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APPLICATIONS DE LA K-THEORIE  
A LA TOPOLOGIE  
DES H-ESPACES FINIS

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# IMPRIMATUR POUR LA THÈSE

Applications de la K-théorie à la topologie  
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# Réalisation topologique de certaines algèbres associées aux algèbres de Dickson

Alain Jeanneret - Ulrich Suter

## Topological realisation of certain algebras associated to the Dickson algebras

*Abstract* - We discuss the topological realisation of certain  $\mathbb{Z}/2$ -algebras  $A(n)$  over the mod 2 Steenrod algebra  $\mathcal{U}(2)$ . If an associative H-space  $X(n)$  satisfies  $H^*(X(n); \mathbb{Z}/2) \cong A(n)$ , the mod 2 cohomology of its classifying space is isomorphic to the algebra of invariants of the canonical  $GL_n(\mathbb{Z}/2)$ -action on a graded polynomial algebra in  $n$  variables of degree 1.

**1. Introduction.** Dans cette note, on envisage de déterminer les entiers  $n \geq 2$  pour lesquels l'algèbre  $A(n)$  définie sous (1) peut être réalisée comme algèbre de cohomologie modulo 2 d'un espace topologique. La  $\mathbb{Z}/2$ -algèbre graduée

$$(1) \quad A(n) = \mathbb{Z}/2 [v_1] / (v_1^4) \otimes E(v_2, v_3, \dots, v_{n-1}) \quad \text{où } \text{degré}(v_k) = 2^n - 2^{n-k} - 1$$

est munie d'une unique structure de module sur l'algèbre de Steenrod  $\mathcal{U}(2)$  imposée par l'égalité  $Sq^{2^{n-1}-1}v_1 = v_1^2$  et les relations d'Adem  $Sq^{2^i-1} = Sq^1$

$Sq^2 \dots Sq^{2^{i-1}}$ . Par exemple,  $A(2) \cong H^*(SO(3); \mathbb{Z}/2)$  et  $A(3) \cong H^*(G_2; \mathbb{Z}/2)$ , où  $G_2$  est le groupe de Lie exceptionnel.

Les algèbres  $A(n)$  sont liées à certaines algèbres de Dickson  $D(n)$  définies comme suit : le groupe linéaire  $GL_n(\mathbb{Z}/2)$  agit canoniquement sur l'algèbre de polynômes graduée  $\mathbb{Z}/2 [t_1, \dots, t_n]$ , où  $\text{degré}(t_k) = 1$  ( $k = 1, \dots, n$ ), et par définition,  $D(n)$  est l'algèbre des invariants. Dickson a démontré que  $D(n) = \mathbb{Z}/2 [w_1, \dots, w_n]$ , où  $\text{degré}(w_k) = 2^n - 2^{n-k}$ . L'algèbre  $D(n)$  est un module sur  $\mathcal{U}(2)$  et la question suivante est naturellement liée au problème de Steenrod:

Existe-t-il un CW-complexe  $Y(n)$  tel que  $H^*(Y(n); \mathbb{Z}/2) \cong D(n)$  ?

Les exemples connus sont :  $D(1) \cong H^*(B\mathbb{Z}/2; \mathbb{Z}/2)$ ,  $D(2) \cong H^*(BSO(3); \mathbb{Z}/2)$  et  $D(3) \cong H^*(BG_2; \mathbb{Z}/2)$ . Dans [ 6 ] L. Smith et M. Switzer montrent qu'un tel espace n'existe pas si  $n \geq 6$ . Les cas où  $n = 4$  et  $n = 5$  sont plus délicats à traiter.

Si  $Y(n)$  existe, on obtient facilement à l'aide de la suite spectrale d'Eilenberg-Moore que l'espace des lacets  $\Omega Y(n) = X(n)$  vérifie :

$$(2) \quad H^*(X(n); \mathbb{Z}/2) \cong A(n).$$

La question précédente peut donc se reformuler en ces termes:

Existe-t-il un H-espace associatif  $X(n)$  vérifiant la condition (2) ?

J. Lin et F. Williams ont répondu par la négative au cas où  $n = 5$  [4]. Leur démonstration utilise la machinerie des opérations cohomologiques secondaires et tertiaires. A l'aide de la K-théorie complexe, il est possible de démontrer, sans faire appel aux opérations cohomologiques d'ordres supérieurs, le résultat plus général suivant :

**Théorème.** *Pour  $n \geq 5$ , l'algèbre  $A(n)$ , définie sous (1), ne peut pas être réalisée comme algèbre de cohomologie modulo 2 d'un espace topologique.*

Notons que ce théorème exclut l'existence de  $X(n)$ , pour  $n \geq 5$ , comme espace topologique et non seulement comme H-espace mais qu'il n'apporte aucune information au cas où  $n = 4$  pour lequel il existe deux résultats contradictoires [ 2 ], [ 5 ].

Le reste de cette note est consacré à l'esquisse de la démonstration du théorème. Elle peut être résumée comme suit : si  $X(n)$  satisfait (2), on constate d'abord que la cohomologie entière de  $\Omega X(n)$  est sans 2-torsion, et on montre ensuite, grâce aux résultats d'Atiyah [ 1 ], que les opérations de Steenrod  $Sq^k$  et l'opération d'Adams  $\Psi^2$  ne sont pas compatibles pour  $\Omega X(n)$ , où  $n \geq 5$ .

Nous n'explicitons que le cas où  $n = 5$ , les autres se traitant de manière identique. Dans la suite de cette note, nous supposons que  $X$  est un CW-complexe fini simplement connexe vérifiant :

$$(3) \quad H^*(X; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_{15}] / (x_{15}^4) \otimes E(x_{23}, x_{27}, x_{29})$$

Un tel complexe a le même type d'homotopie rationnelle que le produit de sphères  $S^{15} \times S^{23} \times S^{27} \times S^{59}$ . Faisant appel à la technique du "mixing homotopy types", nous supposons également que, pour les premiers impairs  $p$ , la cohomologie modulo  $p$  de  $X$  est isomorphe à l'algèbre extérieure  $E_{\mathbb{Z}/p}(z_{15}, z_{23}, z_{27}, z_{59})$ .

Nous tenons à remercier J. Lin avec qui nous avons eu de nombreuses et fructueuses discussions.

## 2. Calculs de cohomologie ordinaire.

Soit  $\rho_* : H^*(X; \mathbb{Z}) \longrightarrow H^*(X; \mathbb{Z}/2)$  l'homomorphisme de réduction modulo 2. A l'aide de la formule des coefficients universels on montre l'existence d'éléments  $y_{15}, y_{23}, y_{27}, z_{59} \in H^*(X; \mathbb{Z})$  pour lesquels  $\rho_*(y_k) = x_k$  ( $k = 15, 23, 27$ ) et  $\rho_*(z_{59}) = x_{29} \cdot x_{15}^2$ ; de plus,

$$H^*(X; \mathbb{Z}) \cong P(y_{15}) \otimes E_{\mathbb{Z}/2}(y_{23}, y_{27}, z_{59}) / (y_{15}^2 \otimes z_{59}),$$

où  $P(y_{15}) = \mathbb{Z}[y_{15}] / (y_{15}^4, 2y_{15}^2) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$  est engendré additivement par  $1, y_{15}, y_{15}^2, y_{15}^3$ .

