Symplectic embeddings of 4-dimensional ellipsoids into polydiscs

Thèse présentée à la Faculté des Sciences
Institut de Mathématiques
Université de Neuchâtel

Pour l'obtention du grade de docteur ès Sciences

par

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Soutenue le 10 décembre 2012

Université de Neuchâtel

2012

UNIVERSITÉ DE
NEUCHÂTEL

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Rue Emile-Argand 11
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Neuchâtel, le 13 décembre 2012

Abstract

Recently, McDuff and Schlenk determined in [19] the function $c_{EB}(a)$, whose value at a is the infimum of the size of a 4-ball, into which the ellipsoid $E(1,a)$ symplectically embeds (here, $a > 1$ is the ratio of the area of the large axis to that of the smaller axis of the ellipsoid). This work is focused on the study of embeddings of 4-dimensional ellipsoids $E(1,a)$ into four-dimensional polydiscs $P(\mu,\mu) = D^2(\mu) \times D^2(\mu)$, where $D^2(\mu)$ denotes the disc in $\mathbb{R}^2$ of area μ, for $a \in [1, \sigma^2]$, where $\sigma^2 = 3 + 2\sqrt{2} \approx 5.83$ is the square of the silver ratio $\sigma := 1 + \sqrt{2}$. The embedding capacity function given by $c_{EC}(a) := \inf \left\{ \mu \;\middle|\; E(1,a) \overset{\text{s}}{\hookrightarrow} P(\mu,\mu) \right\}$ is defined for $a \in [1, \infty)$. As in the case of embeddings into balls, the structure of the graph of $c_{EC}(a)$ is very rich: since symplectic embeddings are volume preserving, we always have $c_{EC}(a) \geq \sqrt{\frac{a}{2}}$ and it is not hard to see that this lower bound is sharp for $a \geq 8$, that is the function $c_{EC}(a)$ is equal to the volume constraint given by $\sqrt{\frac{a}{2}}$ for $a \geq 8$. For a less than the square of the silver ratio $\sigma := 1 + \sqrt{2}$, the function $c_{EC}(a)$ turns out to be piecewise linear, having the form of a stair with an infinite number of steps, converging to $\sqrt{\frac{\sigma^2}{2}}$ for $a \to \sigma^2$. These “stairs” will be determined by the Pell numbers and we thus refer to them as the “Pell stairs”. For the proof, we first translate the embedding problem $E(1,a) \overset{\text{s}}{\hookrightarrow} P(\mu,\mu)$ to a certain ball packing problem of the ball $B(2\mu)$. This embedding problem is then solved by adapting the method from McDuff and Schlenk in [19], which finds all exceptional spheres in blow-ups of the complex projective plane that provide an embedding obstruction. Furthermore we also prove the equivalence of symplectic embeddings of an ellipsoid into a polydisc and the embedding of its decomposition into disjoint balls into a ball, that is $E(a,b) \overset{\text{s}}{\hookrightarrow} P(c,d) \Leftrightarrow B(a,b) \sqcup B(c) \sqcup B(d) \overset{\text{s}}{\hookrightarrow} B(c+d)$.

Keywords. Symplectic embeddings, Pell numbers.

Résumé

Pour un réel $a > 1$, considérons l'ellipsoïde $E(1, a)$ dont le rapport entre la longeur du grand axe avec celle du petit axe vaut a. Dans [19], McDuff et Schlenk étudient la fonction $c_{EB}(a)$ qui associe au nombre a l'infimum de l'ensemble des volumes des boules de dimension 4 dans lesquelles l'ellipsoïde $E(1, a)$ admet un plongement symplectique. Dans ce travail, nous considérons les ellispoides $E(1, a)$ de dimension 4 admettant un plongement dans des polydisques $P(\mu, \mu) = D^2(\mu) \times D^2(\mu)$, où $D^2(\mu)$ est le disque de $\mathbb{R}^2$ d'aire μ, dans le cas où a varie dans l'intervalle $[1, \sigma^2]$ borné supérieurement par $\sigma^2 = 3 + 2\sqrt{2} \approx 5.83$ qui est le carré du nombre d'argent $\sigma := 1 + \sqrt{2}$. Plus précisément, nous étudions la fonction et capacité symplectique $c_{EC}(a)$ qui associe à $a \in [1, \infty)$ l'infimum $c_{EC}(a) := \inf \Big\{ \mu \mid E(1, a) \overset{s}{\hookrightarrow} P(\mu, \mu) \Big\}$. De même que dans le cas de plongements dans des boules, la structure du graphe de $c_{EC}(a)$ s'avère être riche. Etant donné que les plongements symplectiques préservent le volume, il est clair que $c_{EC}(a) \geq \sqrt{\frac{a}{2}}$. De plus on montre facilement que cette borne inférieure est atteinte lorsque $a \geq 8$, c'est-à-dire lorsque la fonction $c_{EC}(a)$ est équivalente à la contrainte de volume donnée par $\sqrt{\frac{a}{2}}$ pour $a \geq 8$. Quand a est inférieur au carré du nombre d'argent $\sigma := 1 + \sqrt{2}$, la fonction $c_{EC}(a)$ est continue par morceaux, plus précisément il s'agit d'une fonction en escalier avec une infinité de marches convergeant vers $\sqrt{\frac{\sigma^2}{2}}$ quand $a \to \sigma^2$. Chaque marche est déterminée via les nombres de Pell, nous les avons donc naturellement apellées "escalier de Pell". Afin de montrer ce résultat, nous commençons par traduire le problème de plongement en un problème précis du empilement de la boule $B(2\mu)$. Il est ensuite résolu en adaptant la méthode développée par McDuff et Schlenk dans [19] qui consiste à trouver toutes les sphères exceptionnelles dans les éclatements du plan projectif complexe qui impliquent une obstruction à l'existence d'un plongement. Nous prouvons également léquivalence entre le plongement symplectique d'un ellipsoïde dans un polydisque et le plongement de sa décomposition en boules disjointes dans une boule, c'est-à-dire $E(a, b) \overset{s}{\hookrightarrow} P(c, d) \Leftrightarrow B(a, b) \sqcup B(c) \sqcup B(d) \overset{s}{\hookrightarrow} B(c + d)$.

Mots clés. Plongements symplectiques, nombres de Pell.

Acknowledgements

This work was partially supported by the FNS (Swiss National Science Foundation) grant 200020-132000.

Special thanks go to Prof. Felix Schlenk for suggesting the problem and for his help. Furthermore I would like to thank Prof. Hansjörg Geiges for introducing me to the field and prof. em. Dusa McDuff and prof. Michael Hutchings for their support during the development of this work.

Moreover, many thanks go to David Frenkel, Dr. Agnès Gadbled, Dr. Muriel Heistercamp and Dr. Athanasios Stylianou for fruitful discussions and their special contributions.

I would also like to thank Yan Huang and Dr. Kolawolé Atchade, Régis Straubhaar and Raphael Wullschleger for creating an efficient and encouraging working environment.

Finally I want to thank my family who have always supported and believed in me.

Per Aspera ad Astra.

Contents

Chapter 1

Introduction

The problem of symplectic embeddings $U \overset{\mathrm{s}}{\hookrightarrow} V$ between two open subsets $U, V \in \mathbb{R}^2$, has been studied much the past three decades. Gromov pin-pointed the difference between volume preserving embeddings and symplectic embeddings. In his famous Nonsqueezing Theorem (for more details see Theorem 2.1.10 in Section 2.1 or [7]) he shows that symplectic embeddings are not only volume preserving, but much more rigid: A ball $B(a)$ (with finite volume) can be embedded into a cylinder $Z(A)$ (with infinite volume) just by *squeezing* (linearly) the ball. But a ball $B(a)$ can be symplectically embedded into a cylinder $Z(A)$ if and only if $a \leq A$ (for an illustration see Figure 1.1).

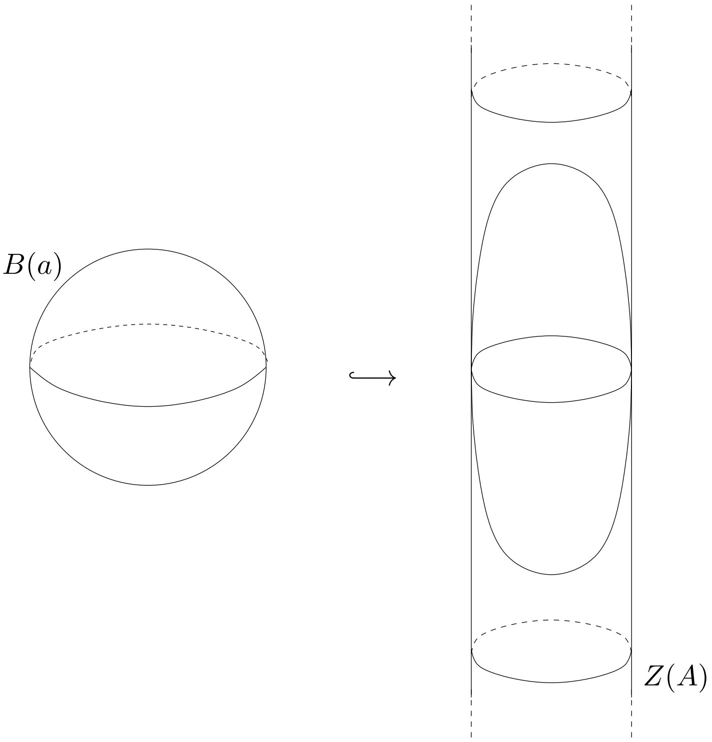

Figure 1.1: For $B(a)$ and $Z(A)$ with $a > A$, Gromov's Nonsqueezing Theorem shows that $B(a)$ embeds into $Z(A)$ but not symplectically.

Thus there is no way to deform a ball using canonical transformations in a way that it fits into a cylinder with smaller cross-section than the cross-section of the original ball. This shows that symplectic embeddings are much more special and *more rigid* than volume preserving embeddings.

In this thesis, we study the symplectic embedding of open symplectic ellipsoids

$$E(a_1, a_2) := \left\{ (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \mid \frac{\pi(x_1^2 + y_1^2)}{a_1} + \frac{\pi(x_2^2 + y_2^2)}{a_2} < 1 \right\},$$

where $0 < a_1 \leq a_2$, into the open cube, that is the polydisc

$$P(\mu, \mu) := D^2(\mu) \times D^2(\mu),$$

where $D^2(\mu)$ is the open disc of area μ. Projecting the ellipsoid $E(a_1, a_2)$ to $\mathbb{R}^2(x_1, y_1)$, $\mathbb{R}^2(x_2, y_2)$ respectively, this projection is a disc of area a_1, a_2 respectively.

We will write $C(\mu) := P(\mu, \mu)$ for the open cube. Both, the ellipsoid and the polydisc, are open subsets of $\mathbb{R}^4$ and thus endowed with the symplectic standard form $\omega_{\text{std}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ (for more details see Section 2.1).

1.1 Approaching the problem

To understand the rigidity of symplectic embeddings better, we fix a domain $V \subset \mathbb{R}^4$ of finite volume and try to determine the **k-th packing number**

$$p_k(V) := \sup \left\{ \frac{k \text{Vol}(B(\mu))}{\text{Vol}(V)} \mid \bigsqcup_k B(\mu) \overset{\text{s}}{\hookrightarrow} V \right\},$$

where $B(\mu) := E(\mu, \mu)$ denotes the open 4-ball of radius $\sqrt{\frac{\mu}{\pi}}$, $\bigsqcup_k B(\mu)$ denotes the disjoint union of k equal 4-balls $B(\mu)$, $\text{Vol}(V)$ denotes the volume $\text{Vol}(V) := \frac{1}{2} \int_V \omega \wedge \omega$, for a symplectic manifold (V, ω), and $U \overset{\text{s}}{\hookrightarrow} V$ denotes the symplectic embedding of U into V.

From Darboux's Theorem it follows that $p_k(V) > 0$ always. If $p_k(V) = 1$, one says that there is a **full packing**, and if $p_k(V) < 1$, one says that there is a **packing obstruction**. If we would only consider volume preserving embeddings instead of symplectic embeddings, then all packing numbers k would always be equal to 1 that is, there are always full packings (see [21, p. 191]).

In important works by Gromov [7], McDuff-Polterovich [17] and Biran [1] all packing numbers of the ball and the cube in $\mathbb{R}^4$ were determined. The result for the cube is

k	1	2	3	4	5	6	7	≥ 8
p_k	$\frac{1}{2}$	1	$\frac{2}{3}$	$\frac{8}{9}$	$\frac{9}{10}$	$\frac{48}{49}$	$\frac{224}{225}$	1

This shows, that there is symplectic rigidity for small k, but none for $k \geq 8$.

Now we look for the smallest polydisc $P(\mu, \mu)$, that is the smallest cube $C(\mu)$, into which the ellipsoid $E(a_1, a_2)$ symplectically embeds. By the scaling property of symplectic embeddings, $E(a_1, a_2)$ symplectically embeds into $C(\mu)$ if and only if $E(\lambda a_1, \lambda a_2)$ symplectically embeds into $C(\lambda\mu)$, we can always assume that $a_1 = 1$ and therefore study the **embedding capacity function**

$$c_{EC}(a) := \inf \left\{ \mu \mid E(1, a) \overset{\text{s}}{\hookrightarrow} C(\mu) \right\},$$

for $a \in [1, \infty)$. From the facts that symplectic embeddings are volume preserving and $c_{EC}(a)$ is a continuous, nondecreasing function, we obtain the lower bound

$$c_{EC}(a) \geq \sqrt{\frac{a}{2}},$$

as $\mathrm{Vol}(E(1,a)) = \frac{a}{2}$ and $\mathrm{Vol}(C(\mu)) = \mu^2$.

Following [15], we will show in Chapter 3 that, if we decompose the ellipsoid $E(1,a)$ into k disjoint equal balls $B(1)$ then the disjoint union $\bigsqcup_k B(1)$ symplectically embeds into $E(1,a)$, that is, $\bigsqcup_k B(1)$ symplectically embeds into $C(\mu)$, whenever the ellipsoid $E(1,a)$ symplectically embeds into $C(\mu)$; the converse is also true. However, our ellipsoid embedding problem now interpolates the problem of packing k equal balls, and we obtain

$$p_k(C) = \frac{k\mathrm{Vol}(E(1,k))}{\mathrm{Vol}(C(c_{EC}(k)))} = \frac{\frac{k}{2}}{(c_{EC}(k))^2}.$$

1.2 Statement of the result

Following the work of McDuff and Schlenk in [19], we will find an analogue to the stairs determining the capacity embedding function $c_{EB}(a)$ of 4-dimensional symplectic ellipsoids into balls. For symplectic embeddings of ellipsoids into balls they found the so called Fibonacci stairs on $[1, \tau^4]$, where τ denotes the golden ratio $\tau := \frac{1+\sqrt{5}}{2}$. Similarly we find a significant point for the embedding capacity function of 4-dimensional symplectic ellipsoids into polydiscs at $a = \sigma^2 = 3 + 2\sqrt{2} \approx 5.83$, where $\sigma := 1 + \sqrt{2}$ denotes the silver ratio. Thus, as there is a relation between the Fibonacci numbers and the golden ratio, we started searching for the analogue of the Fibonacci numbers, the Pell numbers. Thereby we were able to determine the embedding capacity function $c(a) := c_{EC}(a)$ of 4-dimensional symplectic ellipsoids $E(1,a)$ into polydiscs $P(\mu,\mu) = C(\mu)$, on $[1, \sigma^2]$ by the so called Pell stairs (see Figure 1.2).

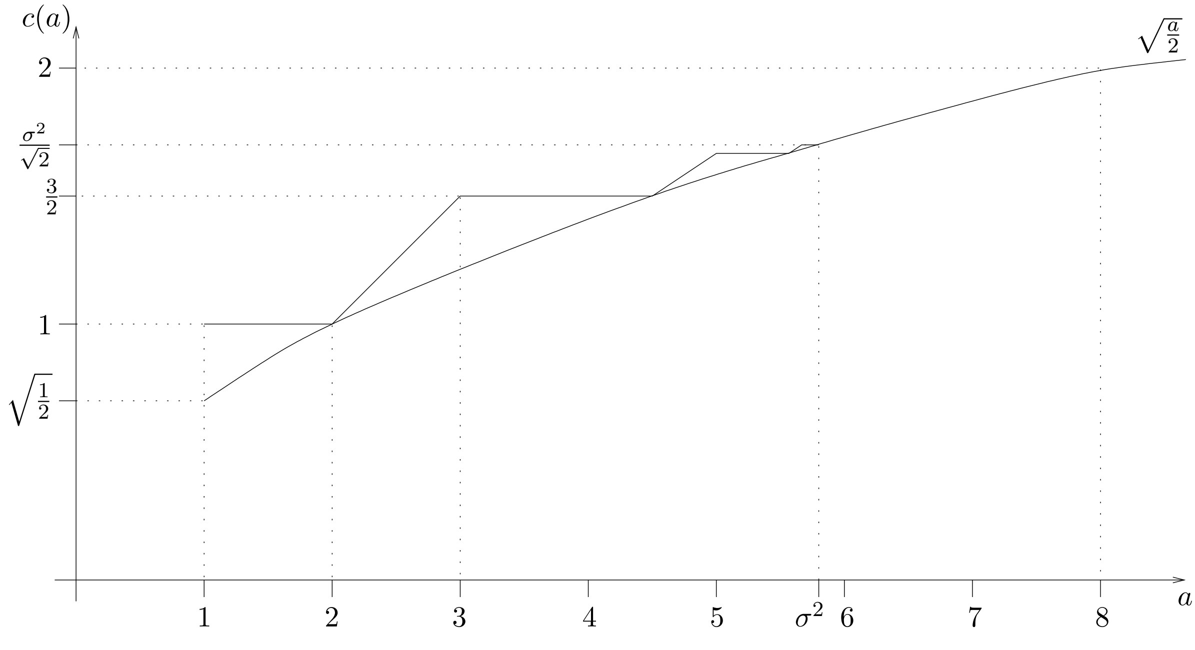

Figure 1.2: The capacity function $c(a)$ and the volume constraint $\sqrt{\frac{a}{2}}$.

In Chapter 6, we will already prove that $c(a)$ is equal to the volume constraint $\sqrt{\frac{a}{2}}$ for $a \geq 8$, that is, symplectic rigidity for the problem of symplectic embedding of $E(1,a)$ into $C(\mu)$ disappears for $a \geq 8$.

To be able to determine the function $c(a)$ for $a \in [1,\sigma^2]$ completely, we will introduce two recursively defined sequences of integers in Section 7.1. The so called *Pell numbers* P_n are defined by

$$P_0 := 0, \qquad P_1 := 1, \qquad P_n := 2P_{n-1} + P_{n-2} \quad \text{for } n \geq 2,$$

and the *half companion Pell numbers* H_n are defined by

$$H_0 := 1, \qquad H_1 := 1, \qquad H_n := 2H_{n-1} + H_{n-2} \quad \text{for } n \geq 2.$$

We thus obtain the infinite sequences

$$P_0 = 0, \quad P_1 = 1, \quad P_2 = 2, \quad P_3 = 5, \quad P_4 = 12, \quad P_5 = 29, \quad \ldots$$

and

$$H_0 = 1, \quad H_1 = 1, \quad H_2 = 3, \quad H_3 = 7, \quad H_4 = 17, \quad H_5 = 41, \quad \ldots .$$

From these sequences P_n and H_n we can define (as in Section 7.2) the sequences

$$\alpha_n := \frac{x_n}{x_{n-1}}, \quad \gamma_n := \frac{x_n^2}{2y_n^2} \quad \text{for } n \geq 1$$

and

$$\beta_n := \frac{y_{n+1}}{y_n}, \quad \delta_n := \frac{2y_{n+1}^2}{x_n^2} \quad \text{for } n \geq 0,$$

where

$$x_n := P_{2n} + P_{2n-1} = H_{2n} \quad \text{and} \quad y_n := P_{2n-1}$$

for $n \geq 1$ and $x_0 := 1$, $y_0 := 1$. We will show that

$$\beta_0 < \delta_0 < \alpha_1 < \gamma_1 < \beta_1 < \ldots < \delta_{n-1} < \alpha_n < \gamma_n < \beta_n < \delta_n < \alpha_{n+1} < \ldots$$

for all $n \geq 1$ and that the sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 0}$, $(\gamma_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 0}$ converge to $\sigma^2 = 3 + 2\sqrt{2}$.

As in [5], we will reduce this notation of four sequences α_n, β_n, γ_n and δ_n to two sequences $(\eta_n)_{n\geq 0}$ and $(\vartheta_n)_{n\geq 0}$, to show the analogy to the two sequences in [19] and to be able to give the reduction later. These two sequences $(\eta_n)_{n\geq 0}$ and $(\vartheta_n)_{n\geq 0}$ will be different for n odd or even, as one already notices in the above definition of α_n, β_n, γ_n and δ_n. They are defined by

$$\eta_n := \begin{cases} \dfrac{2P_{n+1}^2}{H_n^2} & \text{if } n \text{ is even,} \\ \dfrac{H_{n+1}^2}{2P_n^2} & \text{if } n \text{ is odd;} \end{cases} \qquad \vartheta_n := \begin{cases} \dfrac{H_{n+2}}{H_n} & \text{if } n \text{ is even,} \\ \dfrac{P_{n+2}}{P_n} & \text{if } n \text{ is odd.} \end{cases}$$

The first terms in these sequences are

$$\eta_0 = 2 < \vartheta_0 = 3 < \eta_1 = \frac{9}{2} < \vartheta_1 = 5 < \eta_2 = \frac{50}{9} < \vartheta_2 = \frac{17}{3} < \ldots .$$

More generally, for all $n \geq 0$,

$$\ldots < \eta_n < \vartheta_n < \eta_{n+1} < \vartheta_{n+1} < \ldots,$$

and both sequences converge to σ^2, as $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 0}$, $(\gamma_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 0}$ converge to σ^2.

We will prove that

Theorem 1.2.1 *The following hold for the embedding capacity function* $c(a)$*:*

(i) On the interval $\left[1, \sigma^2\right]$,

$$c(a) = \begin{cases} 1 & \text{if } a \in [1,2]\,, \\ \dfrac{1}{\sqrt{2\eta_n}}\, a & \text{if } a \in [\eta_n, \vartheta_n]\,, \\ \sqrt{\dfrac{\eta_{n+1}}{2}} & \text{if } a \in [\vartheta_n, \eta_{n+1}]\,, \end{cases}$$

for all $n \geq 0$ *(see Figure 1.3).*

(ii) For $a \geq 8$ *we have* $c(a) = \sqrt{\frac{a}{2}}$.

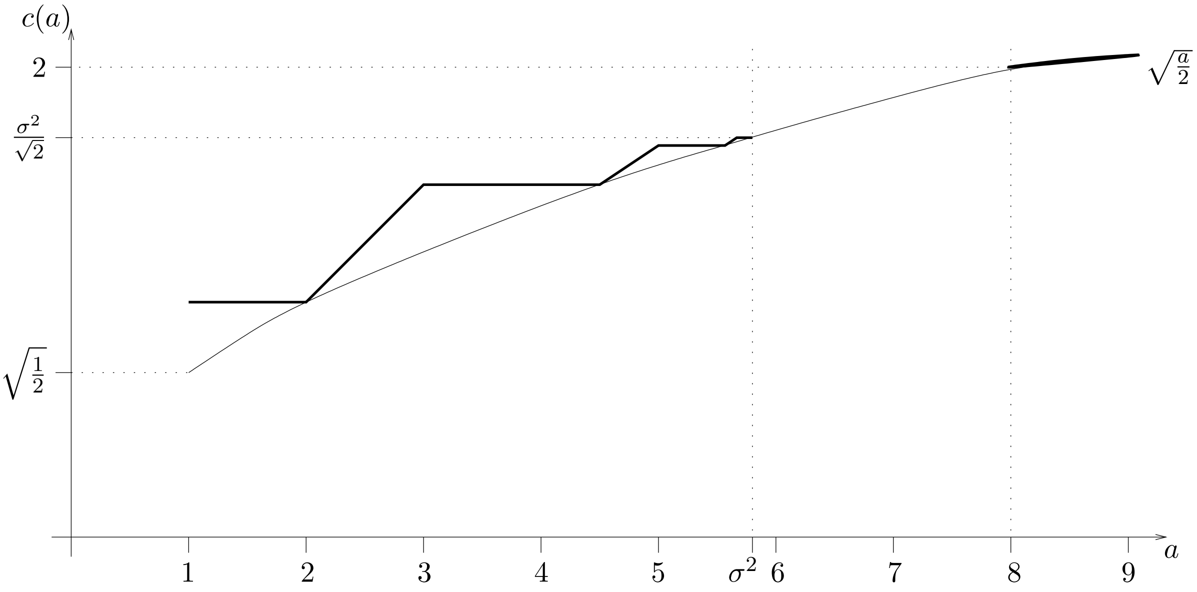

Figure 1.3: The Pell stairs determining the capacity function $c(a)$ on the interval $[1, \sigma^2]$ and the volume constraint $\sqrt{\frac{a}{2}}$ determining $c(a)$ on the interval $[8, \infty)$.

Chapter 2

Preliminaries

This is the foundational chapter about the mathematical setting on which the later chapters are based. In the Section 2.1 we will introduce the definitions of a symplectic manifold and a symplectic embedding, and some of their basic properties. In Section 2.1.3 we will describe the setting of our problem and use the latter definitions. We are then already able to define our problem by means of a function and give a first constraint.

2.1 The mathematical setting

2.1.1 Linear Symplectic Geometry - basic concepts

We start introducing the subject by the definition of a symplectic vector space.

Definition 2.1.1 *A* ***symplectic vector space*** *is a pair* (V, ω) *of a finite, even-dimensional vector space* V *and a bilinear form*

$$\omega : V \times V \to \mathbb{R},$$

satisfying the following two conditions:

skew-symmetry: *For all* $v, w \in V$*,*

$$\omega(v, w) = -\omega(v, w).$$

nondegeneracy: *For every* $v \in V$*,*

$$\omega(v, w) = 0 \ \forall w \in V \quad \Rightarrow \quad v = 0.$$

A bilinear, skew-symmetric, nondegenerate form ω *is called a* ***symplectic form*** *or* ***symplectic structure****.*

Remark 2.1.1.1 The vector space V has to be of even dimension. Otherwise there is a contradiction to the nondegeneracy since a real skew-symmetric matrix of odd dimension must have a kernel.

Example 2.1.2 *The vector space* $\mathbb{R}^{2n}$ *equipped with the standard symplectic structure*

$$\omega_{\text{std}} = \sum_{i=1}^{n} dx_i \wedge dy_i,$$

where $x, y \in \mathbb{R}^n$*, is called the standard symplectic space* $(\mathbb{R}^{2n}, \omega_{\text{std}})$*.*

Definition 2.1.3 *A linear map*

$$\Psi : (V_1, \omega_1) \to (V_2, \omega_2)$$

between symplectic vector spaces (V_1, ω_1) *and* (V_2, ω_2) *is called* ***symplectic*** *if it preserves the symplectic structure, that is*

$$\Psi^* \omega_2 = \omega_2(\Psi \cdot, \Psi \cdot) = \omega_1.$$

2.1.2 Symplectic manifolds

Now let M be a C^∞-smooth manifold without boundary of even dimension $2n$.

Definition 2.1.4 *A* ***symplectic manifold*** *is a pair* (M, ω) *of a* $2n$*-dimensional smooth manifold* M *and a nondegenerate closed* 2*-form* $\omega \in \Omega^2(M)$.

Remark 2.1.4.1 A symplectic form on a surface is an area form. Hence a surface admits a symplectic structure if and only if it is orientable.

Examples 2.1.5 *Let*

$$Z(a) := D^2(a) \times \mathbb{R}^2$$

denote a symplectic cylinder in $\mathbb{R}^4$, *where* $D^2(a)$ *is the open disc of area* a, *let*

$$E(a_1, a_2) := \left\{ (x_1, y_1, x_2, y_1) \in \mathbb{R}^4 \mid \frac{\pi(x_1^2 + y_1^2)}{a_1} + \frac{\pi(x_2^2 + y_2^2)}{a_2} < 1 \right\}$$

denote the ellipsoid with areas a_1, a_2 *respectively, where* $0 < a_1 \leq a_2$, *and let*

$$P(a_1, a_2) := D^2(a_1) \times D^2(a_2)$$

denote the open polydisc. The symplectic form on these subsets of $\mathbb{R}^4$ *is taken to be the restriction of the standard symplectic form* $\omega_{\mathrm{std}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ *as introduced in Example 2.1.2.*

Definition 2.1.6 *Let* (M_1, ω_1) *and* (M_2, ω_2) *be two symplectic manifolds of dimension* $2n$. *Then*

$$\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$$

is called a ***symplectomorphism****, if* φ *is a diffeomorphism and* $\varphi^*(\omega_2) = \omega_1$. *If such a symplectomorphism exists, the two symplectic manifolds* (M_1, ω_1) *and* (M_2, ω_2) *are called* ***symplectomorphic****.*

Example 2.1.7 *The open disc* $D^2(a)$ *and the open square* $(0, a) \times (0, 1)$, *both endowed with* ω_{std} *as subsets of* $\mathbb{R}^2$, *are symplectomorphic. Thus,* $P(a, a) := D^2(a) \times D^2(a)$ *is symplectomorphic to the cube* $C(a) := (0, a) \times (0, 1) \times (0, a) \times (0, 1)$.

Notation 2.1.8 *The ball* $E(a, a)$ *of radius* $\sqrt{\frac{a}{\pi}}$ *is from now on shortly denoted by* $B(a)$ *and the polydisc* $P(a, a)$ *by* $C(a)$.

2.1.3 Symplectic embeddings and the embedding capacity function

Definition 2.1.9 *Let* (M_1, ω_1) *and* (M_2, ω_2) *be two symplectic manifolds of dimension* $2n$. *A* ***symplectic embedding***

$$\varphi : (M_1, \omega_1) \overset{s}{\hookrightarrow} (M_2, \omega_2)$$

is a smooth embedding

$$\varphi : (M_1, \omega_1) \hookrightarrow (M_2, \omega_2)$$

with $\varphi^* \omega_2 = \omega_1$.

Since symplectic embeddings are volume preserving, a necessary condition for the existence of a symplectic embedding $(M_1, \omega_1) \overset{s}{\hookrightarrow} (M_2, \omega_2)$ is that $\mathrm{Vol}(M_1) \leq \mathrm{Vol}(M_2)$. Schlenk proved in [20] that for volume preserving embeddings this is the only condition. However, for symplectic embeddings Gromov proved in [7] the following:

Theorem 2.1.10 (Gromov's Nonsqueezing Theorem) *There exists a symplectic embedding of the ball* $B(a)$ *into the cylinder* $Z(A)$, *if and only if* $a \leq A$.

Remark 2.1.10.1 Notice that the volume of a cylinder $Z(A) = D^2(A) \times \mathbb{R}^2$ is infinite for any $A > 0$. However, Gromov's Nonsqueezing Theorem shows that the ball $B(a)$ embeds symplectically into $Z(A)$ if and only if $a \leq A$.

Example 2.1.11 *There exists a symplectic embedding of the ball* $B(a) = E(a, a)$ *into the cube* $C(A) = P(A, A)$ *if and only if* $a \leq A$.

Definition 2.1.12 *The embedding capacity function* $c : [1, \infty) \to \mathbb{R}$ *is given by*

$$c(a) := \inf \left\{ \mu \mid E(1, a) \overset{s}{\hookrightarrow} C(\mu) \right\}. \tag{2.1}$$

The first necessary restriction for the existence of a symplectic embedding is the **volume constraint**:

Lemma 2.1.13 *The embedding capacity function* $c(a)$ *is bounded by*

$$c(a) \geq \sqrt{\frac{a}{2}}.$$

Proof. Symplectic embeddings preserve the total volume, as they preserve the Euclidean volume form $\frac{1}{2}\omega \wedge \omega$ and are injective. This means in our case, that if $E(1, a) \overset{s}{\hookrightarrow} P(\mu)$, then

$$\mathrm{Vol}(E(1, a)) \leq \mathrm{Vol}(P(\mu)).$$

As $\mathrm{Vol}(E(1, a)) = \frac{a}{2}$ and $\mathrm{Vol}(P(\mu)) = \mathrm{Vol}(D^2(\sqrt{\mu})) \cdot \mathrm{Vol}(D^2(\sqrt{\mu})) = \mu^2$, it follows that $\mu^2 \geq \frac{a}{2}$ and thus the embedding capacity function is bounded by

$$c(a) \geq \sqrt{\frac{a}{2}}.$$

■

More important properties of the embedding capacity function are given in the following lemma.

Lemma 2.1.14 *The embedding capacity function is nondecreasing and continuous. Further, for* $b > a > 0$ *it holds that*

$$a \cdot c(b) \leq b \cdot c(a). \tag{2.2}$$

Proof. The monotonicity comes directly from the definition of $c(a)$. Let $E(1,a) \overset{\mathrm{s}}{\hookrightarrow} P(\mu)$, then also

$$\sqrt{\frac{b}{a}}E(1,a) \overset{\mathrm{s}}{\hookrightarrow} \sqrt{\frac{b}{a}}P(\mu),$$

where $\sqrt{\frac{b}{a}}E(1,a) = E(\frac{b}{a}, b)$ and $\sqrt{\frac{b}{a}}P(\mu) = P(\frac{b}{a}\mu)$. Thus as $E(1,b) \subset \sqrt{\frac{b}{a}}E(1,a)$, we obtain that

$$E(1,b) \overset{\mathrm{s}}{\hookrightarrow} P\left(\frac{b}{a}\mu\right).$$

It follows that $c(b) \leq \frac{b}{a}c(a)$. The continuity of $c(a)$ follows by simple dilation arguments. ■

Lemma 2.1.15 *The function $c(a)$ has the following scaling property: For all $\lambda \geq 1$,*

$$\frac{c(\lambda a)}{\lambda a} \leq \frac{c(a)}{a}.$$

Proof. By definition of $c(a)$, $E(1,a) \overset{\mathrm{s}}{\hookrightarrow} C\left(c(a)+\varepsilon\right)$ for all $\varepsilon > 0$. Since $E(1,a)$ symplectically embeds into $C(A)$ if and only if $E(\lambda, \lambda a)$ symplectically embeds into $C(\lambda A)$, this is equivalent to

$$E(\lambda, \lambda a) \overset{\mathrm{s}}{\hookrightarrow} C\left(\lambda c(a) + \varepsilon\right), \qquad \text{for all } \varepsilon > 0.$$

Since $E(1, \lambda a) \subset E(\lambda, \lambda a)$ when $\lambda > 1$, this implies that

$$E(1, \lambda a) \overset{s}{\hookrightarrow} C\left(\lambda c(a) + \varepsilon\right)$$

for all $\varepsilon > 0$. Thus

$$c(\lambda a) := \inf\left\{A : E(1, \lambda a) \overset{s}{\hookrightarrow} C(A)\right\} \leq \lambda c(a) = \lambda a \frac{c(a)}{a},$$

as claimed. ■

The monotonicity and scaling property of $c(a)$ are enough to determine the function $c(a)$ on the interval $[1, \sigma^2]$, where $\sigma^2 = 3 + 2\sqrt{2}$ and $\sigma := 1 + \sqrt{2}$ is the silver ratio. It turns out that on this interval, $c(a)$ is determined by a step function that we will call the Pell stairs, as it is determined by sequences of combinations of the Pell numbers, which are introduced in Section 7.1.

2.2 Blowing up and down

Here, we first introduce the complex blow-up and blow-down, and then introduce its symplectic analogue, the symplectic blow-up and blow-down. Its applications to symplectic embedding problems are due to McDuff.

The following discussion closely follows [18]. Its focus is on the case $n = 2$ (for more details see [2], [6] and [18]).

2.2.1 Blowing up in the complex category

We first consider the local model, that is blowing up the manifold $\mathbb{C}^n$, and then consider the general case of blowing up a complex manifold M at a point $p_0 \in M$.

The local model

Let $\mathbb{C}^n$ be the n-dimensional complex space and let $\mathbb{C}P^n \simeq \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \sim$, where $\sim$ denotes the equivalence relation

$$x \sim y \Leftrightarrow \exists \lambda \in \mathbb{C}^* \text{ such that } \lambda x = y,$$

be the complex projective space.

Consider the subset $\widetilde{\mathbb{C}^n} \subset \mathbb{C}^n \times \mathbb{C}P^{n-1}$ given by

$$\begin{aligned} \widetilde{\mathbb{C}^n} &:= \left\{ (z, \ell) \,\middle|\, z \in \ell \right\} \subset \mathbb{C}^n \times \mathbb{C}P^{n-1} \\ &= \left\{ (z_1, \dots, z_n; [w_1 : \dots : w_n]) \,\middle|\, w_j z_k = w_k z_j \forall j, k \right\}, \end{aligned}$$

that is $\widetilde{\mathbb{C}^n}$ is the space of pairs (z, ℓ), where $\ell \in \mathbb{C}P^{n-1}$ is a complex line in $\mathbb{C}^n$ and $z \in \ell$ is a point on this line.

There exist two projection maps:

$$\begin{array}{rccc} \mathrm{pr} & : & \widetilde{\mathbb{C}^n} & \longrightarrow & \mathbb{C}P^{n-1} \\ & & (z, \ell) & \longmapsto & \ell \end{array}$$

and

$$\begin{array}{rccc} \Phi & : & \widetilde{\mathbb{C}^n} & \longrightarrow & \mathbb{C}^n \\ & & (z, \ell) & \longmapsto & z. \end{array}$$

For these two projections keep in mind the following picture:

$$\begin{tikzcd}
& \widetilde{\mathbb{C}^n} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \\
& (z, \ell) \arrow[dl, "\Phi"'] \arrow[dr, "\mathrm{pr}"] & \\
z \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & & \ell \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \\
\mathbb{C}^n & & \mathbb{C}P^n
\end{tikzcd}$$

The projection $\mathrm{pr} : \widetilde{\mathbb{C}^n} \longrightarrow \mathbb{C}P^{n-1}$ is the tautological line bundle

$$L = L\left(\mathrm{pr}, \widetilde{\mathbb{C}^n}, \mathbb{C}P^{n-1}\right)$$

over $\mathbb{C}P^{n-1}$. The fibre of L over $[w_1 : \dots : w_n] \in \mathbb{C}P^{n-1}$ is the corresponding line

$$\left\{ (\lambda w_1, \dots, \lambda w_n) \,\middle|\, \lambda \in \mathbb{C} \right\}$$

in $\mathbb{C}^n$.

