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Generalized weights and other coding theoretic invariants

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Abstract

Invariants play a crucial role in coding theory. They are in fact a valuable tool for classifying codes. Specifically, invariants help distinguish between non-equivalent codes. This distinction is relevant because equivalent codes exhibit the same decoding properties. However, determining whether two codes are equivalent is a hard problem.

In this thesis, we focus on a family of invariants called generalized weights. They were introduced in 1977 for linear block codes, but only became widely studied in 1991, when Wei proved that they measure the code robustness against wiretapping. The goal of this work is to define and study similar invariants for other classes of codes, such as codes over finite rings, sum-rank metric codes, and convolutional codes.

The first part of the thesis is dedicated to the notion of support for linear codes over finite commmutative rings. We investigate how to associate a combinatorial object, called latroid, with a code, and we explore which properties and invariants of the code can be recovered from it. We also show that the associated ideal is determined by the associated latroid.

The central chapters of the thesis concern sum-rank metric codes. First, we prove an anticode bound for this class of codes and we classify all the codes attaining this bound, i.e., the optimal anticodes. Then, we define the generalized weights for sum-rank metric codes in terms of optimal anticodes. We study their basic properties and we compute them in the case of maximum sum-rank distance codes.

In the final part of the thesis, we deal with convolutional codes. We give a definition of generalized weights that takes into account the module structure of this family of codes. After studying their basic properties, we prove that they can be computed by an algorithm that terminates in finite time. Then, we give upper and lower bounds for the generalized weights of maximum distance separable and maximum distance profile codes. We also discuss the notion of generalized column distances.

Keywords Coding theory, sum-rank metric codes, convolutional codes, generalized weights, support, MacWilliams' Extension Theorem, extremal codes.

Résumé

Les invariants jouent un rôle crucial dans la théorie des codes. Ils constituent en effet un outil précieux pour leurs classification. Plus précisément, les invariants permettent de distinguer les codes non équivalents. Cette distinction est importante car les codes équivalents présentent les mêmes propriétés de décodage. Cependant, déterminer si deux codes sont équivalents est un problème difficile.

Dans cette thèse, nous nous concentrons sur une famille d'invariants appelés poids généralisés. Ils ont été introduits en 1977 pour les codes linéaires, mais n'ont été largement étudiés qu'en 1991, lorsque Wei a prouvé qu'ils mesuraient la robustesse du code contre la méthode du “wiretapping". L'objectif de ce travail est de définir des invariants similaires pour d'autres classes de codes, tels que les codes sur des anneaux finis, les codes en métrique somme-rang et les codes convolutifs.

La première partie de la thèse est consacrée à la notion de support pour les codes linéaires sur les anneaux commmutatifs finis. Nous y étudions comment associer un objet combinatoire, appelé latroïde, à un code, et nous explorons quelles propriétés et quels invariants du code peuvent être récupérés à partir de cet objet. Nous montrons également que l'idéal associé est déterminé par le latroïde associé.

Les chapitres centraux de la thèse concernent les codes en métrique rang. Tout d'abord, nous prouvons une limite d'anticode pour cette classe de codes et nous classons tous les codes atteignant cette limite, c'est-à-dire les anticodes optimaux. Ensuite, nous définissons les poids généralisés pour les codes à métrique du rang en termes d'anticodes optimaux. Nous étudions leurs propriétés de base et nous les calculons dans le cas des codes MRD.

Dans la dernière partie de la thèse, nous traitons des codes convolutifs. Nous donnons une définition des poids généralisés dans ce contexte qui prend en compte la structure modulaire de cette famille de codes. Après avoir étudié leurs propriétés de base, nous prouvons qu'ils peuvent être calculés par un algorithme qui se termine en temps fini. Ensuite, nous donnons des bornes supérieures et inférieures pour les poids généralisés des codes MDS et MDP. Nous discutons également de la notion de distances de colonne généralisées.

Mots clés Théorie des codes, codes en métrique somme-rang, codes convolutifs, poids généralisés, support, Théorème d'Extension de MacWilliams, codes extrêmes.

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Contents

1. Introduction

Error-correcting codes are a fascinating topic that bridges mathematics, computer science, and engineering. They allow us to detect and correct errors in any form of data, and they are used in various applications, such as wireless and satellite communication, computer networks, and data storage. Frequently, the error correction is done via minimum distance decoding. In this context, a code C is defined as a subspace of a metric space, and the received message m is decoded to an element in C which minimizes the distance between m and C. From a purely theoretical point of view, any metric space could be used to build a code, but in real-world applications we must take into account how fast the code can be encoded and decoded, as well as how much memory is required to transmit the message. Moreover, the choice of the metric space in which to build our code is also influenced by the transmission channel we use and by the type of errors we expect. The most studied codes are the so called linear block codes.

Definition 1.1. An (n, k) linear block code C is an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^n$ of dimension k, where $\mathbb{F}_q^n$ is equipped with the Hamming distance d_H . The latter is defined as the number of entries in which two elements differ, i.e.,

$$d_H(v, w) = |\{1 \leq i \leq n : v_i \neq w_i\}|,$$

for all $v, w \in \mathbb{F}_q^n$. An element of a code C is called codeword.

An important parameter of a linear block code C is the minimum distance $d_{\min}(C)$, or simply d, that is

$$d_{\min}(C) = \min\{d_H(c_1, c_2) : c_1, c_2 \in C\}.$$

By the triangle inequality, we have that given a vector $v \in \mathbb{F}_q^n$ for which there exists $c \in C$ with $d_H(v, c) \leq (d_{\min}(C) - 1)/2$, then that c is unique. The quantity $(d_{\min(C)} - 1)/2$ is called the error-correction capability of the code. This motivates the interest for isometries between codes, since these are the maps that preserve the pairwise distances of codewords, therefore the metric structure of the code, and in particular its error-correction capability. However, the decoding properties of a code highly depend on how the code is embedded in the ambient space. Since the decoding procedure is central in the theory, we focus on the isometries that can be extended to the whole ambient space.

Definition 1.2. Two linear block codes C_1 and C_2 in $\mathbb{F}_q^n$ are said to be equivalent if there exists an isometry $\varphi : \mathbb{F}_q^n \to \mathbb{F}_q^n$ such that $\varphi(C_1) = C_2$.

In [Mac62] MacWilliams showed the following important theorem, that still bears her name today. An elementary proof of this result can be found in [BGG78].

MacWilliams' Extension Theorem. Every isometry φ of linear codes over a finite field $\mathbb{F}_q$ extends to an isometry of the ambient space $\mathbb{F}_q^n$.

This theorem tells us that the distinction between isometric and equivalent codes is irrelevant in the context of linear block codes. However, this is not always the case. For instance, it is known that not every isometry between rank metric codes can be extended to the whole ambient space. We will deeply investigate this case in Chapter 4.

Distinguish non-equivalent codes is a hard task. For instance, for linear block codes, it is known to be at least as hard as the graph isomorphism problem [PR97]. This motivates the interest in constructing new invariants that allow us to distinguish non-equivalent codes. The study of code invariants is of particular relevance nowadays due to its connection with post-quantum cryptography. Often finding an invariant that distinguishes the code used in a cryptographic system from a “random" code results in an attack, see for example [CGGU+14].

This thesis concerns, mainly but not only, a family of invariants called generalized Hamming weights, or more briefly generalized weights. We recall that the Hamming support $\mathrm{supp}(c)$ of an element $c \in C$ is the set of its nonzero coordinates. Given a subcode $\mathcal{D} \subseteq C$, the support $\mathrm{supp}(\mathcal{D})$ of $\mathcal{D}$ is given by the union of the support of the elements contained in $\mathcal{D}$.

Definition 1.3. Given an (n, k) linear block code C, the r-th-generalized Hamming weight of C is defined as

$$d_r(C) = \min\{|\mathrm{supp}(\mathcal{D})| : \mathcal{D} \text{ is an } r\text{-dimensional subcode of } C\},$$

for $1 \leq r \leq k$.

Generalized Hamming weights of linear block codes were first introduced in 1977 [HKM77, Theorem 6.1] in relation to the weight enumerator polynomial. In 1991, they were rediscovered independently by Wei, who proved that they characterize the code robustness against wiretapping [Wei91]. Successively Forney showed a connection with the complexity of the minimal trellis diagram [For94]. Since then, the coding theory community has shown a growing interest in these invariants. The following proposition collects the distinctive properties of generalized weights.

Proposition 1.4. Let $\mathcal{D} \subseteq C \subseteq \mathbb{F}_q^n$ be linear block codes. Then,

1. $d_1(C) = d_{\min}(C)$,
2. $1 \leq d_1(C) < d_2(C) < \cdots < d_k(C) \leq n$,
3. $d_r(C) \leq d_r(\mathcal{D})$, for $1 \leq r \leq k$,
4. $d_1(C), \ldots, d_k(C)$ are invariants under isometries.

Item 1. clarifies the origin of the name “generalized weights”, indeed it comes from the fact that they generalize the concept of minimum distance.

Nowadays, research on generalized Hamming weights is still ongoing in several directions. First of all, a great effort has been made to bound (or to compute explicitly when possible) the generalized Hamming weights of specific families of linear block codes, especially in the case of algebraic geometry codes. The literature includes but is not limited to [BD18, FTW92, GMM+14, GSJV20, HP98, HKY92, JL97, Lee15, Mun94, MR99, SC95, Wei91, WY93, YLFL15].

Second of all, understanding the relation between generalized Hamming weights and different algebraic and combinatorial objects associated to the code is a question of great interest. In [JV13] Johnsen and Verdure proved that the generalized Hamming weight of C are determined by the $\mathbb{N}$-graded Betti numbers of the Stanley–Reisner ring of the simplicial complex whose faces are the independent sets of the matroid associated to the code [JP13]. This relation was further studied, among others, in [GMMCMMP22, GS20, GR22, JPV23, JRV16, JV14, JV21].

A different way of expressing the generalized Hamming weights in terms of an associated algebraic structure is the following. Let C be an (n, k) code with generator matrix G. If G has no proportional columns, then its columns determine the coordinates of n projective points, $X_C = \{P_1, \dots, P_n\}$ in $\mathbb{P}^{k-1}$. Let $I_C \subseteq \mathbb{K}[x_1, \dots, x_n]$ be the defining ideal of X_C. It was observed in [CST+20] that the behavior of the generalized Hamming weights depends on the properties of the ideal. More on this direction can be found in [GLS05, GSMBVV19, Han03, TVT13].

Finally, starting from the original definition, generalized weights were defined also for rank-metric codes [KMU15, MPM17, MP16, OS12, Rav16a], sum-rank metric codes [MP19], convolutional codes [CFN17, CNF19, CFN20, RY97] and codes in the Lee metric [BW23]. The research which constituted the content of this thesis has developed along this stream. We will discuss the generalized weights and related invariants for codes over rings, sum-rank metric codes, and convolutional codes.

1.1. Outline of the thesis

This thesis covers most of the work I have done during my Ph.D. at the Université de Neuchâtel under the guidance of Professor Elisa Gorla. It contains five papers of mine [CMGL+22, GS23a, GGMPS23, GS23b, GS24], a survey paper [GMPS23], and two on-going projects, one with Elisa Gorla, and one with Elisa Gorla and Arthur Bik. This manuscript is organized as follows.

Chapter 2 provides most of the background needed to read the rest of the thesis. Section 2.1 is an introduction to the mathematical theory of sum-rank metric codes mainly based on [GMPS23], while Section 2.2 concerns convolutional codes and it is based on [GS23b]. The two sections are completely independent, so the reader can choose to read only one of them, depending on their interest.

Chapter 3 is devoted to the general theory of supports for linear codes over rings. In Section 3.1 we recall and discuss the definition of support. In Section 3.2 we classify the isometries with respect to a support, and we prove that the generalized weights are invariant under these maps. In Section 3.3, we recall the definition of weight enumerator with respect to a support and other relevant polynomials. In Section 3.4 we show how to associate to an R-linear code a latroid. In Section 3.5 we prove that the Tutte polynomial of a code endowed with the chain support determines the weight enumerator of the code. Finally, in Section 3.6 we exploit the relation between the latroid and the monomial ideal associated to the code. Some of the results of this chapter are part of an ongoing project with Elisa Gorla.

Chapter 4 focuses on the MacWilliams' Extension Theorem for rank-metric codes. In Section 4.1 we classify all the isometries in the sum-rank metric. In Section 4.2 we present an extensive list of obstructions to the Extension Property, providing multiple examples. Section 4.3 establishes the result that we need in Section 4.4 to prove the main result of the chapter, which states that the Extension Property holds for certain isometries of codes generated by elementary matrices. Section 4.1 is based on [CMGL+22], while the rest of the chapter comes from [GS24].

Chapter 5 focuses on the generalized weights in the case of sum-rank metric codes. In Section 5.1 we generalize a result by Meshulam from vector spaces to cosets. This result allows us to prove in Section 5.2 the Anticode Bound for sum-rank metric codes. In the same section we also classify all the optimal anticodes, i.e., the codes attaining this bound. In Section 5.3 we then define the generalized weights using optimal anticodes, in analogy with what happens in the rank-metric case. Finally in Section 5.4, we briefly discuss the weight distribution. This chapter collects results from [CMGL+22] and [GMPS23].

Chapter 6 continues our study on sum-rank metric codes, in particular we focus on MSRD codes. In Section 6.1 we show a generalized Singleton Bound and we explore some properties that are closely related to being MSRD. In Section 6.2 we investigate the generalized weights of MSRD codes and we introduce the concept of r-MSRD codes. Section 6.3 contains some construction of families of MSRD codes. This chapter is based on [CMGL+22, GMPS23, GGMPS23].

Chapter 7 aims to answer the following question: which integer sequences are generalized weights of a linear code? In Section 7.1 we extend the definition of greedy and relative weights to sum-rank metric codes. In Section 7.2, we reply to the question above in the case of linear block codes, while in Section 7.3 we address the sum-rank metric case. This chapter comes from [GGMPS23].

Chapter 8 focuses on generalized weights for convolutional codes. In Section 8.1 we introduce a new family of generalized weights for convolutional codes, which takes into account the module structure of the codes. We explore their basic properties and their relation with other definitions. We also show that they do not satisfy a Wei duality, while they are preserved by the reverse operation. In Section 8.2 we prove that the generalized weights can be computed by an algorithms that terminates in a finite amount of time. Moreover, we show that they are realized by subspaces generated by codewords of minimal support. In Section 8.3 we study the generalized weights of MDS codes. Finally, in Section 8.4 we prove an Anticode Bound, we characterize the optimal anticodes, and we compute their generalized weights. This Chapter is based on [GS23b].

Chapter 9 concerns another family of invariants for convolutional codes, the so-called generalized column weights. These weights are a generalization of the column distances, in the same way as generalized weights are for the minimum distance. In Section 9.1 we introduce and characterize the j-equivalences. In Section 9.2, we discuss the basic properties of the generalized column weights. Finally, in Section 9.3 we compare these weights with other related definitions, and we give bounds for the generalized weights of MDS codes. This chapter comes from [GS23a].

At the end of the dissertation we have enclosed some **appendices** with complementary results. In Appendix A we prove that every path-reduction chain, defined in Chapter 4 has the same length. In Appendix B we discuss a different notion of support for sum-rank metric codes and how the results of Chapter 5 and Chapter 6 extend to this case.

In Chapter 2 we define the convolutional codes as $\mathbb{F}_q[x]$-modules. This is not the classical definition of convolutional codes. In Appendix C we explain why in the context of generalized weights we can assume this definition. Finally, in Appendix D we associate a finitary matroid to a convolutional codes, and we explain how this matroid can be derived from a latroid over a lattice that is not complete. This appendix is part of an on-going project with Elisa Gorla and Arthur Bik.

2. Preliminaries and notations

In this chapter, we cover the relevant background information which is required throughout the thesis. Section 2.1 concerns sum-rank metric codes and is preparatory to Chapter 4, Chapter 5, Chapter 6, and Chapter 7. Even though Chapter 3 is self-contained, to fully appreciate it, it may be helpful to have some basic knowledge of linear block codes. As we will see in Remark 2.2, linear block codes are a particular case of sum-rank metric codes, therefore Section 2.1 can also be useful as an introduction to this family of codes. If the reader is interested only in this particular case, we refer him to [HP03, VL71, vT93]. Section 2.2 focuses on convolutional codes and contains everything one needs to understand Chapter 8 and Chapter 9.

2.1. Sum-rank metric codes

The sum-rank metric has recently attracted attention in Coding Theory due to its applications in reliable and secure multishot network coding [NUF10, MPK19], rate-diversity optimal space-time codes [LK05, SK20], and PMDS codes for repair in distributed storage [CMST21], among others.

For a positive integer r, we denote by $[r]$ the set $\{1,\dots,r\}$. For a prime power q and positive integers $m \geq n$, let $\mathbb{F}_q^{m\times n}$ be the set of $m\times n$ matrices with entries in the finite field $\mathbb{F}_q$. We denote by $\mathrm{rk}(M)$ the rank of a matrix $M \in \mathbb{F}_q^{m\times n}$. Let $\mathbb{M}$ be the $\mathbb{F}_q$-linear vector space

$$\mathbb{M} = \mathbb{F}_q^{m_1\times n_1} \times \ldots \times \mathbb{F}_q^{m_\ell\times n_\ell},$$

where $\ell, m_1, \dots, m_\ell, n_1, \dots, n_\ell$ are positive integers. In order to simplify some statements and improve the readability of the text, we assume that $m_1 \geq \ldots \geq m_\ell$ and $n_i \leq m_i$ for $i \in [\ell]$. Moreover let $n = n_1 + \ldots + n_\ell$ and if $m_1 = \cdots = m_\ell$ we write m in place of m_i. An element $C \in \mathbb{M}$ is called codeword and it can be written as $C = (C_1, \dots, C_\ell)$, with $C_i \in \mathbb{F}_q^{m_i\times n_i}$ for $i \in [\ell]$. The sum-rank weight of C is given by

$$\mathrm{srk}(C) = \sum_{i=1}^{\ell} \mathrm{rk}(C_i).$$

This weight naturally induces a metric on $\mathbb{M}$, called the sum-rank metric. Indeed, let d be the map

$$\begin{array}{rccc} d : & \mathbb{M}\times\mathbb{M} & \longrightarrow & \mathbb{N} \\ & (C,D) & \longmapsto & \mathrm{srk}(C-D), \end{array}$$

then $(\mathbb{M}, d)$ is a metric space.

Definition 2.1. A linear sum-rank metric code C is an $\mathbb{F}_q$-linear subspace of $(\mathbb{M}, d)$.

Throughout the thesis, we often refer to it simply as a code if the metric is clear from the context. We say that a code $C \subseteq \mathbb{M}$ is non-trivial if $C \neq 0, \mathbb{M}$.

Remark 2.2. The sum-rank metric is a natural generalization of both the Hamming metric and the rank metric. Indeed, on the one side if $\ell = 1$, then $\mathbb{M} = \mathbb{F}_q^{m\times n}$ and the sum-rank metric coincides with the rank metric on $\mathbb{M}$. In this case a sum-rank metric code C is a rank-metric code. On the other side if $m_1 = \ldots = m_\ell = 1$, then $\mathbb{M} = \mathbb{F}_q^n$ and the sum-rank metric coincides with the Hamming metric. In this situation, a sum-rank metric code C is a block code endowed with the Hamming metric.

As a consequence, a result proved in the sum-rank metric case holds true also for the rank and the Hamming metric. The mathematical theory of sum-rank metric codes, however, tends to be more complex than that of rank-metric and linear block codes. E.g., there are results that hold for both rank-metric and linear block codes, but do not hold in full generality for sum-rank metric codes. An example of such a result is Wei duality, which we discuss in Section 5.4.

Remark 2.3. It is worth noting that every sum-rank metric code can be viewed as a code with the rank metric in the appropriate ambient space. For instance, consider the space $\mathbb{M}$ and let $\bar{m} = m_1 + \cdots + m_\ell$. We denote by $X(\mathbb{M})$ the set

$$X(\mathbb{M}) = \{(s,t) \in [\bar{m}] \times [n] : \text{if } m_1 + \cdots + m_i < s \leq m_1 + \cdots + m_{i+1} \\ \text{then } n_1 + \cdots + n_i < t \leq n_1 + \cdots + n_{i+1}\}.$$

Let $\mathbb{F}_q^{\bar{m}\times n}[\mathbb{M}]$ be the $\mathbb{F}_q$-linear space of matrices supported on $X(\mathbb{M})$, i.e.,

$$\mathbb{F}_q^{\bar{m}\times n}[\mathbb{M}] = \{M \in \mathbb{F}_q^{\bar{m}\times n} : M_{s,t} \neq 0 \text{ only if } (s,t) \in X(\mathbb{M})\}.$$

Since $\mathbb{F}_q^{\bar{m}\times n}[\mathbb{M}]$ is a subspace of $\mathbb{F}_q^{\bar{m}\times n}$, we can equip it with the metric induced by the rank metric on $\mathbb{F}_q^{\bar{m}\times n}$. Then, the $\mathbb{F}_q$-linear isometry $\iota : \mathbb{M} \to \mathbb{F}_q^{\bar{m}\times n}[\mathbb{M}]$ given by

$$(C_1, \ldots, C_\ell) \longmapsto \begin{pmatrix} C_1 & & \\ & \ddots & \\ & & C_\ell \end{pmatrix}$$

is distance-preserving, i.e., $\mathrm{srk}(C) = \mathrm{rk}(\iota(C))$ for all $C \in \mathbb{M}$. Therefore, a sum-rank metric code $C \subseteq \mathbb{M}$ can be always identified with its image $\iota(C)$, that is a rank-metric space in $\mathbb{F}_q^{\bar{m}\times n}[\mathbb{M}]$.

Now we define two parameters that play a fundamental role in the theory of sum-rank metric codes.

Definition 2.4. The minimum distance of a code $0 \neq \mathcal{C} \subseteq \mathbb{M}$ is

$$d_{\min}(\mathcal{C}) = \min\{\mathrm{srk}(C) : C \in \mathcal{C} \setminus \{0\}\}$$

and the maximum sum-rank distance or maximum sum-rank weight is

$$\mathrm{maxsrk}(\mathcal{C}) = \max\{\mathrm{srk}(C) : C \in \mathcal{C}\}.$$

We say that a codeword $C \in \mathcal{C}$ realizes the minimum distance (respectively, the maximum sum-rank distance) if $d_{\min}(\mathcal{C}) = \mathrm{srk}(C)$ (respectively, $\mathrm{maxsrk}(\mathcal{C}) = \mathrm{srk}(C)$). Notice that a codeword with this property may not be unique.

Another useful code parameter is the covering radius. It has been defined for (non-linear) sum-rank metric codes with $m_1 = \cdots = m_\ell$ in [OLWZ22, Definition 4]. Here, we extend the definition to any code in $\mathbb{M}$.

Definition 2.5. The covering radius of a sum-rank metric code $\mathcal{C} \subseteq \mathbb{M}$ is

$$\rho(\mathcal{C}) = \min\{r \in \mathbb{Z} : d(M, \mathcal{C}) \leq r \text{ for all } M \in \mathbb{M}\},$$

where $d(M, \mathcal{C}) = \min\{d(M, C) : C \in \mathcal{C}\}$.

The following lemma collects some of the basic properties of these parameters.

Lemma 2.6. Let $\mathcal{C} \subseteq \mathcal{D} \subseteq \mathbb{M}$ be sum-rank metric codes. Then

1. $0 \leq d_{\min}(\mathcal{C}), \text{maxsrk}(\mathcal{C}), \rho(\mathcal{C}) \leq n$.
2. $d_{\min}(\mathcal{C}) \geq d_{\min}(\mathcal{D})$ and $\rho(\mathcal{C}) \geq \rho(\mathcal{D})$, while $\text{maxsrk}(\mathcal{C}) \leq \text{maxsrk}(\mathcal{D})$.
3. $d_{\min}(\mathcal{C}) \leq 2\rho(\mathcal{C}) + 1$.

Example 2.7. Consider the space $\mathbb{M} = \mathbb{F}_2^{3\times 2} \times \mathbb{F}_2^{3\times 1} \times \mathbb{F}_2^{2\times 2} \times \mathbb{F}_2^{2\times 1} \times \mathbb{F}_2 \times \mathbb{F}_2$ and let $\mathcal{C} \subseteq \mathbb{M}$ be the following $\mathbb{F}_2$-linear sum-rank metric code

$$\mathcal{C} = \left\{ \left(\begin{pmatrix} a_1 & 0 \\ 0 & a_1 + a_2 \\ a_1 & 0 \end{pmatrix}, \begin{pmatrix} a_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & a_2 \end{pmatrix}, \begin{pmatrix} a_4 \\ 0 \end{pmatrix}, a_3, a_4 \right) : (a_1, a_2, a_3, a_4) \in \mathbb{F}_2^4 \right\}.$$

It is easy to verify that the minimum distance is equal to 2 and it is realized by

$$C = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1, 0 \right),$$

while the maximum sum-rank distance is equal to 7 and it is realized by

$$D = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 1, 1 \right).$$

In order to compute the covering radius, one can consider the following element

$$M = \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 1, 1 \right) \in \mathbb{M}.$$

One can check by direct computation that $d(M, \mathcal{C}) = 6$ and therefore $\rho(\mathcal{C}) \geq 6$. On the other hand we have that $\rho(\mathcal{C}) \leq 6$, since for every element of $\mathbb{M}$ we can find an element of $\mathcal{C}$ with the same last two entries. Therefore $\rho(\mathcal{C}) = 6$.

We discuss now the notion of support in the sum-rank metric. Given an ambient space $\mathbb{M}$, we consider the associated space

$$\mathbb{S} = \mathbb{F}_q^{n_1} \times \mathbb{F}_q^{n_2} \times \cdots \times \mathbb{F}_q^{n_\ell}.$$

We denote by $\mathcal{P}(\mathbb{S})$ the lattice of all subspaces $\mathcal{S}$ of $\mathbb{S}$ such that $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_\ell$ where $\mathcal{S}_i$ is an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^{n_i}$.

Definition 2.8 ([BGLR21, Definition 2.4]). Let $\mathbb{M}$, $\mathbb{S}$ and $\mathcal{P}(\mathbb{S})$ be as above. For an element $C \in \mathbb{M}$, the support of C is

$$\mathrm{supp}(C) = \mathrm{rowsp}(C_1) \times \cdots \times \mathrm{rowsp}(C_\ell) \in \mathcal{P}(\mathbb{S}),$$

where $\mathrm{rowsp}(C_i)$ is the space generated by the rows of C_i over $\mathbb{F}_q$. The support $\mathrm{supp}(\mathcal{C})$ of a code $\mathcal{C}$ is the smallest $\mathcal{S} \in \mathcal{P}(\mathbb{S})$ such that $\mathrm{supp}(C) \subseteq \mathcal{S}$ for all $C \in \mathcal{C}$.

Remark 2.9. If $\mathbb{S} = \mathbb{F}_q \times \cdots \times \mathbb{F}_q$, then this definition of support coincides with the classical one for linear block codes. If $\mathbb{S} = \mathbb{F}_q^n$, then it coincides with the one for $\mathbb{F}_{q^m}$-linear rank-metric codes. In Section 4.1 we define equivalences (i.e., linear isometries) of sum-rank metric codes. We stress that this notion of support is not equivalence-invariant.