Nous allons décrire la cohomologie modulo 2 de  $\Omega X$  au moyen de la suite spectrale d'Eilenberg-Moore. Notre référence, dans ce contexte, est le livre de R. Kane [3, ch. VII] dont nous adoptons aussi les notations. Le terme  $E_2$  de la suite s'obtient à l'aide de la résolution bar  $BH^*(X; \mathbb{Z}/2)$ ; dans notre situation, on obtient

$$E_2 \cong \text{Tor}_{H^*(X; \mathbb{Z}/2)}(\mathbb{Z}/2, \mathbb{Z}/2) \cong E(u_{14}) \otimes \Gamma(u_{22}, u_{26}, u_{28}, u_{58}),$$

où  $u_k = sx_k$  ( $k = 14, 22, 26, 28$ ) sont des éléments "suspension" représentés par  $[x_k]$  et  $u_{58}$  est un élément "transpotent" représenté par  $[x_{15}^2 | x_{15}^2] = [x_{15} | x_{15}^3]$ .

Le degré total de tous les générateurs étant pair, on conclut que la suite spectrale est triviale, i.e.  $E_2 \cong E_\infty$ , et on obtient ainsi les isomorphismes additifs:

$$H^*(\Omega X; \mathbb{Z}/2) \cong E^0(H^*(\Omega X; \mathbb{Z}/2)) \cong E(u_{14}) \otimes \Gamma(u_{22}, u_{26}, u_{28}, u_{58})$$

Lemme 1. Soit  $u_{56}$  l'unique élément non nul dans  $H^{56}(\Omega X; \mathbb{Z}/2)$ . Alors:

$$Sq^2 u_{56} \neq 0.$$

**Démonstration:** La suite spectrale d'Eilenberg-Moore est une suite de modules sur  $\mathcal{U}(2)$  et dans  $\text{Tor}_{H^*(X; \mathbb{Z}/2)}(\mathbb{Z}/2, \mathbb{Z}/2)$  on a l'élément  $u_{56} = \gamma_2(u_{28}) =$

$$[x_{29}|x_{29}] \text{ ainsi que les relations } Sq^2[x_{29}|x_{29}] = \sum_{i+j=2} [Sq^i x_{29} | Sq^j x_{29}] = [x_{15}^2 | x_{15}^2] =$$

$$u_{58} \neq 0.$$

Notons encore que la cohomologie entière de  $\Omega X$  est sans torsion (fait résultant des égalités  $H^{2n+1}(\Omega X; \mathbb{Z}/2) = 0$ , pour  $n \geq 0$ ), et que l'on peut choisir des générateurs  $v_k \in H^k(\Omega X; \mathbb{Z}) \cong \mathbb{Z}$  ( $k = 14, 22, 26, 28, 56$ ) tels que :

$$(4) \quad \rho_*(v_k) = u_k \text{ et } v_{14}^2 = v_{28}, \quad v_{14}^4 = v_{28}^2 = 2v_{56}.$$

### 3. Calculs de K-théorie.

Une étude soignée de la suite spectrale d'Atiyah-Hirzebruch en K-théorie mod 2 et entière pour l'espace  $X$  montre que, dans les deux cas, la seule différentielle non-triviale est  $d_3$ . On obtient ainsi  $E_4 \cong E_\infty \cong E^0(K^*(X))$ , et le calcul explicite fournit que  $K^*(X) \cong E(\xi_{15}, \xi_{23}, \xi_{27}, \xi_{59})$ , les générateurs  $\xi_k$  satisfaisants aux conditions suivantes : dans l'ordre indiqué, les éléments  $\xi_{15}, \xi_{23}, \xi_{27}, \xi_{59}, 2\xi_{59}$  correspondent, dans le sous-quotient  $E_\infty$  de  $H^*(X; \mathbb{Z})$ , aux éléments représentés par  $y_{15}, y_{23}, 2y_{27}, y_{15}^2 \cdot y_{27}, z_{59}$ ; la filtration rationnelle de  $\xi_k$  est égale à  $k$ . On définit  $\eta_k \in \tilde{K}^0(\Sigma X)$  par

$$\eta_k = \sigma(\xi_{k-1}) \quad (k = 16, 24, 28, 60),$$

où  $\sigma : \tilde{K}^1(X) \cong \tilde{K}^0(\Sigma X)$  et on envisage de déterminer l'opération d'Adams  $\Psi^2$  sur ces éléments.

Pour tout CW-complexe  $Y$  et tout entier  $q \geq 0$  soit  $\mathcal{F}^q \tilde{K}(Y) \subseteq \tilde{K}(Y) = \tilde{K}^0(Y)$  le sous-groupe des éléments de filtration  $\geq q$ , c'est-à-dire  $\mathcal{F}^q \tilde{K}(Y) = \text{Ker} \{ \tilde{K}(Y) \longrightarrow \tilde{K}(Y^{q-1}) \}$ . Les éléments  $\eta_k$  ( $k = 16, 24, 28, 60$ ) forment une base du groupe abélien libre  $\tilde{K}(\Sigma X) / \mathcal{F}^{62} \tilde{K}(\Sigma X)$ .

**Lemme 2.** *Pour un choix convenable de  $\eta_{28} \in \tilde{K}(\Sigma X)$  on a dans  $\tilde{K}(\Sigma X) / \mathcal{F}^{62} \tilde{K}(\Sigma X)$  :*

- (i)  $\Psi^2(\eta_{16}) \equiv 2 \eta_{28} \pmod{4}$   
(ii)  $\Psi^2(\eta_{28}) \equiv 0 \pmod{4}$

**Démonstration.** Remarquons préalablement que  $\Psi^2(\eta) \equiv \eta^2 = 0 \pmod{2}$  pour tout  $\eta \in \tilde{K}(\Sigma X)$ . A homotopie près, il existe un sous-complexe  $Z = S^{16} \cup e^{24} \cup e^{28}$  de  $\Sigma X$  qui porte la cohomologie entière jusqu'à la dimension 28. Soit  $i : Z \longrightarrow \Sigma X$  l'inclusion. La K-théorie de  $Z$  est libre à trois générateurs,  $\alpha_{16} = i^*(\eta_{16})$ ,  $\alpha_{24} = i^*(\eta_{24})$  et  $\alpha_{28}$ , où  $2\alpha_{28} = i^*(\eta_{28})$ . Par le résultat d'Atiyah [ 1 ], on obtient  $\Psi^2(\alpha_{16}) = 2^8 \alpha_{18} + 2^4 a \alpha_{24} + 2^2 b \alpha_{28}$ , où  $a \equiv b \equiv 1 \pmod{2}$ , il suit que

$$(5) \quad \Psi^2(\eta_{16}) = 2^8 \eta_{16} + 2^4 a \eta_{24} + 2 b \eta_{28} + 2 c \eta_{60}.$$

Cette dernière égalité implique directement (i) dans le cas où  $c$  est pair; si  $c$  est impair on remplace d'abord le générateur  $\eta_{28}$  par  $\tilde{\eta}_{28} = \eta_{28} + \eta_{60}$ . Pour la démonstration de (ii), on note que  $\Psi^m(\eta_{28}) = m^{14} \eta_{28} + d_m \eta_{60}$ ,  $\Psi^m(\eta_{60}) = m^{30} \eta_{60}$  et on considère l'égalité  $\Psi^3 \Psi^2(\eta_{28}) = \Psi^2 \Psi^3(\eta_{28})$ .

**Remarque :** dans le cas où  $n = 4$  la relation  $\Psi^3 \Psi^2 = \Psi^2 \Psi^3$  implique que  $d_2 \equiv 0 \pmod{2}$  mais ne permet pas d'affirmer que ce coefficient est divisible par 4.