For the zero section σ_0 of L, that is

$$\sigma_0 : \ell \longmapsto (0, \ell),$$

it holds that

$$\sigma_0 = \mathrm{pr}^{-1}([0]) \cong \Phi^{-1}(0) \cong \mathbb{C}P^{n-1}.$$

Definition 2.2.1 *The preimage of Φ at $0 \in \mathbb{C}^n$, that is the zero section L_0 of the line bundle $L\left(\mathrm{pr}, \widetilde{\mathbb{C}^n}, \mathbb{C}P^{n-1}\right)$, is called the* ***exceptional divisor*** *and denoted by*

$$\Sigma := \Phi^{-1}(0)\,(\cong L_0)\,.$$

Note that

$$\Phi\big|_{\widetilde{\mathbb{C}^n}\setminus\Sigma} \;:\; \widetilde{\mathbb{C}^n}\setminus\Sigma \longrightarrow \;\mathbb{C}^n\setminus\{0\}$$

is bijective. Thus we may think of $\widetilde{\mathbb{C}^n}$ as obtained from $\mathbb{C}^n$ by replacing the origin $0 \in \mathbb{C}^n$ by all lines $\ell \in \mathbb{C}^n$ through the origin, i.e. ℓ with $0 \in \ell$, that is $\mathbb{C}P^{n-1}$.

Definition 2.2.2 *The map*

$$\Phi \;:\; \widetilde{\mathbb{C}^n} \longrightarrow \mathbb{C}^n$$

is called the ***blow-up of*** $\mathbb{C}^n$ ***at the origin****.*

Since in this work we are mostly interested in the case $n = 2$, we will now have a closer look at this.

Lemma 2.2.3 *When $n = 2$, the first Chern number of $L\left(\mathrm{pr}, \widetilde{\mathbb{C}^2}, \mathbb{C}P^1\right)$ is -1.*

Proof. Recall that

$$\widetilde{\mathbb{C}^2} = \left\{((x,y),[v:w]) \in \mathbb{C}^2 \times \mathbb{C}P^1 \;\middle|\; xw = yv\right\}.$$

The manifold $\mathbb{C}^2 \times \mathbb{C}P^1$ is covered by the charts

$$\begin{aligned} V_1 &= \{((x,y),[v:1])\} \\ V_2 &= \{((x,y),[1:w])\}, \end{aligned}$$

that cover all of $\mathbb{C}^2 \times \mathbb{C}P^1$ except $\mathbb{C}^2\times$the north pole, $\mathbb{C}^2\times$the south pole respectively.

Thus L is covered by the charts

$$\begin{aligned} U_1 &= \left\{((x,y),[v:1]) \;\middle|\; x = yv\right\} \cong \mathbb{C}^2, \\ U_2 &= \left\{((x,y),[1:w]) \;\middle|\; xw = y\right\} \cong \mathbb{C}^2. \end{aligned}$$

If (y,v) are the coordinates on U_1 and (x,w) are the coordinates on U_2, then if $v \neq 0$, $y = \frac{x}{v} = xw$, that is $\frac{1}{v} = w$, the glueing map is given by

$$x = yv \qquad \text{and} \qquad w = \frac{1}{v} \quad (\text{for } v \neq 0).$$

Over the equator $C = \left\{[1:w] \;\middle|\; |w| = 1\right\}$, we obtain the transition function

$$\begin{array}{ccc} \mathbb{C}\times C & \longrightarrow & \mathbb{C}\times C \\ (x,w) & \longmapsto & \left(xw, \frac{1}{w}\right) \end{array} = (y,v)\,.$$

If we parametrize C with the positively oriented coordinate θ by $e^{2\pi i\theta}$, then the transition function induces the map

$$\theta \;\longmapsto\; \frac{1}{e^{2\pi i\theta}} \;=\; e^{-2\pi i\theta}.$$

The first Chern number is given by the degree of this map, which equals -1. ■

Lemma 2.2.3 is an exercise in [18]. One can find its generalization in [18, Lemma 7.1].

Blowing up a complex manifold at a point

In [4], Cieliebak proves the following:

Lemma 2.2.4 *Every biholomorphism* $\psi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ *with* $\psi(0) = 0$ *lifts to a biholomorphism* $\widetilde{\psi} : \widetilde{\mathbb{C}^n} \longrightarrow \widetilde{\mathbb{C}^n}$ *with* $\widetilde{\psi}(\Sigma) = \Sigma$.

Sketch of the Proof. Cieliebak's proof is based upon the definition of $\widetilde{\psi}$ using the differential to map lines to lines:

$$\widetilde{\psi}(z,\ell) = \begin{cases} (\psi(z), [\psi(z)])\,; & \text{if } z \neq 0, \\ (0, T_0\psi\ell)\,; & \text{if } z = 0, \end{cases}$$

and proves that this map is continuous. With similar arguments as Cieliebak uses for the continuity of $\widetilde{\psi}$, it follows that $\widetilde{\psi}$ is smooth and holomorphic and thus biholomorphic. ■

This allows us now to define the blow-up of a complex manifold at a point.

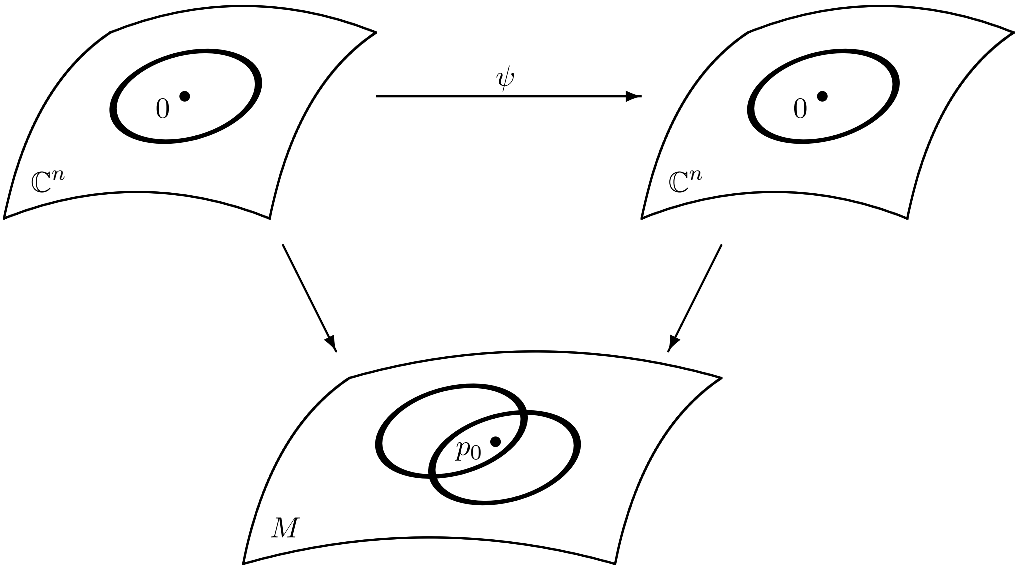

Figure 2.1: Mapping the point p_0 to the origin of two biholomorphic charts of M.

Definition 2.2.5 *The* ***blow-up*** $\widetilde{M}_{p_0}$ *of a complex manifold* M *at* $p_0 \in M$ *is the complex manifold*

$$\widetilde{M}_{p_0} = M \setminus \{p_0\} \cup_\Phi \widetilde{\Delta}$$

obtained by replacing the small disc $\Delta \subset M$, *centered around* $p_0 \in M$, *with* $\widetilde{\Delta}$ *together with the natural projection map*

$$\Phi : \widetilde{M}_{p_0} \longrightarrow M.$$

Remark 2.2.5.1 Constructing the blow-up of M at $p_0 \in M$, that is $\widetilde{M}_{p_0}$, proceeds as follows: Choose a holomorphic coordinate chart that maps p_0 to the origin 0 of $\mathbb{C}^n$ (see Figure 2.1).

The biholomorphism $\psi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ with $\psi(0) = 0$ lifts to the biholomorphism

$$\widetilde{\psi} : \widetilde{\mathbb{C}^n} \longrightarrow \widetilde{\mathbb{C}^n}$$

as described above. For an illustration see Figure 2.2.

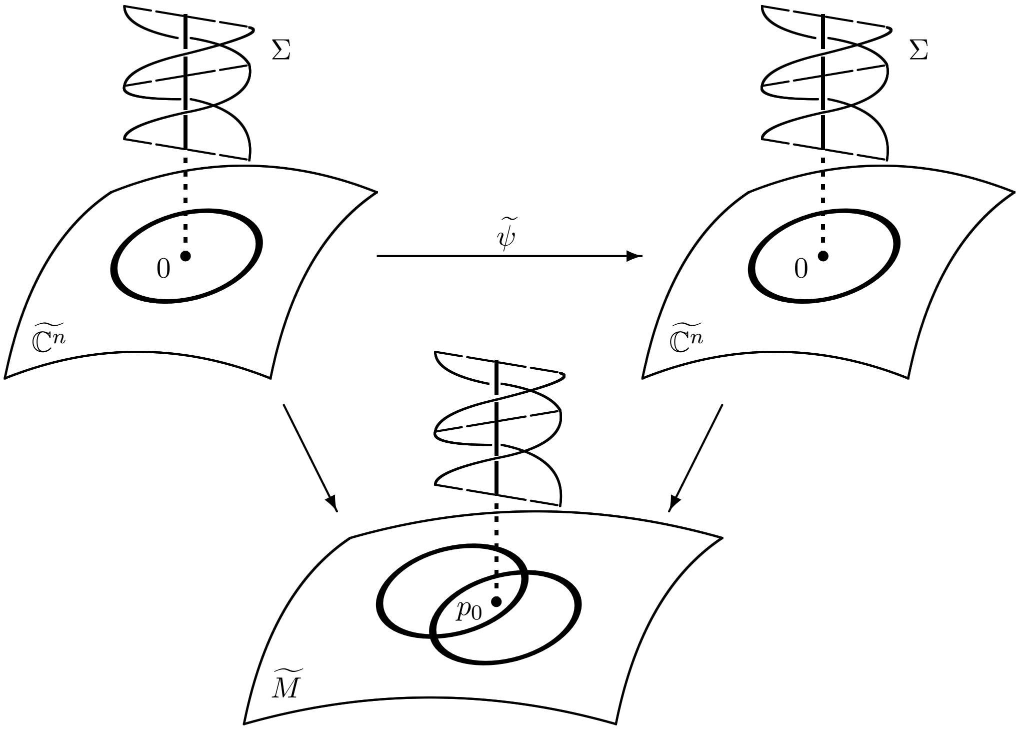

Figure 2.2: The blow-up of M at p_0.

As the projection

$$\Phi : \widetilde{M}_{p_0} \setminus \Sigma \longrightarrow M \setminus \{p_0\},$$

where $\Sigma = \Phi^{-1}(p_0)$, is a diffeomorphism and as $\Sigma = \Phi^{-1}(0) \cong \mathbb{C}P^{n-1}$, the complex blow-up $\widetilde{M}_{p_0}$ is diffeomorphic to the connected sum $M\#\overline{\mathbb{C}P^n}$, i.e.

$$\widetilde{M}_{p_0} \cong M\#\overline{\mathbb{C}P^n},$$

where $\overline{\mathbb{C}P^n}$ denotes the manifold $\mathbb{C}P^n$ with the opposite orientation.

Here $\mathbb{C}P^{n-1}$ carries the opposite orientation of $(\overline{\mathbb{C}P^n})$, because the connected sum of two oriented manifolds M_1 and M_2 is constructed by removing two small discs $\Delta_j \subset M_j$, for $j = 1, 2$, and then identifying the boundaries via a smooth map $\Phi : \partial\Delta_1 \to \partial\Delta_2$, which extends to an orientation-preserving diffeomorphism from a neighbourhood of $\partial\Delta_1$, i.e. the thickened boundary of $M_1 \setminus \partial\Delta_1$, to a neighbourhood of $\partial\Delta_2$, i.e. the thickened boundary of $M_2 \setminus \partial\Delta_2$. This extension must interchange the inner and the outer boundaries of the annuli.

Here $M_1 = M$ and $M_2 = \mathbb{C}P^n$, thus keeping the orientation of M on the thickened boundary $\partial(M \setminus \Delta_1)$, implies that the orientation on the thickened boundary of $\partial(\mathbb{C}P^n \setminus \Delta_2)$ is reversed. For an illustration see Figure 2.3.

Note that the thickened boundary $\partial(\mathbb{C}P^n \setminus \Delta_2) \cong \mathbb{C}P^{n-1}$ equals the normal bundle ν_Σ.

The reversed orientation of $\partial(\mathbb{C}P^n \setminus \Delta_2)$ yields that

$$\widetilde{M}_{p_0} \cong M\#\overline{\mathbb{C}P^n},$$

with $\overline{\mathbb{C}P^n}$ the opposite oriented manifold $\mathbb{C}P^n$.

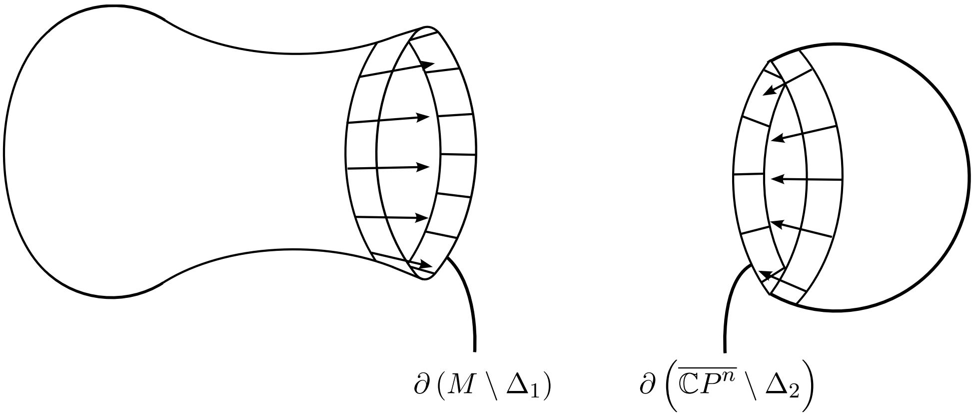

Figure 2.3: The orientations of the thickened boundaries.

The opposite process to blowing up is blowing down, where the exceptional divisor Σ is replaced by a single point p_0. To blow down a manifold, following [18], it is enough to find a copy Σ of $\mathbb{C}P^{n-1}$ whose normal bundle ν_Σ has the correct first Chern class.

For the case $n = 2$, one can blow down any holomorphic copy of $\mathbb{C}P^1$ with self-intersection number -1.

2.2.2 The symplectic blow-up

In Subsection 2.2.1, we introduced the complex blow-up. Thus blowing up a complex manifold can be pictured as removing the interior of a disc around a point p_0 and replacing it by the exceptional divisor Σ, that is diffeomorphic to $\mathbb{C}P^{n-1}$. Accordingly blowing down can be pictured as removing the exceptional divisor and glueing in a ball.

In order to define the symplectic analogue of the complex blow-up, one has to endow the blown up manifold $\widetilde{M}$ with a symplectic form. Thus, the focus lies on what happens to the symplectic form during this procedure of blowing up and down a symplectic manifold.

We closely follow [1], [17] and [18]: Let (M, J, ω) be a Kähler manifold and let

$$\Phi \ : \ \widetilde{M} \ \longrightarrow \ M$$

be the blow-up of M at a point $p_0 \in M$. Then $\Phi^*\omega$ is <u>not</u> a symplectic form on $\widetilde{M}$, as it vanishes on the exceptional divisor:

$$\Phi^*\omega(x)\,(v, w) = \omega(0)\,(D\Phi(0)v, D\Phi(0)w) = 0$$

for $x = (0, \xi)$, $v, w \in T\Sigma$. Let

$$L = \widetilde{\mathbb{C}^n} \subset \mathbb{C}^n \times \mathbb{C}P^{n-1}$$

be the tautological line bundle with the projections

$$\begin{array}{cccc} \mathrm{pr} \ : & \widetilde{\mathbb{C}^n} & \longrightarrow & \mathbb{C}P^{n-1} \\ & (z,\ell) & \longmapsto & \ell \end{array}$$

and

$$\begin{array}{cccc} \Phi \ : & \widetilde{\mathbb{C}^n} & \longrightarrow & \mathbb{C}^n \\ & (z,\ell) & \longmapsto & z \end{array}$$

introduced above. Define

$$B(\delta) := \left\{ z \in \mathbb{C}^n \;\middle|\; |z| < \delta \right\},$$

$$\overline{B(\delta)} := \left\{ z \in \mathbb{C}^n \;\middle|\; |z| \leq \delta \right\},$$

$$L(\delta) := \Phi^{-1}\left(\overline{B(\delta)}\right).$$

Now we define for each $\lambda > 0$ a Kähler form on L by

$$\rho(\lambda) = \Phi^* \omega_0 + \lambda^2 \mathrm{pr}^* \tau_0,$$

where τ_0 is the standard Kähler form on $\mathbb{C}P^{n-1}$, that is

$$\begin{tikzcd}
\left(L, \rho(\lambda)\right) \arrow[r, "\Phi"] \arrow[d, "\mathrm{pr}"] & (\mathbb{C}^n, \omega_0) \\
\left(\mathbb{C}P^{n-1}, \tau_0\right) &
\end{tikzcd}$$

Definition 2.2.6 *An embedding*

$$F \;:\; \mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{C}^n$$

is ***monotone*** *if*

$$F(z) = f(|z|) \cdot \frac{z}{|z|},$$

where $f : (0, \infty) \longrightarrow (0, \infty)$ *is a strictly increasing function.*

Remark 2.2.6.1 In spherical coordinates $(u, r) \in S^{2n-1} \times (0, \infty)$, F can be written as

$$(u, r) \longmapsto (u, f(r)).$$

Lemma 2.2.7 *For every monotone embedding* $F : \mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{C}^n$*, the form* $F^* \omega_0$ *is Kähler.*

For the proof see [18, Lemma 7.10].

Lemma 2.2.8 ([18], Lemma 7.11) *For every* $\lambda, \delta > 0$ *the set* $(L(\delta) - L_0, \rho(\lambda))$ *is symplectomorphic to the spherical shell* $\left(\overline{B\left(\sqrt{\lambda^2 + \delta^2}\right)} - \overline{B(\lambda)}, \omega_0\right)$ *in* $\mathbb{C}^n$*.*

Proof. Identify $L \setminus L_0$ with $\mathbb{C}^n$ via Φ, such that the bundle projection becomes

$$\mathrm{pr} \;:\; \mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{C}P^{n-1},$$

that is

$$\begin{tikzcd}
\left(L \setminus L_0, \rho(\lambda)\right) \arrow[r, "\Phi"] \arrow[dr, "\mathrm{pr}"] & (\mathbb{C}^n \setminus \{0\}, F^* \omega_0) \arrow[r, "F"] \arrow[d, "\mathrm{pr}"] & (\mathbb{C}^n, \omega_0) \\
& \left(\mathbb{C}P^{n-1}, \tau_0\right). &
\end{tikzcd}$$

Define on the bundle

$$\mathrm{pr}\ : \mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{C}P^{n-1}$$

the 1-form

$$\theta := -d\left(\frac{1}{4}\log|z|^2\right) \circ J.$$

Then one can compute that

$$d\left(|z|^2\theta\right) = \omega_0 \qquad \text{and} \qquad d\theta = \mathrm{pr}^*\tau_0.$$

Consider the map

$$\begin{array}{rcccc} F & : & \mathbb{C}^n \setminus \{0\} & \longrightarrow & \mathbb{C}^n \setminus \overline{B(\lambda)} \\ & & z & \longmapsto & \sqrt{|z|^2+\lambda^2}\,\dfrac{z}{|z|}. \end{array}$$

Since θ has no radial component, θ and $F^*\theta$ equal their restrictions to spheres around the origin, but between such spheres F acts as a homothety and θ is invariant under homotheties. Hence, $F^*\theta = \theta$.

It follows now that

$$\begin{aligned} F^*\omega_0 &= F^*\left(d\left(|z|^2\theta\right)\right) \\ &= dF^*\left(|z|^2\theta\right) \\ &= dF^*\left(|z|^2\right) \wedge F^*(\theta) \\ &= d\left(|z|^2+\lambda^2\right)\theta \\ &= d\left(|z|^2\right)\theta + \lambda^2 d\theta \\ &= \omega_0 + \lambda^2\mathrm{pr}^*\tau_0. \end{aligned}$$

And finally

$$\Phi^*\left(F^*\omega_0\right) = \Phi^*\left(\omega_0 + \lambda^2\mathrm{pr}^*\tau_0\right).$$

So $F \circ \Phi$ is the required symplectomorphism. ■

Now let (M,ω) be a symplectic manifold and

$$\psi : \overline{B^{2n}(\lambda)} \overset{\mathrm{s}}{\hookrightarrow} M$$

a symplectic embedding of a closed ball $\overline{B^{2n}(\lambda)}$ into M. As, by definition, a symplectic embedding of a closed ball is the symplectic embedding of an open neighbourhood of the closed ball, ψ extends to a symplectic embedding

$$\psi_0 : B^{2n}(\sqrt{\lambda^2+\delta^2}) \overset{\mathrm{s}}{\hookrightarrow} M$$

for small $\delta > 0$. Now define

$$\widetilde{M} := \left(M \setminus \Psi_0\left(B^{2n}\left(\sqrt{\lambda^2+\delta^2}\right)\right)\right) \cup L(\delta)$$

glued via the symplectomorphism of the above lemma. Define on $\widetilde{M}$ the symplectic form $\widetilde{\omega}_\psi$ by

$$\widetilde{\omega}_\psi = \begin{cases} \omega & \text{on } M \setminus \psi_0\left(B^{2n}\left(\sqrt{\lambda^2+\delta^2}\right)\right), \\ \rho(\lambda) & \text{on } L(\delta). \end{cases}$$

Since on the boundary ω equals ρ, the form $\widetilde{\omega}_\psi$ is well defined.

Definition 2.2.9 *The symplectic manifold* $\left(\widetilde{M}, \widetilde{\omega}_\psi\right)$, *as constructed above, is called the* ***blow up of*** (M,ω) ***of weight*** λ.

Chapter 3

Equivalent symplectic embeddings

Following [5] and [19] we will give an introduction to the basic number theoretical concepts in Section 3.1. Then we explain the canonical decomposition of $E(1,a)$, with $a \geq 1$ rational, into a disjoint union of balls in Section 3.2. We will represent open balls and open polydiscs in a different way than products. These representations will be essential to prove the equivalence of symplectic embeddings of an ellipsoid into a polydisc and the embedding of its decomposition into disjoint balls into a ball (see Proposition 3.3.3).

3.1 Number theoretical concepts: continued fractions and weight expansions

Definition 3.1.1 *Let $a \in \mathbb{Q}_+$, that is $a = \frac{p}{q}$ for some $p, q \in \mathbb{N}$. The* ***weight expansion***

$$\mathbf{w}(a) := (w_1, \dots, w_M)$$

of $a \geq 1$ is defined recursively as

- *$w_1 := 1$ and $w_i \geq w_{i+1}$ for all i,*
- *if $w_i > w_{i+1} = \ldots = w_n$, where $w_0 := a$, then*

$$w_{n+1} = \begin{cases} w_n, & \textit{if } w_{i+1} + \ldots + w_{n+1} = (n-i+1)w_{i+1} \leq w_i \\ w_i - (n-i)w_{i+1}, & \textit{otherwise,} \end{cases}$$

- *the sequence stops at w_n if the above formula gives $w_{n+1} = 0$.*

The i-th entry $w_i(a)$ of the weight expansion $\mathbf{w}(a)$ is called the ***i-th weight of*** *a. The number $\ell(a)$ of nonzero entries of $\mathbf{w}(a)$ is called the* ***length*** *of a. We will write shortly $\mathbf{w}$ for $\mathbf{w}(a)$ if there is no danger of confusion.*

For $p, q \in \mathbb{N}$ coprime, i.e. if $a = \frac{p}{q}$ is given in lowest terms, we will call

$$\mathbf{W}(a) := q \cdot \mathbf{w}(a) \tag{3.1}$$

the ***normalized weight expansion****.*

Example and notation 3.1.2 *Let $a = \frac{50}{9}$. Then the weight expansion is given by*

$$\mathbf{w}\left(\frac{50}{9}\right) = \left(1, 1, 1, 1, 1, \frac{5}{9}, \frac{4}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right).$$

For a visualization see Figure 3.1. Here it is illustrated how one obtains the weight expansion by cutting a rectangle into squares.

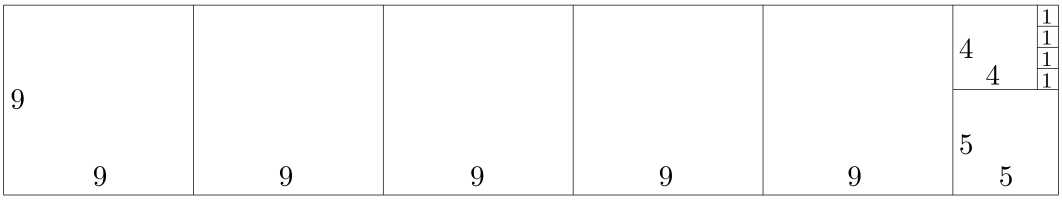

Figure 3.1: The weight expansion for $a = \frac{50}{9}$.

Observing that the weight expansion of $a = \frac{p}{q} \geq 1$ consists of $N+1$ blocks on which the weights w_i of a are constant, we obtain

$$\mathbf{w}(a) := (\underbrace{x_0, \dots, x_0}_{l_0}, \underbrace{x_1, \dots, x_1}_{l_1}, \dots, \underbrace{x_N, \dots, x_N}_{l_N}).$$

Since $a \geq 1$, we have $x_0 = 1$. Thus $x_1 = a - l_0 < 1$ and for all $2 \leq i \leq N$, $x_i = x_{i-2} - l_{i-1} \cdot x_{i-1}$. Moreover, the length of each block gives the **continued fraction expansion** of a, since

$$a = l_0 + \cfrac{1}{l_1 + \cfrac{1}{l_2 + \cfrac{1}{\ddots + \cfrac{1}{l_N}}}} =: [l_0; l_1, \dots, l_N] \, .$$

Thus, one can write the weight expansion of a shortly as

$$\mathbf{w}(a) := \left(x_0^{\times l_0}, x_1^{\times l_1}, \dots, x_N^{\times l_N} \right)$$

and read the continued fraction expansion from the multiplicities:

$$a = [l_0; l_1, \dots, l_N] \, .$$

Example 3.1.3 *The weight expansion of* $a = \frac{50}{9}$ *can be shortly written as*

$$\mathbf{w}\left(\frac{50}{9}\right) = \left(1^{\times 5}, \frac{5}{9}, \frac{4}{9}, \frac{1}{9}^{\times 4}\right),$$

where one can read the continued fraction expansion from the multiplicities:

$$\frac{50}{9} = [5; 1, 1, 4] \, .$$

Indeed,

$$5 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{4}}} = \frac{50}{9}.$$

A very useful tool for our work is the following lemma from McDuff and Schlenk [19, Lemma 1.2.6].

Lemma 3.1.4 (McDuff-Schlenk [19], Lemma 1.2.6) *Let* $\mathbf{w} := (w_1, \dots, w_M)$ *be the weight expansion of* $a = \frac{p}{q} \geq 1$*, where* $a = \frac{p}{q}$ *is given in lowest terms. Then*

(i) $w_M = \frac{1}{q}$,

(ii) $\|\mathbf{w}\|^2 = \langle \mathbf{w}, \mathbf{w} \rangle = \sum_{i=1}^{M} w_i^2 = a$,

(iii) $\sum_{i=1}^{M} w_i = a + 1 - \frac{1}{q}$.

Notation 3.1.5 *For vectors* $\mathbf{v} \in \mathbb{R}^s, \mathbf{w} \in \mathbb{R}^t$*, with* $s \leq t$*, define*

$$\langle \mathbf{v}, \mathbf{w} \rangle := \sum_{i=1}^{s} v_i w_i,$$

i.e. the Euclidean scalar product on $\mathbb{R}^s$*.*

3.2 Several representations of open ellipsoids and open balls

Recall that we defined the open ellipsoid of area a_1, a_2 in Subsection 2.1.2 as

$$E(a_1, a_2) := \left\{ (x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \;\Big|\; \frac{\pi(x_1^2 + y_1^2)}{a_1} + \frac{\pi(x_2^2 + y_2^2)}{a_2} < 1 \right\},$$

where $0 < a_1 \leq a_2$, and the open polydisc

$$P(a_1, a_2) := D^2(a_1) \times D^2(a_2),$$

where $D^2(a)$ is the open disc in $\mathbb{R}^2$ of area a. An open ball in $\mathbb{R}^4$ is denoted by $B(a) := E(a, a)$ and $C(a) := P(a, a)$ denotes the open cube (see Example 2.1.7).

Now let

$$\square(a, b) := (0, a) \times (0, b)$$

denote the open square in $\mathbb{R}^2$. Since the open square $(0, a) \times (0, 1)$ is symplectomorphic to the open discs $D^2(a)$, the open polydisc $P(a, b) := D^2(a) \times D^2(b)$ is symplectomorphic to the open cube

$$(0, a) \times (0, 1) \times (0, b) \times (0, 1) = \square(a, b) \times \square(1, 1) \subset \mathbb{R}^2(x) \times \mathbb{R}^2(y), \tag{3.2}$$

where $\mathbb{R}^2(x)$, $\mathbb{R}^2(y)$ denote the (x_1, x_2)-plane, (y_1, y_2)-plane, respectively. Consider the simplex

$$\triangle(a) := \left\{ (x_1, x_2) \in \mathbb{R}^2(x) \mid x_1, x_2 > 0 \text{ and } x_1 + x_2 < a \right\}.$$

Traynor shows in [22, Proposition 5.2] that the open ball $B(a)$ is symplectomorphic to the product

$$\triangle\,(a) \times \square(1, 1) \subset \mathbb{R}^2(x) \times \mathbb{R}^2(y). \tag{3.3}$$

3.2.1 The image of the moment map

We explain how one can represent a 4-dimensional symplectic manifold by a polytope in $\mathbb{R}^2$. Therefore we first describe this representation for an open polydisc $P(c, d) \subset \mathbb{R}^4$.

Denote by ω_{SF} the Study-Fubini form on the complex projective plane $\mathbb{C}P^2$, normalized by $\int_{\mathbb{C}P^1} \omega_{SF} = 1$. For $\left(\mathbb{C}P^2, a\,\omega_{SF}\right)$ we will write shortly $\mathbb{C}P^2(a)$. Its affine part $\mathbb{C}P^2 \setminus \mathbb{C}P^1$ is symplectomorphic to the open ball $B(a)$:

Remark 3.2.0.1 For $a = \pi$, the embedding

$$z = (z_1, z_2) \;\mapsto\; \left[z_1 : z_2 : \sqrt{1 - |z|^2} \right]$$

is symplectic.

The standard T^2-actions on $\mathbb{C}^2$ and $\mathbb{C}P^2$ are given by

$$\begin{aligned} T^2 :\ \mathbb{C}^2 &\longrightarrow \mathbb{C}^2 \\ (z_1, z_2) &\longmapsto \left(e^{i2\pi\rho_1} z_1, e^{i2\pi\rho_2} z_2\right), \end{aligned}$$

where $\rho_1, \rho_2 \in [0,1]$, and

$$\begin{aligned} T^2 :\ \mathbb{C}P^2 &\longrightarrow \mathbb{C}P^2 \\ [z_0 : z_1 : z_2] &\longmapsto \left[z_0 : e^{i2\pi\rho_1} z_1 : e^{i2\pi\rho_2} z_2\right], \end{aligned}$$

where $[z_0 : z_1 : z_2]$ denote homogeneous coordinates in $\mathbb{C}P^2$.

The corresponding moment maps μ are

$$\begin{aligned} \mu :\ \mathbb{C}^2 &\longrightarrow \mathbb{R}^2_{\geq 0} \\ (z_1, z_2) &\longmapsto \left(\pi|z_1|^2, \pi|z_2|^2\right) \end{aligned}$$

and

$$\begin{aligned} \mu :\ \mathbb{C}P^2 &\longrightarrow \mathbb{R}^2_{\geq 0} \\ [z_0 : z_1 : z_2] &\longmapsto \frac{\left(\pi|z_1|^2, \pi|z_2|^2\right)}{|z_0|^2 + |z_1|^2 + |z_2|^2}. \end{aligned}$$

Following [11], the image of the open ball $B(A)$ is the open triangle

$$\left\{(x, y) \mid x + y < A\right\} \subset \mathbb{R}^2_{\geq 0}.$$

For an illustration see Figure 3.2.

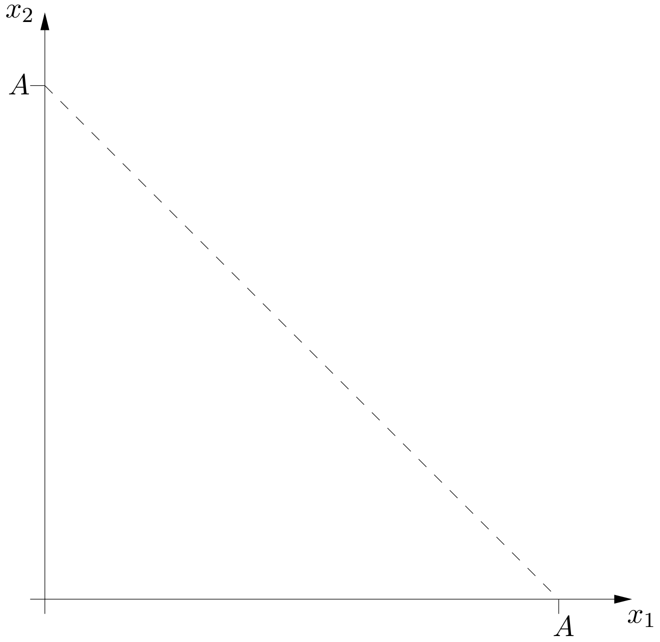

Figure 3.2: The image of an open ball $B(A)$ under the moment map μ in $\mathbb{R}^2_{\geq 0}$.

Remark 3.2.0.2 Permuting the coordinates of $\mathbb{C}P^2$, one can identify neighbourhoods of $[1 : 0 : 0]$, $[0 : 1 : 0]$ and $[0 : 0 : 1]$ in $\mathbb{C}P^2$ with neighbourhoods of 0 in $\mathbb{C}^2$ and thus we obtain the image of a ball of the moment map as in Figure 3.3, see [11].

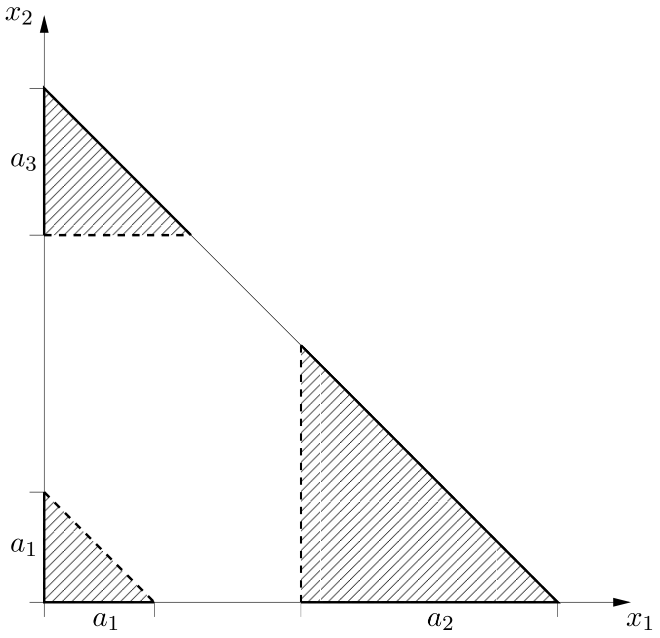

Figure 3.3

Considering now the polydisc $P(c,d)$ under the image of the moment map of the usual T^2-action on $\mathbb{C}^2$, $P(c,d)$ is mapped to the rectangle $[0,c) \times [0,d) \subset \mathbb{R}^2(x)$, see Figure 3.4.

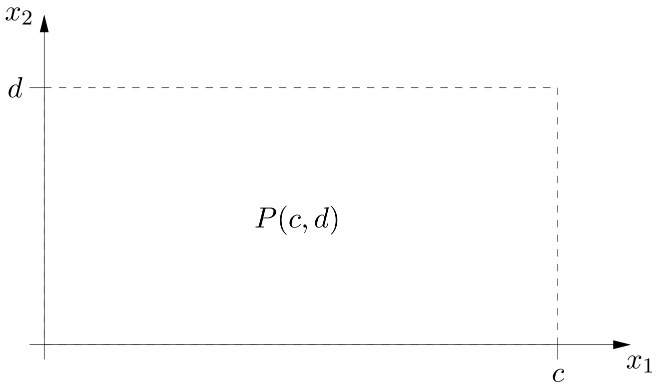

Figure 3.4: The moment image of $P(c,d)$ in $\mathbb{R}^2(x)$.

3.3 Equivalence of symplectic embeddings

McDuff proved in [15] the following

Theorem 3.3.1 (McDuff [15]) *Let $a,b>0$ be two rational numbers. Then there exists a finite sequence $(w_1,\dots,w_M)$ of rational numbers such that the closed ellipsoid $\overline{E(a,b)}$ symplectically embeds into the ball $B(A)$ if and only if the disjoint union of balls $\sqcup_i \overline{B(w_i)}$ symplectically embeds into $B(A)$.*

Notation 3.3.2 *Let $a, b > 0$ such that $\frac{a}{b}$ is rational. Without loss of generality we assume that $\frac{a}{b}$ is given in lowest terms. Then $B(a, b)$ denotes the canonical decomposition of an ellipsoid $E(a, b)$ into M disjoint balls, that is*

$$B(a,b) := \bigsqcup_i B(w_i), \tag{3.4}$$

where w_i denotes the i-th entry of the weight expansion $\mathbf{w}(\frac{a}{b}) = (w_1, \ldots, w_M)$.

For more details on the canonical decomposition of an ellipsoid into disjoint balls by the weight expansion see [15].

We will now prove the following proposition with the help of the constructions in Section 3.2 as in [5].

Proposition 3.3.3 *Let $a, b, c, d > 0$ with $\frac{a}{b}$ rational. Then there exists a symplectic embedding $E(a, b) \overset{s}{\hookrightarrow} P(c, d)$ if and only if there exists a symplectic embedding*

$$B(a,b) \sqcup B(c) \sqcup B(d) \overset{s}{\hookrightarrow} B(c+d).$$

Proof. Let $a, b, c, d > 0$ with $\frac{a}{b}$ rational given in lowest terms. We need to show that

$$E(a,b) \overset{s}{\hookrightarrow} P(c,d) \iff B(a,b) \sqcup B(c) \sqcup B(d) \overset{s}{\hookrightarrow} B(c+d),$$

where $B(a, b)$ denotes the canonical decomposition of an ellipsoid $E(a, b)$ into M disjoint balls:

$$E(a,b) = \bigsqcup_i B(w_i),$$

where w_i denotes the i-th entry of the weight expansion $\mathbf{w}(\frac{a}{b}) = (w_1, \ldots, w_M)$. Given an embedding

$$E(a,b) \overset{s}{\hookrightarrow} P(c,d),$$

decomposing $E(a, b)$ into finitely many disjoint balls $B(a, b)$ (see (3.4)) as in [15], yields that

$$B(a,b) \overset{s}{\hookrightarrow} P(c,d) \tag{3.5}$$

and hence

$$(1-\varepsilon)\overline{B(a,b)} \overset{s}{\hookrightarrow} P(c,d)$$

holds for any $\varepsilon > 0$.