Definition 2.10. For $\mathcal{S} \in \mathcal{P}(\mathbb{S})$ we define the row-support space $\mathcal{V}_\mathcal{S}$ as

$$\mathcal{V}_\mathcal{S} = \{C \in \mathbb{M} : \mathrm{supp}(C) \subseteq \mathcal{S}\}.$$

Clearly, $\mathcal{V}_\mathbb{S} = \mathbb{M}$.

For a code $\mathcal{C} \subseteq \mathbb{M}$ and a subspace $\mathcal{S} \in \mathcal{P}(\mathbb{S})$ the subcode of $\mathcal{C}$ supported on $\mathcal{S}$ is

$$\mathcal{C}(\mathcal{S}) = \mathcal{C} \cap \mathcal{V}_\mathcal{S} = \{C \in \mathcal{C} : \mathrm{supp}(C) \subseteq \mathcal{S}\}.$$

Example 2.11. Let $\mathbb{M}$, $\mathcal{C}$, C and D be as in Example 2.7. We have that $\mathrm{supp}(C) = 0\times\mathbb{F}_2\times0\times0\times\mathbb{F}_2\times0$, $\mathrm{supp}(D) = \langle(1,0)\rangle\times\mathbb{F}_2\times\mathbb{F}_2^2\times\mathbb{F}_2\times\mathbb{F}_2\times\mathbb{F}_2$ and $\mathrm{supp}(\mathcal{C}) = \mathbb{M}$. Let $\mathcal{S} = \langle(1,0)\rangle\times0\times\mathbb{F}_2^2\times0\times0\times0$. Then,

$$\mathcal{C}(\mathcal{S}) = \mathcal{C} \cap \mathcal{V}_\mathcal{S} = \left\langle \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0, 0 \right) \right\rangle_{\mathbb{F}_2}.$$

Let $\mathrm{tr}(M)$ denote the trace of a square matrix M and consider the map $\mathrm{Tr} : \mathbb{M}\times\mathbb{M} \to \mathbb{F}_q$ given by $\mathrm{Tr}(D, C) = \sum_{i=1}^{\ell} \mathrm{tr}(D_i C_i^t)$. Notice that Tr is a non-degenerate symmetric bilinear form. The dual $\mathcal{C}^\perp$ of a sum-rank metric code $\mathcal{C} \subseteq \mathbb{M}$ is the orthogonal subspace of $\mathcal{C}$ in $\mathbb{M}$ with respect to Tr.

Definition 2.12. Let $\mathcal{C} \subseteq \mathbb{M}$ be a code. The dual of $\mathcal{C}$ is

$$\mathcal{C}^\perp = \{D \in \mathbb{M} : \mathrm{Tr}(D, C) = 0 \text{ for all } C \in \mathcal{C}\}.$$

Some of the basic properties that we expect from the duality are trivially true since Tr is a non-degenerate symmetric bilinear form. For instance, $|\mathbb{M}| = |\mathcal{C}||\mathcal{C}^\perp|$ and $(\mathcal{C}^\perp)^\perp = \mathcal{C}$.

2.2. Convolutional codes

Convolutional codes play an important practical role, as they are used extensively to achieve reliable data transmission in digital video, mobile communications, satellite communications, and other applications. Their popularity comes mostly from the fact that maximum-likelihood soft-decision decoding can be performed efficiently on convolutional codes, see e.g. [Vit67, BCJR74]. In spite of the fact that they play a central role in the applications, however, the mathematical theory of convolutional codes is not as well-developed as for other families of codes.

Let $\mathbb{F}_q[x]$ be the ring of univariate polynomials with coefficients in $\mathbb{F}_q$. For $\delta \geq 0$, we denote by $\mathbb{F}_q[x]_{\leq\delta}$ the set of polynomials of degree at most δ. For every positive integers n we write $\mathbb{F}_q[x]^n$ for the direct sum of n copies of $\mathbb{F}_q[x]$. Since $\mathbb{F}_q$ is a field, we have that $\mathbb{F}_q[x]$ is a principal ideal domain. Therefore, every submodule of $\mathbb{F}_q[x]^n$ is a free $\mathbb{F}_q[x]$-module of finite rank $k \leq n$ that admits a finite basis of cardinality k, see [Rotng, Theorem 9.8]. We recall that given a submodule M of $\mathbb{F}_q[x]^n$ a set $B \subseteq M$ is a basis for M if B generates M and B is $\mathbb{F}_q[x]$-linearly independent, that is, for every subset $\{b_1, b_2, \dots, b_n\}$ of B, $r_1b_1+r_2b_2+\cdots+r_nb_n = 0$ implies that $r_1 = r_2 = \cdots = r_n = 0$. From here on, we will only work with submodules of $\mathbb{F}_q[x]^n$, so it will be always possible to fix a basis. For $U \subseteq \mathbb{F}_q[x]^n$ a subset, we denote by $\langle U\rangle_{\mathbb{F}_q[x]} = \langle u \mid u \in U\rangle_{\mathbb{F}_q[x]}$ the $\mathbb{F}_q[x]$-module generated by the elements of U.

Definition 2.13. An (n, k) convolutional code C is an $\mathbb{F}_q[x]$-submodule of $\mathbb{F}_q[x]^n$ of rank k.

We always assume that $C \neq 0$. An element $c(x) \in C$ is an n-tuple $(p_1(x), \dots, p_n(x))$, where

$$p_j(x) = a_{j,0} + a_{j,1}x + \cdots + a_{j,s_j}x^{s_j},$$

for all $j \in \{1, \dots, n\}$. Equivalently, we can express $c(x)$ with a more compact notation as

$$c(x) = \sum_{t=0}^{\deg(c(x))} c[t]x^t,$$

where $\deg(c(x)) = \max_j \deg(p_j(x))$ and for all t we have $c[t] = (a_{1,t}, \dots, a_{n,t}) \in \mathbb{F}_q^n$. The j-th truncation of $c(x)$ is

$$c_{[0,j]}(x) = \sum_{t=0}^{j} c[t]x^t.$$

One can associate to $C = \langle c_1, \dots, c_k\rangle_{\mathbb{F}_q[x]}$ the linear block code

$$C[0] = \langle c_1[0], \dots, c_k[0]\rangle_{\mathbb{F}_q} \subseteq \mathbb{F}_q^n.$$

Here we identify an element of $\mathbb{F}_q$ with the corresponding degree zero polynomial in $\mathbb{F}_q[x]$. Notice that $C[0]$ does not depend on the choice of a system of generators for C and that we have $\dim(C[0]) \leq k$.

In the context of convolutional codes we denote by $\mathrm{wt}_H(c)$ the (Hamming) weight $\mathrm{srk}(c)$ of $c \in \mathbb{F}_q^n$, i.e., $\mathrm{wt}_H(c)$ is the number of non zero components of c. The weight of an element $c(x) \in \mathbb{F}_q[x]^n$ is given by

$$\mathrm{wt}(c(x)) = \sum_{t=0}^{\deg(c(x))} \mathrm{wt}_H(c[t]).$$

Let $c(x) = (p_1(x), \dots, p_n(x))$. Then

$$\mathrm{supp}(c(x)) = \{(j, k) : a_{j,k} \neq 0\}$$

is the support of $c(x)$. Notice that this support is not a support in the sense of Definition 3.4. Clearly, we have $|\mathrm{supp}(c(x))| = \mathrm{wt}(c(x))$. Let $U \subseteq C$ be a subset of C. The support of U is

$$\mathrm{supp}(U) = \bigcup_{c(x)\in U} \mathrm{supp}(c(x)).$$

If $U = \langle c_1(x), \dots, c_h(x)\rangle_{\mathbb{F}_q}$ is an $\mathbb{F}_q$-linear space, then it is easy to show that

$$\mathrm{supp}(U) = \bigcup_{i=1}^{h} \mathrm{supp}(c_i(x)),$$

see e.g. [GR22, discussion after Definition 2.8].

If $C = (c[0] \dots c[\deg(c)]) \in \mathbb{F}_q^{n\times(\deg(c)+1)}$ is the matrix with columns $c[0], \dots, c[\deg(c)]$, then $\mathrm{supp}(c(x))$ simply corresponds to the support of C, i.e., the set of positions of the non zero entries of C. Notice that the columns of C are indexed starting from 0 instead of 1 and the support of a matrix is the set of positions of its nonzero entries.

Let $C_1, C_2 \subseteq \mathbb{F}_q[x]^n$ be convolutional codes. We call C_1 and C_2 isometric if there exists a weight-preserving $\mathbb{F}_q[x]$-isomorphism $\phi : C_1 \to C_2$, that is, ϕ is an isomorphism of $\mathbb{F}_q[x]$-modules and $\mathrm{wt}(c(x)) = \mathrm{wt}(\phi(c(x)))$ for all $c(x) \in C_1$, see [GL09, Definition 3.1]. We will deeply study isometries of convolutional codes in Section 9.1.

A generator matrix of C is a matrix $G(x)$ with entries in $\mathbb{F}_q[x]$ whose rows form a basis of C. We denote by δ the internal degree of a convolutional code C, i.e., the maximum degree of a full size minor of $G(x)$. Since two generator matrices of the same code differ by left multiplication by a unimodular matrix, it follows that δ is independent of the choice of $G(x)$. A convolutional code $C \subseteq \mathbb{F}_q[x]^n$ of rank k and degree δ is an (n, k, δ) convolutional code. Let $G(x) = (p_{i,j}(x))_{i,j}$ be a generator matrix of C. We say that $G(x)$ is row-reduced if $\delta = \sum_{i=1}^{k} \delta_i$, where $\delta_i = \max_{j=1}^{n} \deg(p_{i,j}(x))$. Every code C has a row-reduced generator matrix, see e.g. [Kai80, Example 6.3-2]. It is easy to check that the degrees $\delta_1, \dots, \delta_n$ of the rows of a row-reduced generator matrix of C are invariants of C, i.e., they do not depend on the choice of a row-reduced generator matrix of C. Up to a row permutation, we may assume that $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_k$. The largest degree δ_1 is often called the memory of the code.

A convolutional code C is noncatastrophic if it has a left-prime generator matrix $G(x)$, i.e., a generator matrix $G(x)$ with the property that if $G(x) = H(x)G'(x)$, then $H(x)$ is unimodular. If C has no left-prime generator matrix, we say that C is catastrophic. We refer the interested reader to [LPR21, Section 10.2] for a more complete introduction to convolutional codes in the polynomial setting. The next proposition follows from standard commutative algebra arguments and was first observed in [GLS04]. It provides a useful characterizations of being noncatastrophic.

Proposition 2.14. [GLS04, Proposition 2.2] Let $C \subseteq \mathbb{F}_q[x]^n$ be an (n, k, δ) convolutional code. Then, C is noncatastrophic if and only if for every $r(x) \in \mathbb{F}_q[x] \setminus \{0\}$ and $c(x) \in \mathbb{F}_q[x]^n$ we have that

$$r(x)c(x) \in C \text{ implies } c(x) \in C.$$

The minimum distance or free distance of a convolutional code C is defined as

$$d_{\mathrm{free}}(C) = \min\{\mathrm{wt}(c(x)) : c(x) \in C \setminus \{0\}\}.$$

In [RS99, Theorem 2.2] Smarandache and Rosenthal established an analogue of the Singleton bound for convolutional codes.

Theorem 2.15 (Singleton Bound)**.** Let C be an (n, k, δ) convolutional code. Then

$$d_{\mathrm{free}}(C) \leq (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + 1.$$

Similarly to the case of block codes endowed with the Hamming metric, a code that meets the Singleton bound is called Maximum Distance Separable (MDS). Notice that linear block codes are exactly the convolutional codes with internal degree $\delta = 0$. Coherently, the Singleton bound from Theorem 2.15 coincides with the usual Singleton bound for $\delta = 0$. In particular, a linear block code is MDS if and only if its extension to $\mathbb{F}_q[x]^n$ is MDS of internal degree 0.

The j-th column distance of a convolutional code C is defined as

$$d_j^c(C) = \min\{\mathrm{wt}(c_{[0,j]}(x)) : c(x) \in C \text{ and } c[0] \neq 0\}.$$

In [GLRS06, Proposition 2.2], the authors proved the following bound.

Theorem 2.16. Let C be an (n, k, δ) convolutional code. Then

$$d_j^c(C) \leq (n-k)(j+1) + 1,$$

for all $j \in \mathbb{N}_0$.

A code with $k \neq n$ that achieves this bound for $j = 0, \ldots, L = \left\lfloor \frac{\delta}{k} \right\rfloor + \left\lfloor \frac{\delta}{n-k} \right\rfloor$ is called Maximum Distance Profile (MDP). Finally, C is strongly Maximum Distance Separable (sMDS) if $d_{\mathrm{free}}(C) = d_M^c(C)$, where $M = \left\lfloor \frac{\delta}{k} \right\rfloor + \left\lceil \frac{\delta}{n-k} \right\rceil$. Strongly MDS convolutional codes are a family of MDS convolutional codes and were introduced in [GLRS06].

Let C be an (n, k, δ) convolutional code. The dual code $C^\perp$ is defined as

$$C^\perp = \{d(x) \in \mathbb{F}_q[x]^n \mid d(x)c(x)^T = 0 \text{ for all } c(x) \in C\}.$$

It follows from Proposition 2.14 that, if C is noncatastrophic, then $C^\perp$ is an $(n, n-k, \delta)$ convolutional code and $(C^\perp)^\perp = C$.

3. Codes over finite commutative rings

In order to study linear block codes, it may be useful to associate with them some algebraic or combinatorial objects that partially capture their structure and their basic properties. Understanding which invariants can be determined from this objects is a question of great interest.

For instance, since linear block codes are simply finite vector spaces over finite fields, it is common knowledge that we can associate a matroid to them [Oxl11, Chapter 1]. In [Gre76] Green showed how the weight enumerator of a linear block code is determined by the Tutte polynomial of the associated matroid. More recently, in [JP09b] Jurrius and Pellikaan proved that the Tutte polynomial of the matroid is equivalent to the generalized weight enumerator and to the extended weight enumerator. A similar result was independently proved by Britz in [Bri10]. Starting from the circuits of the matroid, we can also associate a monomial ideal. In [JV13], Johnsen and Verdure proved that the generalized Hamming weight of a linear block code are determined by the $\mathbb{N}$-graded Betti numbers of the associated ideal. This relation was further studied, among others, in [GMMCMMP22, GS20, JRV16, JV14, JV21]. The goal of this chapter is to illustrate how to extend some of these results to linear codes over finite commutative rings. Even though many definitions and many results that we will discuss later in the chapter hold for infinite commutative rings, we prefer to consider only the finite case for the sake of clarity. So, from here on, R will be a finite unitary commutative ring, unless otherwise specified.

3.1. Supports of R-linear codes

Definition 3.1. An R-linear code C is an R-submodule of R^n.

In order to define the generalized weights for R-linear codes we need to define the notion of support on R^n. The general theory of supports over rings was introduced and deeply studied by Gorla and Ravagnani in [GR22]. Here, we limit ourselves to what is necessary for our purposes.

Definition 3.2. An ordered abelian group is a triple $(A, +, \leq)$, where $(A, +)$ is an abelian group and $\leq$ is a partial order on A such that for all $a_1, a_2, a_3 \in A$ $a_1 \leq a_2$ implies $a_1 + a_3 \leq a_2 + a_3$. In particular, we have that

1. $a_1 \leq a_2$ if and only if $0 \leq a_2 - a_1$.
2. If $a_1, a_2 \geq 0$, then $a_1 + a_2 \geq 0$.

Example 3.3. We are mainly interested in the ordered abelian group $(\mathbb{Z}^u, +, \leq)$ with $u \in \mathbb{N}$, where the partial order $\leq$ is defined as follows: $(a_1, \dots, a_u) \leq (a'_1, \dots a'_u)$ if and only if $a_i \leq a'_i$ for $i \in [u]$ in the canonical order of $\mathbb{Z}$.

Definition 3.4. A support on R^n is a function $\text{supp} : R^n \to \mathbb{Z}^u$ such that:

1. $\text{supp}(v) = 0$ if and only if $v = 0$.
2. $\text{supp}(rv) \leq \text{supp}(v)$ for all $r \in R$ and $v \in R^n$.

3. $\mathrm{supp}(v + w) \leq \mathrm{supp}(v) \vee \mathrm{supp}(w)$ for all $v, w \in R^n$

A support is called modular if it satisfies the following property.

4. If $v, w \in R^n$ and $i \in [u]$ satisfy $\mathrm{supp}(v)_i \leq \mathrm{supp}(w)_i$, then there exists $r \in R$ such that $\mathrm{supp}(v + rw)_i < \mathrm{supp}(v)_i$.

Given a support $\mathrm{supp} : R^n \to \mathbb{Z}^u$ we can define a function supp' from the power set of R^n to $\mathbb{Z}^u$ as $\mathrm{supp}'(X) = \bigvee_{x \in X} \mathrm{supp}(x)$. In order to simplify the notation we will denote both these functions by supp and the domain will be clear from the context.

In coding theory the notion of support is closely linked to that of weight. The Hamming support, for instance, arises from the Hamming weight on $\mathbb{F}_q^n$, as already discussed in the Introduction. Notice that the Hamming support is modular. An example of support on $\mathbb{F}_q^n$ that is not modular is given by the function $\tau : \mathbb{F}_q^n \to \mathbb{Z}$ that maps the zero vector to 0 and all the other vectors to 1.

Definition 3.5. The weight of $v \in R^n$ with respect to supp is the 1-norm of the support of v, i.e., $\mathrm{wt}(v) = |\mathrm{supp}(v)|$. The weight of an R-linear code C is defined as $\mathrm{wt}(C) = |\mathrm{supp}(C)|$. The minimum and the maximum weight of a code $0 \neq C \subseteq R^n$ are, respectively,

$$\min \mathrm{wt}(C) = \min \{\mathrm{wt}(v) : v \in C \setminus 0\} \text{ and } \max \mathrm{wt}(C) = \max \{\mathrm{wt}(v) : v \in C\}.$$

It is worth noticing that there exist weights of interest to the community for which the associated "support" does not satisfy the condition of Definition 3.4. For example, the support $\mathrm{supp_L} : \mathbb{Z}_4 \to \mathbb{Z}$, associated to the Lee weight $\mathrm{wt_L} : \mathbb{Z}_4 \to \mathbb{Z}$, is given by $\mathrm{supp_L}(0) = 0$, $\mathrm{supp_L}(1) = \mathrm{supp_L}(3) = 1$, and $\mathrm{supp_L}(2) = 2$. This is not a support according to Definition 3.4, in fact it does not satisfy the second condition since $\mathrm{supp_L}(2) = \mathrm{supp_L}(2 \cdot 1) > \mathrm{supp_L}(1)$. One can easily see that the weight defined in 3.5 is an invariant weight function, but it is not always homogeneous. We refer to [GMFZ13, Section 2] for all these definitions.

Recall that if R is a finite ring, there exist $R_1, \ldots, R_\ell$ finite local rings such that $R \cong R_1 \times \cdots \times R_\ell$, see [AM69, Theorem 8.7]. In particular if R is a principal ideal ring, $R_1, \ldots, R_\ell$ are also principal ideal rings. Notice that from the standpoint of ring theory, R and $R_1 \times \cdots \times R_\ell$ are the same ring, so by abusing notation from here on we will write $R = R_1 \times \cdots \times R_n$. Similarly, we will write $R^n = R_1^n \times \cdots \times R_\ell^n$ and $C = C_1 \times \cdots \times C_\ell$ respectively in place of $R^n \cong R_1^n \times \cdots \times R_\ell^n$ and $C \cong C_1 \times \cdots \times C_\ell$. A finite local commutative principal ideal ring is often called a finite chain ring. If R is a finite chain ring, then any element $r \in R$ is of the form $r = a\alpha^k$, where a is an invertible element and α is a generator of the maximal ideal of R. The following result by Gorla and Ravagnani allows us to reduce the study of supports of rings to that of supports of local rings.

Proposition 3.6 ([GR22, Theorem 2.23])**.** Let $\mathrm{supp} : R^n \to \mathbb{Z}^u$ be a modular support. Up to a permutation of the coordinates of $\mathbb{Z}^u$ we have that $\mathrm{supp} = \mathrm{supp}_1 \times \cdots \times \mathrm{supp}_\ell$, where $\mathrm{supp}_i : R_i^n \to \mathbb{Z}^{u_i}$ for $i \in [\ell]$ and $u_i \in \mathbb{N}$ with $u_1 + \cdots + u_\ell = u$. Moreover, supp_i is a modular support for all $i \in [\ell]$.

Let $\mathrm{supp}_1 : R^{n_1} \to \mathbb{Z}^{u_1}$ and $\mathrm{supp}_2 : R^{n_2} \to \mathbb{Z}^{u_2}$ be two (modular) supports. It is easy to see that the product $\mathrm{supp}_1 \times \mathrm{supp}_2$ is a (modular) support from $R^{n_1+n_2}$ to $\mathbb{Z}^{u_1+u_2}$. A support is called standard if it can be decomposed in product of supports that are supported on a single copy of R.

Definition 3.7. A support supp : $R^n \to \mathbb{Z}^u$ is said to be standard if for each $i \in [n]$ there exist $u_i \in \mathbb{N}$ and a support $\text{supp}_i : R \to \mathbb{Z}^{u_i}$ such that up to permuting the coordinates of $\mathbb{Z}^u$ we have that $\text{supp}((r_1, \ldots, r_n)) = (\text{supp}_1(r_1), \ldots, \text{supp}_n(r_n))$.

We notice that for a standard support supp one has that

$$\text{supp}((r_1, \ldots, r_n)) = \text{supp}((r_1, 0, \ldots, 0)) \vee \cdots \vee \text{supp}((0, \ldots, 0, r_n)).$$

In this chapter we are interested in a specific standard modular support for finite chain rings introduced in [Rav18, Example 26] that is defined as follows.

Definition 3.8. Let R be a finite chain ring with maximal ideal (α). Let k be the smallest positive integer such that $\alpha^k = 0$. Let $\overline{\text{supp}} : R \to \mathbb{Z}$ be the support function given by

$$\overline{\text{supp}}(r) = \min\left\{0 \leq i \leq k : r \in \left(\alpha^{k-i}\right)\right\},$$

for every $r \in R$. The support $\text{supp} = \overline{\text{supp}} \times \cdots \times \overline{\text{supp}} : R^n \to \mathbb{Z}^n$ is called the chain support on R^n.

3.2. Generalized weights

Let $C \subseteq R^n$ be an R-linear code. Since $R = R_1 \times \cdots \times R_\ell$ with R_i finite local ring for all $i \in [\ell]$, we have that $C = C_1 \times \cdots \times C_\ell$, where $C_i \subseteq R_i^n$ is the projection $\pi_i(C)$ of C on the i-th factor of $R^n = R_1^n \times \cdots \times R_\ell^n$ for all $i \in [\ell]$. Following the notation of [GR22], we denote by $\mu(C)$ the least cardinality of a system of generators of a code C. By convention we have $\mu(0) = 0$. Finally, for a code $C = C_1 \times \cdots \times C_\ell \subseteq R^n$, we set $M(C) := \mu(C_1) + \ldots \mu(C_\ell)$. We now have all the necessary elements to state the definition of generalized weights of an R-linear code with respect to a support supp.

Definition 3.9. For $r \in [M(C)]$, the r-th generalized weight of C is given by

$$d_r(C) = \min\left\{\text{wt}(\mathcal{D}) : \mathcal{D} \in S_j(C) \text{ for } j \geq r\right\},$$

where $S_j(C) = \{\mathcal{D} \subseteq C : \mathcal{D}$ is a subcode with a minimal system of generators of cardinality $j\}$.

Notice that the previous definition is well-defined since $S_j(C) \neq \emptyset$ for $j \in [M(C)]$ as proved in [GR22, Theorem 1.8]. When R is a finite field, the cardinality of a minimal system of generators coincides with the dimension of the subcode. Therefore, Definition 3.9 extends Definition 1.3. The next proposition collects some basic properties of generalized weights.

Proposition 3.10 ([GR22, Lemma 2.12])**.** Let $D \subseteq C \subseteq R^n$ be two R-linear codes. Then,

1. $d_1(C) = \min \text{wt}(C)$,
2. $d_r(\mathcal{D}) \geq d_r(C)$ for $r \in [\min\{M(\mathcal{D}), M(C)\}]$,
3. $d_{r+1}(C) \geq d_r(C)$ for $r \in [M(C) - 1]$,
4. $d_r(C) = \min\{|\text{supp}(\mathcal{D})| : D \subseteq C \text{ and } M(\mathcal{D}) \geq r\}$ for $r \in [M(C)]$.

One of the reasons why generalized Hamming weights are very studied is that they are invariant under equivalences. Here, we prove that this is the case also for R-linear codes. We begin defining equivalences between R-linear codes.

Definition 3.11. An isometry between R-linear codes is an R-module isomorphism $\varphi : C_1 \to C_2$ that preserves the weight, i.e., $\mathrm{wt}(v) = \mathrm{wt}(\varphi(v))$ for all $v \in C_1$. Two R-linear codes C_1 and C_2 in R^n are equivalent, if there exists an isometry $\varphi : R^n \to R^n$ that maps C_1 in C_2.

A classical result for the Hamming support states that an isometry from $\mathbb{F}_q^n$ to itself can be expressed as multiplication by a permutation matrix and a diagonal one. In the following, we prove a similar result for codes over principal ideal rings equipped with a standard modular support. We begin by considering the case when R is a finite chain ring.

Lemma 3.12. Let R be a finite chain ring, $\mathrm{supp} = \mathrm{supp}_1 \times \cdots \times \mathrm{supp}_n$ be a standard modular support on R^n, and $\varphi : R^n \to R^n$ be an isometry with respect to supp. Then, there exist a diagonal invertible matrix D and a permutation matrix M such that $\varphi(v) = DMv$ for all $v \in R^n$.

Proof. It is known that an R-module isomorphism from R^n to itself can be expressed as multiplication by a matrix $N = (n_{i,j})$ in $R^{n\times n}$. In order to prove the statement, we want to proceed by induction on n. When $n = 1$, it is trivially true. So assume, we proved the statement for $n-1$. Without loss of generality we assume that $|\mathrm{supp}_1(1)| \leq \cdots \leq |\mathrm{supp}_n(1)|$. Let e_i be an element in the standard basis. Then, an entry of $\varphi(e_i)$ must be invertible, otherwise φ would not be injective. We start by considering e_1. Since we assumed $|\mathrm{supp}_1(1)| \leq |\mathrm{supp}_i(1)|$ for $i > 1$, we conclude that the first column of N has an invertible entry, say the k-th entry, and it is zero everywhere else. Up to multiply by a permutation matrix, we can assume $k = 1$. Consider the vector $v = (-n_{1,2}, n_{1,1}, 0, \ldots, 0)^t \in R^n$. Then, $\mathrm{wt}(\phi(v)) \leq \mathrm{wt}(e_2)$, while $\mathrm{wt}(v) \geq \mathrm{wt}(e_2)$. Since φ is an isometry, we have that $n_{1,2} = 0$. Proceeding in this way, we obtain that the first row of N is different from zero only in the first entry. This implies that φ restricted to $\{0\} \times R^{n-1}$ can be regarded as an isometry of R^{n-1}. We conclude using the inductive hypothesis. □

When R is a principal ideal ring, isometries of R^n can be still expressed as product by a matrix, but describing which matrices represent an isometry is harder. However, we can still classify the isometries of R^n starting from the isometries of finite chain rings that we studied in the previous lemma.