Considérons maintenant l'application canonique  $\varepsilon : \Sigma^2 \Omega X \longrightarrow \Sigma X$ . On définit  $\mu_k \in \tilde{K}(\Omega X)$  ( $k = 14, 22, 26$ ) par  $\sigma^2(\mu_k) = \varepsilon^*(\eta_{k+2})$  et on rappelle que  $2 \sigma^2 \Psi^2(\mu_k) = \varepsilon^* \Psi^2(\eta_{k+2})$ . La cohomologie entière  $H^*(\Omega X; \mathbb{Z})$  étant sans torsion, il en est de même pour  $\tilde{K}(\Omega X) / \mathcal{F}^{60} \tilde{K}(\Omega X)$ . En calculant dans ce dernier groupe on obtient du lemme 2 d'abord que  $\mu_{14}^2 \equiv \Psi^2(\mu_{14}) \equiv \mu_{26}$

(mod 2) et ensuite que  $\mu_{14}^4 \equiv \Psi^2(\mu_{14}^2) \equiv \Psi^2(\mu_{26}) \equiv 0 \pmod{2}$ . Il existe donc un élément  $\omega \in \widetilde{K}(\Omega X)$  tel que

$$\mu_{14}^4 \equiv 2 \omega \pmod{\mathcal{F}^{60}\widetilde{K}(\Omega X)}.$$

La relation (5) implique que  $\Psi^2(\mu_{14}^4) \equiv 2^{28} \mu_{14}^4 \pmod{\mathcal{F}^{60}\widetilde{K}(\Omega X)}$  et on obtient

ainsi :

$$\Psi^2(\omega) \equiv 2^{28} \omega \pmod{\mathcal{F}^{60}\widetilde{K}(\Omega X)}.$$

La suite spectrale d'Atiyah-Hirzebruch de  $\Omega X$  en K-théorie est triviale, i.e.  $H^*(\Omega X; \mathbb{Z}) \cong E^0(K^*(\Omega X))$ . L'élément  $\mu_{14}$  représente  $v_{14}$ , et il suit de (4) que  $\omega$  représente  $v_{56} \in H^{56}(\Omega X; \mathbb{Z})$ . A l'aide du théorème d'Atiyah [ 1 ] on déduit que  $Sq^2 u_{56} = 0$ ; ce qui contredit le lemme 1 et achève la démonstration du théorème.

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## Algebras over the Steenrod Algebra and Finite H-Spaces

Alain Jeanneret

0. Let  $p$  be a prime and  $P = \mathbb{Z}/p[x_1, \dots, x_n]$  be a graded polynomial algebra over the mod  $p$  Steenrod algebra  $\mathcal{U}(p)$ . A fundamental question in algebraic topology is to decide whether  $P$  can be realized as the mod  $p$  cohomology of a topological space. Ideas from the Galois and invariant theories provided a complete answer to the realization problem for the non-modular case (i.e. when  $\deg(x_i) \not\equiv 0 \pmod{p}$ ,  $i = 1, \dots, n$ ) see [A-W]. In contrast the modular case is still not settled.

An interesting family of modular polynomial algebras at the prime 2 is provided by the so called Dickson algebras defined as follows :

The canonical  $GL(q; \mathbb{Z}/2)$  - action on  $(\mathbb{Z}/2)^q$  induces a  $GL(q; \mathbb{Z}/2)$  - action on  $H^*(B(\mathbb{Z}/2)^q; \mathbb{Z}/2) \cong \mathbb{Z}/2[t_1, \dots, t_q]$ , ( $\deg(t_i) = 1$ ,  $i = 1, \dots, n$ ). As the Steenrod algebra acts via the squaring map (which is linear in characteristic 2), the actions of  $W = GL(q; \mathbb{Z}/2)$  and  $\mathcal{U}(2)$  commute. The Dickson algebra is the invariant algebra

$$D(q) = H^*(B(\mathbb{Z}/2)^q; \mathbb{Z}/2)^W$$

which is again an algebra over  $\mathcal{U}(2)$ . Dickson has shown that :

$$D(q) \cong \mathbb{Z}/2[w_1, \dots, w_q], (\deg(w_i) = 2^q - 2^{q-i}).$$

The  $\mathcal{U}(2)$  - action on  $D(q)$  can be described as follows :  $Sq^{2^{q-k-1}} w_k = w_{k+1}$  ( $k = 1, \dots, q-1$ ). A natural question is therefore :

Does there exist a topological space  $Y(q)$  such that  $H^*(Y(q); \mathbb{Z}/2) \cong D(q)$  ?

The well known classical examples are :  $D(1) \cong H^*(B\mathbb{Z}/2; \mathbb{Z}/2)$ ,  $D(2) \cong H^*(BSO(3); \mathbb{Z}/2)$  and  $D(3) \cong H^*(BG_2; \mathbb{Z}/2)$  where  $G_2$  denotes the exceptional

Lie group. L. Smith and R. Switzer have proved in [S-S] that  $Y(q)$  does not exist for  $q \geq 6$ .

Let  $E(x_1, \dots, x_n)$  denote the exterior algebra on  $x_1, \dots, x_n$  and let

$$A(q) = \mathbb{Z}/2[v_1]/(v_1^4) \otimes E(v_2, v_3, \dots, v_{q-1}) \quad (\deg(v_i) = 2^q - 2^{q-i} - 1)$$

be an algebra over  $\mathcal{U}(2)$ . The action of the Steenrod algebra is given by

$$Sq^{2^{q-k-1}} v_k = v_{k+1} \quad (k = 1, \dots, q-2), \quad Sq^1 v_{q-1} = v_1^2.$$

Standard methods show that the cohomology ring of  $X(q) = \Omega Y(q)$  satisfies  $H^*(X(q); \mathbb{Z}/2) \cong A(q)$ . For  $n = 5$ , U. Suter (unpublished), J. Lin and F. Williams [L-W] have shown that  $X(5)$  cannot support an  $H$ -structure. This implies the non realizability of  $D(5)$  as the cohomology algebra of a topological space. In [J-S], U. Suter and the author proved a stronger version of the result of Lin - Williams and Smith - Switzer, namely :

For  $q \geq 5$  the algebra  $A(q)$  cannot be realized as the mod 2 cohomology algebra of a topological space.

The purpose of this note is to give a proof of a further generalization of the last result. Set

$$A_i = A(q) \quad (i = 1, \dots, m)$$

$$B_j = E(w_j) \quad (\deg(w_j) = 2^{n_j} - 1, n_j \geq q-1, j = 1, \dots, n)$$

$$K(q, m, n) = \bigotimes_{i=1}^m A_i \otimes \bigotimes_{j=1}^n B_j \quad (m \geq 1, n \geq 0).$$

We assume that  $\bigotimes_{i=1}^m A_i$  is an  $\mathcal{U}(2)$ -subalgebra of  $K(q, m, n)$  and that  $Sq^1 w_j = Sq^2 w_j = 0$  ( $j = 1, \dots, n$ ).

**Theorem.** For  $q \geq 5$  the  $\mathcal{U}(2)$  - algebras  $K(q, m, n)$  cannot be realized as the mod 2 cohomology of topological spaces.

**Remark.** The hypothesis on the  $\mathcal{U}(2)$  - action on  $K(q, m, n)$  is not essential (see the final remark at the end of this note).

