_i\|_\infty^2\sqrt{4+\sum_{i=1}^n|\lambda_i|a_i+4\|\sum_i e_i^2\|_\infty}.$$

Remark 2.5. *Note that if $g \in L^2(\mu)$, then it is not clear at first glance that $P_t g$ is meaningful. However it is. In order to prove it, set $h_t(y,z) = p_t(y,z)/\varphi(z)$. Due to the properties of $p_t(y,.)$ and φ for all $t > 0$ and $x \in \mathbb{M}$ (see Theorem 2.9, Proposition 2.6 and Definition 2.5), then $h_t(y,.)$ has compact support. Thus, by the Cauchy-Schwarz inequality, we obtain*

$$\mathbb{E}(|g|(Y_t^y)) = \int_{\mathbb{M}} |g|(z)p_t(y,z)dz = \int_{\mathbb{M}} |g|(z)h_t(y,z)\mu(dz) \leqslant \|g\|_{L^2(\mu)}\|h_t(y,.)\|_{L^2(\mu)}. \tag{2.43}$$

Furthermore, we have $P_t g \in L^2(\mu)$. Indeed by Jensen inequality and invariance of μ, we have $\int_{\mathbb{M}}(P_t g)^2(y)\mu(dy) \leqslant \int_{\mathbb{M}} P_t(g^2)(y)\mu(dy) = \int_{\mathbb{M}} g^2(y)\mu(dy) < \infty$.

Since both $\mu(dy)$ and $P_t(y_0, dy)$ have smooth densities with respect to the Lebesgue measure for all $y_0 \in \mathbb{M}$ and in view of the third point of Theorem 2.9, we would hope to get a convergence speed for the total variation norm. Once again the answer is positive as shown by the following theorem.

Theorem 2.11. *For all $z_0 \in \mathbb{M}$ and $t \geqslant 1$,*

$$\|P_t(z_0, dz) - \mu(dz)\|_{TV} \leqslant \sqrt{1+2\eta}\|h(1,z_0,z) - 1\|_{L^2(\mu)}e^{-\lambda(t-1)},$$

where $h(1,z_0,z) = \frac{p_1(z_0,z)}{\varphi(z)}$ and

$$\lambda = \frac{\eta}{1+\eta}\frac{K_1\sigma^2}{1+K_2\sigma^2+K_3\sigma^4},$$

μ is the probability given in Theorem 2.9 and the constants $K_j < \infty$, $j = 1,2,3$, are the same as in Theorem 2.10.

The proofs of Theorem 2.10 and 2.11 are postponed to sections 2.7 and 2.8.

2.6 Proof of Theorem 2.9

We emphasize, from Equation (2.34), that the Kolmogorov operator L can be expressed in Hörmander's form (as a sum of squares):

$$L = \frac{1}{2}\sum_{j=1}^{N} G_j^2 + G_0, \tag{2.44}$$

where $G_j^2(f) = G_j(G_j f)$. The proof mainly relies on classical results recalled in section 2.1.1.

Proof of assertion 1): the Strong Feller Property.

Throughout, we use the following notation. If $\mathcal{N}$ is a smooth manifold (such as $M, \mathbb{M}$ or $\mathbb{R}^m$), $W : \mathcal{C}^\infty(\mathcal{N}) \to \mathcal{C}^\infty(\mathcal{N})$ a linear map (typically a differential operator) and

$f : \mathcal{N} \to \mathbb{R}^n : x \mapsto (f_1(x), \ldots, f_n(x))$ a smooth map, we let $W(f) : \mathcal{N} \to \mathbb{R}^n$ denote the map defined by

$$W(f)(x) = (W(f_1)(x), \ldots, W(f_n)(x)).$$

First of all, let us recall the definition of the so-called *Hörmander condition* to our case.

Definition 2.6. *The dynamics (2.34) satisfies the Hörmander condition if for all $(x, u) \in \mathbb{M}, \mathcal{G}_\infty(x, u)$ spans $T_{(x,u)}\mathbb{M} = T_x M \times \mathbb{R}^n$.*

Remark 2.6. *Note that when $\sigma = 0$, the Hörmander condition is never satisfied since in that case $\mathcal{G}_0$ is reduced to $\{0\}$; hence $\mathcal{G}_\infty = \{0\}$.*

Similarly to the construction used for defining the Hörmander condition, let $\mathcal{A}_0 = \{F_1, \cdots, F_N\}$ and for all $k \geq 1$

$$\mathcal{A}_k = \mathcal{A}_{k-1} \cup \{F_j B,\ B \in \mathcal{A}_{k-1} \text{ and } j = 1, \cdots, N\}, \tag{2.45}$$

where $F_j B$ is the operator on $\mathcal{C}^\infty(M)$ defined by $(F_j B)(f) = F_j(B(f))$.

Let then $\mathcal{A}_\infty = \bigcup_{k\geqslant 0} \mathcal{A}_k$ and for all $x \in M$

$$\mathcal{A}_\infty(x) = \{W(e)(x)\ : W \in \mathcal{A}_\infty\}$$

where $e : M \to \mathbb{R}^n$ is the map defined by $e(x) = (e_1(x), \ldots, e_n(x))$. Note that while $\mathcal{G}_\infty$ is a set of vector fields on $\mathbb{M}$, $\mathcal{A}_\infty$ is a set of differential operators of all orders on $\mathcal{C}^\infty(M)$.

Definition 2.7. *We say that the condition (E') is fulfilled if and only if for all $x \in M$, $\mathcal{A}_\infty(x)$ spans $\mathbb{R}^n$.*

Lemma 2.4. *Suppose $\sigma > 0$. Then, condition (E') implies the Hörmander condition.*

The proof relies on the following lemma.

Lemma 2.5. *Let $\mathfrak{e} : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth function and let $F(x, u) = \begin{bmatrix} A(x) \\ 0 \end{bmatrix}$ and $G(x, u) = \begin{bmatrix} B(x, u) \\ \mathfrak{e}(x) \end{bmatrix}$ be two vector fields on $\mathbb{R}^{m+n}$, where $A : \mathbb{R}^m \to \mathbb{R}^m$ and $B : \mathbb{R}^{m+n} \to \mathbb{R}^m$ are smooth functions. Then*

$$[F, G](x, u) = \begin{bmatrix} [A, B(., u)](x) \\ A(\mathfrak{e})(x) \end{bmatrix},$$

with $B(., u) : \mathbb{R}^m \to \mathbb{R}^m : x \mapsto B(x, u)$

Proof. Let $(x, u) \in \mathbb{R}^m \times \mathbb{R}^n$. We then get that

$$DF(x, u) = \begin{bmatrix} DA & 0 \\ 0 & 0 \end{bmatrix} (x, u)$$

and

$$DG(x,u) = \begin{bmatrix} D_x B & D_u B \\ D\mathfrak{e} & 0 \end{bmatrix} (x,u).$$

Hence

$$\begin{aligned} [F,G](x,u) &= DG(x,u)F(x,u) - DF(x,u)G(x,u) \\ &= \begin{bmatrix} D_x B(x,u)A(x) - DA(x)B(x,u) \\ D\mathfrak{e}(x)A(x) \end{bmatrix} \\ &= \begin{bmatrix} [A, B(.,u)](x) \\ A(\mathfrak{e})(x) \end{bmatrix} \end{aligned} \tag{2.46}$$

as stated. □

Proof of lemma 2.4. Let

$$W = \prod_{j=1}^{l} F_{i_j},\ (i_1, \cdots, i_l) \in \{1, \cdots, N\}^l. \tag{2.47}$$

By definition of G_0, and lemma 2.5 (used in a local chart) it follows that

$$G_W(x,u) := [G_{i_1}, [\cdots, [G_{i_l}, G_0] \cdots]] = \sigma^l \begin{bmatrix} \divideontimes \\ W(e)(x) \end{bmatrix} \tag{2.48}$$

Thus, by hypothesis and the definition of G_j for $j = 1, \cdots, N$,

$$\{G_1(x,u), \cdots, G_N(x,u)\} \cup \{G_W(x,u)\ : W \in \mathcal{A}_\infty\}$$

spans $T_{(x,u)}\mathbb{M}$. This set being a subset of $\mathcal{G}_\infty(x,u)$, this proves the lemma. □

Lemma 2.6. *Suppose that $\{e_1, \ldots, e_n\}$ are eigenfunctions associated to the same nonzero eigenvalue of Δ_M. Then condition (E') holds true.*

Proof. Let $(U, (x_1, \ldots, x_m))$ be a local chart with U an open set in M. Let $D_1, \ldots, D_m$ be the vector fields defined on U by $D_i(f) = \frac{\partial}{\partial x_i} f$. Define $\mathcal{A}_\infty^D$ like $\mathcal{A}_\infty$ by replacing $F_1, \ldots, F_N$ by $D_1, \ldots, D_m$, and set $\mathcal{A}_\infty^D(x) = \{W(e)(x)\ : W \in \mathcal{A}_\infty^D\}$ for all $x \in U$. We claim that $\mathcal{A}_\infty^D(x)$ spans $\mathbb{R}^n$. Suppose to the contrary that there exists some $x^* \in U$ and some vector $t \in \mathbb{R}^n \setminus \{0\}$ such that $\mathcal{A}_\infty^D(x^*) \subset t^\perp$. Let $f(x) = \sum_i t_i e_i(x)$. Then f is an eigenfunction of Δ_M and for all $W \in \mathcal{A}_\infty^D$

$$W(f)(x^*) = W(\sum_{i=1}^{n} t_i e_i)(x^*) = \langle W(e)(x^*), t \rangle = 0.$$

In other words, f vanishes to infinite order at x^*. But by a result of Aronzajn (see [4]), every nonzero eigenfunction of the Laplacian on a C^∞ manifold with C^∞ metric, never vanishes to infinite order. This proves the claim.

It remains to show that $\mathcal{A}_\infty(x)$ spans $\mathbb{R}^n$. Since $F_1(x), \dots, F_N(x)$ span T_xM for all x, there exist smooth real valued maps $\alpha_{ij}, 1 \le i \le m, 1 \le j \le N$, defined on U such that for all $x \in U$ and $1 \le j \le N$

$$D_j(x) = \sum_{k=1}^N \alpha_{j,k}(x) F_k(x).$$

Thus

$$D_j(e)(x) = \sum_{i=1}^N \alpha_{j,i}(x) F_i(e)(x) \in span(\mathcal{A}_\infty(x)).$$

Now, for all $\psi, \xi \in \mathcal{C}^\infty(M)$ and all $H \in \mathcal{A}_\infty$, we have

$$H(\psi\xi)(x) = \psi(x) H(\xi)(x) + \xi(x) H(\psi)(x).$$

Thus,

$$\begin{aligned} D_i D_j(e)(x) &= \sum_{k=1}^N D_i(\alpha_{j,k})(x) F_k(e)(x) + \sum_{k=1}^N \alpha_{j,k}(x) D_i F_k(e)(x) \\ &= \sum_{k=1}^N D_i(\alpha_{j,k})(x) F_k(e)(x) + \sum_{k,l=1}^N \alpha_{j,k}(x)\alpha_{i,l}(x) F_l F_k(e)(x) \in span(\mathcal{A}_\infty(x)) \end{aligned}$$

By recursion, it comes that $\mathcal{A}_\infty^D(x) \subset span(\mathcal{A}_\infty(x))$ and since $\mathcal{A}_\infty^D(x)$ spans $\mathbb{R}^n$, so does $\mathcal{A}_\infty(x)$. □

Lemma 2.7. *Condition (E') holds.*

Proof. Let Λ be the set of distinct eigenvalues of $\{e_1, \cdots, e_n\}$. For $\lambda \in \Lambda$ let $\{e_1^\lambda, \dots, e_{n(\lambda)}^\lambda\} \subset \{e_1, \cdots, e_n\}$ be the set of eigenfunctions having eigenvalue λ and let $e^\lambda = (e_1^\lambda, \dots, e_{n(\lambda)}^\lambda)$.

Let $x \in M$. By Lemma 2.6 there exist $W_1^\lambda, \cdots, W_{n(\lambda)}^\lambda \in \mathcal{A}_\infty$ such that the matrix

$$R_\lambda = (W_i^\lambda(e_j^\lambda)(x))_{1 \leqslant i,j \leqslant n(\lambda)} \tag{2.49}$$

has rank $n(\lambda)$.

Given a polynomial $P(x) = \sum_{j=0}^k \alpha_j x^j$, we let

$$P(\Delta_M) = \sum_{j=0}^k \alpha_j \Delta_M^j, \tag{2.50}$$

where Δ_M^j is the operator defined recursively by $\Delta_M^0 f = f$ and $\Delta_M^{j+1} f = \Delta_M^j(\Delta_M f)$ with $f \in \mathcal{C}^2(M)$. Note that for all $1 \le i \le n(\lambda)$

$$P(\Delta_M)(e_i^\lambda) = P(\lambda) e_i^\lambda. \tag{2.51}$$

Now let $P^\lambda(x) = \prod_{\alpha\in\Lambda;\alpha\neq\lambda}(x-\alpha)$. For $\lambda \in \Lambda$ and $i = 1, \cdots, n(\lambda)$, set

$$H_i^\lambda = W_i^\lambda P^\lambda(\Delta_M). \tag{2.52}$$

Then one has that $H_i^\lambda(e_j^\alpha)(x) = 0$ for $\alpha \neq \lambda$ and $H_i^\lambda(e_j^\lambda)(x) = P^\lambda(\lambda)W_i^\lambda(e_j^\lambda)(x)$. Thus, the matrix

$$H = (H_i^\lambda(e_j^\alpha)(x))_{\lambda\in\Lambda,\, i=1,\cdots,n(\lambda)}$$

can, after a reordering if necessary, be written as a diagonal block matrix $(P^\lambda(\lambda)R_\lambda(x))_{\lambda\in\Lambda}$.

It is then easy to see that H has rank n. □

This later lemma combined with Lemma 2.4 and Theorem 2.1 proves assertion 1).

Proof of assertions 2) and 3). Invariant probability measure and Harris Recurrence

Recall that a probability measure μ is invariant for the semi-group $(P_t)_{t\geqslant 0}$ if

$$\int_{\mathbb{M}} P_t f(y)\mu(dy) = \int_{\mathbb{M}} f(y)\mu(dy)$$

for all $f \in \mathcal{C}_0(\mathbb{M})$.

Existence of an invariant probability measure. We will switch between the two notations $y \in \mathbb{M}$ and $(x,u) \in M \times \mathbb{R}^n$ which represent the same point. Setting

$$L^* = \frac{\sigma^2}{2}\Delta_M + \sum_{k=1}^n a_k u_k div_x(\nabla e_k(x).) - \sum_{k=1}^n e_k(x)\partial_{u_k}. \tag{2.53}$$

we then observe that

$$\begin{aligned} L^*\varphi(y) &= \sum_{k=1}^n a_k u_k div_x(\nabla e_k(x)\varphi(y)) - \sum_{k=1}^n e_k(x)\partial_{u_k}\varphi(y) \\ &= \sum_{k=1}^n a_k u_k \lambda_k e_k(x)\varphi(y) + \sum_{k=1}^n e_k(x) a_k|\lambda_k| u_k\varphi(y) \\ &= 0. \end{aligned}$$

By Propositions 2.7 and 2.8 together with Theorem 2.8, we get for $f \in \mathcal{C}_c^2(\mathbb{M})$

$$\begin{aligned} \int_{\mathbb{M}} (P_t f(y) - f(y))\mu(dy) &= \int_0^t \int_{\mathbb{M}} \mathcal{L}P_s f(y)\mu(dy)ds \\ &= \int_0^t \int_{\mathbb{M}} LP_s f(y)\varphi(dy)ds \end{aligned}$$

Noting that for all $g, h \in \mathcal{C}_c^2(\mathbb{M})$

$$\int_{\mathbb{M}} Lg(y)h(y)dy = \int_{\mathbb{M}} g(y)L^*h(y)dy,$$

we obtain

$$\int_{\mathbb{M}} (P_t f(y) - f(y))\mu(dy) = \int_0^t \int_{\mathbb{M}} P_s f(y) L^* \varphi(dy) ds = 0$$

Since $\mathcal{C}_c^2(\mathbb{M})$ is dense in $\mathcal{C}_0(\mathbb{M})$ for $\|.\|_\infty$, it follows that $\mu(dy) = \varphi(y)dy$ is an invariant probability as stated.

Uniqueness of the invariant probability. In order to do this, we begin by showing that μ is an ergodic probability; that is, if a subset $A \subset \mathbb{M}$ satisfies $P_t \mathbf{1}_A = \mathbf{1}_A\ \mu - a.s$ for all $t \geqslant 0$, then $\mu(A)$ is either 0 or 1.
Let us denote by f the function $P_t \mathbf{1}_A$. Then $f(y) \in \{0, 1\}$ for μ-almost $y \in \mathbb{M}$ and f is continuous by point 1 of Theorem 2.9. Since $\mathbb{M}$ is a connected space and μ has full support, it follows that f is either equal to 0 or 1; and therefore μ is ergodic.
Since two distinct ergodic probabilities are mutual singular, the strong Feller property imply that they must have disjoint support. Since μ has the whole space, which is connected, as support, the uniqueness of μ follows. The second part of the statement is Theorem 4.(i) in [67].

The proof that the process is Harris recurrent follows from the proof's lines of Proposition 5.1 in [67]; which also proves the third point.

2.7 Exponential decay in $L^2(\mu)$

The goal of this section is to prove the exponential decay in the $L^2(\mu)$ norm. The proof heavily relies on the hypocoercitivity method analyzed by M.Grothaus and P.Stilgenbauer in [56] whose roots lie in the series of paper [42], [43] and [55] initiated by J.Dolbeault, C. Mouhot and C. Schmeiser.

We emphasize that in the particular case where $M = \mathbb{S}^d, n = d + 1$ and $(e_j)_{j=1,\cdots,d+1}$ are the eigenfunctions associated to the first non-zero eigenvalue, our model coincides with the one studied in section 3 in [56].

For an operator T on some Hilbert space H, we denote by $D(T)$ its domain and T^* its adjoint. We begin to recall the **Data (D)** and **Hypotheses (H1)-(H4)** introduced in [56]. For convenience we have chosen to replace certain hypotheses from [56] by slightly stronger ones (see the remark 2.8 below) which are sufficient for our purpose.

Definition 2.8. *(**The Data (D)**) Let H be a real Hilbert space and let (P_t) be a strongly continuous semigroup on H with generator $(\mathcal{L}, D(\mathcal{L}))$ and core $D \subset D(\mathcal{L})$. We suppose that*

(i) There exist a closed symmetric operator $(S, D(S))$ and a closed antisymmetric operator $(A, D(A))$ such that $D \subset D(S) \cap D(A)$, $A(D) \subset D$ and $\mathcal{L}_{|D} = S_{|D} - A_{|D}$.

(iii) There exists a closed subspace $F \subset D(S)$ such that $S_{|F} = 0$ and $P(D) \subset D$ where P is the orthogonal projection $P : F \oplus F^\perp \to F : f + g \mapsto f$ for all $(f, g) \in F \times F^\perp$.

By density of $D \subset D(A)$, closedness of A and the fact that $P(D) \subset D \subset D(A)$, AP is closed and densely defined. Hence, by Von Neumann's Theorem, $(AP)^*AP$ is self-adjoint, closed and densely defined. Thus $(I + (AP)^*AP) : D((AP)^*AP) \to H$ is invertible with bounded inverse. Set

$$B_0 = (I + (AP)^*AP)^{-1}(AP)^* \text{ on } D((AP)^*AP). \tag{2.54}$$

In the following we let $(,)_H$ denote the inner product on H and $\|\cdot\|_H$ the associated norm.

Definition 2.9. ***(Hypotheses (H1)-(H4))***

(H1) $PAP_{|D} = 0$

(H2) (Microscopic coercivity). *There exists* $\Lambda_1 > 0$ *such that for all* $f \in D \cap F^\perp$,

$$(-Sf, f)_H \geqslant \Lambda_1 \|f\|_H^2.$$

(H3) (Macroscopic coercivity). *There exists* $\Lambda_2 > 0$ *such that for all* $f \in D((AP)^*(AP)) \cap F$,

$$\|Af\|_H^2 \geqslant \Lambda_2 \|f\|_H^2. \tag{2.55}$$

(H4) (Boundedness of auxiliary operators). *The operators* (B_0S, D) *and* $(B_0A(I-P), D)$ *are bounded and there exists constants* N_1 *and* N_2 *such that for all* $f \in F^\perp \cap D$

(H4, a)

$$\|B_0Sf\|_H \leqslant N_1 \|f\|_H \tag{2.56}$$

and

(H4, b)

$$\|B_0Af\|_H \leqslant N_2 \|f\|_H \tag{2.57}$$

.

If furthermore $(I - PA^2P)(D)$ is dense in H, then conditions $(H3)$ and $(H4, b)$ are implied by the following conditions, as shown by Corollary 2.13 and Proposition 2.15 in [56].

(H3') Equation (2.55) holds for all $f \in D \cap F$.

(H4') b) For all $f \in D \cap F$

$$\|A^2f\|_H \leqslant N_2 \|g\|_H \tag{2.58}$$

where $g = (I - PA^2P)f$.

Theorem 2.12 (Theorem 2 in [43], Theorem 1 in [42], Theorem 2.18 in [56]). *Assume that the assumptions of Definitions 2.8 and 2.9 hold. Then there exist constants* $\kappa_1, \kappa_2 \in (0, \infty)$ *explicitly computable such that for all* $g \in H$ *and* $t \geqslant 0$,

$$\|P_tg\|_H \leqslant \kappa_1 e^{-\kappa_2 t} \|g\|_H \tag{2.59}$$

Remark 2.7. *Following the proof's line of section 3.4 in [42] and the beginning of the proof of Theorem 2.18 in [56], one obtains for any* $\eta > 0$

$$\kappa_1 = \sqrt{\frac{1+\varepsilon_\eta}{1-\varepsilon_\eta}} \leqslant \sqrt{1+2\eta} \text{ and } \kappa_2 = \varepsilon_\eta \frac{\Lambda_2}{4(1+\Lambda_2)}, \tag{2.60}$$

with

$$\varepsilon_\eta = \frac{\eta}{1+\eta} \min(1, \varepsilon_0) \tag{2.61}$$

and

$$\varepsilon_0 = \frac{2\Lambda_2\Lambda_1}{(1+\Lambda_2)(2+(1+N_1+N_2)^2)}. \tag{2.62}$$

Remark 2.8. *In case* (P_t) *is a Markov semigroup with invariant probability* μ, *inducing a strongly continuous semigroup on* $L^2(\mu)$, *a natural choice for* H *is*

$$L_0^2(\mu) = \{f \in L^2(\mu) : \int f d\mu = 0\}.$$

This choice will be adopted later. In this case, conditions $(D6)$ *and* $(D7)$ *from [56] are automatically satisfied and Theorem 2.12 implies that for all* $f \in L^2(\mu)$

$$\|P_t f - \int f d\mu\|_{L^2(\mu)} \leqslant \kappa_1 e^{-\kappa_2 t} \|f - \int f d\mu\|_{L^2(\mu)}.$$

2.7.1 Application to the Proof of Theorem 2.10

Throughout we let

$$H = L_0^2(\mu) := \{f \in L^2(\mathbb{M}, \mu) \ : \int_{\mathbb{M}} f(y)\mu(dy) = 0\}$$

and

$$L_0^2(e^{-\Phi}) = \{f \in L^2(\mathbb{R}^n, e^{-\Phi}) \ : \int_{\mathbb{R}^n} f(u) e^{-\Phi(u)} du = 0\}$$

where μ and Φ are like in Definition 2.5. Both H and $L_0^2(e^{-\Phi})$ are equipped with the associated L^2 inner product and norm.

The map $\imath : L_0^2(e^{-\Phi}) \hookrightarrow H$ defined by $\imath(g)(x,u) = g(u)$ injects isometrically $L_0^2(e^{-\Phi})$ into H. We let

$$F = \imath(L_0^2(e^{-\Phi}))$$

and $P : F \oplus F^\perp \to F$ denote the orthogonal projection onto F. Alternatively P can be defined as

$$(Pf)(x,u) = \int_M f(x,u)\nu(dx). \tag{2.63}$$

Using the notation introduced in section 2.5 we let (P_t) denote the semigroup defined by

$$P_t f(y) = \mathbb{E}(f(Y_t^y))$$

for every bounded Borel map $f : \mathbb{M} \to \mathbb{R}$; where (Y_t^y) stands for the solution to (2.34) with initial condition $Y_0^y = y$.

Lemma 2.8. *(P_t) induces a strongly continuous contraction semigroup on H.*

Proof. By invariance of μ and Jensen inequality P_t defines a bounded operator on H with norm less than 1 (as already proved in Remark 2.5).

Let $\varepsilon > 0$ and $f \in L^2(\mu)$. By density of $\mathcal{C}_0(\mathbb{M})$ in $L^2(\mu)$, there exists $g \in \mathcal{C}_0(\mathbb{M})$ such that $\|f - g\|_{L^2(\mu)} < \varepsilon$. Thus, by the contraction property

$$\begin{aligned} \|P_t f - f\|_{L^2(\mu)} &\leqslant \|P_t f - P_t g\|_{L^2(\mu)} + \|P_t g - g\|_{L^2(\mu)} + \|g - f\|_{L^2(\mu)} \\ &\leqslant 2\varepsilon + \|P_t g - g\|_\infty. \end{aligned}$$

Hence, by Feller continuity of (P_t) (see Lemma 2.3)

$$\limsup_{t\to 0} \|P_t f - f\|_{L^2(\mu)} \leqslant 2\varepsilon.$$

□

Remark 2.9. *Note that the conclusion of Lemma 2.8 hold true for any Feller Markov semigroup having μ as invariant measure. This will be used later.*

Let $(\mathcal{L}, D(\mathcal{L}))$ denote the infinitesimal generator of (P_t) (now seen as a strongly continuous semigroup on H) and let

$$D = \mathcal{C}_c^\infty(\mathbb{M}) \cap H.$$

Proposition 2.9. *There exist a closed symmetric operator $(S, D(S))$ and a closed anti-symmetric operator $(A, D(A))$ such that*

(i) *D is a core for S, A and $\mathcal{L}$ invariant under $S, A, \mathcal{L}$ and P.*

(ii) *$F \subset D(S)$ and $S|_F = 0$.*

(iii) *For all $f \in D$*

$$S(f) = \frac{\sigma^2}{2}\Delta_M f, \tag{2.64}$$

$$A(f) = -G_0(f) = \sum_{i=1}^n a_j u_j (\nabla e_j(x), \nabla_x f)_{TM} - e_j(x)\partial_{u_j} f \tag{2.65}$$

and

$$\mathcal{L}f = Lf = Sf - Af. \tag{2.66}$$

This later proposition shows that conditions of Definition 2.8 are fulfilled.

Let $\eta_1(M) = \eta_1$ denote the *spectral gap* of M. That is

$$\eta_1(M) := \inf\{\int_M |\nabla h|^2 \nu(dx) \ : h \in H^1(M), \ \int_M h^2 \nu(dx) = 1, \ \int_M h\nu(dx) = 0\} \tag{2.67}$$

where $\|h\|^2 = (h, h)_{TM}$ and $(.,.)_{TM}$ is the scalar product on the tangent bundle. By a classical result in spectral geometry, compactness of M ensures that $\eta_1 > 0$ and equals the smallest non zero eigenvalue of $-\Delta_M$.

Proposition 2.10. *Hypotheses (H1)-(H4) in Definition 2.9 hold with*

$$\Lambda_1 = \frac{\eta_1 \sigma^2}{2}, \Lambda_2 = \min_{i=1,\ldots,n} |\lambda_i| a_i,$$

$$N_1 = \frac{\sigma^2}{2} \sum_{j=1}^{n} |\lambda_j|,$$

and

$$N_2 = 2\frac{n}{\min\{|\lambda_j|, j=1,\cdots,n\}} \sup_{i=1,\ldots,n} \|\nabla e_i\|_\infty^2 \sqrt{4 + \sum_{i=1}^{n} |\lambda_i| a_i + 4 \|\sum_i e_i^2\|_\infty}$$

Remark 2.10. *Since $N_1 \geqslant \frac{n\sigma^2}{2}\eta_1$, then $2\Lambda_1 < 2 + (1 + N_1 + N_2)^2$. Hence $\varepsilon_0 < 1$, where ε_0 is defined by (2.62).*

2.7.2 Proof of Propositions 2.9 and 2.10

Proof of Proposition 2.9

We first recall some classical results that will be used throughout.

Proposition 2.11. *(see e.g Corollary 1.6, Proposition 2.1, Proposition 3.1, Proposition 3.3 in [47]) Let K be the generator of a strongly continuous contracting semi-group $(T_t)_t$ on some Banach space $\mathcal{H}$. Then*

1. *K is closed and densely defined.*
2. *The resolvent set of K contains $(0, \infty)$ and $(\lambda I - K)^{-1} g = \int_0^\infty e^{-\lambda t} T_t g dt$, for all $g \in \mathcal{H}$ and $\lambda > 0$.*
3. *A subspace D of $D(K)$ is a core for K if and only if it is dense in $\mathcal{H}$ and $(\lambda I - K)(D)$ is dense in $\mathcal{H}$ for some $\lambda > 0$.*
4. *Let D be a dense subset of $\mathcal{H}$ such that $D \subset D(K)$. If $T_t(D) \subset D$ for all $t \geqslant 0$, then D is a core for K.*

Similarly to (P_t), let (P_t^S) and (P_t^A) be the semigroups respectively induced by the following stochastic and ordinary differential equation on $\mathbb{M}$:

$$dY_t^S = \sum_{j=1}^{N} G_j(Y_t^S) \circ dB_t^j,$$

and

$$\frac{dY_t^A}{dt} = -G_0(Y_t^A). \tag{2.68}$$

Note that (P_t^A) is not merely a semigroup but a group of transformation defined as

$$P_t^A f(y) = (f \circ \psi_t)(y) \tag{2.69}$$

where $\{\psi_t\}$ is the flow induced by (2.68). The proofs given in Lemma 2.3, Proposition 2.7 and Remark 2.9 show that, not only (P_t) but also (P_t^S) and (P_t^A) are Feller, leave $\mathcal{C}_c^2(\mathbb{M})$ invariant and admit μ as invariant probability. Thus, by Remark 2.9 and Proposition 2.11 they induce strongly continuous semigroups on H whose generators, denoted S and A are closed, densely defined and admit $\mathcal{C}_c^2(\mathbb{M}) \cap H$ as a core.

Since for all $f \in F, P_t^S f = f$, assertion (ii) of Proposition 2.9 is satisfied. Furthermore, the definition of $\mathcal{L}, A$ and S easily imply assertion (iii) as well as invariance of D under the generators and under P. The end of the proof is given by the two following lemmas.

Lemma 2.9. *D is a core for $\mathcal{L}, S$ and A.*

Proof. Let G be one of the operators $\mathcal{L}, S$ or A. It is easily checked that for all $f \in C_c^2(\mathbb{M})$

$$\|Af\|_{L^2(\mu)} \leq C\|\nabla f\|_\infty$$

and

$$\|Sf\|_{L^2(\mu)} \leq \frac{\sigma^2}{2}\|\Delta_M f\|_\infty$$

for some $C > 0$ independent of f. Thus G maps continuously the space $\mathcal{C}_c^2(\mathbb{M}) \cap H$ equipped with the $\mathcal{C}^2$ strong topology, into H. By standards approximation results $\mathcal{C}_c^\infty(\mathbb{M})$ is dense into $\mathcal{C}_c^2(\mathbb{M})$ for the $\mathcal{C}^2$ strong topology (see e.g [62], Chapter 2). Since $\mathcal{C}_c^2(\mathbb{M}) \cap H$ is a core for G, $(I - G)(\mathcal{C}_c^2(\mathbb{M}) \cap H)$ is dense in H (see Proposition 2.11). Thus $(I - G)(D)$ is dense in H and D is a core. □

Lemma 2.10. *S is symmetric and $A^* = -A$.*

Proof. Let $f, g \in D$. Then

$$\begin{aligned}(Sf, g)_H &= \frac{\sigma^2}{2}\int\int_M (\Delta_M f) g\nu(dx)e^{-\Phi}d\xi = -\frac{\sigma^2}{2}\int\int_M (\nabla f, \nabla g)_{TM}\nu(dx)e^{-\Phi}d\xi \\ &= \frac{\sigma^2}{2}\int\int_M (\Delta_M g) f\nu(dx)e^{-\Phi}d\xi = (f, Sg)_H\end{aligned}$$

Since D is a core for S, this proves the symmetry of S.
For $f, g \in H$, we obtain from invariance of μ,

$$\begin{aligned}(P_t^A f, g)_H = \int_{\mathbb{M}} (f \circ \psi_t)(y) g(y)\mu(dy) &= \int_{\mathbb{M}} f(\psi_t(y)) g(\psi_{-t} \circ \psi_t(y))\mu(dy) \quad (2.70) \\ &= \int_{\mathbb{M}} f(y)(g \circ \psi_{-t})(y)\mu(dy). \quad (2.71)\end{aligned}$$

Hence $(P_t^A)^* = P_{-t}^A$. In particular, $((P_t^A)^*)$ is strongly continuous and admits $-A$ as infinitesimal generator. Now, when a semigroup and its adjoint are both strongly continuous, the generator of the adjoint equals the adjoint of the generator. This follows for instance from Theorem 1.5 in [85] combined with Proposition 2.11 2. Thus $A^* = -A$. □

Proof of Proposition 2.10

For all $f \in D$ let

$$A_j(f)(x,u) = a_j u_j (\nabla e_j(x), \nabla_x f)_{TM} - e_j(x)\partial_{u_j} f. \tag{2.72}$$

so that $Af = \sum_{j=1}^n A_j f$. Similarly to A, A_j enjoys the same properties as A. In particular, it leaves D invariant and is antisymmetric:

$$(A_j f, g)_{L^2(\mu)} = -(f, A_j g)_{L^2(\mu)}$$

for all $f, g \in D$.

Finally, we introduce the following operators

$$T = (I + (AP)^*(AP))^{-1} \text{ on } H \tag{2.73}$$
$$B_j = -T(PA_j) \text{ on } D \tag{2.74}$$

where I denotes the identity operator. Recall that B_0 was introduced to be the operator

$$B_0 = T(AP)^* \text{ on } D((AP)^* AP).$$

Hypothesis (H1) is immediate because for all $f \in D$, $A_j P f = -e_j(x)\partial_{u_j}(Pf)$ and $\int_M e_j(x)\nu(dx) = 0$, thus $PA_jPf = 0$.