Theorem 3.13. Let $R = R_1 \times \cdots \times R_\ell$ be a principal ideal ring, supp be a standard modular support on R^n, and $\varphi : R^n \to R^n$ be an isometry with respect to supp. Then, for each $i \in [\ell]$ there exists an isometry $\varphi_i : R_i^n \to R_i^n$ such that $\pi_i(\varphi(r))) = \varphi_i(\pi_i(r))$ for every $r \in R^n$, where $\pi_i : R^n \to R_i^n$ is the standard projection.

Proof. This follows from the fact that any R-module isomorphism sends each $0\times\cdots\times R_i^n\times\cdots\times 0$ in itself and from the fact that the restriction of an isometry is an isometry. □

Example 3.14. Consider the free module $\mathbb{Z}_6^2$ over the ring $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$. As support we take the standard modular support supp × supp, where $\mathrm{supp} : \mathbb{Z}_6 \to \mathbb{Z}^2$ is given by $\mathrm{supp}(1) = (1, 1)$, $\mathrm{supp}(2) = (1, 0)$ and $\mathrm{supp}(3) = (0, 1)$, i.e., $\mathrm{supp} = \mathrm{supp}_1 \times \mathrm{supp}_2$, where supp_1 is the Hamming support on $\mathbb{Z}_3$ and supp_2 is the Hamming support on $\mathbb{Z}_2$. Then, one can check by hand that multiplication by the matrix

$$M = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \in \mathbb{Z}_6^{2\times 2}$$

is an isometry $\varphi : \mathbb{Z}_6^2 \to \mathbb{Z}_6^2$. Notice that M is not the product of a permutation matrix and a diagonal one. If we look at the projection on $\mathbb{Z}_2^2$ and $\mathbb{Z}_3^2$ we find that the two isometries φ_1 and φ_2 correspond respectively to the matrices

$$M_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{Z}_2^{2\times 2} \text{ and } M_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in \mathbb{Z}_3^{2\times 2},$$

that are both permutation matrices multiplied by a diagonal one as required by Lemma 3.12.

By Theorem 3.13 we obtain that the generalized weights are a family of invariants.

Corollary 3.15. The generalized weights of an R-linear code are invariant under equivalences.

Proof. Let $\varphi : R^n \to R^n$ be an equivalence between two R-linear codes C_1 and C_2. Consider a minimal system of generators M of a subcode $\mathcal{D}_1$ of C_1. Since φ is an isometry of R^n, $\varphi(M)$ is a minimal system of generators of a subcode $\mathcal{D}_2$ of C_2. In particular, we have that $M(C_1) \geq M(C_2)$. By Theorem 3.13 we obtain that $\mathrm{wt}(\mathcal{D}_1) = \mathrm{wt}(\mathcal{D}_2)$. Therefore, $d_r(C_1) \geq d_r(C_2)$ for $r \in [M(C_2)]$. This is sufficient to conclude since also φ^{-1} is an isometry. □

3.3. Weight enumerator

The weight enumerator is a crucial and extensively researched polynomial in coding theory [VL71, Chapter VI]. It captures most of the interesting properties of a code, and moreover it is also used to study the decoding procedure. For instance, the weight enumerator of a binary code estimates the probability that a received codeword is closer to a different codeword compared to the actual transmitted codeword [JP09a, Section 3]. The goal of this section is to introduce the weight enumerator in our setting, i.e., for linear codes over rings.

Definition 3.16. The homogeneous weight enumerator of an R-linear code $C \subseteq R^n$ is the polynomial

$$W_C(x, y) = \sum_{c\in C} x^{\mathrm{wt}(c)} y^{\mathrm{wt}(R^n)-\mathrm{wt}(c)}.$$

The homogeneous weight enumerator can also be written equivalently as

$$W_C(x, y) = \sum_{w=0}^{\mathrm{wt}(R^n)} A_w x^w y^{\mathrm{wt}(R^n)-w},$$

where $A_w = |\{c \in C : \mathrm{wt}(c) = w\}|$. The list $A_0, \dots, A_{\mathrm{wt}(R^n)}$ is called the weight distribution of C, and it is an invariant of the code. Notice that in the case of the Hamming support we have $\mathrm{wt}(R^n) = n$, and we obtain the classical definition of weight enumerator. In the more general case of standard support, it is important to keep track of what happens in each component. For this reason, we introduce the complete weight enumerator. The name is borrowed from [Ver04, Definition 8.2], but the definition is not the same.

Definition 3.17. For a support $\mathrm{supp} : R^n \to \mathbb{Z}^u$, we define the complete weight enumerator as

$$W_C(\mathbf{x}, \mathbf{y}) = \sum_{c\in C} \mathbf{x}^{\mathrm{supp}(c)} \mathbf{y}^{\overline{\mathrm{supp}(c)}}.$$

where $\mathbf{x}^{\text{supp}(c)} = \prod_{i=1}^{u} x_i^{\text{supp}(c)_i}$, and $\overline{\text{supp}(c)} = \text{supp}(R^n) - \text{supp}(c)$. Starting from the complete weight enumerator, one can recover the homogeneous weight enumerator simply by setting $x_1 = \cdots = x_u = x$ and $y_1 = \cdots = y_u = y$. By straightforward computations we obtain the following result.

Lemma 3.18. Let $R = R_1 \times \cdots \times R_\ell$ be a principal ideal ring, $C = C_1 \times \cdots \times C_\ell$ be an R-linear code, and $\text{supp} = \text{supp}_1 \times \cdots \times \text{supp}_\ell$ be a modular support. Then,

$$W_C(\mathbf{x}_1, \ldots, \mathbf{x}_\ell, \mathbf{y}_1, \ldots, \mathbf{y}_\ell) = \prod_{i=1}^{\ell} W_{C_i}(\mathbf{x}_i, \mathbf{y}_i).$$

In addition to the weight enumerator, we are also interested in the generalized weight enumerator.

Definition 3.19. Let C be an R-linear code. For $0 \leq r \leq M(C)$ the r-th generalized weight enumerator is given by

$$W_C^{(r)}(x, y) = \sum_{r=0}^{M(C)} A_w^{(r)} x^{\text{wt}(R^n)-w} y^w,$$

where $A_w^{(r)} = |\{\mathcal{D} \subseteq C : M(\mathcal{D}) = r \text{ and } \text{wt}(\mathcal{D}) = w\}|$.

While the weight enumerator captures the weight distribution, the generalized weight enumerator captures the generalized weights. Indeed, for $r \in [M(C)]$, we have that

$$d_r(C) = \min\{w : \text{ there exists } j \geq r \text{ such that } A_w^{(j)} \neq 0\}.$$

3.4. Latroids

The goal of this section is to study the relation between supports and latroids. The latter were introduced in [Ver04, Definition 5.6] and generalize matroids, q-matroids, polymatroids and q-polymatroids. Before giving the definition of latroid, we recall that a complete lattice is a partially ordered set in which all subsets have both a supremum $\vee$ and an infimum $\wedge$.

Definition 3.20. Let A be an ordered abelian group, and let $\mathcal{L}$ be a complete lattice. An A-latroid with rank function $\rho : \mathcal{L} \to A$ under length function $\|\cdot\| : \mathcal{L} \to A$ on the lattice $\mathcal{L}$, is a triple $(\rho, \|\cdot\|, \mathcal{L})$ such that:

1. $\rho(0_\mathcal{L}) = \|0_\mathcal{L}\| = 0_A$.
2. $\|\cdot\|$ is strictly increasing, that is, $\|L\| < \|M\|$ for all $L, M \in \mathcal{L}$ with $L < M$.
3. $\|\cdot\|$ is modular, that is, $\|L\| + \|M\| = \|L \vee M\| + \|L \wedge M\|$ for all $L, M \in \mathcal{L}$.
4. ρ is bounded increasing, that is, $0 \leq \rho(M) - \rho(L) \leq \|M\| - \|L\|$ for all $L, M \in \mathcal{L}$ with $L < M$.
5. ρ is submodular, that is, $\rho(L) + \rho(M) \geq \rho(L \vee M) + \rho(L \wedge M)$ for all $L, M \in \mathcal{L}$.

In the next remark we explain how the concept of latroid generalizes the one of matroid.

Remark 3.21. Let E be a finite set. The power set $\mathcal{P}(E)$ of E is a complete lattice with respect to the union and intersection. It is easy to verify that the cardinality function $|\cdot|$ is a strictly increasing modular function on $\mathcal{P}(E)$. Let $\rho : \mathcal{P}(E) \to \mathbb{Z}$ any function for which $(\rho, |\cdot|, \mathcal{P}(E))$ is a $\mathbb{Z}$-latroid. Then, $\{X \subseteq E : |X| - \rho(X) > 0$ and X is minimal with this property$\}$ is the set of circuits of a matroid with ground set E and rank function ρ. This is a direct consequence of [Oxl11, Proposition 11.1.1].

Conversely, let (E, ρ) be a matroid with ground set E and rank function ρ. Then, $(\rho, |\cdot|, \mathcal{P}(E))$ is a $\mathbb{Z}$-latroid. In fact, from the axioms of matroids, we immediately obtain that $\rho(0_{\mathcal{L}}) = 0$, ρ is submodular, and $0 \leq \rho(M) - \rho(L)$ for all $L, M \in \mathcal{P}(E)$ with $L < M$. It remains to prove that $\rho(M) - \rho(L) \leq |M| - |L|$. By the submodularity of ρ, we obtain $\rho(M) \leq \rho(M \setminus L) + \rho(L)$, and by the modularity of the cardinality we conclude

$$\rho(M) - \rho(L) \leq \rho(M \setminus L) \leq |M \setminus L| = |M| - |L|.$$

In the sequel, we show one possible way to associate a latroid to an R-linear code for a given strictly increasing modular function. Let $\mathcal{M}(R^n)$ be the set of all submodules of R^n. We denote by $\mathcal{R}^n$ the set of rectangular sumbmodules of R^n, given by

$$\mathcal{R}^n = \{M = I_1 \times \cdots \times I_n \subseteq R^n : I_i \text{ is an ideal of } R \text{ for all } i \in [n]\}.$$

Notice that $\mathcal{M}(R^n)$ and $\mathcal{R}^n$ are complete lattices with respect to the sum and the intersection, since R is a finite ring. For a code C and a strictly increasing modular function $\|\cdot\| : \mathcal{M}(R^n) \to A$, we define $\rho_C : \mathcal{M}(R^n) \to A$ as

$$\rho_C(M) = \|M\| - \|M \cap C\| \text{ for all } M \in \mathcal{M}(R^n).$$

In the following proposition we will consider the restriction of $\|\cdot\|$ and of ρ_C to the sublattice $\mathcal{R}^n$ of $\mathcal{M}(R^n)$. To simplify the notation, we will not indicate the domain of the functions if it is already clear from the context.

Proposition 3.22. The triple $(\rho_C, \|\cdot\|, \mathcal{R}^n)$ is an A-latroid for every code $C \in \mathcal{R}^n$.

Proof. Let $M_1 < M_2 \in \mathcal{R}^n$. We immediately obtain that

$$\rho_C(M_2) - \rho_C(M_1) \leq \|M_2\| - \|M_1\|,$$

since $M_1 \cap C \leq M_2 \cap C$. Moreover, we have that

$$\begin{aligned}\rho_C(M_1) - \rho_C(M_2) &= \|M_2\| - \|M_1\| - \|M_2 \cap C\| + \|M_1 \cap C\| = \\ &= \|M_2\| - \|M_1\| + (\|M_1\| + \|C\| - \|M_1 + C\|) - (\|M_2\| + \|C\| - \|M_2 + C\|) = \\ &= \|M_2 + C\| - \|M_1 + C\| \geq 0,\end{aligned}$$

and therefore ρ_C is bounded increasing. We have now to prove that ρ_C is a submodular function. So, fix $L_1, L_2 \in \mathcal{R}^n$. Then, by the modularity of the function $\|\cdot\|$ we have that

$$\begin{aligned}\rho_C(L_1) + \rho_C(L_2) &= \|L_1\| + \|L_2\| - \|L_1 \cap C\| - \|L_2 \cap C\| = \\ &= \|L_1 + L_2\| + \|L_1 \cap L_2\| - (\|L_1 \cap L_2 \cap C\| + \|(L_1 \cap C) + (L_2 \cap C)\|) = \\ &\geq \|L_1 + L_2\| + \|L_1 \cap L_2\| - (\|L_1 \cap L_2 \cap C\| + \|(L_1 + L_2) \cap C\|) = \\ &= \rho_C(L_1 \cap L_2) + \rho_C(L_1 + L_2),\end{aligned}$$

where the inequality comes from the fact that $(L_1 \cap C) + (L_2 \cap C) \subseteq (L_1 + L_2) \cap C$. □

It is worth noticing that the the proof of the previous proposition does not depend on the choice of the sublattice $\mathcal{R}^n$. In fact, we can prove the same result for every sublattice of $\mathcal{M}(R^n)$. The reason why we explicitly consider $\mathcal{R}^n$ comes from the following example.

Example 3.23. Let consider the case when R is a finite field $\mathbb{F}_q$, and let C be a linear block code. It is well known that the dimension is a modular function from the set of vector subspaces of $\mathbb{F}_q^n$ to $\mathbb{Z}$. Therefore, by Proposition 3.22 we have that $(\mathcal{R}^n, \dim, \rho_C)$ is a $\mathbb{Z}$-latroid. In this case the rectangular subspaces of $\mathbb{F}_q^n$ are direct products of copies of $\mathbb{F}_q$ and $\{0\}$. In particular, the rectangular subspaces are in bijection with the subsets of $[n]$. Therefore, we can construct an associated matroid proceeding as in Remark 3.21. This matroid is exactly the standard matroid that we associate to a code endowed with the Hamming metric.

We point out that modular functions and modular supports were defined independently in two different contexts. So, even though they are both called modular, they are not the same class of functions. However, there are cases in which modular supports are also modular functions. For instance, we now show that a standard modular support is also a strictly increasing modular function on the lattice $\mathcal{R}^n$, if R is a principal ideal ring. We begin considering the case when R is a finite chain ring.

Lemma 3.24. Let R be a finite chain ring and let $\mathrm{supp} : R^n \to \mathbb{Z}^u$ be a standard support. Then, $\mathrm{supp} : \mathcal{R}^n \to \mathbb{Z}^u$ is a modular function, i.e., $\mathrm{supp}(M_1) + \mathrm{supp}(M_2) = \mathrm{supp}(M_1 + M_2) + \mathrm{supp}(M_1 \cap M_2)$ for all $M_1, M_2 \in \mathcal{R}^n$.

Proof. Directly from the definition of supp we obtain $\mathrm{supp}(M_1) \vee \mathrm{supp}(M_2) \leq \mathrm{supp}(M_1 + M_2)$. For every $m \in M_1 + M_2$ there exists $m_1 \in M_1$ and $m_2 \in M_2$, such that $m = m_1 + m_2$. Therefore, $\mathrm{supp}(m) \leq \mathrm{supp}(m_1) \vee \mathrm{supp}(m_2) \leq \mathrm{supp}(M_1) \vee \mathrm{supp}(M_2)$ and so, $\mathrm{supp}(M_1) \vee \mathrm{supp}(M_2) = \mathrm{supp}(M_1 + M_2)$. This also follows from the fact that $M = \langle m_1, \dots, m_t \rangle$ implies $\mathrm{supp}(M) = \mathrm{supp}(m_1) \vee \cdots \vee \mathrm{supp}(m_t)$.

It is clear that $\mathrm{supp}(M_1 \cap M_2) \leq \mathrm{supp}(M_1) \wedge \mathrm{supp}(M_2)$. Fix $i \in [u]$, since supp is standard, there exist $m_1 = (0, \dots, 0, (m_1)_j, 0, \dots, 0) \in M_1$ and $m_2 = (0, \dots, 0, (m_2)_j, 0, \dots, 0) \in M_2$ such that $\mathrm{supp}(m_1)_i = \mathrm{supp}(M_1)_i$ and $\mathrm{supp}(m_2)_i = \mathrm{supp}(M_2)_i$. Let α be a generator of the maximal ideal of R. Assume without loss of generality that $\mathrm{supp}(m_1)_i \geq \mathrm{supp}(m_2)_i$. Then, there exist r_1, r_2 invertible elements and $k_1 \leq k_2$ such that $(m_1)_j = r_1\alpha^{k_1}$ and $(m_2)_j = r_2\alpha^{k_2}$. So, $m_2 = r_1^{-1} r_2 \alpha^{k_2 - k_1} m_1 \in M_1 \cap M_2$, hence $\mathrm{supp}(M_1 \cap M_2)_i \geq \mathrm{supp}(m_2)_i = \mathrm{supp}(M_2)_i$ and therefore we have that $\mathrm{supp}(M_1 \cap M_2) = \mathrm{supp}(M_1) \wedge \mathrm{supp}(M_2)$. Since $\mathrm{supp}(M_1) + \mathrm{supp}(M_2) = \mathrm{supp}(M_1) \wedge \mathrm{supp}(M_2) + \mathrm{supp}(M_1) \vee \mathrm{supp}(M_2)$, we conclude. □

Proposition 3.25. Let R be a principal ideal ring and let $\mathrm{supp} : R^n \to \mathbb{Z}^u$ be a standard modular support. Then, $\mathrm{supp} : \mathcal{R}^n \to \mathbb{Z}^u$ is a modular function.

Proof. By Proposition 3.6, we know that $R^n = R_1^n \times \cdots \times R_\ell^n$ with $R_1, \dots, R_\ell$ finite chain rings, and $\mathrm{supp} = \mathrm{supp}_1 \times \dots \mathrm{supp}_\ell$, where $\mathrm{supp}_i = R_i^n \to \mathbb{Z}^{u_i}$ is a standard modular support for all $i \in [\ell]$. If M is a rectangular submodule of R^n, then $M = M_1 \times \cdots \times M_\ell$ with M_i rectangular submodules of R_i^n, So, to conclude it is sufficient to apply Lemma 3.24 to each supp_i. □

While in Lemma 3.24 we do not require the support to be modular, Proposition 3.25 does not hold in general without this assumption, as one can see in the next example.

Example 3.26. Consider the ring $\mathbb{Z}_6$ endowed with the Hamming support and let $M_1 = (2)$ and $M_2 = (3)$. Then, we obtain $2 = \mathrm{supp}(M_1) + \mathrm{supp}(M_2) \neq \mathrm{supp}(M_1 + M_2) + \mathrm{supp}(M_1 \cap M_2) = 1$.

Corollary 3.27. Let R be a principal ideal ring and let $\mathrm{supp} : R^n \to \mathbb{Z}^u$ be a standard modular support. Then, the associated weight function is modular.

Proof. Since the direct sum of modular functions is modular, the result follows from Proposition 3.25. □

Notice that there are standard supports that are not strictly increasing functions. For instance, taking Example 3.26, we have that the Hamming support on $\mathbb{Z}_6$ is not strictly increasing. However, in the following proposition we show that all standard modular supports on principal ideal rings are strictly increasing.

Proposition 3.28. Let R be a principal ideal ring and let $\mathrm{supp} : R^n \to \mathbb{Z}^u$ be a standard modular support. Then, $\mathrm{supp} : \mathcal{R}^n \to \mathbb{Z}^u$ is strictly increasing.

Proof. As in the case of Proposition 3.25, it suffices to prove the result for finite chain rings. So, assume that R is a finite chain ring with maximal ideal generated by α, and let $M_1 < M_2$ be two rectangular submodules of R^n. Then, there exist $m_1 = (0, \dots, 0, \alpha^{t_1}, 0, \dots, 0) \in M_1$ and $m_2 = (0, \dots, 0, \alpha^{t_2}, 0, \dots, 0) \in M_2$ with $t_2 < t_1$, such that $\mathrm{supp}(M_1)_i = \mathrm{supp}(m_1)_i$ and $\mathrm{supp}(M_2)_i = \mathrm{supp}(m_2)_i$. Since $m_1 \in M_2$, we have that $\mathrm{supp}(M_1)_i \leq \mathrm{supp}(M_2)_i$. Suppose now that they are equal. On the one side, since supp is a modular support, there is $r \in R$ such that $\mathrm{supp}((0, \dots, 0, \alpha^{t_2} - r\alpha^{t_1}, 0, \dots, 0))_i < \mathrm{supp}((0, \dots, 0, \alpha^{t_2}, 0, \dots, 0))_i$. On the other side, since R is a local ring, we have that $1 - r\alpha^{t_2 - t_1}$ is an invertible element, and so $\mathrm{supp}((0, \dots, 0, \alpha^{t_1} - r\alpha^{t_2}, 0, \dots, 0))_i = \mathrm{supp}((0, \dots, 0, \alpha^{t_1}, 0, \dots, 0))_i$. This is a contradiction, and therefore we conclude that $\mathrm{supp}(M_1)_i < \mathrm{supp}(M_2)_i$, and so $\mathrm{supp}(M_1) < \mathrm{supp}(M_2)$. □

By Proposition 3.25 and Proposition 3.28 we conclude that a standard modular support on a principal ideal ring defines a strictly increasing modular function on $\mathcal{R}^n$. If we try to enlarge the domain and we consider $\mathrm{supp} : \mathcal{M}(R^n) \to \mathbb{Z}^u$, we do not find a modular function. In particular, if we define ρ_C as in Proposition 3.22 the triple $(\rho_C, \mathrm{supp}, \mathcal{R}^n)$ may not be a $\mathbb{Z}^u$-latroid. However, given a standard modular support we can still construct a latroid as follows. We define $\rho_C^{\mathrm{supp}} : \mathcal{R}^n \to \mathbb{Z}^u$ as

$$\rho_C^{\mathrm{supp}}(M) = \mathrm{supp}(M) - \mathrm{supp}\left(\min\{L \in \mathcal{R}^n : M \cap C \subseteq L\}\right) \text{ for all } M \in \mathcal{M}(R^n).$$

Let $\bar{C}$ the smallest rectangular submodule that contains C. It is easy to verify that $\bar{C} = \pi_1(C) \times \dots \times \pi_n(C)$, where π_i is the canonical projection on the i-th entry, and that $M \cap \bar{C} = \min\{L \in \mathcal{R}^n : M \cap C \subseteq L\}$. Following the proof of Proposition 3.22, one can prove that $(\rho_C^{\mathrm{supp}}, \mathrm{supp}, \mathcal{R}^n)$ is a latroid.

3.5. The Chain Support and the Tutte polynomial

The Tutte polynomial was introduced for the first time in [Tut47, Tut54] for graphs and then generalized to matroids in [Cra69]. For a matroid (E, ρ) it is defined as

$$T(\rho, x, y) = \sum_{A \subseteq E} (x-1)^{\rho(E) - \rho(A)} (y-1)^{|A| - \rho(A)}.$$

The Tutte-Whitney rank generating function is obtained from the Tutte polynomial via a change of variable

$$R(\rho, x, y) = T(\rho, x+1, y+1) = \sum_{A \subseteq E} x^{\rho(E)-\rho(A)} y^{|A|-\rho(A)}.$$

In [Ver04], Vertigan defined a generalization of the Tutte-Whitney rank generating function for latroids as follows.

Definition 3.29. The weighted Tutte-Whitney rank generating function of a $\mathbb{Z}^u$-latroid $(\rho, \|\cdot\|, \mathcal{L})$ with $\mathcal{L} \subseteq \mathbb{Z}^u$ is

$$R(\rho, \|\cdot\|, \mathcal{L}, \mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = \sum_{M \in \mathcal{L}} \mathbf{x}^M \mathbf{y}^{M^\perp} \mathbf{u}^{\rho(1_\mathcal{L})-\rho(M)} \mathbf{v}^{\|M\|-\rho(M)}.$$

Since the lattice $\mathcal{L}$ is a sublattice of $\mathbb{Z}^u$, we have that $M^\perp = \overline{M} = 1_\mathcal{L} - M$. Moreover, we observe that the weighted Tutte-Whitney rank generating function fully determines the following function

$$R'(\rho, \|\cdot\|, \mathcal{L}, \mathbf{x}, \mathbf{z}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = \sum_{M \in \mathcal{L}} \mathbf{x}^M \mathbf{z}^{\tilde{M}-M} \mathbf{y}^{M^\perp} \mathbf{u}^{\rho(1_\mathcal{L})-\rho(M)} \mathbf{v}^{\|M\|-\rho(M)},$$

where $\tilde{M} = (M + (1, \dots, 1)) \wedge 1_\mathcal{L}$. Notice that when $\mathcal{L} = \{0,1\}^n$, then $\tilde{M} = (1, \dots, 1)$. In this section we will show how we can recover the weight enumerator of a linear code endowed with the chain support starting from the weighted Tutte-Whitney rank generating function of a suitable associated latroid. Let $C \subseteq R^n$ be an R-linear code with R finite chain ring and let supp be the chain support as defined in Definition 3.8. We define $\mathcal{L}_R$ as the sublattice of $\mathbb{Z}^u$, given by $\mathcal{L}_R = \{\mathrm{supp}(M) : M \in \mathcal{R}^n\}$. We notice that there is a one to one correspondence between $\mathcal{L}_R$ and $\mathcal{R}^n$. Therefore, the function $\rho_C(\mathrm{supp}(M)) = |\mathrm{supp}(M)| - lt(M \cap C)$, where $lt(M)$ is the length of M as R-module, is well defined.

Lemma 3.30. Let R be a finite chain ring and let C be an R-linear code. Then, the triple $(\rho_C, |\cdot|, \mathcal{L}_R)$ defined above is a $\mathbb{Z}$-latroid, called the chain support latroid associated to C.

Proof. See [Ver04, Lemma 5.9]. □

In the following lemma, we recall a useful fact of commutative algebra that we will use in the proof of Theorem 3.32.

Lemma 3.31. Let R be a finite chain ring, and let M be a finitely generated R module. Then, $|M| = |R/(\alpha)|^{lt(M)}$.