Using the presentation given in Section 3.2, we obtain with (3.2) that the open polydisc $P(c, d)$ is symplectomorphic to

$$\square(c,d) \times \square(1,1) \subset \mathbb{R}^2(x) \times \mathbb{R}^2(y).$$

Similarly, using the presentation (3.3) from Section 3.2, the open ball $B(c)$ can be seen as

$$\triangle(c) \times \square(1,1) \subset \mathbb{R}^2(x) \times \mathbb{R}^2(y).$$

As mentioned above, as $B(a, b) \overset{s}{\hookrightarrow} P(c, d)$, also $(1-\varepsilon)\overline{B(a,b)} \overset{s}{\hookrightarrow} P(c, d)$ holds for $\varepsilon > 0$. From the embedding shown in Figure 3.5 we thus obtain the symplectic embedding

$$(1-\varepsilon)\overline{B(a,b)} \sqcup B(c) \sqcup B(d) \overset{s}{\hookrightarrow} B(c+d). \tag{3.6}$$

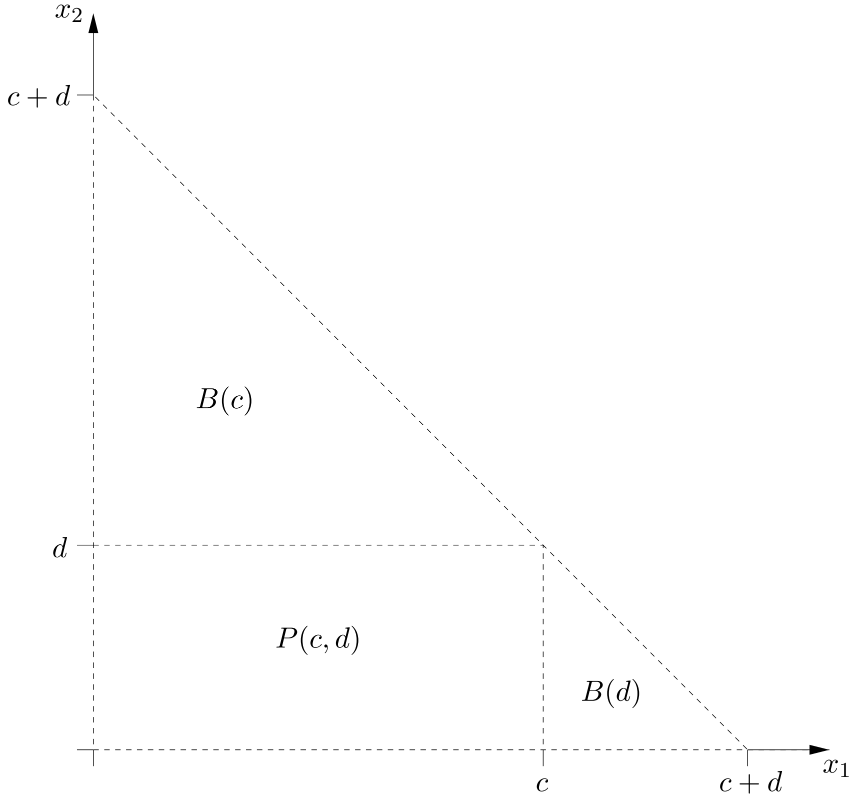

Figure 3.5: $B(c) \sqcup B(d) \sqcup P(c,d) \subset B(c+d)$ represented in $\mathbb{R}^2(x)$.

The symplectic embedding (3.6) holds for every $\varepsilon > 0$. Thus with McDuff's Theorem 1.1 in [15] it follows that

$$B(a,b) \sqcup B(c) \sqcup B(d) \overset{s}{\hookrightarrow} B(c+d)$$

holds.

We assume now that $B(a,b) \sqcup B(c) \sqcup B(d)$ symplectically embeds into $B(c+d)$ and verify that then also

$$E(a,b) \overset{s}{\hookrightarrow} P(c,d)$$

holds.

As $B(a,b) \sqcup B(c) \sqcup B(d) \overset{s}{\hookrightarrow} B(c+d)$, also

$$(1-\varepsilon)\overline{B(a,b)} \sqcup \overline{B(c-\varepsilon)} \sqcup \overline{B(d-\varepsilon)} \overset{s}{\hookrightarrow} B(c+d)$$

for all $\varepsilon > 0$, and then also

$$(1-\varepsilon)\overline{B(a,b)} \sqcup \overline{B(c-\varepsilon)} \sqcup \overline{B(d-\varepsilon)} \overset{s}{\hookrightarrow} \mathbb{C}P^2(c+d).$$

We can assume that the balls $\overline{B(c-\varepsilon)}$ and $\overline{B(d-\varepsilon)}$ lie in $\mathbb{C}P^2(c+d)$ as illustrated in Figure 3.6. According to [14], the space of symplectic embeddings of $\overline{B(c-\varepsilon)} \sqcup \overline{B(d-\varepsilon)}$ into $\mathbb{C}P^2(c+d)$ is connected. Thus, as any such isotopy extends to an ambient symplectic isotopy of $\mathbb{C}P^2(c+d)$, the balls $\overline{B(c-\varepsilon)}$ and $\overline{B(d-\varepsilon)}$ can be moved by an ambient symplectic isotopy of $\mathbb{C}P^2(c+d)$ to the location in Figure 3.6.

From this we can conclude that the image of the disjoint balls $(1-\varepsilon)\overline{B(a,b)}$ must lie over the shaded closed region in Figure 3.6. However, since the balls $\overline{B(c-\varepsilon)}$ and $\overline{B(d-\varepsilon)}$ are closed,

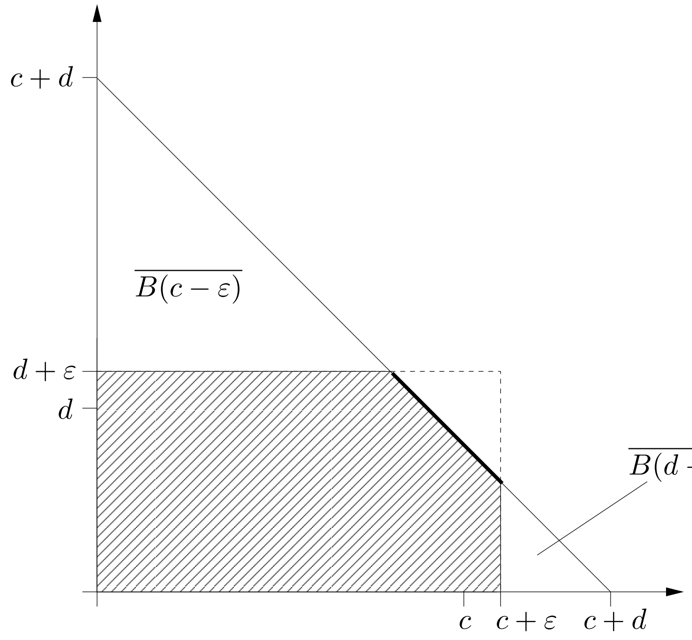

Figure 3.6: The location of $\overline{B(c-\varepsilon)}$ and $\overline{B(d-\varepsilon)}$ in $\mathbb{C}P^2(c+d)$.

the image of $(1-\varepsilon)\overline{B(a,b)}$ under the moment map cannot touch the upper horizontal or the right vertical boundary of the shaded region.

Moreover, according to [17, Remark 2.1.E], we can assume that this image lies in the affine part of $\mathbb{C}P^2(c+d)$, that is the image of the disjoint balls $(1-\varepsilon)\overline{B(a,b)}$ lies in $\mathbb{C}P^2(c+d)\setminus\mathbb{C}P^1$ not touching $\mathbb{C}P^1$ (see Figure 3.7). Thus the image of $(1-\varepsilon)\overline{B(a,b)}$ lies over the shaded region without the dark segment, and hence, by Subsection 3.2.1, in $P(c+\varepsilon,d+\varepsilon)$.

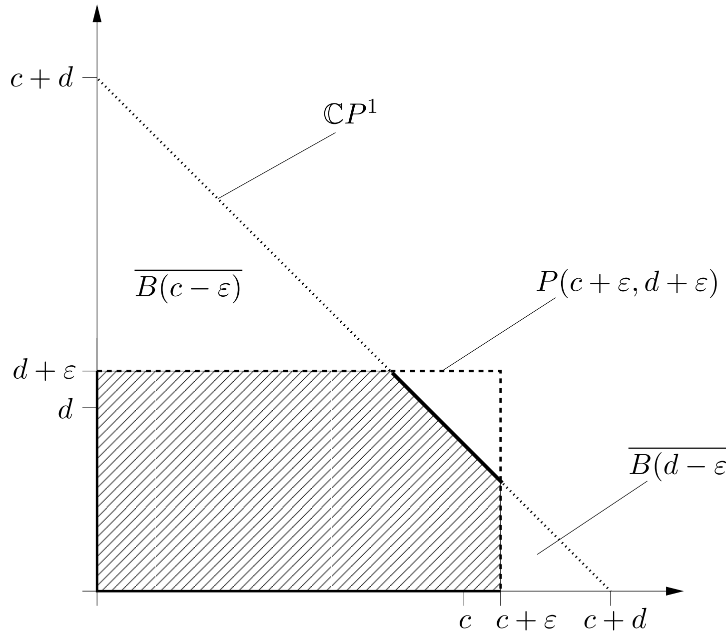

Figure 3.7: The moment picture of $\mathbb{C}P^2(c+d)$ with the affine part $\mathbb{C}P^2(c+d)\setminus\mathbb{C}P^1 \overset{s}{\simeq} B(c+d)$.

Because of the scaling property of symplectic embeddings, we can assume without loss of generality that $c, d \geq 1$. Then

$$P(c+\varepsilon, d+\varepsilon) \subset (1+\varepsilon)P(c,d),$$

and thus

$$(1-\varepsilon)\overline{B(a,b)} \overset{\mathrm{s}}{\hookrightarrow} P(c+\varepsilon, d+\varepsilon) \subset (1+\varepsilon)\, P\,(c,d)\,.$$

Following the same arguments as McDuff for Theorem 1.5 in [15] (McDuff considers embeddings of ellipsoids into open balls), then also

$$(1-\varepsilon)\overline{E(a,b)} \overset{\mathrm{s}}{\hookrightarrow} (1+\varepsilon)P(c,d),$$

hence

$$\left(\frac{1-\varepsilon}{1+\varepsilon}\right)\overline{E(a,b)} \overset{\mathrm{s}}{\hookrightarrow} P(c,d).$$

Finally, using the connectedness of the space of symplectic embeddings of an ellipsoid into $P(c,d)$ (see [15]) we can conclude that

$$E(a,b) \overset{\mathrm{s}}{\hookrightarrow} P(c,d).$$

■

Chapter 4

Relations to ECH-capacities

In this chapter we present a more combinatorial, but non-explicit, way of describing the embedding function $c_{EC}(a)$, that has been found by Hutchings. In [9] he assigned a sequence of real numbers, called the *ECH-capacities*, to each four-dimensional Liouville domain. Among others, he computes the ECH-capacities of ellipsoids, polydiscs and disjoint unions of these in [9].

4.1 Hutchings' ECH-capacities

We will first describe Hutchings ECH-capacities of ellipsoids and polydiscs. Let therefore $a, b \geq 0$. Recall that the ellipsoid $E(a,b)$ and the polydisc $P(a,b)$ are defined as

$$E(a,b) := \left\{ (x_1, y_1, x_2, y_1) \in \mathbb{R}^4 \;\middle|\; \frac{\pi(x_1^2+y_1^2)}{a} + \frac{\pi(x_2^2+y_2^2)}{b} < 1 \right\}$$

and

$$P(a,b) := D^2(a) \times D^2(b),$$

where $D^2(a)$ denotes the 2-disc in $\mathbb{R}^2$.

Let $N_E(a,b)$ denote the sequence that is formed by arranging all numbers of the form $ma+nb$, with $m, n \in \mathbb{N}_0$, in nondecreasing order (with multiplicities). Then for $k \geq 0$, $N_E(a,b)_k$ denotes the k-th entry of the sequence $N_E(a,b)$, counted with repetitions.

In [9], Hutchings proves the following.

Proposition 4.1.1 (Hutchings, Proposition 1.2, [9]) *The k-th ECH-capacity of an ellipsoid is given by*

$$c_{ECH}^k\left(E(a,b)\right) = N_E(a,b)_{k+1},$$

for $k \geq 0$.

Remark 4.1.1.1 The numbers $c_{ECH}^k(E(a,b))$ are called the **ECH-capacities of** $E(a,b)$.

Moreover, for polydiscs, Hutchings proved in [9] that

Theorem 4.1.2 (Hutchings, Theorem 1.4, [9]) *The ECH-capacities of a polydisc are given by*

$$c_{ECH}^k\left(P(a,b)\right) = \min\left\{ma+nb \mid m,n \in \mathbb{N}_0;\; (m+1)(n+1) \geq k+1\right\},$$

for any $k \geq 0$.

Examples 4.1.3 *(a) For $E(1,2)$, the ECH-capacity is*

$$c_{ECH}(E(1,2)) = \{0,1,2,2,3,3,4,4,4,5,5,5,6,6,6,6,7,7,7,7,\dots\}$$

that is, written in short terms,

$$c_{ECH}(E(1,2)) = \left\{0^{\times 1},1^{\times 1},2^{\times 2},3^{\times 2},4^{\times 3},5^{\times 3},6^{\times 4},7^{\times 4},\dots\right\}.$$

The 6-th ECH-capacity is given by

$$c^6_{ECH}(E(1,2)) = \{4\}.$$

(b) The ECH-capacity for the ball $B(1) = E(1,1)$ is

$$\begin{aligned} c_{ECH}(B(1)) &= c_{ECH}(E(1,1)) \\ &= \{0,1,1,2,2,2,3,3,3,3,4,4,4,4,4,5,5,5,5,5,5,6,6,6,6,6,6,6,\dots\}, \end{aligned}$$

that is, written in short terms,

$$c_{ECH}(B(1)) = \left\{0^{\times 1},1^{\times 2},2^{\times 3},3^{\times 4},4^{\times 5},5^{\times 6},6^{\times 7},7^{\times 8},8^{\times 9},9^{\times 10},\dots\right\}.$$

(c) The ECH-capacity for the cube $C(1) = P(1,1)$ is

$$\begin{aligned} c_{ECH}(C(1)) &= c_{ECH}(P(1,1)) \\ &= \{0,1,2,2,3,3,4,4,4,5,5,5,6,6,6,6,7,7,7,7,8,8,8,8,8,9,9,9,9,9,\dots\}, \end{aligned}$$

that is, written in short terms

$$c_{ECH}(C(1)) = \left\{0^{\times 1},1^{\times 1},2^{\times 2},3^{\times 2},4^{\times 3},5^{\times 3},6^{\times 4},7^{\times 4},8^{\times 5},9^{\times 5},\dots\right\}.$$

4.2 The equality of the ECH-capacity of the ellipsoid $E(1,2)$ and the cube $C(1)$

Following [5], we will now prove

Lemma 4.2.1 *The equality*

$$c^k_{ECH}\left(E(1,2)\right) = c^k_{ECH}\left(C(1)\right)$$

holds for all $k \in \mathbb{N}_0$.

Remark 4.2.1.1 In Example 4.1.3, we calculated the ECH-capacities of $E(1,2)$ and $C(1)$:

$$c_{ECH}(E(1,2)) = \left\{0^{\times 1},1^{\times 1},2^{\times 2},3^{\times 2},4^{\times 3},5^{\times 3},6^{\times 4},7^{\times 4},8^{\times 5},9^{\times 5},\dots\right\},$$

$$c_{ECH}(C(1)) \quad = \left\{0^{\times 1},1^{\times 1},2^{\times 2},3^{\times 2},4^{\times 3},5^{\times 3},6^{\times 4},7^{\times 4},8^{\times 5},9^{\times 5},\dots\right\}.$$

For any $k \in \mathbb{N}$, there exists exactly one positive integer $d > 0$, such that

$$\left\lfloor \frac{d+1}{2} \right\rfloor \left\lceil \frac{d+1}{2} \right\rceil \leq k < \left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil, \tag{4.1}$$

where for $x \in \mathbb{R}$,

$$\lfloor\, x\, \rfloor := \max\{n \in \mathbb{N} | n \leq x\}$$

and

$$\lceil\, x\, \rceil := \min\{n \in \mathbb{N} | n \geq x\}$$

are the *floor* and *ceiling function.*

Let us look at the 6-th ECH-capacity:

$$\left\lfloor \frac{d+1}{2} \right\rfloor \left\lceil \frac{d+1}{2} \right\rceil \leq 6 < \left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil . \tag{4.2}$$

The only positive integer $d > 0$ fulfilling (4.2) is $d = 4$:

$$\left\lfloor \tfrac{4+1}{2} \right\rfloor \left\lceil \tfrac{4+1}{2} \right\rceil \leq 6 < \left\lfloor \tfrac{4+2}{2} \right\rfloor \left\lceil \tfrac{4+2}{2} \right\rceil .$$

$$\left\lfloor \tfrac{5}{2} \right\rfloor \left\lceil \tfrac{5}{2} \right\rceil \leq 6 < \left\lfloor \tfrac{6}{2} \right\rfloor \left\lceil \tfrac{6}{2} \right\rceil .$$

$$2 \cdot 3 \leq 6 < 3 \cdot 3.$$

$$6 \leq 6 < 9.$$

Now looking at $c_{ECH}(E(1,2))$ and $c_{ECH}(C(1))$, we see

$$c^6_{ECH}(E(1,2)) = 4 \qquad \text{and} \qquad c^6_{ECH}(C(1)) = 4.$$

Recall, that the first number in the ordered sequences $c^k_{ECH}(E(a,b))$ and $c^k_{ECH}\left(P(a,b)\right)$ is $c^0_{ECH}(E(a,b))$ and $c^0_{ECH}\left(P(a,b)\right)$, respectively, as $k \geq 0$.

To prove Lemma 4.2.1 we will show by induction over $d \in \mathbb{N}$ that

$$c^k_{ECH}\left(E(1,2)\right) = d = c^k_{ECH}\left(C(1)\right),$$

where d is determined as the unique integer fulfilling

$$\left\lfloor \frac{d+1}{2} \right\rfloor \left\lceil \frac{d+1}{2} \right\rceil \leq k < \left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil$$

for $k \in \mathbb{N}_0$ given.

Proof of Lemma 4.2.1. We will prove that for any $k \in \mathbb{N}_0$, $c^k_{ECH}\left(E(1,2)\right)$ and $c^k_{ECH}\left(C(1)\right)$ are both equal to the unique positive integer $d > 0$, such that

$$\left\lfloor \frac{d+1}{2} \right\rfloor \left\lceil \frac{d+1}{2} \right\rceil \leq k < \left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil . \tag{4.3}$$

For $k = 0$ this is clearly true, as $c^0_{ECH}(E(a,b)) = c^0_{ECH}(P(a,b)) = 0$. Thus in the following we will consider $k > 0$ fixed.

We will start by looking at $c^k_{ECH}\left(E(1,2)\right)$. Therefore we analyse the number of pairs of non-negative integers $(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0$, such that

$$m + 2n \leq d, \tag{4.4}$$

that is

$$\#\left\{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid m + 2n \leq d\right\}.$$

Modifying (4.4) to

$$m + 2n \leq 2l,$$

for the case that d is even, and

$$m + 2n \leq 2l + 1,$$

for the case that d is odd, it holds that

$$\#\left\{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid m + 2n = 2l\right\} = \#\left\{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid m + 2n = 2l + 1\right\} = l + 1. \quad (4.5)$$

Now, we suppose that

$$\left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil = \#\left\{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid m + 2n \leq d\right\}, \quad (4.6)$$

holds for $(d-1) \in \mathbb{N}$. Thus for $d = 2l$ even, one concludes with (4.6) and (4.5), that

$$\begin{aligned}
\#\left\{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 : m + 2n \leq d\right\} &= \#\left\{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid m + 2n \leq d - 1\right\} \\
&\quad + \#\left\{(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0 \mid m + 2n = d\right\} \\
&= \left\lfloor \tfrac{(d-1)+2}{2} \right\rfloor \left\lceil \tfrac{(d-1)+2}{2} \right\rceil + (l+1) \\
&= \left\lfloor \tfrac{d+1}{2} \right\rfloor \left\lceil \tfrac{d+1}{2} \right\rceil + (l+1) \\
&= \left\lfloor \tfrac{2l+1}{2} \right\rfloor \left\lceil \tfrac{2l+1}{2} \right\rceil + (l+1) \\
&= l \cdot (l+1) + (l+1) \\
&= (l+1) \cdot (l+1) \\
&= \left\lfloor \tfrac{2l+2}{2} \right\rfloor \left\lceil \tfrac{2l+2}{2} \right\rceil \\
&= \left\lfloor \tfrac{d+2}{2} \right\rfloor \left\lceil \tfrac{d+2}{2} \right\rceil.
\end{aligned}$$

For d odd the argument is the same.

Thus, as equation (4.6) holds for the unique $d \in \mathbb{N}$ that satisfies (4.3) for a fixed $k \in \mathbb{N}$, it follows that $d = c^k_{ECH}(E(1,2))$ for any $k \geq 0$.

Now for $c^k_{ECH}(C(1))$, we have by Theorem 4.1.2 that

$$c^k_{ECH}\left(C(1)\right) = c^k_{ECH}\left(P(1,1)\right) = \min\left\{m + n \mid m, n \in \mathbb{N}_0;\ (m+1)(n+1) \geq k + 1\right\}. \quad (4.7)$$

Again, as mentioned above, we consider $k > 0$ fixed. Let $m_0, n_0 \in \mathbb{N}_0$ be two nonnegative integers such that

$$m_0 + n_0 := \min\left\{m + n \mid (m+1)(n+1) \geq k + 1\right\}. \quad (4.8)$$

Without loss of generality we can assume that $m_0 \geq n_0$. Moreover, we can take $m_0, n_0 \in \mathbb{N}_0$ such that $(m_0 - n_0) \in \{0, 1\}$: Assume that $m_0 = n_0 + c$ with $c \geq 2$ and let $m'_0 = m_0 - 1$ and $n'_0 = n_0 + 1$. Then we get

$$\begin{aligned}
\left(m'_0 + 1\right)\left(n'_0 + 1\right) &= m_0\left(n_0 + 2\right) \\
&= \left(n_0 + c\right)\left(n_0 + 2\right) \\
&= n_0^2 + (c+2)n_0 + 2c \\
&> n_0^2 + (c+2)n_0 + c + 1 \\
&= \left(n_0 + c + 1\right)\left(n_0 + 1\right) \\
&= \left(m_0 + 1\right)\left(n_0 + 1\right) \\
&\geq k + 1.
\end{aligned}$$

Thus, as $m_0' + n_0' = m_0 + n_0$, also (m_0', n_0') realizes the minimum (compare with (4.7)). Now, if $m_0 + n_0$ is even, then $m_0 = n_0$, as $(m_0 - n_0) \in \{0,1\}$, and hence if $m_0 + n_0$ is odd, then $m_0 = n_0 + 1$.

Setting now $d := m_0 + n_0$, for the case that $m_0 + n_0$ is even, that is $n_0 = m_0$, we obtain

$$\left\lfloor \frac{d+1}{2} \right\rfloor \left\lceil \frac{d+1}{2} \right\rceil = \left\lfloor \frac{2m_0+1}{2} \right\rfloor \left\lceil \frac{2m_0+1}{2} \right\rceil = m_0 \cdot (m_0+1)$$

and

$$\left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil = \left\lfloor \frac{2m_0+2}{2} \right\rfloor \left\lceil \frac{2m_0+2}{2} \right\rceil = (m_0+1)^2 .$$

Now as $m_0 + n_0$ was chosen minimal, see (4.8), it follows that

$$\left\lfloor \frac{d+1}{2} \right\rfloor \left\lceil \frac{d+1}{2} \right\rceil = m_0 \cdot (m_0+1) \leq k$$

and

$$\left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil = (m_0+1)^2 > m_0 (m_0+1) \geq k+1 > k.$$

Similarly for the case that $m_0 + n_0$ is odd, that is $n_0 = m_0 - 1$, we obtain that

$$\left\lfloor \frac{d+1}{2} \right\rfloor \left\lceil \frac{d+1}{2} \right\rceil = \left\lfloor \frac{2m_0}{2} \right\rfloor \left\lceil \frac{2m_0}{2} \right\rceil = m_0^2$$

and

$$\left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil = \left\lfloor \frac{2m_0+1}{2} \right\rfloor \left\lceil \frac{2m_0+1}{2} \right\rceil = m_0 (m_0+1) .$$

Now as $m_0 + n_0$ was chosen minimal, see (4.8), it follows that

$$\left\lfloor \frac{d+1}{2} \right\rfloor \left\lceil \frac{d+1}{2} \right\rceil \leq m_0^2 \leq k+1 < k$$

and

$$\left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil = m_0 (m_0+1) \geq k+1 > k.$$

For both cases ($m_0 + n_0$ even and odd) we thus obtain for $m_0, n_0 \in \mathbb{N}_0$ that

$$\left\lfloor \frac{d+1}{2} \right\rfloor \left\lceil \frac{d+1}{2} \right\rceil = \left\lfloor \frac{m_0+n_0+1}{2} \right\rfloor \left\lceil \frac{m_0+n_0+1}{2} \right\rceil \leq m_0 \cdot (m_0+1)$$

and

$$\left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil = \left\lfloor \frac{m_0+n_0+1}{2} \right\rfloor \left\lceil \frac{m_0+n_0+1}{2} \right\rceil \geq m_0 (m_0+1) .$$

Now as $m_0 + n_0$ was chosen minimal, see (4.8), it follows that

$$\left\lfloor \frac{d+1}{2} \right\rfloor \left\lceil \frac{d+1}{2} \right\rceil \leq m_0 \cdot (m_0+1) \leq k$$

and

$$\left\lfloor \frac{d+2}{2} \right\rfloor \left\lceil \frac{d+2}{2} \right\rceil \geq m_0 (m_0+1) \geq k+1 > k.$$

Thus it follows that $d = c_{ECH}^k(C(1))$, for d satisfying (4.3). ■

4.3 Conclusion - generalization

In [10] Hutchings proved the following:

Corollary 4.3.1 (Hutchings, Corollary 11, [10]) *There exists a symplectic embedding of $E(a,b)$ into $P(c,d)$ if and only if*

$$c^k_{ECH}\left(E(a,b)\right) \leq c^k_{ECH}\left(P(c,d)\right), \quad \textit{for all } k \geq 1.$$

Remark 4.3.1.1 This proposition now implies with Proposition 3.3.3 that ECH-capacities form a complete set of invariants for the problem of symplectic embeddings of ellipsoids into polydiscs.

As a further corollary we obtain

Corollary 4.3.2 *The ellipsoid $E(1,a)$ symplectically embeds into the cube $C(A)$ if and only if $E(1,a)$ symplectically embeds into the ellipsoid $E(A,2A)$.*

Proof. By Corollary 4.3.1, it holds that

$$E(1,a) \overset{\mathrm{s}}{\hookrightarrow} C(A) \iff c^k_{ECH}\left(E(1,a)\right) \leq c^k_{ECH}\left(C(A)\right), \quad \text{for all } k \geq 1.$$

By McDuff's proof of the Hofer Conjecture [16], it is true that

$$E(1,a) \overset{\mathrm{s}}{\hookrightarrow} E(A,2A) \iff c^k_{ECH}\left(E(1,a)\right) \leq c^k_{ECH}\left(E(A,2A)\right),$$

for all $k \geq 1$. Now Hutchings' remark on page 6 in [10] (that is Lemma 4.2.1 in Section 4.2), states that

$$c^k_{ECH}\left(C(1)\right) = c^k_{ECH}\left(E(1,2)\right),$$

for all $k \geq 1$. Thus by conformality of capacities, also

$$c^k_{ECH}\left(C(A)\right) = c^k_{ECH}\left(E(A,2A)\right)$$

holds for any $k \geq 1$. ■

Chapter 5

The role of exceptional spheres

Recall from Chapter 2 the volume constraint $c(a) \geq \sqrt{\frac{a}{2}}$ for the embedding capacity function $c(a)$. Here we explain the constraints coming from certain exceptional spheres in blow-ups of $\mathbb{C}P^2$ for the embedding capacity function $c(a)$ as done in [5] and [19]. Since the function $c(a)$ is continuous (see Lemma 2.1.14 in Chapter 2), it suffices to determine $c(a)$ for each rational number $a \geq 1$.

5.1 Reduction to a constraint function which is given by exceptional spheres

We start with the following lemma, which is a special case of Proposition 3.3.3.

Lemma 5.1.1 *Let $a \geq 1$ be a rational number with weight expansion $\mathbf{w}(a) = (w_1, \ldots, w_M)$ and $A > 0$. Then the ellipsoid $E(1, a)$ embeds symplectically into the cube $C(A)$ if and only if there exists a symplectic embedding*

$$B(A) \sqcup B(A) \sqcup_i B(w_i) \overset{s}{\hookrightarrow} B(2A).$$

Proof. See Proposition 3.3.3. ■

With this lemma, we have converted the problem of embedding an ellipsoid into a cube to the problem of embedding a disjoint union of balls into a ball. In [17], the problem of embedding k disjoint balls into a ball was reduced to the question of understanding the symplectic cone of the k-fold blow-up X_k of $\mathbb{C}P^2$.

Let

$$L := \left[\mathbb{C}P^1\right] \in H_2\left(X_k, \mathbb{Z}\right)$$

be the class of a line and let

$$E_1, \ldots, E_k \in H_2\left(X_k, \mathbb{Z}\right)$$

be the homology classes of the exceptional divisors. Their Poincaré duals are denoted by

$$l, e_1, \ldots, e_k \in H^2\left(X_k, \mathbb{Z}\right).$$

Let $-K := 3L - \sum E_i$ be the anti-canonical divisor of X_k, and define the corresponding **symplectic cone**

$$\mathcal{C}_K\left(X_k\right) \subset H^2\left(X_k, \mathbb{Z}\right)$$

as the set of classes represented by symplectic forms ω with first Chern class $c_1\left(M, \omega\right) = -K$.

Theorem 5.1.2 (McDuff-Polterovich, [17]) *The union* $\sqcup_{i=1}^{k}\overline{B\left(w_i\right)}$ *embeds into the ball* $B(\mu)$ *or into* $\mathbb{C}P^2(\mu)$ *if and only if*

$$\mu l-\sum w_i e_i \in \mathcal{C}_K\left(X_k\right).$$

The key of understanding this cone $\mathcal{C}_K\left(X_k\right)$ is the following set

$$\mathcal{E}_k \subset H_2\left(X_k\right),$$

that we define as in [19]:

Definition 5.1.3 *Denote by* $\mathcal{E}_k$ *the set consisting of*

$$(0;-1,0,\ldots,0)$$

and of all tuples

$$(d;\mathbf{m}):=\left(d;m_1,\ldots,m_k\right),$$

with $d\geq 0$ *and* $m_1\geq\ldots\geq m_k\geq 0$*, such that the class*

$$E_{(d;\mathbf{m})}:=dL-\sum m_i E_i\in H_2\left(X_k\right)$$

is represented in X_k *by a symplectically embedded sphere of self-intersection-number* -1*.*

Remark 5.1.3.1 We will often write $\mathcal{E}$ instead of $\mathcal{E}_k$ if there is no danger of confusion.

We then have the following description of $\mathcal{C}_K\left(X_k\right)$.

Proposition 5.1.4 (Li-Li [12], Li-Liu [13]) *It holds that*

$$\mathcal{C}_K\left(X_k\right)=\left\{\alpha\in H^2\left(X_k\right)\ \mid\ \alpha^2>0;\ \alpha\left(E\right)>0;\ \textit{for all } E\in\mathcal{E}_k\right\}.$$

In order to give a characterization of the set $\mathcal{E}_k$, we need the following definition as in [19]:

Definition 5.1.5 *A tuple*

$$(d;\mathbf{m}):=\left(d;m_1,\ldots m_k\right)$$

is said to be ***ordered*** *if the* m_i *are in nonincreasing order. The* ***Cremona transform*** *of an ordered tuple* $(d;\mathbf{m})$ *is defined as*

$$\left(2d-m_1-m_2-m_3;d-m_2-m_3,d-m_1-m_3,d-m_1-m_2,m_4,\ldots,m_k\right).$$

A ***Cremona move*** *of a tuple* $(d;\mathbf{m})$ *is the composition of the Cremona transform of* $(d;\mathbf{m})$ *with any permutation of the new obtained vector* $\mathbf{m}$*.*

Proposition 5.1.6 (McDuff-Schlenk, Prop. 1.2.12 and Rem. 3.3.1 in [19]) *The following hold true:*

(i) All $(d;\mathbf{m})\in\mathcal{E}_k$ *satisfy the two Diophantine equations*

$$\begin{cases}\displaystyle\sum_{i=1}^{k}m_i=3d-1,\\ \displaystyle\sum_{i=1}^{k}m_i^2=d^2+1.\end{cases}$$

(ii) For all distinct $(d;\mathbf{m}),(d';\mathbf{m}')\in\mathcal{E}_k$ *we have*

$$\sum m_i m_i'\leq dd'.$$

(iii) A tuple $(d;\mathbf{m})$ *belongs to* $\mathcal{E}_k$ *if and only if* $(d;\mathbf{m})$ *satisfies the Diophantine equations in (i) and* $(d;\mathbf{m})$ *can be reduced to* $(0;-1,0,\dots,0)$ *by repeated Cremona moves.*

Remark 5.1.6.1 Working directly with Lemma 5.1.1, Theorem 5.1.2 and Proposition 5.1.4, we find, as in [19], that the only constraints for a symplectic embedding $E(1,a)\overset{s}{\hookrightarrow}C(A)$ are the volume constraint

$$A \geq \sqrt{\frac{a}{2}}$$

(compare Lemma 2.1.13) and, for each class $(d;\mathbf{m}) \in \mathcal{E}_k$, that

$$2Ad \geq (m_1+m_2)\,A + \langle (m_3,\dots,m_k)\,, \mathbf{w}(a)\rangle\,. \tag{5.1}$$

One can start from here and use Proposition 5.1.6 to prove Theorem 1.2.1. The analysis becomes, however, rather awkward, since the unknown A appears on both sides of the inequality (5.1).

To improve the situation, we shall apply a base change of $H_2\left(X_k\right)$, and express the elements of $\mathcal{E}$ in a new basis.

Consider the product $S^2\times S^2$ (whose affine part is a cube), and form the M-fold (topological) blow-up $X_M\left(S^2\times S^2\right)$. A basis of $H_2\left(X_M\left(S^2\times S^2\right)\right)$ is given by $S_1,S_2,F_1,\dots,F_M$, where $S_1 := \left[S^2\times\{\text{point}\}\right]$, $S_2 := \left[\{\text{point}\}\times S^2\right]$ and $F_1,\dots,F_M$ are the classes of the exceptional divisors.

Notice that there is a diffeomorphism

$$\varphi\colon X_M\left(S^2\times S^2\right)\to X_{M+1}\left(\mathbb{C}P^2\right),$$

such that the induced map in homology is

$$\begin{array}{rccl} \varphi_*\colon & H_2\left(X_M\left(S^2\times S^2\right)\right) & \longrightarrow & H_2\left(X_{M+1}\left(\mathbb{C}P^2\right)\right) \\ & S_1 & \longmapsto & L-E_1 \\ & S_2 & \longmapsto & L\qquad -E_2 \\ & F_1 & \longmapsto & L-E_1-E_2 \\ & F_i & \longmapsto & \qquad\qquad E_{i+1}. \end{array}$$

The existence of such a φ is straightforward from a moment map picture such as Figure 3.6 in Chapter 3. With respect to the new basis $S_1,S_2,F_1,\dots,F_M$, let an element of $H_2\left(X_M\left(S^2\times S^2\right)\right)$ be denoted as $(d,e;m_1,\dots,m_M)$. Then

$$\varphi_*(d,e;\mathbf{m}) \;=\; (d+e-m_1;d-m_1,e-m_1,m_2,\dots,m_M)\,.$$

In the new basis, the constraint given by a class in $\mathcal{E}$ can be written in a more useful form:

Proposition 5.1.7 *The following hold true:*

(i) All $(d,e;\mathbf{m}) \in \mathcal{E}_M$ *satisfy the two Diophantine equations*

$$\left\{\begin{array}{l} \displaystyle\sum_{i=1}^{M} m_i = 2(d+e)-1, \\ \displaystyle\sum_{i=1}^{M} m_i^2 = 2de+1. \end{array}\right. \tag{5.2}$$

(ii) For all distinct $(d,e;\mathbf{m}),(d',e';\mathbf{m}')\in\mathcal{E}_M$*, we have*

$$\sum_{i=1}^{M} m_i m_i' \leq de' + d'e.$$

(iii) A tuple $(d,e;\mathbf{m})$ *belongs to* $\mathcal{E}_M$ *if and only if* $(d,e;\mathbf{m})$ *satisfies the Diophantine equations of (i) and its image under* φ_* *can be reduced to* $(0;-1,0,\dots,0)$ *by repeated Cremona moves.*

Proof. Let $E\in\mathcal{E}$. The two identities in Proposition 5.1.6 (i) correspond to $c_1(E)=1$ and $E\cdot E=-1$. For $E=dS_1+eS_2-\sum m_iF_i$ these identities become

$$\begin{aligned} c_1(E) &= 2d+2e-\sum m_i = 1,\\ E\cdot E &= -\sum m_i^2 + 2de = -1, \end{aligned}$$

which is proving (i). Claim (ii) of Proposition 5.1.6 corresponds to positivity of intersection of J-holomorphic spheres representing $E,E'\in\mathcal{E}$. For distinct elements $E=(d,e;\mathbf{m})$ and $E'=(d',e';\mathbf{m}')$ in $\mathcal{E}$ we thus have

$$E\cdot E' = de' + ed' - \sum m_i m_i' \geq 0,$$

which is proving (ii). Claim (iii) holds since φ_* is a base change. ■

Proposition 5.1.8 *Let* $a\geq 1$ *be a rational number with weight expansion* $\mathbf{w}(a)=(w_1,\dots,w_M)$*. For* $(d,e;\mathbf{m})\in\mathcal{E}$*, define the constraint*

$$\mu(d,e;\mathbf{m})(a) := \frac{\langle \mathbf{m},\mathbf{w}(a)\rangle}{d+e}.$$

Then

$$c(a) = \sup_{(d,e;\mathbf{m})\in\mathcal{E}} \left\{ \sqrt{\frac{a}{2}}, \mu(d,e;\mathbf{m})(a) \right\}.$$

Proof. By Lemma 5.1.1, $E(1,a)\overset{s}{\hookrightarrow} C(A)$ if and only if

$$B(A)\sqcup B(A)\sqcup_i B(w_i) \overset{s}{\hookrightarrow} B(2A).$$

By Theorem 5.1.2, this is true if and only if

$$(2A)\,l - A\,e_1 - A\,e_2 - \sum_{i=1}^{M} w_i\,e_{i+2} \;\in\; \mathcal{C}_K. \tag{5.3}$$

Denote by $s_1,s_2,f_1,\dots,f_M$ the Poincaré duals of $S_1,S_2,F_1,\dots,F_M$. The base change in cohomology is then

$$\begin{array}{rccc} \varphi^*\colon & H^2\left(X_{M+1}\left(\mathbb{C}P^2\right)\right) & \longrightarrow & H^2\left(X_M\left(S^2\times S^2\right)\right)\\ & l & \longmapsto & s_1+s_2-f_1\\ & e_1 & \longmapsto & s_2-f_1\\ & e_2 & \longmapsto & s_1\quad -f_1\\ & e_i & \longmapsto & f_{i-1}. \end{array}$$

In this new basis of $H^2\left(X_M\left(S^2\times S^2\right)\right)$, (5.3) therefore becomes

$$As_1 + As_2 - \sum_{i=1}^{M} w_i\,f_{i+1} \;\in\; \mathcal{C}_K. \tag{5.4}$$

In view of Proposition 5.1.4, (5.4) translates to the conditions that for all $E := (d, e; \mathbf{m}) \in \mathcal{E}$, we have that

$$2A^2 - \sum w_i^2 > 0$$

and

$$As_1(E) + As_2(E) - \sum w_i f_i(E) \ = \ (d+e)A - \sum m_i w_i \ > \ 0.$$

Recall from Lemma 3.1.4 (ii) that $\sum w_i^2 = a$. We conclude that $E(1,a) \overset{\text{s}}{\hookrightarrow} C(A)$ if and only if

$$A > \sqrt{\frac{a}{2}} \qquad \text{and} \qquad A \ > \ \frac{\sum m_i w_i}{d+e} = \frac{\langle m, \mathbf{w}(a) \rangle}{d+e}, \qquad \text{for all } (d, e; \mathbf{m}) \in \mathcal{E}.$$

This proves the proposition. ■

Remark 5.1.8.1 By the symmetry between d and e in the formula for $\mu(d, e; \mathbf{m})(a)$, we can assume that all elements $(d, e; \mathbf{m}) \in \mathcal{E}$ have $d \geq e$. We will use this convention in the following chapters.