Hypothesis (H2) follows directly from the variational definition of the spectral gap (2.67). Indeed for all $f \in D \cap F^{\perp}$

$$\begin{aligned}-(Sf,f)_{L^2(\mu)} &= -\frac{1}{2}\sigma^2 \int_{\mathbb{R}^n}\int_M (\Delta_M f) f \nu(dx) e^{-\Phi(u)} du \\ &= \frac{1}{2}\sigma^2 \int_{\mathbb{R}^n}\int_M |\nabla_x f|^2 \nu(dx) e^{-\Phi(u)} du \geq \frac{\eta_1}{2}\sigma^2 \|f\|^2_{L^2(\mu)}.\end{aligned}$$

For $k = 1, \ldots, n$ let

$$\alpha_k = |\lambda_k| a_k,$$

so that

$$\Phi(u) = \frac{1}{2}\sum_{k=1}^n \alpha_k u_k^2 + \ln(C(\Phi)).$$

Let (P_t^{OU}) denote the *Ornstein-Uhlenbeck* semi-group on $L_0^2(e^{-\Phi})$ defined as

$$P_t^{OU} f(u) = \int f(e^{-\mathsf{diag}(\alpha_i)t} u + \mathsf{diag}(\sqrt{1 - e^{-2\alpha_i t}})\xi) e^{-\Phi(\xi)} d\xi \tag{2.75}$$

or, equivalently, $P_t^{OU} f(u) = \mathbb{E}(f(U^u(t)))$ where $U^u(t)$ is the solution to the linear equation on $\mathbb{R}^n$

$$dU_i^u(t) = -\alpha_i U_i^u(t) dt + \sqrt{2} dB_t^i, \; i = 1 \ldots n,$$

with initial condition $U^u(0) = u$ and independent Brownian motions $B^1, \ldots, B^n$.

Let L_{OU} denote the generator of (P_t^{OU}) on $L_0^2(e^{-\Phi})$. The set

$$\tilde{D} = \mathcal{C}_c^\infty(\mathbb{R}^n) \cap L_0^2(e^{-\Phi})$$

is a core[2] L_{OU} and for all $f \in \tilde{D}$

$$L_{OU} f = -\langle \nabla\Phi, \nabla f\rangle + \Delta f.$$

The next Lemma is similar to Corollary 2.13 and Proposition 3.13 in [56],

Lemma 2.11. (i) *For all* $f \in F$

$$PA^2 f = \imath \circ L_{OU} \circ \imath^{-1}(f)$$

(ii) *$(I - PA^2P)(D)$ is dense in H.*

(iii) *(H3) holds with* $\Lambda_2 = \min\{\alpha_k : \ k = 1 \dots n\}$

Proof. (i) Let $f \in F \cap D$. Then

$$\begin{aligned} A^2 f &= \sum_{k=1}^n A_k(Af) = \sum_{k=1}^n A_k(\sum_{j=1}^n A_j f) = \sum_{k=1}^n A_k(\sum_{j=1}^n e_j \partial_{u_j} f) \\ &= \sum_{k=1}^n [(\nabla e_k, \sum_{j=1}^n \nabla_x (e_j \partial_{u_j} f))_{TM} a_k u_k - \partial_{u_k}(\sum_{j=1}^n e_j \partial_{u_j} f) e_k] \\ &= \sum_{k,j=1}^n \partial_{u_j} f a_k u_k (\nabla e_k, \nabla e_j)_{TM} - \sum_{k,j=1}^n (\partial^2_{u_j u_k} f) e_j e_k \end{aligned} \tag{2.76}$$

Therefore

$$\begin{aligned} PA^2 f &= \sum_{k,j=1}^n \partial_{u_j} f a_k u_k \int_M (\nabla e_k, \nabla e_j)_{TM} d\nu - \sum_{k,j=1}^n (\partial^2_{u_j u_k} f) \int_M e_j e_k d\nu \\ &= \sum_{j=1}^n \partial_{u_j} f a_j u_j |\lambda_j| - \sum_{j=1}^n (\partial_{u_j u_j} f) = \sum_{j=1}^n \partial_{u_j} f \alpha_j u_j - \sum_{j=1}^n (\partial^2_{u_j u_j} f). \end{aligned} \tag{2.77}$$

This proves the first assertion.

(ii) $(I - PA^2P)(D \cap F^\perp) = D \cap F^\perp$ is dense in $F^\perp$ because $F^\perp = (I-P)(H), (I - P)(D) \subset D \cap F^\perp$ and D is dense. Also, $(I - PA^2P)(D \cap F) = \imath(I - L_{OU})(\tilde{D})$ is dense in F because, $\tilde{D}$ being a core for L_{OU}, $(I - L_{OU})(\tilde{D})$ is dense in $L_0^2(e^{-\Phi})$. This proves (ii).

(iii) Using antisymmetry of A, assertion (i) and the Poincaré inequality for the Gaussian measure $e^{-\Phi(u)}du$ (see e.g [2], chapter 1) we get that for all $f \in F \cap D$,

$$\begin{aligned} \|Af\|_H^2 = \|APf\|_H^2 = (-PA^2Pf, f)_H = -(\imath^{-1}(f), L_{OU}\imath^{-1}(f))_{L_0^2(e^{-\Phi})} \\ \geq \min(\alpha_i)\|\imath^{-1}(f)\|_{L_0^2(e^{-\Phi})} = \min(\alpha_i)\|f\|_H^2. \end{aligned}$$

This proves (H3'), hence (H3). □

[2] This is a classical result and can easily be verified as follows. Formula (2.75) shows that the set $C_b^\infty(\mathbb{R}^n)$ of bounded C^∞ functions with bounded derivatives is stable under (P_t^{OU}); hence a Core by Proposition 2.11. Furthermore for each $f \in C_b^\infty(\mathbb{R}^n)$ it is easy to construct a sequence $f_n \in C_c^\infty(\mathbb{R}^n)$ such that $f_n \to f$ and $L_{OU} f_n \to L_{OU} f$ in $L^2(e^{-\Phi})$.

Lemma 2.12. *For $f \in D \cap F$, we have* $\|Af\|^2_{L^2(\mu)} = \sum_{k=1}^n \|A_k f\|^2_{L^2(\mu)} = \|\nabla f\|^2_{L^2(\mu)}$.

Proof. Let $f \in D \cap F$. Since f does not depend on the x-variable, $A_j f = -e_j \partial_{u_j} f$. The result follows from the fact that the eigenfunctions $(e_j)_{j=1,\cdots,n}$ are orthonormal in $L^2(M, dx)$. □

The next Lemma is inspired from Lemma 2.4 in [56]

Lemma 2.13. *For $j = 1, \cdots, n$ and $f \in D$,*

$$\|B_j f\|_H \leqslant \frac{1}{2}\|(I - P)f\|_H.$$

Proof. The proof is quite similar to the proof of Lemma 2.4 in [56]. Let $f \in D$ and define $g = B_j f$. Thus $g \in D((AP)^* AP)$ and

$$-PA_j f = g + ((AP)^* AP)g. \tag{2.78}$$

Because $(I - PA^2P)(D)$ is dense in H (see Lemma 2.11.(**ii**)), there exists a sequence $(g_n) \subset D$ such that

$$\lim_{n\to\infty} g_n - PA^2Pg_n = g + (AP)^*(AP)g. \tag{2.79}$$

Since $P(D), A(D) \subset D$, it follows from Lemma 2.2 in [56] that

$$-PA^2Pg_n = ((AP)^*(AP))g_n. \tag{2.80}$$

Thus, by continuity of T,

$$\lim_{n\to\infty} g_n = g \tag{2.81}$$

and from (2.80)

$$\lim_{n\to\infty} (AP)^*(AP)g_n = (AP)^*(AP)g. \tag{2.82}$$

Thus, taking the scalar product of (2.78) with respect to g_n on both side provides

$$\lim_{n\to\infty} -(PA_j f, g_n)_H - \|g_n\|^2_H - \|APg_n\|^2_H = 0.$$

Now, using successively antisymmetry of A_j, Cauchy Schwarz (and Young) inequalities and Lemma 2.12,

$$-(PA_j f, g_n)_H = ((I - P)f, A_j P g_n)_H \leq \|(I - P)f\|_H \|A_j P g_n\|_H \tag{2.83}$$

$$\leq \frac{1}{4}\|(I - P)f\|^2_H + \|A_j P g_n\|^2_H \leq \frac{1}{4}\|(I - P)f\|^2_H + \|APg_n\|^2_H \tag{2.84}$$

Thus, letting n tends to ∞, it leads to

$$\|g\|^2_H \leqslant \frac{1}{4}\|(I - P)f\|^2_H. \tag{2.85}$$

□

Lemma 2.14. *(H4 a) holds with* $N_1 = \frac{\sigma^2}{4} \sum_{j=1}^n |\lambda_j|$.

Proof. Let $f \in D \cap F^{\perp}$. Since $\int_{\mathbb{M}} A_j f(y)\mu(dy) = 0$, one has

$$\begin{aligned} -PAf &= \sum_{j=1}^{n} -PA_j f = \sum_{j=1}^{n} P(a_j u_j (\nabla e_j, \nabla_x f)_{TM} - e_j \partial_{u_j} f) \\ &= \sum_{j=1}^{n} [\int_M (\nabla e_j, \nabla_x f)_{TM} a_j u_j d\nu - \int_M e_j \partial_{u_j} f d\nu]. \end{aligned}$$

Since $S(D) \subset D$), then

$$-PASf = -\frac{\sigma^2}{2} \sum_{j=1}^{n} [\int_M (\nabla e_j, \nabla_x \Delta_M f)_{TM} a_j u_j d\nu - \int_M e_j \partial_{u_j} \Delta_M f d\nu].$$

Because

$$\begin{aligned} \int_M (\nabla e_j, \nabla_x \Delta_M f)_{TM} d\nu &= -\int_M \Delta_M e_j \Delta_M f d\nu = -\lambda_j \int_M e_j \Delta_M f d\nu \\ &= \lambda_j \int_M (\nabla e_j, \nabla_x f)_{TM} d\nu \end{aligned}$$

and

$$\int_M e_j \partial_{u_j} \Delta_M f d\nu = \int_M e_j \Delta_M \partial_{u_j} f d\nu = \int_M \Delta_M e_j \partial_{u_j} f d\nu = \lambda_j \int_M e_j \partial_{u_j} f d\nu$$

for all $j = 1, \cdots, n$, it follows that

$$PASf = \frac{\sigma^2}{2} \sum_{j=1}^{n} \lambda_j (PA_j) f.$$

By antisymmetry of A (resp. A_j) and Lemma 2.2 in [56], for all g in D, $(AP)^* g = -PAg$ (resp. $(A_j P)^* f = -PA_j f$). Hence

$$B_0 S f = T(AP)^* S f = -TPASf = \frac{\sigma^2}{2} \sum_{j=1}^{n} \lambda_j B_j f.$$

Applying the triangle inequality, one has

$$\|B_0 S f\|_{L^2(\mu)} \leqslant \frac{\sigma^2}{2} \sum_{j=1}^{n} |\lambda_j| \|B_j f\|_{L^2(\mu)}$$

and the result follows from Lemma 2.13. □

The following estimate can be compared with the a priori estimates obtained in [43] and discussed in Appendix $A1$ of [56] (lemmas A3, A4, A5, A7 and Proposition A6) for a more general elliptic equation. Note, however, that here we provide an elementary proof allowing precise estimates by making use of the Γ and Γ_2 operators combined with the specific form of L_{OU}.

Lemma 2.15. *Let $f \in \tilde{D}$ and*

$$g = (I - L_{OU})f. \tag{2.86}$$

Then

1. $\| |Hess(f)|_2 \|_{L^2(e^{-\Phi})} \leqslant 4\|g\|_{L^2(e^{-\Phi})}$
2. $\| |\nabla\Phi|_2 . |\nabla f|_2 \|_{L^2(e^{-\Phi})} \leqslant 2\sqrt{4 + \sum_{i=1}^n \alpha_i} \|g\|_{L^2(e^{-\Phi})}$,

where $|.|_2$ stands for the usual Euclidean norm and $|Hess(f)|_2^2 = \sum_{ij} |\partial_{u_i u_j} f|^2$.

Proof. From (2.86), we have $f = R_1 g$, where R_1 is the resolvent operator of L_{OU}. Thus

$$\|f\|_{L^2(e^{-\Phi})} \leqslant \|g\|_{L^2(e^{-\Phi})}$$

and

$$\|L_{OU} f\|_{L^2(e^{-\Phi})} \leqslant 2\|g\|_{L^2(e^{-\Phi})}.$$

Let Γ be the *"carré du champs"* operator defined by

$$\Gamma(\psi_1, \psi_2) = \frac{1}{2}[L_{OU}(\psi_1\psi_2) - \psi_2 L_{OU}\psi_1 - \psi_1 L_{OU}\psi_2] \tag{2.87}$$

and

$$\Gamma_2(\psi) = \frac{1}{2} L_{OU}\Gamma(\psi, \psi) - \Gamma(\psi, L_{OU}\psi). \tag{2.88}$$

It is known (see for instance Subsection 5.3.1 in [2]) that

(i) $\Gamma(f, f) = |\nabla f|_2^2$ and

(ii) $\Gamma_2(f) = |Hess(f)|_2^2 + \langle \nabla f, Hess(\Phi)\nabla f\rangle \geqslant |Hess(f)|_2^2$

by positive definiteness of $Hess(\Phi)$. Therefore, by invariance and reversibility of $e^{-\Phi(u)}du$,

$$\begin{aligned} \| |\nabla f|_2 \|_{L^2(e^{-\Phi})}^2 &= \int \Gamma(f, f) e^{-\Phi(u)} du \\ &= \int -f L_{OU} f e^{-\Phi(u)} du \\ &\leqslant \frac{1}{2}\|g\|_{L^2(e^{-\Phi})}^2 \end{aligned} \tag{2.89}$$

and

$$\begin{aligned} \int \Gamma_2(f) e^{-\Phi(u)} du &= \frac{1}{2}\int L_{OU}\Gamma(f, f) e^{-\Phi(u)} du - \int \Gamma(f, L_{OU} f) e^{-\Phi(u)} du \\ &= -\int \Gamma(f, L_{OU} f) e^{-\Phi(u)} du \\ &= -\frac{1}{2}\int L_{OU}(f L_{OU} f) e^{-\Phi(u)} du + \frac{1}{2}\int f L_{OU} L_{OU} f e^{-\Phi(u)} du \\ &\quad + \frac{1}{2}\int (L_{OU} f)^2 e^{-\Phi(u)} du \\ &= \|L_{OU} f\|_{L^2(e^{-\Phi})}^2 \\ &\leqslant 4\|g\|_{L^2(e^{-\Phi})}^2. \end{aligned} \tag{2.90}$$

This last inequality implies (i). Set $h = |\nabla f|_2$ so that $\partial_{u_j} h = \frac{\partial_{u_j} f \partial^2_{u_j} f}{|\nabla f|_2}$. Following the line of the proof of Lemma A.18 in [98] and noting that $\Delta\Phi = \sum_{i=1}^n \alpha_i$, one obtains

$$\int |\nabla\Phi|_2^2 h^2 e^{-\Phi} du \leqslant \sum_{i=1}^n \alpha_i \int h^2 e^{-\Phi} du + 2\sqrt{(\int |\nabla\Phi|_2^2 h^2 e^{-\Phi} du)(\int |\nabla h|_2^2 e^{-\Phi} du)}. \quad (2.91)$$

Using the Young's inequality $2ab \leqslant \delta^2 a^2 + \frac{b^2}{\delta^2}$ with $\delta^2 = 1/2$, one has

$$\int |\nabla\Phi|_2^2 h^2 e^{-\Phi} du \leqslant 2\sum_{i=1}^n \alpha_i \int h^2 e^{-\Phi} du + 4\int |\nabla h|_2^2 e^{-\Phi} du. \quad (2.92)$$

Since

$$\begin{aligned} |\nabla h|_2^2 &= \sum_{j=1}^n \left(\frac{\partial_{u_j} f}{|\nabla f|_2}\right)^2 (\partial^2_{u_j} f)^2 \\ &\leqslant \sum_{j=1}^n (\partial^2_{u_j} f)^2 \\ &\leqslant |Hess(f)|_2^2, \end{aligned} \quad (2.93)$$

we obtain from (2.89) and (2.90)

$$\begin{aligned} \| |\nabla\Phi|_2 . |\nabla f|_2 \|^2_{L^2(e^{-\Phi})} &\leqslant 2(\sum_{i=1}^n \alpha_i) \int |\nabla f|_2^2 e^{-\Phi} du + 4\int |Hess(f)|_2^2 e^{-\Phi} du \\ &\leqslant 4(\sum_{i=1}^n \alpha_i + 4) \|g\|^2_{L^2(e^{-\Phi})}. \end{aligned} \quad (2.94)$$

□

Corollary 2.3. *Hypothesis (H4') b) holds with*

$$N_2 = 2\frac{n}{\min\{|\lambda_j|, j = 1, \ldots, n\}} \sup_{i=1,\ldots,n} \|\nabla e_i\|_\infty^2 \sqrt{4 + \sum_{i=1}^n \alpha_i + 4\|\sum_i e_i^2\|_\infty}$$

Proof. Let $f \in F \cap D$. To shorten notation we identify f and $\imath^{-1}(f) \in \tilde{D}$. Then equation (2.76) and Cauchy-Schwarz inequality implies

$$\begin{aligned} |A^2 f| &\leqslant \sum_{j,k=1}^n |\partial_{u_j} f| |\nabla e_j|_M \frac{\alpha_k}{|\lambda_k|} u_k |\nabla e_k|_M + |\sum_{k,j=1}^n (\partial_{u_j u_k} f) e_j e_k| \\ &\leqslant (\sum_{j=1}^n \partial_{u_j} f| |\nabla e_j|_M)(\sum_{k=1}^n (\alpha_k u_k)|\nabla e_k|_M)\lambda_* + |\sum_{k,j=1}^n (\partial_{u_j u_k} f) e_j e_k| \\ &\leqslant n\lambda_* \sqrt{\sum_{i=1}^n (\partial_{u_i} f |\nabla e_i|_M)^2} \sqrt{\sum_{i=1}^n (\alpha_i u_i)^2 |\nabla e_i|_M^2} + |Hess(f)|_2 (\sum_i e_i^2) \\ &\leqslant n\lambda_* \sup_i \|\nabla e_i\|_\infty^2 |\nabla f|_2 |\nabla\Phi|_2 + |Hess(f)|_2 \|\sum_i e_i^2\|_\infty, \end{aligned}$$

where $\lambda_* = \frac{1}{\min\{|\lambda_j|, j=1,\cdots,n\}}$. The result then follows from the preceding lemma. □

2.8 Exponential decay in the total variation norm

The idea for proving the exponential decay in total variation consists on translating our problem to a setting for which the arguments used for the exponential decay in $L^2(\mu)$ remain valid.

Let $z_0 \in \mathbb{M}$. Since for all $t > 0$, $P_t(z_0, dz) = p_t(z_0, z)dz$ where $p_t(z_0, .)$ is a smooth function and that the invariant probability μ has a smooth density φ, one has

$$\|P_t(z_0, dz) - \mu(dz)\|_{TV} = \int_{\mathbb{M}} |p_t(z_0, z) - \varphi(z)|dz.$$

Because $\varphi > 0$, we can define a function $h(t, z_0, .)$ by

$$h(t, z_0, z) = \frac{p_t(z_0, z)}{\varphi(z)}$$

By Proposition 2.6, $P_t(z_0, dz)$ has a compact support, ie $p_t(z_0, .)$ has a compact support. Hence so does $h(t, z_0, .)$. Moreover the smoothness of φ and $p_t(z_0, .)$ implies the smoothness of $h(t, z_0, .)$. Consequently, $h(t, z_0, .) \in L^2(\mathbb{M}, \mu)$ and

$$\begin{aligned}\int |p_t(z_0, z) - \varphi(z)|dz &= \int |h(t, z_0, z) - 1|\mu(dz) \\ &\leqslant \left(\int (h(t, z_0, z) - 1)^2 \mu(dz)\right)^{\frac{1}{2}} \\ &= \|h(t, z_0, .) - 1\|_{L^2(\mu)}. \end{aligned} \tag{2.95}$$

Since $\int_{\mathbb{M}} h(t, z_0, y)\mu(dy) = 1$ for all t and z_0, we have a similar formulation to the one of Theorem 2.9.

So, in order to give the exponential rate of convergence, we will show that $h(t, z_0, .)$ is solution to the abstract Cauchy problem $\partial_t u(t) = \mathcal{L}_2 u(t)$ in $L^2(\mu)$ where $\mathcal{L}_2$ is an operator for which the arguments used for $\mathcal{L}$ remain valid.
In the following, we denote by h_t (resp. p_t) the function $h_t(z_0, .)$ (resp. $p_t(z_0, .)$)

Since $\partial_t p_t(z_0, .) = L^*(p_t(z_0, .))$ by Theorem 3.(iii) in [67] (recall that L^* is defined by (2.53)), then

$$\begin{aligned}\partial_t h_t &= \frac{\partial_t p_t}{\varphi} \\ &= \frac{L^*(p_t)}{\varphi} \\ &= \frac{\sigma^2}{2}\Delta_M h_t + \sum_{k=1}^n a_k u_k \frac{div_x(\nabla e_k(x) p_t)}{\varphi} - \sum_{k=1}^n \frac{\partial_{u_k} p_t}{\varphi} e_k(x)\end{aligned} \tag{2.96}$$

Because $\partial_{u_k}\varphi = -a_k u_k |\lambda_k| \varphi$,

$$-\frac{\partial_{u_k} p_t}{\varphi} = -\partial_{u_k} h_t + a_k u_k |\lambda_k| h_t.$$

Moreover,

$$\frac{div_x(\nabla e_k(x)p_t)}{\varphi} = \Delta_M(e_k)h_t + (\nabla e_k(x), \nabla_x h_t)_{TM}.$$

Hence,

$$\begin{aligned} \partial_t h_t &= \frac{\sigma^2}{2}\Delta_M h_t + \sum_{k=1}^{n} a_k u_k (\nabla e_k(x), \nabla_x h_t)_{TM} - \sum_{k=1}^{n} \partial_{u_k} h_t e_k(x) \\ &=: L_2 h_t. \end{aligned}$$

Thus, $h_t = T(t-1)h_1$, where $T(t)$ is the semi-group whose infinitesimal generator restricted to $C_c^\infty(\mathbb{M})$ is L_2. Because

$$L_2 = S + \sum_{k=1}^{n} A_k,$$

whereas

$$L = S - \sum_{k=1}^{n} A_k,$$

L_2 is the adjoint operator of L in $L^2(\mu)$. So all the arguments used for proving Theorem 2.10 for L work for L_2.
Applying Theorem 2.10 to L_2 with $g_t = h_{t+1}$ gives the result.

Chapter 3

Self-Repelling diffusions via an infinite dimensional approach

We reproduce in this chapter the paper [14]. It is a joint work with M. Benaïm and I. Ciotir which is published in *Stochastic Partial Differential Equations: Analysis and Computations*, Volume 3, Issue 4 (2015), pages 506-530.

Some redundancies with the Introduction chapter are therefore possible.

Keywords: reinforced process, self-interacting diffusions, stochastic equations in Banach spaces, Feller property, invariant probability measure

MSC: 60K35, 60H10, 60H30

3.1 Introduction

In the present chapter we are interested in stochastic differential equations of the type

$$X_t = x + \int_0^t g\left(X_s\right) ds - \int_0^t \int_0^s f'\left(X_s - X_r\right) dr ds + \beta_t \tag{3.1}$$

where $x \in \mathbb{R}$, β_t is a standard 1D Brownian motion and f is a 2π- periodic function with sufficient regularity. The initial drift profile g shall be chosen in a convenient form detailed below, in order to assure the Markov property of the process.

The motivating example of this equation comes from physics, and more precisely from systems that model the shape of a growing polymer.

A first model was introduced in the framework of random walks by Coppersmith and Diaconis in [34] and intensively studied later (see [10], [41], [83]). The continuous time corresponding processes were also studied under different assumptions on f.

One of the first papers was published by Norris, Rogers and Williams in 1987 and gives a Brownian model with local time drift for self-avoiding random walk, i.e.,

$$X_t = \beta_t - \int_0^t g\left(X_s, L\left(s, X_s\right)\right) ds$$

where $\{L(t,x); t \geqslant 0, x \in \mathbb{R}\}$ is the local time process of X. The main difficulty in this approach is the lack of Markov property (see [82]).

In 1992 Durrett and Rogers studied asymptotic behavior of Brownian polymers. More precisely they are interested in processes of the form

$$X_t = \beta_t + \int_0^t \int_0^s f(X_s - X_r)\, drds$$

where $f(x) = \Psi(x)\, x / \|x\|$, $\Psi(x) \geqslant 0$ (see [44]).

An extended study was also made by Benaïm, Ledoux, Raimond in the series of papers on self interacting diffusions (see [19], [20], [21]).

In a recent paper, Tarrès, Tóth and Valkó proved that a smeared-out version of the local time function from the point of view of the actual position of the process is Markov (see [96])

In the present work we study equation (3.1) following an infinite dimensional approach. In fact we show that, by choosing a particular form for the initial drift profile g and by taking the Fourier development of the function f, the stochastic differential equation becomes equivalent to a system in $\mathbb{R} \times l^2 \times l^2$. Consequently, the problem can be treated by using tools from the theory of stochastic differential equations in infinite dimensions and we show existence and uniqueness of the solution with Markov property.

Then we prove Feller property for the transition semigroup and we show that the system has an invariant probability measure which is explicitly given.

In the sequel, we denote by $C([0,\infty]; H)$ the space of continuous functions from $[0,\infty]$ to the Hilbert space H, by $C_b^k(H)$ the space of bounded functions from H to $\mathbb{R}$ that are k times continuously Fréchet differentiable with bounded derivatives up to order k, and by $L^\infty_{loc}([0,\infty); H)$ the space of functions from $(0,\infty)$ to H which are locally L^∞.

3.2 Equivalence with an infinite dimensional system

Consider the stochastic differential equation

$$X_t = x + \int_0^t g(X_s)\, ds - \int_0^t \int_0^s f'(X_s - X_r)\, drds + \beta_t \tag{3.2}$$

for $x \in \mathbb{R}$ and β_t a standard 1D Brownian motion.

We assume that f is an even, 2π periodical function and sufficiently regular such that the coefficients $(a_n)_n$ of the corresponding Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \tag{3.3}$$

form a positive rapidly decreasing sequence. For reader's convenience, we recall the definition of the space of rapidly decreasing sequences of order k

$$O^k = \left\{ (a_n)_n ; \sum_{n=1}^{\infty} \left(1+n^2\right)^k a_n^2 < \infty \right\}. \tag{3.4}$$

In our case $(a_n)_n$ is assumed to belong at least to O^5 and for that it is sufficient to have f in the Sobolev space $H^5_{2\pi}(\mathbb{R})$ of 2π periodic functions.

We choose an initial drift profile g of the form

$$g(x) = \sum_{n=1}^{\infty} a_n^{1/2} n \left(u_0^{(n)} \sin(nx) + v_0^{(n)} \cos(nx) \right), \tag{3.5}$$

where $\left(u_0^{(n)}\right)_n$ and $\left(v_0^{(n)}\right)_n$ are two arbitrary sequences from l^2.

Since f' and g are both 2π-periodic, $(X_t)_{t\geqslant 0}$ might be interpreted as an angle. Consequently X_t could be identified to the point $(\cos(X_t), \sin(X_t)) \in \mathbb{S}^1$. For more details see for example [42].

By standard computation we see that

$$\begin{aligned} -f'(X_s - X_r) &= \sum_n a_n^{1/2} n \sin(nX_s) \left(a_n^{1/2} \cos(nX_r) \right) \\ &\quad - \sum_n a_n^{1/2} n \cos(nX_s) \left(a_n^{1/2} \sin(nX_r) \right). \end{aligned}$$

If we replace (3.3) and (3.5) in (3.2) and set

$$u_t^{(n)} = u_0^{(n)} + a_n^{1/2} \int_0^t \cos(nX_s)\, ds$$

$$v_t^{(n)} = v_0^{(n)} - a_n^{1/2} \int_0^t \sin(nX_s)\, ds$$

we can rewrite equation (3.2) as a system in the Hilbert space $H = \mathbb{R} \times l^2 \times l^2$ as

$$\begin{cases} X_t = x + \int_0^t \sum_n n \left(a_n^{1/2} \sin(nX_s)\, u_s^{(n)} + a_n^{1/2} \cos(nX_s)\, v_s^{(n)} \right) ds + \beta_t, \\ u_t^{(n)} = u_0^{(n)} + a_n^{1/2} \int_0^t \cos(nX_s)\, ds, \quad n \geq 1, \\ v_t^{(n)} = v_0^{(n)} - a_n^{1/2} \int_0^t \sin(nX_s)\, ds, \quad n \geq 1, \end{cases}$$

or equivalently as a stochastic differential equation in a Hilbert space

$$Y_t = y + \int_0^t F(Y_s)\, ds + \sigma dW_t$$

where the process

$$Y_t = \left(X_t, \left(u_t^{(n)}\right)_n, \left(v_t^{(n)}\right)_n \right) \in H$$

and the operator $F : H \to H$ is defined by

$$F\begin{pmatrix} x \\ \left(u^{(n)}\right)_n \\ \left(v^{(n)}\right)_n \end{pmatrix} \tag{3.6}$$

$$= \begin{pmatrix} \left\langle \left(a_n^{1/2} n \sin(nx)\right)_n, \left(u^{(n)}\right)_n \right\rangle_{l^2} + \left\langle \left(a_n^{1/2} n \cos(nx)\right)_n, \left(v^{(n)}\right)_n \right\rangle_{l^2} \\ \left(a_n^{1/2} \cos(nx)\right)_n \\ -\left(a_n^{1/2} \sin(nx)\right)_n \end{pmatrix}$$

and W_t is a cylindrical Wiener process with values in H and the noise $\sigma = (1,\ 0,\ 0)$ is the *projection on the first coordinate.*

The hypotheses from this section are assumed for the rest of the paper. We shall denote by C a positive constant which might change from line to line.

3.3 Existence and uniqueness of the solution for the infinite dimensional equation

We consider the equation from the previous section

$$\begin{cases} dY_t = F(Y_t)\, dt + \sigma dW_t \\ Y_0 = y \end{cases} \tag{3.7}$$

for an initial condition $y \in \mathbb{R}\times l^2 \times l^2$ and F defined in (3.6).

We can now formulate the existence result.

Proposition 3.1. *Under the assumptions presented above, for each $y \in H$, there is a unique analytically strong solution*

$$Y \in C([0,\infty); H) \cap L^\infty_{loc}(0,\infty;\ H)$$

to equation (3.7).

Moreover, for $T < \infty$, we have that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t|_H^2\right) < \infty.$$

Proof. We study equation (3.7) in the framework of the analytic approach of stochastic differential equations in Banach spaces, and more precisely in the space $H = \mathbb{R} \times l^2 \times l^2$ equipped with the norm

$$\|y\|_H^2 = |x|^2 + \|(u_n)_n\|_{l^2}^2 + \|(v_n)_n\|_{l^2}^2 ,$$

for all $y = (x, (u_n)_n, (v_n)_n) \in \mathbb{R}\times l^2 \times l^2$.

Since the operator F defined before is not Lipschitz in H, we may use Theorem 7.10 from page 198 of [39] in order to get existence of the solution to equation (3.7).

More precisely, we shall prove that the following three conditions are satisfied for the operator F defined in (3.6)

a) F is locally Lipschitz continuous in H

b) F is bounded on bounded subsets of H

c) there exists an increasing function
$$a : \mathbb{R}_+ \to \mathbb{R}_+,$$
such that
$$\langle F(y+\widetilde{y}),\ y^*\rangle \le a(\|\widetilde{y}\|_H)(1+\|y\|_H)$$
for all $y,\ \widetilde{y} \in H$ and $y^* \in \partial\|y\|$, where $\langle .,. \rangle$ is the duality form on H and $\partial\|.\|$ is the subdifferential of the H norm.

We shall first prove a).

Indeed, for all y and $\widetilde{y}$ from H we have that
$$\|F(y) - F(\widetilde{y})\|_H^2$$
$$= \left\| \begin{pmatrix} \sum_n a_n^{1/2} n(\sin(nx)u_n + \cos(nx)v_n - \sin(n\widetilde{x})\widetilde{u}_n - \cos(n\widetilde{x})\widetilde{v}_n) \\ \left(a_n^{1/2}\cos(nx)\right)_n - \left(a_n^{1/2}\cos(n\widetilde{x})\right)_n \\ -\left(a_n^{1/2}\sin(nx)\right)_n + \left(a_n^{1/2}\sin(n\widetilde{x})\right)_n \end{pmatrix} \right\|_H^2$$
$$= \left|\sum_n a_n^{1/2} n(\sin(nx)u_n + \cos(nx)v_n - \sin(n\widetilde{x})\widetilde{u}_n - \cos(n\widetilde{x})\widetilde{v}_n)\right|^2$$
$$+\left\|\left(a_n^{1/2}\cos(nx)\right)_n - \left(a_n^{1/2}\cos(n\widetilde{x})\right)_n\right\|_{l^2}^2$$
$$+\left\|\left(a_n^{1/2}\sin(nx)\right)_n - \left(a_n^{1/2}\sin(n\widetilde{x})\right)_n\right\|_{l^2}^2$$
$$\overset{Denote}{=} T_1 + T_2 + T_3. \tag{3.8}$$

For the first term we see that
$$T_1 \le 2\left|\sum_n a_n^{1/2} n(\sin(nx)u_n - \sin(n\widetilde{x})\widetilde{u}_n)\right|^2$$
$$+2\left|\sum_n a_n^{1/2} n(\cos(nx)v_n - \cos(n\widetilde{x})\widetilde{v}_n)\right|^2$$

$$\leq 4\left|\sum_n a_n^{1/2} n \sin(nx)(u_n - \widetilde{u}_n)\right|^2 + 4\left|\sum_n a_n^{1/2} n (\sin(nx) - \sin(n\widetilde{x}))\widetilde{u}_n\right|^2$$

$$+4\left|\sum_n a_n^{1/2} n \cos(nx)(v_n - \widetilde{v}_n)\right|^2 + 4\left|\sum_n a_n^{1/2} n (\cos(nx) - \cos(n\widetilde{x}))\widetilde{v}_n\right|^2$$

and then, by the Cauchy-Schwarz inequality for the inner product in l^2 and taking into account that $(a_n)_n \in O^5$, we obtain that

$$\begin{aligned}
T_1 &\leq C\left\|\left(a_n^{1/2} n \sin(nx)\right)_n\right\|_{l^2}^2 \left\|(u_n)_n - (\widetilde{u}_n)_n\right\|_{l^2}^2 \\
&\quad + C\left\|\left(a_n^{1/2} n \cos(nx)\right)_n\right\|_{l^2}^2 \left\|(v_n)_n - (\widetilde{v}_n)_n\right\|_{l^2}^2 \\
&\quad + \left((\sum_n a_n^{1/2} n^2 |\widetilde{u}_n|)^2 + (\sum_n a_n^{1/2} n^2 |\widetilde{v}_n|)^2\right) |x - \widetilde{x}|^2 \\
&\leq C\left\|(u_n)_n - (\widetilde{u}_n)_n\right\|_{l^2}^2 + C\left\|(v_n)_n - (\widetilde{v}_n)_n\right\|_{l^2}^2 \\
&\quad + C\left(\|(\widetilde{u}_n)\|_{l^2}^2 + \|(\widetilde{v}_n)\|_{l^2}^2\right)|x - \widetilde{x}|^2 \\
&\leq C(1 + \|(\widetilde{u}_n)\|_{l^2}^2 + \|(\widetilde{v}_n)\|_{l^2}^2) \\
&\quad \times \left(|x - \widetilde{x}|^2 + \left\|(u_n)_n - (\widetilde{u}_n)_n\right\|_{l^2}^2 + \left\|(v_n)_n - (\widetilde{v}_n)_n\right\|_{l^2}^2\right),
\end{aligned}$$

which leads to

$$T_1 \leq C(1 + \|(\widetilde{u}_n)\|_{l^2}^2 + \|(\widetilde{v}_n)\|_{l^2}^2)\, \|y - \widetilde{y}\|_H^2$$

where C is a positive constant depending on $(a_n)_n$ which might change from line to line.