Proof. Since M is finite, there exists a sequence of modules with strict inclusions

$$M = M_0 \supset M_1 \supset \cdots \supset M_{lt(M)},$$

that is a composition series, i.e., M_i/M_{i+1} is a nonzero simple R-module for $0 \leq i < lt(M)$, see [Eis13, Theorem 2.13]. A simple R-module is isomorphic to R/J, where J is a maximal ideal of R. Since R is a local ring, we conclude that $M_i/M_{i+1} \cong R/(\alpha)$ for $0 \leq i < lt(M)$. We conclude by induction on the length of the composition series. □

We can now show that the complete weight enumerator of a code C is determined by the weighted Tutte-Whitney rank generating function of the associated chain support latroid. The proof of the following theorem extends the proof of [Ver04, Theorem 9.4]

Theorem 3.32. Let R be a finite chain ring and let $C \subseteq R^n$ be an R-linear code. Then, the Tutte-Whitney rank generating function of $(\rho_C, |\cdot|, \mathcal{L}_R)$ determines the complete weight enumerator of C. In particular, we have that

$$W_C(\mathbf{x}, \mathbf{y}) = R'\left(\rho_C, \|\cdot\|, \mathcal{L}_R, \mathbf{x}, \frac{\mathbf{y}-\mathbf{x}}{\mathbf{y}}, \mathbf{y}, |R/(\alpha)|, 1\right).$$

Proof. For each $A \in \mathbb{Z}^n$, let $C_A = \{c \in C : \operatorname{supp}(c) \leq A\}$, and let $A_i = A - e_i$ for all $i \in [n]$.

$$\begin{aligned} n_C(A) :=& |\{c \in C : \operatorname{supp}(c) = A\}| = |C_A| - |\bigcup_{i=1}^{n} C_{A_i}| = \\ =& |C_A| - \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} |C_{A_{i_1}} \cap \cdots \cap C_{A_{i_k}}| \right) = \\ =& \sum_{A-(1,\dots,1) \leq B \leq A} (-1)^{|A|-|B|} |C_B|. \end{aligned} \tag{3.5.1}$$

By direct computation one can check that

$$\mathbf{y}^B(\mathbf{y}-\mathbf{x})^{(1,\dots,1)-B} = \sum_{B \subseteq A \subseteq (1,\dots,1)} (-1)^{|A|-|B|} \mathbf{x}^A \mathbf{y}^{(1,\dots,1)-A}, \tag{3.5.2}$$

for all $(0,\dots,0) \leq B \leq (1,\dots,1)$. Let $\tilde{B} = (B + (1,\dots,1)) \wedge \operatorname{supp}(R^n)$. We have that

$$\begin{aligned} \sum_{c \in C} \mathbf{x}^{\operatorname{supp}(c)} \mathbf{y}^{\overline{\operatorname{supp}(c)}} &= \sum_{A \in \mathcal{L}} n_C(A) \mathbf{x}^A \mathbf{y}^{\overline{A}} = \sum_{A \in \mathcal{L}} \left(\sum_{A-(1,\dots,1) \leq B \leq A} (-1)^{|A|-|B|} |C_B| \right) \mathbf{x}^A \mathbf{y}^{\overline{A}} \\ &= \sum_{B \in \mathcal{L}} \left(|C_B| \sum_{B \leq A \leq \tilde{B}} (-1)^{|A|-|B|} \mathbf{x}^A \mathbf{y}^{\overline{A}} \right) = \\ &= \sum_{B \in \mathcal{L}} \left(|C_B| \mathbf{x}^{\tilde{B}-(1,\dots,1)} \mathbf{y}^{\operatorname{supp}(R^n)-\tilde{B}} \sum_{B \leq A \leq \tilde{B}} (-1)^{|A|-|B|} \mathbf{x}^{A-\tilde{B}+(1,\dots,1)} z^{\tilde{B}-A} \right) = \\ &= \sum_{B \in \mathcal{L}} |C_B| \mathbf{x}^{\tilde{B}-(1,\dots,1)} \mathbf{y}^{\operatorname{supp}(R^n)-\tilde{B}} \mathbf{x}^{B-\tilde{B}+(1,\dots,1)} (\mathbf{y}-\mathbf{x})^{\tilde{B}-B} = \\ &= \sum_{B \in \mathcal{L}} |C_B| \mathbf{x}^B \mathbf{y}^{\operatorname{supp}(R^n)-\tilde{B}} (\mathbf{y}-\mathbf{x})^{\tilde{B}-B} = \sum_{B \in \mathcal{L}} |C_B| \mathbf{x}^B \mathbf{y}^{\overline{B}} \left(\frac{\mathbf{y}-\mathbf{x}}{\mathbf{y}} \right)^{\tilde{B}-B}, \end{aligned}$$

where in the decond equality we used Equation (3.5.1) and in the second to last we used Equation (3.5.2). Since R is a finite chain ring and $|B| - \rho_C(B)$ is the length of C_B, by Lemma 3.31 we have that $|C_B| = |R/(\alpha)|^{|B|-\rho_C(B)}$. Combining these results, we finally obtain

$$W_C(\mathbf{x}, \mathbf{y}) = \sum_{B \in \mathcal{L}} |R/(\alpha)|^{|B|-\rho_C(B)} \mathbf{x}^B \mathbf{y}^{\overline{B}} \left(\frac{\mathbf{y}-\mathbf{x}}{\mathbf{y}} \right)^{\tilde{B}-B} = R'\left(\rho_C, \|\cdot\|, \mathcal{L}_R, \mathbf{x}, \frac{\mathbf{y}-\mathbf{x}}{\mathbf{y}}, \mathbf{y}, |R/(\alpha)|, 1\right). \quad \square$$

Remark 3.33. Notice that in Theorem 3.32 we proved that the complete weight enumerator can be obtained from $R'(\rho, \|\cdot\|, \mathcal{L}, \mathbf{u}, \mathbf{z}, \mathbf{v}, \mathbf{x}, \mathbf{y})$. However, as we stated above, this function is determined by the weighted Tutte-Whitney rank generating function. Therefore, it is possible to also write the weight enumerator in terms of the Tutte-Whitney rank generating function, but the formula would not be as concise.

Theorem 3.32 can be generalized to codes over a principal ideal ring R. Let $\mathrm{supp} = \mathrm{supp}_1 \times \cdots \times \mathrm{supp}_\ell$ be the modular support on R^n such that supp_i is the chain support on R_i for $i \in [\ell]$. Each submodule M of R^n decomposes as direct product $M_1 \times \cdots \times M_\ell$ where M_i is a submodule of R_i^n for each $i \in [\ell]$. We define $\rho_C(\mathrm{supp}(M)) = (\rho_{C_1}(\mathrm{supp}(M_1)), \ldots, \rho_{C_\ell}(\mathrm{supp}(M_\ell))$.

Lemma 3.34. Let R be a principal ideal ring and let C be an R-linear code. Then, the triple $(\rho_C, |\cdot|, \mathcal{L}_R)$ defined above is a $\mathbb{Z}^\ell$-latroid, called the chain support latroid associated to C.

Proof. Notice that ρ_C is bounded increasing and submodular if and only if ρ_{C_i} is bounded increasing and submodular for all $i \in [\ell]$. We conclude by applying Lemma 3.30. □

Corollary 3.35. Let R be a principal ideal ring and let $C \subseteq R^n$ be an R-linear code. Then, the Tutte-Whitney rank generating function of $(\rho_C, |\cdot|, \mathcal{L}_R)$ determines the complete weight enumerator of C

$$W_C(\mathbf{x}, \mathbf{y}) = R'\left(\rho_C, \|\cdot\|, \mathcal{L}_R, \mathbf{x}, \frac{\mathbf{y}-\mathbf{x}}{\mathbf{y}}, \mathbf{y}, |R/(\alpha_1)|, \ldots, |R/(\alpha_\ell)|, \mathbf{1}\right)$$

Proof. By Lemma 3.18 and Theorem 3.32 we have

$$W_C(\mathbf{x}_1, \ldots, \mathbf{x}_\ell, \mathbf{y}_1, \ldots, \mathbf{y}_\ell) = \prod_{i=1}^{\ell} W_{C_i}(\mathbf{x}_i, \mathbf{y}_i) = \prod_{i=1}^{\ell} R'\left(\rho_{C_i}, \|\cdot\|, \mathcal{L}_{R_i}, \mathbf{x}_i, \frac{\mathbf{y}_i - \mathbf{x}_i}{\mathbf{y}_i}, \mathbf{y}_i, |R/(\alpha_i)|, 1\right).$$

We conclude by noticing that

$$R'(\rho_C, \|\cdot\|, \mathcal{L}_R, \mathbf{x}_1, \ldots, \mathbf{x}_\ell, \mathbf{z}_1, \ldots, \mathbf{z}_\ell, \mathbf{y}_1, \ldots, \mathbf{y}_\ell, u_1, \ldots, u_\ell, v_1 \ldots, v_\ell) =$$
$$= \prod_{i=1}^{\ell} R(\rho_{C_i}, \|\cdot\|, \mathcal{L}_{R_i}, \mathbf{x}_i, \mathbf{z}_i, \mathbf{y}_i, u_i, v_i).$$
□

3.6. Monomial ideals

We recall that a monomial ideal $I \subseteq S = \mathbb{K}[\mathbf{x}]$ is an ideal which has a system of generators consisting of monomials. Given a modular support supp on R^n we can associate to each nonzero R-linear code C an ideal I_C defined as

$$I_C = (\{\mathbf{x}^{\mathrm{supp}(c)} : c \in C \setminus 0\}) \subseteq S.$$

It was shown in [GR22, Proposition 4.3] that $I_C = (\{\mathbf{x}^{\mathrm{supp}(c)} : c \in \mathrm{Min}(C)\})$, where $\mathrm{Min}(C)$ is the set of codewords with minimal support. The ideal I_C can be recovered from the latroid $(\rho_C^{\mathrm{supp}}, \mathrm{supp}, \mathcal{R}^n)$ defined at the end of Section 3.4. Indeed, we have that

$$\{\mathrm{supp}(c) : c \in \mathrm{Min}(C)\} = \{\mathrm{supp}(M) : \mathrm{supp}(M) - \rho_C^{\mathrm{supp}}(M) > 0 \text{ with } M \in \mathcal{R}^n \text{ and } M \text{ minimal}\}.$$

Notice that proceeding in this way we can always associate an ideal to a latroid. This correspondence is not bijective, since from different latroids we may obtain the same ideal. For instance, let R be a principal ideal ring and let $(\rho_C, |\cdot|, \mathcal{L}_R)$ be the chain support latroid. Then, the associated ideal is again I_C.

From the ideal I_C one can recover some information on the code C. For instance, in [GR22, Theorem 4.4] the authors proved that the graded Betti numbers of the monomial ideal associated to a code determine its generalized weights.

Theorem 3.36 ([GR22, Theorem 4.4]). Let σ be a modular support and let $C \subseteq R^n$ be a code. Assume that either $C \subseteq 0 :_{R^n} J$ or R is a PIR. Let $I_C \subseteq S$ be the monomial ideal associated to C and let $r \in [M(C)]$. Then $M(C)$ is the projective dimension of S/I_C and $d_r(C)$ is the minimum shift (i.e., the minimum degree of a nonzero element) in the r-th free module in a minimal free resolution of S/I_C. In particular, the $\mathbb{N}$-graded Betti numbers of S/I_C determine $M(C)$ and the generalized weights of C.

In [JRV16, Theorem 5.1] the authors proved that the weight enumerator of an $\mathbb{F}_q$-linear block code is determined by the $\mathbb{N}_0$-graded Betti numbers associated with the $\mathbb{N}_0$-graded minimal free resolutions of the ideal of I_C and of its elongations, see [JRV16, Section 2] for a definition. Here, we prove that the weight enumerator of an $\mathbb{F}_q$-linear code $C \subseteq \mathbb{F}_q^n$ is determined by its $\mathbb{N}_0^n$-graded Betti numbers. We will denote by $\beta_{i,X}$ the rank of the free module $S(-X)$ in homological position i.

Theorem 3.37. Let $C \subseteq \mathbb{F}_q^n$ be a linear code. Then,

$$A_w = \sum_{|X|=w, X\leq\{0,1\}^n} \sum_{Y \leq X} (-1)^{|X\setminus Y|} q^{\max\{i : \exists \beta_{i,Z}>0 \text{ and } Z\leq Y\}},$$

for $w \in [n]$ and where $\{\beta_{i,Z}\}$ is the set of the $\mathbb{N}_0^n$-graded Betti numbers that appears in a minimal free resolution of S/I_C.

Proof. By the inclusion–exclusion principle we obtain

$$A_w = \sum_{|X|=w, X\leq\{0,1\}^n} \sum_{Y \leq X} (-1)^{|X\setminus Y|} |C_Y|,$$

where $C_Y = \{c \in C : \mathrm{supp}(c) \leq Y\}$. The dimension of C_y is equal to the projective dimension of S/I_{C_Y}. Fix a minimal free resolution of S/I_C. The subresolution obtained by restricting to the direct summands $S(-X)$ with $X \leq Y$ is a minimal free resolution of S/I_{C_Y}. Therefore, we have $\dim(C_Y) = \max\{i : \exists \beta_{i,Z} > 0 \text{ and } Z \leq Y\}$. Since $|C_Y| = q^{\dim(C_Y)}$, we conclude. □

In the more general case of R-linear codes, the ideal I_C does not determine the weight enumerator, not even in the case of the chain support, as one can see in the following example.

Example 3.38. Consider the $\mathbb{Z}_4$-linear codes C_1 and C_2 in $\mathbb{Z}_4^3$ given by

$$C_1 = \langle(2,1,0),(2,0,1)\rangle_{\mathbb{Z}_4} \text{ and } C_2 = \langle(2,1,0),(0,0,1)\rangle_{\mathbb{Z}_4}.$$

With respect to the chain support, we obtain $I_{C_1} = I_{C_2} = (y,z) \subseteq \mathbb{K}[x,y,z]$. However, we have that $W_{C_1}(x,1) = 1 + 2x + x^2 + 4x^3 + 8x^4$ and $W_{C_2}(x,1) = 1 + 2x + 3x^2 + 5x^3 + 5x^4$.

Determining whether it is possible to associate a more complex ideal with an R-linear code that allows the recovery of the weight enumerator remains an open problem.

4. MacWilliams' Extension Theorem

In the introduction we mentioned the MacWilliams' Extension Theorem, a classical result by Florence Jessie MacWilliams. It shows that every linear isometry between linear block-codes endowed with the Hamming distance can be extended to a linear isometry of the ambient space.

For R-linear codes on an arbitrary standard modular support this result does not hold, as we can see in the following example.

Example 4.1. Let $C_1, C_2 \subseteq \mathbb{F}_2^3$ be two one-dimensional codes generated respectively by the vectors $(1,1,0)$ and $(0,0,1)$. Consider the standard modular support $\sigma = \sigma_1 \times \sigma_2 \times \sigma_3$, where $\sigma_i(1) = 2i$ and $\sigma_i(0) = 0$ for $i \in [3]$. Then, the map $\varphi : C_1 \to C_2$, which sends $(1,1,0)$ to $(0,0,1)$, is an isometry since $\mathrm{wt}((1,1,0)) = \mathrm{wt}((0,0,1)) = 6$. However φ cannot be extended to an isometry of the ambient space.

There are many other cases where isometries of R-linear codes do not extend to the entire ambient space, for instance see [BGL15, Example 6.9]. Despite this, there is still interest in understanding under which hypotheses and for which weights it is possible to prove a MacWilliams' Extension Theorem for R-linear codes. The main result in this direction is due to Wood.

Theorem 4.2 ([Woo99, Theorem 6.3])**.** Let R be a finite Frobenius ring. Suppose that $C \subseteq R^n$ is a right R-linear code, and suppose that $\varphi : C \to R^n$ is a right isometry with respect to the Hamming weight. Then, φ extends to a right isometry of R^n.

In [Woo08, Theorem 2.3] Wood proved that every finite ring that has the extension property for the Hamming weight is a Frobenius ring. The MacWilliams' Extension Theorem for codes over Frobenius rings was also studied in [BGL15, GMFZ13, SZ19, Woo08]. As regards the homogeneous weights, in [CHH96] we find a combinatorial proof of the Extension Theorem for codes over $\mathbb{Z}_m$. Recently, the MacWilliams' Extenstion Theorem was proved in [Dys19, LW19] for codes over $\mathbb{Z}_m$ endowed with the Lee distance. In this chapter, we explore the Extension Property in the setting of sum-rank metric codes, with a focus on the rank-metric case.

Extension Property. Let C_1, C_2 be two sum-rank metric codes in $\mathbb{M}$. An isometry $\varphi : C_1 \to C_2$ satisfies the Extension Property if and only if there exists a linear isometry $\mu : \mathbb{M} \to \mathbb{M}$ such that $\mu|_{C_1} = \varphi$.

It is well known that there exist isometries of sum-rank metric codes that do not satisfy the Extension Property (see [BGL15] and [dlCKWW, Section 7]). We are interested in understanding under which conditions it may be possible to extend an isometry to the whole ambient space and when instead the Extension Property fails. Very little is know in this direction. The results in [GHM+14] imply that isometries between two rank support spaces are extendable. The same result for $\mathbb{F}_{q^m}$-isometries between Galois closed linear subspaces of $\mathbb{F}_{q^m}^n$ was proved by Umberto Martínez-Peñas in [MP16, Theorem 5].

This chapter is organized as follows. In Section 4.1 we study linear isometries of sum-rank metric codes. Such isometries allow us to define the notion of equivalent codes, which allows us to say if a given parameter of a code is a sum-rank invariant. In Section 4.2 we present several counterexample to the MacWilliams' Extension Theorem in the rank-metric. In Section 4.4, we prove the main theorem of this chapter that states that the Extension Property holds for certain isometries of codes generated by elementary matrices. Section 4.3 is dedicated to developing some tools that are used in Section 4.4.

4.1. Isometries in the sum-rank metric

In the first part of this section we characterize the linear isometries of $\mathbb{M}$ and use them to define a notion of equivalence between sum-rank metric codes. In the second part, we will show two examples in which the Extension Property fails.

Definition 4.3. An $\mathbb{F}_q$-linear isometry φ in the sum-rank metric is an $\mathbb{F}_q$-linear homomorphism of $\mathbb{M}$ such that $\mathrm{srk}(\varphi(C)) = \mathrm{srk}(C)$ for all $C \in \mathbb{M}$. Two sum-rank metric codes $\mathcal{C}, \mathcal{D} \subseteq \mathbb{M}$ are equivalent if there is an $\mathbb{F}_q$-linear isometry $\varphi : \mathbb{M} \to \mathbb{M}$ such that $\varphi(\mathcal{C}) = \mathcal{D}$.

The next theorem gives a classification of the isometries of $\mathbb{F}_q^{m\times n}$ with respect to the rank metric.

Theorem 4.4. Let $\varphi : \mathbb{F}_q^{m\times n} \to \mathbb{F}_q^{m\times n}$ be an $\mathbb{F}_q$-linear isometry with respect to the rank metric.

1. If $m \neq n$ then there exist matrices $A \in \mathrm{GL}_m(\mathbb{F}_q)$ and $B \in \mathrm{GL}_n(\mathbb{F}_q)$ such that $\varphi(M) = AMB$ for all $M \in \mathbb{F}_q^{m\times n}$.
2. If $m = n$ then there exist matrices $A, B \in \mathrm{GL}_n(\mathbb{F}_q)$ such that either $\varphi(M) = AMB$ for all $M \in \mathbb{F}_q^{n\times n}$, or $\varphi(M) = AM^tB$ for all $M \in \mathbb{F}_q^{n\times n}$.

We refer the interested reader to [Hua51, Wan62] for a proof of this result. This allows us to characterize the $\mathbb{F}_q$-linear isometries in the sum-rank metric as follows.

Theorem 4.5. Let $\varphi : \mathbb{M} \longrightarrow \mathbb{M}$ be an $\mathbb{F}_q$-linear isometry. Then there is a permutation

$$\sigma : [\ell] \longrightarrow [\ell]$$

with the property that $\sigma(i) = j$ implies $m_i = m_j$ and $n_i = n_j$ and there are rank-metric $\mathbb{F}_q$-linear isometries $\psi_i : \mathbb{F}_q^{m_i\times n_i} \longrightarrow \mathbb{F}_q^{m_i\times n_i}$ for $i \in [\ell]$ such that

$$\varphi(C_1, \dots, C_\ell) = (\psi_1(C_{\sigma(1)}), \dots, \psi_\ell(C_{\sigma(\ell)}))$$

for all $(C_1, \dots, C_\ell) \in \mathbb{M}$.

Proof. For $i \in [\ell]$, let $M_i = 0 \times \dots \times 0 \times \mathbb{F}_q^{m_i\times n_i} \times 0 \times \dots \times 0 \subseteq \mathbb{M}$ where the ith component is the only nonzero one. Let $\{(0, \dots, 0, E_{k,l}, 0, \dots, 0)\}_{(k,l)\in[m_i]\times[n_i]}$ be the standard basis of M_i. Then, for all $(k, l) \in [m_i] \times [n_i]$,

$$\mathrm{srk}(\varphi(0, \dots, 0, E_{k,l}, 0, \dots, 0)) = 1,$$

implying that $\varphi(0, \dots, 0, E_{k,l}, 0, \dots, 0)$ has only one nonzero component for each choice of k and l, say $i_{k,l}$. Further, we notice that for a given $k \in [m_i]$

$$\mathrm{srk}\left(\varphi\left(0, \dots, 0, \sum_{l=1}^{n_i} E_{k,l}, 0, \dots, 0\right)\right) = 1, \tag{4.1.1}$$

and similarly for a given $l \in [n_i]$ we have that

$$\operatorname{srk}\left(\varphi\left(0,\dots,0,\sum_{k=1}^{m_i} E_{k,l},0,\dots,0\right)\right) = 1. \tag{4.1.2}$$

By (4.1.1) we have that

$$\operatorname{srk}\left(\sum_{l=1}^{n_i} \varphi(0,\dots,0,E_{k,l},0,\dots,0)\right) = 1,$$

implying that $i_{k,l}$ does not depend on k. The same argument together with equation (4.1.2) shows that $i_{k,l}$ does not depend on l either. It follows that for all i there is a j such that $\varphi(M_i) \subseteq M_j$. Since φ^{-1} is a linear isometry, it follows from the same argument that that $\varphi^{-1}(M_j) \subseteq M_i$. Hence $\varphi(M_i) = M_j$. In particular, the map that sends i to j is a permutation of $[\ell]$, which we denote by σ^{-1}. Since M_i and M_j have the same weight distribution if and only if $n_i = \operatorname{maxrk}(M_i) = \operatorname{maxrk}(M_j) = n_j$ and $m_i = \dim(M_i)/n_i = \dim(M_j)/n_j = m_j$. Therefore

$$\begin{array}{rccc} \varphi\mid_{M_i}: & M_i & \longrightarrow & M_j \\ & (0,\dots,0,C_i,0,\dots,0) & \longmapsto & (0,\dots,0,\psi_j(C_i),0,\dots,0) \end{array}$$

for $j = \sigma^{-1}(i)$ and for some linear rank-metric isometry $\psi_j : \mathbb{F}_q^{m_j\times n_j} \to \mathbb{F}_q^{m_j\times n_j}$. Hence by linearity

$$\begin{array}{rccc} \varphi: & \mathbb{M} & \longrightarrow & \mathbb{M} \\ & (C_1,\dots,C_\ell) & \longmapsto & (\psi_1(C_{\sigma(1)}),\dots,\psi_\ell(C_{\sigma(\ell)})). \end{array}$$

□

In the next section, we will show many obstructions to the Extension Property in the rank-metric case. It is clear that since we do not have a MacWilliams Extension Theorem for rank-metric codes, we also cannot have a MacWilliams Extension Theorem for sum-rank metric codes. Morever, in the sum-rank metric we have more pathologies than just those coming from the rank metric, as the next examples show.

Example 4.6. Let $\ell = 3, m_1 = n_1 = 3, m_2 = m_3 = n_2 = n_3 = 1$. Let

$$\mathcal{C} = \left\{\left(\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b, c\right) : a,b,c \in \mathbb{F}_q\right\}$$

and

$$\mathcal{D} = \left\{\left(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, 0, 0\right) : a,b,c \in \mathbb{F}_q\right\}.$$

Then $\varphi : \mathcal{C} \to \mathcal{D}$ defined as

$$\varphi\left(\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b, c\right) = \left(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, 0, 0\right)$$

is an $\mathbb{F}_q$-linear isometry between $\mathcal{C}$ and $\mathcal{D}$, which does not extend to an $\mathbb{F}_q$-linear isometry of $\mathbb{M}$

by Theorem 4.5.

Example 4.7. Let $\ell = 2, m_1 = n_1 = m_2 = n_2 = 2$. Let

$$C = \left\{ \left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \right) : a, b, c, d \in \mathbb{F}_q \right\}.$$

Then $\varphi : C \to C$ defined as

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \right) = \left(\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & d \end{pmatrix} \right)$$

is an $\mathbb{F}_q$-linear isometry between C and itself, which does not extend to an $\mathbb{F}_q$-linear isometry of $\mathbb{M}$ by Theorem 4.5.

4.2. Obstructions to the Extension Property

In this section we will focus only on rank-metric codes. Using Theorem 4.4, we can state the Extension Property for rank-metric codes as follows.

Extension Property. Let C_1, C_2 be two linear codes in $\mathbb{F}_q^{m\times n}$. An isometry $\varphi : C_1 \to C_2$ satisfies the Extension Property if and only if there exist two matrices $A \in \mathrm{GL}_m(\mathbb{F}_q)$ and $B \in \mathrm{GL}_n(\mathbb{F}_q)$ such that either $\varphi(M) = AMB$ for all $M \in C_1$, or $\varphi(M) = AM^tB$ for all $M \in C_1$, where the latter case can only happen if $m = n$.

The goal of this section is to discuss several obstructions to the Extension Property in the rank-metric case. A first problem arises from the fact that the transposition is an isometry of the ambient space only in the square case. This makes the composition of the transposition with the natural inclusion of $\iota : \mathbb{F}_q^{m\times m} \hookrightarrow \mathbb{F}_q^{m\times n}$, $m \leq n$, into an $\mathbb{F}_q$-linear isometry of $\iota(\mathbb{F}_q^{m\times m}) \subseteq \mathbb{F}_q^{m\times n}$ with itself, which cannot be extended to $\mathbb{F}_q^{m\times n}$. This is a way of looking at the next example, due to Barra and Gluesing-Luerssen.

Example 4.8 ([BGL15], Example 2.9)**.** Let $C = \{\begin{pmatrix} A & 0 \end{pmatrix} : A \in \mathbb{F}_q^{2\times 2}\} \leq \mathbb{F}_q^{2\times 3}$ and let $\varphi : C \to C$ be the isometry given by $\varphi(\begin{pmatrix} A & 0 \end{pmatrix}) = \begin{pmatrix} A^t & 0 \end{pmatrix}$ for all $A \in \mathbb{F}_q^{2\times 2}$. It is easy to see that it is not possible to extend φ to an isometry of the whole ambient space.

A similar phenomenon happens in the following example, also due to Barra and Gleusing-Luerssen.

Example 4.9 ([BGL15], Example 2.9)**.** Let $C \leq \mathbb{F}_q^{4\times 4}$ be the code given by

$$C = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, B \in \mathbb{F}_q^{2\times 2} \right\}$$

and consider the isometry $\varphi : C \to C$ given by

$$\varphi\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right) = \begin{pmatrix} A & 0 \\ 0 & B^t \end{pmatrix}$$

As before, one can check that φ cannot be extended to an isometry of $\mathbb{F}_q^{4\times 4}$.

In general, the natural inclusion $\iota : \mathbb{F}_q^{m\times m} \times \mathbb{F}_q^{n\times n} \hookrightarrow \mathbb{F}_q^{(m+n)\times(m+n)}$ is an isometry with respect to the sum-rank metric in the domain and the rank metric in the codomain. When composed with the product of the identity on $\mathbb{F}_q^{m\times m}$ and the transposition on $\mathbb{F}_q^{n\times n}$, it yields an isometry of $\iota(\mathbb{F}_q^{m\times m} \times \mathbb{F}_q^{n\times n}) \subseteq \mathbb{F}_q^{(m+n)\times(m+n)}$ with itself, which does not extend to $\mathbb{F}_q^{(m+n)\times(m+n)}$.

We stress that in both examples there is a smaller, natural ambient space to which the isometry can be extended. In fact even more, in those specific examples the isometries are already defined on a smaller ambient space (on which therefore they can be trivially extended). In the first example, the isometry is defined on $\mathbb{F}_q^{2\times 2}$ while in the second example it is defined on $\mathbb{F}_q^{2\times 2}\times\mathbb{F}_q^{2\times 2}$, naturally endowed with the sum-rank metric. In order to avoid such problems, one may want to consider codes that cannot be contained in a smaller ambient space, that is, such that $\mathrm{rowsp}(C) = \mathbb{F}_q^n$ and $\mathrm{colsp}(C) = \mathbb{F}_q^m$.