Chapter 6

Obstructive and perfect classes of $\mathcal{E}$

This chapter is devoted to the analysis of the constraints given in Proposition 5.1.8. This analysis follows the one in [19]. However, several modifications are necessary. Nonetheless, we are now able to determine in Lemma 6.1.1 the embedding capacity function $c(a)$ for $a \in [8, +\infty)$.

6.1 Basic observations for classes $(d, e; \mathbf{m}) \in \mathcal{E}$

We start with the following lemma, which yields that for all $a \geq 8$, $E(1, a)$ fully embeds into $C(A)$ with A satisfying the volume constraint $A \geq \sqrt{\frac{a}{2}}$.

Lemma 6.1.1 *For $a \geq 8$, the embedding capacity function is given by*

$$c(a) = \sqrt{\frac{a}{2}}.$$

Proof. By Proposition 5.1.8 it holds that $c(a) \geq \sqrt{\frac{a}{2}}$. We will show that $c(a) \leq \sqrt{\frac{a}{2}}$ when $a \geq 8$. From (5.2), we obtain

$$\mu(d, e; \mathbf{m})(a) = \frac{\langle \mathbf{m}, \mathbf{w}(a) \rangle}{d+e} = \frac{\sum_{i=1}^{M} m_i \cdot w_i}{d+e} \leq \frac{2(d+e)-1}{d+e} < \frac{2(d+e)}{d+e} = 2 \leq \sqrt{\frac{a}{2}},$$

for all $a \geq 8$ and the lemma is proved. ■

Lemma 6.1.2 *For $M \leq 7$ the only solutions of system (5.2) are* $(0, 0; -1)$,

$$\begin{array}{lll} (1,0;1), & (1,1;1^{\times 3}), & (2,1;1^{\times 5}), \\ (2,2;2,1^{\times 5}), & (3,1;1^{\times 7}), & (3,2;2^{\times 2},1^{\times 5}), \\ (3,3;2^{\times 4},1^{\times 3}), & (4,3;2^{\times 6},1) \textit{ and} & (4,4;3,2^{\times 6}). \end{array}$$

Proof. If $d \geq e = 0$, then system (5.2) reads

$$\begin{cases} 2d - 1 = \sum_{i=1}^{M} m_i \\ 1 = \sum_{i=1}^{M} m_i^2. \end{cases} \tag{6.1}$$

Hence $m_i \in \{0, 1\}$ and $2d - 1 = 1$. The only solution is $(1, 0; 1)$.

From now on we assume that $d \geq e \geq 1$. The Cauchy-Schwarz inequality yields

$$\left(\sum_{i=1}^{M} m_i\right)^2 = \langle(1,\dots,1),\mathbf{m}\rangle^2 \leq \sum_{i=1}^{M} 1 \cdot \sum_{i=1}^{M} m_i^2 = M \cdot \sum_{i=1}^{M} m_i^2, \qquad \text{for all } M \geq 1.$$

Using (5.2) we obtain the inequality

$$\big(2(d+e)-1\big)^2 \leq M \cdot (2de+1)$$

which is equivalent to

$$d^2+e^2 \leq \frac{M-4}{2}\, de + (d+e) + \frac{M-1}{4}. \tag{6.2}$$

Assume $M \leq 4$. Then, since $\frac{M-4}{2} \leq 0$ and $0 \leq \frac{M-1}{4} \leq \frac{3}{4}$, we obtain from (6.2) that

$$d^2+e^2 \leq d+e+\frac{3}{4}.$$

It follows that $d+e \leq 2$ and thus

$$(d,e) \in \{(0,0),(1,0),(1,1)\}.$$

But as $d, e \geq 1$ we only have to consider the case $d = e = 1$, where we obtain from (5.2) the equations

$$\begin{cases} 3 = \displaystyle\sum_{i=1}^{M} m_i \\ 3 = \displaystyle\sum_{i=1}^{M} m_i^2. \end{cases} \tag{6.3}$$

It follows that the only solution of (6.3) is $m_1 = m_2 = m_3 = 1$ and $m_4 = 0$.

We conclude that the only solutions of system (5.2) for $M \leq 4$ are

$$(1,0;1) \quad \text{and} \quad (1,1;1^{\times 3}).$$

Now consider the case $5 \leq M \leq 7$, that is $\frac{1}{2} \leq \frac{M-4}{2} \leq \frac{3}{2}$ and $1 \leq \frac{M-1}{4} \leq \frac{3}{2}$. We obtain from (6.2) that

$$d^2+e^2 \leq \frac{3}{2}\, de + (d+e) + \frac{3}{2}. \tag{6.4}$$

Since $(d-e)^2 \geq 0$ we have

$$\frac{3}{2}\, de \leq \frac{3}{4}(d^2+e^2).$$

Thus it follows from (6.4) that

$$d^2+e^2 \leq 4(d+e)+6,$$

and hence $d, e \leq 4$.

We found that $4 \geq d \geq e$ is necessary for solutions $(d,e;\mathbf{m}) \in \mathcal{E}_7$ of (5.2). It is left to show that in this case there are no other solutions than the ones already stated. One possibility to see this is to do some calculations by hand. Therefore one has to look at the difference of the two equations of (5.2) where

$$(2(d+e)-1)-(2de+1) > 2(d+e)+1 \quad \Longleftrightarrow \quad \sum_{i=1}^{M} m_i(m_i-1) > \sum_{i=1}^{M} m_i$$

must always hold. One sees directly that for given $d, e \in \mathbb{N}$, with $4 \geq d \geq e$, the above inequality yields

$$\sum_{j \geq 2} g(j) \cdot j \cdot (j-1) > \sum_{j \geq 2} g(j) \cdot j, \tag{6.5}$$

where $j, g(j) \in \mathbb{N} \cup \{0\}$ and $g(j)$ is the number of m_i's such that $m_i = j$. Now we have to check all possible combinations of $j, g(j)$ that solve (6.5) for given $d, e \leq 4$ to calculate the rest of the solutions.

Alternatively, one can use the Mathematica script in Section A.1 of the appendix to find the aforementioned solutions. ■

6.2 Obstructive and perfect classes and their properties

Here we will introduce the classes $E = (d, e; \mathbf{m}) \in \mathcal{E}$, that are crucial as they are giving embedding obstructions. We will name these classes *perfect* and *obstructive classes.*

Definition 6.2.1 *A class $E = (d, e; \mathbf{m}) \in \mathcal{E}$ is called **obstructive** if $\mu(d, e; \mathbf{m})(z) > \sqrt{\frac{z}{2}}$ on some nonempty interval $I \subset [1, 8]$. Further we say E is **obstructive at** a, if $\mu(d, e; \mathbf{m})(a) > \sqrt{\frac{a}{2}}$.*

Example 6.2.2 *The only obstructive classes existing in $\mathcal{E}_7$ are $(1, 0; 1)$, $(1, 1; 1^{\times 3})$, $(2, 1; 1^{\times 5})$, $(2, 2; 2, 1^{\times 5})$ and $(4, 4; 3, 2^{\times 6})$, see Figure 6.1.*

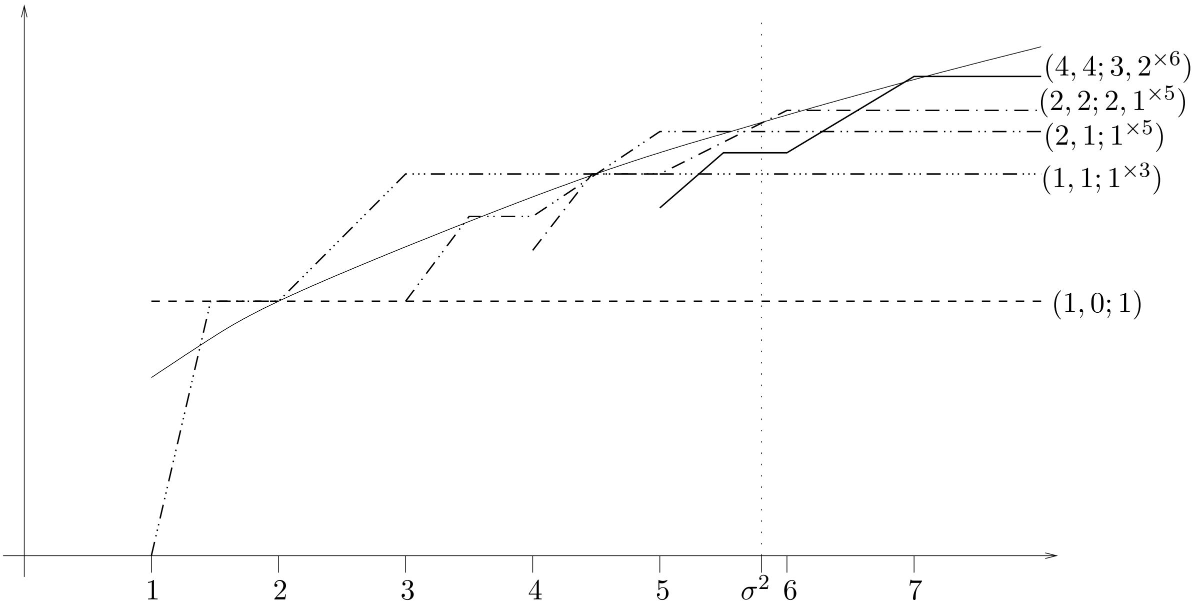

Figure 6.1: $\mu(d, e; \mathbf{m})(a)$ for $(d, e; \mathbf{m}) \in \{(1, 0; 1), (1, 1; 1^{\times 3}), (2, 1; 1^{\times 5}), (2, 2; 2, 1^{\times 5}), (4, 4; 3, 2^{\times 6})\}$.

Lemma 6.2.3 *Assume that $\mu(d, e; \mathbf{m})(a) > \sqrt{\frac{a}{2}}$ for some $a \geq 1$. Then either*

$$d = e \qquad \text{or} \qquad d = e + 1. \tag{6.6}$$

Proof. By the Cauchy-Schwarz inequality we obtain

$$\mu(d, e; \mathbf{m})(a) = \frac{\langle \mathbf{m}, \mathbf{w} \rangle}{d + e} \leq \frac{\|\mathbf{m}\| \cdot \|\mathbf{w}\|}{d + e},$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. As $\sum_{i=1}^{\ell(a)} w_i^2 = a$ (see Lemma 3.1.4 (ii)), and $\sum_{i=1}^{M} m_i^2 = 2de+1$, we obtain that

$$\sqrt{\frac{a}{2}} < \mu(d,e;\mathbf{m})(a) \leq \frac{\sqrt{2de+1}\sqrt{a}}{d+e}. \tag{6.7}$$

If $d = e+k$ for $k \geq 2$, then it follows from $2e^2+2ke+2 < 2e^2+2ke+k^2$, that

$$d+e = 2e+k = \sqrt{4e^2+4ek+k^2} > \sqrt{2\cdot(2e^2+2ek+1)} = \sqrt{2}\cdot\sqrt{2de+1}$$

and hence

$$\frac{\sqrt{2de+1}}{d+e} < \frac{1}{\sqrt{2}} \leq \sqrt{\frac{a}{2}},$$

which is a contradiction to (6.7). ■

Remark 6.2.3.1 It follows from Lemma 6.2.3 that for obstructive classes $(d,e;\mathbf{m})$ it holds that $d=e$ or $d=e+1$. Thus for $d=e$ system (5.2) becomes

$$\begin{cases} 4d-1 = \sum_{i=1}^{M} m_i \\ 2d^2+1 = \sum_{i=1}^{M} m_i^2 \end{cases} \tag{6.8}$$

and for $d=e+1$ we obtain

$$\begin{cases} 4d-3 = \sum_{i=1}^{M} m_i \\ 2d^2-2d+1 = \sum_{i=1}^{M} m_i^2. \end{cases} \tag{6.9}$$

Remark 6.2.3.2 Sometimes things become more clear for obstructive classes if we do not look at the classes

$$(d,d;\mathbf{m}) \quad \text{and} \quad (d,d-1;\mathbf{m}),$$

but we write these classes in the form

$$(d,d;\mathbf{m}) \quad \text{and} \quad (d+\tfrac{1}{2}, d-\tfrac{1}{2};\mathbf{m}).$$

Corollary 6.2.4 *Assume that $\mu(d',e';\mathbf{m}')(a) > \sqrt{\frac{a}{2}}$ for some $a \geq 1$ and let*

$$\Delta := \mu(d',e';\mathbf{m}')(a) - \sqrt{\frac{a}{2}} > 0. \tag{6.10}$$

Then one can list those $(d,e;\mathbf{m}) \in \mathcal{E}$ such that $\mu(d,e;\mathbf{m})(a) > \mu(d',e';\mathbf{m}')(a)$.

Proof. Due to Lemma 6.2.3 we have to consider the two cases $d=e$ and $d=e+1$.

Suppose $d=e$. From (6.7) we have

$$\mu(d,d;\mathbf{m})(a) \leq \frac{\sqrt{2d^2+1}\cdot\sqrt{a}}{2d}$$

and it follows from (6.10), that

$$\Delta + \sqrt{\frac{a}{2}} < \mu(d,d;\mathbf{m})(a) \leq \frac{\sqrt{2d^2+1}\cdot\sqrt{a}}{2d},$$

i.e.

$$\Delta + \sqrt{\frac{a}{2}} < \frac{\sqrt{2d^2+1}\cdot\sqrt{a}}{2d}. \tag{6.11}$$

Setting $L := \frac{\sqrt{2}\Delta}{\sqrt{a}} + 1$, we obtain from (6.11) that

$$d < \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{L^2-1}}. \tag{6.12}$$

Now suppose that $d = e+1$. From (6.10) and (6.7) it follows

$$\Delta + \sqrt{\frac{a}{2}} < \mu(e+1,e;\mathbf{m})(a) \leq \frac{\sqrt{2(e+1)e+1}\cdot\sqrt{a}}{(e+1)+e},$$

which yields for L as defined above that

$$d < \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{\sqrt{L^2-1}}. \tag{6.13}$$

■

Remark 6.2.4.1 It follows from Corollary 6.2.4 that if we already have an obstructive class $(d',e';\mathbf{m}') \in \mathcal{E}$ at a, only finitely many other classes $(d,e;\mathbf{m})$ exist such that $\mu(d,e;\mathbf{m}) > \mu(d',e';\mathbf{m}')$, since d is bounded from above. Hence $c(a) = \mu(d',e';\mathbf{m}')(a)$ if there is no other class $(d,e;\mathbf{m})$ for which (6.12) or (6.13) is fulfilled, and $\mu(d',e';\mathbf{m}') \geq \mu(d,e;\mathbf{m})$.

Definition 6.2.5 *Let $\mathbf{w}(a)$ be the weight expansion of length $\ell(a) := M$ and let $(d,e;\mathbf{m}) \in \mathcal{E}_M$. We define the* ***error vector*** *ε for classes of the form $(d,d;\mathbf{m}) \in \mathcal{E}_M$ by*

$$\mathbf{m} = \frac{\sqrt{2}\,d}{\sqrt{a}}\mathbf{w}(a) + \varepsilon, \tag{6.14}$$

and for classes of the form $(d,d-1;\mathbf{m}) \in \mathcal{E}_M$ by

$$\mathbf{m} = \frac{2d-1}{\sqrt{2a}}\mathbf{w}(a) + \varepsilon. \tag{6.15}$$

The number

$$\|\varepsilon\|^2 = \sum_{i=1}^{M} \varepsilon_i^2$$

is called the ***squared error****.*

Lemma 6.2.6 *Let $a = \frac{p}{q} \geqslant 1$ be a rational number with weight expansion $\mathbf{w}(a)$ and let $(d,e;\mathbf{m}) \in \mathcal{E}$. Then*

(i) $\mu(d,e;\mathbf{m})(a) \leqslant \dfrac{\sqrt{2de+1}\sqrt{a}}{d+e}$. In particular,

$$\mu(d,d;\mathbf{m})(a) \leqslant \sqrt{1+\frac{1}{2d^2}}\sqrt{\frac{a}{2}} \quad \text{and} \quad \mu\left(d+\tfrac{1}{2}, d-\tfrac{1}{2};\mathbf{m}\right)(a) \leqslant \sqrt{1+\frac{1}{4d^2}}\sqrt{\frac{a}{2}},$$

(ii) $\mu(d,e;\mathbf{m})(a) > \sqrt{\frac{a}{2}}$, *if and only if* $\langle \varepsilon, \mathbf{w}(a)\rangle > 0$,

(iii) *If* $\mu(d,d;\mathbf{m})(a) > \sqrt{\frac{a}{2}}$, *respectively* $\mu\left(d+\frac{1}{2}, d-\frac{1}{2};\mathbf{m}\right)(a) > \sqrt{\frac{a}{2}}$, *then* $\langle \varepsilon,\varepsilon\rangle < 1$, *respectively* $\langle\varepsilon,\varepsilon\rangle < \frac{1}{2}$,

(iv) $-\sum_{i=1}^{M} \varepsilon_i = \frac{d+e}{\sqrt{2a}}\left(y(a) - \frac{1}{q}\right) + 1$, *where* $y(a) := a + 1 - 2\sqrt{2a}$.

Proof. (i) By the Cauchy-Schwarz inequality, Proposition 5.1.7 (i) and Lemma 3.1.4, we have

$$(d+e)\mu(d,e;\mathbf{m})(a) = \langle \mathbf{m}, \mathbf{w}(a)\rangle \leqslant \|m\|\,\|\mathbf{w}(a)\| = \sqrt{2de+1}\sqrt{a}.$$

In the case of a class $(d,d;\mathbf{m})$ we find that

$$\mu(d,d;\mathbf{m})(a) \leqslant \frac{\sqrt{2d^2+1}\sqrt{a}}{2d} = \sqrt{\frac{2d^2+1}{2d^2}}\sqrt{\frac{a}{2}} = \sqrt{1+\frac{1}{2d^2}}\sqrt{\frac{a}{2}},$$

and in the case of a class $\left(d+\frac{1}{2}, d-\frac{1}{2};\mathbf{m}\right)$ that

$$\mu\left(d+\tfrac{1}{2}, d-\tfrac{1}{2};\mathbf{m}\right)(a) \leqslant \frac{\sqrt{2d^2+\frac{1}{2}}\,\sqrt{a}}{2d} = \sqrt{1+\frac{1}{4d^2}}\,\sqrt{\frac{a}{2}}.$$

(ii) Since

$$\begin{aligned}\langle \varepsilon, \mathbf{w}(a)\rangle = \left\langle m - \frac{d+e}{\sqrt{2a}}\mathbf{w}(a), \mathbf{w}(a)\right\rangle &= \langle \mathbf{m}, \mathbf{w}(a)\rangle - \frac{d+e}{\sqrt{2a}}\,\|\mathbf{w}(a)\|^2 \\ &= \langle \mathbf{m}, \mathbf{w}(a)\rangle - (d+e)\sqrt{\frac{a}{2}},\end{aligned}$$

we see that $\langle \varepsilon, \mathbf{w}(a)\rangle > 0$ if and only if $\mu(d,e;\mathbf{m})(a) = \frac{\langle \mathbf{m}, \mathbf{w}(a)\rangle}{d+e} > \sqrt{\frac{a}{2}}$.

(iii) For a class $(d,d;\mathbf{m})$,

$$\begin{aligned}2d^2+1 = \langle \mathbf{m},\mathbf{m}\rangle &= \left\langle \sqrt{\frac{2}{a}}\,d\mathbf{w}(a) + \varepsilon, \sqrt{\frac{2}{a}}\,d\mathbf{w}(a)+\varepsilon\right\rangle \\ &= 2d^2 + \frac{2\sqrt{2}}{\sqrt{a}}\,d\,\langle \mathbf{w}(a),\varepsilon\rangle + \langle\varepsilon,\varepsilon\rangle\end{aligned}$$

shows that if $\mu(d,d;\mathbf{m})(a) > \sqrt{\frac{a}{2}}$, then by (ii) $\langle\varepsilon,\varepsilon\rangle < 1$. Similarly, for a class $(d+\frac{1}{2}, d-\frac{1}{2};\mathbf{m})$,

$$2d^2 + \frac{1}{2} = 2d^2 + \frac{2\sqrt{2}}{\sqrt{a}} d\,\langle \mathbf{w}(a),\varepsilon\rangle + \langle\varepsilon,\varepsilon\rangle$$

shows that if $\mu(d+\frac{1}{2}, d-\frac{1}{2};\mathbf{m})(a) > \sqrt{\frac{a}{2}}$, then $\langle\varepsilon,\varepsilon\rangle < \frac{1}{2}$.

(iv) By Proposition 5.1.7 (i) and Lemma 3.1.4, we see that

$$2(d+e) - 1 = \sum_{i=1}^{M} m_i = \frac{d+e}{\sqrt{2a}}\left(a+1-\frac{1}{q}\right) + \sum_{i=1}^{M}\varepsilon_i.$$

Thus

$$-\sum_{i=1}^{M}\varepsilon_i = \frac{d+e}{\sqrt{2a}}\left(a+1-2\sqrt{2a}\right) - \frac{d+e}{q\sqrt{2a}} + 1,$$

from which the result follows. ■

Remark 6.2.6.1 (i) Going back to Definition 6.2.5 of the error vector, it is reasonable to call the vectors $\mathbf{m}$ and $\mathbf{w}(a)$ "almost parallel" if the squared error is smaller than 1. Hence it follows from (iii) of Lemma 6.2.6 that an element of $\mathcal{E}$ is only obstructive at a if the vectors $\mathbf{m}$ and $\mathbf{w}(a)$ are "almost parallel".

(ii) Notice that for the function $y(a)$ defined in (iv) of Lemma 6.2.6 holds that $y(\sigma^2) = 0$.

Remark 6.2.6.2 As before, $\ell(a)$ denotes the length of the weight expansion of a, i.e. of the vector $\mathbf{w}(a)$. In the following $\ell(\mathbf{v})$ will always denote the length of a vector $\mathbf{v}$.

Lemma 6.2.7 *Let $(d, e; \mathbf{m}) \in \mathcal{E}$ and suppose $I \subset (1, 8)$ to be a maximal, nonempty interval where $\mu(d, e; \mathbf{m})(a) > \sqrt{\frac{a}{2}}$ for all $a \in I$. Then*

(i) $\ell(a) \geq \ell(\mathbf{m})$ for all $a \in I$;

(ii) There is a unique $a_0 \in I$ such that $\ell(a_0) = \ell(\mathbf{m})$.

Proof. Take the i-th weight $w_i(a)$ as a function of a. This is a piecewise linear function on I. If there exists no $a' \in I$ such that $\ell(a') < i$, i.e. $w_i(a') = 0$, then $w_i(a)$ is even linear on I. Consider $a \in I$ with $\ell(a) < \ell(\mathbf{m})$. Then it follows from (5.2) that

$$\sum_{i=1}^{\ell(a)} m_i^2 < \sum_{i=1}^{\ell(a)} m_i^2 + 1 \leq \sum_{i=1}^{\ell(\mathbf{m})} m_i^2 = 2de + 1,$$

that is

$$\sum_{i=1}^{\ell(a)} m_i^2 \leq 2de.$$

The Cauchy-Schwarz inequality now yields that

$$\langle \mathbf{m}, \mathbf{w} \rangle \leq \sqrt{\sum_{i=1}^{\ell(a)} m_i^2} \cdot \|\mathbf{w}\| = \sqrt{4de} \cdot \sqrt{\frac{a}{2}}. \tag{6.16}$$

As $\mu(d, e; \mathbf{m})(a) > \sqrt{\frac{a}{2}}$ was an assumption of the lemma, we know by Lemma 6.2.3 that either $d = e$ or $d = e + 1$. In both cases it follows from (6.16) that

$$\mu(d, e; \mathbf{m})(a) \leq \sqrt{\frac{a}{2}}$$

which is a contradiction. Thus we proved (i) of the lemma.

For (ii) we first show the existence of $a_0 \in I$ such that $\ell(a_0) = \ell(\mathbf{m})$. Suppose that $\ell(a) > \ell(\mathbf{m})$ for all $a \in I$, i.e. the function $\mu(d, e; \mathbf{m})(a)$ is linear. But this is a contradiction as $\sqrt{\frac{a}{2}}$ is concave and $I \subset (1, 8)$ is maximal and bounded, i.e. $I = (x, y)$ with $c(x) = \sqrt{\frac{x}{2}}$ and $c(y) = \sqrt{\frac{y}{2}}$, where $x \geq 1$ and $y \leq 8$. Therefore $\ell(a) \leq \ell(\mathbf{m})$. Now, using (i) this yields the existence of $a_0 \in I$, such that $\ell(a_0) = \ell(\mathbf{m})$.

The uniqueness follows with a similar argument as in the proof of [19, Lemma 2.1.3] relying on properties of continued fractions. Let $a, b \in I$, with $a < b$ and $\ell(b) = \ell(a) = \ell(\mathbf{m})$. Then there exists $y \in (a, b) \subset I$, where $\ell(y) < \ell(a)$. Thus $\ell(y) < \ell(\mathbf{m})$, but as $\ell(y) \geq \ell(\mathbf{m})$ for all $y \in I$ this means that $y \notin I$, which leads to a contradiction. ■

Corollary 6.2.8 *Suppose that $c(a) > \sqrt{\frac{a}{2}}$. Then the following hold:*

(i) The set of $(d, e; \mathbf{m})$ such that $c(a) = \mu(d, e; \mathbf{m})(a)$ is finite.

(ii) There exist (possibly equal) elements $(d^\pm, e^\pm; \mathbf{m}^\pm) \in \mathcal{E}$ *and* $\delta > 0$ *such that*

$$c(z) = \begin{cases} \mu(d^-, e^-; \mathbf{m}^-)(z) & \text{for all } z \in (a-\delta, a], \\ \mu(d^+, e^+; \mathbf{m}^+)(z) & \text{for all } z \in [a, a+\delta). \end{cases}$$

(iii) On each of the intervals in (i) there exist rational numbers $s, t \geq 0$, *such that* $c(a) = s + ta$.

Proof. As $c(a) > \sqrt{\frac{a}{2}}$, we know that $c(a)$ is of the form $c(a) = \mu(d, d; \mathbf{m})(a)$ or $c(a) = \mu(d, d-1; \mathbf{m})(a)$ for some $(d, d; \mathbf{m}), (d, d-1; \mathbf{m}) \in \mathcal{E}$. Further, there exists $D \in \mathbb{N}$ such that

$$\sqrt{1 + \frac{1}{2D^2}} \cdot \sqrt{\frac{a}{2}} \leq c(a) \quad \text{and} \quad \sqrt{1 + \frac{1}{(2D-1)^2}} \cdot \sqrt{\frac{a}{2}} \leq c(a) \tag{6.17}$$

respectively, and thus we obtain from (i) of Lemma 6.2.6 that $d \leq D$, i.e. there are only finitely many relevant classes $(d, d; \mathbf{m}), (d, d-1; \mathbf{m}) \in \mathcal{E}$.

Lemma 2.1.14 yields that $c(a)$ is continuous. Consequently for z sufficiently close to a, the bound in (6.17) still holds, that is D does not change. Therefore the relevant classes to determine $c(z)$ for $z \in (a-\delta, a+\delta)$ remain the same. Now if there are two classes $(d^\pm, e^\pm; \mathbf{m}^\pm) \in \mathcal{E}$ determining $c(a)$, then one determines $c(z)$ for $z \in (a-\delta, a]$ and the other one determines $c(z)$ for $z \in (a, a+\delta]$. Thus (i) and (ii) follow.

As we saw in the proof of Lemma 6.2.7 the weights $w_i(a)$ are piecewise linear functions of a and thus $\mu(d, e; \mathbf{m})(a)$ is a piecewise linear function with coefficients in $\mathbb{Q}$, as for all i, $w_i(z) \in \mathbb{Q}$ and $d, e \in \mathbb{N}$. Thus $c(a)$ is a rational piecewise linear function as the supremum of a finite number of rational piecewise linear functions, i.e. $c(z) = s + tz$ with $s, t \in \mathbb{Q}$. Since c is nondecreasing it follows that $t \geq 0$, and $s \geq 0$ since $c(a) \geq \sqrt{\frac{a}{2}}$ for $a \geq 1$. ■

Definition 6.2.9 *A class* $(d, e; \mathbf{m}) \in \mathcal{E}$ *is called* ***perfect*** *if* $\mathbf{m}$ *is a multiple of the weight vector* $\mathbf{w}(a)$ *for some* $a > 1$.

Lemma 6.2.10 *Suppose that* $(d, e; \mathbf{m}) \in \mathcal{E}$ *with* $d = e$ *or* $d = e+1$ *is perfect, that is,* $\mathbf{m} = \kappa \cdot \mathbf{w}(a)$ *for some* $a > 1$. *Then*

(i) $\mu(d, e; \mathbf{m})(a) = c(a) > \sqrt{\frac{a}{2}}$,

(ii) $(d, e; \mathbf{m})$ *with* $\mu(d, e; \mathbf{m})(a) = c(a) > \sqrt{\frac{a}{2}}$ *is unique and*

(iii) for $a = \frac{p}{q} < \sigma^2 = 3 + 2\sqrt{2}$ *it holds that* $\mathbf{m}$ *equals the normalized weight expansion, i.e.* $\kappa = q$.

Proof. (i) Suppose $\mathbf{m} = \kappa \cdot \mathbf{w}(a)$ for some $a > 1$. Then Lemma 3.1.4 (ii) and the second equation of system (5.2) yield

$$\kappa^2 \cdot a = \kappa^2 \cdot \|\mathbf{w}\|^2 = \|\mathbf{m}\|^2 = 2de + 1. \tag{6.18}$$

First suppose the case $d = e$. As $4d^2 < 2d^2 + 10\kappa^2 a$, it follows that $2d < \kappa\sqrt{a}$ and therefore

$$\mu(d, e; \mathbf{m})(a) = \mu(d, d; \mathbf{m})(a) = \frac{\langle \mathbf{m}, \mathbf{w} \rangle}{2d} = \frac{\kappa \|\mathbf{w}\|^2}{2d} = \frac{\kappa \cdot a}{2d} > \frac{\kappa \cdot a}{\kappa\sqrt{a}} = \sqrt{a} > \sqrt{\frac{a}{2}}. \tag{6.19}$$

Now consider the case $d = e + 1$. Then

$$\kappa^2 a = \kappa \|\mathbf{w}\|^2 = \|\mathbf{m}\|^2 = 2de + 1 = 2(e+1)e + 1 = 2e^2 + 2e + 1,$$

that is

$$2\kappa^2 a = 4e^2 + 4e + 2.$$

Thus with $(d+e)^2 = (2e+1)^2 = 4e^2+4e+2$ it follows that

$$\mu(d,e;\mathbf{m})(a) = \mu(e+1,e;\mathbf{m})(a) = \frac{\langle \mathbf{m}, \mathbf{w}\rangle}{2e+1} = \frac{\langle \kappa\mathbf{w}, \mathbf{w}\rangle}{2e+1} = \frac{\kappa\,\|\mathbf{w}\|^2}{2e+1} = \frac{\kappa\cdot a}{2e+1} > \frac{\kappa\cdot a}{\kappa\sqrt{2a}} = \sqrt{\frac{a}{2}}$$

which, together with (6.19), proves (i) of the lemma.

(ii) Assume there exists another class $(d',e';\mathbf{m}') \in \mathcal{E}$ such that $\mu(d',e';\mathbf{m}')(a) = \mu(d,e;\mathbf{m})(a)$. Since $(d',e';\mathbf{m}') \neq (d,e;\mathbf{m})$, positivity of intersection shows that

$$de' + d'e \geq \langle \mathbf{m}, \mathbf{m}'\rangle = \kappa \cdot \langle \mathbf{w}, \mathbf{m}'\rangle$$

and thus

$$\langle \mathbf{w}, \mathbf{m}'\rangle \leq \frac{de'+d'e}{\kappa}. \tag{6.20}$$

We will now show that $(de'+d'e)(d+e) < (2de+1)(d'+e')$. Therefore, by Lemma 6.2.3, we have to analyse four cases as both classes $(d,e;\mathbf{m}), (d',e';\mathbf{m}') \in \mathcal{E}$ are supposed to be obstructive:

(1) For $d=e$ and $d'=e'>1$ we obtain $2d^2d' < 2d^2d' + 2d'$,
(2) for $d=e$ and $d'=e'+1>1$ we get $d^2(2d'-1) < d^2(2d'-1)+2d'-1$,
(3) $d=e+1>1$ and $d'=e'+1$ yield $2d^2d'-2dd'-d^2-d' < 2d^2d'-2dd'-d^2+2d'+d-1$ and
(4) for $d=e+1$ and $d'=e'$ we get $2d^2d'-2dd'+d' < 2d^2d'-2dd'+2d'$.

In (1) we can exclude $d'=e'=1$, i.e. that $(d',e';\mathbf{m}') = (1,1;1^{\times 3})$ as $(d,e;\mathbf{m}) = (1,1;1^{\times 3})$ already and $(d',e';\mathbf{m}') \neq (d,e;\mathbf{m})$. That $c(3) = \mu(1,1;1^{\times 3})(3)$ one can check with the help of Corollary 6.2.4 as $\mu(1,1;1^{\times 3})(3) = \frac{3}{2} > \sqrt{\frac{3}{2}}$.

Thus we obtain $(de'+d'e)(d+e) < (2de+1)(d'+e')$, that is

$$\frac{de'+d'e}{\kappa\cdot(d'+e')} < \frac{2de+1}{\kappa\cdot(d+e)}.$$

This yields with (6.20) that

$$\mu(d',e';\mathbf{m}')(a) = \frac{\langle \mathbf{w}, \mathbf{m}'\rangle}{d'+e'} < \frac{\langle \mathbf{w}, \mathbf{m}\rangle}{d+e} = \mu(d,e;\mathbf{m})(a),$$

which is a contradiction to the assumption and (ii) follows.

(iii) Let $a = \frac{p}{q}$ in lowest terms. As $\mathbf{m} = \kappa\cdot\mathbf{w}(a)$, also $m_M = \kappa\cdot w_M$ and therefore with (i) of Lemma 3.1.4 one gets that $m_M = \kappa\frac{1}{q}$. Thus as $m_M \in \mathbb{Z}$, there exists $s\in\mathbb{Z}$, such that $\kappa = s\cdot q$.

We look at the case $s=1$. With Lemma 3.1.4 (ii) and (iii) the equations of system (5.2) modify to

$$2(d+e)-1 = \sum_{i=1}^{\ell(a)} m_i = \kappa\cdot\sum_{i=1}^{\ell(a)} w_i(a) = \kappa\cdot\left(a+1-\frac{1}{q}\right) = sq\cdot\left(a+1-\frac{1}{q}\right) \tag{6.21}$$

and

$$2de+1 = \sum_{i=1}^{\ell(a)} m_i^2 = \kappa\cdot\sum_{i=1}^{\ell(a)} w_i(a)^2 = \kappa\cdot\|\mathbf{w}\|^2 = \kappa\cdot a = (sq)^2\cdot a. \tag{6.22}$$

As already mentioned above we know by Lemma 6.2.3 that perfect classes are either of the form $(d,d;\mathbf{m})$ or $(d,d-1;\mathbf{m})$. For $s=1$ we thus obtain

$$2(d+e) = q\cdot(a+1) \quad \text{and} \quad 2de+1 = q^2\cdot a$$

that is

$$\sqrt{a} = \frac{\sqrt{2de+1}}{q} \quad \text{and} \quad a = \frac{2(d+e)}{q} - 1. \tag{6.23}$$

We consider the inequality $a^2 - 2\sqrt{2}a + 1 < 0$ for $a \in \left(\sqrt{2}-1, \sqrt{2}+1\right)$ and obtain with (6.23) for $d = e$ that

$$a - 2\sqrt{2}\sqrt{a} + 1 = \frac{4d}{q} - 1 - 2\sqrt{2}\frac{2d^2+1}{q} + 1 = \frac{4d}{q} - \frac{16d^2+1}{q} < 0,$$

and for $d = e + 1$ that

$$a - 2\sqrt{2}\sqrt{a} + 1 = \frac{4d-2}{q} - 1 - 2\sqrt{2}\frac{2d^2-2d+1}{q} + 1 = \frac{4d-2}{q} - \frac{16d^2-16d+8}{q} < 0.$$

Thus $a \in \left(1, 3+2\sqrt{2}\right)$.