Keeping in mind that $(a_n)_n \in O^5$, we can easily see that the second and the third term verify

$$\begin{aligned}
T_2 &= \sum_n \left|a_n^{1/2}(\cos(nx) - \cos(n\widetilde{x}))\right|^2 \\
&\leq \sum_n \left|a_n^{1/2} n (x - \widetilde{x})\right|^2 \\
&\leq \sum_n a_n n^2 |x - \widetilde{x}|^2 \\
&\leq C|x - \widetilde{x}|^2
\end{aligned}$$

and, by a similar argument,
$$T_3 \leq C\left|x-\widetilde{x}\right|^2.$$

Going back to (3.8) we obtain that

$$\begin{aligned} \left\|F(y)-F(\widetilde{y})\right\|_H^2 &\leq C(1+\|(\widetilde{u}_n)\|_{l^2}^2+\|(\widetilde{v}_n)\|_{l^2}^2)\left\|y-\widetilde{y}\right\|_H^2 \\ &\leq C\left(1+\|\widetilde{y}\|_H^2\right)\left\|y-\widetilde{y}\right\|_H^2 \end{aligned} \tag{3.9}$$

where C is a positive constant depending on $(a_n)_n$.

Consequently, for all $y,\widetilde{y}\in B(0,R)$ we obtain that

$$\begin{aligned} \left\|F(y)-F(\widetilde{y})\right\|_H &\leq C\sqrt{(1+\|(\widetilde{u}_n)\|_{l^2}^2+\|(\widetilde{v}_n)\|_{l^2}^2)}\left\|y-\widetilde{y}\right\|_H \\ &\leq \sqrt{C\left(1+\|\widetilde{y}\|_H^2\right)}\left\|y-\widetilde{y}\right\|_H \\ &\leq C\left(R,(a_n)_n\right)\left\|y-\widetilde{y}\right\|_H \end{aligned} \tag{3.10}$$

where $C\left(R,(a_n)_n\right)$ is a positive constant depending on R and $(a_n)_n$, and the proof of the locally Lipschitz property is completed.

For the proof of b) it is sufficient to take $\widetilde{y}=0$ in (3.9). We obtain then

$$\begin{aligned} \left\|F(y)\right\|_H &\leq \left\|F(y)-F(0)\right\|_H+\left\|F(0)\right\|_H \\ &\leq C\left\|y\right\|_H+\left\|\left(a_n^{1/2}\right)_n\right\|_{l^2} \\ &\leq C\left(\left\|y\right\|_H+1\right) \end{aligned} \tag{3.11}$$

where C is a positive constant depending on $(a_n)_n$ which might change from line to line. Consequently, F is bounded on bounded subsets of H.

In order to complete the proof of existence, we still have to prove c) and to this purpose we need to find an increasing function

$$a:\mathbb{R}_+\to\mathbb{R}_+,$$

such that
$$\left\langle F(y+\widetilde{y}),\ y^*\right\rangle \leq a\left(\|\widetilde{y}\|_H\right)\left(1+\|y\|_H\right)$$

for all $y,\ \widetilde{y}\in H$ and $y^*\in\partial\left\|y\right\|$.

For that purpose, we consider the function $a(\alpha)=C\left(1+\alpha\right)$, where C is the constant from (3.9).The constant being positive, the function is clearly increasing on $\mathbb{R}_+$.

Since the subdifferential of the application

$$y\to\frac{1}{2}\left\|y\right\|_H^2$$

is the duality mapping of the space H, and in our case $H = H^*$, we have that

$$\partial \|y\|_H = \begin{cases} \left\{ \dfrac{y}{\|y\|_H} \right\}, & \text{for } y \neq 0 \\ \{\|y\|_H \leq 1\}, & \text{for } y = 0 \end{cases},$$

(see page 72 from [40]).

Since the case $y = 0$ is trivial, we only need to prove that

$$\left\langle F(y + \widetilde{y}), \frac{y}{\|y\|_H} \right\rangle_H \leq a(\|\widetilde{y}\|_H)(1 + \|y\|_H).$$

Indeed, for all $y = \left(x, \left(u^{(n)}\right)_n, \left(v^{(n)}\right)_n\right)$ and $\widetilde{y} = \left(\widetilde{x}, \left(\widetilde{u}^{(n)}\right)_n, \left(\widetilde{v}^{(n)}\right)_n\right)$ in H we have that

$$\begin{aligned} \left\langle F(y + \widetilde{y}), \frac{y}{\|y\|_H} \right\rangle_H &\leq \left\langle F(y + \widetilde{y}) - F(\widetilde{y}), \frac{y}{\|y\|_H} \right\rangle_H \\ &\quad + \left\langle F(\widetilde{y}), \frac{y}{\|y\|_H} \right\rangle_H \\ &\leq C\sqrt{\left(1 + \|\widetilde{y}\|_H^2\right)}\, \|y\|_H + C(\|\widetilde{y}\|_H + 1) \\ &\leq C(1 + \|\widetilde{y}\|_H)(1 + \|y\|_H) \end{aligned}$$

where C is a positive constant depending only on $(a_n)_n$ which might change from line to line. Hence, we obtain

$$\left\langle F(y + \widetilde{y}), \frac{y}{\|y\|} \right\rangle_H \leq a(\|\widetilde{y}\|_H)(1 + \|y\|_H).$$

We have now existence of an unique mild solution. Since in our case the generator of C_0-semigroup is identically zero, a solution is strong if and only if it is mild (see [87]), so we have also existence and uniqueness of a strong solution.

Consequently, the proof of existence and uniqueness is complete.

We shall now prove that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|Y_t\|_H^2\right) < \infty.$$

To this purpose we apply the Itô formula to equation (3.7) with the function

$$y \mapsto \frac{1}{2}\|y\|_H^2$$

and we get

$$\begin{aligned} \frac{1}{2}\|Y(t)\|_H^2 &= \frac{1}{2}\|y\|_H^2 + \int_0^t \langle F(Y(s)), Y(s)\rangle_H ds \\ &\quad + \int_0^t \langle Y(s), \sigma dW_s\rangle_H + \frac{1}{2}\int_0^t |\sigma|^2 ds. \end{aligned} \tag{3.12}$$

We can easily see that

$$\begin{aligned} \int_0^t \langle Y(s), \sigma dW_s\rangle_H &= \int_0^t X(s)\, d\beta_s \\ &\leq \sup_{t\in[0,T]} \left|\int_0^t X(s)\, d\beta_s\right| \end{aligned}$$

and then, by using the Burkholder-Davis-Gundy inequality, we obtain that

$$\mathbb{E}\left(\sup_{r\in[0,t]}\left|\int_0^r X(s)\, d\beta_s\right|\right) \leq C\mathbb{E}\left(\int_0^t |X(s)|^2 ds\right)^{1/2}$$

(see, e.g., [40] page 58).

On the other hand we see that, by (3.11), we get that

$$\begin{aligned} \langle F(Y(s)), Y(s)\rangle_H &\leq \|F(Y(s))\|_H \|Y(s)\|_H \\ &\leq C(\|Y(s)\|_H + 1)\|Y(s)\|_H \\ &\leq C\left(1 + \|Y(s)\|_H^2\right), \end{aligned}$$

where C is a positive constant depending only on $(a_n)_n$ that changes from line to line.

By going back into (3.12) we obtain via the estimates above that

$$\mathbb{E}\left(\sup_{r\in[0,t]}\|Y(r)\|_H^2\right) \leq \|y\|_H^2 + C\mathbb{E}\int_0^t \left(\sup_{r\in[0,s]}\|Y(r)\|_H^2\right) ds + Ct,$$

and finally, by Gronwall's lemma we obtain

$$\mathbb{E}\left(\sup_{t\in[0,T]}\|Y_t\|_H^2\right) \leq Ce^{CT}\left(\|y\|_H^2 + T\right) < \infty$$

and the proof is now complete. □

Remark 3.1. *Note that the solution obtained above has the Markov property. For details see Theorem 9.8 from [39].*

3.4 The Feller property of the transition semigroup

We consider the transition semigroup corresponding to the solution $Y(t,y)$ defined by

$$P_t\varphi(y) = \mathbb{E}\left[\varphi(Y(t,y))\right],$$

for all $\varphi \in B_b(H)$, the space of all bounded and Borel real functions in H, for all $t \geq 0$ and for all $y \in H$.

We intend to prove that the semigroup has the Feller property which means that it maps bounded continuous functions into bounded continuous functions.

Proposition 3.2. *Let $(y_k)_{k\in\mathbb{N}}$ be a sequence of initial conditions from H such that $y_k \to y$ in H for $k \to \infty$. If we denote by*

$$Y_k(t) = \left(X_t^k,\ \left(u_t^{(n)}\right)_n^k,\ \left(v_t^{(n)}\right)_n^k\right)$$

and

$$Y(t) = \left(X_t,\ \left(u_t^{(n)}\right)_n,\ \left(v_t^{(n)}\right)_n\right)$$

the solutions to equation (3.7) corresponding to every y_k and respectively to y, then, for any $t > 0$, we have that

$$\|Y_k(t) - Y(t)\|_H^2 \to 0 \quad \text{as } k \to \infty.$$

In particular we have also that $(Y_t)_{t\geq 0}$ is a Feller process.

Proof. We shall check first the following a priori estimates.

Since

$$u_t^{(n)} = u_0^{(n)} + a_n^{1/2}\int_0^t \cos(nX_s)\,ds$$

we can easily obtain that

$$\begin{aligned} \left\|\left(u_t^{(n)}\right)_n\right\|_{l^2}^2 &\leq 2\left\|\left(u_0^{(n)}\right)_n\right\|_{l^2}^2 + 2t^2\left\|\left(a_n^{1/2}\right)_n\right\|_{l^2}^2 \\ &\leq C\left(\left(u_0^{(n)}\right)_n\right)(1+t)^2, \end{aligned} \tag{3.13}$$

where $C\left(\left(u_0^{(n)}\right)_n\right)$ is a constant which might change from line to line, depending on the initial condition $\left(u_0^{(n)}\right)_n$.

Of course, by the same argument, we get that

$$\left\|\left(v_t^{(n)}\right)_n\right\|_{l^2}^2 \leq C\left(\left(v_0^{(n)}\right)_n\right)(1+t)^2. \tag{3.14}$$

By taking the inner product in H between the difference

$$\frac{d}{dt}\left(Y_k\left(t\right)-Y\left(t\right)\right)=F\left(Y_k\left(t\right)\right)-F\left(Y\left(t\right)\right)$$

and $\left(Y_k\left(t\right)-Y\left(t\right)\right)$ and keeping in mind that

$$\left\langle \frac{d}{dt}\left(Y_k\left(t\right)-Y\left(t\right)\right),\left(Y_k\left(t\right)-Y\left(t\right)\right)\right\rangle_H=\frac{d}{dt}\left(\frac{1}{2}\left\|\left(Y_k\left(t\right)-Y\left(t\right)\right)\right\|_H^2\right),$$

we get that

$$\begin{aligned}\left\|Y_k\left(t\right)-Y\left(t\right)\right\|_H^2 &= \left\|y_k-y\right\|_H^2 \\ &\quad +2\int_0^t\left\langle F\left(Y_k\left(s\right)\right)-F\left(Y\left(s\right)\right),Y_k\left(s\right)-Y\left(s\right)\right\rangle_H ds.\end{aligned} \tag{3.15}$$

We can see by (3.10) that

$$\begin{aligned}&\left\langle F\left(Y_k\left(s\right)\right)-F\left(Y\left(s\right)\right),Y_k\left(s\right)-Y\left(s\right)\right\rangle_H \\ \leq\ &\left\|F\left(Y_k\left(s\right)\right)-F\left(Y\left(s\right)\right)\right\|_H\left\|Y_k\left(s\right)-Y\left(s\right)\right\|_H \\ \leq\ &C\sqrt{(1+\left\|\left(u_s^{(n)}\right)\right\|_{l^2}^2+\left\|\left(v_s^{(n)}\right)\right\|_{l^2}^2)}\left\|Y_k\left(s\right)-Y\left(s\right)\right\|_H^2\end{aligned}$$

and then, by (3.13) and (3.14) we see that

$$\begin{aligned}&\left\langle F\left(Y_k\left(s\right)\right)-F\left(Y\left(s\right)\right),Y_k\left(s\right)-Y\left(s\right)\right\rangle_H \\ \leq\ &C\left(\left(u_0^{(n)}\right)_n,\left(v_0^{(n)}\right)_n\right)\left(1+s\right)\left\|Y_k\left(s\right)-Y\left(s\right)\right\|_H^2,\end{aligned}$$

where C is a positive constant which might depend on $\left(a_n\right)_n$ and also on the initial condition $y=\left(x,\left(u_0^{(n)}\right)_n,\left(v_0^{(n)}\right)_n\right)$.

Finally, from (3.15) we have that

$$\begin{aligned}&\left\|Y_k\left(t\right)-Y\left(t\right)\right\|_H^2 \\ =\ &\left\|y_k-y\right\|_H^2+2\int_0^t\left\langle F\left(Y_k\left(s\right)\right)-F\left(Y\left(s\right)\right),Y_k\left(s\right)-Y\left(s\right)\right\rangle_H ds \\ \leq\ &\left\|y_k-y\right\|_H^2+C\left(y\right)\int_0^t\left(1+s\right)\left\|Y_k\left(s\right)-Y\left(s\right)\right\|_H^2 ds\end{aligned}$$

where C is a positive constant depending on $(a_n)_n$ and also on the initial condition $y = \left(x, \left(u_0^{(n)}\right)_n, \left(v_0^{(n)}\right)_n\right)$.

Then, by Gronwall's lemma, we obtain that

$$\|Y_k(t) - Y(t)\|_H^2 \leq e^{C(y)\left(t+t^2\right)} \|y_k - y\|_H^2 .$$

Let $\varphi : H \to \mathbb{R}$ be a bounded and continuous function. Since L^2 convergence implies a convergence in probability, we then have that $\varphi(Y_k(t)) \to \varphi(Y(t))$ in probability (see Lemma 3.3 in [69]).

Consequently,

$$\lim_{k\to\infty} \mathbb{E}\varphi(Y_k(t)) = \mathbb{E}\varphi(Y(t)), \quad \text{for any fixed } t > 0,$$

which is actually

$$\lim_{k\to\infty} P_t\varphi(y_k) = P_t\varphi(y), \quad \text{for any fixed } t > 0,$$

and then we have proved the Feller property. □

Remark 3.2. *Let $\mathcal{A} = \mathbb{R} \times O^1 \times O^1$, with O^1 defined by (3.4). It easily follows from the definition of $(u_t^{(n)})_n$ and $(v_t^{(n)})_n$ that*

$$y \in \mathcal{A} \Leftrightarrow Y(t) \in \mathcal{A} \text{ for all } t \geqslant 0,$$

where $Y(t)$ is the solution of equation (3.7) with initial condition $Y(0) = y$. This makes $1_{\mathcal{A}}$ invariant under P_t (i.e., $P_t 1_{\mathcal{A}} = 1_{\mathcal{A}}$).
Hence the process $(Y_t)_t$ is not strongly Feller.

3.5 The invariant measure of the transition semigroup

In this section we shall prove existence of an invariant measure for the transition semigroup corresponding to the equation on $\mathbb{S}^1 \times l^2 \times l^2$

$$\begin{cases} dY_t = F(Y_t)\,dt + \sigma dW_t \\ Y_0 = y \end{cases} \tag{3.16}$$

with initial condition $y \in \mathbb{R}\times l^2 \times l^2$, where $\mathbb{S}^1$ is identified to $\mathbb{R}/2\pi\mathbb{Z}$.

A probability μ on H is said to be an invariant measure for the transition semigroup $(P_t)_t$ iff

$$\int_H P_t\varphi(y)\,\mu(dy) = \int_H \varphi(y)\,\mu(dy), \tag{3.17}$$

for all measurable and bounded function φ.

By standard arguments (see Theorem 1.2, page 8 from [27] and relation (1.5) at page 2 of [37]) it is sufficient that (3.17) holds for all $\varphi \in C_b(H)$.

Existence of an invariant measure of the transition semigroup

We consider the measure

$$\mu(dy) = \frac{dx}{2\pi} \otimes \prod_{n\geq 1} N\left(0, \frac{1}{n^2}\right) du_n \otimes \prod_{n\geq 1} N\left(0, \frac{1}{n^2}\right) dv_n \tag{3.18}$$

where $N\left(0, \frac{1}{n^2}\right)$ is the normal distribution. The form of μ is inspired from the finite dimensional case (see [16]).

First, the fact that μ is a probability measure on H is clearly explained in Exercise 2.1.8. from [87].

We intend to prove that μ is an invariant measure of $(P_t)_t$ on $\mathbb{S}^1 \times l^2 \times l^2$ by using the strong convergence of a Galerkin type approximation.

To this purpose, we consider that

$$H = H_N \times l^2 \times l^2$$

where $H_N = \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$, and

$$\Pi_N : H \to H_N \times \{0\}^\infty \times \{0\}^\infty$$

be defined by

$$\Pi_N \left(x, (u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}\right) = \left(x, (u_n)_{n=1}^N \times \{0\}^\infty, (u_n)_{n=1}^N \times \{0\}^\infty\right).$$

Obviously, the following stochastic equation on H_N

$$\begin{cases} dY_t^{(N)} = \Pi_N \left(F\left(Y_t^{(N)}\right)\right) dt + \sigma dW_t \\ Y_0^{(N)} = \Pi_N y \end{cases} \tag{3.19}$$

can be treated by classical results for the solvability of SDE in finite-dimension. Consequently, equation (3.19) has a unique strong solution.

We can now prove the following preliminary result.

Lemma 3.1. *Under the assumptions given before, the sequence of solutions* $\left(Y^{(N)}\right)_N$ *to equations (3.19) converges strongly in* H *to the solution* Y *to equation (3.7). More precisely we have that*

$$\lim_{N\to\infty} \sup_{0\leqslant t\leqslant T} \left\|Y^{(N)}(t) - Y(t)\right\|_H^2 = 0,$$

for all $T > 0$ *and* $\omega \in \Omega$.

Proof. By taking the inner product between $Y^{(N)}(t) - Y(t)$ and the difference

$$\frac{d}{dt}\left(Y^{(N)}(t) - Y(t)\right) = \left(\Pi_N F\left(Y^{(N)}(s)\right) - F(Y(s))\right)$$

we obtain that

$$\begin{aligned}
&\left\|Y^{(N)}(t)-Y(t)\right\|_H^2 \\
= &\ \|\Pi_N y-y\|_H^2 \\
&+2\int_0^t \left\langle \Pi_N F\left(Y^{(N)}(s)\right)-F(Y(s)),Y^{(N)}(s)-Y(s)\right\rangle_H ds \\
= &\ \|\Pi_N y-y\|_H^2 \\
&+2\int_0^t \left\langle \Pi_N F\left(Y^{(N)}(s)\right)-F\left(Y^{(N)}(s)\right),Y^{(N)}(s)-Y(s)\right\rangle_H ds \\
&+2\int_0^t \left\langle F\left(Y^{(N)}(s)\right)-F(Y(s)),Y^{(N)}(s)-Y(s)\right\rangle_H ds.
\end{aligned}$$

We can easily see that

$$\begin{aligned}
&\left\langle \Pi_N F\left(Y^{(N)}(s)\right)-F\left(Y^{(N)}(s)\right),Y^{(N)}(s)-Y(s)\right\rangle_H \\
\leq &\ \left\|\left(0,\left(a_n^{1/2}\cos\left(nX^{(N)}(s)\right)\right)_{n>N},-\left(a_n^{1/2}\sin\left(nX^{(N)}(s)\right)\right)_{n>N}\right)\right\|_H \\
&\times\left\|Y^{(N)}(s)-Y(s)\right\|_H \\
\leq &\ C\left\|\left(a_n^{1/2}\right)_{n>N}\right\|_{l^2}^2+\left\|Y^{(N)}(s)-Y(s)\right\|_H^2
\end{aligned}$$

and, by arguing as in Proposition 3.2, we have that

$$\begin{aligned}
&\left\langle F\left(Y^{(N)}(s)\right)-F(Y(s)),Y^{(N)}(s)-Y(s)\right\rangle_H \\
\leq &\ C(y)(1+s)\left\|Y^{(N)}(s)-Y(s)\right\|_H^2.
\end{aligned}$$

where C is a positive constant depending on $(a_n)_n$ and also on the initial condition $y=$

$\left(x,\left(u_0^{(n)}\right)_n,\left(v_0^{(n)}\right)_n\right)$.

We obtain, for $0 \leqslant t \leqslant T$, that

$$\begin{aligned}\left\|Y^{(N)}(t)-Y(t)\right\|_H^2 &\leq \|\Pi_N y-y\|_H^2+Ct\left\|\left(a_n^{1/2}\right)_{n>N}\right\|_{l^2}^2\\ &\quad +C(y)\int_0^t(1+s)\left\|Y^{(N)}(s)-Y(s)\right\|_H^2 ds.\\ &\leq \|\Pi_N y-y\|_H^2+CT\|(a_n^{1/2})_{n>N}\|_{l^2}^2\\ &\quad +C(y)(1+T)\int_0^t\|Y^{(N)}(s)-Y(s)\|_H^2 ds.\end{aligned}$$

By using Gronwall's lemma we deduce

$$\left\|Y^{(N)}(t)-Y(t)\right\|_H^2 \leq \left(\|\Pi_N y-y\|_H^2+CT\left\|\left(a_n^{1/2}\right)_{n>N}\right\|_{l^2}^2\right)e^{C(y,T)t}$$

and since

$$\lim_{N\to\infty}\|\Pi_N y-y\|_H^2=0$$

and

$$\lim_{N\to\infty}\left\|\left(a_n^{1/2}\right)_{n>N}\right\|_{l^2}^2=0$$

we can conclude the proof of this result. □

Proposition 3.3. *Under the assumptions presented above, the probability μ defined in (3.18) is an invariant measure of the transition semigroup $(P_t)_t$ of (3.16) on H.*

Proof. We define the measure

$$\begin{aligned}\mu_\infty^N(dy) &= \frac{dx}{2\pi}\otimes\prod_{n=1}^N N\left(0,\frac{1}{n^2}\right)du_n\otimes\prod_{n>N}\delta_0(du_n)\\ &\quad\otimes\prod_{n=1}^N N\left(0,\frac{1}{n^2}\right)dv_n\otimes\prod_{n>N}\delta_0(dv_n)\\ &\overset{Denote}{=}\frac{dx}{2\pi}\otimes\mu^N\left(d(u_n)_{n=1}^N\right)\otimes\mu^{N+}\left(d(u_n)_n\right)\\ &\quad\otimes\mu^N\left(d(v_n)_{n=1}^N\right)\otimes\mu^{N+}\left(d(v_n)_n\right),\end{aligned}$$

where δ_0 is the Dirac measure on $\mathbb{R}$.

Step I: We prove that

$$\mu_\infty^N \underset{N\to\infty}{\longrightarrow}\mu$$

for the topology of weak convergence, i.e.,

$$\mu_\infty^N\varphi \underset{N\to\infty}{\longrightarrow}\mu\varphi,\quad\forall\varphi\in C_b(H).$$

Let $\varphi \in C_b(H)$ and denote by $\varphi_N = \varphi(\Pi_N)$.

We can easily see that

$$\begin{aligned}\int_H \varphi(y)\,\mu_\infty^N(dy) &= \int_{H_N}\int_{l^2\times l^2} \varphi\left(y^N, y'\right)\mu^N\left(dy^N\right)\mu^{N+}(dy') \\ &= \int_{H_N} \varphi\left(y^N, 0, .., 0, ...\right)\mu^N\left(dy^N\right)\end{aligned}$$

and that

$$\int_{H_N} \varphi\left(y^N, 0, .., 0, ...\right)\mu^N\left(dy^N\right) = \int_H \varphi_N(y)\,\mu_\infty^N(dy) = \int_H \varphi_N(y)\,\mu(dy).$$

This leads to

$$\int_H \varphi(y)\,\mu_\infty^N = \int_H \varphi_N(y)\,\mu(dy).$$

Since

$$\lim_{N\to\infty} \Pi_N(y) = y$$

and keeping in mind that φ is bounded continuous, we have via Lebesgue dominated convergence theorem that

$$\lim_{N\to\infty}\int_H \varphi(\Pi_N(y))\,\mu(dy) = \int_H \varphi(y)\,\mu(dy),$$

and consequently

$$\lim_{N\to\infty}\int_H \varphi(y)\,\mu_\infty^N(dy) = \int_H \varphi(y)\,\mu(dy),$$

i.e.,

$$\mu_\infty^N \underset{N\to\infty}{\longrightarrow} \mu.$$

Step II: We show that μ is an invariant measure for the transition semigroup. Let P_t^N be the transition semigroup corresponding to (3.19). We take

$$\begin{aligned}\int_H P_t\varphi(y)\,\mu(dy) &= \int_H \left(P_t\varphi(y) - P_t^N\varphi(\Pi_N y)\right)\mu(dy) \\ &\quad + \int_H \left(P_t^N\varphi(\Pi_N y)\right)\mu(dy) \\ &\overset{Denote}{=} \varepsilon_N + \int_H \left(P_t^N\varphi(\Pi_N y)\right)\mu(dy).\end{aligned} \tag{3.20}$$

By the same arguments developed in [16] one can prove that μ^N is an invariant measure for P_t^N. So we obtain that

$$\begin{aligned}\int_H \left(P_t^N\varphi(\Pi_N y)\right)\mu(dy) &= \int_{H_N} P_t^N\varphi\left(y^N, 0, .., 0, ...\right)\mu^N\left(dy^N\right) \\ &= \int_{H_N} \varphi\left(y^N, 0, .., 0, ...\right)\mu^N\left(dy^N\right) \\ &= \int_H \varphi(y)\,\mu_\infty^N(dy).\end{aligned}$$

On the other hand we have that

$$P_t\varphi(y) - P_t^N\varphi(\Pi_N y) = \mathbb{E}\left(\varphi(Y_t) - \varphi_N\left(Y_t^{(N)}\right)\right).$$

By Lemma 3.1, we have

$$Y_t^{(N)} \to Y_t \text{ a.s.}$$

for $N \to \infty$, and thus for $\varphi \in C_b(H)$, we obtain that

$$\varphi(Y_t^{(N)}) \to \varphi(Y_t) \text{ a.s. }.$$

Because $\varphi(Y_t^{(N)})$ is bounded, this leads to

$$P_t^N\varphi(\Pi_N y) \to P_t\varphi(y),$$

for $N \to \infty$ by the Dominated Convergence Theorem.

Since $Y_t^{(N)}$ is a Feller process, we have that $P_t^{(N)}\varphi \in C_b(H)$ for all $\varphi \in C_b(H)$, and we get via the Lebesgue Dominated Convergence Theorem

$$\varepsilon_N = \int_H \left(P_t\varphi(y) - P_t^N\varphi(\Pi_N y)\right)\mu(dy) \to 0,$$

for $N \to \infty$.

Going back to (3.20) and passing to the limit for $N \to \infty$ we get that

$$\int_H P_t\varphi(y)\,\mu(dy) = \lim_{N\to\infty}\int_H \varphi(y)\,\mu_\infty^N(dy) = \int_H \varphi(y)\,\mu(dy).$$

The existence of an invariant measure is now completely proved. □

3.6 On the uniqueness of the invariant measure

In this section, we intend to give an important feature for the Kolmogorov operator L. Keeping in mind that σ is the projection on the first coordinate, we have

$$L\varphi(y) = \frac{1}{2}\partial_{xx}\varphi(y) + \partial_x\varphi(y)\sum_{n=1}^{\infty} n a_n^{1/2}\left(v_n\cos(nx) + u_n\sin(nx)\right) \tag{3.21}$$
$$+\sum_{n=1}^{\infty} a_n^{1/2}\cos(nx)\,\partial_{u_n}\varphi(y) - \sum_{n=1}^{\infty} a_n^{1/2}\sin(nx)\,\partial_{v_n}\varphi(y)$$

for $\varphi \in C_b^2(H)$ the class of all bounded functions which are twice Fréchet differentiable and whose derivatives are bounded.

We recall that the Kolmogorov operator associated to (3.16) is obtained by using Itô formula to function φ in $C_b^2(H)$ (for details see Theorem 5.4.2 from page 72 of [40]).

Set $S\varphi(y) = \frac{1}{2}\partial_{xx}\varphi(y)$ and $A\varphi(y) := L\varphi(y) - S\varphi(y)$.

Lemma 3.2. *For two functions ψ and φ in $C_b^2(H)$, we have*

$$\int_H S\varphi(y)\psi(y)\mu(dy) = \int_H \varphi(y)S\psi(y)\mu(dy) = -\frac{1}{2}\int_H \partial_x\varphi(y)\partial_x\psi(y)\mu(dy) \tag{3.22}$$

and

$$\int_H A\varphi(y)\psi(y)\mu(dy) = -\int_H \varphi(y)A\psi(y)\mu(dy) \tag{3.23}$$

Proof. Let $\varphi \in C_b^2(H)$. It is trivial that φ, $S\varphi \in L^2(H,\mu)$ by definition of $C_b^2(H)$. We shall start by proving that $A\varphi \in L^2(H,\mu)$.

From the definition of A, we have

$$\begin{aligned} A\varphi\,(y) &= \partial_x\varphi(y)(\langle(na_n^{1/2}\cos(nx))_{n\geqslant 1}, v\rangle_{l^2} + \langle(na_n^{1/2}\sin(nx))_{n\geqslant 1}, u\rangle_{l^2}) \\ &\quad +\langle(a_n^{1/2}\cos(nx))_{n\geqslant 1}, \nabla_u\varphi(y)\rangle_{l^2} + \langle(-a_n^{1/2}\sin(nx))_{n\geqslant 1}, \nabla_v\varphi(y)\rangle_{l^2} \end{aligned}$$

Therefore, by using the inequality $(\sum_{j=1}^n x_j)^2 \leqslant n\sum_{j=1}^n x_j^2$ with $n=4$, we obtain

$$\begin{aligned} A\varphi\,(y)^2 &\leqslant 4\partial_x\varphi(y)^2(\langle(na_n^{1/2}\cos(nx))_{n\geqslant 1}, v\rangle_{l^2}^2 + \langle(na_n^{1/2}\sin(nx))_{n\geqslant 1}, u\rangle_{l^2}^2) \\ &\quad +4\langle(a_n^{1/2}\cos(nx))_{n\geqslant 1}, \nabla_u\varphi(y)\rangle_{l^2}^2 + 4\langle(-a_n^{1/2}\sin(nx))_{n\geqslant 1}, \nabla_v\varphi(y)\rangle_{l^2}^2 \\ &\leqslant C(1+\|u\|_{l^2}^2 + \|v\|_{l^2}^2) \end{aligned}$$

where the last inequality is obtained by the Cauchy-Schwarz inequality and C is a constant depending on $(a_n)_n$ and on the upper bounds of the derivatives of φ. Hence $A\varphi \in L^2(H,\mu)$.

Since μ has $\mathbb{S}^1 \times l^2 \times l^2$ as support, we may extend φ to $\widetilde{H} = \mathbb{S}^1 \times \mathbb{R}^\infty \times \mathbb{R}^\infty$ by the same expression.

Therefore

$$\begin{aligned} \int_H S\varphi(y)\psi(y)\mu(dy) &= \int_{\widetilde{H}} S\varphi(y)\psi(y)\mu(dy) \\ &= \int_{\widetilde{H}} \frac{1}{2}\partial_{xx}\varphi\,(y)\,\psi\,(y)\,\mu\,(dy) \\ &= \int_{\mathbb{R}^\infty\times\mathbb{R}^\infty}\int_{\mathbb{S}^1} \frac{1}{4\pi}\partial_{xx}\varphi\,(y)\,\psi\,(y)\,dx\; N\,(0,Q)\,(d\,(u_n)_n)\;\; N\,(0,Q)\,(d\,(v_n)_n) \\ &= -\frac{1}{2}\int_H \partial_x\psi(y)\partial_x\varphi(y)\mu(dy), \end{aligned}$$

where $N\,(0,Q)\,(d\,(u_n)_n) = \prod_{n\geq 1} N\left(0,\frac{1}{n^2}\right)du_n$ and similarly for $N\,(0,Q)\,(d\,(v_n)_n)$. This proves (3.22).

Furthermore

$$\begin{aligned}
&\int_H A\varphi(y)\,\psi(y)\,\mu(dy)\\
=&\int_{\widetilde{H}} A\varphi(y)\,\psi(y)\,\mu(dy)\\
=&\int_{\widetilde{H}}\left(\partial_x\varphi(y)\sum_{n=1}^{\infty} n a_n^{1/2}\left(v_n\cos(nx)+u_n\sin(nx)\right)\right)\psi(y)\,\mu(dy)\\
&+\int_{\widetilde{H}}\sum_{n=1}^{\infty} a_n^{1/2}\cos(nx)\,\partial_{u_n}\varphi(y)\,\psi(y)\,\mu(dy)\\
&-\int_{\widetilde{H}}\sum_{n=1}^{\infty} a_n^{1/2}\sin(nx)\,\partial_{v_n}\varphi(y)\,\psi(y)\,\mu(dy).
\end{aligned}$$

For the term

$$\int_{\widetilde{H}}\sum_{n=1}^{\infty} n a_n^{1/2} v_n\cos(nx)\,\partial_x\varphi(y)\,\psi(y)\,\mu(dy)$$

we compute

$$\int_{\mathbb{S}^1}\cos(nx)\,\psi(y)\,\partial_x\varphi(y)\,dx=-\int_{\mathbb{S}^1}\partial_x\left(\psi(y)\cos(nx)\right)\varphi(y)\,dx$$

$$=-\int_{\mathbb{S}^1}\partial_x(\psi(y))\cos(nx)\,\varphi(y)\,dx+\int_{\mathbb{S}^1} n\sin(nx)\,\psi(y)\,\varphi(y)\,dx \tag{3.24}$$

and then we get

$$\begin{aligned}
&\int_{\widetilde{H}}\sum_{n=1}^{\infty} n a_n^{1/2} v_n\cos(nx)\,\partial_x\varphi(y)\,\psi(y)\,\mu(dy)\\
=&\int_{\widetilde{H}}\sum_{n=1}^{\infty} n^2 a_n^{1/2} v_n\sin(nx)\,\varphi(y)\,\psi(y)\,\mu(dy)\\
&-\int_{\widetilde{H}}\sum_{n=1}^{\infty} n a_n^{1/2} v_n\cos(nx)\,\partial_x\psi(y)\,\varphi(y)\,\mu(dy)
\end{aligned}$$

Moreover for the term

$$-\int_{\widetilde{H}}\sum_{n=1}^{\infty} a_n^{1/2}\sin(nx)\,\partial_{v_n}\varphi(y)\,\psi(y)\,\mu(dy)$$

we have

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_{v_n} \varphi(y) \psi(y) e^{-\frac{n^2}{2} v_n^2} dv_n \\
= &-\int_{\mathbb{R}} \varphi(y) \partial_{v_n} \left(\psi(y) e^{-\frac{n^2}{2} v_n^2} \right) dv_n \\
= &-\int_{\mathbb{R}} \varphi(y) \partial_{v_n} (\psi(y)) e^{-\frac{n^2}{2} v_n^2} dv_n \\
&+ \int_{\mathbb{R}} \varphi(y) \psi(y) n^2 v_n e^{-\frac{n^2}{2} v_n^2} dv_n,
\end{aligned} \tag{3.25}$$

which yields

$$\begin{aligned}
&-\int_{\widetilde{H}} \sum_{n=1}^{\infty} a_n^{1/2} \sin(nx) \partial_{v_n} \varphi(y) \psi(y) \mu(dy) \\
= &\int_{\widetilde{H}} \sum_{n=1}^{\infty} a_n^{1/2} \sin(nx) \partial_{v_n} \psi(y) \varphi(y) \mu(dy) \\
&-\int_{\widetilde{H}} \sum_{n=1}^{\infty} n^2 a_n^{1/2} v_n \sin(nx) \psi(y) \varphi(y) \mu(dy)
\end{aligned}$$

Similarly to (3.24) and (3.25) we get

$$\begin{aligned}
&\int_{\widetilde{H}} \sum_{n=1}^{\infty} n a_n^{1/2} u_n \sin(nx) \partial_x \varphi(y) \psi(y) \mu(dy) \\
= &-\int_{\widetilde{H}} \sum_{n=1}^{\infty} n^2 a_n^{1/2} u_n \cos(nx) \varphi(y) \psi(y) \mu(dy) \\
&-\int_{\widetilde{H}} \sum_{n=1}^{\infty} n a_n^{1/2} u_n \sin(nx) \partial_x \psi(y) \varphi(y) \mu(dy)
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
&\int_{\widetilde{H}} \sum_{n=1}^{\infty} a_n^{1/2} \cos(nx) \partial_{u_n} \varphi(y) \psi(y) \mu(dy) \\
= &\int_{\widetilde{H}} \sum_{n=1}^{\infty} n^2 a_n^{1/2} u_n \cos(nx) \varphi(y) \psi(y) \mu(dy) \\
&-\int_{\widetilde{H}} \sum_{n=1}^{\infty} a_n^{1/2} \cos(nx) \partial_{u_n} \psi(y) \varphi(y) \mu(dy)
\end{aligned} \tag{3.27}$$

Putting (3.24) to (3.27) altogether gives (3.23). □

An easy consequence of the result above is the following.