This result is related to the question of connectivity of finite  $H$  - spaces in the following way. Let us first recall a result of J. Lin :

**Theorem.**(see [L]) *Let  $X$  be a  $14$  - connected finite  $H$  - space with  $H_*(X; \mathbb{Z}/2)$  associative, then  $\xi(H^{15}(X; \mathbb{Z}/2)) \neq 0$  where  $\xi : H^n(X; \mathbb{Z}/2) \longrightarrow H^{2n}(X; \mathbb{Z}/2)$  is the squaring map.*

The algebras  $K(5, m, n)$  satisfy the condition  $\xi(K(5, m, n)^{15}) \neq 0$ , so they are not ruled out by Lin's theorem. Our methods show however that they cannot even be the cohomology algebras of topological spaces.

We shall give the proof of the theorem for  $q = 5$  only, the other cases can be treated exactly in the same way.

I am grateful to U. Suter for his guidance and his help. I would also like to take the opportunity to thank J. Lin for the many useful discussions I had with him.

**Notations.** Throughout the paper we set

$$A_i = \mathbb{Z}/2[x_{15}^{(i)}]/((x_{15}^{(i)})^4) \otimes E(x_{23}^{(i)}, x_{27}^{(i)}, x_{29}^{(i)}) \quad (\deg(x_k^{(i)}) = k, i = 1, \dots, m)$$

$$B_j = E(w_j) \quad (\deg(w_j) = 2^{n_j} - 1, n_j \geq 4, j = 1, \dots, n)$$

$$K = \bigotimes_{i=1}^m A_i \otimes \bigotimes_{j=1}^n B_j$$

We assume that there exists a simply connected CW - complex  $X$  such that  $H^*(X; \mathbb{Z}/2) \cong K$ . We shall show that the Adams operations and the action of the

Steenrod algebra are not compatible on  $\Omega X$ . This will rule out the existence of such a space.

1. In this section we shall describe the cohomology of  $\Omega X$ , the loop space on  $X$ , and give some results on the  $\mathbb{Z}_{(2)}$  - cohomology of  $X$  where  $\mathbb{Z}_{(2)}$  stands for the ring of integers localized at the prime 2.

Our main device to obtain the mod 2 cohomology of  $\Omega X$  is the Eilenberg-Moore spectral sequence (E-Mss for short). Though we adopt the notation of [K] let us recall briefly some properties of the E-Mss. It is a second quadrant spectral sequence of  $\mathbb{Q}(2)$  - modules with

$$E_2 \cong \text{Tor}_{H^*(X; \mathbb{Z}/2)}(\mathbb{Z}/2, \mathbb{Z}/2) \text{ and } E_\infty \cong E^0(H^*(\Omega X; \mathbb{Z}/2))$$

( $E^0(H)$  is the graded module associated to a filtration of  $H$ ). Let  $H$  denote the algebra  $E(x)$  or  $\mathbb{Z}/2[x]/(x^4)$ , and let  $\bar{H}$  denote the augmentation ideal of  $H$ . A straightforward computation, using the bar construction, shows that :

$$\text{Tor}_{E(x)}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \Gamma(sx) \text{ and } \text{Tor}_{\mathbb{Z}/2[x]/(x^4)}(\mathbb{Z}/2, \mathbb{Z}/2) \cong E(sx) \otimes \Gamma(tx)$$

The element  $sx$ , called suspension element has bidegree  $(-1, \text{deg}(x))$  and is represented (via the bar construction) by  $[x] \in \bar{H}$ , whereas  $tx$ , called transpotence element has bidegree  $(-2, \text{deg}(x))$  and is represented by  $[x^2|x^2]$  or by  $[x|x^3] \in \bar{H} \otimes \bar{H}$  (these two cycles differ from a boundary and therefore represent the same element in the  $E_2$  term). As usual  $\Gamma(u)$  is the divided polynomial algebra on  $u$ . We also mention the following property of  $\text{Tor}$  :

$$\text{Tor}_{N \otimes M}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \text{Tor}_N(\mathbb{Z}/2, \mathbb{Z}/2) \otimes \text{Tor}_M(\mathbb{Z}/2, \mathbb{Z}/2).$$

The E-Mss for  $X$  can then be computed. We obtain easily :

$$E_2 \cong \text{Tor}_K(\mathbb{Z}/2, \mathbb{Z}/2) \cong \bigotimes_{i=1}^m C_i \otimes \bigotimes_{j=1}^n D_j$$

where

$$C_i \cong E(u_{14}^{(i)}) \otimes \Gamma(u_{22}^{(i)}, u_{26}^{(i)}, u_{28}^{(i)}, u_{58}^{(i)}) \quad (i = 1, \dots, m)$$

$$D_j \cong \Gamma(v_j) \quad (j = 1, \dots, n)$$

with  $u_{2n}^{(i)} = sx_{2n+1}^{(i)}$  ( $n = 7, 11, 13, 14$ ),  $u_{58}^{(i)} = tx_{15}^{(i)}$  and  $v_j = sw_j$ .

The differentials of the E-Mss raise the total degree by one. As  $E_2$  is concentrated in even total degree the spectral sequence is trivial, i.e.  $E_2 \cong E_\infty$ , and so we get the additive isomorphisms :

$$H^*(\Omega X; \mathbb{Z}/2) \cong E^0(H^*(\Omega X; \mathbb{Z}/2)) \cong \bigotimes_{i=1}^m C_i \otimes \bigotimes_{j=1}^n D_j.$$

For each  $i = 1, \dots, m$  there is an element  $u_{56}^{(i)} \in H^{56}(\Omega X; \mathbb{Z}/2)$  represented by  $[x_{29}^{(i)} | x_{29}^{(i)}] = \gamma_2(u_{28}^{(i)})$  in  $E^0(H^*(\Omega X; \mathbb{Z}/2)) \cong E_2$ . As  $\mathcal{U}(2)$  acts on  $E_2$  via the Cartan formula, one gets (with some abuse of notations) that  $Sq^2 u_{56} = Sq^2 [x_{29}^{(i)} | x_{29}^{(i)}] = \sum_{i+j=2} [Sq^i x_{29}^{(i)} | Sq^j x_{29}^{(i)}] = [x_{15}^2 | x_{15}^2] = [x_{15} | x_{15}^3] = u_{58} \neq 0$ . So we

have proved the following result :

**Lemma 1.** *The element  $u_{56}^{(i)} \in H^{56}(\Omega X; \mathbb{Z}/2)$ , represented by  $[x_{29}^{(i)} | x_{29}^{(i)}]$  in  $E^0(H^*(\Omega X; \mathbb{Z}/2))$  ( $i = 1, \dots, m$ ) satisfies :*

$$Sq^2 u_{56}^{(i)} \neq 0.$$

An immediate consequence of the fact that  $H^{2n+1}(\Omega X; \mathbb{Z}/2) = 0$  is the next lemma. Let  $\rho_* : H^*(X; \mathbb{Z}/2) \longrightarrow H^*(X; \mathbb{Z}/2)$  denote the modulo 2 reduction homomorphism.