Now let $s \in \mathbb{Z} \setminus \{1\}$. First we consider the case $d = e$. Therefore equations (6.21) and (6.22) modify to

$$4d - 1 = sq \cdot \left(a + 1 - \frac{1}{q}\right) \tag{6.24}$$

and

$$2d^2 + 1 = (sq)^2 \cdot a. \tag{6.25}$$

The latter directly yields by assuming the opposite, that $2 \nmid s$. Assuming that $s \mid d$, it follows that $s \mid 4d$. But from (6.24) one sees that $s \mid (4d-1)$, which leads to a contradiction, i.e. $s \nmid d$. Adding (6.24) and (6.25) we obtain that $s \mid 2d(2+d)$ and thus $s \mid (2+d)$. It follows that $s \mid k_1(4d-1) + k_2(2+d)$, that is for $k_1 = 2$ and $k_2 = 1$, one gets that $s \mid 9$. As $s \leq 0$ directly leads to a contradiction of (6.24) and we assumed that $s \neq 1$, it must hold that $s = 3$ or $s = 9$. Now, suppose $s = 3$. Adding (6.24) and (6.25), one obtains that

$$\frac{(3p+3q-2)^2}{8} = 9pq - 1.$$

Taking into consideration that $a = \frac{p}{q}$, the latter transforms into

$$a^2 - \left(\frac{4}{3p} + 6\right)a + 1 = 0,$$

that is

$$a = \frac{2 + 9p \pm \sqrt{1 + 9p + 18p^2}}{3p}.$$

From this it follows that $a \notin \left[1, 3+2\sqrt{2}\right]$. We proceed similarly for the case $s = 9$.

For $d = e + 1$ equations (6.21) and (6.22) modify to

$$4d - 3 = sq \cdot \left(a + 1 - \frac{1}{q}\right) \tag{6.26}$$

and

$$2d^2 - 1 = (sq)^2 \cdot a. \tag{6.27}$$

Subtracting (6.27) three times from (6.26) yields that $s \mid 2d(2-3d)$. If $s \mid 2d$ then

$$s \mid k_1(2d) + k_2(4d-3),$$

i.e. for $k_1 = 2$ and $k_2 = -1$ this yields $s \mid 3$. If $s \mid (2-3d)$ then

$$s \mid k_1(4d-3) + k_2(2-3d),$$

i.e. for $k_1 = -3$ and $k_2 = -4$ we obtain $s \mid 1$, that is $s = 1$, which is a contradiction to our assumption $s \in \mathbb{Z} \setminus \{1\}$. ■

Lemma 6.2.11 *Let $(d, e; \mathbf{m}) \in \mathcal{E}$ be such that $\mu(d, e; \mathbf{m})(a) > \sqrt{\frac{a}{2}}$. Let $J = \{k, \ldots, k+s-1\}$ be a block of $s \geq 2$ consecutive integers such that the $w_i(a)$ are equal for all $i \in J$. Then we have the three following possibilities: Either*

(1) $m_k = \ldots = m_{k+s-1}$, or

(2) $m_k - 1 = m_{k+1} = \ldots = m_{k+s-1}$, or

(3) $m_k = \ldots = m_{k+s-2} = m_{k+s-1} + 1$.

Moreover, there is at most one block of length $s \geq 2$ where the m_i are not all equal, and if such a block J exists, then

$$\sum_{i \in J} \varepsilon_i^2 \geq \frac{s-1}{s}.$$

The proof is similar to the proof of [19, Lemma 2.1.7].

Corollary 6.2.12 *If a class of the form $(d+\frac{1}{2}, d-\frac{1}{2}; \mathbf{m})$ is obstructive, then the m_i are constant on each block.*

Proof. Suppose there exists a block J of length $s \geq 2$ on which the $w_i(a)$ are not all equal. Then by Lemma 6.2.11,

$$\sum_{i \in J} \varepsilon_i^2 \geq \frac{s-1}{s} \geq \frac{1}{2},$$

which contradicts Lemma 6.2.6 (iii), which states that $\sum \varepsilon_i^2 < \frac{1}{2}$. ■

Lemma 6.2.13 *Let $(d, e; \mathbf{m}) \in \mathcal{E}$ be an obstructive class at some rational number $a \geq 1$ with $\ell(a) = \ell(\mathbf{m})$. Let $w_{k+1}, \ldots, w_{k+s}$ be a block which is not the first block of $\mathbf{w}(a)$.*

1. *If the block is not the last one, then*

$$|m_k - (m_{k+1} + \ldots + m_{k+s+1})| < \sqrt{s+2}.$$

If the block is the last one, then

$$|m_k - (m_{k+1} + \ldots + m_{k+s})| < \sqrt{s+1}.$$

2. *It is always the case that*

$$m_k - \sum_{i=k+1}^{M} m_i < \sqrt{M-k+1}.$$

The proof is similar to the one of [19, Lemma 2.1.8].

Proposition 6.2.14 *Let $(d, e; \mathbf{m}) \in \mathcal{E}$ be an obstructive class at a point $a = \frac{p}{q} \in \mathbb{Q}$ written in lowest terms with $\ell(a) = \ell(\mathbf{m})$. Let m_M be the last nonzero entry of the vector $\mathbf{m}$ and let I be the maximal open interval containing a such that $\mu(d, e; \mathbf{m})(a) > \sqrt{\frac{a}{2}}$. Then there exist integers $A < p$ and $B < (m_M + 1)q$ such that*

$$(d+e)\mu(d, e; \mathbf{m})(z) = \begin{cases} A + Bz & \text{if } z \leq a, z \in I, \\ (A + m_M p) + (B - m_M q)z & \text{if } z \geq a, z \in I. \end{cases}$$

Again, the proof is similar to the one of [19, Proposition 2.3.2].

Chapter 7

The Pell numbers

In this chapter we will define classes for points which are determined by sequences that, themselves, are described by the Pell numbers. We will first formally introduce the sequence of the Pell numbers and the sequence of the half companion Pell numbers, in Section 7.1. With these we then define four more sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 0}$, $(\gamma_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 0}$ that are determined by the Pell numbers and half companion Pell numbers, in Section 7.2. In Section 7.3 we simplify this description of four sequences to two sequences, $(\eta_n)_{n\geq 0}$ and $(\vartheta_n)_{n\geq 0}$, and determine the associated classes $E(\eta_n)$ and $E(\vartheta_n)$ for $n \geq 0$.

7.1 The Pell numbers and their identities

The infinite sequence $(P_n)_{n\geq 0}$, recursively defined by

$$\begin{aligned} P_0 &:= 0 \\ P_1 &:= 1 \\ P_n &:= 2P_{n-1} + P_{n-2} \quad \text{for } n \geq 2 \end{aligned} \tag{7.1}$$

is known as the sequence of **Pell numbers**. The unique solution of the equation $\sigma^2 = 2\sigma + 1$, that is $\sigma = 1 + \sqrt{2}$ is also known as the **silver ratio** and one has

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \sigma^2 = 3 + 2\sqrt{2}.$$

We now summarize some basic properties of Pell numbers.

Proposition 7.1.1 *The following holds true:*

(i) Two consecutive Pell numbers are coprime, i.e. the greatest common divisor

$$\mathrm{GCD}(P_{2n}, P_{2n-1}) = 1 \text{ and } \mathrm{GCD}(P_{2n+1}, P_{2n}) = 1$$

respectively.

(ii) The Pell number analogue of Cassini's identity for the Fibonacci numbers is

$$P_{n-1}P_{n+1} - P_n^2 = (-1)^n. \tag{7.2}$$

(iii) The relation

$$P_{r+s} = P_r P_{s+1} + P_{r-1} P_s \tag{7.3}$$

holds for all $r \geq 1$, $s \geq 0$.

Proof. First of all, one can show by induction that $2 \mid P_{2n}$ but $2 \nmid P_{2n+1}$, using the fact that $P_{2n} = 2P_{2n-1} + P_{2n-2}$. Next, one proves the statement in the following way: suppose it is true for n, that is $\mathrm{GCD}(P_{2n}, P_{2n-1}) = 1$. Now assume that

$$\mathrm{GCD}(P_{2n+2}, P_{2n+1}) = k \neq 1. \tag{7.4}$$

Applying $P_{2n} = 2P_{2n-1} + P_{2n-2}$ on P_{2n+2}, we obtain that

$$P_{2n+2} = 2P_{2n+1} + P_{2n}$$

and thus with (7.4), that $k \mid P_{2n}$. Now applying $P_{2n} = 2P_{2n-1} + P_{2n-2}$ another time, one gets

$$P_{2n+2} = 5P_{2n} + 2P_{2n-1},$$

that is $k \mid P_{2n-1}$ as $k \mid P_{2n+2}$ and $k \mid P_{2n}$. Hence $k \mid \mathrm{GCD}(P_{2n}, P_{2n-1})$. This contradicts the induction hypothesis and the statement follows for all $n \geq 1$. The proof of the other equation in (i) follows similarly.

The proof of (ii) can be found in [8], while (iii) has been proven in [3]. ■

Remark 7.1.1.1 Replacing P_{n+1} by $2P_n + P_{n-1}$ in (ii) of Proposition 7.1.1 yields directly that

$$2P_{n-1}P_n = P_n^2 - P_{n-1}^2 + (-1)^n. \tag{7.5}$$

Moreover, changing in (ii) of Proposition 7.1.1 the indices from n to $2n + 2$ we obtain that

$$P_{2n+1}P_{2n+3} - 1 = P_{2n+2}^2. \tag{7.6}$$

Proof of relation (7.6) of Remark 7.1.1.1. We prove that $P_{2n+1}P_{2n+3} - 1 = P_{2n+2}^2$ for $n \geq 0$ by induction over n: For $n = 0$, we have $P_1P_3 - 1 = P_2^2$, that is $5 - 1 = 2^2$ which is true. Suppose it is true for $(n - 1)$, i.e.

$$\begin{aligned} P_{2n}^2 &= P_{2n-1}P_{2n+1} - 1 \\ &= P_{2n+1}(P_{2n-1} + 2P_{2n} - 2P_{2n}) - 1 \\ &= P_{2n+1}(P_{2n+1} - 2P_{2n}) - 1 \\ &= P_{2n+1}^2 - 2P_{2n}P_{2n+1} - 1. \end{aligned}$$

This is equivalent to $P_{2n}^2 + 2P_{2n}P_{2n+1} = P_{2n+1}^2 - 1$ and thus

$$P_{2n}^2 + 4P_{2n}P_{2n+1} + 4P_{2n+1}^2 = 5P_{2n+1}^2 + 2P_{2n}P_{2n+1} - 1.$$

The left hand side of this equation equals to

$$(P_{2n} + 2P_{2n+1})^2 = P_{2n+2}^2,$$

while the right hand side equals

$$P_{2n+1}(P_{2n+1} + 2(2P_{2n+1} + P_{2n})) - 1 = P_{2n+1}(P_{2n+1} + 2P_{2n+2}) - 1 = P_{2n+1}(P_{2n+3}) - 1$$

and thus $P_{2n+1}P_{2n+3} - 1 = P_{2n+2}^2$. ■

Definition 7.1.2 *The sequence* $(Q_n)_{n\geq 0}$ *of the* ***companion Pell numbers*** *or* ***Pell-Lucas numbers*** *is defined recursively by*

$$\begin{aligned} Q_0 &:= 2 \\ Q_1 &:= 2 \\ Q_n &:= 2Q_{n-1} + Q_{n-2} \quad \text{for } n \geq 2 \end{aligned} \tag{7.7}$$

and the sequence $(H_n)_{n\geq 0}$ *of the* ***half companion Pell numbers*** *by*

$$\begin{aligned} H_0 &:= 1 \\ H_1 &:= 1 \\ H_n &:= 2H_{n-1} + H_{n-2} \quad \text{for } n \geq 2. \end{aligned} \tag{7.8}$$

Remark 7.1.2.1 (i) One directly sees that $H_n = \frac{1}{2}Q_n$ for all $n \in \mathbb{N}$.

(ii) The Pell numbers and the Pell-Lucas numbers can also be expressed by

$$P_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}$$

and

$$Q_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$$

for $n \in \mathbb{N}$.

Remark 7.1.2.2 There are two relations between the Pell numbers and the half companion Pell numbers.

(i) One is that

$$H_n = P_n + P_{n-1} \tag{7.9}$$

for all $n \geq 1$.

(ii) The other one is coming from Pell's equation $x^2 - ky^2 = \pm 1$, where $k \in \mathbb{N}$. It holds that every pair (H_n, P_n) solves Pell's equation for $k = 2$, that is

$$H_n^2 - 2P_n^2 = \pm 1$$

for all $n \in \mathbb{N}$. More precisely the pairs (H_n, P_n) solve the equation $H_n^2 - 2P_n^2 = 1$ for n even and $H_n^2 - 2P_n^2 = -1$ for n odd. Thus we obtain that

$$H_n + \sqrt{2}P_n = \left(1+\sqrt{2}\right)^n$$

and

$$H_n - \sqrt{2}P_n = \left(1-\sqrt{2}\right)^n$$

respectively.

Lemma 7.1.3 *For all* $m \geqslant 0$ *and* $k \leqslant 2m$ *holds that*

$$P_{2m-k} = (-1)^{k+1}\left(P_k H_{2m} - H_k P_{2m}\right). \tag{7.10}$$

Proof. One can prove this lemma by a straightforward induction over k: Using (7.1) and (ii) of Remark 7.1.2.2 one gets for $k = 1$ that

$$\begin{aligned} P_{2m-1} &= H_{2m} - P_{2m} \\ &= P_1 H_{2m} - H_1 P_{2m} \end{aligned}$$

and for $k = 2$ that

$$\begin{aligned} P_{2m-2} &= P_{2m} - 2P_{2m-2} \\ &= P_{2m} - 2\left(P_{2m} + P_{2m-1}\right) \\ &= 3P_{2m} - 2H_{2m} \\ &= (-1)P_2 H_{2m} - H_2 P_{2m}. \end{aligned}$$

Now let $k \geq 2$ and let the statement of the lemma be true for $k-2$ and $k-1$. Then by the induction hypothesis and with (7.1) and (7.8) it follows that

$$\begin{aligned} P_{2m-k} &= P_{2m-k+2} - 2P_{2m-k+1} \\ &= P_{2m-(k-2)} - 2\left(P_{2m} + P_{2m-(k-1)}\right) \\ &= (-1)^{(k-2)-1}\left(P_{k-2}H_{2m} - H_{k-2}P_{2m}\right) \\ &\quad - 2 \cdot (-1)^{(k-1)-1}\left(P_{k-1}H_{2m} - H_{k-1}P_{2m}\right) \\ &= (-1)^{k-1}\left(\left(P_k - 2P_{k-1}\right)H_{2m} - \left(H_k - 2H_{k-1}\right)P_{2m}\right) \\ &\quad + 2 \cdot (-1)^{k-1}\left(P_{k-1}H_{2m} - H_{k-1}P_{2m}\right) \\ &= (-1)^{k-1}\left(\left(P_k - 2P_{k-1} + 2P_{k-1}\right)H_{2m} \right. \\ &\quad \left. - \left(H_k - 2H_{k-1} + 2H_{k-1}\right)P_{2m}\right) \\ &= (-1)^{k-1}\left(P_k H_{2m} - H_k P_{2m}\right). \end{aligned}$$

■

7.2 Special sequences determined by the Pell numbers

We are now able to define and discuss the sequences that will finally determine the points $a \in [1, \sigma^2]$ that give the steps, that is the corner points of the Pell stairs. The latter, as we will see in Chapter 8, determine the embedding capacity function on the interval $[1, \sigma^2]$.

We define the sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 0}$, $(\gamma_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 0}$ in the following way:

$$\alpha_n := \frac{x_n}{x_{n-1}} \quad \text{and} \quad \gamma_n := \frac{x_n^2}{2y_n^2} \quad \text{for } n \geq 1 \tag{7.11}$$

and

$$\beta_n := \frac{y_{n+1}}{y_n} \quad \text{and} \quad \delta_n := \frac{2y_{n+1}^2}{x_n^2} \quad \text{for } n \geq 0, \tag{7.12}$$

where

$$x_n := P_{2n} + P_{2n-1} = H_{2n} \quad \text{and} \quad y_n := P_{2n-1} \quad \text{for } n \geq 1 \tag{7.13}$$

and

$$x_0 := 1, \qquad y_0 := 1.$$

We first illustrate some fundamental properties.

Lemma 7.2.1 *The following hold true:*

(i) The elements of the sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 0}$ represented by x_n and y_n as in (7.11) and (7.12) are in the form of lowest terms, i.e.

$$\operatorname{GCD}(x_n, x_{n-1}) = \operatorname{GCD}(y_{n+1}, y_n) = 1.$$

(ii) For elements of $(x_n)_{n\geq 0}$ *and* $(y_n)_{n\geq 0}$ *we have the following identities:*

$$\begin{aligned} 4x_n^2-2 &= y_{2n}+y_{2n+1}=2x_{2n}, && (7.14)\\ 4y_{n+1}^2-1 &= x_{2n+1}, && (7.15)\\ (y_n+y_{n+1})^2 &= 8y_ny_{n+1}-4, && (7.16)\\ x_n^2+2y_n^2 &= 4y_nx_n-1 && (7.17)\\ 2y_{n+1}^2+x_n^2 &= 4y_{n+1}x_n-1. && (7.18)\end{aligned}$$

Proof. One can show (i) by induction with the help of (7.1) and (i) of Proposition 7.1.1. For the equalities in (ii) we will do the calculations explicitly: We begin by verifying the first equation of (ii) with the help of (7.1), (7.3), (7.5) and (7.13). To that end, we proceed as follows:

$$\begin{aligned} y_{2n}+y_{2n+1} &= P_{4n-1}+P_{4n+1}\\ &= P_{2n}^2+P_{2n-1}^2+P_{2n}^2+P_{2n+1}^2\\ &= 2P_{2n}^2+P_{2n-1}^2+P_{2n+1}^2\\ &= 2P_{2n}^2+P_{2n-1}^2+4P_{2n}^2+4P_{2n}P_{2n-1}+P_{2n-1}^2\\ &= 6P_{2n}^2+4P_{2n}P_{2n-1}+2P_{2n-1}^2\\ &= 4P_{2n}^2+8P_{2n}P_{2n-1}+4P_{2n-1}^2+2\left(P_{2n}^2-2P_{2n}P_{2n-1}-P_{2n-1}^2\right)\\ &= 4P_{2n}^2+8P_{2n}P_{2n-1}+4P_{2n-1}^2+2\left(-(-1)^{2n}\right)\\ &= 4P_{2n}^2+8P_{2n}P_{2n-1}+4P_{2n-1}^2-2\\ &= 4\left(P_{2n-1}+P_{2n}\right)^2-2\\ &= 4x_n^2-2.\end{aligned}$$

Thus the equality $y_{2n}+y_{2n+1}=4x_n^2-2$ yields with (7.1) and (7.13) that

$$\begin{aligned} x_{2n}^2 &= \left(P_{4n-1}+P_{4n}\right)^2\\ &= \left(P_{4n-1}+\frac{1}{2}\left(P_{4n+1}-P_{4n-1}\right)\right)^2\\ &= \left(\frac{1}{2}P_{4n-1}+\frac{1}{2}P_{4n+1}\right)^2\\ &= \frac{1}{4}\left(P_{4n-1}+P_{4n+1}\right)^2\\ &= \frac{1}{4}\left(y_{2n}+y_{2n+1}\right)^2=\frac{1}{4}\left(4x_n^2-2\right)^2,\end{aligned}$$

i.e. $2x_{2n}=4x_n^2-2$, and the first equality of (ii) is proven.

Using (7.1), (7.3), (7.5) and (7.13) we calculate

$$\begin{aligned} 4y_{n+1}^2-1 &= 4P_{2n+1}^2-1\\ &= 4P_{2n+1}^2-(-1)^{2n}\\ &= 4P_{2n+1}^2-2P_{2n-1}P_{2n}-P_{2n-1}^2+P_{2n}^2\\ &= 4P_{2n+1}^2+P_{2n}^2-P_{2n-1}(2P_{2n}+P_{2n-1})\\ &= 4P_{2n+1}^2+P_{2n}^2-P_{2n-1}P_{2n+1}\\ &= 4P_{2n+1}^2+P_{2n}^2-(P_{2n+1}-2P_{2n})P_{2n+1}\\ &= 4P_{2n+1}^2+P_{2n}^2-P_{2n+1}^2+2P_{2n}P_{2n+1}\end{aligned}$$

$$
\begin{aligned}
&= P_{2n+1}^2 + P_{2n}^2 + 2P_{2n+1}^2 + 2P_{2n}P_{2n+1} \\
&= P_{2n+1}^2 + P_{2n}^2 + P_{2n+1}(2P_{2n} + 2P_{2n+1}) \\
&= P_{2n+1}^2 + P_{2n}^2 + P_{2n+1}P_{2n} + P_{2n+1}P_{2n+2} \\
&= P_{4n+1} + P_{4n+2} \\
&= x_{2n+1},
\end{aligned}
$$

which leads to the second equality of (ii). With the help of the equalities (7.1), (7.5) and (7.13) we obtain

$$\begin{aligned}(y_n + y_{n+1})^2 - 8y_n \cdot y_{n+1} &= P_{2n-1}^2 + 2P_{2n-1}P_{2n+1} + P_{2n+1}^2 - 8P_{2n-1}P_{2n+1}\\ &= P_{2n-1}^2 - 6P_{2n-1}P_{2n+1} + P_{2n+1}^2\\ &= P_{2n-1}^2 - 12P_{2n-1}P_{2n} - 6P_{2n-1}^2 + 4P_{2n}^2 + 4P_{2n}P_{2n-1} + P_{2n-1}^2\\ &= -4P_{2n-1}^2 - 8P_{2n-1}P_{2n} + 4P_{2n}^2\\ &= 4(P_{2n}^2 - P_{2n-1}^2) - 8P_{2n-1}P_{2n}\\ &= 4(2P_{2n-1}P_{2n} - (-1)^{2n}) - 8P_{2n-1}P_{2n}\\ &= 8P_{2n-1}P_{2n} - 8P_{2n-1}P_{2n} - 4\\ &= -4.\end{aligned}$$

To show the fourth equality we use (7.5) and (7.13) and compute

$$\begin{aligned}x_n^2 + 2y_n^2 &= 2P_{2n-1}^2 + (P_{2n-1} + P_{2n})^2\\ &= 3P_{2n-1}^2 + 2P_{2n-1}P_{2n} + P_{2n}^2\\ &= 3P_{2n-1}^2 + 2P_{2n-1}P_{2n} + 2P_{2n-1}P_{2n} + P_{2n-1}^2 - 1\\ &= 4P_{2n-1}(P_{2n-1} + P_{2n}) - 1\\ &= 4y_nx_n - 1.\end{aligned}$$

Using (7.1), (7.13) and the just proven equality we obtain that

$$\begin{aligned}2y_{n+1}^2 + x_n^2 &= 2(y_{n+1} + 2x_ny_n - y_n^2) - 1\\ &= 2(P_{2n+1}^2 + 2(P_{2n-1} + P_{2n})P_{2n-1} - P_{2n-1}^2) - 1\\ &= 2(P_{2n+1}^2 + P_{2n-1}(2P_{2n} + P_{2n-1})) - 1\\ &= 2(P_{2n+1}^2 + P_{2n-1}P_{2n+1}) - 1\\ &= 2(P_{2n+1}(2P_{2n} + 2P_{2n-1})) - 1\\ &= 2(2P_{2n+1}(P_{2n} + P_{2n-1})) - 1\\ &= 4y_{n+1}x_n - 1.\end{aligned}$$

■

Proposition 7.2.2 *Let $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 0}$, $(\gamma_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 0}$ be the four sequences constructed above. Then*

(i) $\beta_0 < \delta_0 < \alpha_1 < \gamma_1 < \beta_1 < \ldots < \delta_{n-1} < \alpha_n < \gamma_n < \beta_n < \delta_n < \alpha_{n+1} < \ldots$ *for all* $n \geq 1$,

(ii) the sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 0}$, $(\gamma_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 0}$ converge to the square of the silver ratio $\sigma^2 = 3 + 2\sqrt{2}$ for $n \to \infty$.

Proof. As $\delta_0 = 2 < 3 = \alpha_1$, we show that $\delta_n < \alpha_{n+1}$ for $n \geq 1$. With (7.1), (7.11), (7.12) and (7.13) we obtain

$$\begin{aligned}x_nx_{n+1} &= (7P_{2n} + 3P_{2n-1})(P_{2n-1} + P_{2n})\\ &= 3P_{2n-1}^2 + 10P_{2n-1}P_{2n} + 7P_{2n}^2\\ &> 2P_{2n-1}^2 = 2y_{n+1}^2,\end{aligned}$$

thus $2y_{n+1}^2x_n < x_{n+1}x_n^2$, which is equivalent to $\delta_n = \frac{2y_{n+1}^2}{x_n^2} < \frac{x_{n+1}}{x_n} = \alpha_{n+1}$.

The other three inequalities of (i) follow similarly using (7.1), (7.11), (7.12) and (7.13).

Now we look at the convergence of the sequence $(\alpha_n)_{n\geq 1}$. To that end we obtain with (7.1), (7.11) and (7.13) that

$$\alpha_n = \frac{x_n}{y_n} \cdot \frac{y_n}{x_{n-1}} = \frac{P_{2n-1}+P_{2n}}{P_{2n-1}} \cdot \frac{P_{2n-1}}{P_{2n-3}+P_{2n-2}} = \left(1+\frac{P_{2n}}{P_{2n-1}}\right) \cdot \left(\frac{P_{2n-2}}{P_{2n-1}} + \frac{P_{2n-3}}{P_{2n-1}}\right)^{-1}.$$

As $\frac{P_{2n}}{P_{2n-1}} \to 1+\sigma$ for $n \to \infty$, the sequence $(\alpha_n)_{n\geq 1}$ converges to

$$(1+\sigma)\cdot\left(\frac{1}{\sigma}+\frac{1}{\sigma^2}\right) = (1+\sigma)\cdot\frac{\sigma^2}{(1+\sigma)} = \sigma^2$$

for $n \to \infty$. The convergence of $(\beta_n)_{n\geq 0}$, $(\gamma_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 0}$ follows similarly. ■

7.3 Determining the classes of $(\eta_n)_{n\geq 0}$ and $(\vartheta_n)_{n\geq 0}$ coming from the Pell numbers

Using the sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 0}$, $(\gamma_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 0}$ defined in the Section 7.2 we now describe the sequence $\ldots < \alpha_n < \gamma_n < \beta_n < \delta_n < \ldots$ by two instead of four sequences, i.e. $\ldots < \vartheta_n < \eta_n < \ldots$. Finally we determine the classes $E(\eta_n)$ and $E(\vartheta_n)$ for $n \geq 0$.

We define the sequence $(\eta_n)_{n\geq 0}$ by

$$\eta_n := \begin{cases} \dfrac{2P_{n+1}^2}{H_n^2} =: \dfrac{p_n}{q_n} & \text{if } n \text{ is even or } 0, \\ \dfrac{H_{n+1}^2}{2P_n^2} =: \dfrac{p_n}{q_n} & \text{if } n \text{ is odd.} \end{cases} \tag{7.19}$$

Set $\mathbf{W}'(\eta_n) = q_n \mathbf{w}(\eta_n)$ with adding an extra 1 at the end. Define the classes $E(\eta_n)$ by

$$E(\eta_n) := \begin{cases} \left(P_{n+1}H_n, P_{n+1}H_n; \mathbf{W}'(\eta_n)\right) & \text{if } n \text{ is even or } 0, \\ \left(P_n H_{n+1}, P_n H_{n+1}; \mathbf{W}'(\eta_n)\right) & \text{if } n \text{ is odd.} \end{cases} \tag{7.20}$$

For instance,

$$\begin{aligned} E(\eta_0) &= \left(1,1;1^{\times 3}\right), \\ E(\eta_1) &= \left(3,3;2^{\times 4},1^{\times 3}\right), \\ E(\eta_2) &= \left(15,15;9^{\times 5},5,4,1^{\times 5}\right), \\ E(\eta_3) &= \left(85,85;50^{\times 5},39,11^{\times 3},6,5,1^{\times 6}\right). \end{aligned}$$

Moreover, we define the sequence $(\vartheta_n)_{n\geq 0}$ by

$$\vartheta_n := \begin{cases} \dfrac{H_{n+2}}{H_n} =: \dfrac{p_n}{q_n} & \text{if } n \text{ is even or } 0, \\ \dfrac{P_{n+2}}{P_n} =: \dfrac{p_n}{q_n} & \text{if } n \text{ is odd.} \end{cases}$$

Set $\mathbf{W}(\vartheta_n) = q_n \mathbf{w}(\vartheta_n)$. Then the classes $E(\vartheta_n)$ are defined by

$$E(\vartheta_n) := \begin{cases} \left(\frac{1}{4}(H_n+H_{n+2}), \frac{1}{4}(H_n+H_{n+2}); \mathbf{W}(\vartheta_n)\right) & \text{if } n \text{ is even or } 0, \\ \left(\frac{1}{4}(P_n+P_{n+2})+\frac{1}{2}, \frac{1}{4}(P_n+P_{n+2})-\frac{1}{2}; \mathbf{W}(\vartheta_n)\right) & \text{if } n \text{ is odd.} \end{cases} \tag{7.21}$$

For $1=\frac{H_1}{H_0}$ we define the class

$$E(1)=E\left(\frac{H_1}{H_0}\right):=\left(\frac{1}{4}\left(P_0+P_2\right)+\frac{1}{2},\frac{1}{4}\left(P_0+P_2\right)-\frac{1}{2};\mathbf{W}\left(1\right)\right).$$

For instance,

$$\begin{aligned} E\left(1\right)&=\left(1,0;1\right),\\ E\left(\vartheta_0\right)&=\left(1,1;1^{\times 3}\right),\\ E\left(\vartheta_1\right)&=\left(2,1;1^{\times 5}\right),\\ E\left(\vartheta_2\right)&=\left(5,5;3^{\times 5},2,1^{\times 2}\right),\\ E\left(\vartheta_3\right)&=\left(9,8;5^{\times 5},4,1^{\times 4}\right).\end{aligned}$$

Remark 7.3.0.1 The sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 0}$, $(\gamma_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 0}$ really shorten to the two sequences $(\eta_n)_{n\geq 0}$ and $(\vartheta_n)_{n\geq 0}$, as

$$\begin{aligned} 1 &= \beta_0, \\ \eta_{2n} &= \delta_n \quad \text{for } n\geq 0, \\ \eta_{2n-1} &= \gamma_n \quad \text{for } n\geq 1, \end{aligned}$$

and

$$\begin{aligned} \vartheta_{2n} &= \alpha_{n+1} \quad \text{for } n\geq 0, \\ \vartheta_{2n-1} &= \beta_n \quad \text{for } n\geq 1, \end{aligned}$$

as one can directly check.

As for the sequences $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 0}$, $(\gamma_n)_{n\geq 1}$ and $(\delta_n)_{n\geq 0}$, it follows with Remark 7.3.0.1 that the following proposition holds:

Proposition 7.3.1 *Let $(\eta_n)_{n\geq 0}$ and $(\vartheta_n)_{n\geq 0}$ be the sequences defined above. Then*

(i) $1<\eta_0<\vartheta_0<\eta_1<\vartheta_1<\ldots<\eta_{n-1}<\vartheta_{n-1}<\eta_n<\vartheta_n<\eta_{n+1}<\ldots$ *for all $n\geq 0$,*

(ii) the sequences $(\eta_n)_{n\geq 0}$ and $(\vartheta_n)_{n\geq 0}$ converge to the square of the silver ratio $\sigma^2=3+2\sqrt{2}$ for $n\to\infty$.

Chapter 8

The embedding capacity function $c(a)$ on $\left[1, \sigma^2\right]$

The goal of this chapter is to prove part (i) of Theorem 1.2.1. In Chapter 7 we introduced the sequence of the Pell numbers and the sequence of the half companion Pell numbers and with these the sequences $(\eta_n)_{n\geq 0}$ and $(\vartheta_n)_{n\geq 0}$ that, as we show here, are determining the embedding capacity function $c(a)$. We will then show in Section 8.3 that the classes $E\left(\eta_n\right)$ belong to $\mathcal{E}$ and in Section 8.5 that the classes $E\left(\vartheta_n\right)$ belong to $\mathcal{E}$.

We proceed in both cases as in [5]: In Section 8.2 and Section 8.4 we first show that the classes of η_n and ϑ_n satisfy the Diophantine equations (5.2), which are given in (i) of Proposition 5.1.7, for all $n \geq 0$. Then we will show in each case that the image under φ_* of each of the classes can be reduced to $(0; -1)$ by repeated Cremona moves, in Section 8.3 and Section 8.5. Thus we can explicitly determine the embedding capacity function $c(a)$ for $a \in \left[1, \sigma^2\right]$ in Section 8.6, as it follows by (iii) of Proposition 5.1.7 that the classes $E\left(\vartheta_n\right)$ and $E\left(\vartheta_n\right)$ belong to $\mathcal{E}$ for all $n \geq 0$ and thus with Proposition 5.1.8 that

$$c(a) = \sup_{(d,e;\mathbf{m})\in\mathcal{E}} \left\{ \sqrt{\frac{a}{2}}, \mu(d,e;\mathbf{m})(a) \right\}.$$

First we will give an idea for the more complex part, that is, to show the reducibility of $\varphi_*\left(E(\eta_n)\right)$ and $\varphi_*\left(E(\eta_n)\right)$ for $n \geq 0$ to $(0; -1)$, in Section 8.1. Here we reduce some classes explicitly to give a first idea how the reduction is working.

8.1 Explicit reductions for some of the classes of $(\eta_n)_{n\geq 0}$ and $(\vartheta_n)_{n\geq 0}$

Although there are separate sections that are dedicated to each sequence $(\eta_n)_{n\geq 0}$, $(\vartheta_n)_{n\geq 0}$ respectively, we will now outline the idea of proving that the image of the classes $E(\eta_n)$ and $E(\vartheta_n)$ under φ_* reduces to $(0; 1, 0, \ldots, 0)$ by repeated Cremona moves, by considering the first few elements of both sequences $(\eta_n)_{n\geq 0}$ and $(\vartheta_n)_{n\geq 0}$ and showing the reduction for a few cases explicitly. Let

$$1 < \eta_0 = 2 < \vartheta_0 = 3 < \eta_1 = \frac{9}{2} < \vartheta_1 = 5 < \eta_2 = \frac{50}{9} < \vartheta_2 = \frac{17}{3} < \eta_3 = \frac{289}{50} < \vartheta_3 = \frac{29}{5}.$$

With (7.20) and (7.21) we determine the classes of $1, \eta_0, \vartheta_0, \eta_1, \vartheta_1, \eta_2, \vartheta_2, \eta_3, \vartheta_3$:

$$\begin{array}{rcrcl} & & E\left(1\right) & = & \left(1, 0; 1\right), \\ E\left(\eta_0\right) & = & E\left(2\right) & = & \left(1, 1; 1^{\times 3}\right) \end{array}$$

$$\begin{aligned}
E(\vartheta_0) &= E(3) &&= \left(1,1;1^{\times 3}\right),\\
E(\eta_1) &= E\left(\tfrac{9}{2}\right) &&= \left(3,3;2^{\times 4},1^{\times 3}\right)\\
E(\vartheta_1) &= E(5) &&= \left(2,1;1^{\times 5}\right),\\
E(\eta_2) &= E\left(\tfrac{50}{9}\right) &&= \left(15,15;9^{\times 5},5,4,1^{\times 5}\right)\\
E(\vartheta_2) &= E\left(\tfrac{17}{3}\right) &&= \left(5,5;3^{\times 5},2,1^{\times 2}\right),\\
E(\eta_3) &= E\left(\tfrac{289}{50}\right) &&= \left(85,85;50^{\times 5},39,11^{\times 3},6,5,1^{\times 6}\right)\\
E(\vartheta_3) &= E\left(\tfrac{29}{5}\right) &&= \left(9,8;5^{\times 5},4,1^{\times 4}\right).
\end{aligned}$$

We will denote the Cremona transform of a class by $\to$ and by $\rightsquigarrow$ the reordering of a class. Recall, that

$$\varphi_*(d,e;\mathbf{m}) \;=\; (d+e-m_1;d-m_1,e-m_1,m_2,\dots,m_M)$$

and that the Cremona transform was defined in Definition 5.1.5 by

$$(2d-m_1-m_2-m_3;d-m_2-m_3,d-m_1-m_3,d-m_1-m_2,m_4,\dots,m_k)\,.$$

Now, we will explicitly reduce the classes of 1, η_0, ϑ_0, η_1 ϑ_1, η_2 and ϑ_2:

$$\begin{aligned}
\varphi_*(E(1)) &= \varphi_*((1,0;1))\\
&= (1+0-1;1-1,0-1)\\
&\rightsquigarrow (0;-1)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_*(E(\eta_0)) &= \varphi_*(E(2))\\
&= \varphi_*\left(\left(1,1;1^{\times 3}\right)\right)\\
&= \left(1;0,0,1^{\times 2}\right)\\
&\rightsquigarrow \left(1;1^{\times 2}\right)\\
&\to (0;0,0,-1)\\
&\rightsquigarrow (0;-1)\,.
\end{aligned}$$

Since

$$E(\vartheta_0) = \left(1,1;1^{\times 3}\right) = E(\eta_0),$$

it is reduced just like $\varphi_*(E(\eta_0))$. We have

$$\begin{aligned}
\varphi_*(E(\eta_1)) &= \varphi_*\left(E\left(\frac{9}{2}\right)\right)\\
&= \varphi_*\left(\left(3,3;2^{\times 4},1^{\times 3}\right)\right)\\
&= \left(3+3-2;3-2,3-2,2^{\times 3},1^{\times 3}\right)\\
&\rightsquigarrow \left(4;2^{\times 3},1^{\times 5}\right)\\
&\to \left(2\cdot 4-2-2-2;4-2-2,4-2-2,4-2-2,1^{\times 5}\right)\\
&\rightsquigarrow \left(2;1^{\times 5}\right)\\
&\to \left(2\cdot 2-1-1-1;2-1-1,2-1-1,2-1-1,1^{\times 2}\right)\\
&\rightsquigarrow \left(1;1^{\times 2}\right),
\end{aligned}$$

and

$$\begin{aligned}
\varphi_*(E(\vartheta_1)) &= \varphi_*(E(5))\\
&= \varphi_*\left(\left(2,1;1^{\times 5}\right)\right)\\
&= \left(2+1-1;2-1,1-1,1^{\times 4}\right)\\
&\rightsquigarrow \left(2;1^{\times 5}\right),
\end{aligned}$$

which reduces to $(0;-1)$ as we already saw in the calculation of $\varphi_*\left(E\left(\eta_1\right)\right)$. Moreover

$$\begin{aligned}
\varphi_*\left(E(\eta_2)\right) &= \varphi_*\left(E\left(\frac{50}{9}\right)\right) \\
&= \varphi_*\left(\left(15,15;9^{\times 5},5,4,1^{\times 5}\right)\right) \\
&= \left(15+15-9;15-9,9^{\times 4},5,4,1^{\times 5}\right) \\
&\rightsquigarrow \left(21;9^{\times 4},6^{\times 2},5,4,3^{\times 3},1^{\times 5}\right) \\
&\rightarrow \left(2\cdot 21-9-9-9;21-9-9,21-9-9,21-9-9,9,6^{\times 2},5,4,1^{\times 5}\right) \\
&\rightsquigarrow \left(15;9,6^{\times 2},5,4,3^{\times 3},1^{\times 5}\right) \\
&\rightarrow \left(2\cdot 15-9-6-6;15-6-6,15-9-6,15-9-6,5,4,3^{\times 3},1^{\times 5}\right) \\
&\rightsquigarrow \left(9;5,4,3^{\times 4},1^{\times 5}\right) \\
&\rightarrow \left(2\cdot 9-5-4-3;9-4-3,9-5-3,9-5-4,3^{\times 3},1^{\times 5}\right) \\
&\rightsquigarrow \left(6;3^{\times 3},2,1^{\times 6}\right) \\
&\rightarrow \left(2\cdot 6-3-3-3;6-3-3,6-3-3,6-3-3,2,1^{\times 6}\right) \\
&\rightsquigarrow \left(3;2,1^{\times 6}\right) \\
&\rightarrow \left(2\cdot 3-2-1-1;3-1-1,3-2-1,3-2-1,1^{\times 4}\right) \\
&\rightsquigarrow \left(2;1^{\times 5}\right),
\end{aligned}$$

which reduces to $(0;-1)$ as we already saw in the calculation of $\varphi_*\left(E\left(\eta_1\right)\right)$. Finally

$$\begin{aligned}
\varphi_*\left(E(\vartheta_2)\right) &= \varphi_*\left(E\left(\frac{17}{3}\right)\right) \\
&= \varphi_*\left(\left(5,5;3^{\times 5},2,1^{\times 2}\right)\right) \\
&= \left(7;2,2,3^{\times 4},2,1^{\times 2}\right) \\
&\rightsquigarrow \left(7;3^{\times 4},2^{\times 3},1^{\times 2}\right) \\
&\rightarrow \left(5;1,1,1,3,2^{\times 3},1^{\times 2}\right) \\
&\rightsquigarrow \left(5;3,2^{\times 3},1^{\times 5}\right) \\
&\rightarrow \left(3;1,0,0,2,1^{\times 5}\right) \\
&\rightsquigarrow \left(3;2,1^{\times 6}\right) \\
&\rightarrow \left(2;1,0,0,1^{\times 4}\right),
\end{aligned}$$

which reduces to $(0;-1)$ by repeated Cremona moves as we have already shown in the calculation of $\varphi_*\left(E\left(\eta_1\right)\right)$.