Corollary 3.1. *For a function $\varphi \in C_b^2(H)$, we have*

$$\int_H L\varphi(y)\,\varphi(y)\,\mu(dy) = -\frac{1}{2}\int_H |\partial_x \varphi(y)|^2\,\mu(dy).$$

Furthermore, if φ is such that $L\varphi = 0$, then φ is constant on H.

Proof. Let $\varphi \in C_b^2(H)$. By Lemma 3.2, we have

$$\int_H A\varphi(y)\varphi(y)\mu(dy) = -\int_H \varphi(y)A\varphi(y)\mu(dy);$$

hence $\int_H A\varphi(y)\varphi(y)\mu(dy) = 0$.

Thus

$$\begin{aligned}\int_H L\varphi(y)\,\varphi(y)\,\mu(dy) &= \int_H S\varphi(y)\,\varphi(y)\,\mu(dy)\\ &= -\frac{1}{2}\int_H |\partial_x\varphi(y)|^2\,\mu(dy).\end{aligned}\tag{3.28}$$

Assume now that φ satisfies $L\varphi = 0$. Then, by (3.28), we obtain

$$0 = -\frac{1}{2}\int_H |\partial_x\varphi(y)|^2\,\mu(dy).$$

Since μ has full support on H and $\partial_x\varphi$ is continuous, it follows that

$$\partial_x\varphi \equiv 0,$$

i.e., φ is independent of the x variable on H.

Therefore

$$\begin{aligned}0 &= L\varphi\left(x, (u_n)_n, (v_n)_n\right) \\ &= \sum_{n=1}^{\infty} a_n^{1/2}\cos(nx)\,\partial_{u_n}\varphi(y) - \sum_{n=1}^{\infty} a_n^{1/2}\sin(nx)\,\partial_{v_n}\varphi(y)\end{aligned}\tag{3.29}$$

for all $\left(x, (u_n)_n, (v_n)_n\right) \in H$.

Since $\{(\cos nx)_n, (\sin nx)_n\}$ forms an orthogonal basis of $L^2(\mathbb{S}^1, dx)$, the relation (3.29) forces to have

$$\partial_{u_n}\varphi = 0 = \partial_{v_n}\varphi, \quad \text{for all } n \geq 1$$

on H, because a_n is supposed to be strictly positive. Consequently, φ is a constant on H. □

Set $L^* = S - A$. Then, by applying Lemma 3.2, one can check that

$$\int_H L\varphi(y)\psi(y)\mu(dy) = \int_H \varphi(y)L^*\psi(y)\mu(dy)$$

for all $\varphi, \psi \in C_b^2(H)$.

Let ν be any invariant probability measure of (3.7).

We shall explain why we believe that ν should be identical to the measure μ defined in (3.18), which would prove uniqueness of the invariant probability, as well as the ergodicity of μ.
By the Lebesgue's decomposition theorem, there exists a positive function $g \in L^1(H, \mu)$ and a measure ν_s which is singular to μ, such that

$$\nu = g\mu + \nu_s.$$

Since μ and ν are both invariant for (3.7), it follows that $g\mu$ and ν_s are also invariant. We can now formulate the following result.

Proposition 3.4. *Assume that the function g defined above lies in $C_b^4(H)$. Then g is constant.*

Proof. Since $g\mu$ is invariant, we obtain that

$$\begin{aligned} 0 = \int_H Lg(y)(g\mu)(dy) &= \int_H Lg(y)g(y)\mu(dy) \\ &= -\frac{1}{2}\int_H \left|\partial_x g\left(y\right)\right|^2 \mu\left(dy\right) \end{aligned} \tag{3.30}$$

by Corollary 3.1.

Therefore we deduce that $\partial_x g \equiv 0$ by the continuity of $\partial_x g$ and the full support of μ. Hence

$$\begin{aligned} Lg(y) &= \sum_{n=1}^{\infty} a_n^{1/2} \cos\left(nx\right) \partial_{u_n} g\left(y\right) - \sum_{n=1}^{\infty} a_n^{1/2} \sin\left(nx\right) \partial_{v_n} g\left(y\right) \\ &= -L^* g(y). \end{aligned}$$

By the Cauchy-Schwarz inequality and the definition of the space $C_b^4(H)$, it is clear that Lg is bounded and consequently that $Lg \in C_b^2(H)$ as well as L^*g. Therefore, by application of (3.30) with L^*g in place of g, we get

$$\begin{aligned} 0 = \int_H L(L^*g)(y)(g\mu)(dy) &= \int_H L(L^*g)(y)g(y)\mu(dy) \\ &= \int_H (L^*g)^2(y)\mu(dy), \end{aligned}$$

which leads to $L^*g \equiv 0$ and so does Lg. By Corollary 3.1, we get that g is constant. □

A straightforward consequence is the following result.

Corollary 3.2. *If ν is absolutely continuous with respect to μ and such that its Radon-Nikodym derivative lies in $C_b^4(H)$, then $\nu = \mu$.*

Remark 3.3.
1. *The proposition still holds true for $g \in C_b^2(H)$ since L is well defined on $C_b^{2,1,1}(H)$, the set of bounded functions which are twice differentiable in x, and once differentiable in u and v and such that these partial derivatives are bounded.*
2. *If the function g in the proposition has bounded support then $g \equiv 0$ and so ν is singular to μ.*

3.7 Conclusion

In this work, we aim to generalize the setting of [16] to the infinite dimensional case, at least for the case of the unit circle. Since our non-linear operator F is neither Lipschitz nor monotone we could not directly apply classic results in the sense that we had to prove some additional properties which hold for F.

As mentioned at the beginning of the Section 3.5, we succeed to prove that a natural generalization of the invariant measure in the finite dimensional case was indeed an invariant measure in our setting.

However, we were not yet able to obtain its uniqueness, while in [16] it is the case. This is due to the fact that we could not use Hörmander's like condition to get the strong Feller property which was the main argument in the finite dimensional case. So at this point, a first question is

1. Do we have uniqueness for the invariant measure?

Thanks to Corollary 3.1, we think that it might be the case.

If this is not true, a second open question would be

2. Is μ an ergodic measure, which means that, if $A \in \mathcal{B}(H)$ is such that $P_t 1_A = 1_A$, then $\mu(A) \in \{0, 1\}$?

As mentioned above, the strong Feller property was proved in [16], while in our case, it does not hold (see remark 3.2). On the other hand, the question of having asymptotically strong Feller property is still open. More precisely, in order to ensure that all our computations make sense, we had to choose our coefficients $(a_n)_n$ in O^5; so the question can be formulated as

3. If $(a_n)_n \in \cap_{k\geqslant 1} O^k$ for example, do we have the asymptotic strong Feller property? If yes, can we weaken the assumption on the sequence $(a_n)_n$?

Finally, in the case of positive answer to this last question, the answer for the first will be positive since μ has full support.

Chapter 4

Self-Attracting diffusions on a sphere and application to a periodic case

This chapter is based on the paper [53] which is published in *Electronic Communication of Probability*, volume 21, Paper 53 (2016).

In the first section, we give a review of the notion that playing a crucial role in the convergence result of the paper and recall a useful result for a time-inhomogeneous Ornstein-Uhlenbeck Process. The next sections are the content of the paper. Therefore some redundancies with the Introduction chapter are possible.

Keywords: Reinforced processes, self-interacting diffusions, asymptotic pseudotrajectories, rate of convergence.

MSC primary: 60K35, 60G17, 60J60

Contents

4.1 Basic tools

4.1.1 The Asymptotic Pseudotrajectories

In this subsection, we introduce the *theory of the Asymptotic Pseudotrajectories* developed by M. Benaïm and M.W. Hirsch in the nineties. It is based on the article [17] and on the lecture note [11].

Let (M, d) be a metric space and Φ a semiflow; that is

$$\Phi : \mathbb{R}_+ \times M \to M : (t, x) \mapsto \Phi(t, x) = \Phi_t(x)$$

is a continuous map such that

$$\Phi_0 = Id \text{ and } \Phi_{t+s} = \Phi_t \circ \Phi_s$$

for all $s, t \in \mathbb{R}_+$. For a continuous function $X : [0, \infty) \to M$, its limit set is

$$L(X) = \bigcap_{t \geqslant 0} \overline{X([t, \infty))}.$$

Definition 4.1. *A continuous function $X : \mathbb{R}_+ \to M$ is an asymptotic pseudotrajectory for Φ if*

$$\lim_{t\to\infty} \sup_{0\leqslant h\leqslant T} d(X_{t+h}, \Phi_h(X_t)) = 0 \tag{4.1}$$

for any $T > 0$. In other words, it means that for each fixed $T > 0$, the curve $X : [0, T] \to M : h \mapsto X_{t+h}$ shadows the Φ-trajectory over the interval $[0, T]$ with arbitrary accuracy for sufficiently large t.
If X is a continuous random process, then X is an almost-surely asymptotic pseudotrajectory for Φ if (4.1) holds almost-surely.

In the context of this thesis, a good way to picture an asymptotic pseudotrajectory for a flow Φ is to view it as the flow induced by some homogeneous ODE and X as the solution of some differential equation which is a perturbation of this ODE, whose intensity tends to 0 sufficiently fast as t converges to infinity. For relations to stochastic algorithms, we refer for instance to [11], [13], [15], [48] or [49] and for reinforced processes to [10], [12], [23] or [72].

Since the long term behaviour of X is highly related to the flow, one would expect that $L(X)$ keeps some properties of ω-limit sets if X has compact closure (for instance invariance by the flow, connectivity or compactness). Before giving some results on $L(X)$, let us recall some notion of dynamical system that can be found in [11].

Definition 4.2. *A subset $A \subset M$ is an attractor for Φ if*

i. A is nonempty, compact and invariant.

ii. A has a neighbourhood $W \subset M$ such that $\lim_{t\to\infty} d(\Phi_t(x), A) = 0$ uniformly in $x \in W$.

Its basin of attraction is the set of points $x \in M$ such that $\lim_{t\to\infty} d(\Phi_t(x), A) = 0$. Furthermore, if $A \neq M$, A is called proper attractor.

Definition 4.3. *Let $A \subset M$ be a nonempty set.*

1. *Let $p, q \in A$ and $\delta, T > 0$. We say that there is (δ, T)- pseudo orbit for Φ from p to q if there exists $y_0, \cdots, y_k \in M$ and $0 < t_0 < \cdots < t_k$ with $t_k \geqslant T$ such that*

   (a) $d(y_0, p) < \delta$,

   (b) $d((\Phi)_{t_i}(y_i), y_{i+1}) < \delta$, $i = 0, \cdots, k$;

   (c) $y_k = q$.

2. *A is chain recurrent for Φ if for every $p \in A$ nd for every $\delta, T > 0$, there exists a (δ, T)-pseudo orbit from p to p and internally chain recurrent if A is invariant for Φ and is chain recurrent for $\Phi_{|A}$.*

3. *K is chain transitive for Φ if for every $p, q \in A$ nd for every $\delta, T > 0$, there exists a (δ, T)- pseudo orbit from p to q and internally chain transitive if A is invariant for Φ and is chain transitive for $\Phi_{|A}$.*

These different definitions are relied by the following results

Proposition 4.1. *(Bowen, 1975 [29], Proposition 5.3 in [11]) Let $A \subset M$ be a nonempty set. The following assertion are equivalent.*

1. *A is internally chain transitive.*

2. *A is connected and internally chain recurrent.*

3. *A is a compact invariant set for Φ and $\Phi_{|A}$ admits no proper attractor.*

Corollary 4.1. *(Corollary 5.4 in [11]) If an internally chain transitive set meets the basin of an attractor A, then it is contained in A.*

We can now state the first result about asymptotic pseudotrajectories.

Theorem 4.1. *(Theorem 5.7 in [11]) Suppose that $X([0, \infty))$ has compact closure in M. Then $L(X)$ is internally chain transitive.*

In particular, we have

Theorem 4.2. *(Theorem 1.2 in [17]) Suppose that $X([0, \infty))$ has compact closure in M. Let A be an attractor for Φ with basin W. If $X_{t_k} \in W$ for some sequence $t_k \to \infty$, then $L(X) \subset A$.*

Let us end this subsection by explaining how to relate stochastic differential equations with asymptotic pseudotrajectories. Let $(x_t^z)_{t\geqslant 0}$ be the solution of the SDE

$$dx_t^z = g(x_t^z)dt + h(t, x_t^z)dB_t,\ x_0^z = z \in \mathbb{R} \tag{4.2}$$

where $(B_t)_{t\geqslant 0}$ is a Brownian motion, $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function and $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ a continuous function.

The next Theorem gives a sufficient condition on h in order to ensure that $(x_t^z)_{t\geqslant 0}$ is with probability 1 an asymptotic pseudotrajectory for the flow induced by the ODE

$$\dot{y} = g(y). \tag{4.3}$$

Theorem 4.3. *(Proposition 4.1 in [17], Proposition 4.6 in [11]) Assume there exists a non-increasing function $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that $h^2(t, x) \leqslant \varepsilon(t)$ for all (t, x) and such that*

$$\forall k > 0,\ \int_0^\infty \exp(-k/\varepsilon(t))dt < \infty^1. \tag{4.4}$$

Then, for all $z \in \mathbb{R}$, $(x_t^z)_{t\geqslant 0}$ is with probability 1 an asymptotic pseudotrajectory for the flow induced by (4.3).

Remark 4.1. *The same result holds if $(x_t)_{t\geqslant 0}$ solves the SDE*

$$dx_t = g(x_t)dt + h(t, x_t)dB_t + \delta(t)h_2(x_t)dt,$$

where h_2 is a bounded function and δ is a random adapted function with $\lim_{t\to\infty} \delta(t) = 0$ almost surely.

4.1.2 Rate of convergence for a time-inhomogeneous Ornstein-Uhlenbeck Process

We reproduce in this subsection the appendix of [53], where we study the long-term behaviour of the following SDE

$$dX_t = g_t dW_t + \mu_t dt - (1 + \alpha)\lambda t^\alpha X_t dt, \tag{4.5}$$

where $t \mapsto \mu_t$ is a deterministic constant, $\alpha = 1$ and $(g_t)_{t\geqslant 0}$ is an adapted process bounded by 1. Here, $(W_t)_{t\geqslant 0}$ stands for a real Brownian motion and $\lambda > 0$.

Proposition 4.2. *Let X_t be the solution of (4.5) with initial condition $X_0 = x$. Assume that $(g_t)_{t\geqslant 0}$ and $(\mu_t)_{t\geqslant 0}$ are adapted processes bounded by some deterministic constant K and let $\alpha \geqslant 0$. Then*

$$X_t = O\Big(t^{-\alpha/2}\sqrt{\log(t)}\Big)\ a.s.$$

[1] For example $\varepsilon(t) = O(1/(\log(t))^\alpha)$ with $\alpha > 1$.

Proof. The solution of Equation (4.5) with initial condition $X_0 = x$ is

$$\begin{aligned} X_t &= e^{-\lambda t^{1+\alpha}}\Big(x + \int_0^t e^{\lambda s^{1+\alpha}} g_s dW_s + \int_0^t e^{\lambda s^{1+\alpha}} \mu_s ds\Big) \\ &= e^{-\lambda t^{1+\alpha}}\Big(x + M_t + \int_0^t e^{\lambda s^{1+\alpha}} \mu_s ds\Big). \end{aligned} \tag{4.6}$$

Since

$$e^{-\lambda t^{1+\alpha}}\Big|\int_0^t e^{\lambda s^{1+\alpha}} \mu_s ds\Big| \leqslant K e^{-\lambda t^{1+\alpha}} \int_0^t e^{\lambda s^{1+\alpha}} ds = O\Big(t^{-\alpha}\Big), \tag{4.7}$$

then, on the event $\{\langle M\rangle_\infty := \int_0^\infty e^{2\lambda s^{1+\alpha}} g_s^2 ds < \infty\}$, the result is immediate because in that case, M_t converges almost surely. In the sequel, we assume that we are on the event $\{\langle M\rangle_\infty = \infty\}$.

By the Dubins-Schwarz Theorem (see Theorem 4.6 in [70], Chapter 3) with the law of Iterated Logarithm for Brownian motion (see Theorem 9.23 in [70], Chapter 2), we have

$$M_t = O\Big(\sqrt{\langle M\rangle_t \log\log(\langle M\rangle_t)}\Big). \tag{4.8}$$

By the hypothesis on g_t, we have

$$\langle M\rangle_t = \int_0^t e^{2\lambda s^{1+\alpha}} g_s^2 ds \leqslant K^2 \int_0^t e^{2\lambda s^{1+\alpha}} ds.$$

Therefore,

$$\langle M\rangle_t = O\Big(e^{2\lambda t^{1+\alpha}} t^{-\alpha}\Big). \tag{4.9}$$

The desired result follows from (4.6)-(4.9). □

4.2 Introduction

In this chapter, we are interested in the asymptotic behaviour of the solution of the stochastic differential equation (SDE)

$$dX_t = \nu \circ dW_t(X_t) - a \int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t) ds dt, \qquad X_0 = x \in \mathbb{S}^n, \tag{4.10}$$

where $\nu > 0$, $a \in \mathbb{R}$, $\circ$ stands for the Stratonovich differential, $(W_t(.))_{t\geqslant 0}$ is a Brownian vector field on $\mathbb{S}^n$, $\nabla_{\mathbb{S}^n}$ is the gradient on $\mathbb{S}^n$ and $V_y(x) = \langle x, y\rangle$ where $\langle ., .\rangle$ is the canonical scalar product on $\mathbb{R}^{n+1}$.

Let us start with a short heuristic description of the process. First of all, observe that for $x, y \in \mathbb{S}^n$, we have

$$\|x - y\|^2 = 2 - 2\langle x, y\rangle = 2 - 2\cos(D(x, y)), \tag{4.11}$$

where $D(., .)$ is the geodesic distance on $\mathbb{S}^n$ and $\|.\|$ is the standard Euclidean norm on $\mathbb{R}^{n+1}$. If $a > 0$, the drift term points in a direction that tends to increase the distance between X_t and its past positions. In other words X_t *is repelled by its past.* It follows from a more general result proved in [16] dealing with self repelling diffusions on a compact manifold that

Theorem 4.4. *(Theorem 5, [16], Benaïm, Gauthier) If $a > 0$, the law of X_t converges to the uniform law on $\mathbb{S}^n$.*

If $a < 0$, X_t *is attracted by its past* and one may expect localization.The goal of this chapter is to prove such a result.

Theorem 4.5. *If $a < 0$, there exists a random variable $X_\infty \in \mathbb{S}^n$ such that*

$$\|X_t - X_\infty\| = \begin{cases} O\big(t^{-1/2}\sqrt{\ln(t)}\big) & \text{if } n = 1 \\ O\big((\frac{\ln(t)}{t})^{1/4}\big) & \text{otherwise} \end{cases}.$$

We point out that the self interacting diffusion (4.10) has already received some attention in 2002 by M.Benaïm, M.Ledoux and O.Raimond ([19]), *but* in the normalized case; that is, when $\int_0^t V_{X_s}(X_t)ds$ is replaced by $\frac{1}{t}\int_0^t V_{X_s}(X_t)ds$. The interpretation is therefore different. While the drift term of (4.14) can be "seen" as a summation over $[0,t]$ of the interaction between the current position X_t and its position at time s and thus an *accumulation of the interacting force*, their drift is then an *average of the interacting force.* The asymptotic behaviour is then given by the following Theorem.

Theorem 4.6. *(Theorem 4.5, [19], Benaïm, Ledoux, Raimond) For $a \neq 0$, let $(X_t)_{t\geqslant 0}$ be the solution of the SDE*

$$dX_t = \circ dW_t(X_t) - \frac{a}{t}\int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X_0 = x \in \mathbb{S}^n. \tag{4.12}$$

Set $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}ds$.

1. *If $a \geqslant -(n+1)/2$, then μ_t converges almost surely (for the topology of weak* convergence) toward the Riemannian probability measure on $\mathbb{S}^n$.*

2. *If $a < -(n+1)/2$, then there exists a random variable $\varsigma \in \mathbb{S}^n$ such that μ_t converges almost surely toward the measure*

$$\mu_{c,\varsigma}(dx) = \frac{\exp(\beta(a)\langle x, \varsigma\rangle)}{Z_a},$$

   where Z_a is the normalization constant, $\beta(a)$ is the unique positive solution to the implicit equation

$$2a\Lambda_n'(\beta) + \beta = 0,$$

   where $\Lambda_n(\beta) = \log(\int_0^\pi \exp(-\beta\cos(x))\lambda_n(dx))$ and $\lambda_n(dx) = \frac{(\sin(x))^{n-1}}{\int_0^\pi (\sin(x))^{n-1}dx}dx$.

An intermediate framework between those considered in Theorem 4.5 and Theorem 4.6 is to add a time-dependent weight $g(t)$ to the normalized case that increases to infinity, but "not too fast", when time increases. In that case, O.Raimond proved the following Theorem.

Theorem 4.7. *(Theorem 3.1, [89], Raimond) Let $(X_t)_{t\geqslant 0}$ be the solution of the SDE*

$$dX_t = \circ dW_t(X_t) - \frac{g(t)}{t}\int_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X_0 = x \in \mathbb{S}^n, \tag{4.13}$$

where g is an increasing function such that $\lim_{t\to\infty} g(t) = \infty$. Assume that there exists positive constants c, t_0 such that for $t \geqslant t_0$, $g(t) \leqslant c\log(t)$ and $|g'(t)| = O(t^{-\gamma})$, with $\gamma \in]0,1]$.

Then, there exists a random variable X_∞ in $\mathbb{S}^n$ such that almost surely, $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}ds$ converges weakly towards δ_{X_∞}.

As an easy application of Theorem 4.5, we obtain the almost-sure convergence with a rate of convergence of the solution of the real-valued SDE

$$d\vartheta_t = \nu dW_t + a\int_0^t \sin(c(\vartheta_t - \vartheta_s))dsdt, \qquad \vartheta_0 = 0, \tag{4.14}$$

where $(W_t)_t$ is a real Brownian motion, $a < 0$ and $\nu, c > 0$.

In 1995, M.Cranston and Y. Le Jan proved an almost-sure convergence result in [36] in the cases where $a\sin(cx)$ is replaced by $f(x) = ax$ (linear case) or $f(x) = a \times sgn(x)$ (constant case) with $a < 0$. This last case was extended in all dimension by O.Raimond in [88] in 1997. A few years later, S.Herrmann and B.Roynette weakened the condition of the profile function f around 0 and were still able to get almost-sure convergence (see [60]) for the solution of the stochastic differential equation

$$d\vartheta_t = \nu dW_t + \int_0^t f(\vartheta_t - \vartheta_s)dsdt. \tag{4.15}$$

Rate of convergence were given in [61] by S. Herrmann and M. Scheutzow. For the linear case, they proved that the optimal rate of convergence is $O\big(t^{-1/2}\sqrt{\log(t)}\big)$ (Proposition 4 in [61]).

However, a common fundamental property of these three papers lies in the fact that the associated profile function f is monotone.

4.2.1 Reformulation of the problem

From now on, we assume that $a < 0$ and that n is fixed. Since the values of ν and a do not play any particular role, we assume without loss of generality that $\nu = 1$ and $a = -1$. Thus (4.10) becomes

$$dX_t = \circ dW_t(X_t) + \int_0^t \nabla_{\mathbb{S}^n} V_{X_s}(X_t)dsdt, \qquad X_0 = x \in \mathbb{S}^n \tag{4.16}$$

with $V_y(x) = \langle x, y\rangle =: V(x,y)$. Because the law of the process $(X_t)_{t\geqslant 0}$ is the same for any Brownian vector field on $\mathbb{S}^n$, we assume from now on and without loss of generality that $W_t(x) = B_t - \langle x, B_t\rangle x$, where $(B_t)_{t\geqslant 0}$ is a standard Brownian motion on $\mathbb{R}^{n+1}$.

Since V satisfies Hypothesis 1.3 and 1.4 in [19], then (4.16) admits a unique strong solution by Proposition 2.5 in [19]. We recall that for a function $F : \mathbb{R}^{n+1} \to \mathbb{R}$, we have

$$\nabla_{\mathbb{S}^n}(F_{|\mathbb{S}^n})(x) = \nabla_{\mathbb{R}^{n+1}}F(x) - \langle x, \nabla_{\mathbb{R}^{n+1}}F(x)\rangle x;\ x \in \mathbb{S}^n. \tag{4.17}$$

For $x \in \mathbb{S}^n$, we let $u \mapsto P(x,u)$ be the orthogonal projection on $T_x\mathbb{S}^n$ given by

$$P(x,u) = u - \langle x, u\rangle x.$$

Following the same idea as in [16], we set $U_t := \int_0^t X_s ds \in \mathbb{R}^{n+1}$ in order to get the SDE on $\mathbb{S}^n \times \mathbb{R}^{n+1}$:

$$\begin{cases} dX_t = P(X_t, \circ dB_t + U_t dt) \\ dU_t = X_t dt \end{cases} \tag{4.18}$$

with initial condition $(X_0, U_0) = (x, 0)$.

Remark 4.2. *We have* $P(X_t, \circ dB_t + U_t dt) = P(X_t, dB_t + U_t dt) - \frac{n}{2}X_t dt,$

The paper is organised as follows. In Section 4.3, we present the detailed strategy used for proving Theorem 4.5 and prove the application to a periodic case whereas the more technical proofs are presented in Sections 4.4 and 4.5.

4.3 Guideline of the proof of Theorem 4.5

Set $R_t = \|U_t\|$ and define $V_t \in \mathbb{S}^n$ and $C_t \in [-1,1]$ as follows:

$$V_t = \begin{cases} U_t/R_t \text{ if } R_t > 0 \\ X_t \text{ otherwise} \end{cases} \tag{4.19}$$

and

$$C_t = \langle V_t, X_t\rangle. \tag{4.20}$$

With these notations, we have

$$R_t C_t = \langle U_t, X_t\rangle. \tag{4.21}$$

Since the coordinates functions

$$e_j : \mathbb{S}^n \subset \mathbb{R}^{n+1} \to \mathbb{R} : x \mapsto x_j, \text{ for } j = 1, \cdots, n+1,$$

are eigenfunctions for the Laplacian operator on $\mathbb{S}^n$ associated to the eigenvalue $-n$ (see Chapter 3, Section C in [26]), then by Lemma 3 and Lemma 5 in section 4 of [16], the system (4.18) satisfies the Hörmander condition (also called condition (E) in [16] and [67]).

Thus, for all $t > 0$, the law of (X_t, U_t) has a smooth density with respect to the Lebesgue measure on $\mathbb{S}^n \times \mathbb{R}^{n+1}$ (see Theorem 3.(i) in [67]). Hence for all $t > 0$,

$$\mathbb{P}\Big(C_t^2 = 1 \text{ or } R_t = 0\Big) = \mathbb{P}\Big(U_t \text{ is parallel to } X_t\Big) = 0. \tag{4.22}$$

Since $\int_0^t \langle P(X_s, dB_s), V_s\rangle$ is a martingale whose quadratic variation is $\int_0^t (1 - C_s^2)ds$, then the process $(W_t)_{t\geqslant 0}$ defined by $W_0 = 0$ and, for $t > 0$, by

$$W_t = \int_0^t \mathbf{1}_{\{C_s^2<1 \text{ and } R_s>0\}} \frac{\langle P(X_s, dB_s), V_s\rangle}{\sqrt{1 - C_s^2}}, \tag{4.23}$$

is a standard Brownian motion on $\mathbb{R}$.

Lemma 4.1. *$((C_t, R_t))_{t\geqslant 0}$ is solution to*

$$\begin{cases} dC_t = \sqrt{1 - C_t^2}dW_t + [(R_t + \frac{1}{R_t})(1 - C_t^2) - \frac{n}{2}C_t]dt \\ dR_t = C_t dt \end{cases} \tag{4.24}$$

whenever $R_t > 0$.

Proof. Since $dR_t^2 = 2\langle U_t, dU_t\rangle = 2R_tC_tdt$, then, as long as $R_t > 0$, we have

$$dR_t = C_t dt. \tag{4.25}$$

Hence,

$$dV_t = \frac{1}{R_t}(X_t - C_tV_t)dt. \tag{4.26}$$

Therefore, by Itô's formula

$$\begin{aligned} dC_t &= \langle X_t, dV_t\rangle + \langle V_t, P(X_t, dB_t + U_tdt)\rangle - \frac{n}{2}\langle V_t, X_t\rangle dt \\ &= \sqrt{1 - C_t^2}dW_t + (R_t + \frac{1}{R_t})(1 - C_t^2)dt - \frac{n}{2}C_tdt. \end{aligned} \tag{4.27}$$

□

A first important result, whose proof is postponed to Section 4.4, is

Lemma 4.2. *One has $\liminf_{t\to\infty} \frac{R_t}{\sqrt{t}} \geqslant \frac{2}{\sqrt{n}}$ almost surely.*

From this lemma, we prove in Section 4.5

Lemma 4.3. *The processes $(C_t)_{t\geqslant 0}$ and $(\frac{R_t}{t})_{t>0}$ converge almost surely to 1. Furthermore*

$$|C_t - 1| = \begin{cases} O\big(\frac{\ln(t)}{t}\big) & \text{if } n = 1 \\ O\big(\sqrt{\frac{\ln(t)}{t}}\big) & \text{otherwise} \end{cases}.$$

Thanks to this lemma, we obtain

Lemma 4.4. *V_t converges almost surely.*

Proof. Since

$$\|X_t - C_t V_t\| = \sqrt{1 - C_t^2}, \tag{4.28}$$

it follows from Lemma 4.3

$$\frac{1}{R_t}\|X_t - C_t V_t\| = O\big(t^{-5/4}\ln^{1/4}(t)\big), \tag{4.29}$$

which is an integrable quantity. Hence the result follows from (4.26) and (4.29). □

We can now prove the main result.

Proof of Theorem 4.5. By Lemmas 4.3 and 4.4, there exists a random variable $X_\infty \in \mathbb{S}^n$ such that $\lim_{t\to\infty} C_t V_t = X_\infty$.

The rate of convergence follows from the triangle inequality, (4.26), (4.28), (4.29) and Lemma 4.3. □

As an application of Theorem 4.5, we have the following result.

Theorem 4.8. *Let $(\vartheta_t)_{t\geqslant 0}$ be the solution of the SDE*

$$d\vartheta_t = \nu dW_t + a\int_0^t \sin(c(\vartheta_t - \vartheta_s))dsdt, \qquad \vartheta_0 = 0, \tag{4.30}$$

where $(W_t)_t$ is a real Brownian motion, $a < 0$ and $\nu, c > 0$. Then there exists a random variable ϑ_∞ such that $|\vartheta_t - \vartheta_\infty| = O(\sqrt{\frac{\ln(t)}{t}})$.

Proof. First of all (4.30) admits a unique strong solution because the function sin(.) is Lipschitz continuous (see for example Proposition 1 in [60]).

Set $\vartheta_t^{(c)} = c\vartheta_t$. Hence $(\vartheta_t^{(c)})_{t\geqslant 0}$ solves the SDE

$$d\vartheta_t^{(c)} = c\nu dW_t + ac\int_0^t \sin(\vartheta_t^{(c)} - \vartheta_s^{(c)})dsdt, \qquad \vartheta_0^{(c)} = 0. \tag{4.31}$$

Letting $X_t = \big(\cos(\vartheta_t^{(c)}), \sin(\vartheta_t^{(c)})\big)$, it follows that $(X_t)_{t\geqslant 0}$ is a solution of (4.10) when $n = 1$. Because $ac < 0$, there exists, by Theorem 4.5, $X_\infty \in \mathbb{S}^1$ such that

$$\|X_t - X_\infty\| = O\big(t^{-1/2}\sqrt{\log(t)}\big).$$

The result follows from the continuity of $t \mapsto \vartheta_t^{(c)}$. □

4.4 Proof of Lemma 4.2

Set $M_t = -2\int_0^t R_s\sqrt{1 - C_s^2}dW_s$, where W_t is defined by (4.23), and let

$$\mathcal{E}_M(t) = \exp\big(M_t - \frac{1}{2}\langle M\rangle_t\big). \tag{4.32}$$

Because

$$\langle M\rangle_t = 4\int_0^t R_s^2(1-C_s^2)ds \leqslant 4\int_0^t R_s^2 ds \leqslant 4\int_0^t s^2 ds = \frac{4t^3}{3}, \tag{4.33}$$

M_t satisfies the Novikov Condition (see [70], Chapter V, section D, page 198). Therefore $\mathcal{E}_M(t)$ is a positive martingale having 1 as expectation. Thus, it converges almost surely to a nonnegative integrable random variable $\mathcal{E}_M(\infty)$.

Hence, there exists a random variable $K<\infty$, such that almost surely, for all $t\geqslant 0$,

$$\ln(\mathcal{E}_M(t)) \leqslant 2K.$$

By Itô's formula and Lemma 4.1, we have

$$\begin{aligned} d(R_tC_t) &= C_t^2dt + R_t\sqrt{1-C_t}dW_t + (R_t^2+1)(1-C_t^2)dt - \frac{n}{2}R_tC_tdt \\ &= -\frac{1}{2}dM_t + \frac{1}{4}d\langle M\rangle_t - \frac{n}{2}R_tC_tdt + dt. \end{aligned} \tag{4.34}$$

Since $dR_t^2 = 2R_tC_tdt$, we obtain

$$\begin{aligned} R_tC_t + \frac{n}{4}R_t^2 &= -\frac{1}{2}\ln(\mathcal{E}_M(t)) + t \\ &\geqslant t - K. \end{aligned} \tag{4.35}$$

Because $C_t \leqslant 1$, we have for $t > K$

$$\frac{n}{2}R_t \geqslant -1 + \sqrt{n(t-K)}. \tag{4.36}$$

This completes the proof.