**Lemma 2.** *The algebra  $H^*(\Omega X; \mathbb{Z}_{(2)})$  is torsion free and there are generators  $s_k^{(i)} \in H^k(\Omega X; \mathbb{Z}_{(2)})$  ( $i = 1, \dots, m$  and  $k = 14, 22, 26, 28, 56$ ) with*

$$\rho_*(s_k^{(i)}) = u_k^{(i)} \text{ and } (s_{14}^{(i)})^2 = s_{28}^{(i)}, (s_{14}^{(i)})^4 = (s_{28}^{(i)})^2 = 2s_{56}^{(i)},$$

*and there exist  $t_j \in H^{2^{n_j}-i}(\Omega X; \mathbb{Z}_{(2)})$  ( $j = 1, \dots, n$ ) with  $\rho_*(t_j) = v_j$ .*

**Proof.** The only point to be checked is the assertion concerning the multiplicative structure. As the E-Mss is a spectral sequence of  $\mathcal{Q}(2)$  - modules, we obtain from the  $\mathcal{Q}(2)$  - action on  $K$  :

$$\text{Sq}^{14} u_{14}^{(i)} = (u_{14}^{(i)})^2 = u_{28}^{(i)} \text{ and } \text{Sq}^{28} u_{28}^{(i)} = (u_{28}^{(i)})^2 = 0.$$

Therefore we can choose  $s_{14}^{(i)}, s_{28}^{(i)}$  ( $i = 1, \dots, m$ ) such that  $(s_{14}^{(i)})^2 = s_{28}^{(i)}$  and  $(s_{14}^{(i)})^4 = (s_{28}^{(i)})^2 \equiv 0 \pmod{2}$ . Using the Hopf algebra structure on  $H^*(\Omega X; \mathbb{Z}_{(2)})$  and the fact that  $s_{14}^{(i)}$  is primitive, one can show there exists  $s_{56}^{(i)}$  such that  $(s_{14}^{(i)})^4 = (s_{28}^{(i)})^2 = 2s_{56}^{(i)}$ . Moreover  $\rho_*(s_{56}^{(i)}) = u_{56}^{(i)}$  ( $i = 1, \dots, m$ ).

We end this section with a description of the  $\mathbb{Z}_{(2)}$  - cohomology of  $X$ . Using the universal coefficient formula and the fact that  $\text{Sq}^1$  detects  $\mathbb{Z}/2$  - summands, one shows the existence of free generators  $y_k^{(i)} \in H^k(X; \mathbb{Z}_{(2)})$  ( $i = 1, \dots, m; k = 15, 23, 27, 59$ ) and  $z_j \in H^{2^{n_j}-i}(X; \mathbb{Z}_{(2)})$  ( $j = 1, \dots, n$ ) such that :

$$\rho_*(y_k^{(i)}) = x_k^{(i)} \quad (k = 15, 23, 27), \rho_*(y_{59}^{(i)}) = x_{29}^{(i)} \cdot (x_{15}^{(i)})^2 \text{ and } \rho_*(z_j) = w_j;$$

moreover we have the isomorphism of  $\mathbb{Z}_{(2)}$  - modules :

$$H^*(X; \mathbb{Z}_{(2)}) \cong \bigotimes_{i=1}^n \{P(y_{15}^{(i)}) \otimes E_{\mathbb{Z}/2}(y_{23}^{(i)}, y_{27}^{(i)}, y_{59}^{(i)}) / ((y_{15}^{(i)})^2 \otimes y_{59}^{(i)})\}$$

$$\bigotimes_{j=1}^n E_{\mathbb{Z}_{(2)}}(z_j)$$

where  $P(y_{15}^{(i)}) = \mathbb{Z}_{(2)}[y_{15}^{(i)}] / ((y_{15}^{(i)})^4, 2(y_{15}^{(i)})^2) \cong \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$  is additively generated by  $1, y_{15}^{(i)}, (y_{15}^{(i)})^2, (y_{15}^{(i)})^3$ . Note in particular that the two torsion of  $H^*(X; \mathbb{Z}_{(2)})$  has only order two.

2. This section contains the computation of the K - theory of X. Let us first fix the notation and recall some basic facts about the K - theory with coefficients. If Y is a finite CW - complex,  $K^*(Y)$  denotes the  $\mathbb{Z}/2$  - graded ordinary complex K - theory [A-H]. For  $G = \mathbb{Z}_{(2)}$  or  $\mathbb{Z}/2$  the K - theory with coefficients in G, denoted by  $K^*(Y; G)$ , is defined in [A-T].  $K^*(Y; G)$  is again a  $\mathbb{Z}/2$  - graded algebra. Moreover it satisfies  $\tilde{K}^0(S^n; G) = 0$  if n is odd and  $\tilde{K}^0(S^n; G) \cong G$  if n is even.

The Atiyah-Hirzebruch spectral sequence (A-Hss for short) is a first and fourth quadrant spectral sequence of algebras with

$$E_2(Y; G) \cong H^*(Y; K^*(\text{pte}; G)) \text{ and } E_\infty(Y; G) \cong E^0(K^*(Y; G)).$$

The modulo 2 reduction homomorphism induces a morphism of spectral sequences  $\rho_* : E_r(Y; \mathbb{Z}_{(2)}) \longrightarrow E_r(Y; \mathbb{Z}/2)$ . Finally, for  $G = \mathbb{Z}/2$  the differential  $d_3^{\mathbb{Z}/2}$  is given by

$$d_3^{\mathbb{Z}/2} = Sq^2 Sq^1 + Sq^1 Sq^2.$$

Our first step towards the determination of  $K^*(X; \mathbb{Z}_{(2)})$  is the computation of  $K^*(X; \mathbb{Z}/2)$ . Recall that in this case  $E_2 \cong E_3 \cong H^*(X; \mathbb{Z}/2)$ . Using the  $\mathcal{U}(2)$  - action on K ( see section 0) we infer that :

$$d_3(x_{15}^{(i)}) = d_3(x_{23}^{(i)}) = d_3(x_{29}^{(i)}) = 0 \text{ and } d_3(x_{27}^{(i)}) = (x_{15}^{(i)})^2 \quad (i = 1, \dots, m)$$

(1)

$$d_3(w_j) = 0 \quad (j = 1, \dots, n).$$

As  $d_3$  is a derivation we readily get :

$$E_4 \cong \bigotimes_{i=1}^m E(x_{15}^{(i)}, x_{23}^{(i)}, x_{29}^{(i)}, x_{27}^{(i)}(x_{15}^{(i)})^2) \otimes \bigotimes_{j=1}^n E(w_j)$$

where  $x_k$  still denotes (for simplicity of notation) the element of  $E_4$  represented by  $x_k \in \ker d_3 \subset E_3$ . For dimensional reasons ( $\dim_{\mathbb{Z}/2} E_4(X; \mathbb{Z}/2) = \dim_{\mathbb{Q}} H^*(X; \mathbb{Q})$ ) the spectral sequence collapses after  $E_4$ , i.e.  $E_4 \cong E_\infty$  and hence

$$K^*(X; \mathbb{Z}/2) \cong \bigotimes_{i=1}^m E(\theta_{15}^{(i)}, \theta_{23}^{(i)}, \theta_{29}^{(i)}, \theta_{57}^{(i)}) \otimes \bigotimes_{j=1}^n E(\tau_j)$$

where  $\tau_j$  and  $\theta_k^{(i)}$  are represented by the obvious classes in  $E^0(K^*(X; \mathbb{Z}/2)) \cong E_4$ .

From the universal coefficient formula for K-theory and the observation that  $\dim_{\mathbb{Z}/2} K^*(X; \mathbb{Z}/2) = \dim_{\mathbb{Q}} H^*(X; \mathbb{Q})$  we conclude that  $K^*(X; \mathbb{Z}/2)$  is torsion free.