We now discuss a different obstruction to the Extension Property. Let φ be an isometry of $\mathbb{F}_q^{m\times n}$. Then for every $C \leq \mathbb{F}_q^{m\times n}$ we have that

$$\dim(\mathrm{rowsp}(C)) = \dim(\mathrm{rowsp}(\varphi(C))) \text{ and } \dim(\mathrm{colsp}(C)) = \dim(\mathrm{colsp}(\varphi(C))). \tag{4.2.3}$$

Therefore, in order to be extendable, an isometry must satisfy this property. The next example shows that not all linear isometries do.

Example 4.10. Let $C_1, C_2 \in \mathbb{F}_2^{2\times 3}$ be the codes

$$C_1 = \left\langle \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle \quad C_2 = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\rangle$$

and let $\varphi : C_1 \to C_2$ be the $\mathbb{F}_2$-linear map given by

$$\varphi\left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \varphi\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since C_1 and C_2 are codes of constant rank 2, then φ is an isometry. One can easily see that $\dim(\mathrm{rowsp}(C_1)) = 2$ while $\dim(\mathrm{rowsp}(C_2)) = 3$. In particular, φ cannot be extended to an isometry of $\mathbb{F}_2^{2\times 3}$.

The last example motivates us to look at isometries $\varphi : C_1 \to C_2 \leq \mathbb{F}_q^{m\times n}$ with the following property, which implies (4.2.3).

Property 1. There exist $A \in \mathrm{GL}_m(\mathbb{F}_q)$ and $B \in \mathrm{GL}_n(\mathbb{F}_q)$ such that

$$\mathrm{rowsp}(\varphi(C)) = \mathrm{rowsp}(CB) \text{ and } \mathrm{colsp}(\varphi(C)) = \mathrm{colsp}(AC)$$

for all $C \in C_1$.

Notice that none of the isometries considered in Examples 4.8, 4.9 and 4.10 satisfy Property 1. While Property 1 is necessary for the Extension Property to hold, it is not sufficient, as the next example shows.

Example 4.11. In [dlCKWW16, Example 1] the authors exhibit three distinct equivalence classes of MRD codes in $\mathbb{F}_2^{4\times 4}$ with minimum distance 4. Any $\mathbb{F}_2$-linear map between codes in different equivalent classes is an isometry, since each nonzero element has rank 4. Moreover, each of these maps satisfy Property 1 with $A = B = \mathrm{Id}$. A proof that these codes do not satisfy the Extension Property appeared in the first arXiv version of the same paper as [dlCKWW, Example 7.1].

The obstruction to the Extension Property in Example 4.11 can be seen as coming from the interaction between the linear structure of the code and the group structure of the code without the zero matrix. More precisely, if C is a vector space of square matrices and $C \setminus \{0\}$ is a subgroup of the general linear group, then every $\mathbb{F}_q$-linear isomorphism from C to itself is a linear isometry. Moreover, if it fixes the identity and it has the Extension Property, then it is either a group homomorphism or a group antihomomorphism. Therefore, any $\mathbb{F}_q$-linear isomorphism from C to itself which fixes the identity and is not a group homomorphism or a group antihomomorphism cannot have the Extension Property.

Example 4.12. Let $P \in \mathrm{GL}_n(\mathbb{F}_q)$ of order $q^n - 1$, let $Q = P^{q-1}$. Let $C = \mathbb{F}_q[P] = \langle P \rangle \cup \{0\} \subseteq \mathbb{F}_q^{n\times n}$. Every nonzero element of C has rank n, hence any $\mathbb{F}_q$-linear isomorphism of C with itself is an isometry. Both P and Q are linearly independent from the identity matrix Id, so there is a linear isometry $\varphi : C \to C$ with $\varphi(\mathrm{Id}) = \mathrm{Id}$ and $\varphi(P) = Q$. If φ has the Extension Property, then either $\varphi(M) = AMA^{-1}$ or $\varphi(M) = AM^tA^{-1}$ for some $A \in \mathrm{GL}_n(\mathbb{F}_q)$. Therefore $Q = \varphi(P) \in \{APA^{-1}, AP^tA^{-1}\}$, however Q has order $q^{n-1} + q^{n-2} + \ldots + 1$, while APA^{-1} and AP^tA^{-1} have order $q^n - 1$.

Even when $C \setminus \{0\}$ is not a group, an isometry on a set of square matrices which fixes the identity and for which the Extension Property holds needs to be multiplicative. This constitutes an obstruction to the Extension Property, since not every linear isometry is multiplicative.

Example 4.13. Let $C \in \mathbb{F}_2^{3\times 3}$ be the code given by

$$C = \left\{ 0, \mathrm{Id}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

and let $\varphi : C \to C$ be the isometry of C with itself that fixes the identity matrix and swaps the other two matrices.

Suppose that φ can be extended to an isometry of the whole ambient space. Then, there are $A, B \in \mathrm{GL}_3(\mathbb{F}_2)$ such that either $\varphi(C) = ACB$ for all $C \in C$ or $\varphi(C) = AC^tB$ for all $C \in C$. Since $\varphi(\mathrm{Id}) = \mathrm{Id}$, we have that $AB = \mathrm{Id}$ and so $B = A^{-1}$. Therefore, we obtain that

$$\begin{aligned} \varphi\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) &= \varphi\left(\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)\varphi\left(\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The map φ sends an element of rank 2 to an element of rank 1, contradicting the assumption that φ is an isometry. We conclude that φ does not have the Extension Property. Notice however that φ satisfies Property 1 with

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Property 1 suggests to look at codes generated by rank-one elements. In fact, if C is a rank-one element with row space $\langle u \rangle$ and column space $\langle v \rangle$, then $\varphi(C)$ is a rank-one element with row space $\langle uB \rangle$ and column space $\langle Av \rangle$. Therefore, φ determines Av and uB up to a scalar multiple. This simple observation allows us to prove the next result.

Proposition 4.14. Let $C_1, C_2 \leq \mathbb{F}_2^{m\times n}$ and let $\varphi : C_1 \to C_2$ be an isometry which satisfies Property 1. If C_1 is generated by elements of rank 1, then φ is extendable.

Proof. Since φ has Property 1, then $\varphi(C)$ and ACB have the same row and column space for all $C \in \mathcal{C}$. Over $\mathbb{F}_2$ this give that $A^{-1}\varphi(C)B^{-1} = C$ for every $C \in C_1$ of rank 1. If C_1 is generated by elements of rank 1, we conclude by linearity that $A^{-1}\varphi(C)B^{-1} = C$ for all $C \in C_1$. □

Even for $\mathcal{C}$ generated by elements of rank 1, the Extension Property may fail if we do not require Property 1.

Example 4.15. Let $\mathcal{C} \subseteq \mathbb{F}_2^{2\times 3}$ be the linear code generated by

$$C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Let $\varphi : \mathcal{C} \to \mathcal{C}$ be the linear map given by $\varphi(C_i) = C_i$ for $i = 1, 2, 3$ and $\varphi(C_4) = C_4 + C_3$. One can verify that φ is an isometry that cannot be extended to the whole ambient space, since it does not satisfy Property 1.

One may wonder whether the failure of the Extension Property is due to the fact that the code is small compared to the ambient space. The next example shows that this is not the case.

Example 4.16. Starting from the code $\mathcal{C}$ from the previous example, for each $n > 3$ we construct a code $\mathcal{C}_n \in \mathbb{F}_2^{2\times n}$ given by

$$\mathcal{C}_n = \left\{ \begin{pmatrix} A & C \end{pmatrix} : A \in \mathbb{F}_2^{2\times(n-3)}, C \in \mathcal{C} \right\}.$$

Let $\varphi_n : \mathcal{C}_n \to \mathcal{C}_n$ be the linear map given by $\varphi_n \begin{pmatrix} A & 0 \end{pmatrix} = A$ for $A \in \mathbb{F}_2^{2\times(n-3)}$ and $\varphi_n \begin{pmatrix} 0 & C \end{pmatrix} = \varphi(C)$. Again, φ_n is an isometry that cannot be extended to the whole ambient space. Moreover, notice that

$$\lim_{n\to\infty} \frac{\dim(\mathcal{C}_n)}{\dim\left(\mathbb{F}_2^{2\times n}\right)} = \lim_{n\to\infty} \frac{2n-2}{2n} = 1.$$

This show that there exist non-extendable isometries defined on codes, whose dimension comes arbitrarily close to that of the ambient space.

We state the analogous result of Proposition 4.14 for arbitrary q as an open question.

Question 4.17. Let $C_1, C_2 \leq \mathbb{F}_q^{m\times n}$ and let $\varphi : C_1 \to C_2$ be an isometry which satisfies Property 1. If C_1 is generated by elements of rank 1, then the same is true for C_2. If this is the case, does φ have the Extension Property?

Theorem 4.18 provides a positive answer to Question 4.17, for codes which are generated by elementary matrices.

Let $1 \le i \le m$ and $1 \le j \le n$. We denote by $E_{i,j}$ the matrix in $\mathbb{F}_q^{m\times n}$ that has 1 in position (i, j) and 0 everywhere else. We call these matrices elementary. We now state the main result of this chapter, which we will prove in Section 4.4.

Theorem 4.18. Let $C = \langle E_{i_1,j_1}, \dots, E_{i_k,j_k}\rangle \le \mathbb{F}_q^{m\times n}$ be a code generated by k elementary matrices. Let $\varphi : C \to C$ be an isometry such that for all $1 \le h \le k$ one has $\varphi(E_{i_h,j_h}) = \alpha_h E_{i_h,j_h}$ for some $\alpha_h \in \mathbb{F}_q^*$. Then φ satisfies the Extension Property.

The next example shows that the statement of the Theorem 4.18 fails, if the code is generated by non-elementary, rank-one matrices.

Example 4.19. Let $q \neq 2$ and let $C \in \mathbb{F}_q^{2\times 4}$ the code generated by the following elements of rank 1:

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Let $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$ and let $\varphi : C \to C$ be the linear map given by $\varphi(C_i) = C_i$ for $1 \le i \le 4$ and $\varphi(C_5) = \alpha C_5$. One can check that φ is an isometry and that it does not have the Extension Property. In fact, φ does not satisfy Property 1, since $\mathrm{rowsp}(C_5 - C_2) \le \mathrm{rowsp}(\sum_{i=1}^5 C_i)$ but $\mathrm{rowsp}(\varphi(C_5 - C_2)) \cap \mathrm{rowsp}(\varphi(\sum_{i=1}^5 C_i)) = \{0\}$. Notice that, since φ does not satisfies Property 1, it does not yield a negative answer to Question 4.17. In addition, this example shows that it does not suffice in general to check Property 1 on a system of generators of the code.

4.3. Matrix paths

In this section we establish some preliminary result which we will use in the proof of Theorem 4.18. We start by introducing the notion of path in a matrix. From here on, let $m, n \ge 2$.

Definition 4.20. Let $M \in \mathbb{F}_q^{m\times n}$ be a matrix. A path π of length $k \in \mathbb{N}$ in M is a finite ordered sequence of positions of nonzero entries $((i_1, j_1), (i_2, j_2), \dots (i_k, j_k))$ such that two consecutive elements share either the first or the second component and $(i_h, j_h) \neq (i_s, j_s)$ for $h \neq s$.

A path π of length at least 4 is closed if the first and the last entries share a component. The support $\mathrm{supp}(\pi)$ of a path π is the set of elements of π. A path π is simple if no three entries of π share a component.

These definitions are borrowed from graph theory. Indeed, one can naturally associate to every $M \in \mathbb{F}_q^{m\times n}$ a finite graph $G_M = (V_M, E_M)$, such that V_M is the set of positions of the nonzero entries of M and two vertices in V_M are connected by an edge in E_M if and only if the corresponding entries lay on a common line (that is, a common row or column). The notions of path and closed path from Definition 4.20 correspond to the usual definitions in graph theory. A path is simple if the subgraph of G_M induced by the set of vertices in the path does not contain any clique.

We are mainly interested in closed simple paths. We begin by establishing some of their basic properties. First notice that, up to a cyclic permutation and to reversing the order, every simple path is determined by its support. Moreover, in the next lemma we see that the entries corresponding to the elements of a closed simple path are contained in a square submatrix with exactly two nonzero elements in each row and column.

Lemma 4.21. Let $M \in \mathbb{F}_q^{m\times n}$ be a matrix. The entries of M corresponding to the elements of a closed simple path are contained in a square submatrix with exactly two nonzero elements in each row and column.

Proof. Let $\pi = ((i_1, j_1), (i_2, j_2), \dots (i_k, j_k))$ be a closed simple path in M. By definition, each line of M contains at most two nonzero entries whose position belongs to the support of π. Suppose by contradiction that there exists a line in M which contains exactly one nonzero entry in position (i_h, j_h). If $1 < h < k$, then the three elements $(i_{h-1}, j_{h-1}), (i_h, j_h), (i_{h+1}, j_{h+1})$ have either the first or the second coordinate in common. If $h = 1$, the same is true for (i_1, j_1), $(i_2, j_2), (i_k, j_k)$. If $h = k$, the same holds for $(i_1, j_1), (i_{k-1}, j_{k-1}), (i_k, j_k)$. In each case, π is not simple. We conclude that the entries of M corresponding to the elements of a closed simple path are contained in a $m' \times n'$ submatrix with exactly two nonzero elements in each row and column. In particular, it must be that $2m' = 2n'$ and so $m' = n'$. □

The next proposition ensures that in every matrix with enough nonzero entries there is a closed simple path.

Proposition 4.22. Let $m, n \geq 2$ and let $M \in \mathbb{F}_q^{m\times n}$ be a matrix with at least $m+n$ nonzero entries. Then there is a closed simple path in M.

Proof. We proceed by induction on $m + n$. If $m + n = 4$ then $m = n = 2$ and all the entries of the matrix are nonzero and so trivially we have a closed simple path.

Suppose now that $m + n > 4$. If there exists a row in which there is at most one nonzero entry, then $m > 2$. By Lemma 4.21 no closed simple path can contain the position of that entry. Therefore, one may erase that row from M and obtain a matrix of size $(m-1)\times n$ which contains the same paths as M. Similarly, one may erase any column of M which contain a single nonzero entry without affecting the paths contained in M.

By eliminating all rows and columns of M which contain at most one nonzero entry, we reduce to a matrix which contains at least two nonzero entries in each row and column. Notice that the operation of canceling any rows and columns of M which contain at most one nonzero entry preserves the property that the matrix has at least as many nonzero entries as the sum of its number of rows and its number of columns. We can now build a closed simple path as follows. Starting from an arbitrary nonzero entry, move along the correspondent row and select another nonzero entry. Then move along the column of last nonzero entry picked and select another nonzero entry. Proceed in this way, alternating between rows and columns. At every step, we find a nonzero entry different from the last one that was picked, since we supposed that in each line we have at least two nonzero entries. Since the number of lines is finite, after k steps we must choose an entry on a line where there is already one entry which was picked at a step h with $1 \leq h < k - 1$. As soon as that happens, we choose that entry. The positions of the entries that we have picked are the support of a closed simple path in M. □

Remark 4.23. The result in Proposition 4.22 is optimal, in the sense that there are matrices in $\mathbb{F}_q^{m\times n}$ with $m + n - 1$ nonzero entries that do not contain any closed simple path. An example is given by

$$M = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{F}_q^{m\times n}.$$

Definition 4.24. Let $m, n \geq 2$ and $M \in \mathbb{F}_q^{m\times n}$. We say that a matrix $M' \in \mathbb{F}_q^{m\times n}$ is a path-reduction, or just a reduction, of M if it is obtained from M by changing to zero a nonzero entry that belong to a closed simple path.

A matrix $M \in \mathbb{F}_q^{m\times n}$ is path-irreducible, or just irreducible, if does not contain any closed simple path.

Let $M_1, \dots, M_\ell \in \mathbb{F}_q^{m\times n}$. We say that $(M_1, \dots, M_\ell)$ is a path-reduction chain if for every $1 \leq i < \ell$, M_{i+1} is a reduction of M_i and M_ℓ is irreducible.

Since in a closed simple path there are at least four entries and a matrix may have more than one closed simple path, a matrix may have several path-reductions. We illustrate the situation in the next simple example.

Example 4.25. Consider the matrix $M \in \mathbb{F}_2^{3\times 5}$ given by

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The path $((1,1),(1,4),(2,4),(2,2),(3,2),(3,1))$ is closed and simple. Replacing any of the ones in M yields a reduction of M. In particular

$$M' = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad M'' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

are reductions of M. Notice that both M' and M'' are irreducible.

The next corollary is an immediate consequence of Proposition 4.22.

Corollary 4.26. Let $M \in \mathbb{F}_q^{m\times n}$. If M is irreducible, than M has at most $m + n - 1$ nonzero entries.

Given a matrix $M \in \mathbb{F}_q^{m\times n}$, it is always possible to find a path-reduction chain starting from M. In fact, one can simply apply consecutive reductions. Since M has a finite number of nonzero entries, one obtains an irreducible matrix in a finite number of steps.

Proposition 4.27. Let $M \in \mathbb{F}_q^{m\times n}$. Then there exists a path-reduction chain $(M_1, \dots, M_\ell)$ such that $M_1 = M$.

Notice that one can find more than one path-reduction chain starting with the same matrix M. In Appendix C we prove that each path-reduction chain with $M_1 = M$ has the same length.

Example 4.28. Let $M \in \mathbb{F}_2^{3\times 3}$ be the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Both

$$\left(\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right),$$

and

$$\left(\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right)$$

are path-reduction chains starting with M.

4.4. Proof of Theorem 4.18

In order to clarify the structure of the proof of Theorem 4.18, we enclose part of it in two technical lemmas. The first one shows under which conditions two maps coincide on a closed simple path.

Lemma 4.29. Let $((i_1, j_1), \ldots, (i_k, j_k))$ be a closed simple path in $\sum_{h=1}^{k} E_{i_h,j_h} \in \mathbb{F}_q^{m\times n}$. Let $\varphi, \psi : \langle E_{i_1,j_1}, \ldots, E_{i_k,j_k} \rangle \to \langle E_{i_1,j_1}, \ldots, E_{i_k,j_k} \rangle$ two rank-preserving linear maps such that $\varphi(E_{i_h,j_h}) = s_h E_{i_h,j_h}$ and $\psi(E_{i_h,j_h}) = t_h E_{i_h,j_h}$, where $s_1, \ldots, s_k, t_1, \ldots, t_k \in \mathbb{F}_q^*$. If $s_h = t_h$ for $1 \le h < k$, then $s_k = t_k$.

Proof. For $a \in \mathbb{F}_q^*$, consider the matrix

$$M_a = \left(\sum_{h=1}^{k-1} E_{i_h,j_h} \right) + a E_{i_k,j_k}.$$

Since $((i_1, j_1), \ldots, (i_k, j_k))$ is a closed simple path, by Lemma 4.21, k is even and the nonzero entries of M_a are contained in a square submatrix of size $k/2$, whose determinant is a linear function of a. Hence there exists $\bar{a} \in \mathbb{F}_q^*$ such that $\mathrm{rk}(M_{\bar{a}}) = k/2 - 1$ and $\mathrm{rk}(M_a) = k/2$ for all $a \in \mathbb{F}_q \setminus \{\bar{a}\}$.

Let M be the matrix given by

$$M = \left(\sum_{h=1}^{k-1} s_h^{-1} E_{i_h,j_h} \right) + \bar{a} s_k^{-1} E_{i_k,j_k}.$$

By assumption $\mathrm{rk}(\psi(M)) = \mathrm{rk}(M) = \mathrm{rk}(\varphi(M)) = k/2 - 1$. Moreover, if $s_h = t_h$ for $1 \le h < k$, then

$$\psi(M) = \left(\sum_{h=1}^{k-1} E_{i_h,j_h} \right) + t_k \bar{a} s_k^{-1} E_{i_k,j_k}.$$

By the uniqueness of $\bar{a}$ we conclude that $\bar{a} = t_k \bar{a} s_k^{-1}$, hence $t_k = s_k$. □

The next lemma establish the Extension Property in a special case.

Lemma 4.30. Let $(i_1, j_1), \dots, (i_k, j_k)$ be k distinct elements in $\{1, \dots, m\} \times \{1, \dots, n\}$. Let $\varphi : \langle E_{i_1,j_1}, \dots, E_{i_k,j_k} \rangle \to \langle E_{i_1,j_1}, \dots, E_{i_k,j_k} \rangle \subseteq \mathbb{F}_q^{m\times n}$ be a rank-preserving linear map such that $\varphi(E_{i_h,j_h}) = s_h E_{i_h,j_h}$, where $s_1, \dots, s_k \in \mathbb{F}_q$. If the matrix $M = \sum_{h=1}^k E_{i_h,j_h}$ is irreducible, then there are two diagonal invertible matrices $A \in \mathbb{F}_q^{m\times m}$ and $B \in \mathbb{F}_q^{n\times n}$ such that

$$\varphi(C) = ACB$$

for all $C \in \langle E_{i_1,j_1}, \dots, E_{i_k,j_k} \rangle$.

Proof. We build the matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ step by step. Let $h = 1$ and set $a_{i_1,i_1} = 1$ and $b_{j_1,j_1} = s_1$. This guarantees that $AE_{i_1,j_1}B = s_1 E_{i_1,j_1}$. At each subsequent step, choose $h \in \{1, \dots, k\}$ among those that have not been previously chosen and such that either a_{i_h,i_h} or b_{i_h,i_h} has been assigned a value, if such an h exists. If a_{i_h,i_h} was already assigned a value, set $b_{j_h,j_h} = a_{i_h,i_h}^{-1} s_h$. If b_{j_h,j_h} was already assigned a value, set $a_{i_h,i_h} = b_{j_h,j_h}^{-1} s_h$.

Notice that at most one among a_{i_h,i_h} and b_{j_h,j_h} can already have an assigned value. Indeed, assume by contradiction that both a_{i_h,i_h} and b_{j_h,j_h} are fixed. Then there exist two simple paths $(\alpha_1, \dots, \alpha_u)$ and $(\beta_1, \dots, \beta_v)$ such that $\alpha_1 = \beta_1 = (i_1, j_1)$, $\alpha_u = \beta_v = (i_h, j_h)$ and $\alpha_{u-1} \neq \beta_{v-1}$. Let $z > 1$ be the smallest index such that $\alpha_z \neq \beta_z$. Let N be the inclusion-minimal submatrix of M whose support contains $\{\alpha_{z-1}, \dots, \alpha_u, \beta_z, \dots, \beta_{v-1}\}$. Let d, e be such that N has size $d \times e$. Notice that $d, e \geq 2$, since α_{z-1}, α_z, and α_u are not aligned. If β_z and α_z are not aligned, then every line of N contains at least two nonzero entries. Otherwise, α_{z-1}, α_z, and β_z are aligned, then any line that does not pass through the position α_{z-1} contains at least two nonzero entries of N. Therefore, in both cases, we have $2\max\{d, e\}$ nonzero entries in a submatrix of size $d \times e$. Since $d + e \leq 2\max\{d, e\}$, by Proposition 4.22 there exists a closed simple path in N, contradicting the irreducibility of M.

If no such h exists, choose any h among those that have not been previously chosen and set $a_{i_h,i_h} = 1$ and $b_{j_h,j_h} = s_h$. When all values of h have been considered, set to 1 all the entries on the diagonal of A and B which have not been assigned a value yet. □

Remark 4.31. The matrix M in Lemma 4.30 is irreducible, which by Corollary 4.26 implies that $\dim(\langle E_{i_1,j_1}, \dots, E_{i_k,j_k} \rangle) \leq m+n-1$. Notice that $m+n-1$ is the number of degree of freedom of the pair of matrices A, B.

We conclude the section with the proof of Theorem 4.18.

Proof of Theorem 4.18. If $m = 1$ or $n = 1$, any injective linear map is a linear isometry and the statement holds. Suppose therefore that $m, n \geq 2$ and let $M = \sum_{h=1}^k E_{i_h,j_h}$. By Proposition 4.27 there exists a path-reduction chain $(M, M_2, \dots, M_\ell)$ with M_ℓ irreducible. Consider the subset $R \subseteq \{1, \dots, k\}$ such that $M_\ell = \sum_{r\in R} E_{i_r,j_r}$. By Lemma 4.30 there are two invertible matrices A, B such that

$$AE_{i_r,j_r}B = \varphi(E_{i_r,j_r}),$$

for all $r \in R$. Following the path-reduction chain and applying $\ell - 1$ times Lemma 4.29, we have that $AE_{i_h,j_h}B = \varphi(E_{i_h,j_h})$, for $1 \leq h \leq k$. By linearity we conclude that $\varphi(C) = ACB$ for all $C \in \mathcal{C}$. □

5. Generalized weights for sum-rank metric codes

In this chapter, we introduce the generalized weights for sum-rank metric codes. Our definition is based on optimal anticodes for the sum-rank metric, in line with the rank-metric case [Rav16a]. To this end, we provide in Theorem 5.12 an Anticode Bound for the sum-rank metric, which extends the Hamming-metric Anticode Bound (Theorem 8.49) and the rank-metric Anticode Bound [Rav16b, Prop. 47]. We then provide in Theorem 5.22 a classification of optimal anticodes in the sum-rank metric, that is, codes attaining the sum-rank metric Anticode Bound. Recently in [BGLR22], a different Anticode Bound was given for the sum-rank metric. However, our bound is sharper and the resulting optimal anticodes lead to a definition of generalized sum-rank weights that satisfy desirable properties, whereas generalized weights based on anticodes as in [BGLR22] do not recover the minimum sum-rank distance of the code.

This chapter is organized as follows. In Section 5.1, we study and lower bound the maximum rank of cosets of a linear rank-metric code, extending results from Meshulam [Mes85] to cosets. Using these results, we provide in Section 5.2 our Anticode Bound for sum-rank metric codes and we provide an explicit description and classification of optimal anticodes for the sum-rank metric. In Section 5.3, we use optimal anticodes to define and obtain the main properties of generalized sum-rank weights. Finally in Section 5.4, we briefly discuss the weight distribution and other related invariants.

5.1. Maximal rank in cosets of rank-metric codes

In this section we provide lower bounds for the maximum rank of a coset of a rank-metric code. Our strategy is inspired by that used by Meshulam in [Mes85] and extends it to cosets of a vector space.

Let $\prec$ be the lexicographic order on $\mathbb{N}\times\mathbb{N}$ and let

$$\begin{array}{rcl}\phi : \mathbb{F}_q^{m\times n} & \to & \mathbb{N}\times\mathbb{N} \\ M & \mapsto & \min_{\prec}\{(i,j) : M(i,j)\neq 0\}.\end{array}$$

Definition 5.1. For a collection $\mathcal{M}=\{M_1,\dots,M_d\}$ of matrices in $\mathbb{F}_q^{m\times n}$, we define a matrix M whose entry in position (i,j) is

$$M(i,j)=\begin{cases}1 & \text{if } (i,j)=\phi(M_k) \text{ for some } k\in[d],\\ 0 & \text{otherwise.}\end{cases}$$

Denote by $\rho(\mathcal{M})$ the minimal number of lines in M which cover all ones in M, where a line of a matrix is either a row or a column.