4.5 Proof of Lemma 4.3

The proof is divided into two parts.
Proof of the convergence:

First we prove that C_t converges almost surely to 1. Recall that

$$dC_t = \sqrt{1-C_t^2}dW_t + [(R_t + \frac{1}{R_t})(1-C_t^2) - \frac{n}{2}C_t]dt. \tag{4.37}$$

Define $\alpha(t) = (\frac{3}{2}t)^{\frac{2}{3}}$ so that $\dot\alpha(t) = \alpha^{-\frac{1}{2}}(t)$. Set $Z_t = C_{\alpha(t)}$ and $M_t = W_{\alpha(t)}$. Thus $(M_t)_{t\geqslant 0}$ is a martingale with respect to the filtration $\mathcal{G}_t = \sigma\{W_s \mid 0\leqslant s\leqslant \alpha(t)\}$, whose quadratic variation at time t is $\alpha(t) = \int_0^t(\sqrt{\dot\alpha(s)})^2ds$.
Define $B_t^{(\alpha)} = \int_0^t \frac{dM_s}{\sqrt{\dot\alpha(s)}}$, so that $(B_t^{(\alpha)})_{t\geqslant 0}$ is a Brownian motion. Then

$$Z_t = \int_0^t \sqrt{\dot\alpha(s)}\sqrt{1-Z_s^2}dB_s^{(\alpha)} + \int_0^t \frac{R_{\alpha(s)} + \frac{1}{R_{\alpha(s)}}}{\sqrt{\alpha(s)}}(1-Z_s^2)ds - \frac{n}{2}\int_0^t \dot\alpha(s)Z_sds. \tag{4.38}$$

For $y \in [-1,1]$ and $\sigma \geqslant 0$, let $(Y_t^{\sigma,y})_{t\geqslant\sigma}$ be the solution to the SDE on $[-1,1]$

$$\begin{cases} dY_t^{\sigma,y} = \sqrt{\dot\alpha(t)}\sqrt{1-(Y_t^{\sigma,y})^2}dB_t^{(\alpha)} + [\frac{1}{\sqrt{n}}\big(1-(Y_t^{\sigma,y})^2\big) - \frac{n}{2}\dot\alpha(t)Y_t^{\sigma,y}]dt \\ Y_\sigma^{\sigma,y} = y \end{cases} . \tag{4.39}$$

We divide the proof of the convergence in two steps. In the first one, we prove that for all $y \in [-1,1]$ and $\sigma \geqslant 0$, $Y_t^{\sigma,y}$ converges almost surely to 1; and then prove the convergence of Z_t to 1 in the second one.

Step I: Let $y \in [-1,1]$ and assume without loss of generality $\sigma = 0$. In order to lighten the notation, we omit the superscripts y and σ in $Y_t^{\sigma,y}$ during this step. We start by proving that Y_t is an almost sure asymptotic pseudotrajectory for the flow induced by the ODE

$$\dot x = \frac{1}{\sqrt{n}}(1-x^2). \tag{4.40}$$

In order to achieve it, we use Theorem 4.3. Since $x \mapsto (1-x^2)$ is Lipschitz continuous on $[-1,1]$ and that $Y_t \in [-1,1]$ for all $t \geqslant 0$, it remains to verify the hypothesis concerning the noise term.

Set

$$\varepsilon(t) = \dot\alpha(t) = (\frac{3}{2}t)^{-\frac{1}{3}}. \tag{4.41}$$

It is then obvious that $\varepsilon(t)$ satisfies (4.4). Because $Y_t \in [-1,1]$ for all $t \geqslant 0$, it is clear that the conditions in Remark 4.1 are satisfied. Consequently, by Theorem 4.3, $(Y_t)_t$ is an almost sure asymptotic pseudotrajectory for the flow induced by (4.40).

Because $\{1\}$ is an attractor for the flow induced by (4.40) with basin $]-1,1]$ and that almost surely $Y_t \in]-1,1]$ infinitely often, then

$$\lim_{t\to\infty} Y_t = 1 \text{ a.s} \tag{4.42}$$

by Theorem 4.2.

Step II: Our goal is to prove

$$\mathbb{P}(\lim_{t\to\infty} Z_t = 1) = 1. \tag{4.43}$$

Define the stopping times $\sigma_0 = 0$,

$$\tau_j = \inf\big(t > \sigma_{j-1} \mid \frac{R_{\alpha(t)}}{\sqrt{\alpha(t)}} = \frac{1}{\sqrt{n}}\big), \quad j \geqslant 1 \tag{4.44}$$

and

$$\sigma_j = \inf\big(t > \tau_j \mid \frac{R_{\alpha(t)}}{\sqrt{\alpha(t)}} = \frac{3}{2\sqrt{n}}\big), \quad j \geqslant 1 \tag{4.45}$$

with the convention $\inf\emptyset = +\infty$.

By Lemma 4.2, we have

$$\mathbb{P}\big(\bigcup_{j\geqslant 1}\{\tau_j = \infty\}\big) = 1 \text{ and for all } j \geqslant 1,\ \mathbb{P}\big(\sigma_j < \infty \mid \tau_j < \infty\big) = 1. \tag{4.46}$$

Let us start by estimating $\mathbb{P}(\lim_{t\to\infty} Z_t = 1,\ \tau_{j+1} = \infty \mid \sigma_j < \infty)$. For $s \in [\sigma_j, \tau_{j+1}]$, we have

$$\frac{R_{\alpha(s)} + \frac{1}{R_{\alpha(s)}}}{\sqrt{\alpha(s)}} \geqslant \frac{1}{\sqrt{n}}.$$

So, by Ikeda-Watanabe's comparison result (see Theorem 1.1, Chapter VI in [68]),

$$\mathbb{P}\big(Z_{(t+\sigma_j)\wedge\tau_{j+1}} \geqslant Y^{\sigma_j, Z_{\sigma_j}}_{(t+\sigma_j)\wedge\tau_{j+1}},\ \forall t \geqslant 0 \mid \sigma_j < \infty\big) = 1. \tag{4.47}$$

As a consequence, we have

$$\begin{aligned}\mathbb{P}\big(\lim_{t\to\infty} Z_t = 1,\ \tau_{j+1} = \infty \mid \sigma_j < \infty\big) &\geqslant \mathbb{P}\big(\lim_{t\to\infty} Y^{\sigma_j, Z_{\sigma_j}}_{t+\sigma_j} = 1,\ \tau_{j+1} = \infty \mid \sigma_j < \infty\big) \\ &= \mathbb{P}\big(\tau_{j+1} = \infty \mid \sigma_j < \infty\big). \qquad (4.48)\end{aligned}$$

where the last equality follows from Step I.

Since $(\{\tau_j = \infty\})_{j\geqslant 1}$ is an increasing family of events such that

$$\{\tau_j = \infty\} = \mathcal{N}_j \cup \bigcup_{k=0}^{j-1} \{\tau_{k+1} = \infty,\ \sigma_k < \infty\},$$

where $\mathcal{N}_j$ is an event of probability 0, we obtain from (4.46) and (4.48)

$$\begin{aligned}\mathbb{P}\big(\lim_{t\to\infty} Z_t = 1\big) &= \sum_{j\geqslant 0} \mathbb{P}\big(\lim_{t\to\infty} Z_t = 1,\ \tau_{j+1} = \infty \text{ and } \sigma_j < \infty\big) \\ &\geqslant \sum_{j\geqslant 0} \mathbb{P}\big(\tau_{j+1} = \infty \text{ and } \sigma_j < \infty\big) \\ &= 1. \qquad (4.49)\end{aligned}$$

Consequently, C_t converges almost surely to 1. Therefore, so does

$$\frac{R_t}{t} = \frac{1}{t}\int_0^t C_s ds. \tag{4.50}$$

Proof of the rate of convergence:

Set $\sigma_0 = 0$ and define the stopping times

$$\tau_j = \inf\big(t > \sigma_{j-1} \mid C_t = 0 \text{ or } \frac{R_t}{t} = \frac{1}{2}\big), \quad j \geqslant 1 \tag{4.51}$$

and

$$\sigma_j = \inf\big(t > \tau_j \mid C_t \geqslant \frac{1}{2} \text{ and } \frac{R_t}{t} \geqslant \frac{3}{4}\big), \quad j \geqslant 1 \tag{4.52}$$

with the convention $\inf \emptyset = +\infty$. So, by the previous part,

$$\mathbb{P}\big(\bigcup_{j\geqslant 1}\{\tau_j = \infty\}\big) = 1 \text{ and for all } j \geqslant 1,\ \mathbb{P}\big(\sigma_j < \infty \mid \tau_j < \infty\big) = 1. \tag{4.53}$$

Case $n \geqslant 2$: Set $Z_t = 1 - C_t$ and define the process $(\vartheta_t)_{t\geqslant 0}$ by $\vartheta_0 = 2$, $\vartheta_t = Z_t - Z_{\tau_j} + \vartheta_{\tau_j}$ for $t \in [\tau_j, \sigma_j]$ and

$$\vartheta_t = \vartheta_{\sigma_j} - \int_{\sigma_j}^t \sqrt{1 - C_s^2} dW_s - \frac{1}{2}\int_{\sigma_j}^t s\vartheta_s ds + \frac{n}{2}(t - \sigma_j) \tag{4.54}$$

for $t \in [\sigma_j, \tau_{j+1}]$. Thanks to (4.37), one can also write Z_t, for $\sigma_j \leqslant t \leqslant \tau_{j+1}$,

$$Z_t = Z_{\sigma_j} - \int_{\sigma_j}^t \sqrt{1 - C_s^2} dW_s - \int_{\sigma_j}^t \Big((R_s + \frac{1}{R_s})(1 + C_s) + \frac{n}{2}\Big) Z_s ds + \frac{n}{2}(t - \sigma_j). \tag{4.55}$$

Moreover, for such times t, we have

$$\Big(R_t + \frac{1}{R_t}\Big)(1 + C_t) \geqslant \frac{t}{2}.$$

Hence, from Ikeda-Watanabe comparison's result

$$\mathbb{P}(Z_t \leqslant \vartheta_t,\ \forall t \geqslant 0) = 1. \tag{4.56}$$

Since $1 - C_t^2 \in [0, 1]$, we have by Proposition 4.2

$$\mathbb{P}\Big(\vartheta_t = O\big(t^{-1/2}\sqrt{\ln(t)}\big), \tau_{j+1} = \infty \mid \sigma_j < \infty\Big) = \mathbb{P}\Big(\tau_{j+1} = \infty \mid \sigma_j < \infty\Big). \tag{4.57}$$

By the same argumentation as in Step II of the proof of the convergence, one obtains

$$1 - C_t = O\big(t^{-1/2}\sqrt{\ln(t)}\big)\ a.s. \tag{4.58}$$

Case $n = 1$: Set $\theta_t = \arccos(C_t) \in [0, \pi]$. Then, as long as $R_t > 0$, θ_t solves

$$d\theta_t = -dW_t - (R_t + \frac{1}{R_t})\sin(\theta_t)dt + dL_t(0) - dL_t(\pi), \tag{4.59}$$

where $(L_t(0))_{t\geqslant 0}$ (resp. $(L_t(\pi))_{t\geqslant 0}$) is a process of finite variation that increases when $\theta_t = 0$ (resp. $\theta_t = \pi$).

Because C_t converges almost surely to 1, then, from the second order Taylor expansion of cos(.) about 0, an estimate of its rate of convergence is given by the one of θ_t^2 to 0.

Set $\Theta_t = \theta_t^2$. Then, as long as $R_t > 0$, it solves

$$d\Theta_t = -2\sqrt{\Theta_t} dW_t - 2(R_t + \frac{1}{R_t})\sqrt{\Theta_t}\sin(\sqrt{\Theta_t})dt + dt - 2\sqrt{\Theta_t} dL_t(\pi). \tag{4.60}$$

Note that $L_t(\pi)$ increases only when $\Theta_t = \pi^2$.

Following the same methodology as for the case $n \geqslant 2$, define a process $(\Psi_t)_{t\geqslant 0}$ as follows: $\Psi_0 = \pi^2$, $\Psi_t = \Theta_t - \Theta_{\tau_j} + \Psi_{\tau_j}$ if $t \in [\tau_j, \sigma_j]$ and for $t \in [\sigma_j, \tau_{j+1}]$,

$$\Psi_t = \Psi_{\sigma_j} - 2\int_{\sigma_j}^t \sqrt{\Psi_s} dW_s - 2\int_{\sigma_j}^t \frac{s}{\pi}\Psi_s ds + (t - \sigma_j). \tag{4.61}$$

Because for $t \in [\sigma_j, \tau_{j+1}]$, we have

$$(R_t + \frac{1}{R_t}) \geqslant \frac{t}{2} \text{ and } \sqrt{\Theta_t}\sin(\sqrt{\Theta_t}) \geqslant \frac{2}{\pi}\Theta_t,$$

it follows from Ikeda-Watanabe's comparison result

$$\mathbb{P}(\Theta_t \leqslant \Psi_t, \ \forall t \geqslant 0) = 1. \tag{4.62}$$

Since $(\Psi_{t\wedge\tau_{j+1}})_{t\geqslant\sigma_j}$ has the same law as $(Z^2_{t\wedge\tau_{j+1}})_{t\geqslant\sigma_j}$, where $(Z_t)_{t\geqslant\sigma_j}$ is the solution of the SDE

$$dZ_t = dW_t - \frac{t}{\pi}Z_t dt, \ Z_{\sigma_j} = \theta_{\sigma_j}, \tag{4.63}$$

it follows from Proposition 4.2

$$\mathbb{P}\Big(\Psi_t = O\big(t^{-1}\ln(t)\big), \tau_{j+1} = \infty \mid \sigma_j < \infty\Big) = \mathbb{P}\Big(\tau_{j+1} = \infty \mid \sigma_j < \infty\Big).$$

Arguing like in Step II of the proof of the convergence, one obtains

$$\Theta_t = O\big(t^{-1}\ln(t)\big) \ a.s.$$

Thus

$$1 - C_t = O\big(t^{-1}\ln(t)\big) \ a.s.$$

Remark 4.3. *Following the proof of the rate of convergence from the case $n = 1$, one proves that the rate of convergence to 1 of the solution of the SDE*

$$\begin{cases} dC_t^{(n)} = \sqrt{n}\sqrt{1-(C_t^{(n)})^2}dW_t + [(R_t + \frac{1}{R_t})\big(1-(C_t^{(n)})^2\big) - \frac{n}{2}C_t^{(n)}]dt \\ C_0^{(n)} = y \end{cases} \tag{4.64}$$

is $O\big(t^{-1}\ln(t)\big)$. Therefore, we conjecture that so does C_t for any $n \geqslant 2$.

Chapter 5

Strongly self-interacting processes on the circle

We reproduce in this chapter the content of the preprint [54] which is currently submitted to *Stochastics: An International Journal of Probability and Stochastic Processes*. Therefore some redundancies with the Introduction chapter are possible. It is a joint work with Pierre Monmarché and answers a question raised to me by Olivier Raimond.

Contrary to the previous chapters, this one deals also with a Self-Interacting Velocity Jump Process, which lies in the family of Piecewise Deterministic Markov Processes (PDMP). To our knowledge, it is the first time that a PDMP with self-interaction is studied.

Keywords: Self-Interacting Markov processes, diffusions, PDMP, ergodicity, almost-sure convergence.

MSC primary: 60K35, 60J25, 60H10, 60J75, 60J60

Contents

5.1 Introduction

Our aim is to study the effect of the addition of a self-interaction mechanism to two initially Markovian dynamics. The first one is the classical Fokker-Planck diffusion $X \in \mathbb{R}$ that solves the SDE

$$dX_t \quad = \quad dB_t - V'(X_t)dt,$$

where $(B_t)_{t\geqslant 0}$ is a standard Brownian motion on $\mathbb{R}$. Namely X is the Markov process with generator

$$Lf(x) \quad = \quad \frac{1}{2}f''(x) - V'(x)f'(x).$$

We recall the generator of a Markov process $(Z_t)_{t\geq 0}$ is formally defined by

$$Lf(z) \quad = \quad (\partial_t)_{|t=0}\, \mathbb{E}\left(f(Z_t) \mid Z_0 = z\right).$$

The second one is the velocity jump process $(X, Y) \in \mathbb{R} \times \{-1, 1\}$ which is the piecewise deterministic Markov process (PDMP) introduced in [78] with generator

$$Lf(x,y) \quad = \quad y\partial_x f(x,y) + \left(\lambda + (yV'(x))_+\right)\left(f(x,-y) - f(x,y)\right)$$

where $\lambda > 0$ is constant and $(\ )_+$ denotes the positive part (see [78] and Section 5.2.2 for a trajectory definition of the dynamic). In both cases, if we suppose that the potential V is sufficiently coercive at infinity, X is ergodic and its law converges to the Gibbs measure with density proportional to e^{-V}. Note that when the rate of jump λ goes to infinity and time is correctly accelerated, the velocity Gibbs process (more precisely its first coordinate) converges to the Fokker-Planck diffusion (see [52]).

In both cases we want to replace the potential $V(X_t)$ by a self-interacting potential

$$V_t(X_t) \quad = \quad \int_0^t W(X_t, X_s)ds$$

where W is a symmetric interaction potential. In other words $V_t(X_t)$ depends both on the current position X_t and the (non-normalized) occupation measure $\int_0^t \delta_{X_s} ds$. This is a strong self-interaction, by contrast with the weak self-interaction such as studied in [19] where the self-interacting potential is a function of X_t and of the normalized occupation measure $\frac{1}{t}\int_0^t \delta_{X_s} ds$.

Self-Interacting processes belong to the family of *path-dependent* processes. The particularity of such processes is their lack of Markov property since the past modifies the environment that drives the particle. New phenomena may arise in their long time behaviour, which would be impossible without the path-dependency.

A first example of strong self-interaction is the linear one, that correspond to $W(x,y) = \frac{1}{2}(x-y)^2$. M.Cranston and Y.Le Jan proved in 1995 (see [36]) the almost sure convergence of the solution of the SDE

$$dX_t = dB_t - \int_0^t (X_t - X_s)ds. \tag{5.1}$$

Later, S.Herrmann and B.Roynette extended this result to a broader class $W(x,y) = V(x-y)$ with V convex (see [60]). In the case of the circle, the first author obtained the same result for $W(x,y) = -\cos(x-y)$ (see [53]). In all these cases the particle is attracted by its past.

In [16], M.Benaïm and the first author considered the repulsive case, in which the particle is repelled by its past trajectory. More precisely they studied a self-repelling diffusion on a compact manifold where W can be decomposed as

$$W(x,y) = \sum_{i=1}^{n} a_i e_i(x) e_i(y)$$

with the a_i's being positive numbers and the e_i's being eigenfunctions of the Laplace operator on the manifold. The basic example on the circle would be $W(x,y) = \cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$. This assumption on the e_i's yields an explicit formula for the invariant measure of the Markov process $\left(X_t, \left(\int_0^t e_i(X_s) ds \right)_{i=1,\cdots,n} \right)$.

The aim of the present work is to investigate the case where the e_i's are not eigenfunctions of the Laplace operator. On the other hand we restrict the study (in dimension 1) to the case $n = 1$, namely we take a potential of the form

$$W(x,y) = F(x)F(y)$$

with moreover F smooth and 2π-periodic, so that we consider $x \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Following [16], we set

$$U_t = \int_0^t F(X_s) ds, \tag{5.2}$$

which reduces the study of the non-Markovian process to the study of some Markov process on an extended space. This restriction should be seen as a first step toward the analysis of the more general situation.

As a consequence, in this paper we study the Markov processes (X, U) on $\mathbb{S}^1 \times \mathbb{R}$ and (X, U, Y) on $\mathbb{S}^1 \times \mathbb{R} \times \{-1, 1\}$ with respective generators

$$L_1 f(x,u) = \frac{1}{2} \partial_x^2 f(x,u) - uF'(x) \partial_x f(x,u) + F(x) \partial_u f(x,u) \tag{5.3}$$

and

$$L_2 f(x,u,y) = y \partial_x f(x,u,y) + F(x) \partial_u f(x,u,y) + \left(\lambda + (yuF'(x))_+ \right) \left(f(x,u,-y) - f(x,u,y) \right) \tag{5.4}$$

In both cases we call X the position, U the auxiliary variable and, in the case of the velocity jump process, Y the velocity. We work under the following assumption:

- The function $F : \mathbb{S}^1 \to \mathbb{R}$ is non-constant, smooth, changes signs, and $F'(x) = 0$ implies $F(x) \neq 0$. Moreover for all $x \in \mathbb{S}^1$ there exists $k \geq 1$ such that $F^{(k)}(x) \neq 0$. In particular the critical points of F are isolated points.

Throughout this chapter, we consider the discrete sets

$$\begin{aligned} M(F,+) &= \{x \in \mathbb{S}^1 \mid x \text{ is a local maximum of } F \text{ and } F(x) > 0\} \\ M(F,-) &= \{x \in \mathbb{S}^1 \mid x \text{ is a local maximum of } F \text{ and } F(x) < 0\} \\ m(F,+) &= \{x \in \mathbb{S}^1 \mid x \text{ is a local minimum of } F \text{ and } F(x) > 0\} \\ m(F,-) &= \{x \in \mathbb{S}^1 \mid x \text{ is a local minimum of } F \text{ and } F(x) < 0\} \end{aligned}$$

and $\mathcal{M} = M(F,-) \cup m(F,+)$. Recall the total variation distance between two probability laws μ and ν is

$$d_{TV}(\mu,\nu) = \inf\{\mathbb{P}(\Xi_1 \neq \Xi_2),\ Law(\Xi_1) = \mu,\ Law(\Xi_2) = \nu\}$$

and a measure μ is said invariant for a Markov process $(Z_t)_{t\geq 0}$ if $\{Law(Z_0) = \mu\}$ implies $\{\forall t \geq 0, Law(Z_t) = \mu\}$. We say that the law of $(Z_t)_{t\geq 0}$ converges exponentially fast to μ in the total variation sense if there exist $C, \rho > 0$, that may depend on the law of Z_0, such that

$$d_{TV}(Law(Z_t), \mu) \leq Ce^{-\rho t}.$$

Our main result is the following:

Theorem 5.1.

1. *If $\mathcal{M} = \emptyset$, then each of the the processes (X, U) with generator (5.3) and (X, U, Y) with generator (5.4) admits a unique invariant measure. If the law of U_0 admits an exponential moment then the process converges exponentially fast in the total variation distance sense to this invariant measure.*

2. *If $\mathcal{M} \neq \emptyset$, then, in both cases, the position X_t almost surely converges as t goes to infinity to a point of $\mathcal{M}$. Any point of $\mathcal{M}$ has a positive probability to be the limit of X.*

Before proceeding to its proof, let us mention why this result may be expected. Suppose that, at some time, $U > 0$. Then, as long as U is large enough, the force $U_t F'(X_t)$ tends to confine X close to the minima of F. If these minima are all negative, while X stays in their neighbourhood, U decreases, up to some point where it becomes negative. From then the effect of the force is reversed, X is attracted by the maxima of F, and the same mechanism comes into play with U and F changed to $-U$ and $-F$. In some sense X and U have then an inhibitory effect one on the other.

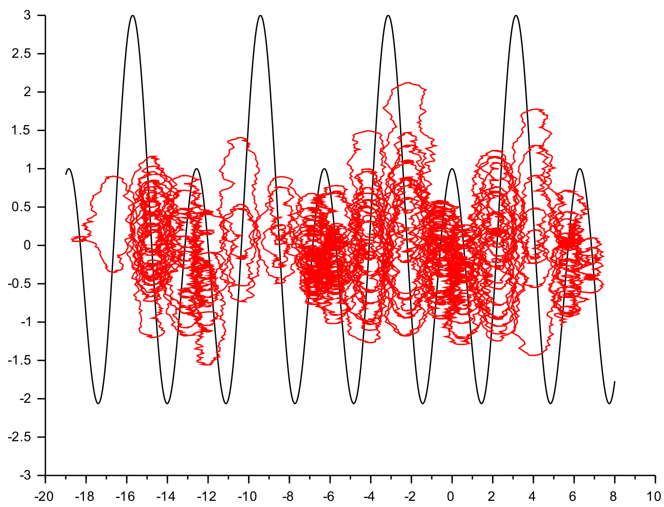

Figure 5.1: In black, the function $f(x) = -cos(x) + 2cos(2x)$ and in red a trajectory of (X_t, U_t) with initial condition $(0, 0)$. In that case $\mathcal{M} = \emptyset$

On the other hand if X falls in the neighbourhood of a positive minimum of F while $U > 0$ (the case of a negative maximum with $U < 0$ being symmetric) then, as long as it stays there, U increases, which make it more and more unlikely for X to escape away from the minimum, so that eventually there is a positive probability that X never leaves and U goes to infinity. This is reminiscent of the annealing problem (see [93] for the diffusion and [78] for the velocity jump process) where U_t is replaced by a deterministic $(\beta_t)_{t\geq 0}$, called the inverse temperature. It is classical that in this case, if β increases faster than logarithmically then X will eventually stay trapped forever in the cusp of a local minima. Yet, in our present case, as long as X stays close to a positive minimum, U increases linearly in time.

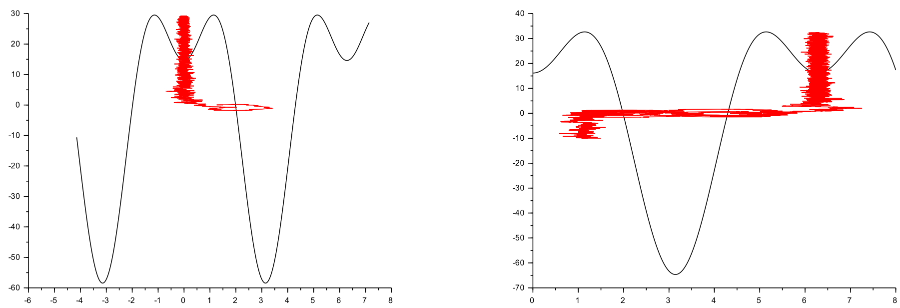

Figure 5.2: In black, the function $f(x) = \big(5cos(x) - 3cos(2x)\big)/2$. In that case $\mathcal{M} \neq \emptyset$.
Left: in red a trajectory of (X_t, U_t) with initial condition $(2, 0)$.
Right: in red a trajectory of (X_t, U_t) with initial condition $(2, -10)$

Remark 5.1.

1. *The particular form of the interacting potential $W(x,y) = F(x)F(y)$ implies that W is a Mercer Kernel, which means the particle is repulsed by its past (see [16]). We could also consider the case $W(x,y) = -F(x)F(y)$. Following the proof of Theorem 5.1, it is not hard to see that in this case X_t almost surely converges as t goes to infinity to a point of $\mathcal{M}' = m(F,-) \cup M(F,+)$ which, as soon as F is not constant and changes signs, is non-empty.*

2. *If F does not change signs, then, depending on the sign, U_t converges either to ∞ or to $-\infty$ linearly fast. Therefore, Proposition 5.1 and Proposition 5.4 imply the almost-sure convergence of X_t respectively either to a local minimum or to a local maximum of F.*

We made the choice to write as much as possible notations, results and proofs which are common to both processes, isolating only the few lemmas that deal with the specific technical difficulties of each case. Our arguments are based on bounds for some hitting times of the processes which are established in Section 5.2. From them we show in Section 5.3 that, when $\mathcal{M}$ is empty, the time for the processes to return to compact sets is short (i.e. in a time with exponential moments). Section 5.4 is devoted to some uniform bounds of the transition kernel of the processes over compact sets, and Section 5.5 to the proof of Theorem 5.1.

5.2 Hitting times

In this section, for a redaction purpose, we will hide the dependency on U of the evolution of X. More precisely we will consider the (inhomogeneous in time) diffusion

$$dX_t = dB_t - g(t)F'(X_t)dt \tag{5.5}$$

for any Lipschitz function g and, similarly, the inhomogeneous PDMP (X,Y) with generator

$$L_t f(x,y) = y\partial_x f(x,y) + \left(\lambda + (g(t)yF'(x))_+\right)(f(x,-y) - f(x,y)) \tag{5.6}$$

where the generator of an inhomogeneous Markov process Z is by definition

$$L_t f(z) = (\partial_s)_{|s=0}\mathbb{E}\left(f(Z_{t+s} \mid Z_t = z\right).$$

Note the processes considered in Theorem 5.1 are particular cases of those defined here.

Let $A = m(F,+) \cup m(F,-)$ be the set of minima of F, and $\delta \leq -\frac{1}{3}\max\{F(x) : x \in m(F,-)\}$ be positive and small enough so that

- for all $x \in A$, denoting by $I_x^\delta = [z_l, z_r]$ the connected component of $\{F \leq F(x) + 2\delta\}$ containing x, then F decreases on $[z_l, x]$ and increases on $[x, z_r]$.

- there exists $\kappa > 0$ such that for all $x \in A$ and $\eta \in [0, \delta]$,

$$d(x, B_x^\eta) \geqslant \kappa\sqrt{\eta},$$

where $B_x^\eta = \{z \in I_x^\delta,\ F(z) = F(x) + \eta\}$.

Finally, let

$$B^\eta = \bigcup_{x\in A} B_x^\eta \qquad \text{and} \qquad C^\eta = \left(\bigcup_{x\in A} I_x^\eta\right)^c.$$

In other words C^η is the complementary of a neighbourhood of the minima of F and B^η is a set of intermediary points from A to C^η. These sets (for $\eta = \delta$) are represented in the figure below. Note that the choice of δ ensures that if $\mathcal{M} = \emptyset$ then C^δ contains $\{F \geq -\delta\}$.

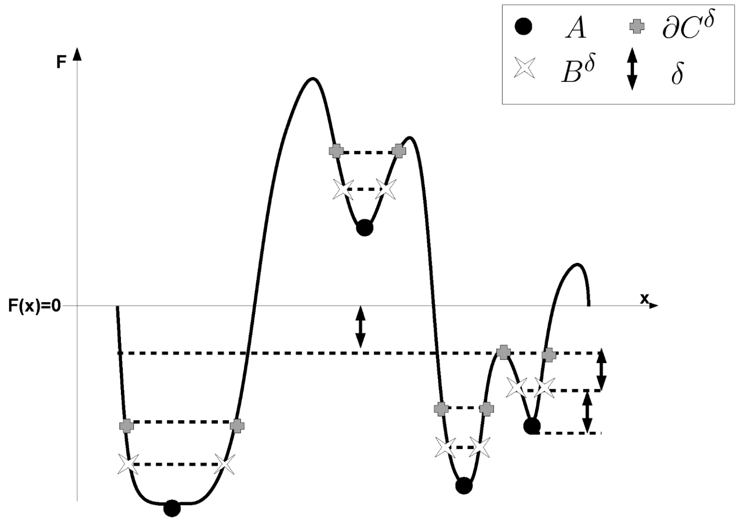

Figure 5.3: Starting from a minimum in A, the process has to cross an intermediary point of B^δ halfway before reaching C^δ. The energy level difference from A to B^δ, from B^δ to C^δ or (for a negative minimum) from ∂C^δ to $\{F = 0\}$ is always at least δ.

For $x \in \mathbb{S}^1$ and $D \subset \mathbb{S}^1$ we write

$$\begin{aligned} T_{x\to D} &= \inf\{t \geq 0, X_t \in D \mid X_0 = x\} \\ q_{x\to D} &= \mathbb{P}\left(X \text{ reaches } D \text{ before } A \mid X_0 = x\right) \end{aligned}$$

For two real random variables V, W, recall that V is said to be stochastically smaller than W, denoted by $V \overset{sto}{\leq} W$ (or equivalently $W \overset{sto}{\geq} V$), if for all $r \in \mathbb{R}$

$$\mathbb{P}(V > r) \ \leq \ \mathbb{P}(W > r).$$

If V and W have same law we write $V \overset{law}{=} W$.

The aim of this section is to prove the following:

Proposition 5.1. *There exist a constant $K > 1$ and nonnegatives random variables S and R with some finite positive exponential moments, such that for all $M > 1$ and $\eta \in (0, \delta]$ with $M\eta > 1$, for all Lipschitz function $g \geq M$, if X is defined by* (5.5) *or if (X, Y) is defined by the generator* (5.6)*, the following holds:*

$$\begin{aligned}
\forall x \in B^\eta, \qquad & q_{x\to C^\eta} \;\leq\; KMe^{-\eta M} && (5.7)\\
\forall x \in A, \qquad & T_{x\to B^\eta} \;\overset{sto}{\geq}\; R\sqrt{\eta}. && (5.8)\\
\forall x \in \mathbb{S}^1, \qquad & T_{x\to A} \;\overset{sto}{\leq}\; S && (5.9)
\end{aligned}$$

Remark 5.2. *In the case of the velocity jump process (X, Y), note that these bounds are uniform over the initial velocity Y_0.*

The meaning of these bounds is the following. Suppose the auxiliary variable U (whose role here is played by an arbitrary function g) stays for some time above a given level $M > 1$. Then the position X will fall in a local minima of F within a time shorter than S, which does not depend on M (i.e. a high U can only accelerate the hitting time of A). Then to climb back up to an intermediary point of B^η, it takes a time $R\sqrt{\eta}$, which is again uniform on $M > 1$. From B^η, the probability to escape from the neighbourhood of the minimum in one attempt (namely to reach C^η before having fallen back to A, the bottom) is of order $e^{-\eta M}$, which is a classical metastability result (see [28, 78] for instance) if g is thought as an inverse temperature, since η is the potential barrier to overcome.

The proof of Proposition 5.1 is split in the two next subsections since the arguments are different for each dynamic. Note that in several proofs we will make assumptions like $x_1 \leq x_2$ where x_1 and x_2 are in $\mathbb{S}^1$, which will make sense since at these times we will only be concerned by the behaviour of the processes on given simply connected intervals of $\mathbb{S}^1$.