For η_3 and η_4, one can calculate and check the reduction just the way it has been done above. The reducibility of $E(\eta_n)$ under φ_*, for $n\geq 5$, is then proven in Section 8.3.

Furthermore, from the examples above, one can see, that in order to understand the reducibility by repeated Cremona moves, one must find the way the classes under φ_* can be reduced. One is led to conjecture that the classes $E\left(\vartheta_n\right)$ under φ_* reduce in several steps to each other. Indeed, as we will show in Section 8.5, the classes $E\left(\vartheta_{2m+2}\right)$ reduce to $E\left(\vartheta_{2m}\right)$ under φ_*, for all $m\geq 1$, by four Cremona moves and, similarly, $E\left(\vartheta_{2m+3}\right)$ reduce to $E\left(\vartheta_{2m+1}\right)$ under φ_*, for all $m\geq 1$, by four Cremona moves.

8.2 The classes $E\left(\eta_n\right)$ satisfy the Diophantine equations

In this section we prove that the classes $E(\eta_n)$ satisfy the Diophantine equations of Chapter 5.

Lemma 8.2.1 *The classes* $E(\eta_n)$ *satisfy the Diophantine equations (5.2).*

Proof. We will prove this separately for n even and odd. In both cases, we use Lemma 3.1.4 and the relation

$$-P_{2m}^2 + 2P_{2m}P_{2m-1} + P_{2m-1}^2 = 1 \tag{8.1}$$

which can be easily deduced from (7.5), given in Remark 7.1.1.1, for $n = 2m$.

In (7.20), the classes $E(\eta_n)$ are given as:

$$E(\eta_n) := \begin{cases} \left(P_{n+1}H_n, P_{n+1}H_n; \mathbf{W}'(\eta_n)\right) & \text{if } n \text{ is even,} \\ \left(P_nH_{n+1}, P_nH_{n+1}; \mathbf{W}'(\eta_n)\right) & \text{if } n \text{ is odd.} \end{cases}$$

Recall that $\mathbf{W}'(\eta_n) = q_n \cdot \mathbf{w}\left(\frac{p_n}{q_n}\right)$ with adding an extra 1 at the end, for $\eta_n = \frac{p_n}{q_n}$. Thus for $n = 2m$, we obtain with (7.1), (7.9) and (8.1) that

$$\begin{aligned}
\sum m_i &= H_{2m}^2 \sum w_i + 1 \\
&= H_{2m}^2 \left(\frac{2P_{2m+1}^2}{H_{2m}^2} + 1 - \frac{1}{H_{2m}^2}\right) + 1 \\
&= 2P_{2m+1}^2 + h_{2m}^2 - 1 + 1 \\
&= 2\left(2P_{2m} + P_{2m-1}\right)^2 + \left(P_{2m} + P_{2m-1}\right)^2 \\
&= 9P_{2m}^2 + 10P_{2m-1}P_{2m} + 3P_{2m-1}^2 \\
&= 8P_{2m}^2 + 12P_{2m-1}P_{2m} + 4P_{2m-1}^2 - \left(-P_{2m}^2 + 2P_{2m-1}P_{2m} + P_{2m-1}^2\right) \\
&= 8P_{2m}^2 + 12P_{2m-1}P_{2m} + 4P_{2m-1}^2 - (-1)^{2m} \\
&= 4P_{2m+1}H_{2m} - 1 \\
&= 2(d+e) - 1
\end{aligned}$$

and

$$\begin{aligned}
\sum m_i^2 &= H_{2m}^4 \sum w_i^2 + 1 \\
&= H_{2m}^4 \frac{2P_{2m+1}^2}{H_{2m}^2} + 1 \\
&= 2P_{2m+1}^2 H_{2m}^2 + 1 \\
&= 2de + 1.
\end{aligned}$$

Moreover, for $n = 2m - 1$, again with (7.9) and (8.1) that

$$\begin{aligned}
\sum m_i &= 2P_{2m-1}^2 \sum w_i + 1 \\
&= 2P_{2m-1}^2 \left(\frac{H_{2m}^2}{2P_{2m-1}^2} + 1 - \frac{1}{2P_{2m-1}^2}\right) + 1 \\
&= H_{2m}^2 + 2P_{2m-1}^2 - 1 + 1 \\
&= \left(P_{2m} + P_{2m-1}\right)^2 + 2P_{2m-1}^2 \\
&= P_{2m}^2 + 2P_{2m-1}P_{2m} + 3P_{2m-1}^2 \\
&= 4P_{2m-1}P_{2m} + 4P_{2m-1}^2 - \left(-P_{2m}^2 + 2P_{2m-1}P_{2m} + P_{2m-1}^2\right) \\
&= 4P_{2m-1}\left(P_{2m} + P_{2m-1}\right) - (-1)^{2m} \\
&= 4P_{2m-1}H_{2m} - 1 \\
&= 2(d+e) - 1
\end{aligned}$$

and

$$\begin{aligned}\sum m_i^2 &= 4P_{2m-1}^4 \sum w_i^2 + 1 \\ &= 4P_{2m-1}^4 \frac{H_{2m}^2}{2P_{2m-1}^2} + 1 \\ &= 2P_{2m-1}^2 H_{2m}^2 + 1 \\ &= 2de + 1.\end{aligned}$$

This proves the lemma. ■

8.3 The classes $E\left(\eta_n\right)$ belong to $\mathcal{E}$

In this Section we prove that the classes $E(\eta_n)$ belong to $\mathcal{E}$ for all $n \geq 0$, that is by (iii) of Proposition 5.1.7, we show that their image under φ_*, that is

$$\varphi_*(d, e; \mathbf{m}) \;=\; \left(d + e - m_1; d - m_1, e - m_1, m_2, \ldots, m_M\right),$$

can be reduced to $(0; -1, 0, \ldots, 0)$ by repeated Cremona moves. To that end, we follow the strategy presented in [5]. The fact that the classes satisfy the Diophantine equations has already been proven in Section 8.2.

8.3.1 The classes $E\left(\eta_{2m}\right)$ reduce to $(0; -1)$

In the following, we reduce the classes $E\left(\eta_{2m}\right)$ for $m \geq 3$. Therefore we first determine the continued fraction expansion, as well as the form of the classes of η_{2m} for $m \geq 3$ in Lemma 8.3.1. To show that the image of the classes $E(\eta_{2m})$ under φ_* reduce to $(0; -1)$ by standard Cremona moves, we first compute $\varphi_*\left(E(\eta_{2m})\right)$ in Lemma 8.3.3. The reduction itself will be shown in four steps (Lemmas 8.3.4, 8.3.5, 8.3.6 and 8.3.7).

Lemma 8.3.1 *The continued fraction expansion of η_{2m} for $m \geq 3$ is*

$$\left[5; \{1, 4\}^{\times(m-1)}, 1, 1, 3, 1, \{4, 1\}^{\times(m-1)}\right].$$

Moreover, with

$$u_j := \left(2H_j - P_j\right) H_{2m} + H_j P_{2m},$$

for $j \geq 1$, holds that

$$\begin{aligned}E\left(\eta_{2m}\right) = \Bigg(& P_{2m+1}H_{2m}, P_{2m+1}H_{2m}; \\ & \left(\frac{1}{2}u_{2m}\right)^{\times 5}, u_{2m-1}, \left(\frac{1}{2}u_{2m-2}\right)^{\times 4}, \ldots, u_3, \left(\frac{1}{2}u_2\right)^{\times 4}, \\ & u_1, H_{2m} + \frac{1}{2}P_{2m}, \left(\frac{1}{2}P_{2m}\right)^{\times 3}, P_{2m-1}, \\ & \left(\frac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, \ldots, \left(\frac{1}{2}P_2\right)^{\times 4}, P_1, 1\Bigg).\end{aligned}$$

Proof. Since (η_n) is an increasing sequence converging to $\sigma^2 < 6$ and $\eta_2 = \frac{50}{9} > 5$, the first term of $\mathbf{W}'\left(\eta_{2m}\right)$ is

$$\left(q_{2m}\right)^{\times 5} = \left(H_{2m}^2\right)^{\times 5} = \left(\frac{1}{2}u_{2m}\right)^{\times 5}$$

for any $m \geq 3$.

To determine the next terms, we first prove that for all $k \geq 1$,

$$u_{2k+1} > \frac{1}{2} u_{2k} > u_{2k-1}. \tag{8.2}$$

Indeed, using (7.1), (7.8) and (7.9), we obtain that

$$\begin{aligned}
u_{2k+1} &= \left(2H_{2k+1} - P_{2k+1}\right) H_{2m} + H_{2k+1} P_{2m} \\
&= \left(P_{2k+1} + 2P_{2k}\right) H_{2m} + H_{2k+1} P_{2m} \\
&= \left(4P_{2k} + P_{2k-1}\right) H_{2m} + \left(2H_{2k} + H_{2k-1}\right) P_{2m} \\
&> \left(\frac{1}{2} P_{2k} + P_{2k-1}\right) H_{2m} + \frac{1}{2} H_{2k} P_{2m} \\
&= \frac{1}{2}\left(2P_{2k} + 2P_{2k-1} - P_{2k}\right) H_{2m} + \frac{1}{2} H_{2k} P_{2m} \\
&= \frac{1}{2}\left(2H_{2k} - P_{2k}\right) H_{2m} + \frac{1}{2} H_{2k} P_{2m} \\
&= \frac{1}{2} u_{2k} \\
&= \frac{1}{2}\left(2H_{2k} - P_{2k}\right) H_{2m} + \frac{1}{2} H_{2k} P_{2m} \\
&= \left(H_{2k} - \frac{1}{2} P_{2k}\right) H_{2m} + \frac{1}{2}\left(2H_{2k-1} + H_{2k-2}\right) P_{2m} \\
&= \left(P_{2k} + P_{2k-1} - P_{2k-1} - \frac{1}{2} P_{2k-2}\right) H_{2m} + \frac{1}{2}\left(2H_{2k-1} + H_{2k-2}\right) P_{2m} \\
&= \left(P_{2k} - \frac{1}{2} P_{2k-2}\right) H_{2m} + \frac{1}{2}\left(2H_{2k-1} + H_{2k-2}\right) P_{2m} \\
&= \left(2P_{2k-1} + P_{2k-2} - \frac{1}{2} P_{2k-2}\right) H_{2m} + \frac{1}{2}\left(2H_{2k-1} + H_{2k-2}\right) P_{2m} \\
&= \left(2P_{2k-1} + \frac{1}{2} P_{2k-2}\right) H_{2m} + \frac{1}{2}\left(2H_{2k-1} + H_{2k-2}\right) P_{2m} \\
&= \left(P_{2k-1} + \frac{5}{2} P_{2k-2} + P_{2k-3}\right) H_{2m} + \left(H_{2k-1} + \frac{1}{2} H_{2k-2}\right) P_{2m} \\
&> \left(P_{2k-1} + 2P_{2k-2}\right) H_{2m} + H_{2k-1} P_{2m} \\
&= \left(2\left(P_{2k-1} + P_{2k-2}\right) - P_{2k-1}\right) H_{2m} + H_{2k-1} P_{2m} \\
&= \left(2H_{2k-1} - P_{2k-1}\right) H_{2m} + H_{2k-1} P_{2m} \\
&= u_{2k-1}.
\end{aligned}$$

From (7.19), the second term of $\mathbf{W}'\left(\eta_{2m}\right)$ is

$$p_{2m} - 5q_{2m} = 2P_{2m+1}^2 - 5H_{2m}^2.$$

Using (7.1), (7.9) and (8.2),

$$\begin{aligned}
2P_{2m+1}^2 - 5H_{2m}^2 &= 2\left(2P_{2m} + P_{2m-1}\right)^2 - 5\left(P_{2m} + P_{2m-1}\right)^2 \\
&= 3P_{2m}^2 - 2P_{2m} P_{2m-1} - 3P_{2m-1}^2 \\
&= 3P_{2m}\left(2P_{2m-1} + P_{2m-2}\right) - 2P_{2m} P_{2m-1} - 3P_{2m-1}^2 \\
&= 6P_{2m-1} P_{2m} + 3P_{2m-2} P_{2m} - 2P_{2m} P_{2m-1} - 3P_{2m-1}^2 \\
&= 4P_{2m-1} P_{2m} + 3P_{2m-2} P_{2m} - 3P_{2m-1}^2
\end{aligned}$$

$$\begin{aligned}
&= 4P_{2m-1}P_{2m} + 3P_{2m-2}P_{2m} + P_{2m-1}^2 - 4P_{2m-1}^2 + 2P_{2m-1}P_{2m} \\
&= 2P_{2m-1}P_{2m} + 3P_{2m-2}P_{2m} + P_{2m-1}^2 + 2P_{2m-1}\left(P_{2m} - 2P_{2m-1}\right) \\
&= 2P_{2m}P_{2m-1} + 3P_{2m}P_{2m-2} + P_{2m-1}^2 + 2P_{2m-1}P_{2m-2} \\
&= \left(P_{2m-1} + 2P_{2m-2}\right)\left(P_{2m} + P_{2m-1}\right) + \left(P_{2m-1} + P_{2m-2}\right)P_{2m} \\
&= \left(2H_{2m-1} - P_{2m-1}\right)H_{2m} + H_{2m-1}P_{2m} \\
&= u_{2m-1} \\
&< \frac{1}{2}u_{2m}.
\end{aligned}$$

Now for all $k \geq 1$, it follows with (7.1), (7.8) and (8.2) that

$$\begin{aligned}
u_{2k+1} - 4\left(\frac{1}{2}u_{2k}\right) &= \left(2H_{2k+1} - P_{2k+1} - 4H_{2k} + 2P_{2k}\right)H_{2m} + \left(H_{2k+1} - 2H_{2k}\right)P_{2m} \\
&= \left(2\left(H_{2k+1} - 2H_{2k}\right) + 2P_{2k} - P_{2k+1}\right)H_{2m} + \left(H_{2k+1} - 2H_{2k}\right)P_{2m} \\
&= \left(2H_{2k-1} - P_{2k-1}\right)H_{2m} + H_{2k-1}P_{2m} \\
&= u_{2k-1} \\
&< \frac{1}{2}u_{2k}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}u_{2k} - u_{2k-1} &= \left(H_{2k} - \frac{1}{2}P_{2k} - 2H_{2k-1} + P_{2k-1}\right)H_{2m} \\
&\qquad + \left(\frac{1}{2}H_{2k} - H_{2k-1}\right)P_{2m} \\
&= \frac{1}{2}\left(2H_{2k-2} - P_{2k-2}\right)H_{2m} + \frac{1}{2}H_{2k-2}P_{2m} \\
&= \frac{1}{2}u_{2k-2} \\
&< u_{2k-1}.
\end{aligned}$$

Thus the first terms of $\mathbf{W}'(\eta_{2m})$ are

$$\left(\frac{1}{2}u_{2m}\right)^{\times 5}, u_{2m-1}, \left(\frac{1}{2}u_{2m-2}\right)^{\times 4}, \ldots, u_3, \left(\frac{1}{2}u_2\right)^{\times 4}, u_1.$$

The next terms are $H_{2m} + \frac{1}{2}P_{2m}, \left(\frac{1}{2}P_{2m}\right)^{\times 3}, P_{2m-1}$. Indeed, by (7.1) and (7.8),

$$\frac{1}{2}u_2 - u_1 = H_{2m} + \frac{1}{2}P_{2m} < H_{2m} + P_{2m} = u_1,$$

$$u_1 - \left(H_{2m} + \frac{1}{2}P_{2m}\right) = \frac{1}{2}P_{2m} < H_{2m} + \frac{1}{2}P_{2m}$$

and

$$H_{2m} + \frac{1}{2}P_{2m} - 3\left(\frac{1}{2}P_{2m}\right) = P_{2m-1} < \frac{1}{2}P_{2m}.$$

Notice that for all $k \geq 1$,

$$P_{k+1} > \frac{1}{2}P_k > P_{k-1},$$

and thus by (7.1)

$$P_{2k+1} - 4\left(\frac{1}{2}P_{2k}\right) = P_{2k-1} < \frac{1}{2}P_{2k}$$

and

$$\frac{1}{2}P_{2k} - P_{2k-1} = \frac{1}{2}P_{2k-2} < P_{2k-1}$$

holds.

This proves that the last terms of $\mathbf{W}'(\eta_{2m})$ are $\left(\frac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, \dots, \left(\frac{1}{2}P_2\right)^{\times 4}, P_1, 1,$ with the last 1 added by the definition of $\mathbf{W}'(\eta_{2m})$. ∎

Let us now introduce some notation in order to simplify the expressions of the classes.

Definition 8.3.2 *Set*

$$A_k^m := \left(\left(\frac{1}{2}u_{2k}\right)^{\times 4}, u_{2k-1}, (P_{2k}H_{2m})^{\times 2}, \left(\frac{1}{2}u_{2k-2}\right)^{\times 4}, u_{2k-3}, \dots, \left(\frac{1}{2}u_2\right)^{\times 4}, u_1, \right.$$
$$\left. H_{2m} + \frac{1}{2}P_{2m}, \left(\frac{1}{2}P_{2m}\right)^{\times 3}, P_{2m-1} \right),$$

$$B_k^m := \left(\left(\frac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, \dots, \left(\frac{1}{2}P_{2m-2k+2}\right)^{\times 4}, P_{2m-2k+1}, \right.$$
$$\left. \left(\frac{1}{2}P_{2m-2k}\right)^{\times 8}, (P_{2m-2k-1})^{\times 2}, \dots, \left(\frac{1}{2}P_2\right)^{\times 8}, (P_1)^{\times 2}, 1 \right),$$

$$V_k^m := \left(P_{2k+1}H_{2m} + H_{2k}P_{2m}; A_k^m, B_k^m\right).$$

Thus A_k^m has the structure

$$\left[4, 1, 2, \{4,1\}^{\times(k-1)}, 1, 3, 1\right]$$

and B_k^m has the structure

$$\left[\{4,1\}^{\times(k-1)}, \{8,2\}^{\times(m-k)}, 1\right].$$

Here we use the convention that if $k = m$, B_m^m has the structure

$$\left[\{4,1\}^{\times(m-1)}, 1\right]$$

and that if $k = 1$, B_1^m has the structure

$$\left[\{8,2\}^{\times(m-1)}, 1\right].$$

The structure of the reduction process of a class $E(\eta_{2m})$ will be the following. First, we compute in Lemma 8.3.3 that the image of $E(\eta_{2m})$ under φ_* is

$$V_m^m = \left(P_{2m+1}H_{2m} + H_{2m}P_{2m}; A_m^m, B_m^m\right).$$

Then, we reduce V_m^m in Lemma 8.3.4 and Lemma 8.3.5 to

$$V^m := \left(H_{2m}; H_{2m} - \frac{1}{2}P_{2m}, \left(\frac{1}{2}P_{2m}\right)^{\times 3}, (P_{2m-1})^{\times 2}, B_1^m \right)$$

in $4(m-2)+8$ Cremona moves. Finally, we show in Lemma 8.3.6 and Lemma 8.3.7 that V^m reduces to $(0;-1)$ in $5(m-2)+8$ moves.

Remark 8.3.2.1 Since the Cremona transform of a class $(d;\mathbf{m})$ only modifies the first 3 entries of the vector $\mathbf{m}$, we will write some of the first entries of the classes and will abbreviate the other terms by $(*)$. It is important to notice that the terms denoted by $(*)$ will always be left invariant during the reduction process. Then, each time after applying the Cremona transform, we will reorder the entries of $\mathbf{m}$. We will not always reorder the entries in decreasing order, but this will have no consequence on the reduction because we will reorder them in a way such that the first 3 entries of the vector $\mathbf{m}$ will always be the 3 biggest entries in decreasing order. We will write down each step of the reduction, that is the class obtained after applying the Cremona transform and reordering. But sometimes when the reordering will not be obvious, we will write the intermediate step before reordering as we have done in detail in Section 8.1. Again we will denote the Cremona transform of a class by $\rightarrow$ and the reordering of a class by $\rightsquigarrow$. As we will freely use the three relations (7.1), (7.8) and (7.10) we recall them:

$$P_n = 2P_{n-1} + P_{n-2} \qquad \text{and} \qquad H_n = P_n + P_{n-1}, \qquad \text{for } n \geq 0,$$

and

$$P_{2m-k} = (-1)^{k+1}\left(P_k H_{2m} - H_k P_{2m}\right), \qquad \text{for } m \geq 0 \ \text{ and } \ k \leq 2m.$$

Lemma 8.3.3 *The image of* $E\left(\eta_{2m}\right)$ *by* φ_* *is the class*

$$\varphi_*\left(E\left(\eta_{2m}\right)\right) = \left(P_{2m+1}H_{2m} + H_{2m}P_{2m}; A_m^m, B_m^m\right) = V_m^m.$$

Proof. The first terms of $E\left(\eta_{2m}\right)$ are $\left(P_{2m+1}H_{2m}, P_{2m+1}H_{2m}; \frac{1}{2}u_{2m}, (*)\right)$. Since $\frac{1}{2}u_{2m} = H_{2m}^2$, we get

$$\begin{aligned}
\varphi_*\left(E\left(\eta_{2m}\right)\right) &= \left(2P_{2m+1}H_{2m} - H_{2m}^2; \left(P_{2m+1}H_{2m} - H_{2m}^2\right)^{\times 2}, (*)\right) \\
&= \left(P_{2m+1}H_{2m} + H_{2m}P_{2m}; \left(P_{2m}H_{2m}\right)^{\times 2}, (*)\right).
\end{aligned}$$

After reordering this class, we see that we indeed obtain V_m^m as required. ■

Lemma 8.3.4 *For all* $3 \leq k \leq m$, V_k^m *reduces to* V_{k-1}^m *in 4 Cremona moves.*

Proof. We have

$$\begin{aligned}
V_k^m \;=\; & \Bigg(P_{2k+1}H_{2m} + H_{2k}P_{2m}; \frac{1}{2}\left(\left(2H_{2k} - P_{2k}\right)H_{2m} + H_{2k}P_{2m}\right)^{\times 4}, \\
& \quad \left(2H_{2k-1} - P_{2k-1}\right)H_{2m} + H_{2k-1}P_{2m}, \left(P_{2k}H_{2m}\right)^{\times 2}, (*)\Bigg) \\
\rightarrow\; & \Bigg(\left(-H_{2k} + \frac{7}{2}P_{2k}\right)H_{2m} + \frac{1}{2}H_{2k}P_{2m}; \left(H_{2k-1}H_{2m}\right)^{\times 3}, \\
& \quad \frac{1}{2}\left(\left(2H_{2k} - P_{2k}\right)H_{2m} + H_{2k}P_{2m}\right), \left(2H_{2k-1} - P_{2k-1}\right)H_{2m} + H_{2k-1}P_{2m}, \\
& \quad \left(P_{2k}H_{2m}\right)^{\times 2}, (*)\Bigg) \\
\rightsquigarrow\; & \Bigg(\left(-H_{2k} + \frac{7}{2}P_{2k}\right)H_{2m} + \frac{1}{2}H_{2k}P_{2m}; \frac{1}{2}\left(\left(2H_{2k} - P_{2k}\right)H_{2m} + H_{2k}P_{2m}\right), \\
& \quad \left(2H_{2k-1} - P_{2k-1}\right)H_{2m} + H_{2k-1}P_{2m}, \left(P_{2k}H_{2m}\right)^{\times 2}, \\
& \quad \left(H_{2k-1}H_{2m}\right)^{\times 3}, (*)\Bigg) \\
\rightarrow\; & \Bigg(\frac{3}{2}P_{2k}H_{2m} + \frac{1}{2}H_{2k-2}P_{2m}; \left(2H_{2k} - \frac{5}{2}P_{2k}\right)H_{2m} + \frac{1}{2}H_{2k-2}P_{2m}, P_{2k-2}H_{2m}, \\
& \quad P_{2k-1}H_{2m} - H_{2k-1}P_{2m}, P_{2k}H_{2m}, \left(H_{2k-1}H_{2m}\right)^{\times 3}, (*)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
\rightsquigarrow\quad & \Bigg(\frac{3}{2}P_{2k}H_{2m}+\frac{1}{2}H_{2k-2}P_{2m};P_{2k}H_{2m},\left(H_{2k-1}H_{2m}\right)^{\times 3},\\
& \qquad \left(2H_{2k}-\frac{5}{2}P_{2k}\right)H_{2m}+\frac{1}{2}H_{2k-2}P_{2m},P_{2k-2}H_{2m},P_{2m-(2k-1)},(*)\Bigg)\\
\to\quad & \Bigg(2P_{2k-1}H_{2m}+H_{2k-2}P_{2m};\left(2H_{2k}-\frac{5}{2}P_{2k}\right)H_{2m}+\frac{1}{2}H_{2k-2}P_{2m},\\
& \qquad \left(-\frac{1}{2}P_{2k-2}H_{2m}+\frac{1}{2}H_{2k-2}P_{2m}\right)^{\times 2},H_{2k-1}H_{2m},\\
& \qquad \left(2H_{2k}-\frac{5}{2}P_{2k}\right)H_{2m}+\frac{1}{2}H_{2k-2}P_{2m},P_{2k-2}H_{2m},P_{2m-(2k-1)},(*)\Bigg)\\
\rightsquigarrow\quad & \Bigg(2P_{2k-1}H_{2m}+H_{2k-2}P_{2m};H_{2k-1}H_{2m},\left(\left(2H_{2k}-\frac{5}{2}P_{2k}\right)H_{2m}+\frac{1}{2}H_{2k-2}P_{2m}\right)^{\times 2},\\
& \qquad P_{2k-2}H_{2m},\left(\frac{1}{2}P_{2m-(2k-2)}\right)^{\times 2},P_{2m-(2k-1)},(*)\Bigg)\\
\to\quad & \Bigg(P_{2k-1}H_{2m}+H_{2k-2}P_{2m};P_{2k-2}H_{2m},\left(-\frac{1}{2}P_{2k-2}H_{2m}+\frac{1}{2}H_{2k-2}P_{2m}\right)^{\times 2},\\
& \qquad P_{2k-2}H_{2m},\left(\frac{1}{2}P_{2m-(2k-2)}\right)^{\times 2},P_{2m-(2k-1)},(*)\Bigg)\\
\to\quad & \Bigg(P_{2k-1}H_{2m}+H_{2k-2}P_{2m};\left(P_{2k-2}H_{2m}\right)^{\times 2},\left(\frac{1}{2}P_{2m-(2k-2)}\right)^{\times 4},P_{2m-(2k-1)},(*)\Bigg).
\end{aligned}
$$

Now, after reordering this last class, we obtain V^m_{k-1} as required. ■

Lemma 8.3.5 *V^m_2 reduces in 8 Cremona moves to the class*

$$V^m := \left(H_{2m};H_{2m}-\frac{1}{2}P_{2m},\left(\frac{1}{2}P_{2m}\right)^{\times 3},\left(P_{2m-1}\right)^{\times 2},B^m_1\right).$$

Proof. We have

$$
\begin{aligned}
V^m_2 \quad = \quad & \Bigg(29H_{2m}+17P_{2m};\left(11H_{2m}+\frac{17}{2}P_{2m}\right)^{\times 4},9H_{2m}+7P_{2m},\left(12H_{2m}\right)^{\times 2},\\
& \left(2H_{2m}+\frac{3}{2}P_{2m}\right)^{\times 4},H_{2m}+P_{2m},H_{2m}+\frac{1}{2}P_{2m},(*)\Bigg)\\
\to\quad & \Bigg(25H_{2m}+\frac{17}{2}P_{2m};11H_{2m}+\frac{17}{2}P_{2m},9H_{2m}+7P_{2m},\left(12H_{2m}\right)^{\times 2},\left(7H_{2m}\right)^{\times 3},\\
& \qquad \left(2H_{2m}+\frac{3}{2}P_{2m}\right)^{\times 4},H_{2m}+P_{2m},H_{2m}+\frac{1}{2}P_{2m},(*)\Bigg)\\
\to\quad & \Bigg(18H_{2m}+\frac{3}{2}P_{2m};12H_{2m},\left(7H_{2m}\right)^{\times 3},4H_{2m}+\frac{3}{2}P_{2m},\left(2H_{2m}+\frac{3}{2}P_{2m}\right)^{\times 4},\\
& \qquad 2H_{2m},H_{2m}+P_{2m},H_{2m}+\frac{1}{2}P_{2m},P_{2m-3},(*)\Bigg)\\
\to\quad & \Bigg(10H_{2m}+3P_{2m};7H_{2m},\left(4H_{2m}+\frac{3}{2}P_{2m}\right)^{\times 2},\left(2H_{2m}+\frac{3}{2}P_{2m}\right)^{\times 4},\\
& \qquad 2H_{2m},H_{2m}+P_{2m},H_{2m}+\frac{1}{2}P_{2m},\left(\frac{1}{2}P_{2m-2}\right)^{\times 2},P_{2m-3},(*)\Bigg)
\end{aligned}
$$

$$\begin{aligned}
&\to \Bigg(5H_{2m}+3P_{2m};\left(2H_{2m}+\frac{3}{2}P_{2m}\right)^{\times 4},\left(2H_{2m}\right)^{\times 2},H_{2m}+P_{2m},H_{2m}+\frac{1}{2}P_{2m},\\
&\qquad\left(\frac{1}{2}P_{2m-2}\right)^{\times 2},P_{2m-3},(*)\Bigg)\\
&\to \Bigg(4H_{2m}+\frac{3}{2}P_{2m};2H_{2m}+\frac{3}{2}P_{2m},\left(2H_{2m}\right)^{\times 2},H_{2m}+P_{2m},H_{2m}+\frac{1}{2}P_{2m},\\
&\qquad\left(H_{2m}\right)^{\times 3},\left(\frac{1}{2}P_{2m-2}\right)^{\times 4},P_{2m-3},(*)\Bigg)\\
&\to \Bigg(2H_{2m}+\frac{3}{2}P_{2m};H_{2m}+P_{2m},H_{2m}+\frac{1}{2}P_{2m},\frac{3}{2}P_{2m},\left(H_{2m}\right)^{\times 3},\left(\frac{1}{2}P_{2m-2}\right)^{\times 4},\\
&\qquad P_{2m-3},(*)\Bigg)\\
&\to \left(2H_{2m};\left(H_{2m}\right)^{\times 3},H_{2m}-\frac{1}{2}P_{2m},P_{2m-1},\left(\frac{1}{2}P_{2m-2}\right)^{\times 4},P_{2m-3},(*)\right)\\
&\to \left(H_{2m};H_{2m}-\frac{1}{2}P_{2m},P_{2m-1},\left(\frac{1}{2}P_{2m-2}\right)^{\times 4},P_{2m-3},(*)\right).
\end{aligned}$$

After reordering this last class, we obtain V^m as required. ∎

Lemma 8.3.6 *For all $m \geq 3$, V^m reduces in 5 Cremona moves to V^{m-1}.*

Proof. We have

$$\begin{aligned}
V^m &= \left(H_{2m};H_{2m}-\frac{1}{2}P_{2m},\left(\frac{1}{2}P_{2m}\right)^{\times 3},\left(P_{2m-1}\right)^{\times 2},\left(\frac{1}{2}P_{2m-2}\right)^{\times 8},(*)\right)\\
&\to \left(H_{2m}-\frac{1}{2}P_{2m};\frac{1}{2}P_{2m},\left(P_{2m-1}\right)^{\times 3},\left(\frac{1}{2}P_{2m-2}\right)^{\times 8},(*)\right)\\
&\to \left(\frac{1}{2}P_{2m};P_{2m-1},\left(\frac{1}{2}P_{2m-2}\right)^{\times 9},(*)\right)\\
&\to \left(P_{2m-1};P_{2m-1}-\frac{1}{2}P_{2m-2},\left(\frac{1}{2}P_{2m-2}\right)^{\times 7},(*)\right)\\
&\to \left(P_{2m-1}-\frac{1}{2}P_{2m-2};H_{2m-2},\left(\frac{1}{2}P_{2m-2}\right)^{\times 5},(*)\right)\\
&\to \left(H_{2m-2};H_{2m-2}-\frac{1}{2}P_{2m-2},\left(\frac{1}{2}P_{2m-2}\right)^{\times 3},(*)\right).
\end{aligned}$$

After reordering this class, we obtain V^{m-1} as required. ∎

Lemma 8.3.7 *V^2 reduces in 8 Cremona moves to $(0;-1)$.*

Proof. We have

$$\begin{aligned}
V^2 &= \left(17;11,6^{\times 3},5^{\times 2},1^{\times 11}\right)\\
&\to \left(11;6,5^{\times 3},1^{\times 11}\right)\\
&\to \left(6;5,1^{\times 12}\right)
\end{aligned}$$

$$
\begin{aligned}
&\to \left(5;4,1^{\times 10}\right)\\
&\to \left(4;3,1^{\times 8}\right)\\
&\to \left(3;2,1^{\times 6}\right)\\
&\to \left(2;1^{\times 5}\right)\\
&\to \left(1;1^{\times 2}\right)\\
&\to (0;-1)
\end{aligned}
$$

which proves the claim. ■

8.3.2 The classes $E\left(\eta_{2m-1}\right)$ reduce to $(0;-1)$

In the following, we reduce the classes $E\left(\eta_{2m-1}\right)$ for $m \geq 3$. Therefore we first determine the continued fraction expansion, as well as the form of the classes of η_{2m-1} for $m \geq 3$ in Lemma 8.3.8. To show that the image of the classes $E(\eta_{2m-1})$ under φ_* reduces to $(0;-1)$ by standard Cremona moves, we first compute $\varphi_*\left(E(\eta_{2m-1})\right)$ in Lemma 8.3.10. The reduction itself will be shown in four steps (Lemmas 8.3.11, 8.3.12, 8.3.13 and 8.3.14).