A set of positions $\{(i_1,j_1),\dots,(i_r,j_r)\}$ of entries in a matrix is independent if for all $h\neq k$, $h,k\in[r]$ one has $i_h\neq i_k$ and $j_h\neq j_k$. König's Theorem relates the cardinality of an independent set of positions of a zero-one matrix to the minimum number of lines containing all the nonzero entries.

Theorem 5.2 (König's Theorem, [Kön31, Szá20]). If the entries of a rectangular matrix are zeros and ones, then the minimum number of lines containing all the entries equal to one is equal to the maximum cardinality of an independent set of positions corresponding to nonzero entries.

In [Mes85], Meshulam uses König's Theorem to establish a lower bound for the maximum rank of a matrix in a given vector space. In this section, we extend Meshulam's result from vector spaces of matrices to cosets. We start with a preliminary result.

Lemma 5.3. Let $D_1, \dots, D_r, A \in \mathbb{F}_q^{r\times r}$ such that for all $1 \leq i \leq r$, the first $i-1$ rows of D_i are zero and the ith row is the ith standard basis vector. Then there are $x_1, \dots, x_r \in \{0, 1\}$ such that

$$\operatorname{rk}\left(A + \sum_{i=1}^{r} x_i D_i\right) = r.$$

Proof. We proceed by induction on r. The case $r = 1$ is trivial. Assume $r > 1$. For $i \in [r-1]$ let $D_i' = D_i([r-1], [r-1])$. By the induction hypothesis, there exist $x_1, \dots, x_{r-1} \in \{0, 1\}$ such that the matrix $A([r-1], [r-1]) + \sum_{i=1}^{r-1} x_i D_i'$ is non-singular. Since $D_r(i, j) = 0$ for all $(i, j) \neq (r, r)$ and $D_r(r, r) = 1$, by expanding with respect to the bottom row we obtain

$$\det\left(A + \sum_{i=1}^{r-1} x_i D_i + D_r\right) = \det\left(A + \sum_{i=1}^{r-1} x_i D_i\right) + (-1)^{r+1} \det\left(A([r-1], [r-1]) + \sum_{i=1}^{r-1} x_i D_i'\right).$$

The last summand is nonzero, therefore

$$A + \sum_{i=1}^{r-1} x_i D_i + D_r \text{ and } A + \sum_{i=1}^{r-1} x_i D_i$$

cannot both be singular. □

The next theorem extends the main result of [Mes85] from vector spaces to cosets.

Theorem 5.4. Let $A \in \mathbb{F}_q^{m\times n}$ and let $\mathcal{M} = \{M_1, \dots, M_d\} \subseteq \mathbb{F}_q^{m\times n}$. Then there exist $x_1, \dots, x_d \in \{0, 1\}$ such that

$$\operatorname{rk}(A + x_1 M_1 + \cdots + x_d M_d) \geq \rho(\mathcal{M}).$$

Proof. Let $\rho(\mathcal{M}) = r$. By Theorem 5.2 there exist $i_1, \dots, i_r \in [d]$ such that $\{\phi(M_{i_j}) : j \in [r]\}$ is independent. Let $\phi(M_{i_j}) = (s_j, l_j)$ for $j \in [r]$, then both $S = \{s_1, \dots, s_r\}$ and $L = \{l_1, \dots, l_r\}$ have cardinality r.

We shall prove the theorem by showing that $A(S, L) + \langle B_1, \dots, B_r\rangle$ contains a non-singular matrix, where $B_j = M_{i_j}(S, L)$. We may assume that $s_1 < s_2 < \cdots < s_r$. Let σ be the permutation on $[r]$ for which $l_{\sigma(1)} < \cdots < l_{\sigma(r)}$. Denote the jth row of B_j by b_j.

Clearly the first $j-1$ rows of B_j are zero, $B_j(j, s) = 0$ for $s \in [\sigma^{-1}(j)-1]$ and $B_j(j, \sigma^{-1}(j)) \neq 0$. Let $C \in \mathbb{F}_q^{r\times r}$ be the matrix with rows $b_1, \dots b_r$. Notice that C is non-singular, since we can obtain an upper triangular matrix with nonzero entries on the diagonal by permuting the rows of C.

Let $D_j = B_j C^{-1}$ for $j \in [r]$. It easy to check that the first $j-1$ rows of D_j are zero and the jth row is the jth standard basis vector, for all $j \in [r]$.

By Lemma 5.3 we have that $A(S,L)C^{-1} + \sum_{j=1}^{r} x_j D_j$ is non-singular for some $x_1, \dots, x_r \in \{0,1\}$. Therefore

$$A(S,L) + \sum_{j=1}^{r} x_j B_j = \left(A(S,L)C^{-1} + \sum_{j=1}^{r} x_j D_j \right) C$$

is also non-singular. This implies that

$$\operatorname{rk}(A + \sum_{j=1}^{r} x_j M_{i_j}) \geq r.$$

□

A theorem by Meshulam [Mes85, Theorem 2] states that if $\mathcal{V} \subseteq \mathbb{F}_q^{m\times m}$ is an $\mathbb{F}_q$-linear subspace of $\dim(\mathcal{V}) > mt$, then $\mathcal{V}$ contains a matrix of rank at least $t+1$. This result is easily generalized to $m \times n$ matrices. The next theorem extends Meshulam's results to cosets, i.e. sets of the form $A + \mathcal{V}$, where $\mathcal{V} \subseteq \mathbb{F}_q^{m\times n}$ is $\mathbb{F}_q$-linear and $A \in \mathbb{F}_q^{m\times n}$. The theorem was first shown by C. de Seguins Pazzis, see [dSP10, Corollary 2].

Theorem 5.5. Let $0 \leq t < n$ and let $\mathcal{V} \subseteq \mathbb{F}_q^{m\times n}$ be an $\mathbb{F}_q$-linear subspace of $\dim(\mathcal{V}) > mt$. Let $A \in \mathbb{F}_q^{m\times n}$. Then there exists $B \in \mathcal{V}$ such that

$$\operatorname{rk}(A + B) \geq t + 1.$$

Moreover, if $\{B_1, \dots, B_{mt+d}\}$ is a basis of $\mathcal{V}$, $d = \dim(\mathcal{V}) - mt$, then B can be chosen of the form $B = \sum_{i=1}^{mt+d} x_i B_i$ with $x_i \in \{0,1\}$.

Proof. Let $\dim(\mathcal{V}) = mt + d$ with $d > 0$ and choose a basis $\{B_1, \dots, B_{mt+d}\}$ of $\mathcal{V}$. By performing Gaussian elimination, we may assume that $\phi(B_1), \dots, \phi(B_{mt+d})$ are distinct. Since a line in a matrix cover at most m entries we cannot cover $\phi(B_1), \dots, \phi(B_{mt+d})$ by less than $(mt+d)/m$ lines. Therefore,

$$\rho(\{B_1, \dots, B_{mt+d}\}) \geq t + 1.$$

Theorem 5.4 implies that there exists $B \in \mathcal{V}$ of the desired form, such that $\operatorname{rk}(A+B) \geq t+1$. □

Results on vector spaces are a special case of those on cosets. For example, the Anticode Bound for rank-metric codes is a direct consequence of Theorem 5.5.

Theorem 5.6 (Anticode Bound, [Rav16a])**.** Let $C \subseteq \mathbb{F}_q^{m\times n}$ be a rank-metric code. Then

$$\dim(C) \leq m \operatorname{maxrk}(C).$$

If $A \in \mathcal{V}$ and $\mathcal{V}$ is a linear space, then $A + \mathcal{V} = \mathcal{V}$ and there exist linear spaces $\mathcal{V} \subseteq \mathbb{F}_q^{m\times n}$ such that $\dim(\mathcal{V}) = mt$ and $\operatorname{rk}(A) \leq t$ for all $A \in \mathcal{V}$. Such linear spaces appear in the coding theory literature under the name of optimal anticodes. We now show that if $A \notin \mathcal{V}$, that is if $A + \mathcal{V} \neq \mathcal{V}$, then every $\mathcal{V}$ of $\dim(\mathcal{V}) = mt$ contains a B such that $\operatorname{rk}(A+B) > t$. For odd q this is an immediate consequence of Theorem 5.4, as we show in the next corollary. In Theorem 5.11 we prove the same result for any q. We choose to include Corollary 5.7, since the proof is immediate.

Corollary 5.7. Let $0 \leq t < n$ and let $\mathcal{V} \subseteq \mathbb{F}_q^{m\times n}$ be an $\mathbb{F}_q$-linear subspace of dimension $\dim(\mathcal{V}) = mt$. Let $A \in \mathbb{F}_q^{m\times n} \setminus \mathcal{V}$. If q is odd, then there exists $B \in \mathcal{V}$ such that

$$\mathrm{rk}(A+B) \geq t+1.$$

Proof. Let $\{B_1, \dots, B_{mt}\}$ be a basis of $\mathcal{V}$ and let $\bar{\mathcal{V}} = \langle A\rangle + \mathcal{V}$. Since $A \notin \mathcal{V}$, then $\dim(\bar{\mathcal{V}}) = mt+1$. By Theorem 5.5 there are $x_0, \dots, x_{mt} \in \{0,1\}$ such that

$$\mathrm{rk}\left(A + x_0 A + \sum_{i=1}^{mt} x_i B_i\right) \geq t+1.$$

Multiplying by $(1+x_0)^{-1}$ we find a matrix of the form $A+B$ with $A \notin \mathcal{V}$, $B \in \mathcal{V}$ such that $\mathrm{rk}(A+B) \geq t+1$. □

The following two lemmas will be used in the proof of Theorem 5.11.

Lemma 5.8. Let $f : \mathbb{F}_q^{r\times r} \to \mathbb{F}_q$ be a linear form that is constant on $\mathrm{GL}_r(\mathbb{F}_q)$. Suppose that either $r > 1$ or $q \neq 2$. Then $f = 0$.

Proof. Since f is linear, there exist $a_{i,j} \in \mathbb{F}_q$, $i, j \in [r]$, such that

$$f(X) = \sum_{1\leq i,j\leq r} a_{i,j} x_{i,j}$$

for any $X = (x_{i,j}) \in \mathbb{F}_q^{r\times r}$. If $r = 1$ and $q \neq 2$, let $1 \neq \alpha \in \mathbb{F}_q^*$. Then $f(\alpha) = f(1) - f(1-\alpha) = 0$, hence $f = 0$. If $r > 1$, fix $(k,l) \in [r]\times[r]$. Let $B = (b_{i,j})$ be a permutation matrix such that $b_{k,l} = 0$. Let $\bar{B} = B + E_{k,l}$. Both B and $\bar{B}$ are non-singular, so $f(B) = f(\bar{B})$. Therefore $f(E_{k,l}) = 0$ by linearity. Since this is the case for every $(k,l) \in [r]\times[r]$, we conclude that $f = 0$. □

Lemma 5.9. Let $n \geq 2$ and $m > 2$. Let $\mathcal{V} \subseteq \mathbb{F}_2^{m\times n}$ be an $\mathbb{F}_2$-linear subspace such that $\dim(\mathcal{V}) = m$. Let $A \in \mathbb{F}_2^{m\times n} \setminus \mathcal{V}$. Then there exists $B \in \mathcal{V}$ such that

$$\mathrm{rk}(A+B) \geq 2.$$

Proof. For $v \in \mathbb{F}_q^m$ and $w \in \mathbb{F}_q^m$, denote by $v \otimes w$ the $m \times n$ matrix whose entry in position (i,j) is $v_i w_j$. If $\mathrm{rk}(A) \geq 2$, then the statement holds with $B = 0$. If $\mathrm{rk}(A) = 1$, then up to equivalence we may assume that $A = E_{1,1} = e_1 \otimes e_1$. If $\mathrm{maxrk}(\mathcal{V}) = 1$, then $\mathcal{V}$ is an optimal anticode and the statement holds. If $\mathrm{maxrk}(\mathcal{V}) > 2$, then there exists $B \in \mathcal{V}$ with $\mathrm{rk}(B) > 2$. Hence $\mathrm{rk}(A+B) \geq 2$, since A has rank 1. Therefore, it suffices to prove the statement for $\mathrm{maxrk}(\mathcal{V}) = 2$.

First suppose that there are two different elements $V_1, V_2 \in \mathcal{V}$ of rank 1. Write $V_1 = v_1 \otimes w_1$ and $V_2 = v_2 \otimes w_2$ for some $v_1, v_2 \in \mathbb{F}_2^m$ and $w_1, w_2 \in \mathbb{F}_2^n$. If $\mathrm{rk}(A+V_1) = \mathrm{rk}(A+V_2) = 1$, then either $v_1 = e_1$ or $w_1 = e_1$ and either $v_2 = e_1$ or $w_2 = e_1$. If either $v_1 = e_1$ and $w_2 = e_1$, or $w_1 = e_1$ and $v_2 = e_1$, then $\mathrm{rk}(A+V_1+V_2) = 2$, since $V_1, V_2 \neq A$. If instead $v_1 = v_2 = e_1$, then e_1, w_1, w_2 are linearly independent and every matrix in $\langle A, A+V_1, A+V_2\rangle$ has rank 1. Let $B \in \mathcal{V}$ be an element of rank two. Then one of the vectors $e_1, e_1+w_1, e_1+w_2 \notin \mathrm{rowsp}(B)$. Therefore, there exists $C \in \{A, A+V_1, A+V_2\}$ such that $\mathrm{rk}(C+B) = \dim(\mathrm{rowsp}(C+B)) \geq 2$. In the case where $w_1 = w_2 = e_1$, we proceed similarly using the column space.

Suppose now that in $\mathcal{V}$ there is at most one element of rank 1. Then every linear combination with an element of maximum rank in $\mathcal{V}$ has again maximum rank. Hence, since $\dim(\mathcal{V}) > 2$, there are two linearly independent elements B_1, B_2 such that $\mathrm{rk}(B_1) = \mathrm{rk}(B_2) = \mathrm{rk}(B_1 + B_2) = 2$. If $\mathrm{rk}(A + B_1) = \mathrm{rk}(A + B_2) = \mathrm{rk}(A + B_1 + B_2) = 1$, then

$$B_1 = e_1 \otimes e_1 + e_2 \otimes e_2,\ B_2 = e_1 \otimes e_1 + v_2 \otimes w_2,$$

$$B_1 + B_2 = e_1 \otimes e_1 + v_3 \otimes w_3,$$

possibly after applying a code equivalence that fixes A. Since

$$B_2 = e_1 \otimes e_1 + v_2 \otimes w_2 = e_2 \otimes e_2 + v_3 \otimes w_3$$

and

$$B_1 + B_2 = e_1 \otimes e_1 + v_3 \otimes w_3 = e_2 \otimes e_2 + v_2 \otimes w_2$$

have rank 2, then $v_2, v_3 \notin \{e_1, e_2\}$. Moreover

$$e_1 \otimes e_1 + e_2 \otimes e_2 = v_2 \otimes w_2 + v_3 \otimes w_3,$$

hence $\langle v_2, v_3 \rangle = \langle e_1, e_2 \rangle$. The only possibility is that $v_2 = v_3 = e_1 + e_2$, but this contradicts the assumption that $B_1 = v_2 \otimes w_2 + v_3 \otimes w_3$ has rank 2. Therefore, one among $A + B_1, A + B_2, A + B_1 + B_2$ has rank at least 2. □

The next example we show that the condition $m > 2$ in Lemma 5.9 is necessary. The example is essentially the same as the example that appears below Theorem 2 in [dSP10].

Example 5.10. Consider the 2-dimensional space $\mathcal{V} \subseteq \mathbb{F}_2^{2\times 2}$ given by

$$\mathcal{V} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle$$

and let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \notin \mathcal{V}$. Then $\max_{B\in\mathcal{V}}\{\mathrm{rk}(A + B)\} = 1$.

The next theorem generalizes Corollary 5.7 to any q. It was first shown by C. de Seguins Pazzis, see [dSP10, Corollary 2].

Theorem 5.11. Let $0 \le t < n$ and let $\mathcal{V} \subseteq \mathbb{F}_q^{m\times n}$ be an $\mathbb{F}_q$-linear subspace such that $\dim(\mathcal{V}) = mt$. Let $A \in \mathbb{F}_q^{m\times n} \setminus \mathcal{V}$. If either $t \neq 1$ or $m \neq 2$ or $q \neq 2$ or $\mathrm{rk}(A) \neq 1$, then there exists $B \in \mathcal{V}$ such that

$$\mathrm{rk}(A + B) \ge t + 1.$$

Proof. If $t = 0$, then $\mathcal{V} = 0$ and the thesis is readily verified. Suppose that $t \ge 1$ and let $\mathcal{M} = \{M_1, \dots, M_{mt}\}$ be a basis of $\mathcal{V}$. Up to a change of basis, we may assume without loss of generality that $\phi(M_i) \neq \phi(M_j)$ if $i \neq j$. In particular, $\rho(\mathcal{M}) \ge t$. If $\rho(\mathcal{M}) \ge t + 1$, then we conclude by Theorem 5.4.

Suppose that $\rho(\mathcal{M}) = t$. Up to code equivalence, we may assume that the t lines that cover $\phi(M_i)$ for all i are the first t columns. If $t = 1$ and $\mathrm{rk}(A) \ge 2$, then let $B = 0$. If $t = 1$, $\mathrm{rk}(A) = 1$, $q \neq 2$ and there exists $B \in \mathcal{V}$ with $\mathrm{rk}(B) \ge 2$, then either $\mathrm{rk}(A + B) \ge 2$ or $\mathrm{rk}(A + 2B) \ge 2$. If $t = 1$, $\mathrm{rk}(A) = 1$, $q \neq 2$, and $\mathrm{maxrk}\,\mathcal{V} = 1$, then $\mathcal{V}$ is an optimal anticode and the result follows easily. If $q = 2$ we conclude by Lemma 5.9, since $m \neq 2$.

Suppose now that $\rho(\mathcal{M}) = t \geq 2$. For every $t+1 \leq l \leq n$ and every $k \in [m]$ there exists a linear form $f_{k,l} \in \mathbb{F}_q[x_{i,j} \mid (i,j) \in [m] \times [t]]$ such that $\mathcal{V}$ is equal to

$$\{(x_{u,v})_{u,v} \in \mathbb{F}_q^{m\times n} : x_{k,l} = f_{k,l}(x_{i,j}) \ \forall k \in [m], l \in [n] \setminus [t]\}.$$

Assume without loss of generality that the entry of M_i in position $\phi(M_i)$ is 1. Then $f_{k,l}$ is obtained by writing a matrix of $\mathcal{V}$ as $\sum_{(i,j)\in[m]\times[t]} x_{i,j} M_{(i-1)t+j}$. Assume that $\max_{B\in\mathcal{V}} \mathrm{rk}(A+B) = t$ for some $A \in \mathbb{F}_q^{m\times n}$. It suffices to show that $A \in \mathcal{V}$. Up to reducing A modulo $\mathcal{V}$, we may assume without loss of generality that $a_{i,j} = 0$ for $(i,j) \in [m] \times [t]$. Fix $(k,l) \in [m] \times [n]$ with $l \geq t+1$. Let $X = (x_{i,j})_{i,j} \in \mathcal{V}$. We have that

$$x_{k,l} + a_{k,l} = f_{k,l}(x_{i,j}) + a_{k,l}.$$

Let $L = [t]$ and let S be a subset of $[m] \setminus \{k\}$ of cardinality t. Let $x_{i,j} = 0$ for $i \notin S$ and $j \in L$. For any choice of $(x_{i,j})_{i\in S, j\in L}$ such that $X(S,L) + A(S,L) = X(S,L)$ is invertible, one has

$$0 = x_{k,l} + a_{k,l} = f_{k,l}(x_{i,j}) + a_{k,l}, \tag{5.1.1}$$

since every matrix in $(A+\mathcal{V})(S \cup \{k\}, L \cup \{l\})$ has rank smaller than or equal to t. Lemma 5.8 together with (5.1.1) implies that $a_{k,l} = 0$. This proves that $A \in \mathcal{V}$. □

5.2. Anticode Bound and optimal anticodes

In this section we prove an Anticode Bound for sum-rank metric codes. Our bound improves the bound from [BGLR22, Theorem 2.2] and generalizes Theorem 8.49.

Theorem 5.12 (Anticode Bound)**.** Let $C \subseteq \mathbb{M}$ be an $\mathbb{F}_q$-linear subspace. Then

$$\dim(C) \leq \max_{C\in C} \left\{ \sum_{i=1}^{\ell} m_i \mathrm{rk}(C_i) \right\}. \tag{5.2.2}$$

In particular, if $m_1 = \ldots = m_\ell = m$, then

$$\dim(C) \leq m \,\mathrm{maxsrk}(C).$$

Proof. We proceed by induction on ℓ. If $\ell = 1$, then C is a rank-metric code, the sum-rank metric coincides with the rank metric, and the statement is Theorem 5.6.

Let $\ell > 1$. Let π be the canonical projection from $\mathbb{M}$ onto $\mathbb{F}_q^{m_1\times n_1} \times \cdots \times \mathbb{F}_q^{m_{\ell-1}\times n_{\ell-1}}$ and let π_ℓ be the canonical projection from $\mathbb{M}$ onto $\mathbb{F}_q^{m_\ell \times n_\ell}$. Define $\mathcal{A} = \pi(C)$ and $\mathcal{B} = \pi_\ell(\pi^{-1}(0) \cap C)$ and let $\tilde{C} = \mathcal{A} \times \mathcal{B}$. Since $\dim(\pi^{-1}(0) \cap C) = \dim(\pi_\ell(\pi^{-1}(0) \cap C)) = \dim(\mathcal{B})$, we have that

$$\dim(C) = \dim(\mathcal{A}) + \dim(\pi^{-1}(0) \cap C) = \dim(\tilde{C}).$$

By the induction hypothesis there is $(C_1, \ldots, C_{\ell-1}) \in \mathcal{A}$ such that

$$\sum_{i=1}^{\ell-1} m_i \mathrm{rk}(C_i) \geq \dim(\mathcal{A}) = \dim(C) - \dim(\mathcal{B}).$$

Let $C_\ell \in \pi_\ell(\mathcal{C})$ such that $(C_1, \dots, C_\ell) \in \mathcal{C}$. By Theorem 5.5 there is a $B \in \mathcal{B}$ such that

$$\mathrm{rk}(C_\ell + B) \geq \left\lceil \frac{\dim(\mathcal{B})}{m_\ell} \right\rceil .$$

Therefore

$$\sum_{i=1}^{\ell-1} m_i \mathrm{rk}(C_i) + m_\ell \mathrm{rk}(C_\ell + B) \geq \dim(\mathcal{C}) - \dim(\mathcal{B}) + m_\ell \left\lceil \frac{\dim(\mathcal{B})}{m_\ell} \right\rceil \geq \dim(\mathcal{C}).$$

Since $(C_1, \dots, C_{\ell-1}, C_\ell) \in \mathcal{C}$ and $B \in \mathcal{B}$, then the element $(C_1, \dots, C_{\ell-1}, C_\ell + B) \in \mathcal{C}$. This concludes the proof. □

Optimal sum-rank metric anticodes may now be defined as the codes which meet the Anticode Bound.

Definition 5.13. A sum-rank metric code $\mathcal{C} \subseteq \mathbb{M}$ is an optimal anticode if

$$\dim(\mathcal{C}) = \max_{C \in \mathcal{C}} \left\{ \sum_{i=1}^{\ell} m_i \mathrm{rk}(C_i) \right\} .$$

Remark 5.14. In [BGLR22], the authors give a definition of r-anticode for r a non-negative integer. In [BGLR22, Theorem 2.2] they establish an upper bound for the dimension of an r-anticode. For a given $\mathcal{C} \subseteq \mathbb{M}$ and $r = \mathrm{maxsrk}(\mathcal{C})$, [BGLR22, Theorem 2.2] yields

$$\dim(\mathcal{C}) \leq \max \left\{ \sum_{i=1}^{\ell} m_i u_i : \sum_{i=1}^{\ell} u_i = r, u_i \leq n_i \text{ for all } i \right\} . \tag{5.2.3}$$

Notice that our Anticode Bound is tighter than (5.2.3), since for all $C = (C_1, \dots, C_\ell) \in \mathcal{C}$ there exist $u_1, \dots, u_\ell \in \mathbb{Z}$ such that $\sum_{i=1}^{\ell} u_i = \mathrm{maxsrk}(\mathcal{C})$ and $\mathrm{rk}(C_i) \leq u_i \leq n_i$ for all i. In particular, all codes that meet bound (5.2.3) also meet our Anticode Bound. Moreover the bounds are different, as one can easily check by comparing Theorem 5.22 in this chapter and [BGLR22, Corollary 3.8]. In [BGLR22, Definition 2.3], the authors define optimal anticodes as those that meet the bound (5.2.3). In particular, an optimal anticode according to [BGLR22] is an optimal anticode according to Definition 5.13, but the converse is not true in general. For example, the code $0 \times \mathbb{F}_2 \subseteq \mathbb{F}_2^{2\times 2} \times \mathbb{F}_2$ is an optimal anticode according to Definition 5.13, but it does not meet (5.2.3).

A simple computation allows one to show that if $\mathcal{C}_i \subseteq \mathbb{F}_q^{m_i \times n_i}$ is an optimal anticode with respect to the rank metric for $i \in [\ell]$, then $\mathcal{C}_1 \times \cdots \times \mathcal{C}_\ell \subseteq \mathbb{M}$ is an optimal anticode with respect to the sum-rank metric. Moreover, one has the following.

Proposition 5.15. Let $\mathcal{C} \subseteq \mathbb{M}$ be an optimal anticode and assume that $m_1 = \ldots = m_\ell = m$. For $i \in [\ell]$ let $\pi_i : \mathbb{M} \to \mathbb{F}_q^{m_i \times n_i}$ be the canonical projection. The following are equivalent:

1. $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_\ell$ and $\mathcal{C}_i$ is an optimal rank-metric anticode for $i \in [\ell]$.
2. $\mathrm{maxsrk}(\mathcal{C}) = \sum_{i=1}^{\ell} \mathrm{maxrk}(\pi_i(\mathcal{C}))$.

Proof. (1) $\implies$ (2) follows from a simple computation.
(2) $\implies$ (1) Clearly, $C \subseteq \prod_{i=1}^{\ell} \pi_i(C)$, so

$$m\,\mathrm{maxsrk}(C) = \dim(C) \leq \sum_{i=1}^{\ell} \dim(\pi_i(C)) \leq \sum_{i=1}^{\ell} m\,\mathrm{maxrk}(\pi_i(C)).$$

Since $\mathrm{maxsrk}(C) = \sum_{i=1}^{\ell} \mathrm{maxrk}(\pi_i(C))$, we have that

$$\dim(C) = \dim\left(\prod_{i=1}^{\ell} \pi_i(C)\right)$$

and

$$\dim(\pi_i(C)) = m\,\mathrm{maxrk}(\pi_i(C)).$$

Therefore $C = \prod_{i=1}^{\ell} \pi_i(C)$ and C_i is an optimal rank-metric anticode for all $i \in [\ell]$. □

We will prove that optimal anticodes in the sum-rank metric are generated by their elements of maximum sum rank. We start by proving the result in the special case of rank-metric anticodes.

Lemma 5.16. Let $C \subseteq \mathbb{F}_q^{m\times n}$ be an optimal anticode. Then C is generated by its elements of maximum rank.