5.2.1 For the diffusion

Proof of Inequality (5.7) *in the diffusion case.* Consider the diffusion defined by (5.5) with $g \geq M$ and $X_0 = x \in B^\eta$. Since $g \geq M$, it follows from Ikeda-Watanabe's comparison result [68, Theorem 1.1, Chapter VI] that

$$q_{x\to C^\eta} \leqslant \mathbb{P}\left(\widetilde{X} \text{ hits } C^\eta \text{ before } A \mid \widetilde{X}_0 = x\right) := \tilde{q}_{x\to C^\eta},$$

where $\widetilde{X}$ solves the SDE

$$d\widetilde{X}_t = dB_t - MF'\left(\widetilde{X}_t\right) dt.$$

Its scale function is defined by

$$\begin{aligned}
p(y) &= \int_x^y \exp(-2\int_x^z -MF'(s)ds)dz\\
&= \int_x^y e^{2M(F(z)-F(x))}dz.
\end{aligned}$$

Let $x_0 \in A$ and $x_1 \in C^\eta$ be such that F is monotonous on the interval between x_0 and x_1 that contains x. Suppose without loss of generality that $x_0 < x < x_1$. By [70, Proposition 5.22, Chapter 5.5],

$$\tilde{q}_{x \to C^\eta} = \frac{p(x_1) - p(x)}{p(x_1) - p(x_0)} \leq \frac{2\pi e^{2M\eta}}{\int_{x_0}^{x_1} e^{2M(F(z)-F(x))} dz}$$

where we used the local monotonicity of F. On the other hand,

$$\begin{aligned} \int_{x_0}^{x_1} e^{2M(F(z)-F(x))} dz &\geqslant \frac{1}{2M\|F'\|_\infty} \int_{x_0}^{x_1} 2MF'(z) e^{2M(F(z)-F(x))} dz \\ &= \frac{1}{2M\|F'\|_\infty} \left(e^{4M\eta} - 1\right). \end{aligned}$$

Therefore, as $M\eta > 1$,

$$q_{x \to C^\eta} \leqslant 4\pi M \|F'\|_\infty \frac{e^{2M\eta}}{e^{4M\eta} - 1} \leqslant 8\pi M \|F'\|_\infty e^{-2M\eta}.$$

□

To prove the two other assertions of Proposition 5.1, we need the following comparison result:

Lemma 5.1. *Let x_0 be a local extrema of F and $\varepsilon > 0$ be such that F' is monotonous on $J_\varepsilon := (x_0 - \varepsilon, x_0 + \varepsilon)$. Consider X the diffusion defined by (5.5), with $g \geq M > 1$, starting at $X_0 = x \in J_\varepsilon$, and W a standard Brownian motion. Denote by*

$$\chi^\varepsilon(x) = \inf\{t > 0,\ X_t \notin J_\varepsilon\} \quad \text{and} \quad \iota^\varepsilon = \inf\{t > 0,\ |W_t| = \varepsilon\}$$

the respective exit time from J_ε of X and $x_0 + W$. Then:

1. *if x_0 is a local maximum of F, for all $x \in J_\varepsilon$,*

$$\chi^\varepsilon(x) \overset{sto}{\leqslant} \iota^\varepsilon.$$

2. *if x_0 is a local minimum of F,*

$$\chi^\varepsilon(x_0) \overset{sto}{\geqslant} \iota^\varepsilon.$$

Proof. First, note that by symmetry the exit time from J_ε of $x + W$ has the same law as the exit time of $x + 2(x_0 - x) + W$, and since the process $x_0 + W$ necessarily crosses x or $x + 2(x_0 - x)$ before leaving J_ε, the exit time of $x_0 + W$ is stochastically greater than the one of $x + W$ for any $x \in J_\varepsilon$.

Consider $\Theta = (X - x_0)^2$, which solves

$$d\Theta_t = 2\sqrt{\Theta_t} d\widetilde{B}_t + dt + 2g(t)\big((X_t - x_0)F'(X_t)\big)dt,$$

where $\widetilde{B}_t = \int_0^t sign(X_s - x_0)dB_s$ is still a standard Brownian motion. Then $\left(X_0 - x_0 + \widetilde{B}\right)^2$ is a weak solution of

$$dZ_t \;=\; 2\sqrt{Z_t}d\widetilde{B}_t + dt.$$

When x_0 is a maximum (resp. minimum) of F, $(x - x_0)F'(x)$ is non-positive (resp. non-negative) on J_ε, so that by Ikeda-Watanabe's comparison result, $\Theta_t \geqslant Z_t$ (resp. $\Theta_t \leqslant Z_t$) up to the first time where Θ reaches ε^2. As a conclusion, when x_0 is a maximum, Θ reaches ε^2 before Z, and thus in a time stochastically greater than i^ε, and when x_0 is a minimum, Θ reaches ε^2 after Z and the latter happens at a time with law i^ε if the starting point is x_0. □

Proof of Inequality (5.8) *in the diffusion case.* Recall that there exists a constant $\kappa > 0$ such that for all $x \in A$ and $\eta < \delta$, $d(x, B_x^\eta) \geqslant \kappa\sqrt{\eta}$. From Lemma 5.1 and the Brownian motion's scaling property,

$$T_{x\to B^\eta} \geqslant \chi(\kappa\sqrt{\eta}, x) \overset{sto}{\geqslant} \iota^{\kappa\sqrt{\eta}} \overset{law}{=} \eta^{\frac{1}{4}}\iota^\kappa.$$

The fact that ι^κ has an exponential moment is a consequence of [33, Theorem 2]. □

Proof of Inequality (5.9) *in the diffusion case.* For a given small enough $\varepsilon > 0$, denote by

$$E^\varepsilon = \bigcup_{x \in M(F,+)\cup M(F,-)} (x-\varepsilon, x+\varepsilon)$$

the set of points which are at a distance less than ε from a maximum of F. Let X be the diffusion defined by (5.5) with $g \geq M$. We apply the following procedure:

1. If, at some time, $X_t \in E^\varepsilon$, wait until it leaves E^ε, which according to the first part of Lemma 5.1 happens in a time stochastically smaller than ι^ε.

2. If at some time t_0, X leaves E^ε, compare it with $X_{t_0} + B$ where B is the Brownian motion that drives the SDE (5.5). More precisely by Ikeda-Watanabe's comparison result, $F(X_t) \leqslant F(X_{t_0} + B_t)$ up to the time where either X or B reach an extremum of F.

3. Wait until B reaches an extrema of F. If this is a maximum, go back to the first step. If this is a minimum then necessarily, at this time, X has already crossed this minimum, stop the procedure.

Note that, ε being fixed, the probability that $x_0 + B$ reaches a maximum rather than a minimum is bounded above by some $p < 1$ which is uniform over all $x_0 \in \partial E_\varepsilon$. Hence the number of iteration of the procedure is stochastically less than a geometric random variable G with parameter p. Conditionally to whether the Brownian motion reaches a minimum or a maximum in step 3, the law of the duration of the third step is different, but in either cases it is stochastically smaller than $\iota^{2\pi}$. Therefore the total duration of one iteration of the procedure is stochastically smaller than ι^C for some constant $C > 0$, independently from whether this is the last iteration or not. Let $(\iota_k)_{k\geq 0}$ be i.i.d copies of ι^C, independent from G.

We have obtained that for all $x \in \mathbb{S}^1$,

$$T_{x\to A} \overset{sto}{\leqslant} \sum_{k=0}^{G} \iota_k$$

so that

$$\mathbb{E}\left(e^{cT_{x\to A}}\right) \leq \mathbb{E}\left(\left(\mathbb{E}\left(e^{c\iota_0}\right)\right)^G\right)$$

which is finite for c small enough. □

5.2.2 For the velocity jump process

This subsection is devoted to the proof Proposition 5.1 in the PDMP case, namely for the inhomogeneous Markov process (X, Y) with generator (5.6). First we construct a trajectory of the process (X, Y) in the following way: consider two independent i.i.d. sequences of standard (with mean 1) exponential random variables $(E_i)_{i\in\mathbb{N}}$ and $(F_i)_{i\in\mathbb{N}}$. Set $T_0 = 0$ and suppose the process has been defined up to some time T_k independently from $(E_i, F_i)_{i\geq k}$. Let

$$\begin{aligned} \theta_1 &= \inf\left\{t > 0, \int_0^t g(T_k + s)(Y_{T_k} F'(X_{T_k} + sY_{T_k}))_+ \mathrm{d}s > E_k\right\}, \\ \theta_2 &= \frac{1}{\lambda} F_k, \end{aligned}$$

and $T_{k+1} = T_k + \theta_1 \wedge \theta_2$, which is the next jump time. If $T_{k+1} = T_k + \theta_1$ we say that the jump is due to the landscape, else we say it is due to the constant rate λ. In either cases, set $X_t = X_{T_k} + (t - T_k)Y_{T_k}$ for all $t \in [T_k, T_{k+1}]$, $Y_t = Y_{T_k}$ for all $t \in [T_k, T_{k+1})$ and $Y_{T_{k+1}} = -Y_{T_k}$. Thus by induction the process is defined up to time T_n for all n. Note that even if, depending on g, the rate of jump may not be bounded, two jumps due to the landscape cannot be arbitrarily close (since at such a jump time, $yF'(x)$ becomes non-positive), so that there cannot be infinitely many jumps in a finite time and $T_n \to \infty$ as $n \to \infty$.

Proof of Inequality (5.7) *in the PDMP case.* We mainly have to adapt to our inhomogeneous settings the proof of [78, Proposition 4.1]. Without loss of generality, we consider the following configuration: $x_0 \in A$, $x_1 \in B^\eta$ and $x_2 \in \partial C^\eta$ with $x_0 < x_1 < x_2$, and F is increasing on $[x_0, x_2]$.

Let $M \geqslant 1$ and $\mathcal{L}_M$ be the set of Lipschitz functions $g \geqslant M$. For all $x \in [x_1, x_2]$, set

$$\eta_x = \sup_{g\in\mathcal{L}_M} \mathbb{P}((X, Y) \text{ reaches } (x_2, 1) \text{ before } (x, -1) \ \mid (X_0, Y_0) = (x, 1)).$$

where the supremum runs over the function g that appears in the generator (5.6) of the process (X, Y).

Consider a process (X, Y) with generator (5.6) with some function $g \in \mathcal{L}_M$. For a small $\varepsilon > 0$, suppose that $(X_0, Y_0) = (x - \varepsilon, 1)$.

Then the probability that X goes from $x - \varepsilon$ to x without any jump is less than $1 - \varepsilon\,(MF'(x) + \lambda) + \underset{\varepsilon\to 0}{o}(\varepsilon)$ and the probability it reaches $(x, 1)$ before $(x - \varepsilon, -1)$ but with at least one jump is of order ε^2 as $\varepsilon \to 0$.

If the process has reached $(x, 1)$, it has a probability less than η_x to reach $(x_2, 1)$ before having fallen back to $(x, -1)$. Nevertheless, if indeed it has fallen back to $(x, -1)$, it has a probability $\varepsilon\lambda + \underset{\varepsilon\to 0}{o}(\varepsilon)$ to jump before reaching $(x - \varepsilon, -1)$, in which case it reaches again $(x, 1)$ with probability $1 + \underset{\varepsilon\to 0}{o}(1)$. In this latter case, it reaches $(x_2, 1)$ before $(x - \varepsilon, -1)$ with probability less than $\eta_x + \underset{\varepsilon\to 0}{o}(1)$. Thus everything boils down to

$$\begin{aligned}\eta_{x-\varepsilon} &\leqslant (1 - \varepsilon\,(MF'(x) + \lambda))\,\eta_x\,(1 + \varepsilon\lambda) + \underset{\varepsilon\to 0}{o}(\varepsilon)\\ &= (1 - \varepsilon MF'(x))\,\eta_x + \underset{\varepsilon\to 0}{o}(\varepsilon).\end{aligned}$$

Together with $\eta_{x_2} = 1$, it yields $\eta_x \leqslant e^{-M(F(x_2)-F(x))}$, and in particular $\eta_{x_1} \leqslant e^{-\eta M}$.

Let

$$r_y = \sup_{g\in\mathcal{L}_M} \mathbb{P}((X, Y) \text{ reaches } (x_2, 1) \text{ before } (x_0, -1) \mid (X_0, Y_0) = (x_1, y)).$$

Starting from $(x_1, -1)$ and until the process either jumps or reaches $(x_0, -1)$, we have $YF'(X) < 0$ so that, whatever the function g in (5.6) is, there cannot be any jump due to the landscape during this time. On the other hand if $\theta_2 > 2\pi$, which happens with probability $e^{-2\lambda\pi}$, there is also no jump due to the constant rate during this time, so that

$$\mathbb{P}\left((X, Y) \text{ reaches } (x_1, 1) \text{ before } (x_0, -1) \mid (X_0, Y_0) = (x_1, -1)\right) \leqslant 1 - e^{-2\lambda\pi}.$$

Thus it means that $r_{-1} \leq \left(1 - e^{-2\lambda\pi}\right) r_1$. Therefore

$$r_1 \leqslant \eta_{x_1} + \left(1 - e^{-2\lambda\pi}\right) r_1$$

and finally

$$q_{x_1\to C^\eta} \leqslant \max(r_1, r_{-1}) \leqslant e^{2\lambda\pi} e^{-\eta M}.$$

□

Proof of Inequality (5.8) *in the PDMP case.* Since $|Y| = 1$, the time needed to reach B^η from A is deterministically larger than $d(A, B^\eta) \geqslant \kappa\sqrt{\eta}$. □

Proof of Inequality (5.9) *in the PDMP case.* Suppose that, at some point in the construction of a trajectory, $\theta_2 > 4\pi$, which happens with probability $e^{-4\lambda\pi}$. If there is also no jump due to the landscape in the meanwhile, X covers the whole circle and in particular reaches A in a time less than 2π. On the other hand if there is a jump due to the landscape before time 2π, the velocity turns to its opposite, and from then and up to the hitting time of A, $YF'(X) < 0$, so that in the meanwhile there cannot be another jump due to the landscape: A is attained in a time less than 4π.

It means that as soon as $\theta_2 > 4\pi$, X reaches A in a time less than 4π, so that starting from any point of $\mathbb{S}^1$, X reaches A in a time stochastically smaller than $4\pi G$ where G is a geometric variable with parameter $e^{-4\lambda\pi}$. □

5.3 Stability

In this section we consider either $Z = (X, U)$ or $Z = (X, U, Y)$ such as in Theorem 5.1, and we are interested in the time of return of Z to compact sets. More precisely for $M > 1$ we write

$$\tau_M = \inf\{t > 0,\ |U_t| \leqslant M\}$$

and we want to prove that τ_M admits exponential moments. The constant K and the random variables R, S appearing along this section are those given by Proposition 5.1.

Lemma 5.2. *Suppose $\mathcal{M} = \emptyset$. Let $M > 1$ be such that $KMe^{-\delta M} < 1$, and let $(S_i)_{i\in\mathbb{N}}$, $(R_i)_{i\in\mathbb{N}}$ and $(G_i)_{i\in\mathbb{N}}$ be independent i.i.d. sequences where S_0 (resp. R_0) is a copy of S (resp. $\sqrt{\delta}R$) and G_0 has geometric law with parameter $KMe^{-\delta M}$. For $t \geq 0$, let*

$$N_t = \inf\left\{n \in \mathbb{N},\ \sum_{k=1}^{G_0+\cdots+G_n} R_k \geq t\right\}.$$

Then for all $t > 0$ and for any initial condition Z_0 with $U_0 > M$,

$$\int_0^{t\wedge\tau_M} \mathbb{1}_{\{F(X_s)\geq -\delta\}} ds \overset{sto}{\leq} \sum_{k=0}^{N_t} S_k.$$

Proof. While $t \leqslant \tau_M$, the estimates of Proposition 5.1 hold for X. In particular, independently from its initial condition, the process reaches A in a time stochastically smaller than S_0. Then it takes at least a time R_1 to climb back to B^δ. From there, it reaches C^δ with probability less that $KMe^{-\delta M}$, else it falls back to A. Therefore it remains a time stochastically greater than $\sum_{k=1}^{G_0} R_k$ in $(C^\delta)^c = \{F \leqslant -\delta\}$ before reaching C^δ. When this finally occurs, the process falls again back to A after a time less than S_1 (independently from what occurred before it had reached C). We call this an excursion in C^δ. After n excursions, the processes has stayed at least a time $\sum_{k=1}^{G_0+\cdots+G_n} R_k$ in $\{F \leqslant -\delta\}$, which implies in particular that at time t there have been stochastically less than N_t excursions. Thus during a time t, the time spent in C^δ is stochastically less than $\sum_{k=0}^{N_t} S_k$. □

Recall that from Cramer's Theorem (see e.g [91, Chapter 2.4] with the exercise 2.28 in it), if $(V_i)_{i\geq 0}$ is an i.i.d. sequence of non-negative variables with some finite positive exponential moments, then $\left(\frac{1}{n}\sum_{i=0}^n V_i\right)_{n\geq 0}$ satisfies a Large Deviation Principle, in the sense there exists $c_1, c_2 > 0$ such that for all $n \geq 0$,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=0}^n V_i - \mathbb{E}(V)\right| > \frac{1}{2}\right) \leq c_1 e^{-c_2 n}.$$

Proposition 5.2. *Suppose $\mathcal{M} = \emptyset$. Then for $M > 0$ large enough there exist $\zeta > 0$ and $C < \infty$ such that for all initial condition z_0*

$$\mathbb{E}_{z_0}\left(e^{\zeta\tau_M}\right) \leq e^{C\zeta|u_0|}.$$

Proof. We only treat the case $u_0 > M > 1$ (the case $u_0 < -M$ is obtained by changing U and F to their opposite). Moreover we suppose $KMe^{-\delta M} < 1$. From

$$U_t \leq u_0 - \delta t + (\delta + \max F) \int_0^t \mathbb{1}_{\{F(X_s) \geq -\delta\}} \mathrm{d}s$$

we deduce

$$\begin{aligned} \mathbb{P}(\tau_M > t) &= \mathbb{P}\left(\tau_M > t \text{ and } \int_0^t \mathbb{1}_{\{F(X_s)\geq -\delta\}}\mathrm{d}s > \frac{M - u_0 + \delta t}{\delta + \max F}\right) \\ &\leq \mathbb{P}\left(\sum_{k=0}^{N_t} S_k > \frac{M - u_0 + \delta t}{\delta + \max F}\right) \end{aligned}$$

where N_t is defined in Lemma 5.2. For any $a, b \in \mathbb{N}$,

$$\left\{\sum_{i=0}^{b} G_i \geq a \quad \text{and} \quad \sum_{i=1}^{a} R_i \geq t\right\} \subset \{N_t \leq b\}$$

and

$$\left\{N_t \leq b \quad \text{and} \quad \sum_{i=0}^{b} S_i \leq \frac{M - U_0 + \delta t}{\delta + \max F}\right\} \subset \left\{\sum_{i=0}^{N_t} S_i \leq \frac{M - u_0 + \delta t}{\delta + \max F}\right\},$$

which implies

$$\mathbb{P}(\tau_M > t) \leq \mathbb{P}\left(\sum_{i=0}^{b} G_i < a \quad \text{or} \quad \sum_{i=1}^{a} R_i < t \quad \text{or} \quad \sum_{i=0}^{b} S_i > \frac{M - u_0 + \delta t}{\delta + \max F}\right).$$

Applied with $a = a_t = \lceil \frac{2t}{\mathbb{E}(R_1)} \rceil$ and $b = b_t = \lceil \frac{2a_t}{\mathbb{E}(G_1)} \rceil$, it implies

$$\begin{aligned} \mathbb{P}(\tau_M > t) &\leq \mathbb{P}\left(\frac{1}{b_t}\sum_{i=1}^{b_t} \frac{G_i}{\mathbb{E}(G_i)} < \frac{1}{2}\right) + \mathbb{P}\left(\frac{1}{a_t}\sum_{i=1}^{a_t} \frac{R_i}{\mathbb{E}(R_i)} < \frac{1}{2}\right) \\ &\quad + \mathbb{P}\left(\sum_{i=0}^{b_t} S_i > \frac{M - u_0 + \delta t}{\delta + \max F}\right). \end{aligned}$$

For $t \geqslant \mathbb{E}(R_1)$, we have

$$b_t \leqslant \left(2KMe^{-\delta M}\left(\frac{2t}{\mathbb{E}(R_1)} + 1\right) + 1\right) \leqslant \frac{7KMe^{-\delta M}}{\mathbb{E}(R_1)} t.$$

Therefore, for $t \geqslant 1 + \mathbb{E}(R_1) + 2\frac{1}{\delta}u_0$ and M large enough,

$$\begin{aligned} \mathbb{P}\left(\sum_{i=0}^{b_t} S_i > \frac{M - u_0 + \delta t}{\delta + \max F}\right) &\leqslant \mathbb{P}\left(\frac{1}{b_t}\sum_{i=0}^{b_t} S_i > \frac{\delta t}{4(\delta + \max F) b_t}\right) \\ &\leqslant \mathbb{P}\left(\frac{1}{b_t}\sum_{i=0}^{b_t} S_i > \frac{\delta \mathbb{E}(R_1) e^{\delta M}}{28KM(\delta + \max F)}\right) \\ &\leqslant \mathbb{P}\left(\frac{1}{b_t}\sum_{i=0}^{b_t} \frac{S_i}{\mathbb{E}(S_i)} > 2\right). \end{aligned}$$

For such large M and t, the Large Deviation Principle satisfied by R, S and G_0 implies

$$\mathbb{P}\Big(\tau_M > t\Big) \leqslant \beta e^{-\rho t}$$

for some $\beta, \rho > 0$ which do not depend on u_0. The proof is completed with

$$\mathbb{E}(e^{\zeta \tau_M}) = 1 + \zeta \int_0^\infty e^{\zeta s} \mathbb{P}(\tau_M > s) ds.$$

□

Remark 5.3. *The statement of the proposition remains valid for any initial condition z_0 such that $u_0 > 0$ (resp. $u_0 < 0$) under the weaker assumption $m(F,+) = \emptyset$ (resp. $M(F,-) = \emptyset$).*

5.4 Transition kernel bounds

In this section we still consider either $Z = (X, U)$ or $Z = (X, U, Y)$ such as in Theorem 5.1, and we call E its state space, namely either $\mathbb{S}^1 \times \mathbb{R}_+$ or $\mathbb{S}^1 \times \mathbb{R}_+ \times \{-1, 1\}$. We aim to prove the following local Doeblin condition holds:

Proposition 5.3. *Let $\mathcal{K}$ be a compact set of E. There exist $t_0 > 0$, $0 < c < 1$ and a probability measure ν on E such that for all $z \in \mathcal{K}$, for all Borel set D,*

$$\mathbb{P}\left(Z_{t_0} \in D \mid Z_0 = z\right) \quad \geq \quad c\nu(D).$$

For the diffusion process, this classically follows from an hypoellipticy argument. By contrast, note that the velocity jump process is not regularizing, in the sense its transition kernel is never absolutely continuous with respect to the Lebesgue measure (at all time there is a positive probability that the process hasn't jumped yet). However the Doeblin condition can still be obtained from some controllability property and a partial regularization.

Since, again, the arguments are different for both processes, we split the proof of Proposition 5.3 in two paragraphs.

5.4.1 For the diffusion

In this subsection we consider the process $Z = (X, U)$ induced by the generator (5.3), namely the solution of the SDE

$$\left\{ \begin{array}{rcl} dX_t &=& dB_t - U_t F'(X_t) dt \\ dU_t &=& F(X_t) dt. \end{array} \right. \tag{5.10}$$

Lemma 5.3. *For all $z_0 \in \mathbb{S}^1 \times \mathbb{R}$ and $t > 0$, the transition kernel $\mathbb{P}\left(Z_t \in \cdot \mid Z_0 = z_0\right)$ admits a smooth density with respect to the Lebesgue measure and its support is $\mathbb{S}^1 \times [u_0 + (\min F)t, u_0 + (\max F)t]$.*

Proof. For $(x,u) \in \mathbb{S}^1 \times \mathbb{R}$, set

$$G_0(x,u) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } G_1(x,u) = \begin{pmatrix} -uF'(x) \\ F(x) \end{pmatrix}.$$

The Lie-bracket of G_0, G_1 is the vector field $[G_0, G_1]$ given by

$$[G_0,G_1](x,u) = DG_1(x,u)G_0(x,u) - DG_0(x,u)G_1(x,u) = \partial_x G_1(x,u) = \begin{pmatrix} -uF''(x) \\ F'(x) \end{pmatrix}.$$

So by iteration, we have

$$\underbrace{[G_0,[G_0,\dots[G_0}_{k \text{ times}},G_1]\dots]](x,u) = \partial_x^{(k)} G_1(x,u) = \begin{pmatrix} -uF^{(k+1)}(x) \\ F^{(k)}(x) \end{pmatrix}.$$

Therefore, by our non-degeneracy assumption on F, the SDE (5.10) satisfies everywhere the Hörmander condition (see for instance [57]), which gives the first part of the proposition. For the second part, first note that for $z = (x_0, u_0) \in \mathbb{S}^1 \times \mathbb{R}$,

$$\Big((X_t, U_t)\Big)_{t \geqslant 0} \subset \mathbb{S}^1 \times [u_0 + (\min F)t, u_0 + (\max F)t].$$

Now let $((x_s, u_s))_{s \geqslant 0}$ denotes the solution of the ordinary differential equation

$$\begin{cases} \dot{x} &= v(t) - uF'(x) \\ \dot{u} &= F(x) \end{cases}$$

with initial condition $(x(0), u(0)) = z$ and where $s \mapsto v(s)$ is a piecewise constant function. Given $z' = (x', u') \in \mathbb{S}^1 \times \Big(u_0 + (\min F)t, u_0 + (\max F)t\Big)$, we aim to build a function v such that $(x(t), u(t))$ is arbitrarily close to z'. Let $\varepsilon \in (0,t)$ be arbitrary small and $t_0, t_1 \geqslant 0$ such that $t_0 + t_1 = t$ and $u' - u_0 = t_0(\min F) + t_1(\max F)$.

First, choose $v_0 \in \mathbb{R}$ such that $v(s) = v_0$ for all $s \in [0, \varepsilon]$ and $x(\varepsilon) \in \{y \in \mathbb{S}^1 \ s.t\ F(y) = \min F\}$ and let $v(s) = 0$ for $s \in (\varepsilon, t_0]$. Then, pick $v_1 \in \mathbb{R}$ such that $v(s) = v_1$ for all $s \in (t_0, t_0 + \varepsilon]$ and $x(t_0 + \varepsilon) \in \{y \in \mathbb{S}^1 \ s.t\ F(y) = \max F\}$ and let $v(s) = 0$ for $s \in (t_0 + \varepsilon, t - \varepsilon]$. Finally, choose $v_2 \in \mathbb{R}$ such that $v(s) = v_2$ for all $s \in (t - \varepsilon, t]$ and $x(t) = x'$.

Note that $u(t) = u' + \underset{\varepsilon \to 0}{o}(1)$. The Stroock-Varadhan support's Theorem concludes. □

Proof of Proposition 5.3 in the diffusion case. Denoting by $p_t(\cdot,\cdot)$ the transition density given by Lemma 5.3, let $z_1, z_2 \in E$ be such that $p_{t_1}(z_1, z_2) > 0$ for some $t_1 > 0$. By continuity, there exist neighbourhood I_1 and I_2 of respectively z_1 and z_2 such that the infimum of p_{t_1} over $I_1 \times I_2$ is $c_1 > 0$.

Let $\mathcal{K}$ be a compact set and let t_0 be large enough so that

$$I_1 \cap \left(\mathbb{S}^1 \times \bigcap_{(x,u) \in \mathcal{K}} [u + (\min F)t_0, u + (\max F)t_0] \right)$$

has a non-empty interior. The continuity of p_{t_0} and the compactness of $\mathcal{K}$ imply

$$c_0 := \inf_{z\in\mathcal{K}} \mathbb{P}(Z_{t_0} \in I_1 \mid Z_0 = z) > 0.$$

Let ν be the uniform measure on I_2, namely $\nu(D) = \frac{\lambda(D\cap I_2)}{\lambda(I_2)}$ for any Borel set D of E. Then for all $z \in \mathcal{K}$,

$$\begin{aligned}\mathbb{P}\left(Z_{t_0+t_1} \in D \mid Z_0 = z\right) &\geqslant \mathbb{P}\left(Z_{t_0+t_1} \in D \mid Z_{t_0} \in I_1\right)\mathbb{P}\left(Z_{t_0} \in I_1 \mid Z_0 = z\right)\\ &\geqslant c_0 c_1 \lambda(I_2)\nu(D).\end{aligned}$$

□

5.4.2 For the velocity jump process

In this subsection we consider the process $Z = (X, Y, U)$ with generator (5.4). The construction of a trajectory is similar to the one exposed in Section 5.2.2, except from these slight modifications: in the definition of θ_1, $g(T_k+s)$ is replaced by $U_{T_k}+\int_0^s F\left(X_{T_k} + uY_{T_k}\right) du$ and between the two jump times T_k and T_{k+1}, U is defined by $U_t = \int_{T_k}^t F(X_s)ds$.

We start with a controllability result.

Lemma 5.4. *Let $\mathcal{K}$ and $\mathcal{V}$ respectively be a compact and open set of $\mathbb{S}^1 \times \mathbb{R} \times \{-1, 1\}$. Then there exists $t_0 > 0$ such that*

$$\inf_{z\in\mathcal{K}} \mathbb{P}\left(Z_{t_0} \in \mathcal{V} \mid Z_0 = z\right) \quad > \quad 0.$$

Proof. The boundedness of F implies that for $t > 0$, there exists a compact set $\mathcal{K}_2$ such that for all $s < t$ and for all $z_0 \in \mathcal{K}$, if $Z_0 = z_0$ then $Z_s \in \mathcal{K}_2$. Hence results from [18] apply even if our whole state space is not compact. In particular, the process is Feller, and because K is compact we only need to prove that there exists t_0 such that

$$\mathbb{P}\left(Z_{t_0} \in \mathcal{V} \mid Z_0 = z\right) > 0$$

for all $z \in \mathcal{K}$. Let $z_0 = (x_0, y_0, u_0) \in \mathcal{K}$ and $z_1 = (x_1, y_1, u_1) \in \mathcal{V}$. We proceed in three steps.

First, suppose that we can deterministically choose a piecewise constant velocity $y(t) \in \{-1, 0, 1\}$, from which $(x(t), u(t))$ is defined by an initial condition and by the ODE

$$\begin{pmatrix}\dot{x}\\ \dot{u}\end{pmatrix} = \begin{pmatrix}y\\ F(x)\end{pmatrix}. \tag{5.11}$$

For a sufficiently large time t_0, we can build a path between z_0 and z_1 as follows. If $u_1 > u_0$ (resp. $u_1 < u_0$), choose a non-zero velocity to bring x_0 to a point $x^* \in M(F, +)$ (resp. $x^* \in m(F, -)$). Then, pick the zero velocity and wait until u reaches the value $u_1-\int_{x^*}^{x_1} F(s)\mathrm{d}s$. Next, with the velocity $y = 1$, bring x^* to a point x^{**} such that $F(x^{**}) = 0$ and wait up to the time $t_0 - |x^{**} - x_1|$. Finally, with the velocity $y = 1$, push x^{**} to x_1, and set the velocity to y_1 at time t_0.

In a second instance, we can choose a deterministic $y(t) \in \{-1, 1\}$ such that the solution of the system (5.11) starting from z_0 is arbitrarily close to z_1 at time t_0. To ensure this, we simply approximate the case $y = 0$ in the previous step by sufficiently fast and balanced jumps between -1 and 1.

Finally, we consider the PDMP starting from z_0. Since the random jump times have positive density, the PDMP follows arbitrarily closely a trajectory as described in the second step with positive probability. Hence, given any neighbourhood of z_1, the PDMP has positive probability to be in it at time t_0, which concludes.

□

Proof of Proposition 5.3 in the PDMP case. Consider the following vector fields:

$$G_{-1}(x,u) = \begin{pmatrix} -1 \\ F(x) \end{pmatrix} \text{ and } G_1(x,u) = \begin{pmatrix} 1 \\ F(x) \end{pmatrix}.$$

Then their difference is

$$G_1 - G_{-1} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

so that the Lie bracket $[G_1 - G_{-1}, G_1](x,u)$ is

$$[G_1 - G_{-1}, G_1] = 2\partial_x G_1(x,u) = \begin{pmatrix} 0 \\ 2F'(x) \end{pmatrix}$$

Since F is not constant and smooth, there exists some x such that $F'(x) \neq 0$, at which point the rank of $(G_1 - G_{-1}, [G_1 - G_{-1}, G_1])$ is 2.

According to [18, Theorem 4.4], it implies that there exist a non-empty open set $\mathcal{U}$, a probability measure ν and $t_1, c > 0$ such that $\forall z \in \mathcal{U}$,

$$\mathbb{P}\left(Z_{t_1} \in \cdot \mid Z_0 = z\right) \;\geqslant\; c\,\nu(\cdot).$$

Thus for any $z \in \mathcal{K}$ and any Borel set D,

$$\begin{aligned}\mathbb{P}\left(Z_{t_0+t_1} \in D \mid Z_0 = z\right) &\geq \mathbb{P}\left(Z_{t_0} \in \mathcal{U} \mid Z_0 = z\right) \times \inf_{z' \in \mathcal{U}} \mathbb{P}\left(Z_{t_0+t_1} \in D \mid Z_{t_0} = z'\right) \\ &\geq \left(\inf_{z' \in \mathcal{K}} \mathbb{P}\left(Z_{t_0} \in \mathcal{U} \mid Z_0 = z'\right)\right) c\,\nu(D)\end{aligned}$$

and Lemma 5.4 concludes. □

5.5 Proof of the main theorem

In this section we consider either $Z = (X, U)$ or $Z = (X, U, Y)$ such as in Theorem 5.1, and we call E the state space, namely either $\mathbb{S}^1 \times \mathbb{R}$ or $\mathbb{S}^1 \times \mathbb{R} \times \{-1, 1\}$.

5.5.1 Ergodicity when $\mathcal{M} = \emptyset$

Proof of point 1 of Theorem 5.1. Let $\mathcal{K} = \{z \in E,\ |u| \leqslant M\}$ where M is large enough so that Proposition 5.2 holds, and $\mathcal{K}' = \{z \in E,\ |u| \leqslant M+1\}$. Let $h_0 = 0$ and

$$\begin{aligned} s_k &= \inf\{r > h_k,\ Z_r \in \mathcal{K}\} \\ h_{k+1} &= \inf\{r > s_k,\ Z_r \notin \mathcal{K}'\} \end{aligned}$$

The boundary $\partial\mathcal{K}$ being compact and the processes being Feller, the embedded Markov chain $(Z_{s_k})_{k\geq 1}$ admits an invariant measure μ_e. For a Borel set D,

$$\mu(D) = \int \mathbb{E}\left(\int_{s_0}^{s_1} \mathbb{1}_{Z_s \in D} ds \ \middle|\ Z_0 = z_0 \right) d\mu_e(z_0)$$

defines a measure which is invariant for Z (see [71, Proof of Theorem 4.1]), and has finite mass (the expectation of $s_1 - s_0$ being bounded uniformly on $z_0 \in \partial K$ from Proposition 5.2), which we suppose normalized to 1. Note that from the controllability results proven in Section 5.4, the support of μ is equal to E.

The uniqueness of the invariant measure and the exponential convergence to equilibrium are both obtained from a classical coupling argument. Let $\mu_0 = Law(Z_0)$, and let t_0, ν and c be given by Proposition 5.3. Wait until the time τ_M (which according to Proposition 5.2 is almost surely finite, and moreover, has a finite exponential moment if so does μ_0). At time τ_M, consider two random variables Z'_{τ_M} and Θ distributed according to μ and ν respectively.

If $Z'_{\tau_M} \in \mathcal{K}$ (which happens with positive probability), we define simultaneously two processes Z and Z' with the same generator (either (5.3) or (5.4)) in such a way that $Z_{\tau_M + t_0} = \Theta = Z'_{\tau_M + t_0}$ with probability c, in which case we say the coupling is a success and from then we let Z' evolve according to its Markov dynamics (i.e. either the diffusion or the PDMP one) and set $Z_t = Z'_t$ for all $t \geq \tau_M + t_0$.