We are now ready to calculate the K-theory of  $X$  with  $\mathbb{Z}/2$  - coefficients. The modulo 2 reduction homomorphism  $\rho_* : H^*(X; \mathbb{Z}/2) \longrightarrow H^*(X; \mathbb{Z}/2)$  is an injection on the torsion subgroup (see the end of section 1) and since  $\text{image}(d_r) \subset \text{torsion}(E_r)$  we obtain using (1) that

$$d_3(y_{15}^{(i)}) = d_3(y_{23}^{(i)}) = d_3(y_{59}^{(i)}) = 0, \quad d_3(y_{27}^{(i)}) = (y_{15}^{(i)})^2 \text{ and } d_3(z_j) = 0.$$

We then compute (2) :

$$E_4(X; \mathbb{Z}(2)) \cong \bigotimes_{i=1}^m \{E_{\mathbb{Z}(2)}(y_{15}^{(i)}, y_{23}^{(i)}, 2y_{27}^{(i)}) \oplus E_{\mathbb{Z}(2)}(y_{15}^{(i)}, y_{23}^{(i)}, y_{27}^{(i)}) y_{59}^{(i)} \\ \otimes (y_{15}^{(i)})^2 \cdot y_{27}^{(i)} E_{\mathbb{Z}(2)}(y_{15}^{(i)}, y_{23}^{(i)})\} \otimes \bigotimes_{j=1}^n E_{\mathbb{Z}(2)}(z_j)$$

Observe that  $\rho_* : E_4(X; \mathbb{Z}(2)) \longrightarrow E_4(X; \mathbb{Z}/2)$  is again injective on the torsion subgroup. Since  $d_r^{\mathbb{Z}/2} = 0$  for  $r \geq 4$  we infer that  $d_r = 0$  for  $r \geq 4$ , and therefore

$$(3) \quad E_4(X; \mathbb{Z}(2)) \cong E_\infty(X; \mathbb{Z}(2)) \cong E^0(K^*(X; \mathbb{Z}(2))).$$

Before giving the explicit description of  $K^*(X; \mathbb{Z}(2))$  we recall some relations between the  $K$ -theory and the rational cohomology.

Let  $\xi \in K^*(Y; \mathbb{Z}(2))$  and let  $\text{ch}(\xi) = \sum_{q \geq 0} \text{ch}_q(\xi)$ ,  $\text{ch}_q(\xi) \in H^q(X; \mathbb{Q})$ , be

its Chern character.

**Definition.** *The rational filtration of  $\xi$ ,  $\text{rat.filt.}(\xi)$ , equals  $p$  if  $\text{ch}_q(\xi) = 0$  for  $q = 0, \dots, p-1$  and  $\text{ch}_p(\xi) \neq 0$ . If  $\xi$  is a torsion element, we set  $\text{rat.filt.}(\xi) = \infty$ . For  $\xi \in K^*(Y; \mathbb{Z}(2))$  we denote by  $\text{ch}_\xi$  the first non vanishing component of  $\text{ch}(\xi)$ , i.e.  $\text{ch}_\xi = \text{ch}_p(\xi)$  with  $p = \text{rat.filt.}(\xi)$ .*

The following result is not difficult to prove :

**Lemma 3.** *Let  $Y$  be a finite CW - complex. There exist free elements  $\xi_1, \dots, \xi_n$  providing a basis of  $K^*(Y; \mathbb{Z}(2)) / (\text{torsion})$  such that  $\text{ch}_{\xi_1}, \dots, \text{ch}_{\xi_n}$  is a homogenous basis of  $H^*(X; \mathbb{Q})$ .*

From (2), (3) and lemma 3 we obtain the main proposition of this section.

**Proposition.** *There exist elements  $\xi_{15}^{(i)}, \xi_{23}^{(i)}, \xi_{27}^{(i)}, \xi_{59}^{(i)}$  ( $i = 1, \dots, m$ ) and  $v_j$*

*( $j = 1, \dots, n$ ) such that :*

$$K^*(X; \mathbb{Z}_{(2)}) \cong \bigotimes_{i=1}^m E(\xi_{15}^{(i)}, \xi_{23}^{(i)}, \xi_{27}^{(i)}, \xi_{59}^{(i)}) \otimes \bigotimes_{j=1}^n E(v_j)$$

where  $\text{rat.filt.}(\xi_k^{(i)}) = k$  and  $\text{rat.filt.}(v_j) = 2^{n_j} - 1$ . Moreover  $\xi_{15}^{(i)}, \xi_{23}^{(i)}, \xi_{27}^{(i)}, \xi_{59}^{(i)}$ ,  $2\xi_{59}^{(i)}$  are represented by  $y_{15}^{(i)}, y_{23}^{(i)}, 2y_{27}^{(i)}, (y_{15}^{(i)})^2 \cdot y_{27}^{(i)}, y_{59}^{(i)}$  whereas  $v_j$  is represented by  $z_j$  in  $E^0(K^*(X; \mathbb{Z}_{(2)}))$ .

3. This section contains the description of the action of the Adams operations on some generators of  $\widetilde{K}(\Sigma X; \mathbb{Z}_{(2)})$  and the proof of the main theorem. As we are going to use it many times, we first recall a generalisation of an important result of Atiyah [At].

Let  $Y$  be a finite 2 - torsion free CW - complex (i.e.  $H^*(Y; \mathbb{Z}_{(2)})$  without 2 - torsion). The submodule of elements of filtration  $\geq 2q$ ,  $\mathcal{F}_{2q} \widetilde{K}(Y; \mathbb{Z}_{(2)}) \subseteq \widetilde{K}^0(Y; \mathbb{Z}_{(2)}) = \widetilde{K}(Y; \mathbb{Z}_{(2)})$  is defined by  $\mathcal{F}_{2q} \widetilde{K}(Y; \mathbb{Z}_{(2)}) = \text{Ker}\{\widetilde{K}(Y; \mathbb{Z}_{(2)}) \longrightarrow \widetilde{K}(Y^{(2n-1)}; \mathbb{Z}_{(2)})\}$  where  $Y^{(2n-1)}$  is the  $(2n-1)$  skeleton of  $Y$ . As  $Y$  is 2 - torsion free the A-Hss with  $\mathbb{Z}_{(2)}$  - coefficients collapses and we may identify  $H^*(Y; \mathbb{Z}/2)$  with  $E^0(K^*(Y; \mathbb{Z}_{(2)})) \otimes \mathbb{Z}/2$ ; if  $\alpha \in \mathcal{F}_{2q} \widetilde{K}(Y; \mathbb{Z}_{(2)})$  we denote the corresponding element of  $H^{2q}(Y; \mathbb{Z}/2)$  by  $\bar{\alpha}$ . We are now ready to state the  $\mathbb{Z}_{(2)}$  - localized version of Atiyah's result.

**Theorem.**(see [H]) *Let  $Y$  be a finite, 2 - torsion free CW-complex and  $\alpha \in \mathcal{F}_{2q} \widetilde{K}(Y; \mathbb{Z}_{(2)})$ . Then there exist elements  $\alpha_i \in \mathcal{F}_{2q+2i} \widetilde{K}(Y; \mathbb{Z}_{(2)})$  ( $i = 0, \dots, q$ ) such that*

$$\Psi^2(\alpha) = \sum_{i=0}^q 2^{q-i} \alpha_i.$$

Moreover  $\bar{\alpha}_i = \text{Sq}^{2i} \bar{\alpha}$  for  $i = 0, \dots, q$ .