Lemma 8.3.8 *The continued fraction expansion of* η_{2m-1} *is*

$$\left[5;\{1,4\}^{\times(m-2)},1,3,1,1,\{4,1\}^{\times(m-1)}\right].$$

Moreover, if $v_j := \left(2H_j+P_j\right)H_{2m}-H_jP_{2m}$*, then*

$$
\begin{aligned}
E\left(\eta_{2m-1}\right) = \Bigg(& P_{2m-1}H_{2m}, P_{2m-1}H_{2m};\\
&\left(\frac{1}{2}v_{2m-2}\right)^{\times 5}, v_{2m-3}, \left(\frac{1}{2}v_{2m-4}\right)^{\times 4}, \ldots, v_3, \left(\frac{1}{2}v_2\right)^{\times 4},\\
&v_1, \left(H_{2m}-\frac{1}{2}P_{2m}\right)^{\times 3}, \frac{1}{2}P_{2m}, P_{2m-1},\\
&\left(\frac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, \ldots, \left(\frac{1}{2}P_2\right)^{\times 4}, P_1, 1\Bigg).
\end{aligned}
$$

Proof. The first terms of $\mathbf{W}'\left(\eta_{2m-1}\right)$ are $\left(\frac{1}{2}v_{2m-2}\right)^{\times 5}$ as $5<\eta_{2m-1}<6$ for $m \geq 2$ and since with (7.1) and (7.9) we obtain

$$
\begin{aligned}
\frac{1}{2}v_{2m-2} &= \frac{1}{2}\left(\left(2H_{2m-2}+P_{2m-2}\right)H_{2m}-H_{2m-2}P_{2m}\right)\\
&= \frac{1}{2}\left(\left(3P_{2m-2}+2P_{2m-3}\right)\left(P_{2m}+P_{2m-1}\right)-\left(P_{2m-2}+P_{2m-3}\right)P_{2m}\right)\\
&= \frac{1}{2}\left(\left(P_{2m-1}+P_{2m-2}+P_{2m-3}\right)\left(P_{2m}+P_{2m-1}\right)-\left(P_{2m-2}+P_{2m-3}\right)P_{2m}\right)\\
&= \frac{1}{2}\left(P_{2m-1}P_{2m}+P_{2m-1}^2+\left(P_{2m-2}+P_{2m-3}\right)P_{2m-1}\right)\\
&= \frac{1}{2}\left(P_{2m-1}^2+\left(2P_{2m-1}+2P_{2m-2}+P_{2m-3}\right)P_{2m-1}\right)\\
&= \frac{1}{2}\left(P_{2m-1}^2+3P_{2m-1}P_{2m-1}\right)
\end{aligned}
$$

$$= 2P_{2m-1}^2.$$

Before determining the next terms, we prove that

$$v_{2k+1} > \frac{1}{2}v_{2k} > v_{2k-1} \tag{8.3}$$

for all $k \geq 1$. Indeed using (7.1) and (7.9)

$$\begin{aligned}
v_{2k+1} &= \left(H_{2k+1} + P_{2k+1}\right) P_{2m} + \left(2H_{2k+1} + P_{2k+1}\right) P_{2m-1} \\
&= \left(2P_{2k+1} + P_{2k}\right) P_{2m} + \left(3P_{2k+1} + 2P_{2k}\right) P_{2m-1} \\
&= \left(5P_{2k} + 2P_{2k-1}\right) P_{2m} + \left(8P_{2k} + 3P_{2k-1}\right) P_{2m-1} \\
&> \left(P_{2k} + \frac{1}{2}P_{2k-1}\right) P_{2m} + \left(\frac{3}{2}P_{2k} + P_{2k-1}\right) P_{2m-1} \\
&= \frac{1}{2}\left(H_{2k} + P_{2k}\right) P_{2m} + \frac{1}{2}\left(2H_{2k} + P_{2k}\right) P_{2m-1} \\
&= \frac{1}{2}v_{2k} \\
&= \frac{1}{2}\left(H_{2k} + P_{2k}\right) P_{2m} + \frac{1}{2}\left(2H_{2k} + P_{2k}\right) P_{2m-1} \\
&= \frac{1}{2}\left(P_{2k-1} + 2P_{2k}\right) P_{2m} + \frac{1}{2}\left(2P_{2k-1} + 3P_{2k}\right) P_{2m-1} \\
&= \frac{1}{2}\left(P_{2k-1} + 2\left(2P_{2k-1} + P_{2k-2}\right)\right) P_{2m} \\
&\quad + \frac{1}{2}\left(2\left(2P_{2k-2} + P_{2k-3}\right) + 3\left(2P_{2k-1} + P_{2k-2}\right)\right) P_{2m-1} \\
&= \frac{1}{2}\left(5P_{2k-1} + 2P_{2k-2}\right) P_{2m} + \frac{1}{2}\left(7P_{2k-2} + 2P_{2k-3} + 6P_{2k-1}\right) P_{2m-1} \\
&= \left(\frac{5}{2}P_{2k-1} + P_{2k-2}\right) P_{2m} + \left(3P_{2k-1} + \frac{7}{2}P_{2k-2} + P_{2k-3}\right) P_{2m-1} \\
&> \left(2P_{2k-1} + P_{2k-2}\right) P_{2m} + \left(3P_{2k-1} + 2P_{2k-2}\right) P_{2m-1} \\
&= \left(H_{2k-1} + P_{2k-1}\right) P_{2m} + \left(2H_{2k-1} + P_{2k-1}\right) P_{2m-1} \\
&= v_{2k-1}.
\end{aligned}$$

So the next term of $\mathbf{W}'\left(\eta_{2m-1}\right)$ is $H_{2m}^2 - 5\left(2P_{2m-1}^2\right) = v_{2m-3} < \frac{1}{2}v_{2m-2}$. Moreover, for all $k \geq 1$, we obtain with (7.1), (7.8), (7.9) and (8.3) that

$$\begin{aligned}
v_{2k+1} - 4\left(\frac{1}{2}v_{2k}\right) &= \left(2H_{2k+1} + P_{2k+1}\right) H_{2m} - H_{2k+1}P_{2m} - 2\left(\left(2H_{2k} + P_{2k}\right) H_{2m} - H_{2k}P_{2m}\right) \\
&= \left(2H_{2k+1} + P_{2k+1} - 4H_{2k} - 2P_{2k}\right) H_{2m} - \left(H_{2k+1} - 2H_{2k}\right) P_{2m} \\
&= \left(2\left(H_{2k+1} - 2H_{2k}\right) + P_{2k+1} - 2P_{2k}\right) H_{2m} - H_{2k-1}P_{2m} \\
&= \left(2H_{2k-1} + P_{2k-1}\right) H_{2m} - H_{2k-1}P_{2m} \\
&= v_{2k-1} \\
&< \frac{1}{2}v_{2k}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}v_{2k} - v_{2k-1} &= \left(H_{2k} + \frac{1}{2}P_{2k} - 2H_{2k-1} - P_{2k-1}\right) H_{2m} - \left(\frac{1}{2}H_{2k} - H_{2k-1}\right) P_{2m} \\
&= \left(H_{2k} - 2H_{2k-1} + \frac{1}{2}\left(P_{2k} - 2P_{2k-1}\right)\right) H_{2m} - \frac{1}{2}\left(H_{2k} - 2H_{2k-1}\right) P_{2m}
\end{aligned}$$

$$\begin{aligned}
&= \left(H_{2k-2}+\frac{1}{2}P_{2k-2}\right)H_{2m}-\frac{1}{2}\left(H_{2k}-2H_{2k-1}\right)P_{2m}\\
&= \frac{1}{2}\left(\left(2H_{2k-2}+P_{2k-2}\right)H_{2m}-H_{2k-2}P_{2m}\right)\\
&= \frac{1}{2}v_{2k-2}\\
&< v_{2k-1}.
\end{aligned}$$

This proves that the first terms of $\mathbf{W}'\left(\eta_{2m-1}\right)$ are

$$\left(\frac{1}{2}v_{2m-2}\right)^{\times 5}, v_{2m-3}, \left(\frac{1}{2}v_{2m-4}\right)^{\times 4}, \ldots, v_3, \left(\frac{1}{2}v_2\right)^{\times 4}, v_1.$$

The next three terms are $\left(H_{2m}-\frac{1}{2}P_{2m}\right)^{\times 3}, \frac{1}{2}P_{2m}, P_{2m-1}$ since

$$\frac{1}{2}v_2 - v_1 = H_{2m}-\frac{1}{2}P_{2m} < 3H_{2m}-P_{2m} = v_1,$$

$$v_1 - 3\left(H_{2m}-\frac{1}{2}P_{2m}\right) = \frac{1}{2}P_{2m} < H_{2m}-\frac{1}{2}P_{2m}$$

and

$$H_{2m}-\frac{1}{2}P_{2m}-\frac{1}{2}P_{2m} = P_{2m-1} < \frac{1}{2}P_{2m}.$$

Since the last terms are the same as those of $\mathbf{W}'\left(\eta_{2m}\right)$, the lemma is proved. ■

Let us introduce again some notation.

Definition 8.3.9 *Set*

$$\begin{aligned}
\hat{A}_k^m := \Bigg(&\left(\frac{1}{2}v_{2k-2}\right)^{\times 4}, v_{2k-3}, \left(H_{2k-1}H_{2m}-H_{2k-1}P_{2m}\right)^{\times 2}, \left(\frac{1}{2}v_{2k-4}\right)^{\times 4}, v_{2k-5}, \ldots,\\
&\left(\frac{1}{2}v_2\right)^{\times 4}, v_1, \left(H_{2m}-\frac{1}{2}P_{2m}\right)^{\times 3}, \frac{1}{2}P_{2m}, P_{2m-1}\Bigg),
\end{aligned}$$

$$\begin{aligned}
\hat{B}_k^m := \Bigg(&\left(\frac{1}{2}P_{2m-2}\right)^{\times 4}, P_{2m-3}, \ldots, \left(\frac{1}{2}P_{2m-2k+2}\right)^{\times 4}, P_{2m-2k+1},\\
&\left(\frac{1}{2}P_{2m-2k}\right)^{\times 8}, \left(P_{2m-2k-1}\right)^{\times 2}, \ldots, \left(\frac{1}{2}P_2\right)^{\times 8}, \left(P_1\right)^{\times 2}, 1\Bigg),
\end{aligned}$$

$$\hat{V}_k^m := \left(P_{2k}H_{2m}+H_{2k-1}P_{2m}; \hat{A}_k^m, \hat{B}_k^m\right).$$

Note that $\hat{B}_k^m$ is actually equal to the vector B_k^m that we used in the reduction of the classes $E\left(\eta_{2m}\right)$. Here, $\hat{A}_k^m$ has the structure

$$\left[4,1,2,\{4,1\}^{\times(k-2)},3,1,1\right]$$

and $\hat{B}_k^m$ has the structure

$$\left[\{4,1\}^{\times(k-1)},\{8,2\}^{\times(m-k)},1\right].$$

We use again the convention that if $k=m$, $\hat{B}_m^m$ has the structure $\left[\{4,1\}^{\times(m-1)},1\right]$ and that if $k=1$, $\hat{B}_1^m$ has the structure $\left[\{8,2\}^{\times(m-1)},1\right]$.

Remark 8.3.9.1 Notice that we will now use again the notation introduced in Section 8.1 and stated in Remark 8.3.2.1, that is the Cremona transform of a class is denoted by → and the reordering of a class by ⇝. In the following we use again frequently the three relations (7.1), (7.8) and (7.10), without specifying them. We recall them here:

$$P_n = 2P_{n-1} + P_{n-2} \qquad \text{and} \qquad H_n = P_n + P_{n-1}, \qquad\qquad \text{for } n \geq 0,$$

and

$$P_{2m-k} = (-1)^{k+1}\left(P_k H_{2m} - H_k P_{2m}\right), \qquad \text{for } m \geq 0 \ \text{ and } \ k \leq 2m.$$

Lemma 8.3.10 *The image of* $E\left(\eta_{2m-1}\right)$ *by* φ_* *is the class*

$$\varphi_*\left(E\left(\eta_{2m-1}\right)\right) = \left(P_{2m}H_{2m} - H_{2m-1}P_{2m}; \hat{A}_m^m, \hat{B}_m^m\right) = \hat{V}_m^m.$$

Proof. The first terms of $E\left(\eta_{2m-1}\right)$ are $\left(P_{2m-1}H_{2m}, P_{2m-1}H_{2m}; \frac{1}{2}v_{2m-2}, (*)\right)$. Since $\frac{1}{2}v_{2m-2} = 2P_{2m-1}^2$, we get

$$\begin{aligned}
\varphi_*\left(E\left(\eta_{2m-1}\right)\right) &= \varphi_*\left(\left(P_{2m-1}H_{2m}, P_{2m-1}H_{2m}; \frac{1}{2}v_{2m-2}, (*)\right)\right) \\
&= \left(2P_{2m-1}H_{2m} - 2P_{2m-1}^2; \left(P_{2m-1}H_{2m} - 2P_{2m-1}^2\right)^{\times 2}, (*)\right) \\
&= \left(P_{2m}H_{2m} - H_{2m-1}P_{2m}; \left(H_{2m-1}H_{2m} - H_{2m-1}P_{2m}\right)^{\times 2}, (*)\right).
\end{aligned}$$

After reordering, this last class is $\left(P_{2m}H_{2m} - H_{2m-1}P_{2m}; \hat{A}_m^m, \hat{B}_m^m\right) = \hat{V}_m^m$ as required. ∎

Lemma 8.3.11 *For all* $3 \leq k \leq m$, $\hat{V}_k^m$ *reduces to* $\hat{V}_{k-1}^m$ *in 4 Cremona moves.*

Proof. We have

$$\begin{aligned}
\hat{V}_k^m \ = \ & \Big(P_{2k}H_{2m} - H_{2k-1}P_{2m}; \frac{1}{2}\left(\left(2H_{2k-2} + P_{2k-2}\right)H_{2m} - H_{2k-2}P_{2m}\right)^{\times 4}, \\
& \quad \left(2H_{2k-3} + P_{2k-3}\right)H_{2m} - H_{2k-3}P_{2m}, \left(H_{2k-1}H_{2m} - H_{2k-1}P_{2m}\right)^{\times 2}, (*)\Big) \\
\to \ & \left(\left(H_{2k-2} + \frac{9}{2}P_{2k-2}\right)H_{2m} - \left(\frac{1}{2}H_{2k-2} + 4P_{2k-2}\right)P_{2m};\right. \\
& \quad \left(2P_{2k-2}H_{2m} - 2P_{2k-2}P_{2m}\right)^{\times 3}, \frac{1}{2}\left(\left(2H_{2k-2} + P_{2k-2}\right)H_{2m} - H_{2k-2}P_{2m}\right), \\
& \quad \left.\left(2H_{2k-3} + P_{2k-3}\right)H_{2m} - H_{2k-3}P_{2m}, \left(H_{2k-1}H_{2m} - H_{2k-1}P_{2m}\right)^{\times 2}, (*)\right) \\
\to \ & \left(\left(H_{2k-2} + \frac{9}{2}P_{2k-2}\right)H_{2m} - \left(\frac{1}{2}H_{2k-2} + 4P_{2k-2}\right)P_{2m};\right. \\
& \quad \frac{1}{2}\left(\left(2H_{2k-2} + P_{2k-2}\right)H_{2m} - H_{2k-2}P_{2m}\right), \left(2H_{2k-3} + P_{2k-3}\right)H_{2m} - H_{2k-3}P_{2m}, \\
& \quad \left.\left(H_{2k-1}H_{2m} - H_{2k-1}P_{2m}\right)^{\times 2}, \left(2P_{2k-2}H_{2m} - 2P_{2k-2}P_{2m}\right)^{\times 3}, (*)\right) \\
\to \ & \left(\left(H_{2k-2} + \frac{7}{2}P_{2k-2}\right)H_{2m} - \left(\frac{1}{2}H_{2k-2} + 4P_{2k-2}\right)P_{2m};\right. \\
& \quad \left(H_{2k-2} - \frac{1}{2}P_{2k-2}\right)H_{2m} - \frac{1}{2}H_{2k-2}P_{2m}, H_{2k-3}H_{2m} - H_{2k-3}P_{2m}, \\
& \quad P_{2k-1}H_{2m} - H_{2k-1}P_{2m}, H_{2k-1}H_{2m} - H_{2k-1}P_{2m}, \\
& \quad \left.\left(2P_{2k-2}H_{2m} - 2P_{2k-2}P_{2m}\right)^{\times 3}, (*)\right)
\end{aligned}$$

$$\begin{aligned}
\to\ & \bigg(\left(H_{2k-2} + \frac{7}{2}P_{2k-2} \right) H_{2m} - \left(\frac{1}{2}H_{2k-2} + 4P_{2k-2} \right) P_{2m}; \\
& \quad H_{2k-1}H_{2m} - H_{2k-1}P_{2m}, \left(2P_{2k-2}H_{2m} - 2P_{2k-2}P_{2m}\right)^{\times 3}, \\
& \quad \left(H_{2k-2} - \frac{1}{2}P_{2k-2} \right) H_{2m} - \frac{1}{2}H_{2k-2}P_{2m}, H_{2k-3}H_{2m} - H_{2k-3}P_{2m}, \\
& \quad P_{2m-(2k-1)}, (*) \bigg) \\
\to\ & \bigg(P_{2k-1}H_{2m} - 2P_{2k-2}P_{2m}; \left(H_{2k-2} - \frac{1}{2}P_{2k-2} \right) H_{2m} - \frac{1}{2}H_{2k-2}P_{2m}, \\
& \quad \left(-\frac{1}{2}P_{2k-2}H_{2m} + \frac{1}{2}H_{2k-2}P_{2m} \right)^{\times 2}, 2P_{2k-2}H_{2m} - 2P_{2k-2}P_{2m}, \\
& \quad \left(H_{2k-2} - \frac{1}{2}P_{2k-2} \right) H_{2m} - \frac{1}{2}H_{2k-2}P_{2m}, H_{2k-3}H_{2m} - H_{2k-3}P_{2m}, \\
& \quad P_{2m-(2k-1)}, (*) \bigg) \\
\to\ & \bigg(P_{2k-1}H_{2m} - 2P_{2k-2}P_{2m}; 2P_{2k-2}H_{2m} - 2P_{2k-2}P_{2m}, \\
& \quad \left(\left(H_{2k-2} - \frac{1}{2}P_{2k-2} \right) H_{2m} - \frac{1}{2}H_{2k-2}P_{2m} \right)^{\times 2}, H_{2k-3}H_{2m} - H_{2k-3}P_{2m}, \\
& \quad \left(\frac{1}{2}P_{2m-(2k-2)} \right)^{\times 2}, P_{2m-(2k-1)}, (*) \bigg) \\
\to\ & \bigg(P_{2k-2}H_{2m} - H_{2k-3}P_{2m}; H_{2k-3}H_{2m} - H_{2k-3}P_{2m}, \\
& \quad \left(-\frac{1}{2}P_{2k-2}H_{2m} + \frac{1}{2}H_{2k-2}P_{2m} \right)^{\times 2}, H_{2k-3}H_{2m} - H_{2k-3}P_{2m}, \\
& \quad \left(\frac{1}{2}P_{2m-(2k-2)} \right)^{\times 2}, P_{2m-(2k-1)}, (*) \bigg) \\
\to\ & \bigg(P_{2k-2}H_{2m} - H_{2k-3}P_{2m}; \left(H_{2k-3}H_{2m} - H_{2k-3}P_{2m}\right)^{\times 2}, \left(\frac{1}{2}P_{2m-(2k-2)} \right)^{\times 4}, \\
& \quad P_{2m-(2k-1)}, (*) \bigg).
\end{aligned}$$

Now, after reordering this last class, we obtain $\hat{V}^m_{k-1}$ as required. ∎

Lemma 8.3.12 *$\hat{V}^m_2$ reduces in 5 Cremona moves to the class*

$$\hat{V}^m := \left(H_{2m} - \frac{1}{2}P_{2m}; \frac{1}{2}P_{2m}, \left(P_{2m-1}\right)^{\times 3}, \hat{B}^m_1 \right).$$

Proof. We have

$$\begin{aligned}
\hat{V}^m_2 \ =\ & \bigg(12H_{2m} - 7P_{2m}; \left(4H_{2m} - \frac{3}{2}P_{2m} \right)^{\times 4}, 3H_{2m} - P_{2m}, \left(7H_{2m} - 7P_{2m}\right)^{\times 2}, \\
& \quad \left(H_{2m} - \frac{1}{2}P_{2m} \right)^{\times 3}, (*) \bigg) \\
\to\ & \bigg(12H_{2m} - \frac{19}{2}P_{2m}; 4H_{2m} - \frac{3}{2}P_{2m}, 3H_{2m} - P_{2m}, \left(7H_{2m} - 7P_{2m}\right)^{\times 2},
\end{aligned}$$

$$
\begin{aligned}
&\left.(4H_{2m}-4P_{2m})^{\times 3},\left(H_{2m}-\frac{1}{2}P_{2m}\right)^{\times 3},(*)\right)\\
\to\ &\left(10H_{2m}-\frac{19}{2}P_{2m};7H_{2m}-7P_{2m},(4H_{2m}-4P_{2m})^{\times 3},2H_{2m}-\frac{3}{2}P_{2m},\right.\\
&\left.\left(H_{2m}-\frac{1}{2}P_{2m}\right)^{\times 3},P_{2m-1},P_{2m-3},(*)\right)\\
\to\ &\left(5H_{2m}-4P_{2m};4H_{2m}-4P_{2m},\left(2H_{2m}-\frac{3}{2}P_{2m}\right)^{\times 2},\left(H_{2m}-\frac{1}{2}P_{2m}\right)^{\times 3},\right.\\
&\left.P_{2m-1},\left(\frac{1}{2}P_{2m-2}\right)^{\times 2},P_{2m-3},(*)\right)\\
\to\ &\left(2H_{2m}-P_{2m};\left(H_{2m}-\frac{1}{2}P_{2m}\right)^{\times 3},(P_{2m-1})^{\times 2},\left(\frac{1}{2}P_{2m-2}\right)^{\times 4},P_{2m-3},(*)\right)\\
\to\ &\left(H_{2m}-\frac{1}{2}P_{2m};\left(H_{2m}-\frac{1}{2}P_{2m}\right)^{\times 3},(P_{2m-1})^{\times 2},\left(\frac{1}{2}P_{2m-2}\right)^{\times 4},P_{2m-3},(*)\right).
\end{aligned}
$$

After reordering this last class, we obtain $\hat{V}^m$ as required. ■

Lemma 8.3.13 *For all $m \geq 3$, $\hat{V}^m$ reduces in 5 Cremona moves to $\hat{V}^{m-1}$.*

Proof. We have

$$
\begin{aligned}
\hat{V}^m &= \left(H_{2m}-\frac{1}{2}P_{2m};\frac{1}{2}P_{2m},(P_{2m-1})^{\times 3},\left(\frac{1}{2}P_{2m-2}\right)^{\times 8},(*)\right)\\
&\to \left(\frac{1}{2}P_{2m};P_{2m-1},\left(\frac{1}{2}P_{2m-2}\right)^{\times 9},(*)\right)\\
&\to \left(P_{2m-1};P_{2m-1}-\frac{1}{2}P_{2m-2},\left(\frac{1}{2}P_{2m-2}\right)^{\times 7},(*)\right)\\
&\to \left(P_{2m-1}-\frac{1}{2}P_{2m-2};H_{2m-2},\left(\frac{1}{2}P_{2m-2}\right)^{\times 5},(*)\right)\\
&\to \left(H_{2m-2};H_{2m-2}-\frac{1}{2}P_{2m-2},\left(\frac{1}{2}P_{2m-2}\right)^{\times 3},(*)\right)\\
&\to \left(H_{2m-2}-\frac{1}{2}P_{2m-2};\frac{1}{2}P_{2m-2},P_{2m-3},(*)\right).
\end{aligned}
$$

After reordering this class, we obtain $\hat{V}^{m-1}$ as required. ■

Lemma 8.3.14 *$\hat{V}^2$ reduces in 7 Cremona moves to $(0;-1)$.*

Proof. We have

$$
\begin{aligned}
\hat{V}^2 &= \left(11;6,5^{\times 3},1^{\times 11}\right)\\
&\to \left(6;5,1^{\times 12}\right)\\
&\to \left(5;4,1^{\times 10}\right)\\
&\to \left(4;3,1^{\times 8}\right)
\end{aligned}
$$

$$\begin{aligned}
&\to \left(3;2,1^{\times 6}\right)\\
&\to \left(2;1^{\times 5}\right)\\
&\to \left(1;1^{\times 2}\right)\\
&\to (0;-1)\,,
\end{aligned}$$

which proves the claim. ■

8.4 The classes $E\left(\vartheta_n\right)$ satisfy the Diophantine equations

In this Section we show that the classes $E(\vartheta_n)$ satisfy the Diophantine equations (5.2).

Lemma 8.4.1 *The classes $E\left(\vartheta_n\right)$ satisfy the Diophantine equations (5.2).*

Proof. We will prove this separately for n even and odd. In both cases, we use Lemma 3.1.4. In (7.21), the classes $E\left(\vartheta_n\right)$ are given as:

$$E\left(\vartheta_n\right) := \begin{cases} \left(\frac{1}{4}\left(H_n+H_{n+2}\right), \frac{1}{4}\left(H_n+H_{n+2}\right); \mathbf{W}\left(\vartheta_n\right)\right) & \text{if } n \text{ is even,}\\ \left(\frac{1}{4}\left(P_n+P_{n+2}\right)+\frac{1}{2}, \frac{1}{4}\left(P_n+P_{n+2}\right)-\frac{1}{2}; \mathbf{W}\left(\vartheta_n\right)\right) & \text{if } n \text{ is odd.}\end{cases}$$

Thus, for $n=2m$ we obtain

$$\begin{aligned}
\sum m_i &= H_{2m}\sum w_i\\
&= H_{2m}\left(\frac{H_{2m+2}}{H_{2m}}+1-\frac{1}{H_{2m}}\right)\\
&= H_{2m+2}+H_{2m}-1\\
&= 2\left(\frac{1}{4}\left(H_{2m}+H_{2m+2}\right)+\frac{1}{4}\left(H_{2m}+H_{2m+2}\right)\right)-1\\
&= 2(d+e)-1.
\end{aligned}$$

With (7.2), (7.5) and the relation (7.9) given in (i) of Remark 7.1.2.2 it follows that

$$\begin{aligned}
\left(H_{2m}^2-6H_{2m+2}H_{2m}+H_{2m+2}^2\right) &= \left(H_{2m}-H_{2m+2}\right)^2-4H_{2m}H_{2m+2}\\
&= \left(P_{2m}+P_{2m-1}-P_{2m+1}-P_{2m+2}\right)^2\\
&\qquad -4\left(P_{2m}+P_{2m-1}\right)\left(P_{2m+2}+P_{2m+1}\right)\\
&= \left(-2P_{2m}-2P_{2m+1}\right)^2\\
&\qquad -4\left(P_{2m+1}-P_{2m}\right)\left(3P_{2m+1}+P_{2m}\right)\\
&= 8P_{2m}^2+16P_{2m}P_{2m+1}-8P_{2m+1}^2\\
&= 8\cdot\left(P_{2m}^2-P_{2m+1}^2+2P_{2m}P_{2m+1}\right)\\
&= 8\cdot\left(P_{2m}^2-P_{2m+1}^2+P_{2m+1}^2-P_{2m}^2+(-1)^{2m+1}\right)\\
&= -8
\end{aligned}$$

and thus

$$\sum m_i^2 = H_{2m}^2\sum w_i^2$$

$$\begin{aligned}
&= H_{2m}^2\left(\frac{H_{2m+2}}{H_{2m}}\right)\\
&= H_{2m}H_{2m+2}\\
&= H_{2m}H_{2m+2}+\frac{1}{8}\left(H_{2m}^2-6H_{2m+2}H_{2m}+H_{2m+2}^2\right)+1\\
&= \frac{1}{8}\left(H_{2m}^2+2H_{2m+2}H_{2m}+H_{2m+2}^2\right)+1\\
&= 2\left(\frac{1}{4}\left(H_{2m}+H_{2m+2}\right)\cdot\frac{1}{4}\left(H_{2m}+H_{2m+2}\right)\right)+1\\
&= 2de+1.
\end{aligned}$$

Moreover, for $n = 2m-1$,

$$\begin{aligned}
\sum m_i &= P_{2m-1}\sum w_i\\
&= P_{2m-1}\left(\frac{P_{2m+1}}{P_{2m-1}}+1-\frac{1}{P_{2m-1}}\right)\\
&= P_{2m+1}+P_{2m-1}-1\\
&= 2\left(\frac{1}{4}\left(P_{2m+1}+P_{2m-1}\right)+\frac{1}{2}+\frac{1}{4}\left(P_{2m+1}+P_{2m-1}\right)-\frac{1}{2}\right)-1\\
&= 2(d+e)-1.
\end{aligned}$$

Again using (7.2), (7.5) and the relation (7.9) given in (i) of Remark 7.1.2.2 we obtain that

$$\begin{aligned}
P_{2m-1}^2-6P_{2m-1}P_{2m+1}+P_{2m+1}^2 &= P_{2m-1}^2-6\left(P_{2m}^2+(-1)^{2m}\right)+P_{2m+1}^2\\
&= P_{2m-1}^2-6P_{2m}^2-6+\left(2P_{2m}+P_{2m-1}\right)^2\\
&= P_{2m-1}^2-6P_{2m}^2-6+4P_{2m}^2+4P_{2m}P_{2m-1}+P_{2m-1}^2\\
&= 2P_{2m-1}^2-2P_{2m}^2+4P_{2m}P_{2m-1}-6\\
&= 2P_{2m-1}^2-2P_{2m}^2+2\cdot\left(2P_{2m}P_{2m-1}\right)-6\\
&= 2P_{2m-1}^2-2P_{2m}^2+2\cdot\left(P_{2m}^2-P_{2m-1}^2+(-1)^{2m}\right)-6\\
&= 2P_{2m-1}^2-2P_{2m}^2+2P_{2m}^2-2P_{2m-1}^2+2-6\\
&= -4
\end{aligned}$$

holds, and thus

$$\begin{aligned}
\sum m_i^2 &= P_{2m-1}^2\sum w_i^2+1\\
&= P_{2m-1}^2\left(\frac{P_{2m+1}}{P_{2m-1}}\right)\\
&= P_{2m-1}P_{2m+1}\\
&= P_{2m-1}P_{2m+1}+\frac{1}{8}\left(P_{2m-1}^2-6P_{2m-1}P_{2m+1}+P_{2m+1}^2\right)+\frac{1}{2}\\
&= \frac{1}{8}\left(P_{2m-1}^2+2P_{2m-1}P_{2m+1}+P_{2m+1}^2\right)+\frac{1}{2}\\
&= 2\left(\frac{1}{16}\left(P_{2m-1}+P_{2m+1}\right)^2-\frac{1}{4}\right)+1\\
&= 2\left(\left(\frac{1}{4}\left(P_{2m-1}+P_{2m+1}\right)+\frac{1}{2}\right)\cdot\left(\frac{1}{4}\left(P_{2m-1}+P_{2m+1}\right)-\frac{1}{2}\right)\right)+1\\
&= 2de+1.
\end{aligned}$$

This proves the lemma. ■

8.5 The classes $E(\vartheta_n)$ belong to $\mathcal{E}$

In this Section we prove that the classes $E(\vartheta_n)$ belong to $\mathcal{E}$ for all $n \geq 2$. Again we will do this in two steps as in Section 8.3: By (iii) of Proposition 5.1.7, we show that the image of $E(\vartheta_n)$ under φ_*, that is

$$\varphi_*(d,e;\mathbf{m}) \;=\; (d+e-m_1; d-m_1, e-m_1, m_2, \dots, m_M)\,,$$

can be reduced to $(0;-1,0,\dots,0)$ by repeated Cremona moves. The fact that the classes satisfy the Diophantine equations has already been proven in the previous Section 8.4.

In Subsection 8.5.1 we will show this for the case that n is even and in Subsection 8.5.2 we show this for the case that n is odd.

Corollary 8.5.1 *For all $n \geqslant 0$, the classes $E(\vartheta_n)$ belong to $\mathcal{E}$.*

Let us first introduce some notations in order to simplify the expressions of the classes.

Definition 8.5.2 *For all $k \geqslant 1$, set*

$$c_{2k-1} := \left[5; \{1,4\}^{\times(k-1)}, 1\right],$$

$$c_{2k} := \left[5; \{1,4\}^{\times k}\right]$$

and

$$v_k(j) := \left[5; \{1,4\}^{\times(k-1)}, 1, j\right].$$

Lemma 8.5.3 *For all $j, k \geqslant 1$, the following relations, given in lowest terms, hold:*

$$c_{2k-1} = \frac{\frac{1}{2}P_{2k+2}}{\frac{1}{2}P_{2k}}, \qquad c_{2k} = \frac{P_{2k+3}}{P_{2k+1}} \qquad \textit{and} \qquad v_k(j) = \frac{\frac{1}{2}jP_{2k+2} + P_{2k+1}}{\frac{1}{2}jP_{2k} + P_{2k-1}}.$$

Proof. We use the fact that if $[a_0; a_1, \dots, a_M]$ is a continued fraction and $\frac{p_k}{q_k} := [a_0; a_1, \dots, a_k]$ is its k-th convergent written in lowest terms, then for any real number x,

$$[a_0; a_1, \dots, a_k, x] = \frac{xp_k + p_{k-1}}{xq_k + q_{k-1}}, \tag{8.4}$$

written in lowest terms.

We first prove the equality for c_{2k-1} and c_{2k} by induction over k. For $k = 1$ we obtain that

$$c_1 = [5;1] = 6 = \frac{6}{1} = \frac{\frac{1}{2}P_4}{\frac{1}{2}P_2}$$

and

$$c_2 = [5;1,4] = \frac{29}{5} = 5 + \frac{1}{1+\frac{1}{4}} = \frac{P_5}{P_3}.$$

Assume that both equations hold for $k-1$, that is for $k \geq 2$ it holds that

$$c_{2(k-1)-1} = c_{2k-3} = \left[5; \{1,4\}^{\times(k-2)}, 1\right] = \frac{\frac{1}{2}P_{2k}}{\frac{1}{2}P_{2k-2}} \tag{8.5}$$

and

$$c_{2(k-1)} = c_{2k-2} = \left[5; \{1,4\}^{\times(k-1)}\right] = \frac{P_{2k+1}}{P_{2k-1}}. \tag{8.6}$$

Thus with (8.5) the $(2k-2)$-th convergent of

$$c_{2k-1} = \left[5; \{1,4\}^{\times(k-1)}, 1\right]$$

equals

$$\left[5; \{1,4\}^{\times(k-1)}\right] = \frac{\frac{1}{2}P_{2k}}{\frac{1}{2}P_{2k-2}} =: \frac{p_{2k-2}}{q_{2k-2}} \tag{8.7}$$

and with (compare 8.6) the $(2k-3)$-th convergent of c_{2k-1} equals

$$\frac{p_{2k-3}}{q_{2k-3}} = \frac{P_{2k+1}}{P_{2k-1}}. \tag{8.8}$$

Now using (7.1), (8.7), (8.8) and (8.4) yield

$$\begin{aligned}
c_{2k-1} &= \left[5; \{1,4\}^{\times(k-1)}, 1\right] \\
&= \frac{1 \cdot p_{2k-2} + p_{2k-3}}{1 \cdot q_{2k-2} + q_{2k-3}} \\
&= \frac{1 \cdot \frac{1}{2}P_{2k} + P_{2k+1}}{1 \cdot \frac{1}{2}P_{2k-2} + P_{2k-1}} \\
&= \frac{\frac{1}{2}\left(2P_{2k+1} + P_{2k}\right)}{\frac{1}{2}\left(2P_{2k-1} + P_{2k-2}\right)} \\
&= \frac{\frac{1}{2}P_{2k+2}}{\frac{1}{2}P_{2k}}.
\end{aligned}$$

Similarly the $(2k-1)$-th convergent of

$$c_{2k} = \left[5; \{1,4\}^{\times k}\right]$$

is

$$c_{2k-1} = \left[5; \{1,4\}^{\times(k-1)}, 1\right] = \frac{\frac{1}{2}P_{2k+2}}{\frac{1}{2}P_{2k}} =: \frac{p_{2k-1}}{q_{2k-1}}$$

and its $(2k-2)$-th convergent is

$$c_{2k-2} = \left[5; \{1,4\}^{\times(k-1)}\right] = \frac{P_{2k+1}}{P_{2k-1}}. =: \frac{p_{2k-2}}{q_{2k-2}}$$

Thus we obtain with (7.1) and (8.4) that

$$\begin{aligned}
c_{2k} &= \left[5; \{1,4\}^{\times k}\right] \\
&= \frac{1 \cdot p_{2k-1} + p_{2k-2}}{1 \cdot q_{2k-1} + q_{2k-2}} \\
&= \frac{1 \cdot \frac{1}{2}P_{2k+2} + P_{2k+1}}{1 \cdot \frac{1}{2}P_{2k} + P_{2k-1}} \\
&= \frac{\frac{1}{2}\left(2P_{2k+2} + P_{2k+1}\right)}{\frac{1}{2}\left(P_{2k} + P_{2k-1}\right)} \\
&= \frac{P_{2k+3}}{P_{2k+1}}.
\end{aligned}$$

Now we show the equation for $v_k(j)$. By Definition 8.5.2, $v_k(j)$ is determined as

$$v_k(j) = \left[5; \{1,4\}^{\times(k-1)}, 1, j\right].$$

As its $(2k-1)$-th convergent is

$$\left[5; \{1,4\}^{\times(k-1)}, 1\right] = c_{2k-1} = \frac{\frac{1}{2}P_{2k+2}}{\frac{1}{2}P_{2k+2}} =: \frac{p_{2k-1}}{q_{2k-1}}$$

and its $(2k-2)$-th convergent is

$$\left[5; \{1,4\}^{\times(k-1)}\right] = c_{2k} = \frac{P_{2k+3}}{P_{2k+1}} =: \frac{p_{2k-2}}{q_{2k-2}},$$

this and (8.4) yield that

$$\begin{aligned} v_k(j) &= \left[5; \{1,4\}^{\times(k-1)}, 1, j\right] \\ &= \frac{j \cdot p_{2k-1} + p_{2k-2}}{j \cdot \frac{1}{2}q_{2k-1} + q_{2k-2}} \\ &= \frac{j \cdot \frac{1}{2}P_{2k+2} + P_{2k+3}}{j \cdot \frac{1}{2}P_{2k+2} + P_{2k+3}} \end{aligned}$$

and the lemma is proven. ∎

Remark 8.5.3.1 From Lemma 8.5.3 follows that for all $k \geq 1$ the continued fraction $v_k(2)$, $v_k(4)$ respectively, is the one of ϑ_{2k}, ϑ_{2k+1} respectively, as

$$v_k(2) = \frac{P_{2k+2} + P_{2k+1}}{P_{2k} + P_{2k-1}} = \frac{H_{2k+2}}{H_{2k}} = \vartheta_{2k}$$

and

$$v_k(4) = \frac{2P_{2k+2} + P_{2k+1}}{2P_{2k} + P_{2k-1}} = \frac{P_{2k+3}}{P_{2k+1}} = \vartheta_{2k+1}$$

for all $k \geq 0$.

8.5.1 The classes $E(\vartheta_{2m})$ reduce to $(0; -1)$

With the same notation as in Section 8.4, we now prove that

Lemma 8.5.4 *The classes* $E(\vartheta_{2m})$ *reduce to* $(0;-1)$ *for all* $m \geq 1$.

Proof. The proof is by induction over m. For the initial step $m = 1$, we have

$$\vartheta_2 = \frac{17}{3} \qquad \text{and} \qquad 3 \cdot \mathbf{w}(\vartheta_2) = 3 \cdot \mathbf{w}\left(\frac{17}{3}\right) = \left(3^{\times 5}, 2, 1^{\times 2}\right).$$

Thus

$$E(\vartheta_2) = \left(5, 5; 3^{\times 5}, 2, 1^{\times 2}\right).$$

We now show that under φ_* this class reduces to $(0; -1)$:

$$\varphi_*\left(E(\vartheta_2)\right) = \varphi_*\left(E\left(\frac{17}{3}\right)\right)$$

$$\begin{aligned}
&= \varphi_*\left(\left(5,5;3^{\times 5},2,1^{\times 2}\right)\right)\\
&= \left(7;2,2,3^{\times 4},2,1^{\times 2}\right)\\
&\rightsquigarrow \left(7;3^{\times 4},2^{\times 3},1^{\times 2}\right)\\
&\rightarrow \left(5;1,1,1,3,2^{\times 3},1^{\times 2}\right)\\
&\rightsquigarrow \left(5;3,2^{\times 3},1^{\times 5}\right)\\
&\rightarrow \left(3;1,0,0,2,1^{\times 5}\right)\\
&\rightsquigarrow \left(3;2,1^{\times 6}\right)\\
&\rightarrow \left(2;1,0,0,1^{\times 4}\right)\\
&\rightsquigarrow \left(2;1^{\times 5}\right)\\
&\rightarrow \left(1;0,0,0,1^{\times 2}\right)\\
&\rightsquigarrow \left(1;1^{\times 2}\right).
\end{aligned}$$

The class $\left(1;1^{\times 2}\right)$ reduces to $(0;-1)$ by repeated Cremona moves as we have already shown in Section 8.1.

We turn now to the general case. We will freely use the definitions of the Pell numbers (7.1) and the Half companion Pell numbers (7.8) given in Definition 7.1.2, as well as the fact that for all $n \geqslant 0$,

$$H_n = P_n + P_{n-1}$$

as in (i) of Remark 7.1.2.2.

Suppose that the image of the class $E\left(\vartheta_{2m}\right)$ under φ_* reduces to $(0;-1)$ and let us show that $\varphi_*\left(E\left(\vartheta_{2(m+1)}\right)\right) = \varphi_*\left(E\left(\vartheta_{2m+2}\right)\right)$ also reduces to $(0;-1)$.