If the coupling is a failure (which happens with probability $1 - c$), we wait until Z enters $\mathcal{K}$ again. Note that at time $\tau_M + t_0$, Z is necessarily at most at distance $t_0 \times \|F\|_\infty$ from $\mathcal{K}$, hence its next time of return to $\mathcal{K}$ has some finite exponential moments that do not depend on μ_0. Once Z has reached $\mathcal{K}$ we try a new coupling, and so on as long as the coupling fails.

Let T be the first instance the coupling succeeds. Since μ is invariant for the dynamics, (Z, Z') is a coupling between $Law(Z_t)$ and μ, and

$$d_{TV}\left(Law(Z_t), \mu\right) \leqslant \mathbb{P}\left(T > t\right) \underset{t\to\infty}{\longrightarrow} 0.$$

Uniqueness of the invariant measure is obtained by taking $Law(Z_0)$ invariant. Moreover, if μ_0 has some finite exponential moments, so does T, and the Chernoff's Inequality concludes:

$$\mathbb{P}\left(T > t\right) \leqslant e^{-\rho t}\mathbb{E}\left(e^{\rho T}\right).$$

□

5.5.2 Localization when $\mathcal{M} \neq \emptyset$

Proposition 5.4. *Suppose $m(F,+) \neq \emptyset$. Then there exist $p > 0$ and $M > 0$ (which does not depend on Z_0) such that if $X_0 = x_0 \in m(F,+)$ and $U_0 \geq M$, then*

$$\mathbb{P}\left(X_t \underset{t\to\infty}{\longrightarrow} x_0\right) \;\; \geqslant \;\; p.$$

Proof. For $j \geqslant 0$, define

$$\eta_j = \frac{4\ln(2+j)}{2+j} \wedge \delta,$$

set $c = \max\{\frac{1}{F(x)},\ x \in m(F,+)\}$, $S_0 = 0$ and define the following stopping times:

$$\begin{aligned}
\tau_{j+1} &= \inf\left\{t > S_j,\ X_t \in C^{\eta_{j+1}}\right\},\\
\tilde{S}_{0,j} &= S_j,\\
\tilde{T}_{k,j} &= \inf\left\{t > \tilde{S}_{k-1,j},\ X_t \in B^{\eta_{j+1}}\right\} \wedge (\tilde{S}_{k-1,j} + c) \wedge \tau_{j+1},\ k \geqslant 1,\\
\tilde{S}_{k,j} &= \inf\left\{t > \tilde{T}_{k,j},\ X_t \in A\right\} \wedge \tau_{j+1},\ k \geqslant 1.
\end{aligned}$$

Let

$$N_j \;\; = \;\; \inf\left\{k \in \mathbb{N},\ \tilde{S}_{k,j} \geqslant S_j + c \text{ or } \tilde{S}_{k,j} = \tau_{j+1}\right\}$$

and $S_{j+1} = \tilde{S}_{N_j,j}$.

Let us give some intuition on these definitions. The connected component of $(C^{\eta_j})^c$ that contains x_0 is a neighbourhood of x_0 whose diameter goes to 0 as j goes to ∞. At time τ_j, the process has escaped from this neighbourhood. For $t \leq \tau_j$, the process makes possibly many oscillations near x_0. When such an oscillation is large enough for the process to reach B^{η_j} (this is at a time $\widetilde{T}_{k,j}$ for some k), we consider this is the beginning of an attempt to leave $(C^{\eta_j})^c$. If this attempt fails, the process falls back to x_0 (this is $\widetilde{S}_{k,j}$). While X makes those attempts to escape, time goes by, so that U increases: after a time c, U has increased at least by 1. Next time X falls back to x_0 (this is S_{j+1}), we shrink the neighbourhood, namely from then we consider that the process escapes if it reaches $C^{\eta_{j+1}}$. From S_j to S_{j+1}, there have been N_j attempts to leave. The sequence η is scaled so that there is in fact a positive probability that the process never escape from the shrinking neighbourhood that collapses at infinity to $\{x_0\}$.

Let us write these ideas more precisely. Note that as long as $S_{j+1} < \tau_{j+1}$,

$$S_{j+1} - S_j \geqslant c \qquad \text{and} \qquad U_t \geqslant M + j$$

for $t \geqslant S_j$. We take M large enough so that $(M+j)\,\eta_j > 1$ for all $j \in \mathbb{N}$. Therefore, from Proposition 5.1, for all $k \geq 1$,

$$\mathbb{P}(\tilde{S}_{k,j} = \tau_{j+1} |\, \tilde{T}_{k,j} < \tau_{j+1}) \;\; \leqslant \;\; K(j+M)e^{-(j+M)\eta_j}.$$

It implies that $\left(\mathbb{1}_{\tilde{S}_{(i\wedge N_j),j}<\tau_{j+1}} + (i\wedge N_j)K(j+M)e^{-(j+M)\eta_j}\right)_{i\geqslant 0}$ is a submartingale. Thus,

$$\begin{aligned}\mathbb{P}(S_{j+1}<\tau_{j+1}|\, S_j<\tau_j) &= 1+\mathbb{E}(\mathbb{1}_{S_{j+1}<\tau_{j+1}} - \mathbb{1}_{S_j<\tau_j}|\, S_j<\tau_j)\\ &\geqslant 1-K(j+M)e^{-(j+M)\eta_{j+1}}\mathbb{E}(N_j|\, S_j<\tau_j). \qquad (5.12)\end{aligned}$$

From Proposition 5.1, we have

$$\tilde{S}_{k+1,j}-\tilde{S}_{k,j} \overset{sto}{\geqslant} \sqrt{\eta_j}R.$$

Hence, considering a sequence $(R_i)_{i\in\mathbb{N}}$ of i.i.d random variables distributed like R

$$\begin{aligned}N_j &\overset{sto}{\leqslant} \inf\left\{n\geqslant 1,\ \sqrt{\eta_j}\sum_{i=1}^{n}R_i\geqslant c\right\}\\ &\leqslant \left\lceil\frac{2c}{\mathbb{E}(R_1)\sqrt{\eta_j}}\right\rceil+\inf\left\{n\geqslant 1,\ \frac{1}{n}\sum_{i=1}^{n}R_i\geqslant\frac{\mathbb{E}(R_1)}{2}\right\}.\end{aligned}$$

Because R satisfies a Large Deviation Principle,

$$\begin{aligned}\mathbb{E}(N_j|S_j<\tau_j) &\leqslant \left\lceil\frac{2c}{\mathbb{E}(R_1)\sqrt{\eta_j}}\right\rceil+\sum_{n\geqslant 1}\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^{n}R_i\leqslant\frac{\mathbb{E}(R_1)}{2}\right)\\ &\leqslant \frac{K'}{\sqrt{\eta_j}}\end{aligned}$$

for some constant K' which does not depend on j, nor M. Thus (5.12) is now

$$\mathbb{P}(S_{j+1}<\tau_{j+1}|\, S_j<\tau_j) \geqslant 1-\frac{K'K}{\sqrt{\eta_j}}(j+M)e^{-(j+M)\eta_{j+1}}.$$

Take M large enough so that the right-hand side is positive for all $j\in\mathbb{N}$. Then by induction

$$\begin{aligned}\mathbb{P}(S_{j+1}<\tau_{j+1}) &= \mathbb{P}(S_{j+1}<\tau_{j+1}\mid S_j<\tau_j)\ \mathbb{P}(S_j<\tau_j)\\ &\geqslant \prod_{i=0}^{j+1}\left(1-\frac{K'K}{\sqrt{\eta_i}}(i+M)e^{-(i+M)\eta_{i+1}}\right).\end{aligned}$$

As $(\{S_j<\tau_j\})_{j\geqslant 1}$ is a decreasing family of events,

$$\begin{aligned}\mathbb{P}(S_j<\tau_j\ \forall j\in\mathbb{N}) &= \lim_{j\to\infty}\mathbb{P}(S_j<\tau_j)\\ &\geqslant \prod_{j=0}^{\infty}\left(1-\frac{K'K}{\sqrt{\eta_j}}(j+M)e^{-(j+M)\eta_{j+1}}\right)\\ &= \exp\left(\sum_{j\geqslant 0}\ln\left(1-\frac{K'K}{\sqrt{\eta_j}}(j+M)e^{-(j+M)\eta_{j+1}}\right)\right).\end{aligned}$$

For j large enough,

$$\frac{K'K}{\sqrt{\eta_j}}(j+M)e^{-(j+M)\eta_{j+1}} \quad \leqslant \quad \frac{1}{j^2}$$

and

$$\ln\left(1-\frac{K'K}{\sqrt{\eta_j}}(j+M)e^{-(j+M)\eta_{j+1}}\right) \quad \geqslant \quad -\frac{1}{2j^2}$$

which means $\mathbb{P}(S_j < \tau_j \ \forall j \in \mathbb{N}) > p > 0$ where p does not on depend Z_0. Yet,

$$\{S_j < \tau_j \ \forall j \in \mathbb{N}\} = \{\forall j \in \mathbb{N}, \ \forall s \geq S_j, \ X_s \in I_{x_0}^{\eta_j}\}$$

and the S_j's are all a.s. finite, which concludes. □

Remark 5.4. *The proof even gives a speed of convergence. Indeed we can see that* $S_{j+1} \overset{sto}{\leqslant} S_j + c + \sqrt{\delta}R$*, so that the* S_j*'s grow linearly to infinity. From the non-degeneracy assumption on* F*, there exist* $n \in \mathbb{N}$ *and* $c > 0$ *such that the diameter of* $I_{x_0}^{\eta_j}$ *is less than* $c\eta_j^{\frac{1}{n}}$*, depending on the first derivative of* F *at* x_0 *to be non-zero (if* F *is a Morse function,* $n = 2$*). It means when there is convergence, it occurs at least at a speed of order* $\left(\frac{\ln t}{t}\right)^{\frac{1}{n}}$.

Proof of point 2 of Theorem 5.1: First note that by changing U and F to their opposites, Proposition 5.4 also says that if $M(F,-) \neq \emptyset$ then there exist $p, M > 0$ such that if $U_0 < -M$ and $X_0 \in M(F,-)$ then X_t converges to x_0 with probability at least p.

For $M > 0$ large enough, $\varepsilon > 0$ small enough and $x \in \mathcal{M}$, let

$$\begin{aligned}
\mathcal{V}_x^\varepsilon &= \{z' \in E \text{ s.t. } |x' - x| < \varepsilon \text{ and } u' \times sign\,(F(x)) > M\} \\
\mathcal{V}^\varepsilon &= \bigcup_{x \in \mathcal{M}} \mathcal{V}_x^\varepsilon.
\end{aligned}$$

When ε is fixed, for M large enough, if the process starts in $\mathcal{V}_x^\varepsilon$, from Inequality (5.7) (which is written for $x \in m(F,+)$ but by symmetry, again, also holds for $x \in M(F,-)$) it has a probability at least $\frac{1}{2}$ to hit $\mathcal{V}_x^0$ before leaving $\mathcal{V}_x^{2\varepsilon}$. Then from Proposition 5.4, X has a probability at least p to converge to x.

Let

$$\mathcal{K} = \{z \in E, \ |u| \leqslant M\}.$$

As long as the process is in the complementary of $\mathcal{K} \cup \mathcal{V}^\varepsilon$, the conclusion of Proposition 5.2 (together with Remark 5.3) holds so that, denoting by τ_D the first hitting time of a set D, we have

$$\mathbb{P}(\tau_{\mathcal{V}^\varepsilon} \wedge \tau_{\mathcal{K}} < \infty | Z_0 = z) = 1.$$

for all $z \in E$ (more precisely: suppose $U_0 > M$, the case $U_0 < -M$ being symmetric. Then we can define a potential $\widetilde{F}$ which is equal to F away from $m(F,+)$ and which have no positive minimum, from which we can define an associated process $\widetilde{Z}$ with the initial condition $\widetilde{Z}_0 = Z_0$ such that $\widetilde{Z}_t = Z_t$ as long as $Z \notin \mathcal{K} \cup \mathcal{V}^\varepsilon$. Then $\widetilde{Z}$, to which Proposition 5.2 applies, hits $\mathcal{K}$ in a finite time).

On the other hand, by Lemmas 5.4 (for the PDMP) and 5.3 (for the diffusion), there exists $t_0 > 0$ such that for all $x \in \mathcal{M}$,

$$\inf_{z \in \mathcal{K}} \mathbb{P}\left(Z_{t_0} \in \mathcal{V}_x^{\varepsilon} \mid Z_0 = z\right) \quad > \quad 0.$$

It therefore follows that for any $z \in E$,

$$\mathbb{P}(\tau_{\mathcal{V}^0} < \infty | Z_0 = z) = 1$$

and moreover

$$\mathbb{P}(X_{\tau_{\mathcal{V}^0}} = x) > 0$$

for all $x \in \mathcal{M}$. Proposition 5.4 concludes. $\square$

Chapter 6

Open problems and partial results

The goal of this chapter is to present some problems that the previous chapters arose and the partial results that we could obtain. This chapter is divided in two sections that can be read independently.

In the first one, we consider a natural generalisation of chapters 2 and 4 by investigating the long term behaviour of the solution of the SDE

$$dX_t = dB_t - \int_0^t F'(X_t - X_s)dsdt, \tag{6.1}$$

where $F(x) = \sum_{j=1}^n a_k \cos(kx)$ and $a_k \neq 0$ for all $k = 1, \cdots, n$. We aim to present the different behaviours that can be expected according to the values of the coefficients a_k.

The spirit of the second section differs from the first section and the previous chapters. Instead of investigating the behaviour of a particle that interacts with its own past, we consider two different particles that interact each other via the past of the other. Namely, we are interested in the long term behaviour of $X_t = (X_1(t), X_2(t))$ which solves the SDE

$$dX_t = dB_t - \nabla F_t(X_t)dt, \tag{6.2}$$

where B is a two dimensional Brownian motion and F_t is the function

$$F_t(x_1, x_2) = a \int_0^t V(x_1 - X_2(s))ds + b \int_0^t V(x_2 - X_1(s))ds$$

for some constant $a, b \neq 0$ and potential function V.

Contents

6.1 A mixture of self-Attracting/Repelling Diffusions

In this section, we aim to analyse the solution's long term behaviour of the SDE

$$dX_t = dB_t - \int_0^t F'(X_t - X_s)dsdt, \tag{6.3}$$

where $(B_t)_t$ is a real valued Brownian motion and $F(x) = \sum_{k=1}^n a_k \cos(kx)$, with $n \in \mathbb{N}^*$ and $a_k \in \mathbb{R}^*$ for all $k = 1, \cdots, n$.

In the case of the circle, this is a generalisation of the study done in chapters 2 and 3 when $a_k > 0$ and the one done in chapter 4 as $n = 1$ and $a_1 < 0$. Therefore, we assume throughout this section that $n \geqslant 2$ and there exists $j = 1, \cdots, n$ such that $a_j < 0$. For technical, we also assume that

$$\sum_{j=1}^n j^2 a_j \neq 0 \text{ and } \sum_{j=1}^n a_j \neq 0.$$

From numerical simulations and some heuristic arguments, we deeply believe in the following conjecture which is at the root of this section.

Conjecture 6.1. *We have*

1. *If* $\sum_{j=1}^n j^2 a_j < 0$, *then* X_t *converges almost surely.*

2. *If* $\sum_{j=1}^n j^2 a_j > 0$ *and* $\sum_{j=1}^n a_j < 0$, *then* $\frac{1}{t}\int_0^t \delta_{e^{iX_s}} ds$ *does not converge to the uniform law on the circle.*

Indeed, assume that X_t localizes. Then, as explained in the Introduction chapter, the most important feature of the drift function is the shape of F around 0. Since $F'(0) = 0$ and $F''(0) = -\sum_{j=1}^n j^2 a_j$, then, in the first case of the assumption 0 is a local minima of F. Hence, F pushes X_t to localize and the slope of F around 0 becomes steeper. This explains why convergence should hold.

When $\sum_{j=1}^n j^2 a_j > 0$, X is pushed to places where it has spent less times. So, a natural expectation would be to say that the renormalized occupation measure $\frac{1}{t}\int_0^t \delta_{X_s} ds$ should converge to the uniform distribution on the circle. However, the fact that $\sum_{j=1}^n a_j < 0$ implies that the attracting force is not too weak with respect to the repulsive one. Thus, it slowdowns the particle's escape. As a consequence, the most visited part "follows" X_t.

We divided this section in two subsections. In the first one, we present the results we could obtain whereas in the second one, we investigated the possible long term behaviour and address our open questions.

6.1.1 The partial results

Following the same idea as in [16], set $U_j(t) = \int_0^t \cos(jX_s)ds$ and $V_j(t) = \int_0^t \sin(jX_s)ds$. With these new variables, we obtain the following system.

$$\begin{cases} dX_t = dB_t + \sum_{j=1}^n ja_j\Big(\sin(jX_t)U_j(t) - \cos(jX_t)V_j(t)\Big)dt \\ dU_j(t) = \cos(jX_t)dt,\ j = 1, \dots, n. \\ dV_j(t) = \sin(jX_t)dt,\ j = 1, \dots, n. \end{cases} \tag{6.4}$$

From Lemma 2.4 and Lemma 2.7 from Chapter 2, a first result is

Proposition 6.1. *The system (6.4) satisfies the Hörmander condition. In particular it has smooth density function with respect to the Lebesgue measure.*

As for non-degenerated diffusion and despite the degeneracy of (6.4), we can classify its qualitative behaviour in term of transience or recurrence. To shorten the notation, we write $Y_t = (X_t, U_1(t), V_1(t), \cdots, U_n(t), V_n(t))$.

Definition 6.1. *(Definition 3.1 and Discussion between Corollary 4.1 and Lemma 4.3 at page 703, [74], Kliemann)*

1. *A point $y = (x, u_1, v_1, \cdots, u_n, v_n)$ is transient if there exists an open neighbourhood $V(y)$, $\mathbb{P}(Y_t \in V(y)$ finitely often $|Y_0 = y) = 1$. A set A is transient if all $x \in A$ are transient. If A is the whole state space, we say that the system is transient.*

2. *A point $y = (x, u_1, v_1, \cdots, u_n, v_n)$ is recurrent if for all open neighbourhood $V(y)$, $\mathbb{P}(Y_t \in V(y) i.o|Y_0 = y) = 1$. A set A is recurrent if all $x \in A$ are recurrent. If A is the whole state space, we say that the system is recurrent.*

3. *Y_t is positive recurrent in a recurrent set A, if for all open set $V \subset A$, $\mathbb{E}(\sigma_U|Y_0 = y) < \infty$ for all $y \in A$. Here σ_U is the time of first entrance in U.*

Our first result is

Theorem 6.1. *The system (6.4) is either transient or recurrent. Moreover, it is positive recurrent if and if there exists an invariant probability measure.*

Before turning to the theorem's proof, we recall the notion of *control set* for our setting.[1].

Definition 6.2. *A function $c : [0, \infty) \to \mathbb{R}$ is called admissible control function if it is piecewise constant. We denote by $\mathcal{C}$ the set of such functions.*

[1]See also [3] or [74]

Given an admissible control function $c \in \mathcal{C}$, we denote by $\phi(t, y, c)$ the solution of the deterministic control system

$$\begin{cases} \dot{x}_t = c(t) + \sum_{j=1}^n j a_j (\sin(jx) u_j - \cos(jx) v_j) \\ \dot{u}_j(t) = \cos(jx), \ j = 1, \dots, n \\ \dot{v}_j(t) = \sin(jx), \ j = 1, \dots, . \end{cases} \tag{6.5}$$

with initial condition y. Its positive orbit starting from y, and denoted by $O^+(y)$, is

$$O^+(y) = \{z \text{ such that there exists } c \in \mathcal{C} \text{ and } t \geqslant 0 \text{ so that } \phi(t, y, c) = z\}.$$

We emphasize that the values of $a_1, \cdots, a_n$ does not matter as long as they are nonzero.

Definition 6.3. *A set C is called invariant control set if*

(i) for all $y, z \in C$, $z \in \overline{O^+(y)}$,

(ii) if there exists a set $B \supset C$ satisfying (i), then $B = C$.

(iii) For all $y \in C$, $\overline{O^+(y)} = \overline{C}$.

Proof of Theorem 6.1: By Theorem 2.9 in Chapter 2 and Lemma 4.1 in [74], the whole state space is an invariant control set. The result follows from Theorem 3.2 and Theorem 4.1 in [74][2].

■

At this level, we introduce the following new variables

$$C_j(t) = U_j(t)\cos(jX_t) + V_j(t)\sin(jX_t) = \left\langle \begin{pmatrix} U_j(t) \\ V_j(t) \end{pmatrix}, \begin{pmatrix} \cos(jX_t) \\ \sin(jX_t) \end{pmatrix} \right\rangle \tag{6.6}$$

and

$$S_j(t) = \sin(jX_t)U_j(t) - \cos(jX_t)V_j(t) = \left\langle \begin{pmatrix} U_j(t) \\ V_j(t) \end{pmatrix}, \begin{pmatrix} \sin(jX_t) \\ -\cos(jX_t) \end{pmatrix} \right\rangle. \tag{6.7}$$

The interest of these variables is twofold. First of all, if X_t converges almost surely to some random variable X_∞, then we know that the limit of $(C_j(t)/t, S_j(t)/t)$ is $(1, 0)$ for all $j = 1, \cdots, n$. Therefore, it provides a deterministic necessary condition.

The second interest is the following. If $\bar{\eta}_t$ is the potential viewed by the particle function defined by

$$\bar{\eta}_t(x) = F_t(x + X_t) = \int_0^t \sum_{k=1}^n a_k \cos(k(x + X_t - X_s)) ds,$$

then

$$\bar{\eta}_t(x) = \sum_{k=1}^n a_k \Big(C_k(t)\cos(kx) - S_k(t)\sin(kx) \Big).$$

[2] see also the diagram in the introduction of that paper

By Itô's formula, these new variables satisfy the SDE:

$$d\begin{pmatrix} C_1(t) \\ S_1(t) \\ C_2(t) \\ S_2(t) \\ \vdots \\ C_n(t) \\ S_n(t) \end{pmatrix} = \begin{pmatrix} -S_1(t) \\ C_1(t) \\ -2S_2(t) \\ 2C_2(t) \\ \vdots \\ -nS_n(t) \\ nC_n(t) \end{pmatrix} \big(\circ\, dB_t + (\sum_{k=1}^{n} ka_k S_k(t))dt \big) + \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} dt. \tag{6.8}$$

where $\circ$ stands for the Stratonovich integral.

Remark 6.1.

1. *We have*

$$jS_j(t) \circ dB_t = jS_j(t)dB_t + \frac{j^2}{2}C_j(t)dt$$

and

$$jC_j(t) \circ dB_t = jC_j(t)dB_t - \frac{j^2}{2}S_j(t)dt.$$

2. *Set* $R_j^2(t) := C_j^2(t) + S_j^2(t) (= U_j^2(t) + V_j^2(t))$. *Then* $R_j^2(t) - R_j^2(0) = 2\int_0^t C_j(u)du$.

We can now state and prove our main Theorem.

Theorem 6.2. *We have the following properties*

1. *If* $\sum_{j=1}^n a_j < 0$, *the system (6.4) is transient.*
2. *We have*

$$\limsup_{t\to\infty} \frac{1}{t^2} \sum_{j=1}^{n} j^2 a_j R_j^2(t) \leqslant 0\, a.s.$$

In particular, if $\sum_{j=1}^n j^2 a_j > 0$, *then with probability 1,* X_t *does not converge.*

3. *We have* $\liminf_{t\to\infty} \left| \sum_{j=1}^n \frac{ja_jS_j(t)}{t} \right| = 0$.

Proof. Let M_t be the martingale[3] defined by

$$M_t = -\int_0^t (\sum_{k=1}^n ka_kS_k(u))dB_u,$$

so that its quadratic variation is

$$\langle M \rangle_t = \int_0^t (\sum_{k=1}^n ka_kS_k(u))^2 du. \tag{6.9}$$

[3]For all $j = 1, \cdots, n$ and all $t > 0$, we have $|S_j(t)| \leqslant t$.

So, from (6.8) and Remark 6.1, we have the following key relation

$$\sum_{j=1}^{n} a_j C_j(t) + \frac{1}{4}\sum_{j=1}^{n} j^2 a_j R_j^2(t) = M_t - \langle M\rangle_t + (\sum_{j=1}^{n} a_j)t. \tag{6.10}$$

Hence

$$\exp\Big(2(\sum_{j=1}^{n} a_j C_j(t) + \frac{1}{4}\sum_{j=1}^{n} j^2 a_j R_j^2(t) - (\sum_{j=1}^{n} a_j)t)\Big) = \exp\Big(2M_t - \frac{1}{2}\langle 2M\rangle_t\Big)$$

As in [53] (see also the proof of Lemma 4.2 in Chapter 4.4), it follows from the definition of $S_j(t)$ and Equation (6.4) that $2M_t$ satisfies the Novikov condition (see [70]). Hence, the right hand side term is an exponential martingale; and therefore converges almost surely to an integrable nonnegative random variable. Thus there exists a random variable $K < \infty$ almost-surely such that for all $t > 0$

$$\sum_{j=1}^{n} a_j C_j(t) + \frac{1}{4}\sum_{j=1}^{n} j^2 a_j R_j^2(t) \leqslant K + \sum_{j=1}^{n} a_j t. \tag{6.11}$$

This proves statement 1.

From Remark 6.1, the definition of $C_j(t)$ and Equation (6.4), the left hand side term of (6.10) is at most quadratic in t as it converges to infinity. Because $\lim_{t\to\infty}\frac{M_t}{\langle M\rangle_t} = 0$ on the event $\{\lim_{t\to\infty}\langle M\rangle_t = \infty\}$, we obtain that $\langle M\rangle_t$ is at most quadratic in t as it goes to infinity. Consequently,

$$\limsup_{t\to\infty}\frac{1}{t^2}\sum_{j=1}^{n} j^2 a_j R_j^2(t) \leqslant 0 \ a.s.$$

If $X_t(\omega)$ converges, then for all $j \in \{1,\cdots,n\}$, $R_j(t,\omega)/t$ converges to 1. So

$$\lim_{t\to\infty}\frac{1}{t^2}\sum_{j=1}^{n} j^2 a_j R_j^2(t,\omega) = \sum_{j=1}^{n} j^2 a_j.$$

This completes the proof of statement 2.

Since $\langle M\rangle_t$ grows at most quadratically in time, then statement 3 follows from (6.9). □

As a corollary, we have

Corollary 6.1. *1. If $a_j < 0$ for all $j = 1,\ldots,n$, then there exists $\kappa > 0$ such that*

$$\liminf_{t}\frac{\sum_{k=1}^{n} R_k(t)}{\sqrt{t}} \geqslant \kappa.$$

2. *Assume* $n = 2$, $a_1 + a_2 < 0$ *and* $a_1 a_2 < 0$. *For* $j \in \{1, 2\}$ *such that* $a_j < 0$, *we have*

$$\liminf_{t\to\infty} \frac{R_j(t)}{\sqrt{t}} \geqslant \kappa \text{ and } \limsup_{t\to\infty} \frac{R_{3-j}(t)}{R_j(t)} \leqslant 1$$

for some constant $\kappa > 0$.

Proof. The result follows from (6.11) and the equivalences of different weighted norms on $\mathbb{R}^{2n}$. □

6.1.2 The possibles limit behaviours

In this subsection, we aim to identify the possible limit behaviours. First of all, let us recall the observation that we did when the variables C_j and S_j were introduced.

- If X_t converges to some random variable, then for all $j = 1, \cdots, n$, $(\frac{C_j(t)}{t}, \frac{S_j(t)}{t})$ converges to $(1, 0)$ as t tends to infinity.
- If $\frac{1}{t}\int_0^t \delta_{e^{iX_s}} ds$ converges to the uniform law on the circle, then for all $j = 1, \cdots, n$, $(\frac{C_j(t)}{t}, \frac{S_j(t)}{t})$ converges to $(0, 0)$.

For $j = 1, \cdots, n$ and $t > 0$, let

$$c_j(t) := \frac{C_j(t)}{t},\ s_j(t) := \frac{S_j(t)}{t} \text{ and } r_j(t) := \frac{R_j(t)}{t}.$$

Remark 6.2. *If the initial condition of (6.4) is* $(x, 0, \cdots, 0)$, *then the initial condition of (6.8) is* $(0, \cdots, 0)$. *Hence, it follows that* $(C_1(dt), S_1(dt), \cdots, C_n(dt), S_n(dt)) \simeq (dt, 0, \cdots, dt, 0)$. *Thus, we may assume that* $\big(c_1(0), s_1(0), \cdots, c_n(0), s_n(0)\big) = \big(1, 0, \cdots, 1, 0\big)$.

From Itô's Formula and (6.8), $(c_1(t), s_1(t), \cdots, c_n(t), s_n(t))$ is solution of the inhomogeneous SDE

$$d\begin{pmatrix} c_1(t)\\ s_1(t)\\ c_2(t)\\ s_2(t)\\ \vdots\\ c_n(t)\\ s_n(t)\end{pmatrix} = F_1(c_t, s_t)[\circ dB_t + t(\sum_{k=1}^n k a_k s_k(t))dt] + \frac{1}{t}F_2(c_t, s_t)dt, \tag{6.12}$$

where

$$F_1(c, s) = \begin{pmatrix} -s_1\\ c_1\\ \vdots\\ -ns_n\\ nc_n\end{pmatrix} \text{ and } F_2(c, s) = \begin{pmatrix} 1 - c_1\\ -s_1\\ \vdots\\ 1 - c_n\\ -s_n\end{pmatrix}.$$

From Remark 6.1, we have

Lemma 6.1. $\frac{d}{dt}r_j^2(t) = \frac{2}{t}\Big(c_j - r_j^2\Big)$. *In particular, for all* $\alpha \in \mathbb{R}^n$,

$$d\Big(\sum_{j=1}^n \alpha_j r_j^2(t)\Big) = \frac{2}{t}\Big((\sum_{j=1}^n \alpha_j c_j(t)) - (\sum_{j=1}^n \alpha_j r_j^2(t))\Big)dt.$$

The main consequences of the lemma and which justifies the utilisation of $c_j(t)$ and $s_j(t)$ are

- The compact space $\bar{B}\Big((0,0),1\Big)^n \subset \mathbb{R}^{2n}$ is positively invariant for the SDE (6.12) [4]. Hence, if $L(c,s)$ denotes the limit set of $\big(c_1(t), s_1(t), \cdots, c_n(t), s_n(t)\big)$, then it is not empty
- For all $\alpha \in \mathbb{R}^n$, $\Big(\sum_{j=1}^n \alpha_j r_j^2(t)\Big)_t$ is bounded. Thus

$$L(c,s) \cap \Big\{(c,s) \in \bar{B}\Big((0,0),1\Big)^n \mid \sum_{j=1}^n \alpha_j c_j = \sum_{j=1}^n \alpha_j r_j^2\Big\} \neq \emptyset.$$

Concerning the limit set, we can say a little bit more.

Proposition 6.2. *The limit set* $L(c,s)$ *is a non-empty, compact and connected subset of*

$$\Big\{(c_1, s_1, \cdots, c_n, s_n) \in \bar{B}\Big((0,0),1\Big)^n \ s.t \ \sum_{j=1}^n j^2 a_j r_j^2 \leqslant 0\Big\}$$

which is internally chain transitive for the flow Φ *induced by the ODE*

$$\begin{cases} \dot{c}_j(t) = j(\sum_{k=1}^n k a_k s_k(t))(-s_j(t)), \ \ j = 1, \dots, n \\ \dot{s}_j(t) = j(\sum_{k=1}^n k a_k s_k(t)) c_j(t) \ \ j = 1, \dots, n. \end{cases} \tag{6.13}$$

Moreover

$$L(c,s) \cap \Big\{(c,s) \in \bar{B}\Big((0,0),1\Big)^n \mid \sum_{j=1}^n j a_j s_j = 0\Big\} \neq \emptyset.$$

Proof. For $j = 1, \cdots, n$, let $(\bar{c}_j(t), \bar{s}_j(t)) = (c_j(\sqrt{2t}), s_j(\sqrt{2t}))$ and $(W_t)_{t \geqslant 0}$ be the Brownian motion defined by $W_t = \int_0^t (2s)^{1/4} dB_{\sqrt{2s}}$.

Hence $\Big(\bar{c}(t), \bar{s}(t)\Big) \overset{notation}{=} \Big(\bar{c}_1(t), \bar{s}_1(t), \cdots, \bar{c}_n(t), \bar{s}_n(t)\Big)$ solves the SDE

$$\begin{cases} d\bar{c}_j(t) = -j\bar{s}_j(t)[(2t)^{-1/4}dW_t + (\sum_{k=1}^n k a_k \bar{s}_k(t))dt] - \frac{j^2}{2}(2t)^{-1/2}\bar{c}_j(t)dt + \frac{1}{2t}(1 - \bar{c}_j(t))dt \\ d\bar{s}_j(t) = j\bar{c}_j(t)[(2t)^{-1/4}dW_t + (\sum_{k=1}^n k a_k \bar{s}_k(t))dt] - \frac{j^2}{2}(2t)^{-1/2}\bar{s}_j(t)dt - \frac{\bar{s}_j(t)}{2t}dt \end{cases}, \tag{6.14}$$

for $j = 1, \cdots, n$. Since it lives in a compact state space, it follows from Remark 4.1 (see chapter 4) that $(\bar{c}_1(t), \bar{s}_1(t), \cdots, \bar{c}_n(t), \bar{s}_n(t))$ is an asymptotic pseudotrajectory for the flow Φ. Thus $L(c,s)$ is internally chain transitive by Theorem 4.1. Hence, the result follows from Theorem 6.2. □

[4] Here $\bar{B}\Big((0,0),1\Big)$ stands for the closed unit ball in $\mathbb{R}^2$ equipped with the Euclidean norm.

Remark 6.3.

1. *Following the lines from the proof of Proposition 4.1 in [17], one can even prove that for all* $T > 0$

$$\lim_{t\to\infty} \sup_{0\leqslant u\leqslant T} \|\Big(\bar{c}(t+u),\bar{s}(t+u)\Big) - \varphi_u^t\Big(\big(\bar{c}(t),\bar{s}(t)\big)\Big)\| = 0 a.s,$$

where $\varphi_\cdot^{t_0}(c,s)$ *is the solution of the inhomogeneous ODE*

$$\begin{cases} \dot{\mathfrak{c}}_j(t) = j(\sum_{k=1}^n k a_k \mathfrak{s}_k(t))(-\mathfrak{s}_j(t)) + \frac{1}{2t}(1-\mathfrak{c}_j), & j=1,\ldots,n \\ \dot{\mathfrak{s}}_j(t) = j(\sum_{k=1}^n k a_k \mathfrak{s}_k(t))\mathfrak{c}_j(t) - \frac{1}{2t}\mathfrak{s}_j & j=1,\ldots,n. \end{cases} \tag{6.15}$$

with initial time $t_0 > 0$ *and starting point* (c,s).

2. *For all initial condition* (c,s,t_0), *we have* $\lim_{t\to\infty} T(\varphi_t^{t_0}(c,s)) \leqslant \sum_{j=1}^n a_j$, *where* T *is the linear application*

$$T(c,s) \overset{notation}{=} T(c_1,s_1,\cdots,c_n,s_n) = \sum_{j=1}^n a_j c_j.$$

Indeed, a simple derivation with respect to t *gives*

$$\frac{d}{dt}\Big(\sum_{j=1}^n a_j\mathfrak{c}_j\Big) = -\Big(\sum_{k=1}^n k a_k \mathfrak{s}_k(t)\Big)^2 + \frac{1}{2t}\Big(\sum_{j=1}^n a_j - \big(\sum_{j=1}^n a_j\mathfrak{c}_j\big)\Big).$$

Note that $\Big\{(c,s)\in\bar{B}\Big((0,0),1\Big)^n \mid \sum_{j=1}^n a_j c_j \leqslant \sum_{j=1}^n a_j\Big\}$ *is positively invariant for* $\varphi_\cdot^\cdot(.,.)$.