In order to state a first lemma about the Adams operations we define  $\eta_{\kappa} \in \widetilde{K}(\Sigma X; \mathbb{Z}_{(2)})$  by  $\eta_{\kappa}^{(i)} = \sigma(\xi_{\kappa-1}^{(i)})$  ( $i = 1, \dots, m$ ,  $\kappa = 16, 24, 28, 60$ ) and  $\zeta_j \in \widetilde{K}(\Sigma X; \mathbb{Z}_{(2)})$  by  $\zeta_j = \sigma(v_j)$  ( $j = 1, \dots, n$ ), where  $\sigma : \widetilde{K}^{-1}(X; \mathbb{Z}_{(2)}) \cong \widetilde{K}^0(\Sigma X; \mathbb{Z}_{(2)})$  is the suspension isomorphism.

**Lemma 4.** *There exist a choice of generators  $\xi_{15}^{(i)}, \xi_{23}^{(i)}, \xi_{27}^{(i)}$  ( $i = 1, \dots, m$ ) such that in  $\widetilde{K}(\Sigma X; \mathbb{Z}_{(2)}) / \mathcal{F}_{32} \widetilde{K}(\Sigma X; \mathbb{Z}_{(2)})$  the following relation holds :*

$$\Psi^2(\eta_{16}^{(i)}) \equiv 2\eta_{28}^{(i)} \pmod{4}.$$

**Proof.** Up to homotopy there is a subcomplex  $Y = \times_{i=1}^m \{S^{16} \cup e^{24} \cup e^{28}\}_i$  of  $\Sigma X$  (and inclusion  $i : Y \longrightarrow \Sigma X$ ) which carries the cohomology with  $\mathbb{Z}_{(2)}$ -coefficients of  $\Sigma X$  up to dimension 28. The K - theory of  $Y$  is free on generators  $\beta_{16}^{(i)} = i^*(\eta_{16}^{(i)})$ ,  $\beta_{24}^{(i)} = i^*(\eta_{24}^{(i)})$  and  $\beta_{28}^{(i)}$ , where  $2\beta_{28}^{(i)} = i^*(\eta_{28}^{(i)})$  ( $i = 1, \dots, m$ ).

The last equality comes from the fact that the A-Hss of  $Y$  collapses, whereas for  $\Sigma X$  we have  $d_3(y_{27}^{(i)}) = (y_{15}^{(i)})^2$  ( $i = 1, \dots, m$ ). Atiyah's result now implies :

$$\Psi^2(\beta_{16}^{(i)}) = 2^8 \beta_{16}^{(i)} + 2^4 a^{(i)} \beta_{24}^{(i)} + 2^2 b^{(i)} \beta_{28}^{(i)} \quad (i = 1, \dots, m),$$

where  $a^{(i)}$  and  $b^{(i)} \equiv 1 \pmod{2}$ . So we obtain :

$$(4) \quad \Psi^2(\eta_{16}^{(i)}) = 2^8 \eta_{16}^{(i)} + 2^4 \eta_{24}^{(i)} + 2 \eta_{28}^{(i)} \pmod{\mathcal{F}_{32} \widetilde{K}(\Sigma X; \mathbb{Z}_{(2)})}.$$

It is clear that the lemma follows immediatly from (4).

The second lemma about the Adams operations concerns the K - theory of  $\Omega X$ . Let us denote by  $\varepsilon : \Sigma^2 \Omega X \longrightarrow \Sigma X$  the suspension of the evaluation map

and define  $\mu_k^{(i)} \in \widetilde{K}(\Omega X; \mathbb{Z}_{(2)})$  by  $\sigma^2(\mu_k^{(i)}) = \varepsilon^*(\eta_{k+2}^{(i)})$  ( $i = 1, \dots, m$  and  $k = 14, 22, 26$ ) and  $\lambda_j \in \widetilde{K}(\Omega X; \mathbb{Z}_{(2)})$  by  $\sigma^2(\lambda_j) = \varepsilon^*(v_j)$  ( $j = 1, \dots, n$ ).

**Lemma 5.** *In  $\widetilde{K}(\Omega X; \mathbb{Z}_{(2)}) / \mathcal{F}_{60} \widetilde{K}(\Omega X; \mathbb{Z}_{(2)})$  the following relations hold :*

- (i)  $\Psi^2(\mu_{26}^{(i)}) \equiv 0 \pmod{2}$  ( $i = 1, \dots, m$ )
- (ii)  $\Psi^2(\lambda_j) \equiv 0 \pmod{2}$  (for each  $j$  such that  $n_j = 5$ ).

**Proof.** Recall first that the suspension of decomposable elements is in the kernel of  $\varepsilon^*$ , therefore we obtain in  $\widetilde{K}(\Sigma^2 \Omega X; \mathbb{Z}_{(2)}) / \mathcal{F}_{62} \widetilde{K}(\Sigma^2 \Omega X; \mathbb{Z}_{(2)})$  (for  $i = 1, \dots, m$  and  $j$  such that  $n_j = 5$ ) :

$$\Psi^p(\varepsilon^*(\eta_{28}^{(i)})) = p^{14} \varepsilon^*(\eta_{28}^{(i)}) + \sum_{n_j=5} a_p^{(i,j)} \varepsilon^*(v_j) + \sum_{k=1}^m b_p^{(i,k)} \varepsilon^*(\eta_{60}^{(k)})$$

$$\Psi^p(\varepsilon^*(v_j)) = p^{16} \varepsilon^*(v_j) + \sum_{k=1}^m c_p^{(j,k)} \varepsilon^*(\eta_{60}^{(k)})$$

$$\Psi^p(\varepsilon^*(\eta_{60}^{(i)})) = p^{30} \varepsilon^*(\eta_{60}^{(i)})$$

Equating the coefficient of  $\varepsilon^*(\eta_{60}^{(k)})$  in  $\Psi^3 \Psi^2(\varepsilon^*(v_j)) = \Psi^2 \Psi^3(\varepsilon^*(v_j))$  gives :

$$(5) \quad c_2^{(j,k)} \equiv 0 \pmod{2^{12}} \quad (k = 1, \dots, m \text{ and } j \text{ such that } n_j = 5).$$

The same method applied to the coefficient of  $\varepsilon^*(v_j)$  in  $\Psi^3 \Psi^2(\varepsilon^*(\eta_{28}^{(i)})) =$

$\Psi^2 \Psi^3(\varepsilon^*(\eta_{28}^{(i)}))$  implies :

$$(6) \quad a_2^{(j,j)} \equiv 0 \pmod{2^{11}} \quad (i = 1, \dots, m \text{ and } j \text{ such that } n_j = 5).$$

Equalities (5), (6) and the coefficient of  $\varepsilon^*(\eta_{60}^{(k)})$  in  $\Psi^3\Psi^2(\varepsilon^*(\eta_{28}^{(i)})) = \Psi^2\Psi^3(\varepsilon^*(\eta_{28}^{(i)}))$  give :

$$(7) \quad b_2^{(i,k)} \equiv 0 \pmod{2^5} \quad (i \text{ and } k = 1, \dots, m).$$

In view of (5), (6) and (7) we have (for  $i = 1, \dots, m$  and  $j$  such that  $n_j = 5$ ) in  $\widetilde{K}(\Sigma^2\Omega X; \mathbb{Z}_{(2)}) / \mathcal{F}_{62}\widetilde{K}(\Sigma^2\Omega X; \mathbb{Z}_{(2)})$  :

$$(8) \quad \Psi^2(\varepsilon^*(\eta_{28}^{(i)})) \equiv \Psi^2(\varepsilon^*(v_j)) \equiv 0 \pmod{4}.$$

Recall that the Adams operations do NOT commute with the Bott periodicity [Ad], the actual relation is :

$$\Psi^2(\sigma^2(\alpha)) = 2\sigma^2\Psi^2(\alpha).$$

As  $K^*(\Omega X; \mathbb{Z}_{(2)})$  is torsion free (by lemma 2  $H^*(\Omega X; \mathbb{Z}_{(2)})$  is torsion free and thus the A-Hss is trivial), the relation (8) forces  $\Psi^2(\mu_{26}^{(i)}) \equiv \Psi^2(\lambda_j) \equiv 0 \pmod{2}$  in  $\widetilde{K}(\Omega X; \mathbb{Z}_{(2)}) / \mathcal{F}_{60}\widetilde{K}(\Omega X; \mathbb{Z}_{(2)})$  and the lemma is proved.