Proof. Let $t = \mathrm{maxrk}(C)$, then $\dim(C) = mt$. Up to code equivalence we may assume that C consists of all matrices whose rowspace is contained in $\langle e_1, \ldots, e_t\rangle$, where $e_1, \ldots, e_t \in \mathbb{F}_q^n$ are the first t elements of the standard basis. Therefore it suffices to prove the statement for $C = \mathbb{F}_q^{m\times t}$. Let $\{E_{i,j}\}_{1\leq i\leq m, 1\leq j\leq t}$ be the standard basis of $\mathbb{F}_q^{m\times t}$. Let $I = \sum_{i=1}^{t} E_{i,i} \in \mathbb{F}_q^{m\times t}$. For each $(i, j) \in [m] \times [t]$ there exists a permutation matrix $S_{i,j} \in \mathbb{F}_q^{m\times m}$ such that $(S_{i,j}I)_{i,j} = 0$. Therefore one can write $E_{i,j} = (S_{i,j}I + E_{i,j}) - S_{i,j}I$, with $\mathrm{rk}(S_{i,j}I) = \mathrm{rk}(S_{i,j}I + E_{i,j}) = t$. This implies that $\{S_{i,j}I + E_{i,j}, S_{i,j}I\}_{1\leq i\leq m, 1\leq j\leq t}$ is a set of matrices of rank t which generates $\mathbb{F}_q^{m\times t}$. □

The next observations will be useful in order to extend the result of Lemma 5.16 to optimal anticodes in the sum-rank metric.

Lemma 5.17. Let $m \geq 2$ and let $C \subseteq \mathbb{F}_2^{m\times n}$ be an optimal rank-metric anticode of $\mathrm{maxrk}(C) = t$. Then every element of C of rank t can be written as the sum of two elements of C of rank t.

Proof. Up to code equivalence we may assume that C consists of all matrices whose rowspace is contained in $\langle e_1, \ldots, e_t\rangle$, where $e_1, \ldots, e_t \in \mathbb{F}_2^n$ are the first t elements of the standard basis. Therefore, it suffices to show that every element of full rank in $\mathbb{F}_2^{m\times t}$ can be written as the sum of two elements of $\mathbb{F}_2^{m\times t}$ of full rank. Let $C = (c_1, \ldots, c_t) \in \mathbb{F}_2^{m\times t}$ be the matrix whose columns are $c_1, \ldots, c_t \in \mathbb{F}_2^m$. Assume that $\mathrm{rk}(C) = t$. If $t = 1$, let $\tilde{C} \in C \setminus \{C, 0\}$. Notice that $\tilde{C}$ exists, since $m \geq 2$. Then $\tilde{C}, C + \tilde{C}$ are elements of rank 1 and $C = \tilde{C} + (C + \tilde{C})$. If t is even, then $C = C_1 + C_2$ where

$$\begin{aligned} C_1 &= (c_1 + c_2, c_1, c_3 + c_4, c_3, \ldots, c_{t-1} + c_t, c_{t-1}), \\ C_2 &= (c_2, c_1 + c_2, c_4, c_3 + c_4, \ldots, c_t, c_{t-1} + c_t). \end{aligned}$$

If $t \neq 1$ is odd, then $C = C_1 + C_2$ where

$$\begin{aligned} C_1 &= (c_1 + c_2, c_3, c_1, c_4 + c_5, c_4, \ldots, c_{t-1} + c_t, c_{t-1}), \\ C_2 &= (c_2, c_3 + c_2, c_1 + c_3, c_5, c_4 + c_5, \ldots, c_t, c_{t-1} + c_t). \end{aligned}$$

Since C_1 and C_2 have the same column space as C, they have full rank. □

Theorem 5.18. Let $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_\ell \subseteq \mathbb{M}$, where $\mathcal{C}_i$ is an optimal rank-metric anticode for all $i \in [\ell]$. If either $m_{\ell-1} \geq 2$ or $q \neq 2$, then $\mathcal{C}$ is generated by its elements of maximum sum-rank.

Proof. Let $C = (C_1, \ldots, C_\ell) \in \mathcal{C}$ be such that

$$\sum_{i=1}^{\ell} m_i \mathrm{rk}(C_i) = \max_{D \in \mathcal{C}} \left\{ \sum_{i=1}^{\ell} m_i \mathrm{rk}(D_i) \right\}.$$

Since $\mathcal{C}$ is a product, then C_i is an element of maximum rank in $\mathcal{C}_i$ for all $1 \leq i \leq \ell$. If $q \neq 2$, let $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$. Then $(0, \ldots, 0, C_i, 0, \ldots, 0)$ is an element of

$$\langle (C_1, \ldots, C_\ell), (C_1, \ldots, C_{i-1}, \alpha C_i, C_{i+1}, \ldots, C_\ell) \rangle.$$

Therefore $\mathcal{C}$ is generated by its element of maximum sum-rank, since each $\mathcal{C}_i$ is generated by its elements of maximum rank by Lemma 5.16.

If $q = 2$ and $i \neq \ell$, then by Lemma 5.17 there exist $C_i', C_i'' \in \mathcal{C}_i$ of maximum rank such that $C_i = C_i' + C_i''$. Let $C' = (C_1, \ldots, C_{i-1}, C_i', C_{i+1}, \ldots, C_\ell)$ and $C'' = (C_1, \ldots, C_{i-1}, C_i'', C_{i+1}, \ldots, C_\ell)$. Then

$$(0, \ldots, 0, C_i, 0, \ldots, 0) \in \langle C', C'' \rangle.$$

Since C and $(0, \ldots, 0, C_i, 0, \ldots, 0)$, $i \in [\ell - 1]$, belong to the subcode of $\mathcal{C}$ generated by its codewords of maximum sum-rank, then also $(0, \ldots, 0, C_\ell)$ does. Therefore $\mathcal{C}$ is generated by its element of maximum sum-rank. □

Example 5.19. For $\ell \geq 2$ and $m_{\ell-1} = 1$, the code $\mathcal{C} = 0 \times \ldots \times 0 \times \mathbb{F}_2 \times \mathbb{F}_2$ is an optimal anticode, which is not generated by its unique element $(0, \ldots, 0, 1, 1)$ of maximum sum-rank.

The next result on generating sets of optimal binary anticodes in the Hamming metric will also be useful.

Lemma 5.20. Let $\mathcal{C} \subseteq \mathbb{F}_2^\ell$ be an optimal anticode of $\dim(\mathcal{C}) = t \geq 1$. Then $\mathcal{C}$ is generated by its elements of weight t and $t - 1$.

Proof. Let G be a generator matrix of $\mathcal{C}$ and assume that G is in reduced row echelon form. Denote by $g_1, \ldots, g_t$ the rows of G. Let $v = g_1 + \ldots + g_t$. Then the vectors $v, v + g_1, \ldots, v + g_t$ have weight $t - 1$ or t and are a system of generators of $\mathcal{C}$, since $g_i = v + (v + g_i)$ for all i. □

The following technical lemma will be used in the proof of Theorem 5.22.

Lemma 5.21. Let $q = 2$, $\ell \geq 2$, $m_1 = n_1 = 2$, and let $k = \max\{i \in [\ell] \mid m_i > 1\}$. Let $\mathcal{C} \subseteq \mathbb{M}$, $\mathcal{A} = \pi(\mathcal{C})$, $\mathcal{B} = \pi_1(\pi^{-1}(0) \cap \mathcal{C})$, where $\pi : \mathbb{M} \to \mathbb{F}_2^{m_2 \times n_2} \times \cdots \times \mathbb{F}_2^{m_\ell \times n_\ell}$ and $\pi_1 : \mathbb{M} \to \mathbb{F}_2^{m_1 \times n_1}$ are the canonical projections. If $\dim(\mathcal{B}) = 2$ and $\mathcal{A} = \prod_{i=2}^{k} \mathcal{C}_i \times \mathcal{C}'$ for optimal anticodes $\mathcal{C}' \subseteq \mathbb{F}_2^{\ell-k}$ and $\mathcal{C}_i \subseteq \mathbb{F}_2^{m_i \times n_i}$ for all $i \in [k] \setminus \{1\}$, then one of the following holds:

(i) $\mathcal{B}$ is an optimal anticode,

(ii) There is $B \in \mathcal{B}$ and $C = (C_1, \dots, C_\ell) \in C$ with

$$\sum_{i=2}^{k} m_i \mathrm{rk}(C_i) + \mathrm{wt}(C_{k+1}, \dots, C_\ell) \geq \sum_{i=2}^{k} m_i \,\mathrm{maxrk}(C_i) + \mathrm{maxwt}(C') - 1,$$

such that

$$\mathrm{rk}(B + C_1) = 2.$$

Proof. If $\mathrm{maxrk}(\mathcal{B}) = 1$, then $\mathcal{B}$ is an optimal anticode. Assume therefore that $\mathrm{maxrk}(\mathcal{B}) = 2$. Let G be a generator matrix of C' and assume that G is in reduced row echelon form and that $\dim(C') = t$. If $t = 0$, then let $D = (D_2, \dots, D_k), E = (E_2, \dots, E_k) \in \prod_{i=2}^{k} C_i$ be codewords such that each component of $D, E, D+E$ has maximal rank. Such matrices exist, since each C_i is an optimal anticode. Let $D_1, E_1 \in C_1$ be such that $(D_1, \dots, D_k, 0), (E_1, \dots, E_k, 0) \in C$. If one of D_1, E_1, and $D_1 + E_1$ is zero, then we conclude by taking B of rank 2. If there is a rank 2 element among D_1, E_1, and $D_1 + E_1$, then we conclude by taking $B = 0$. If $D_1, E_1, D_1 + E_1$ all have rank 1, then again we easily conclude. In fact, either $\langle D_1, E_1 \rangle \cap \mathcal{B} \neq 0$, or

$$|(D_1 + \mathcal{B}) \cup (E_1 + \mathcal{B}) \cup (D_1 + E_1 + \mathcal{B})| = 12,$$

but in $\mathbb{F}_2^{2\times 2}$ we have only 9 elements of rank 1.

Suppose now that $t \geq 1$ and let $g_1, \dots, g_t, v$ as in the proof of Lemma 5.20. For every $i \in [t]$, there exists $G_1^i \in \mathbb{F}_2^{2\times 2}$ such that $G^i = (G_1^i, 0, \dots, 0, g_i) \in C$. If for every $i \in [t]$, $G_1^i \in \mathcal{B}$, then $C = \mathcal{D} \times C'$, where $\mathcal{D} \subseteq \mathbb{F}_2^{m_1 \times n_1} \times \ldots \times \mathbb{F}_2^{m_k \times n_k}$. Therefore we reduce to the situation $t = 0$, which we treated above. Hence we assume without loss of generality that $G_1^1 \notin \mathcal{B}$. Let $C = \sum_{i=1}^{t} G^i = (C_1, 0, \dots, 0, v)$ and $D = (D_1, D_2, \dots, D_k, 0, \dots, 0)$ such that D_i has max rank in C_i for $i \in [k] \setminus \{1\}$. If either $D_1 + C_1$ or $D_1 + C_1 + G_1^1$ belongs to $\mathcal{B}$, then we conclude. If $D_1 + C_1$,$D_1 + C_1 + G_1^1 \notin \mathcal{B}$, then since $G_1^1 \notin \mathcal{B}$, we have that

$$(D_1 + C_1 + G_1^1 + \mathcal{B}) \cap (D_1 + C_1 + \mathcal{B}) = \emptyset,$$

and

$$((D_1 + C_1 + G_1^1 + \mathcal{B}) \cup (D_1 + C_1 + \mathcal{B})) \cap (\langle G_1^1 \rangle + \mathcal{B}) = \emptyset.$$

Notice that $\mathbb{F}_2^{2\times 2}$ consists of the zero matrix, 9 elements of rank 1, and 6 elements of rank 2. In $\langle G_1^1 \rangle + \mathcal{B}$ there are at least two elements of rank 1, since $\dim(\langle G_1^1 \rangle + \mathcal{B}) = 3$. Therefore, in $(D_1 + C_1 + G_1^1 + \mathcal{B}) \cup (D_1 + C_1 + \mathcal{B})$ there must be at least an element of rank 2. We conclude, since the elements $D + C$ and $D + C + G^1$ satisfy the condition from (ii). □

In the next theorem we show that the optimal anticodes in the sum-rank metric are products of optimal anticodes in the rank metric and an optimal anticode in the Hamming metric.

Theorem 5.22. Let $k = 0$ if $m_1 = 1$ and $k = \max\{i \in [\ell] \mid m_i > 1\}$ otherwise. A code $C \subseteq \mathbb{M}$ is an optimal anticode if and only if there is an optimal anticode $C' \subseteq \mathbb{F}_q^{\ell-k}$ and optimal anticodes $C_i \subseteq \mathbb{F}_q^{m_i \times n_i}$ for all $i \in [k]$ such that $C = \prod_{i=1}^{k} C_i \times C'$.

Proof. Assume that $C' \subseteq \mathbb{F}_q^{\ell-k}$ is an optimal Hamming-metric anticode and $C_i \subseteq \mathbb{F}_q^{m_i \times n_i}$ are optimal rank-metric anticodes for $i \in [k]$. It is straightforward to prove that $C = \prod_{i=1}^{k} C_i \times C' \subseteq$

$\mathbb{M}$ is an optimal anticode. Further, the statement of the theorem holds for $m_1 = \ldots = m_\ell = 1$. Therefore, we may assume that $m_1 > 1$, hence also $k \geq 1$. We proceed by induction on ℓ. For $\ell = 1$, the theorem holds trivially.

We suppose that the theorem holds for $\ell - 1$ and we prove it for $\ell > 1$. Let $\mathcal{A} = \pi(C)$, $\mathcal{B} = \pi_1(\pi^{-1}(0) \cap C)$, and $\tilde{C} = \mathcal{B} \times \mathcal{A}$. As in the proof of Theorem 5.12, we have

$$\dim(C) = \dim(\mathcal{B}) + \dim(\mathcal{A}) \leq m_1 \mathrm{rk}(C_1 + B) + \sum_{i=2}^{\ell} m_i \mathrm{rk}(C_i), \tag{5.2.4}$$

where $(C_1, \ldots, C_\ell) \in C$ is such that $(C_2, \ldots, C_\ell)$ maximizes $\sum_{i=2}^{\ell} m_i \mathrm{rk}(C_i)$ on $\mathcal{A}$ and $m_1 \mathrm{rk}(C_1) \geq \dim(\mathcal{B})$. Since $(C_1, C_2, \ldots, C_\ell) \in C$ and C is an optimal anticode, then (5.2.4) is an equality. In particular,

$$m_1 \mathrm{rk}(C_1) = \dim(\mathcal{B}) \quad \text{and} \quad \sum_{i=2}^{\ell} m_i \mathrm{rk}(C_i) = \dim(\mathcal{A}). \tag{5.2.5}$$

This proves that $\mathcal{A}$ is an optimal anticode. Therefore, by the induction hypothesis, there is an optimal anticode $C' \subseteq \mathbb{F}_q^{\ell-k}$ and optimal anticodes $C_i \subseteq \mathbb{F}_q^{m_i \times n_i}$ for $2 \leq i \leq k$ such that $\mathcal{A} = \prod_{i=2}^{k} C_i \times C'$.

We claim that $C_1 \in \mathcal{B}$. In fact, if $C_1 \notin \mathcal{B}$ and either $\dim(\mathcal{B}) \neq 2$ or $m \neq 2$ or $q \neq 2$ or $n \neq 2$ or $\mathrm{rk}(C_1) \neq 1$, then by Theorem 5.11 there exists $B \in \mathcal{B}$ such that

$$\mathrm{rk}(C_1 + B) > \dim(\mathcal{B})/m_1 = \mathrm{rk}(C_1).$$

Since $B \in \mathcal{B}$, then $(C_1 + B, C_2, \ldots, C_\ell) \in C$. However, this contradicts the optimality of C, since $m_1 \mathrm{rk}(C_1 + B) + \sum_{i=2}^{\ell} m_i \mathrm{rk}(C_i) > \sum_{i=1}^{\ell} m_i \mathrm{rk}(C_i) = \dim(C)$. This proves that $C_1 \in \mathcal{B}$, so $C_1 + B \in \mathcal{B}$, hence $\mathcal{B}$ is an optimal anticode by (5.2.5). If $\dim(\mathcal{B}) = 2$, $m = n = 2$, $q = 2$, and $\mathrm{rk}(C_1) = 1$, then by Lemma 5.21 either $\mathcal{B}$ is an optimal anticode, or there exists $B \in \mathcal{B}$ and $\bar{C} = (\bar{C}_1, \ldots, \bar{C}_\ell)$ such that $(\bar{C}_1 + B, \ldots, \bar{C}_\ell) \in C$ and

$$m_1 \mathrm{rk}(\bar{C}_1 + B) + \sum_{i=2}^{\ell} m_i \mathrm{rk}(\bar{C}_i) \geq 4 + \dim(\mathcal{A}) - 1 = \dim(C) + 1.$$

This is a contradiction, since C is an optimal anticode. We conclude that also in this case $\mathcal{B}$ is an optimal anticode. In addition, our arguments show that, if $(C_1, \ldots, C_\ell) \in C$ is such that $(C_2, \ldots, C_\ell)$ maximizes $\sum_{i=2}^{\ell} m_i \mathrm{rk}(C_i)$, then $C_1 \in \mathcal{B}$. Hence $(0, C_2, \ldots, C_\ell) \in C$.

In order to conclude the proof, it suffices to show that $C = \mathcal{B} \times \mathcal{A}$. Since $C \supseteq \mathcal{B} \times 0$, it suffices to show that $C \supseteq 0 \times \mathcal{A}$. If either $k \geq \ell - 1$ or $q \neq 2$, then $0 \times \mathcal{A}$ is generated by its element of maximum sum-rank by Theorem 5.18. Since these belong to C, we have that $0 \times \mathcal{A} \subseteq C$. Therefore, assume that $k \leq \ell - 2$ and $q = 2$. Let $2 \leq i \leq k$. By Lemma 5.17, if $C_i \in \mathcal{C}_i$ is an element of maximum rank, then $C_i = D_i + D_i'$ for some $D_i, D_i' \in \mathcal{C}_i$ of maximum rank. Hence $(0, \ldots, 0, C_i, 0, \ldots, 0)$ is equal to

$$(0, D_2, \ldots, D_k, D) + (0, D_2', \ldots, D_k', D)$$

where $D_j = D_j' \in \mathcal{C}_j$ is an element of maximum rank for any $j \in \{2, \ldots, k\} \setminus \{i\}$ and D is an element of maximum rank of C'. Since $(0, D_2, \ldots, D_k, D)$, $(0, D_2', \ldots, D_k', D)$ are elements of maximum sum-rank in $0 \times \mathcal{A}$, they belong to C. This proves that, for any $2 \leq i \leq k$, if C_i has maximum rank among the elements of C_i, then

$$(0, \dots, 0, C_i, 0, \dots, 0) \in C. \tag{5.2.6}$$

Since C_i is generated by its elements of maximum rank by Lemma 5.16, then

$$0 \times \dots \times 0 \times C_i \times 0 \times \dots \times 0 \subseteq C$$

for all $2 \leq i \leq k$.

In addition, it follows from (5.2.6) that $(0, \dots, 0, D) \in C$ for any $D \in C'$ of maximum Hamming weight. We claim that $0 \times \dots \times 0 \times C' \subseteq C$. Let t be the maximum weight of a codeword in C' and let $D' \in C'$ be an element of weight $t-1$. By Lemma 5.20 it suffices to show that $(0, \dots, 0, D') \in C$. Let $(D_2, \dots, D_k, D') \in \mathcal{A}$ with $\mathrm{rk}(D_i) = \mathrm{maxrk}(C_i)$ for $2 \leq i \leq k$. Let D_1 be such that $(D_1, D_2, \dots, D_k, D') \in C$. Since $0 \times C_2 \times \dots \times C_k \times 0 \subseteq C$, then $(D_1, 0, \dots, 0, D') \in C$. If $D_1 \in \mathcal{B}$ the claim follows, since $(D_1, 0, \dots, 0) \in \mathcal{B} \times 0 \subseteq C$. If $D_1 \notin \mathcal{B}$, then since $\mathcal{B}$ is an optimal anticode, there exists $B \in \mathcal{B}$ such that $\mathrm{rk}(B + D_1) \geq \mathrm{maxrk}(\mathcal{B}) + 1$. Then the element $(B + D_1, D_2, \dots, D_k, D') \in C$ has sum rank

$$\begin{aligned} m_1 \mathrm{rk}(B + D_1) + \sum_{j=2}^{k} m_j \mathrm{rk}(D_j) + \mathrm{wt}(D') &\geq m_1(\mathrm{maxrk}(\mathcal{B}) + 1) + \sum_{j=2}^{k} m_j \,\mathrm{maxrk}(C_j) + t - 1 \\ &= dim(C) + m_1 - 1 > \dim(C), \end{aligned}$$

where $\mathrm{wt}(D')$ denotes the Hamming weight of D', and the inequality follows from the assumption that $m_1 > 1$. This contradicts the assumption that C is an optimal anticode, completing the proof of the claim and of the theorem. □

Example 5.23. Denote by $\mathrm{rowsp}(M)$ the row-space of a matrix M. The optimal anticodes in $\mathbb{F}_q^{5\times 2} \times \mathbb{F}_q^{4\times 3}$ are exactly the codes of the form

$$\{(A, B) \mid \mathrm{rowsp}(A) \subseteq U, \mathrm{rowsp}(B) \subseteq V\}$$

for some $U \subseteq \mathbb{F}_q^2$, $V \subseteq \mathbb{F}_q^3$ vector subspaces.

The next result is an easy consequence of Theorem 5.22.

Corollary 5.24. Assume that either $q \neq 2$ or $m_{\ell-2} \geq 2$. An $\mathbb{F}_q$-linear space $C \subseteq \mathbb{M}$ is an optimal anticode if and only if for all $i \in [\ell]$ there is $C_i \subseteq \mathbb{F}_q^{m_i \times n_i}$ optimal anticode such that $C = \prod_{i=1}^{\ell} C_i$.

Proof. By Theorem 5.22 $C = \prod_{i=1}^{k} C_i \times C'$, where $C' \subseteq \mathbb{F}_q^{\ell-k}$ is an optimal anticode, $k = \max\{i \in [\ell] \mid m_i > 1\}$, and $C_i \subseteq \mathbb{F}_q^{m_i \times n_i}$ are optimal anticodes for all $i \in [k]$. If $q \neq 2$, then C' is a product of zeroes and copies of $\mathbb{F}_q$ by [Rav16a, Proposition 9]. If $q = 2$ and $\ell - k \leq 2$, the same is true by direct inspection. □

We conclude this section with a proof that the dual of an optimal anticode in the sum-rank metric is an optimal anticode, if $q \neq 2$ or $m_{\ell-2} > 1$.

Proposition 5.25. Let $q \neq 2$ or $m_{\ell-2} > 1$. Then $\mathcal{A} \subseteq \mathbb{M}$ is an optimal anticode if and only if $\mathcal{A}^{\perp} \subseteq \mathbb{M}$ is an optimal anticode.

Proof. The dual of an optimal anticode in the rank-metric is an optimal anticode by [Rav16b, Theorem 54]. The result now follows from Corollary 5.24, after observing that the dual of a product is the product of the duals. □

Notice that Corollary 5.24 and Proposition 5.25 cannot be extended to the case $q = 2$ and $m_{\ell-2} = 1$, since for $n \geq 3$ there exist optimal anticodes in $\mathbb{F}_2^n$ which are not products of zeroes and copies of $\mathbb{F}_2$, and whose dual is not an optimal anticode.

Example 5.26. Let $n \geq 3$ be odd and let $C \subseteq \mathbb{F}_2^n$ be the even-weight code. Then C is an optimal anticode since $\dim(C) = n - 1 = \mathrm{maxwt}(C)$. Its dual $C^\perp$ is the repetition code, which is not an optimal anticode since $\dim(C^\perp) = 1 < n = \mathrm{maxwt}(C^\perp)$.

The next corollary is immediate, after observing that every optimal anticode in the rank metric is equivalent to a standard optimal anticode, see e.g. [Gor21, Section 3].

Corollary 5.27. For $i \in [\ell]$ let $\mathcal{A}_i \subseteq \mathbb{F}_q^{m_i \times n_i}$ be an optimal anticode and let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_\ell \subseteq \mathbb{M}$. Then $\mathcal{A}$ is equivalent to

$$\prod_{i=1}^{\ell} \langle E_{k,l} \mid k \in [m_i], l \in [u_i] \rangle,$$

where $u_i = \mathrm{maxrk}(\mathcal{A}_i)$.

5.3. Generalized weights

In this section we define generalized weights in the sum-rank metric and establish some of their basic properties, including a weak monotonicity along the lines of the corresponding result for rank-metric codes. In addition, we prove that they satisfy Wei's Duality if $m_1 = \ldots = m_\ell$. For general m_i's, we show by means of an example that the generalized weights of a code do not determine those of its dual, hence Wei's Duality cannot hold.

Definition 5.28. Let $C \subseteq \mathbb{M}$ be a sum-rank metric code. For each $r \in [\dim(C)]$, we define the r-th generalized sum-rank weight of C as

$$\begin{aligned} d_r(C) = \min\{ \mathrm{maxsrk}(\mathcal{A}) : \mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_\ell \text{ where } \\ \mathcal{A}_i \subseteq \mathbb{F}_q^{m_i \times n_i} \text{ are optimal anticodes and } \dim(C \cap \mathcal{A}) \geq r\}. \end{aligned}$$

Notice that if $m_1 = \cdots = m_\ell = m$, then

$$\begin{aligned} d_r(C) = \frac{1}{m} \min\{ \dim(\mathcal{A}) : \mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_\ell \text{ where } \\ \mathcal{A}_i \subseteq \mathbb{F}_q^{m \times n_i} \text{ are optimal anticodes and } \dim(C \cap \mathcal{A}) \geq r\}. \end{aligned} \tag{5.3.7}$$

Remark 5.29. We could have defined $d_r(C)$ to be

$$d'_r(C) = \min\{\mathrm{maxsrk}(\mathcal{A}) : \mathcal{A} \text{ an optimal anticode and } \dim(C \cap \mathcal{A}) \geq r\}.$$

For either $q \neq 2$ or $m_{\ell-2} > 1$ we have that $d_r(C) = d'_r(C)$ as, by Corollary 5.24, $\mathcal{A}$ is an optimal anticode if and only if $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_\ell$ for $\mathcal{A}_i$ optimal anticode in $\mathbb{F}_q^{m_i \times n_i}$. In the case $q = 2$ and $m_{\ell-2} = 1$ one has

$$d'_r(C) \leq d_r(C).$$

Notice moreover that $d_r(C)$ recovers the Hamming weights, since the cardinality of a support of a code is the minimum dimension of a code which contains it and is a product of copies of $\mathbb{F}_q$ and zeros. If $q = 2$, then $d'_r(C)$ does not recover the Hamming weights, as there are optimal binary anticodes which are not a product of copies of $\mathbb{F}_2$ and zeros. See also the example following Theorem 10 in [Rav16a].

Remark 5.30. It follows from the definition that the generalized weights are invariant under code equivalence.

As an example, we compute the generalized weights of optimal anticodes.