Our first questions are

Question 6.1.
- *Do we have* $L(c,s) \subset \Big\{(c,s)\in\bar{B}\Big((0,0),1\Big)^n \mid \sum_{j=1}^n a_j c_j \leqslant \sum_{j=1}^n a_j\Big\}$ *almost-surely?*
- *Is* $L(c,s)$ *reduced to one point?*

I believe that the answers to these questions are positive.

Assuming that $L(c,s)$ is reduced to one point, the next step consists on looking for candidates. In order to ensure that the process might converge to these points, a condition is that they have to lay in the set of point such that $F_1(c,s)$ and $F_2(c,s)$ are parallel[5].

Proposition 6.3. *$F_1(c,s)$ is parallel to $F_2(c,s)$ if and only if there exists $z\in\mathbb{R}\cup\{\infty\}$ such that $c_j = \frac{1}{1+j^2z^2}$ and $s_j = \frac{jz}{1+j^2z^2}$ for all j.*

[5] It is here that we use the advantage of the Stratonovich integral since the Brownian term acts along the vector field F_1

Proof. The "if" part is obvious. Let us prove the "only if" part.

Assume that $F_1(c,s)$ is parallel to $F_2(c,s)$. Then for all $j=1,\cdots,n$, $c_j = c_j^2 + s_j^2$. Hence there exists $z_j \in \mathbb{R}\cup\{\infty\}$ such that

$$c_j = \frac{1}{1+z_j^2} \text{ and } s_j = \frac{z_j}{1+z_j^2}.$$

On the other hand, one also obtains

$$c_1 s_j = j c_j s_1.$$

This last equality implies $z_j = j z_1$. □

Remark 6.4. *If $c_j = \frac{1}{1+j^2z^2}$ and $s_j = \frac{jz}{1+j^2z^2}$ for all $j=1,\cdots,n$, then $F_2(c,s) = -zF_1(c,s)$.*

By Theorem 6.2 and Proposition 6.3, the set of limit candidates is

$$\begin{aligned} LC \;:=\; & \Big\{\Big(\frac{1}{1+z^2},\frac{z}{1+z^2},\cdots,\frac{1}{1+n^2z^2},\frac{nz}{1+n^2z^2}\Big) : \sum_{k=1}^n \frac{k^2a_kz}{1+k^2z^2} = 0 \text{ and } z\in\mathbb{R}\cup\{\infty\}\Big)\Big\} \\ & \bigcap\Big\{\Big(\frac{1}{1+z^2},\frac{z}{1+z^2},\cdots,\frac{1}{1+n^2z^2},\frac{nz}{1+n^2z^2}\Big) : \sum_{k=1}^n \frac{k^2a_k}{1+k^2z^2} \leqslant 0 \text{ and } z\in\mathbb{R}\cup\{\infty\}\Big)\Big\}\end{aligned}$$

We point out that *this set is disjoint and finite.*

Example 6.1. *If $n=2$, then LC is given by*

1. $LC = \Big\{\Big(0,0,\cdots,0,0\Big),\Big(1,0,\cdots,1,0\Big)\Big\}$ *if* (a_1+a_2), $(a_1+4a_2)<0$.
2. $LC = \Big\{\Big(0,0,\cdots,0,0\Big)\Big\}$ *if* (a_1+a_2), $(a_1+4a_2)>0$
3. $LC = \Big\{\Big(0,0,\cdots,0,0\Big),\Big(1,0,\cdots,1,0\Big),\Big(\frac{1}{1+z^2},\frac{z}{1+z^2},\cdots,\frac{1}{1+n^2z^2},\frac{nz}{1+n^2z^2}\Big)\Big\}$ *with* $z=\pm\sqrt{|\frac{a_1+4a_2}{a_1+a_2}|}$ *if* $(a_1+a_2)\times(a_1+4a_2)<0$ *and* $(a_1+4a_2)<0$.
4. $LC = \Big\{\Big(0,0,\cdots,0,0\Big),\ \Big(\frac{1}{1+z^2},\frac{z}{1+z^2},\cdots,\frac{1}{1+n^2z^2},\frac{nz}{1+n^2z^2}\Big)\Big\}$ *with* $z=\pm\sqrt{|\frac{a_1+4a_2}{a_1+a_2}|}$ *if* $(a_1+a_2)\times(a_1+4a_2)<0$ *and* $(a_1+4a_2)>0$.

Since the set of points such that $F_1(c,s)$ is parallel to $F_2(c,s)$ is a curve, it would be great if we could "read" on it the long term behaviour of $(c_1(t),s_1(t),\cdots,c_n(t),s_n(t))$. Assume that there exists $Z_t\in\mathbb{R}\cup\{\infty\}$ such that $c_j(t) = \frac{1}{1+j^2Z_t^2}$ and $s_j(t) = \frac{jZ_t}{1+j^2Z_t^2}$. Hence, we have from (6.12),

$$d\begin{pmatrix} c_1(t)\\ s_1(t)\\ \vdots \\ c_n(t)\\ s_n(t)\end{pmatrix} = F_1(c_t,s_t)\circ[dB_t + \Big(t\sum_{k=1}^n \frac{k^2a_k}{1+k^2Z_t^2} - \frac{1}{t}\Big)Z_tdt]\ j=1,\ldots,n.$$

Question 6.2. *Is the behaviour of $(c_1(t), s_1(t), \cdots, c_n(t), s_n(t))$ partially described by the SDE*

$$dZ_t = dB_t + \big(t\sum_{k=1}^{n} \frac{k^2 a_k}{1+k^2 Z_t^2} - \frac{1}{t}\big) Z_t dt \quad ? \tag{6.16}$$

Set $g(z) = \sum_{k=1}^{n} \frac{k^2 a_k z}{1+k^2 z^2}$. Then g is a smooth and odd Lipschitz function such that $g'(0) = \sum_{j=1}^{n} j^2 a_j$ and $\lim_{Z\to\infty} z g(z) = \sum_{j=1}^{n} a_j$.

Proposition 6.4. *If $\sum_{j=1}^{n} a_j < 0$, then Z_t converges almost-surely to the set*

$$Lim = \Big\{z \in \mathbb{R} \ s.t \ g(z) = 0 \ and \ g'(z) < 0\Big\}.$$

In particular, $\Big(\frac{1}{1+j^2 Z_t^2}, \frac{j Z_t}{1+j^2 Z_t^2}\Big)$ does not converges to $\Big(0,0\Big)$.

Proof. We proceed in three steps.
Step 1 (auxiliary process):
Let $(Y_t)_{t\geqslant 0}$ be the solution of the SDE

$$dY_t = dB_t + t g(Y_t) dt, \tag{6.17}$$

with initial condition $Y_0 = Z_0$. By the same kind argument as in the proof of Lemma 4.3 in Chapter 4, it can be shown that $(Y_{\sqrt{2t}})_{t\geqslant 0}$ is an asymptotic pseudotrajectory for the flow induced by the ODE

$$\dot{z} = g(z). \tag{6.18}$$

Then, notice that *Lim* is an attractor for the ODE (6.18) whose basin of attraction is $\mathbb{R}\backslash\{z \in \mathbb{R} \text{ s.t } g(z) = 0 \text{ and } g'(z) > 0\}$. Set

$$z^* = \sup\{z > 0 \text{ s.t } g(z) = 0\},$$

so that $A = [-z^*, z^*]$ is a global attractor. For $(y_0, t_0) \in \mathbb{R} \times (0, \infty)$, let

$$\tau(y_0, t_0) = \inf\{t > 0 \text{ s.t } Y_{t+t_0}^{y_0,t_0} \in A\}.$$

Let $|z_0| > z^*$ and $t_0 \geqslant \frac{1}{g(z^*)z^*}$.

Let $\tilde{B}_t = \int_0^t \Big(\mathbb{1}_{\{Y_s>0\}} - \mathbb{1}_{\{Y_s\leqslant 0\}}\Big) dB_s$. Then, by Lévy's characterization, $\Big(\tilde{B}_t\Big)_{t\geqslant 0}$ is a Brownian motion.

By Itô's formula, we have

$$\begin{aligned} dY_t^2 &= 2Y_t dB_t + 2t g(Y_t) Y_t + dt \\ &= 2\sqrt{Y_t^2} d\tilde{B}_t + 2t \sum_{k=1}^{n} k^2 a_k \frac{Y_t^2}{1+k^2 Y_t^2} + dt \end{aligned}$$

Hence, by Ikeda-Watanabe's comparison result,

$$\mathbb{P}\Big(Y_{t+t_0}^2 \leqslant K_t \ \forall t \in [0, \tau(z_0, t_0)]\Big) = 1,$$

where $\Big(K_t\Big)_{t\geqslant 0}$ solves

$$dK_t = 2\sqrt{K_t}d\tilde{B}_t + dt. \tag{6.19}$$

Since $(K_t)_{t\geqslant 0} \overset{\mathcal{L}}{=} (W_t^2)_{t\geqslant 0}$, where W is a real-valued Brownian motion, we obtain that $\tau(z_0, t_0)$ is finite almost-surely. Hence, by Proposition 7.4.(iii) and Theorem 6.10 from [11]. Y_t converges almost-surely to Lim.

Step 2 (boundedness of the process):

Let $\bar{B}_t = \int_0^t \Big(\mathbb{1}_{\{X_s>0\}} - \mathbb{1}_{\{X_s\leqslant 0\}}\Big)dB_s$, so that $(\bar{B}_t)_{t\geqslant 0}$ is a real valued Brownian motion by Lévy's characterization. Let $(\mathfrak{U}_t)_{t\geqslant 0}$ be the solution of the SDE

$$d\mathfrak{U}_t = 2\sqrt{\mathfrak{U}_t}d\bar{B}_t + 2t\sum_{k=1}^{n} k^2 a_k \frac{\mathfrak{U}_t}{1+k^2\mathfrak{U}_t} + dt,$$

with initial condition $\mathfrak{U}_0 = Z_0^2$. Since, by Itô's formula,

$$dZ_t^2 = 2\sqrt{Z_t^2}d\bar{B}_t + 2t\sum_{k=1}^{n} k^2 a_k \frac{Z_t^2}{1+k^2 Z_t^2} - \frac{2Z_t^2}{t} + dt,$$

then by Ikeda-Watanabe's comparison result,

$$\mathbb{P}\Big(Z_t^2 \leqslant \mathfrak{U}_t \ \forall t \geqslant 0\Big) = 1.$$

Because $(\mathfrak{U}_t)_{t\geqslant 0} \overset{\mathcal{L}}{=} (Y_t^2)_{t\geqslant 0}$, then $\mathfrak{U}_t$ converges almost-surely to a finite quantity by the first step. Hence $(Z_t)_{t\geqslant 0}$ is almost-surely bounded.

Step 3 (conclusion):

Since $(Z_t)_{t\geqslant 0}$ is almost-surely bounded, then it can be shown that $(Z_{\sqrt{2t}})_{t\geqslant 0}$ is an asymptotic pseudotrajectory for the flow induced by (6.18). Thus, by Theorem 4.2, Z_t converges almost-surely to Lim. □

A partial complementary result is

Proposition 6.5. *Assume that $g(z) > 0$ for all $z > 0$. Then,*

$$\mathbb{P}\Big(\lim_{t\to\infty} |Z_t|\Big) = \infty) = 1.$$

Proof. Let $t_0 > 0$ sufficiently large such that

$$z(t_0) = \sup\{z > 0 \text{ s.t } \sum_{k=1}^{n} \frac{k^2 a_k}{1+k^2 z^2} = \frac{1}{t_0}\}$$

exists. We emphasize that $\lim_{t_0\to\infty} = \infty$ and $\big(f(z) - \frac{z}{t_0}\big)\big(z - z(t_0)\big) > 0$ for all positive $z \neq z(t_0)$.

Let $\Big(Y_t^{t_0}\Big)_{t\geqslant t_0}$ be the solution of

$$dY_t^{t_0} = dB_t + tg(Y_t^{t_0})dt - \frac{Y_t^{t_0}}{t_0}dt \tag{6.20}$$

with initial condition $Y_{t_0}^{t_0} = Z_{t_0}$. Then, by a similar argumentation as for Proposition 6.4's proof, we can define a process $(U_t)_{t\geqslant t_0}$ such that $\mathbb{P}\Big(U_t \leqslant Z_t^2,\ \forall t \geqslant t_0\Big) = 1$ and $\Big(U_t\Big)_{t\geqslant t_0} \stackrel{\mathcal{L}}{=} \Big((Y_t^{t_0})^2\Big)_{t\geqslant t_0}$.

Moreover, following the lines of the first step of Proposition 6.4, we obtain that $|Y_t^{t_0}|$ converges almost surely to $z(t_0)$. Hence, for all t_0 sufficiently large,

$$\liminf_{t\to\infty} |Z_t| \geqslant z(t_0) \text{ a.s.}$$

Letting t_0 go to ∞ provides the result. □

It is now time to state our conjectures. In view of Proposition 6.3, the first one is

Conjecture 6.2. *On the event* $\Big\{L(c,s) = \big\{(0,0,\cdots,0,0)\big\}\Big\}$, $\frac{1}{t}\int_0^t \delta_{e^{iX_s}} ds$ *converges to the uniform law on the circle.*

This statement is the converse one from the second observation that we did at the beginning of the subsection. In view of Proposition 6.3, Proposition 6.4 and Proposition 6.5, the second conjecture is

Conjecture 6.3.
1. *If* $\sum_{j=1}^n a_j < 0$, *then with probability one* $(c_j(t), s_j(t))$ *does not converge to* $(0,0)$ *for all* $j = 1,\cdots,n$.
2. *If* $g(z) > 0$ *for all* $z > 0$, *then with probability one* $(c_j(t), s_j(t))$ *converges to* $(0,0)$ *for all* $j = 1,\cdots,n$.

Since the noise term acts only along the vector field F_1, an another reason why I believe that the first point of the conjecture is true comes from the following result

Proposition 6.6. *Consider the non-homogeneous ordinary differential equation*

$$d\begin{pmatrix} \mathfrak{c}_1(t)\\ \mathfrak{s}_1(t)\\ \mathfrak{c}_2(t)\\ \mathfrak{s}_2(t)\\ \vdots\\ \mathfrak{c}_n(t)\\ \mathfrak{s}_n(t)\end{pmatrix} = F_1(\mathfrak{c}_t,\mathfrak{s}_t)t(\sum_{k=1}^n k a_k \mathfrak{s}_k(t))F_1(\mathfrak{c}_t,\mathfrak{s}_t) + \frac{1}{t}F_2(\mathfrak{c}_t,\mathfrak{s}_t). \tag{6.21}$$

Then for any initial condition, we have $\limsup_{t\to\infty}\sum_{j=1}^n a_j \mathfrak{c}_j(t) \leqslant \sum_{j=1}^n a_j$.

Moreover, denoting by φ *its flow, we have for any* $T>0$,

$$\lim_{t\to\infty}\sup_{s\in[t,t+T]} \|\big(c_1(\sqrt{2s}), s_1(\sqrt{2s}),\cdots,c_n(\sqrt{2s}), s_n(\sqrt{2s})\big) - \varphi\big(\mathfrak{c}(\sqrt{2t}),\mathfrak{s}(\sqrt{2t})\big)\| = 0.$$

Proof. The first part easily follows from differentiating $t\mapsto \sum_{j=1}^n a_j\mathfrak{c}_j(t)$, whereas the second follows by first doing the change of $t\mapsto\sqrt{2t}$ and then by copying the lines of the proof of Proposition 4.1 from [17]. □

In view of Theorem 6.2, our second and third conjecture is

Conjecture 6.4. *If $\sum_{j=1}^n j^2 a_j < 0$, then for all $j = 1, \cdots, n$, $(c_j(t), s_j(t))$ converges to $(1, 0)$ with positive probability. If in addition LC reduces to $\Big\{\Big(0, 0, \cdots, 0, 0\Big), \Big(1, 0, \cdots, 1, 0\Big)\Big\}$ (in particular $\sum_{j=1}^n a_j < 0$), then this convergence holds almost surely.*

If, as we believe, this conjecture is true, then we can prove the convergence of X_t.

Proposition 6.7. *X_t converges almost surely on the event*

$$\Big\{L(c, s) = \big\{(1, 0, \cdots, 1, 0)\big\}\Big\}.$$

Proof. We assume that the event $\Big\{L(c, s) = \big\{(1, 0, \cdots, 1, 0)\big\}\Big\}$ holds with positive probability. So, from Theorem 6.2[6], $\sum_{j=1}^n j^2 a_j < 0$. From now on, we work on that event.

From equation (6.12), we obtain, in the Itô's formulation,

$$\begin{aligned} d\Big(\sum_{j=1}^n j a_j s_j(t)\Big) &= \sum_{j=1}^n j^2 a_j c_j(t) dB_t - \frac{1}{2}\sum_{j=1}^n j^3 a_j s_j(t) dt - \frac{1}{t}\Big(\sum_{j=1}^n j a_j s_j(t)\Big) dt \\ &\quad + t\Big(\sum_{j=1}^n j^2 a_j c_j(t)\Big)\Big(\sum_{j=1}^n j a_j s_j(t)\Big) dt \end{aligned} \tag{6.22}$$

By the assumption, it holds on the event $\Big\{L(c, s) = \big\{(1, 0, \cdots, 1, 0)\big\}\Big\}$ that

$$\lim_{t\to\infty}\Big(\sum_{j=1}^n j^2 a_j c_j(t)\Big) = \sum_{j=1}^n j^2 a_j < 0 \text{ and } \lim_{t\to\infty} \frac{1}{2}\sum_{j=1}^n j^3 a_j s_j(t) + \frac{1}{t}\Big(\sum_{j=1}^n j a_j s_j(t)\Big) = 0.$$

Therefore, following Theorem 3.2's proof of the rate of convergence of [53], we obtain

$$\sum_{j=1}^n j a_j s_j(t) = \mathcal{O}\Big(\sqrt{\frac{\ln(t)}{t}}\Big). \tag{6.23}$$

For $j = 1, \cdots, n$, let $(R_j(t), \theta_j(t)$ be the polar coordinates of $\Big(U_j(t), V_j(t)\Big)$, where $t \mapsto \theta_j(t) \in \mathbb{R}$ is defined to be continuous.

So, from (6.6) and (6.7), we have

$$\Big(C_j(t), S_j(t)\Big) = \Big(R_j(t)\cos\big(jX_t - \theta_j(t)\big), R_j(t)\sin\big(jX_t - \theta_j(t)\big)\Big).$$

Since $t \mapsto U_j(t)$, $t \mapsto V_j(t)$ and $t \mapsto R_j(t)$ are differentiable in t, so does $t \mapsto \theta_j(t)$. Deriving $\frac{U_j(t)}{R_j(t)}$ and $\frac{V_j(t)}{R_j(t)}$ with respect to t, we obtain

$$\frac{d}{dt}\theta_j(t) = \frac{\sin\big(jX_t - \theta_j(t)\big)}{R_j(t)} = \frac{t}{R_j^2(t)} s_j(t), \tag{6.24}$$

[6] and the assumption $\sum_{j=1}^n j^2 a_j \neq 0$

By assumption, $\lim_{t\to\infty} \frac{R_j(t)}{t} = 1$. So, together with (6.23), it implies the convergence of $\sum_{j=1}^n ja_j\theta_j(t)$.

By assumption, $\Big(cos\big(jX_t - \theta_j(t)\big), sin\big(jX_t - \theta_j(t)\big)\Big)$ converges to $(1,0)$. Thus $jX_t - \theta_j(t)$ converges almost-surely to an element of $2\pi\mathbb{Z}$ by continuity of $jX_t - \theta_j(t)$.

Therefore,

$$(\sum_{j=1}^{n} j^2 a_j)X_t = \sum_{j=1} ja_j\theta_j(t) + \sum_{j=1} ja_j(jX_t - \theta_j(t))$$

converges almost-surely on the event $\Big\{L(c,s) = \big\{(1,0,\cdots,1,0)\big\}\Big\}$. □

6.2 Interacting with the past of the other

The existence of this section is due to a question of P. Tarrès about the long term behaviour of two particles living on the unit circle and interacting with the past of the other. Unfortunately, we could not obtain full satisfactory results in that case, but we could have some in the linear one.

This section is divided in two subsections: the linear case and the periodic case.

6.2.1 The linear case

In this subsection, we investigate the long term behaviour of the solution of the reinforced SDE

$$\begin{cases} dX_t = \frac{1}{\sqrt{2}}dB_1(t) - a\int_0^t X_t - Y_s ds dt \\ dY_t = \frac{1}{\sqrt{2}}dB_2(t) - b\int_0^t Y_t - X_s ds dt \end{cases} \tag{6.25}$$

with initial condition $(X_0, Y_0) = (x, y)$. Here (B_1, B_2) is a two dimensional Brownian motion and $a, b \in \{-1, 1\}$. The interpretation of the constant is the following. If $a = 1$ (resp. $b = 1$), then X_t (resp. Y_t) is attracted by the motion $(Y_u)_{u\in[0,t]}$ (resp. $(X_u)_{u\in[0,t]}$). If $a = -1$, then X_t wants to get away from $(Y_u)_{u\in[0,t]}$.

Before working with the possible situations, let us fix some notations. Set

$$Z_t = X_t - Y_t,\ \tilde{Z}_t = X_t + Y_t$$

and

$$W_1(t) = \frac{1}{\sqrt{2}}dB_1(t) + \frac{1}{\sqrt{2}}dB_2(t),\ W_2(t) = \frac{1}{\sqrt{2}}dB_1(t) - \frac{1}{\sqrt{2}}dB_2(t).$$

In addition, let $U_t = \int_0^t Z_s ds$.

Notice that $t \mapsto (W_1, W_2)$ is a two dimensional Brownian motion.

Case a=b=1

If both are attracted to each other, then one expects to have convergence of X_t and Y_t to the same limit. This is the purpose of the following Theorem.

Theorem 6.3. *Let $(X_t, Y_t)_t$ be the solution of (6.25) with $a = b = 1$. Then X_t and Y_t converge almost-surely to the same limit.*

Proof. Since $a = b$, $(\tilde{Z}_t, Z_t)$ solves

$$\begin{cases} d\tilde{Z}_t = dW_1(t) - \int_0^t (\tilde{Z}_t - \tilde{Z}_s) ds dt \\ dZ_t = dW_2(t) - \int_0^t (Z_t + Z_s) ds dt \end{cases}. \tag{6.26}$$

From Theorem 1 in [35], $\tilde{Z}_t$ converges almost surely to some random variable $\tilde{Z}_\infty$ which is finite almost surely. It remains to show that Z_t converges almost surely to 0.

From the definition of U_t and (6.26), we have

$$Z_t^2 = (x-y)^2 + 2\int_0^t Z_s dW_2(s) - 2\int_0^t s Z_s^2 ds + t - U_t^2. \tag{6.27}$$

Because

$$\int_0^t s Z_s^2 ds \geqslant \int_0^t Z_s^2 ds$$

for all t sufficiently large, we obtain

$$t \geqslant U_t^2.$$

In particular, $V_t = \frac{U_t}{t}$ converges to 0 almost surely. From (6.26), we obtain

$$Z_t = (x-y)e^{-\frac{t^2}{2}} + e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}} dW_2(s) - e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}} V_s ds. \tag{6.28}$$

Since $e^{-t^2} \int_0^t e^{s^2} ds$ converges to 0 as t goes to infinity, then by the Dubins-Schwarz Theorem and the Law of Iterated Logarithm, $e^{-\frac{t^2}{2}} \int_0^t e^{\frac{s^2}{2}} dW_2(s)$ converges almost-surely to 0. Therefore, so does Z_t and thus X_t and Y_t converge almost surely to $\tilde{Z}_\infty/2$. □

The important fact in the proof is that $a = b$. So a natural question is

Question 6.3. *Does the statement of Theorem 6.3 remain true if $a \neq b$?*

Case a=b=-1

If both are repelled to each other, then one expects that X_t and Y_t get away from each other as far as possible. We have the following Theorem.

Theorem 6.4. *Let $(X_t, Y_t)_t$ be the solution of (6.25) with $a = b = -1$. Then $|X_t - Y_t|$ converges almost surely to ∞.*

Proof. Since $a = b$, we have from the definition of U_t and Z_t,

$$Z_t^2 = (x-y)^2 + 2\int_0^t Z_s dW_2(s) + 2\int_0^t s Z_s^2 ds + t + U_t^2. \tag{6.29}$$

Because

$$\int_0^t sZ_s^2 ds \geqslant \int_0^t Z_s^2 ds$$

for all t sufficiently large, we deduce that

$$\liminf_{t\to\infty} \frac{Z_t^2}{t} \geqslant 1.$$

□

Remark 6.5. *I think that either* $\lim_{t\to\infty} X_t = \infty$ *and* $\lim_{t\to\infty} Y_t = -\infty$ *or* $\lim_{t\to\infty} X_t = -\infty$ *and* $\lim_{t\to\infty} Y_t = \infty$.

Indeed, proceeding like in [35], we have a semi-explicit expression for X_t *and* Y_t*; namely*

$$X_t = x + \frac{1}{\sqrt{2}} \int_0^t \Big(1 + se^{-\frac{s^2}{2}} \int_s^t e^{\frac{u^2}{2}} du\Big) dB_1(s) + G(t)$$

and

$$Y_t = y + \frac{1}{\sqrt{2}} \int_0^t \Big(1 + se^{-\frac{s^2}{2}} \int_s^t e^{\frac{u^2}{2}} du\Big) dB_2(s) - G(t)$$

with

$$G(t) = \int_0^t U_s \Big(1 + se^{-\frac{s^2}{2}} \int_s^t e^{\frac{u^2}{2}} du\Big) ds.$$

As in the attracting case, the fact that $a = b$ is very important to make the proof works. Hence the same question holds.

Question 6.4. *Does the statement of Theorem 6.4 remain true if* $a \neq b$*?*

Case a=1 and b=-1

This case is the most difficult to treat since X_t is attracted by $(Y_u)_{u\in[0,t]}$, whereas Y_t is repelled from $(X_u)_{u\in[0,t]}$. it would not be surprising to have that Y_t goes either to $+\infty$ or to $-\infty$ and X_t follows Y_t.

Question 6.5. *Is this the true behaviour of* (X_t, Y_t)*?*

Setting $Z_t = X_t - Y_t$ and $\tilde{Z}_t = X_t + Y_t$, a more interesting question might be

Question 6.6. *What is the behaviour of* $(Z_t, \tilde{Z}_t)$*?*

In the previous subsections, we had to analyse the behaviour of Z_t and/or $\tilde{Z}_t$ in order to described the behaviour of (X_t, Y_t). Here, $(Z_t, \tilde{Z}_t)$ solves the reinforced SDE

$$\begin{cases} dZ_t = dW_2(t) - t\tilde{Z}_t dt + \int_0^t \tilde{Z}_s ds dt \\ d\tilde{Z}_t = dW_1(t) - tZ_t dt - \int_0^t Z_s ds dt \end{cases}. \tag{6.30}$$

Letting $U_t = \int_0^t \tilde{Z}_s ds$ and $V_t = \int_0^t Z_s ds$, we obtain from Itô's formula

$$Z_t^2 = \tilde{Z}_t^2 + U_t V_t + 2M_t$$

and

$$2Z_t\tilde{Z}_t = U_t^2 - V_t^2 + 2N_t - 2\int_0^t s(Z_s^2 + \tilde{Z}_s^2) ds,$$

where $M_t = \int_0^t Z_s dW_2(s) + \tilde{Z}_s dW_1(s)$ and $N_t = \int_0^t \tilde{Z}_s dW_2(s) + Z_s dW_1(s)$.

6.2.2 The periodic case

This case was the initial case of interest. However, the study is more complicated as for the linear case as explained below. Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Let $X_t = (X_1(t), X_2(t)) \in \mathbb{T}^2$ be the solution of the SDE

$$dX_t = dB_t - \nabla F_t(X)dt, \tag{6.31}$$

where (B_t) is a two-dimensional Brownian motion and F_t is the two variables function defined by

$$F_t(x_1, x_2) = a \int_0^t \cos(x_1 - X_2(s))ds + b \int_0^t \cos(x_2 - X_1(s))ds. \tag{6.32}$$

As in the linear case, if $a > 0$ (resp. $b > 0$), then X_t (resp. Y_t) wants to maximize its distance with the past positions of Y (resp. X) until time t. So, if $ab > 0$, one is expecting to observe a localization result, either at the same place ($a < 0$) or at two antipodals positions ($a > 0$).

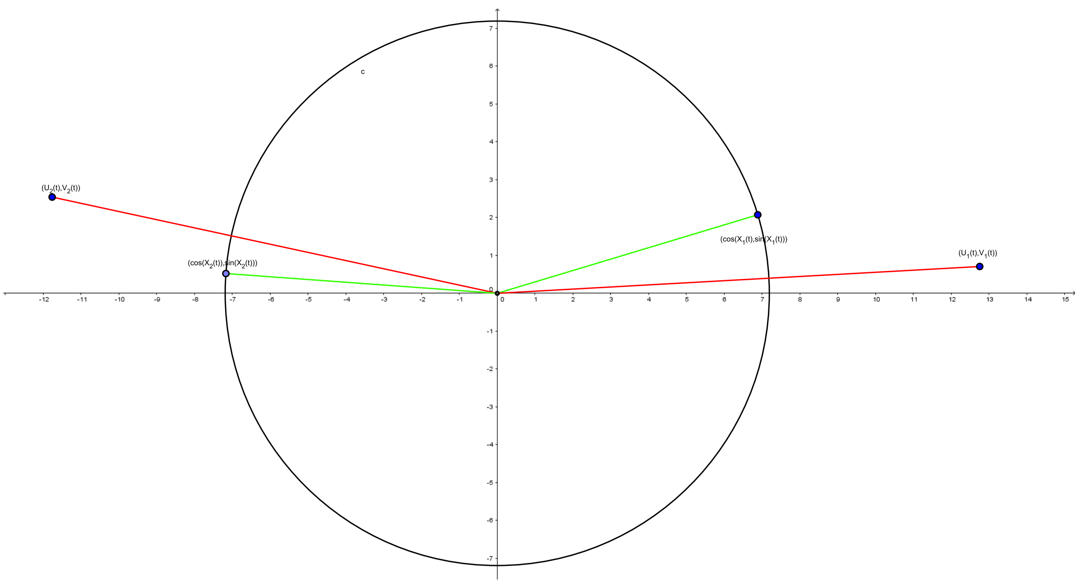

Figure 6.1: diagram of how the particles should behave when they are both repelled by the past of the other one.

Following the same idea as in the previous chapters, set, for $j \in \{1, 2\}$, $U_j(t) = \int_0^t \cos(X_j(s))ds$

and $V_j(t) = \int_0^t \sin(X_j(s))ds$. Hence, (6.31) is translated to the SDE on $\mathbb{T}^2 \times \mathbb{R}^4$

$$\begin{cases} dX_1(t) = dB_1(t) + a\Big(U_2(t)\sin(X_1(t)) - V_2(t)\cos(X_1(t))\Big)dt \\ dX_2(t) = dB_2(t) + b(\Big(U_1(t)\sin(X_2(t)) - V_1(t)\cos(X_2(t))\Big)dt \\ dU_1(t) = \cos(X_1(t))dt \\ dV_1(t) = \sin(X_1(t))dt \\ dU_2(t) = \cos(X_2(t))dt \\ dV_2(t) = \sin(X_2(t))dt \end{cases} \tag{6.33}$$

Set $Z_t = X_1(t) - X_2(t)$, so that

$$\cos(Z_t) = \Big\langle \Big(\cos(X_1(t)), \sin(X_1(t))\Big), \Big(\cos(X_2(t)), \sin(X_2(t))\Big)\Big\rangle.$$

In chapter 3, we introduced (in the framework of the circle) new variables that corresponded to the scalar product between the accumulation variables $(\int_0^t \cos(X_s)ds, \int_0^t \sin(X_s)ds)$ with $(\cos(X_t), \sin(X_t))$ and $(\sin(X_t), -\cos(X_t))$ in order to study the long term behaviour of X_t. Here, we follow the same idea, but because the interaction is with the past of the other, the scalar product is taken with respect of the accumulation variable from the other particle; namely we define

$$C_1(t) = U_2(t)\cos(X_1(t)) + V_2(t)\sin(X_1(t)) \text{ and } S_1(t) = U_2(t)\sin(X_1(t)) - V_2(t)\cos(X_1(t)),$$

as well as

$$C_2(t) = U_1(t)\cos(X_2(t)) + V_1(t)\sin(X_2(t)) \text{ and } S_2(t) = U_1(t)\sin(X_2(t)) - V_1(t)\cos(X_2(t)).$$

Applying Itô's formula to $(C_1(t), S_1(t))$, we obtain

$$d\begin{pmatrix} C_1(t) \\ S_1(t) \end{pmatrix} = \begin{pmatrix} -S_1(t) \\ C_1(t) \end{pmatrix} \circ dB_1(t) + aS_1(t)\begin{pmatrix} -S_1(t) \\ C_1(t) \end{pmatrix} dt + \begin{pmatrix} \cos(Z_t) \\ \sin(Z_t) \end{pmatrix} dt \tag{6.34}$$

From this point, it is easy to see that if one can prove that $\cos(Z_t)$ converges to 1 when $a, b < 0$ or to -1 when $a, b > 0$, then the techniques used in [53] (see also Chapter 4) imply the almost-sure convergence of $X_1(t)$ and $X_2(t)$.

Conjecture 6.5. *If $a, b < 0$, $\cos(Z_t)$ converges almost surely to 1 and if $a, b > 0$, $\cos(Z_t)$ converges almost surely to -1.*

Similarly to the linear case, this conjecture seems to be the key in the resolution of this problem.

However, following the proof of Theorem 6.2, we are still able to exhibit a relation if $a = b$.

Proposition 6.8. *Assume that $a = b$ and define*

$$\begin{aligned} M_t &= -a\int_0^t \Big(U_2(u)\sin(X_1(u)) - V_2(u)\cos(X_1(u))\Big)dB_1(u) \\ &\quad -a\int_0^t \Big(U_1(u)\sin(X_2(u)) - V_1(u)\cos(X_2(u))\Big)dB_2(u) \end{aligned}$$

and

$$H_t = U_2(t)\cos(X_1(t)) + V_2(t)\sin(X_1(t)) + U_1(t)\cos(X_2(t)) + V_1(t)\sin(X_2(t)).$$

Then

$$aH_t + \frac{a}{2}\Big(U_1(t)U_2(t) + V_1(t)V_2(t)\Big) = 2a\int_0^t \cos(Z_s)ds + M_t - \langle M\rangle_t. \tag{6.35}$$

This proposition follows from Itô's Formula and equation (6.33). In chapter 4, this "equivalent" equation was sufficient to give all the informations that we needed. However, it might not be the case here as seen in section 6.1. So a natural question is

Question 6.7. *Does Equation (6.35) provide sufficiently informations in order to study the behaviour of* $\cos(Z_t)$*?*

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