The last part of this section is devoted to the proof of the main theorem. In  $\widetilde{K}(\Omega X; \mathbb{Z}_{(2)}) / \mathcal{F}_{30}\widetilde{K}(\Omega X; \mathbb{Z}_{(2)})$  we have by lemma 4 :

$$(9) \quad (\mu_{14}^{(i)})^2 \equiv \Psi^2(\mu_{14}^{(i)}) \equiv \mu_{26}^{(i)} \pmod{2} \quad (i = 1, \dots, m).$$

Equality (4) can be written more precisely as :

$$(10) \quad \begin{aligned} \Psi^2(\varepsilon^*(\eta_{16}^{(i)})) &= 2^8 \varepsilon^*(\eta_{16}^{(i)}) + 2^4 \varepsilon^*(\eta_{24}^{(i)}) + 2 \varepsilon^*(\eta_{28}^{(i)}) + \sum_{n_j=5} d_j^{(i)} \varepsilon^*(v_j) \\ &+ \sum_{k=1}^m c_k^{(i)} \varepsilon^*(\eta_{60}^{(k)}) \pmod{\mathcal{F}_{62}\widetilde{K}(\Sigma^2\Omega X; \mathbb{Z}_{(2)})}. \end{aligned}$$

Therefore congruence (9), equality (10) and lemma 5 imply that :

$$(\mu_{14}^{(i)})^4 \equiv \Psi^2((\mu_{14}^{(i)})^2) \equiv \Psi^2(\mu_{26}^{(i)}) \equiv 0 \pmod{2} \quad (i = 1, \dots, m)$$

in  $\widetilde{K}(\Omega X; \mathbb{Z}_{(2)}) / \mathcal{F}_{60} \widetilde{K}(\Omega X; \mathbb{Z}_{(2)})$ . So, there exists an element  $\omega^{(i)} \in \widetilde{K}(\Omega X; \mathbb{Z}_{(2)})$  ( $i = 1, \dots, m$ ) such that  $(\mu_{14}^{(i)})^4 = 2\omega^{(i)} \pmod{\mathcal{F}_{60} \widetilde{K}(\Omega X; \mathbb{Z}_{(2)})}$ .

With (4) or (10) we get :

$$\Psi^2(\omega^{(i)}) \equiv 2^{28} \omega^{(i)} \pmod{\mathcal{F}_{60} \widetilde{K}(\Omega X; \mathbb{Z}_{(2)})}.$$

As the A-Hss for  $\Omega X$  collapses,  $H^*(\Omega X; \mathbb{Z}_{(2)}) \cong E^0(K^*(\Omega X; \mathbb{Z}_{(2)}))$  and thus  $\mu_{14}^{(i)}$  is represented by  $s_{14}^{(i)}$  ( $i = 1, \dots, m$ ), it follows from lemma 2 that  $\omega^{(i)}$  is represented by  $s_{56}^{(i)} \in H^{56}(\Omega X; \mathbb{Z}_{(2)})$  ( $i = 1, \dots, m$ ). Using one more time Atiyah's result (for the element  $\omega^{(i)}$ ) we obtain that  $Sq^2 u_{56}^{(i)} = 0$  for each  $i = 1, \dots, m$  which contradicts lemma 1 and thus proves the main theorem.

**Final remark.** As mentioned in the introduction, the hypothesis on the  $\mathcal{Q}(2)$  - action on  $K$  is not essential. In fact every  $\mathcal{Q}(2)$  - action on  $K$  can always be described in the following way :

The relations  $Sq^{15} x_{15}^{(i)} = (x_{15}^{(i)})^2$  ( $i = 1, \dots, m$ ),  $Sq^{15} = Sq^1 Sq^2 Sq^4 Sq^8$  and the linear independence of  $\{(x_{15}^{(i)})^2 : i = 1, \dots, m\}$  allow us to change (if necessary) the generators  $x_{23}^{(i)}, x_{27}^{(i)}, x_{29}^{(i)}$  such that the following Steenrod connections hold :

$$Sq^8 x_{15}^{(i)} = x_{23}^{(i)}, Sq^4 x_{23}^{(i)} = x_{27}^{(i)}, Sq^2 x_{27}^{(i)} = x_{29}^{(i)}, Sq^1 x_{29}^{(i)} = (x_{15}^{(i)})^2 \quad (i = 1, \dots, m).$$

Next let  $h : X \longrightarrow \prod_{i=1}^m K(\mathbb{Z}/2; 15)$  be the map such that in mod 2 cohomology

$h^*(t_{15}^{(i)}) = x_{15}^{(i)}$  ( $i = 1, \dots, m$ ) ( $t_{15}^{(i)}$  being the canonical generator of the  $i^{\text{th}}$  factor of  $\bigotimes_{i=1}^m H^*(K(\mathbb{Z}/2; 15); \mathbb{Z}/2)$ ). It is not difficult to prove that  $h^*$  factors through  $\bigotimes_{i=1}^m A_i$ :

$$\begin{array}{ccc}
 \bigotimes_{i=1}^m H^*(K(\mathbb{Z}/2; 15); \mathbb{Z}/2) & \longrightarrow & H^*(X; \mathbb{Z}/2) \\
 & \searrow & \nearrow \\
 & \bigotimes_{i=1}^m A_i &
 \end{array}$$

Therefore with the above choice of generators,  $A_i$  ( $i = 1, \dots, m$ ) becomes an  $\mathcal{U}(2)$ -subalgebra of  $K$ . Moreover it is easy to show that  $Sq^1 w_j = Sq^2 w_j = 0$  for each  $j$  such that  $n_j = 4, 5$ . As the argument to prove the theorem does not imply filtrations greater than 58 in  $K^*(\Omega X; \mathbb{Z}_{(2)})$ , it is possible to prove it without any hypothesis on the  $\mathcal{U}(2)$ -action on  $K$ . This can be achieved in the following way: Let us perform the changes mentioned above. Lemma 1 and 2 can then be obtained similarly. Next, we consider a subcomplex  $Y$  of  $X$  which carries the  $\mathbb{Z}_{(2)}$ -homology of  $X$  up to dimension 61 (homology approximation). The inclusion  $Y \longrightarrow X$  induces a mod 2 cohomology isomorphism up to dimension 62 (note that  $H^{62}(X; \mathbb{Z}/2) = 0$ ). One checks then that the results of the final part of section 1, concerning the  $\mathbb{Z}_{(2)}$ -cohomology of  $X$ , persist up to dimension 62. The argument in  $K$ -theory is valid for  $Y$  in place of  $X$ .

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### Liste des publications

1. Réalisation topologique de certaines algèbres associées aux algèbres de Dickson
2. Algebras over the Steenrod algebra and finite H-spaces

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