In (7.21), the class $E\left(\vartheta_{2m+2}\right) = E\left(v_{m+1}(2)\right)$ is determined by

$$E\left(\vartheta_{2m+2}\right) = \left(\frac{1}{4}\left(H_{2m+2}+H_{2m+4}\right), \frac{1}{4}\left(H_{2m+2}+H_{2m+4}\right); \mathbf{W}\left(\vartheta_{2m+2}\right)\right).$$

As

$$\begin{aligned}
H_{2m+2}+H_{2m+4} &= P_{2m+2}+P_{2m+1}+P_{2m+4}+P_{2m+3}\\
&= 3P_{2m+3}+2P_{2m+2}+P_{2m+1}\\
&= 4P_{2m+3},
\end{aligned}$$

we thus obtain with $\vartheta_{2m+2} = \frac{H_{2m+4}}{H_{2m+2}}$, Lemma 8.5.3 and Remark 8.5.3.1 that the first terms of $E\left(\vartheta_{2m+2}\right)$ are

$$E\left(\vartheta_{2m+2}\right) = \left(P_{2m+3}, P_{2m+3}; H_{2m+2}^{\times 5}, 2H_{2m+1}, H_{2m}^{\times 4}, 2H_{2m-1}, (*)\right),$$

where $(*)$ stands for all the next terms. The image of $E\left(\vartheta_{2m+2}\right)$ under φ_* is

$$\begin{aligned}
\varphi_*\left(E\left(\vartheta_{2m+2}\right)\right) &= \varphi_*\left(\left(P_{2m+3}, P_{2m+3}; H_{2m+2}^{\times 5}, 2H_{2m+1}, H_{2m}^{\times 4}, 2H_{2m-1}, (*)\right)\right)\\
&= \big(P_{2m+3}+P_{2m+3}-H_{2m+2}\,;\\
&\qquad P_{2m+3}-H_{2m+2}, P_{2m+3}-H_{2m+2}, H_{2m+2}^{\times 4}, 2H_{2m+1}, H_{2m}^{\times 4}, 2H_{2m-1}, (*)\big)\\
&= \left(H_{2m+3}; P_{2m+2}, P_{2m+2}, H_{2m+2}^{\times 4}, 2H_{2m+1}, H_{2m}^{\times 4}, 2H_{2m-1}, (*)\right)\\
&\rightsquigarrow \left(H_{2m+3}; H_{2m+2}^{\times 4}, 2H_{2m+1}, P_{2m+2}^{\times 2}, H_{2m}^{\times 4}, 2H_{2m-1}, (*)\right).
\end{aligned}$$

To finish the proof, we will show that $\varphi_*\left(E\left(\vartheta_{2m+2}\right)\right)$ reduces to $\varphi_*\left(E\left(\vartheta_{2m}\right)\right)$ in four steps:

$$\begin{aligned}
\varphi_*\left(E\left(\vartheta_{2m+2}\right)\right) &= \left(H_{2m+3}; H_{2m+2}^{\times 4}, 2H_{2m+1}, P_{2m+2}^{\times 2}, H_{2m}^{\times 4}, 2H_{2m-1}, (*)\right)\\
&\to \left(3H_{2m+1}+2P_{2m+1}; H_{2m+2}, 2H_{2m+1}, P_{2m+2}^{\times 2}, H_{2m+1}^{\times 3}, H_{2m}^{\times 4}, 2H_{2m-1}, (*)\right)\\
&\to \left(2H_{2m+1}+P_{2m+1}; P_{2m+2}, H_{2m+1}^{\times 3}, P_{2m+1}, H_{2m}^{\times 4}, 2H_{2m-1}, P_{2m}, (*)\right)\\
&\to \left(P_{2m+2}; H_{2m+1}, P_{2m+1}^{\times 2}, H_{2m}^{\times 4}, 2H_{2m-1}, P_{2m}, (*)\right)\\
&\to \left(H_{2m+1}; H_{2m}^{\times 4}, 2H_{2m-1}, P_{2m}^{\times 2}, (*)\right)\\
&= \varphi_*\left(E\left(\vartheta_{2m}\right)\right).
\end{aligned}$$

It is important to note that $(*)$ was left invariant during the whole reduction process. So the last class is precisely $\varphi_*\left(E\left(\vartheta_{2m}\right)\right)$. ■

8.5.2 The classes $E\left(\vartheta_{2m-1}\right)$ reduce to $(0;-1)$

Lemma 8.5.5 *The classes* $E\left(\vartheta_{2m-1}\right)$ *reduce to* $(0;-1)$ *for all* $k \geq 1$.

Proof. The proof is by induction over m. For the initial step $m=1$, we have

$$\vartheta_1 = 5 \qquad \text{and} \qquad \mathbf{w}\left(\vartheta_1\right) = \mathbf{w}(5) = \left(1^{\times 5}\right).$$

Thus

$$\begin{aligned}
\varphi_*\left(E(\vartheta_1)\right) &= \varphi_*\left(E(5)\right)\\
&= \varphi_*\left(\left(2,1;1^{\times 5}\right)\right)\\
&= \left(2+1-1; 2-1, 1-1, 1^{\times 4}\right)\\
&\rightsquigarrow \left(2;1^{\times 5}\right).
\end{aligned}$$

The class $\left(1;1^{\times 2}\right)$ reduces to $(0;-1)$ by repeated Cremona moves as we have already shown in Section 8.1.

We turn now to the general case. We will freely use the definitions of the Pell numbers (7.1) and the Half companion Pell numbers (7.8) given in Definition 7.1.2, as well as the fact that for all $n \geqslant 0$,

$$H_n = P_n + P_{n-1}$$

as in (i) of Remark 7.1.2.2.

Suppose that the image of the class $E\left(\vartheta_{2m+1}\right)$ under φ_* reduces to $(0;-1)$ and let us show that $\varphi_*\left(E\left(\vartheta_{2(m+1)+1}\right)\right) = \varphi_*\left(E\left(\vartheta_{2m+3}\right)\right)$ also reduces to $(0;-1)$.

In (7.21), the class $E\left(\vartheta_{2m+3}\right) = E\left(v_{m+1}(4)\right)$ is determined by

$$E\left(\vartheta_{2m+3}\right) = \left(\frac{1}{4}\left(P_{2m+3}+P_{2m+5}\right)+\frac{1}{2}, \frac{1}{4}\left(P_{2m+3}+P_{2m+5}\right)-\frac{1}{2}; \mathbf{W}\left(\vartheta_{2m+3}\right)\right).$$

As

$$\begin{aligned}
P_{2m+3}+P_{2m+5} &= P_{2m+3}+2P_{2m+4}+P_{2m+3}\\
&= 2\left(P_{2m+4}+P_{2m+3}\right)\\
&= 2H_{2m+4},
\end{aligned}$$

we thus obtain with $\vartheta_{2m+3}=\frac{P_{2m+5}}{P_{2m+3}}$, Lemma 8.5.3 and Remark 8.5.3.1 that the first terms of $E\left(\vartheta_{2m+3}\right)$ are

$$E\left(\vartheta_{2m+3}\right)=\left(\frac{1}{2}H_{2m+4}+\frac{1}{2},\frac{1}{2}H_{2m+4}-\frac{1}{2};P_{2m+3}^{\times 5},2P_{2m+2},P_{2m+1}^{\times 4},2P_{2m},(*)\right),$$

where $(*)$ stands for all the next terms. The image of $E\left(\vartheta_{2m+3}\right)$ under φ_* is

$$\begin{aligned}
\varphi_*\left(E\left(\vartheta_{2m+3}\right)\right)&=\varphi_*\left(\left(\frac{1}{2}H_{2m+4}+\frac{1}{2},\frac{1}{2}H_{2m+4}-\frac{1}{2};P_{2m+3}^{\times 5},2P_{2m+2},P_{2m+1}^{\times 4},2P_{2m},(*)\right)\right)\\
&=\Bigg(\frac{1}{2}H_{2m+4}+\frac{1}{2}+\frac{1}{2}H_{2m+4}-\frac{1}{2}-P_{2m+3};\\
&\qquad\frac{1}{2}H_{2m+4}+\frac{1}{2}-P_{2m+3},\frac{1}{2}H_{2m+4}-\frac{1}{2}-P_{2m+3},\\
&\qquad P_{2m+3}^{\times 4},2P_{2m+2},P_{2m+1}^{\times 4},2P_{2m},(*)\Bigg)\\
&=\Bigg(H_{2m+4}-P_{2m+3};\\
&\qquad\frac{1}{2}H_{2m+4}-P_{2m+3}+\frac{1}{2},\frac{1}{2}H_{2m+4}-P_{2m+3}-\frac{1}{2},\\
&\qquad P_{2m+3}^{\times 4},2P_{2m+2},P_{2m+1}^{\times 4},2P_{2m},(*)\Bigg)\\
&\rightsquigarrow\Bigg(P_{2m+3};P_{2m+3}^{\times 4},2P_{2m+2},\frac{1}{2}H_{2m+3}+\frac{1}{2},\frac{1}{2}H_{2m+3}-\frac{1}{2},\\
&\qquad P_{2m+1}^{\times 4},2P_{2m},(*)\Bigg).
\end{aligned}$$

To finish the proof, we will show that $\varphi_*\left(E\left(\vartheta_{2m+2}\right)\right)$ reduces to $\varphi_*\left(E\left(\vartheta_{2m}\right)\right)$ in four steps:

$$\begin{aligned}
\varphi_*\left(E\left(\vartheta_{2m+3}\right)\right)&=\Bigg(P_{2m+3};P_{2m+3}^{\times 4},2P_{2m+2},\frac{1}{2}H_{2m+3}+\frac{1}{2},\frac{1}{2}H_{2m+3}-\frac{1}{2},\\
&\qquad P_{2m+1}^{\times 4},2P_{2m},(*)\Bigg)\\
&\to\Bigg(P_{2m+3}+2P_{2m+2};P_{2m+3},2P_{2m+2},\frac{1}{2}H_{2m+3}+\frac{1}{2},\frac{1}{2}H_{2m+3}-\frac{1}{2},P_{2m+2}^{\times 3},\\
&\qquad P_{2m+1}^{\times 4},2P_{2m},(*)\Bigg)\\
&\to\Bigg(\frac{1}{2}P_{2m+3}+\frac{3}{2}P_{2m+2}-\frac{1}{2};\frac{1}{2}H_{2m+3}-\frac{1}{2},P_{2m+2}^{\times 3},\frac{1}{2}H_{2m+2}-\frac{1}{2},\\
&\qquad P_{2m+1}^{\times 4},2P_{2m},\frac{1}{2}H_{2m+1}-\frac{1}{2},(*)\Bigg)\\
&\to\left(\frac{1}{2}H_{2m+3}-\frac{1}{2};P_{2m+2},\left(\frac{1}{2}H_{2m+2}-\frac{1}{2}\right)^{\times 2},P_{2m+1}^{\times 4},2P_{2m},\frac{1}{2}H_{2m+1}-\frac{1}{2},(*)\right)\\
&\to\left(P_{2m+1};P_{2m+1}^{\times 4},2P_{2m},\frac{1}{2}H_{2m+1}+\frac{1}{2},\frac{1}{2}H_{2m+1}-\frac{1}{2},(*)\right)\\
&=\varphi_*\left(E\left(\vartheta_{2m+1}\right)\right).
\end{aligned}$$

Since $(*)$ was left invariant during the whole reduction process, the last class is precisely $\varphi_*\left(E\left(\vartheta_{2m+1}\right)\right)$. ■

8.6 Proving the Pell stair Theorem

With Section 8.2 and Section 8.3, we are now able to prove (ii) of Theorem 1.2.1:

Corollary 8.6.1 *On the interval* $\left[1,\sigma^2\right]$,

$$c(a)=\begin{cases} 1 & \text{if } a\in[1,2]\,,\\ \dfrac{1}{\sqrt{2\eta_n}}\,a & \text{if } a\in[\eta_n,\vartheta_n]\,,\\ \sqrt{\dfrac{\eta_{n+1}}{2}} & \text{if } a\in[\vartheta_n,\eta_{n+1}]\,,\end{cases}$$

for all $n\geq 0$.

Proof. Since for all $n\geq 0$, $E\left(\vartheta_n\right)$ is a perfect class, we know by Lemma 6.2.10 that $c\left(\vartheta_n\right)=\mu\left(E\left(\vartheta_n\right)\right)\left(\vartheta_n\right)$. Hence

$$c\left(\vartheta_n\right)=\sqrt{\frac{\eta_{n+1}}{2}}$$

for all $n\geq 0$. Indeed, for n even, we compute

$$\begin{aligned} c\left(\vartheta_n\right) &= \frac{2H_n\left\langle \mathbf{w}\left(\vartheta_n\right),\mathbf{w}\left(\vartheta_n\right)\right\rangle}{H_{n+2}+H_n}=\frac{2H_n\vartheta_n}{H_{n+2}+H_n}=\frac{2H_{n+2}}{H_{n+2}+H_n}\\ &= \frac{2\left(P_{n+2}+P_{n+1}\right)}{P_{n+2}+P_{n+1}+P_n+P_{n-1}}=\frac{2\left(P_{n+2}+P_{n+1}\right)}{4P_{n+1}}=\frac{H_{n+2}}{2P_{n+1}}=\sqrt{\frac{\eta_{n+1}}{2}}, \end{aligned}$$

and for n odd,

$$\begin{aligned} c\left(\vartheta_n\right) &= \frac{2P_n\left\langle \mathbf{w}\left(\vartheta_n\right),\mathbf{w}\left(\vartheta_n\right)\right\rangle}{P_{n+2}+P_n}=\frac{2P_n\vartheta_n}{P_{n+2}+P_n}=\frac{2P_{n+2}}{P_{n+2}+P_n}\\ &= \frac{P_{n+2}}{\frac{1}{2}\left(P_{n+2}-P_n\right)+P_n}=\frac{P_{n+2}}{P_{n+1}+P_n}=\frac{P_{n+2}}{H_n}=\sqrt{\frac{\eta_{n+1}}{2}}. \end{aligned}$$

Furthermore,

$$c\left(\eta_n\right)=\sqrt{\frac{\eta_n}{2}} \tag{8.9}$$

for all $n\geq 0$. Indeed, for n even, we have

$$\mu\left(E\left(\eta_n\right)\right)\left(\eta_n\right)=\frac{H_n^2\eta_n}{2P_{n+1}H_n}=\frac{H_n\eta_n}{2P_{n+1}}=\frac{P_{n+1}}{H_n}=\sqrt{\frac{\eta_n}{2}}.$$

Thus for all $(d,e;\mathbf{m})\in\mathcal{E}$ distinct from $E\left(\eta_n\right)$, we get by Proposition 5.1.7 (ii) that

$$P_{n+1}H_n(d+e)\geq\left\langle \mathbf{m},\mathbf{W}'\left(\eta_n\right)\right\rangle\geq H_n^2\left\langle \mathbf{m},\mathbf{w}\left(\eta_n\right)\right\rangle,$$

and hence

$$\mu(d,e;\mathbf{m})\left(\eta_n\right)=\frac{\left\langle \mathbf{m},\mathbf{w}\left(\eta_n\right)\right\rangle}{d+e}\leq\frac{P_{n+1}H_n}{H_n^2}=\frac{P_{n+1}}{H_n}=\sqrt{\frac{\eta_n}{2}}.$$

Next, for n odd, we have

$$\mu\left(E\left(\eta_n\right)\right)\left(\eta_n\right)=\frac{2P_n^2\eta_n}{2P_nH_{n+1}}=\frac{P_n\eta_n}{H_{n+1}}=\frac{H_{n+1}}{2P_n}=\sqrt{\frac{\eta_n}{2}}.$$

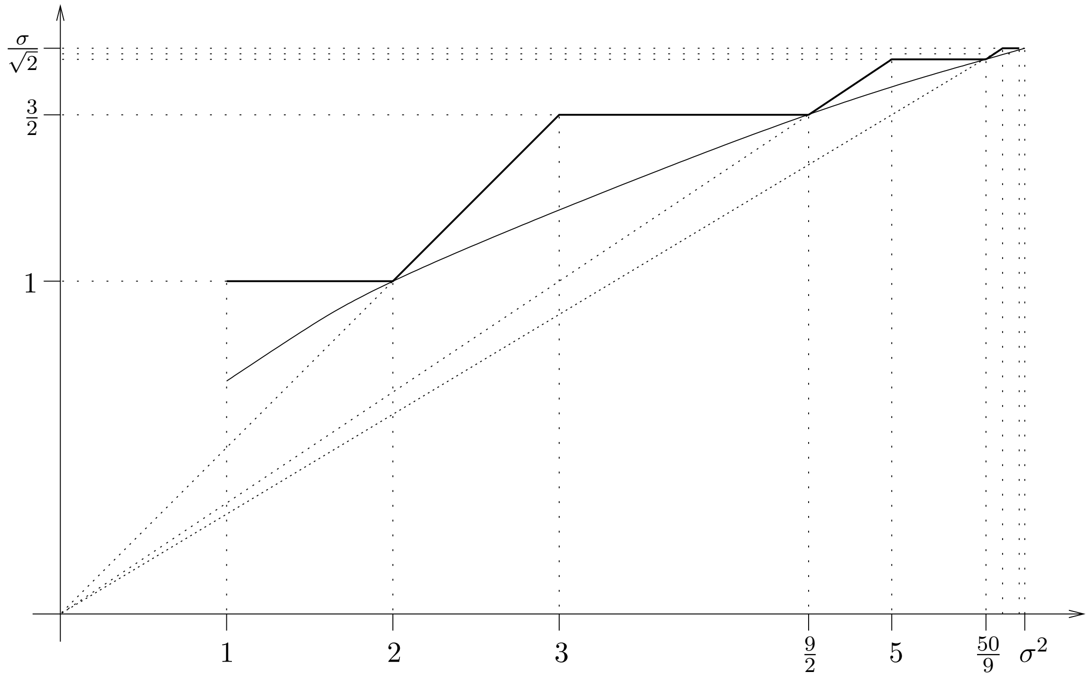

Figure 8.1: Graph of $c(a)$ on $[1,\sigma^2]$.

Thus for all $(d,e;\mathbf{m})\in\mathcal{E}$ distinct from $E(\eta_n)$, we get

$$P_nH_{n+1}(d+e)\geq\left\langle\mathbf{m},\mathbf{W}'(\eta_n)\right\rangle\geq 2P_n^2\left\langle\mathbf{m},\mathbf{w}(\eta_n)\right\rangle,$$

and hence

$$\mu(d,e;\mathbf{m})(\eta_n)=\frac{\langle\mathbf{m},\mathbf{w}(\eta_n)\rangle}{d+e}\leq\frac{P_nH_{n+1}}{2P_n^2}=\frac{H_{n+1}}{2P_n}=\sqrt{\frac{\eta_n}{2}}.$$

Thus, by Proposition 5.1.8, we have $c(\eta_n)=\sqrt{\frac{\eta_n}{2}}$ for all $n\geq 0$ as required. Since c is nondecreasing, we get that $c(a)=\sqrt{\frac{\eta_{n+1}}{2}}$ for $a\in[\vartheta_n,\eta_{n+1}]$. Moreover, we have that for all $n\geq 0$

$$\frac{c(\vartheta_n)}{\vartheta_n}=\frac{c(\eta_n)}{\eta_n}.$$

Indeed, for n even,

$$\frac{c(\vartheta_n)}{\vartheta_n}=\frac{\sqrt{\frac{\eta_{n+1}}{2}}}{\vartheta_n}=\frac{H_n}{2P_{n+1}}=\frac{\sqrt{\frac{\eta_n}{2}}}{\eta_n}=\frac{c(\eta_n)}{\eta_n},$$

and for n odd,

$$\frac{c(\vartheta_n)}{\vartheta_n}=\frac{\sqrt{\frac{\eta_{n+1}}{2}}}{\vartheta_n}=\frac{P_n}{H_{n+1}}=\frac{\sqrt{\frac{\eta_n}{2}}}{\eta_n}=\frac{c(\eta_n)}{\eta_n}.$$

Hence, by the scaling property of Lemma 2.1.14, the function c has to be linear on $[\eta_n,\vartheta_n]$ and thus

$$c(a)=\frac{1}{\sqrt{2\eta_n}}a$$

for $a\in[\eta_n,\vartheta_n]$. For $a\in[1,2]$, the classes $E(1)$ and $E(2)=E(\eta_0)$ are perfect classes. Hence (8.9) yields

$$c(\eta_0)=c(2)=\sqrt{\tfrac{2}{2}}=1$$

and

$$c(1)=\mu(E(1))(1)=\mu((1,0;1))=\frac{\langle(1,0;1),(1)\rangle}{1}=1.$$

Since the function c is nondecreasing on $[1,2]$, we get that $c(a)=1$ for $a\in[1,2]$. ∎

Chapter 9

Outlook

In this work we have proven fundamental properties for symplectic embeddings of 4-dimensional ellipsoids into polydiscs and determined the embedding capacity function $c_{EC}(a)$ for $a \in [1, \sigma^2]$. In [5] the embedding capacity function $c_{EC}(a)$ is not only determined for $a \in [1, \sigma^2]$, but also for $a \in (\sigma^2, \infty)$, that is Theorem 1.2.1 is expanded to

Theorem 9.1.2 *(i) On the interval* $\left[1, \sigma^2\right]$,

$$c(a) = \begin{cases} 1 & \text{if } a \in [1,2]\,, \\ \dfrac{1}{\sqrt{2\alpha_n}}\, a & \text{if } a \in [\alpha_n, \beta_n]\,, \\ \sqrt{\dfrac{\alpha_{n+1}}{2}} & \text{if } a \in [\beta_n, \alpha_{n+1}]\,, \end{cases}$$

for all $n \geq 0$.

(ii) On the interval $\left[\sigma^2, 7\frac{1}{32}\right]$ *we have* $c(a) = \sqrt{\frac{a}{2}}$ *except on seven disjoint intervals, where c is piecewise linear.*

(iii) For $a \geq 7\frac{1}{32}$ *we have* $c(a) = \sqrt{\frac{a}{2}}$.

Remark 9.1.2.1 Recall the ECH capacities of $B(1)$ and $C(1)$ from Example 4.1.3:

$$c_{ECH}\left(B(1)\right) = \left(0^{\times 1}, 1^{\times 2}, 2^{\times 3}, 3^{\times 4}, 4^{\times 5}, 5^{\times 6}, 6^{\times 7}, 7^{\times 8}, 8^{\times 9}, 9^{\times 10}, \ldots\right),$$
$$c_{ECH}\left(C(1)\right) = \left(0^{\times 1}, 1^{\times 1}, 2^{\times 2}, 3^{\times 2}, 4^{\times 3}, 5^{\times 3}, 6^{\times 4}, 7^{\times 4}, 8^{\times 5}, 9^{\times 5}, \ldots\right).$$

One sees that the sequence $c_{ECH}\left(C(1)\right)$ is obtained from $c_{ECH}\left(B(1)\right)$ by some sort of doubling. This is reminiscent to the doubling in the definition of the Pell numbers: The Fibonacci and Pell numbers are defined recursively by

$$F_{n+1} = F_n + F_{n-1}, \qquad P_{n+1} = 2P_n + P_{n-1},$$

and while the Fibonacci numbers determine the infinite stairs of the function $c_{EB}(a)$ for $a \leq \tau^4$ (with τ the golden ratio, see [19]), the Pell numbers determine the infinite stairs of the function $c_{EC}(a)$ for $a \leq \sigma^2$. This reminiscence may, however, be a coincidence. Indeed, for the ellipsoid $E(1,3)$ the sequence

$$c_{ECH}\left(E(1,3)\right) = \left(0^{\times 1}, 1^{\times 1}, 2^{\times 1}, 3^{\times 2}, 4^{\times 2}, 5^{\times 2}, 6^{\times 3}, 7^{\times 3}, 8^{\times 3}, 9^{\times 4}, \ldots\right)$$

is obtained from $c_{ECH}(B(1))$ by some sort of tripling, but the beginning of the function describing the embedding problem $E(1,a) \overset{\text{s}}{\hookrightarrow} E(A, 3A)$ seems not to be given in terms of numbers defined by $G_{n+1} = 3G_n + G_{n-1}$.

Appendix A

Mathematica scripts

A.1 Computing obstructive classes

In this Section we describe the Mathematica scripts we used in the proof of Lemma 6.1.2 to compute the values of the function $\mu(d,e;\mathbf{m})(a)$ for some $(d,e;\mathbf{m}) \in \mathcal{E}$. We know that for obstructive classes $(d,e;\mathbf{m}) \in \mathcal{E}$ it is necessary that $d = e$ or $d = e+1$, so as before we will always consider both cases and look at the systems (6.8) and (6.9) respectively. The scripts `ObstructiveClassdd[a_, d_]` (for the case $d = e$) and `ObstructiveClassde[a_, d_]` (for the case $d = e+1$) yield for a given d all obstructive classes $(d,e;\mathbf{m}) \in \mathcal{E}$ at a point $a \geq 1$.

First we used the script `W[a_]` from [19] that gives the weight expansion of a value $a \in \mathbb{Q}$, using the continued fraction expansion of a.

```
W[a_] := Module[{aa=a,M,i=2,L,u,v},
                M = ContinuedFraction[aa];
                L = Table[1, {j,M[[1]]}];
                {u,v} = {1,aa-Floor[aa]};
                While[i <= Length[M],
                      L = Join[L, Table[ v, {j,M[[i]]}] ];
                      {u,v} = {v,u - M[[i]] v};
                      i++];
                Return[L] ]
```

For example `W[3]` yields the weight expansion `{1,1,1}`.

We also used the scripts `Solutions[x_, y_]` and `Solutions[x_, y_, z_]` from [19] where $\mathbf{m}$ is determined in the following way: for given x and y, where x and y equal the left hand sides of the two equations of (5.2) `Solutions[x_, y_, z_]` computes $\mathbf{m} = (m_1, \dots, m_M)$ recursively. Therefore it is only used that from classes $(d,e;\mathbf{m}) \in \mathcal{E}$ the two equations of (5.2) must be fulfilled.

```
Solutions[x_, y_] := Solutions[x, y, Min[x, Floor[Sqrt[y]]]]

Solutions[x_, y_, z_] :=
 Module[{X = x, Y = y, Z = z, i, m, K, j, V, L = {}},
  If[X^2 < Y, L = {}];
  If[X^2 == Y, If[X > Y, L = {}, L = {{X}}]];
  If[X^2 > Y, i = 1;
   m = Min[Floor[Sqrt[Y]], Z];
   While[i <= m, K = Solutions[X - i, Y - i^2, i];
```

```
    j = 1;
    While[j <= Length[K], V = Prepend[K[[j]], i];
     L = Append[L, V];
     j++];
    i++]];
  Return[Union[L]]]
```

To simplify matters, we determine four functions that give the left hand sides of the equations of the systems (6.8) and (6.9) for given d.

```
f1[d_] := 4 d - 1
g1[d_] := 2 d^2 + 1
f2[d_] := 4 d - 3
g2[d_] := 2 d^2 - 2 d + 1
```

For instance for $d = 3$, `Solutions[f1[3], g1[3]]` yields

```
{{2,2,2,1,1,1,2},{2,2,2,2,1,1,1},
{2,1,1,1,1,1,1,3},{3,1,1,1,1,1,1,2},{3,2,1,1,1,1,1,1}}
```

and `Solutions[f2[3], g2[3]]` yields

```
{{2,1,1,1,1,1,2},{2,2,1,1,1,1,1}}.
```

Now we construct the scripts `ObstructiveClassdd[a_,d,_]` and `ObstructiveClassde[a_,d,_]` that return all obstructive classes $(d, e; \mathbf{m}) \in \mathcal{E}$ for given d, that is all solutions $\mathbf{m}(d)$ of `Solutions[x_, y_]` for given d, such that $(d, e; \mathbf{m}(d))$ is obstructive at a.

```
ObstructiveClassdd[a_, d_] := Module[{aa = a, dd = d, Q = {}, j, V},
  Kd = Solutions[f1[dd], g1[dd]];
  For[ j = 1, j <= Length[Kd], j++,
   m = Min[Length[Kd[[j]]], Length[W[aa]]];
   Trunc = Kd[[j]][[1 ;; m]].W[aa][[1 ;; m]];
   Mu = Trunc/(dd + dd);
   If [Mu > Sqrt[aa/2],
    V = {Kd[[j]], Mu};
    Q = Append[Q, V]
    ]
   ];
  Return [Q]]

ObstructiveClassde[a_, d_] := Module[{aa = a, dd = d, Q = {}, j, V},
  Kd = Solutions[f2[dd], g2[dd]];
  For[ j = 1, j <= Length[Kd], j++,
   m = Min[Length[Kd[[j]]], Length[W[aa]]];
   Trunc = Kd[[j]][[1 ;; m]].W[aa][[1 ;; m]];
   Mu = Trunc/(dd + (dd - 1));
   If [Mu > Sqrt[aa/2],
    V = {Kd[[j]], Mu};
    Q = Append[Q, V]
    ]
   ];
  Return [Q]]
```

`ObstructiveClassdd[3, 1]` gives `{{{1, 1, 1}, 3/2}}` and `ObstructiveClassde[3, 1]` leads to $\{\}$. It is also useful to know for which classes $\mu(d,e;\mathbf{m})(a) = \sqrt{\frac{a}{2}}$. Therefore one can replace the if-loop: `If [Mu > Sqrt[aa/2]`, by: `If [Mu >= Sqrt[aa/2]`. For example for `a := 2` we obtain exactly two solutions:

`{1,1{{{1,1,1},1}}}` and `{1,0{{{1},1}}}`.

We will always obtain exactly two classes for $a \in \gamma_n, \delta_n$.

As we want to check for many classes of $\mathcal{E}$ if they are obstructive at a point a, we constructed the following.

```
a := 3
Z := 10
For[d = 1, d <= Z, d++,
 Print["{", d, ",", d, ObstructiveClassdd[a, d], "}", ",", "{", d, ",", d - 1,
   ObstructiveClassde[a, d], "}"]]
```

which leads to

```
{1,1{{{1,1,1},3/2}}},{1,0{}}
{2,2{}},{2,1{}}
{3,3{}},{3,2{}}
{4,4{}},{4,3{}}
{5,5{}},{5,4{}}
{6,6{}},{6,5{}}
{7,7{}},{7,6{}}
{8,8{}},{8,7{}}
{9,9{}},{9,8{}}
{10,10{}},{10,9{}}
```

as the solutions of System (5.2).

A.2 Testing the reducibility of classes

Here we give and explain the Mathematica scripts one can use to show that the classes in $\mathcal{E}_7$ are reducible.

To simplify the script, we will write a tuple $(f;\mathbf{n})$ and a triple $(d,e;\mathbf{m})$ as a vector v. Indeed, in doing so we lose the accentuation of the first entry, first two entries respectively, but being careful when sorting the vectors not to mix the elements, there is no reason to worry.

The standard Cremona move for a vector v representing a tuple, that is the image of a class $(d,e;\mathbf{m})$ under φ_*, of [19] as described in Chapter 5, is expressed in Mathematica as the function `CMV[v_]`:

```
 CMV[v_] :=
 Join[{2*v[[1]] - v[[2]] - v[[3]] - v[[4]]},
  Sort[Join[{v[[1]] - v[[3]] - v[[4]], v[[1]] - v[[2]] - v[[4]],
     v[[1]] - v[[2]] - v[[3]]}, v[[5 ;; Length[v]]]], Greater]]
```

The following script `Reduction[v_]` describes two big steps: first the vector v representing a class $(d, e; \mathbf{m})$ is transformed into a sorted vector representing $\varphi_*((d, e; \mathbf{m}))$, then the standard Cremona move is applied until one of the elements

$$(0; -1), (1; 1, 1), (2; 1^{\times 5}), (3; 2, 1^{\times 6}),$$
$$(4; 2^{\times 3}, 1^{\times 5}), (5; 2^{\times 2}, 1, 1), (6; 3, 2^{\times 7})$$

of $\mathcal{E}_8$ (compare [19, Lemma 1.2.7]) is obtained.

```
Reduction[v_] :=
Module[{W =
   Join[{v[[1]] + v[[2]] - v[[3]]},
    Sort[Join[{v[[2]] - v[[3]], v[[1]] - v[[3]]},
      v[[4 ;; Length[v]]]], Greater]], L = {},},
 While[Length[W] < 10, Return[W]; Break[]];
 If [W[[10]] == 0, L = W];
 If[W[[10]] > 0, W = CMV[W];
  While[W[[10]] > 0, W = CMV[W]];
  L = W];
 Return[L]]
```

We will now show how the Mathematica script `Reduction[v_]` is applied to a class $(d, e; \mathbf{m})$ with the help of an example. Therefore we take

$$(d, e; \mathbf{m}) := (169, 169; 99, 99, 99, 99, 99, 82, 17, 17, 17, 17, 14, 3, 3, 3, 3, 2, 1, 1)$$

that is the vector

$$v = \{169, 169, 99, 99, 99, 99, 99, 82, 17, 17, 17, 17, 14, 3, 3, 3, 3, 2, 1, 1\}.$$

Now `Reduction[169,169,99,99,99,99,99,82,17,17,17,17,14,3,3,3,3,2,1,1]` yields the vector

```
{3, 2, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```

which represents the class $(3; 2, 1^{\times 6})$ in $\mathcal{E}_8$ of [19, Lemma 1.2.7.].

Reducibility of classes of elements of the special sequences. Now for elements of the sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 0}$ we have a script, that directly computes the corresponding class for given n. Therefore it is necessary to first express the elements of the sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 0}$ as functions depending on n:

```
x[n_] := Simplify[((1 + Sqrt[2])^(2 n - 1) - (1 - Sqrt[2])^(2 n - 1))/(2 Sqrt[2])
+ ((1 + Sqrt[2])^(2 n) - (1 - Sqrt[2])^(2 n))/(2 Sqrt[2])]

y[n_] := Simplify[((1 + Sqrt[2])^(2 n - 1)
- (1 - Sqrt[2])^(2 n - 1))/(2 Sqrt[2])]

a[n_] := Simplify[x[n]/x[n - 1]]

b[n_] := Simplify[y[n + 1]/y[n]]
b[0] := 1
```

For given n, the corresponding class of an element of $(\alpha_n)_{n\geq 1}$ or $(\beta_n)_{n\geq 0}$ is determined by the functions `EA[n_]` and `EB[n_]` respectively, where the distinction of cases for elements of $(\beta_n)_{n\geq 0}$ for n even or odd is already taken into account:

```
EA[n_] := Simplify[Join[{y[n], y[n]}, x[n - 1]*W[a[n]]]]

EB[n_] := Simplify[If[Divisible[n, 2],
    Join[{x[n/2]^2, x[n/2]^2 - 1}, y[n]*W[b[n]]],
    Join[{2*y[(n + 1)/2]^2, 2*y[(n + 1)/2]^2 - 1}, y[n]*W[b[n]]]]]
```

Here `W[a_]` is the in Section A.1 of the appendix described Mathematica script of [19], that gives the weight expansion of a value $a \in \mathbb{Q}$.

Now, applying `Reduction[v_]` to `EA[n_]` and `EB[n_]` respectively and preserving the minimum and upper limit for n, that is $m \leq n \leq X$, we obtain for

```
 m := 1
 X := 8
 For[m = 1, m <= X, m++, Print[a[m], ",", Reduction[EA[m]]]]
```

that

```
 3,{1,1,1,0,0}
 17/3,{3,2,1,1,1,1,1,1,0,0}
 99/17,{3,2,1,1,1,1,1,1,0,0,0,0,0,0,0}
 577/99,{3,2,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0}
 3363/577,{3,2,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
 19601/3363,{3,2,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
 114243/19601,{3,2,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0}
 665857/114243,{3,2,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0,0,0,0,0,0}
```

and for

```
 m := 0
 X := 7
 For[m = 1, m <= X, m++, Print[b[m], ",", Reduction[EB[m]]]]
```

we get that

```
 5,{2,1,1,1,1,1,0}
 29/5,{3,2,1,1,1,1,1,1,0,0,0,0}
 169/29,{3,2,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0}
 985/169,{3,2,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
 5741/985,{3,2,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
 33461/5741,{3,2,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0}
 195025/33461,{3,2,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
0,0,0,0,0,0}.
```

Bibliography

[1] P. Biran, *Symplectic packing in dimension* 4, Geom. Funct. Anal., 7 (1997), p. 420-437

[2] R. Bott, L. W. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics, 82. Springer-Verlag, New York-Berlin, 1982

[3] B. Bradie, *Extensions and refinements of some properties of sums involving pell numbers*, Missouri J. Math. Sci. 22 (2010), no. 1, 37-43

[4] K. Cieliebak, *Lectures on Symplectic Geometry*, lecture notes, `http://www.mathematik.uni-muenchen.de/~kai/classes/SympGeo11/vorlesung.html`

[5] D. Frenkel, D. Müller, *Symplectic embeddings of* 4*-dimensional ellipsoids into cubes*, arXiv: 1210.2266v1

[6] P. Griffiths, J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics. Wiley-Interscience (John Wiley & Sons), New York, 1978.

[7] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math., 82 (1985), p. 307-347

[8] A. F. Horadam, *Pell Identities*, Fibonacci Quart. 9 (1971), no. 3, p. 245-252, 263

[9] M. Hutchings, *Quantitative embedded contact homology*, J. Differential Geom. 88 (2011), no. 2, p. 231-266

[10] M. Hutchings, *Recent progress on symplectic embedding problems in four dimensions*, Proc. Natl. Acad. Sci. USA 108 (2011), no. 20, p. 8093-8099

[11] Y. Karshon, *Appendix to [17]*, Invent. Math. 115 (1994), no. 1, p. 431-434

[12] B.-H. Li, T.-J. Li, *Symplectic genus, minimal genus and diffeomorphisms*, Asian J. Math. 6 (2002), no. 1, p. 123-144

[13] T.-J. Li, A. K. Liu, *Uniqueness of symplectic canonical class, surface cone and symplectic cone of* 4*-manifolds with* $b^+ = 1$, J. Differential Geom. 58 (2001), no. 2, p. 331-370

[14] D. McDuff, *From symplectic deformation to isotopy*, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 85-99, First Int. Press Lect. Ser., I, Int. Press, Cambridge, MA, 1998

[15] D. McDuff, *Symplectic embeddings of* 4*-dimensional ellipsoids*, J. Topol. 2 (2009), no. 1, p. 1-22

[16] D. McDuff, *The Hofer conjecture on embedding symplectic ellipsoids*, J. Differential Geom. 88 (2011), no. 3, p. 519-532

[17] D. McDuff, L. Polterovich, *Symplectic packings and algebraic geometry*, Invent. Math. 115 (1994), no. 3, p. 405-434

[18] D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, Second edition. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998

[19] D. McDuff, F. Schlenk, *The embedding capacity of* 4*-dimensional symplectic ellipsoids*, Ann. of Math. (2) 175 (2012), no. 3, p. 1191-1282

[20] F. Schlenk, *Volume preserving embeddings of open subsets of* $\mathbb{R}^n$ *into manifolds*, Proc. Amer. Math. Soc. 131 (2003), no. 6, p. 1925-1929

[21] F. Schlenk, *Embedding problems in symplectic geometry*, de Gruyter Expositions in Mathematics, 40. Walter de Gruyter GmbH & Co. KG, Berlin, 2005

[22] L. Traynor, *Symplectic packing constructions*, J. Differential Geom. 42 (1995), no. 2, 411-429