Example 5.31. For $i \in [\ell]$ let $\mathcal{A}_i \subseteq \mathbb{F}_q^{m_i \times n_i}$ be an optimal anticode and let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_\ell \subseteq \mathbb{M}$ with $\dim \mathcal{A}_i = m_i u_i$. By Corollary 5.27 and the previous remark, $d_r(\mathcal{A}) = d_r(\mathcal{A}')$ for $r \in [\dim(C)]$, where $\mathcal{A}' = \prod_{i=1}^{\ell} \langle E_{k,l} \mid (k,l) \in [m_i] \times [u_i] \rangle$. Let $j \in [\ell]$, $0 \leq \delta \leq u_j - 1$, $r = \sum_{i=1}^{j-1} m_i u_i + m_j \delta$. Then

$$d_{r+1}(\mathcal{A}) = \ldots = d_{r+m_j}(\mathcal{A}) = u_1 + \ldots + u_{j-1} + \delta + 1.$$

Lemma 5.32. Let $m_1 \geq \ldots \geq m_\ell \in \mathbb{N}$, $u_1, \ldots, u_\ell, u'_1, \ldots, u'_\ell \in \mathbb{R}_{\geq 0}$ such that $\sum_{i=1}^{\ell} u_i = \sum_{i=1}^{\ell} u'_i$ and such that there exists k with $u_i \geq u'_i$ for all $1 \leq i \leq k$ and $u_i \leq u'_i$ for all $k < i \leq \ell$, then $\sum_{i=1}^{\ell} m_i u_i \geq \sum_{i=1}^{\ell} m_i u'_i$.

Proof. Since $\sum_{i=1}^{k}(u_i - u'_i) = \sum_{i=k+1}^{\ell}(u'_i - u_i)$ and $m_1 \geq \ldots \geq m_\ell$, then

$$\sum_{i=1}^{k} m_i(u_i - u'_i) \geq m_k \sum_{i=1}^{k}(u_i - u'_i) \geq m_{k+1} \sum_{i=k+1}^{\ell}(u'_i - u_i) \geq \sum_{i=k+1}^{\ell} m_i(u'_i - u_i),$$

which proves the thesis. □

In the next proposition we establish some basic properties of generalized weights. This generalizes Proposition 1.4 to sum-rank metric codes. Moreover, notice that in the case $m_1 = \ldots = m_\ell$ one gets inequalities of the same form as those in [Rav16a, Theorem 30].

Proposition 5.33. Let $0 \neq C \subseteq \mathcal{D} \subseteq \mathbb{M}$, then:

1. $d_1(C) = d(C)$,
2. $d_r(C) \leq d_s(C)$ for $1 \leq r \leq s \leq \dim(C)$,
3. $d_r(C) \geq d_r(\mathcal{D})$ for $r \in [\dim(C)]$,
4. $d_{\dim(C)}(C) \leq n_1 + \cdots + n_\ell$,
5. $d_{r+n_1 m_1+\cdots+n_{j-1}m_{j-1}+\delta m_j}(C) \geq d_r(C) + n_1 + \cdots + n_{j-1} + \delta$ for $j \in [\ell]$, $r \in [\dim(C) - (n_1 m_1 + \cdots + n_{j-1} m_{j-1} + \delta m_j)]$, and $0 \leq \delta \leq n_j - 1$.

Proof. 1. Let $C = (C_1, \ldots, C_\ell) \in C$ be an element of minimum sum-rank. Let $\mathcal{A}_i$ be an optimal anticode of $\dim(\mathcal{A}_i) = m_i \mathrm{rk}(C_i)$ containing C_i and let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_\ell$. Then $C \cap \mathcal{A} \neq 0$, hence $d_1(C) \leq d(C)$. To prove that they are equal, observe that if $\mathcal{A}'$ is an optimal anticode with $\mathrm{maxsrk}(\mathcal{A}') < d(C)$, then $\mathcal{A}' \cap C = 0$.

2., 3., and 4. follow directly from the definition.

5. Let $s = r + n_1m_1 + \cdots + n_{j-1}m_{j-1} + \delta m_j$. Let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_\ell$ be an optimal anticode such that $\dim(C \cap \mathcal{A}) \geq s$ and $d_s(C) = \operatorname{maxsrk}(\mathcal{A})$. For $i \in [\ell]$, write $\dim(\mathcal{A}_i) = m_iu_i$. Since

$$\sum_{i=1}^{\ell} m_iu_i = \dim(\mathcal{A}) \geq \dim(C \cap \mathcal{A}) \geq s > n_1m_1 + \cdots + n_{j-1}m_{j-1} + \delta m_j$$

and $m_1 \geq \cdots \geq m_\ell$, then $d_s(C) = u_1 + \ldots + u_\ell > n_1 + \cdots + n_{j-1} + \delta$ by Lemma 5.32. Let $v_1, \ldots, v_\ell$ be such that $n_1 + \cdots + n_{j-1} + \delta = v_1 + \cdots + v_\ell$ and $v_i \leq u_i$ for $i \in [\ell]$. We have that $n_1m_1 + \cdots + n_{j-1}m_{j-1} + \delta m_j \geq v_1m_1 + \cdots + v_\ell m_\ell$, since $m_1 \geq \cdots \geq m_\ell$. For all $i \in [\ell]$ there exist optimal anticodes $\mathcal{A}'_i \subseteq \mathcal{A}_i$ of $\dim(\mathcal{A}'_i) = m_i(u_i - v_i)$. Let $\mathcal{A}' = \mathcal{A}'_1 \times \cdots \times \mathcal{A}'_\ell$, then

$$\dim(C \cap \mathcal{A}') \geq s - (v_1m_1 + \cdots + v_\ell m_\ell) \geq s - (n_1m_1 + \cdots + n_{j-1}m_{j-1} + \delta m_j) = r$$

hence

$$d_r(C) \leq \sum_{i=1}^{\ell}(u_i - v_i) = d_s(C) - (n_1 + \cdots + n_{j-1} + \delta). \qquad \square$$

From parts 4. and 5. of Proposition 5.33, we easily obtain the following Singleton-type bound. This bound will be improved in Theorem 6.2.

Corollary 5.34. Let $j \in [\ell]$, $0 \leq \delta \leq n_j - 1$, $0 \leq s \leq m_j - 1$, and let $C \subseteq \mathbb{M}$ be a non-trivial code of

$$\dim(C) = \sum_{i=1}^{j-1} m_in_i + \delta m_j + s.$$

Then

$$d(C) \leq \sum_{i=j}^{\ell} n_i - \delta + \begin{cases} 1 & \text{if } s = 0 \\ 0 & \text{else.} \end{cases}$$

The next lemma will be useful in the next chapter for computing the generalized weights of an MSRD code.

Lemma 5.35. Let $C \subseteq \mathbb{M}$ be a code and let $k \in [\ell]$, $r + m_k \in [\dim(C)]$. If

$$d_{r+m_k}(C) > \sum_{i=1}^{k-1} n_i$$

then

$$d_{r+m_k}(C) \geq d_r(C) + 1.$$

Proof. Let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_\ell$ be an optimal anticode such that $\operatorname{maxsrk}(\mathcal{A}) = d_{r+m_k}(C)$ and $\dim(C \cap \mathcal{A}) \geq r + m_k$. We claim that there exists $k \leq j \leq \ell$ such that $\mathcal{A}_j \neq 0$. In fact, if this were not the case, then

$$\sum_{i=1}^{k-1} n_i \geq \operatorname{maxsrk}(\mathcal{A}) = d_{r+m_k}(C).$$

Let $\mathcal{A}' \subseteq \mathcal{A}$ be an optimal anticode such that

$$\dim(\mathcal{A}') = \dim(\mathcal{A}) - m_j$$

and

$$\mathrm{maxsrk}(\mathcal{A}') = \mathrm{maxsrk}(\mathcal{A}) - 1.$$

One has

$$\dim(C \cap \mathcal{A}') \geq \dim(C \cap \mathcal{A}) - m_j \geq r + m_k - m_j \geq r,$$

hence

$$d_r(C) \leq \mathrm{maxsrk}(\mathcal{A}') = d_{r+m_k}(C) - 1.$$

□

The next theorem extends Wei's Duality Theorem [Wei91, Theorem 3] and [Rav16a, Corollary 38]. Let $m_1 = \ldots = m_\ell = m$ and let $C \subseteq \mathbb{M}$ be a sum-rank metric code. For any $r \in \mathbb{Z}$ define

$$T_r(C) = \{d_{r+sm}(C) : s \in \mathbb{Z}, r + sm \in [\dim(C)]\},$$
$$\overline{T}_r(C) = \left\{n + 1 - d_{r+sm}(C) : s \in \mathbb{Z}, r + sm \in [\dim(C)]\right\}.$$

The same arguments as in [Rav16a, Corollary 38] together with Proposition 5.33 prove the next theorem.

Theorem 5.36. Let $m_1 = \ldots = m_\ell = m$, $r \in [m]$, and let $C \subseteq \mathbb{M}$ be a sum-rank metric code. Then

$$T_r(C^\perp) = [n] \backslash \overline{T}_{r+\dim(C)}(C).$$

In particular the generalized weights of a sum rank metric code C determine the generalized weights of $C^\perp$.

The next two example show that the generalized weights of a code do not determine those of its dual for arbitrary m_i's.

Example 5.37. Consider the codes $C_1, C_2 \subseteq \mathbb{F}_2^{2\times 2} \times \mathbb{F}_2$ given by

$$C_1 = \left\{\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, a\right) : a \in \mathbb{F}_2\right\} \text{ and } C_2 = \left\{\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, 0\right) : a \in \mathbb{F}_2\right\}.$$

The corresponding duals are

$$C_1^\perp = \left\{\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, 0\right) : (a,b,c,d) \in \mathbb{F}_2^4\right\} \text{ and } C_2^\perp = \left\{\left(\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}, a\right) : (a,b,c,d) \in \mathbb{F}_2^4\right\}.$$

One has that $d_1(C_1) = d_1(C_2) = 1$, while $d_4(C_1^\perp) = 2$ and $d_4(C_2^\perp) = 3$.

Example 5.38. Let $C_1, C_2 \subseteq \mathbb{F}_2^{3\times 1} \times \mathbb{F}_2^{2\times 2}$ be given by

$$C_1 = 0 \times \mathbb{F}_2^{2\times 2}$$
$$C_2 = \left\{\left(\begin{pmatrix} a \\ b \\ 0 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}\right) : (a,b,c,d) \in \mathbb{F}_2^4\right\}.$$

One can check that $d_1(C_i) = d_2(C_i) = 1$ and $d_3(C_i) = d_4(C_i) = 2$ for $i = 1, 2$. The corresponding duals

$$C_1^\perp = \mathbb{F}_2^{3\times 1} \times 0$$
$$C_2^\perp = \left\{ \left(\begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ b & c \end{pmatrix} \right) : (a, b, c) \in \mathbb{F}_2^3 \right\}$$

have different generalized weights, as $d_3(C_1^\perp) = 1$ and $d_3(C_2^\perp) = 2$.

Remark 5.39. Notice that the first code in the previous example is an optimal anticode, while the second one is not, as its first component is not an optimal rank-metric anticode. Therefore, the example also shows that in the sum-rank metric there exist codes which have the same dimension and generalized weights as an optimal anticode, without being one. This is in contrast with codes endowed with the rank metric or the Hamming metric, where a code which has the same dimension and generalized weights as an optimal anticode is an optimal anticode.

Remark 5.40. There is another simple situation in which the generalized weights of the dual code are determined by numerical data on the original code. Let $C = C_1 \times \ldots \times C_\ell$, then the generalized weights of C satisfy

$$d_r(C) = \min \left\{ \sum_{i=1}^{\ell} d_{r_i}(C_i) : \sum_{i=1}^{\ell} r_i = r, r_i \in [\dim(C_i)] \right\}.$$

The generalized weights of the rank-metric codes $C_1, \ldots, C_\ell$ determine those of $C_1^\perp, \ldots, C_\ell^\perp$, hence they determine the generalized weights of $C^\perp$.

We conclude this section with a result on the generalized weights of a code which is $\mathbb{F}_{q^m}$-linear or, more generally, $\mathbb{F}_{q^k}$-linear. Let $k = \gcd\{m_1, \ldots, m_\ell\}$. As $k \mid m_i$ for all $i \in [\ell]$, then $\mathbb{F}_{q^{m_1}}^{n_1} \times \cdots \times \mathbb{F}_{q^{m_\ell}}^{n_\ell}$ is a vector space over $\mathbb{F}_{q^k}$. For $i \in [\ell]$, let $\Gamma_i = \{\gamma_{1,i}, \ldots, \gamma_{m_i,i}\}$ be a basis of $\mathbb{F}_{q^{m_i}}$ over $\mathbb{F}_q$. For every $w \in \mathbb{F}_{q^{m_i}}^{n_i}$ define $\Gamma_i(w) \in \mathbb{F}_q^{m_i \times n_i}$ via the identity

$$\begin{pmatrix} \gamma_{1,i} & \ldots & \gamma_{m_i,i} \end{pmatrix} \Gamma_i(w) = w.$$

For every $v = (v_1, \ldots, v_\ell) \in \mathbb{F}_{q^{m_1}}^{n_1} \times \cdots \times \mathbb{F}_{q^{m_\ell}}^{n_\ell}$, define $\Gamma(v) \in \mathbb{M}$ as

$$(\Gamma(v))_i = \Gamma_i(v_i).$$

Let $\mathcal{V} \subseteq \mathbb{F}_{q^{m_1}}^{n_1} \times \cdots \times \mathbb{F}_{q^{m_\ell}}^{n_\ell}$ be a vector space over $\mathbb{F}_{q^k}$. The set $\Gamma(\mathcal{V}) = \{\Gamma(v) : v \in \mathcal{V}\}$ is the sum-rank metric code associated to $\mathcal{V}$ with respect to $\{\Gamma_1, \ldots, \Gamma_\ell\}$. We say that $\Gamma(\mathcal{V})$ is $\mathbb{F}_{q^k}$-linear, see also [Gor21, Definition 11.1.3]. In the next theorem we extend the result in [Rav16a, Theorem 28] to the sum-rank metric case. The statement in particular applies to $\mathbb{F}_{q^m}$-linear codes in the case when $m_1 = \ldots = m_\ell = m$.

Theorem 5.41. Let $k = \gcd\{m_1, \ldots, m_\ell\}$, let $\mathcal{V} \subseteq \mathbb{F}_{q^{m_1}}^{n_1} \times \cdots \times \mathbb{F}_{q^{m_\ell}}^{n_\ell}$ be an $\mathbb{F}_{q^k}$-linear vector space with $\dim_{\mathbb{F}_{q^k}}(\mathcal{V}) = t$. If $m_i > n_i$ for $i \in [\ell]$, then

$$d_{kr+1}(\Gamma(\mathcal{V})) = \ldots = d_{k(r+1)}(\Gamma(\mathcal{V}))$$

for $0 \leq r < t$.

Proof. Write C for $\Gamma(\mathcal{V})$. By Proposition 5.33, $d_{kr+1}(C) \leq \ldots \leq d_{k(r+1)}(C)$. Therefore it suffices to show that $d_{kr+1}(C) = d_{k(r+1)}(C)$. Since $m_i > n_i$ for $i \in [\ell]$, $\mathcal{A}$ is an $\mathbb{F}_{q^k}$-linear code and so $C \cap \mathcal{A}$ is $\mathbb{F}_{q^k}$-linear too. Since the dimension over $\mathbb{F}_q$ of an $\mathbb{F}_{q^k}$-linear vector space is divisible by k, if $\dim(C \cap \mathcal{A}) \geq kr + 1$, then $\dim(C \cap \mathcal{A}) \geq k(r + 1)$. Therefore we conclude that $d_{kr+1}(C) \geq d_{k(r+1)}(C)$. □

Remark 5.42. Although the condition that $m > n$ is missing in the statement of [Rav16a, Theorem 28], it is necessary for the result to hold. In fact, [GLJ22, Example 6.15] is a counterexample to the statement of [Rav16a, Theorem 28] for square matrices.

5.4. Weight distribution

Another important family of invariants of sum-rank metric codes is the sum-rank distribution. The results in this section are not original but we include them for the sake of completeness, since they are related with what we have studied in Section 3.3 for R-linear codes.

Definition 5.43 ([BGLR21, Definition 2.8])**.** Let $C \subseteq \mathbb{M}$ be a sum-rank metric code. For a non-negative integer r, let

$$W_r(C) = |\{C \in C : \mathrm{srk}(C) = r\}| .$$

The sequence $(W_r(C))_{r>0}$ is the sum-rank distribution of C.

The notion of sum-rank distribution generalizes that of weight distribution for linear block codes and rank-metric codes. In contrast to what happens in those two cases, we do not have MacWilliams Identities for sum-rank metric codes, as was first observed in [BGLR21, Example 5.2]. The following is a simple example of this phenomenon.

Example 5.44. Let $C_1, C_2 \subseteq \mathbb{F}_2^{2\times 2} \times \mathbb{F}_2$ be as in Example 5.37. Then, C_1 and C_2 have the same sum-rank distribution. Indeed, $W_0(C_1) = W_0(C_2) = W_1(C_1) = W_1(C_2) = 1$ and $W_1(C_i) = W_1(C_i) = 0$ for $i > 1$. Instead for $C_1^\perp, C_2^\perp$ we have $W_1(C_1^\perp) = 9$ and $W_1(C_2^\perp) = 6$.

It is nevertheless possible to define other partitions of a sum-rank metric code for which there exists a relation between the distribution of a code and the one of its dual.

Definition 5.45 ([BGLR21, Definition 2.8])**.** For a vector $v \in \mathbb{Z}_{\geq 0}^{\ell}$, let

$$W_v(C) = |\{C \in C : \mathrm{rk}(C_i) = v_i \text{ for all } i \in [\ell]\}| .$$

The list $(W_v(C))_{v\in\mathbb{Z}_{\geq 0}^{\ell}}$ is called the rank-list distribution of C.

In the next example we compute the rank-list distribution of the codes of Example 5.37.

Example 5.46. Let $C_1, C_2 \subseteq \mathbb{F}_2^{2\times 2} \times \mathbb{F}_2$ be as in Example 5.37. The rank-list distribution of C_1 is given by $W_{(0,0)}(C_1) = W_{(0,1)}(C_1) = 1$ and $W_u(C_1) = 0$ when $u \neq (0,0), (0,1)$. The rank-list distribution of $C_1^\perp$ is $W_{(0,0)}(C_1^\perp) = 1$, $W_{(1,0)}(C_1^\perp) = 9$, $W_{(2,0)}(C_1^\perp) = 6$ and 0 in all the remaining cases. The rank-list distribution of C_2 is $W_{(0,0)}(C_1) = W_{(1,0)}(C_1) = 1$ and $W_u(C_1) = 0$ when $u \neq (0,0), (1,0)$. The rank-list distribution of $C_2^\perp$ is $W_{(0,0)}(C_2^\perp) = 1$, $W_{(1,0)}(C_2^\perp) = 5$, $W_{(0,1)}(C_2^\perp) = 1$, $W_{(2,0)}(C_2^\perp) = 2$, $W_{(1,1)}(C_2^\perp) = 5$, $W_{(2,1)}(C_2^\perp) = 2$ and 0 in the remaining cases.

The next theorem is the analogue of the MacWilliams Identities for the rank-list distribution. Recall that the q-ary Gaussian coefficient of $a, b \in \mathbb{Z}$ is defined as

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{cases} 0 & \text{if } a < 0,\ b < 0,\ \text{or } b > a \\ 1 & \text{if } b = 0 \text{ and } a \geq 0 \\ \frac{(q^a-1)(q^{a-1}-1)\cdots(q^{a-b+1}-1)}{(q^b-1)(q^{b-1}-1)\cdots(q-1)} & \text{otherwise.} \end{cases}$$

Theorem 5.47 ([BGLR21, Theorem 5.5])**.** Let $C \subseteq \mathbb{M}$ be a code. Then

$$W_v(C^\perp) = \frac{1}{|C|} \sum_{u \in \mathbb{Z}_{\geq 0}^\ell} W_u(C) \sum_{w \leq v} q^{\sum_{i=1}^\ell m_i w_i} \prod_{i=1}^{\ell} (-1)^{v_i - w_i} q^{\binom{v_i - w_i}{2}} \begin{bmatrix} n_i - u_i \\ w_i \end{bmatrix}_q \begin{bmatrix} n_i - w_i \\ v_i - w_i \end{bmatrix}_q$$

for $v \in \mathbb{Z}_{\geq 0}^\ell$.

The next theorem is the sum-rank metric analogue of the binomial moments of MacWilliams identities.

Theorem 5.48 ([BGLR21, Theorem 5.6])**.** Let $C \subseteq \mathbb{M}$ be a code. Then

$$\sum_{u \in \mathbb{Z}_{\geq 0}^\ell} W_u(C) \prod_{i=1}^{\ell} \begin{bmatrix} n_i - u_i \\ v_i - u_i \end{bmatrix}_q = \frac{|C|}{q^{\sum_{i=1}^\ell m_i(n_i - v_i)}} \sum_{u \in \mathbb{Z}_{\geq 0}^\ell} W_u(C^\perp) \prod_{i=1}^{\ell} \begin{bmatrix} n_i - u_i \\ v_i \end{bmatrix}_q$$

for $v \in \mathbb{Z}_{\geq 0}^\ell$.

Notice that for $\ell = 1$ the previous equality becomes

$$\sum_{i=0}^{r} W_i(C) \begin{bmatrix} n - i \\ r - i \end{bmatrix}_q = \frac{|C|}{q^{m(n-r)}} \sum_{i=0}^{n-r} W_i(C^\perp) \begin{bmatrix} n - i \\ r \end{bmatrix}_q,$$

for $r \in [n]$. This is exactly the identity proved in [Rav16b, Theorem 31] for rank metric codes. Moreover in the case $m_1 = \cdots = m_\ell = 1$ we obtain

$$\sum_{\substack{u \in \{0,1\}^\ell \\ u \leq v}} W_u(C) = \frac{|C|}{q^{\ell - \mathrm{wt}(v)}} \sum_{\substack{u \in \{0,1\}^\ell \\ u \leq (1,\ldots,1) - v}} W_u(C^\perp)$$

for $v \in \{0,1\}^\ell$. Summing over all v with $\mathrm{wt}(v) = g$, one finds

$$\sum_{i=0}^{g} W_i(C) \binom{\ell - i}{\ell - g} = \frac{|C|}{q^{\ell - g}} \sum_{i=0}^{\ell - g} W_i(C^\perp) \binom{\ell - i}{g},$$

which is the equation given in [HP03, equation (M1), page 257] for linear block codes.

We conclude this section by discussing an additional definition of distribution for sum-rank metric codes.

Definition 5.49 ([BGLR21, Definition 2.8])**.** Let $C \subseteq \mathbb{M}$ be a code. For $\mathcal{S} \in \mathcal{P}(\mathbb{S})$, let

$$W_{\mathcal{S}}(C) = \left| \{ C \in C : \mathrm{supp}(C) = \mathcal{S} \} \right|.$$

The list $(W_{\mathcal{S}}(C))_{\mathcal{S}\in\mathcal{P}(\mathbb{S})}$ is the support distribution of C.

Similarly to the rank-list distribution, the support distribution of a code determines that of its dual.

Theorem 5.50 ([BGLR21, Theorem 5.4]). Let $C \subseteq \mathbb{M}$ be a code. Let $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_\ell \in \mathcal{P}(\mathbb{S})$ and $v = (\dim(\mathcal{S}_1), \ldots, \dim(\mathcal{S}_\ell))$. Then

$$W_{\mathcal{S}}(C^\perp) = \frac{1}{|C|} \sum_{\mathcal{H}\in\mathcal{P}(\mathbb{M})} W_{\mathcal{H}}(C) \sum_{u\leq v} q^{\sum_{i=1}^{\ell} m_i u_i} \prod_{i=1}^{\ell} (-1)^{v_i-u_i} q^{\binom{v_i-u_i}{2}} \begin{bmatrix} \dim(\mathcal{H}_i \cap \mathcal{S}_i) \\ u_i \end{bmatrix}_q .$$

6. Maximum sum-rank metric distance codes

Several constructions of sum-rank metric codes exist in the literature. The first constructions were mainly of convolutional codes, see [NS18] for a survey and references. In this chapter we are interested in studying sum-rank metric codes with a large minimum distance. A trivial Singleton Bound on the minimum sum-rank distance of linear codes may be immediately derived from the classical Singleton Bound on minimum Hamming distance [MP18, Prop. 34]. Any code attaining the Singleton Bound for the rank metric (i.e., any maximum rank-distance (MRD) code, including Gabidulin codes [Del78, Gab85, Rot91]) also attains it for the sum-rank metric, that is, it is also a maximum sum-rank distance (MSRD) code. However, the parameters of MRD codes (including the matrix sizes) are very strongly restricted. Furthermore, their decoding algorithms are over finite fields whose sizes are exponential in the code length (i.e., the total number of columns), making such decoding algorithms slow for large parameters. What makes the study of MSRD codes interesting is that there are MSRD codes not coming from MRD codes and attaining a wider range of parameters, including codes [MP18] with decoding algorithms over finite fields of sub-exponential size [MPK19]. Since then, other families of sum-rank metric codes have been found and studied [MP21, MPK19, Ner22, BGLR22, BGLR21]. However, previous works, with the exception of [BGLR22, BGLR21], consider sum-rank metric codes where the number of columns and/or rows are equal at different positions. A general Singleton Bound for arbitrary numbers of columns and rows was given in [BGLR21, Th. 3.2], together with corresponding MSRD codes for certain parameter ranges [BGLR21, Sec. 7].

In Section 6.1 we define MSRD and we studied their main properties. In Section 6.2 we compute their generalized weights and we introduce the notion of r-MSRD codes. Finally, in Section 6.3 we show how to construct MSRD codes for certain parameters.

6.1. Singleton Bound for sum-rank metric codes

Let $\mu \in [n]$. We denote by $\mathbb{A}(\mu)$ the set of optimal anticodes of the form $\mathcal{A} = \mathcal{A}_1 \times \ldots \times \mathcal{A}_\ell \subseteq \mathbb{M}$, with $\mathcal{A}_i \subseteq \mathbb{F}_q^{m_i \times n_i}$ optimal rank-metric anticode for all $i \in [\ell]$ and $\text{maxsrk}(\mathcal{A}) = \sum_{i=1}^{\ell} \text{maxrk}(A_i) = \mu$. The next result follows from Lemma 5.32.

Lemma 6.1. Let $\mu \in [n]$ and write $\mu = \sum_{i=1}^{j-1} n_i + \delta = \sum_{i=l+1}^{\ell} n_i + \delta'$ for some $j, l \in [\ell]$, $\delta \in [n_j]$, and $\delta' \in [n_l]$. Then

$$\min_{\mathcal{A} \in \mathbb{A}(\mu)} \dim(\mathcal{A}) = \sum_{i=l+1}^{\ell} m_i n_i + \delta' m_l$$

and

$$\max_{\mathcal{A} \in \mathbb{A}(\mu)} \dim(\mathcal{A}) = \sum_{i=1}^{j-1} m_i n_i + \delta m_j.$$

Moreover, if

$$\min_{\mathcal{A}\in\mathbb{A}(\mu)} \dim(\mathcal{A}) = \max_{\mathcal{A}\in\mathbb{A}(\mu)} \dim(\mathcal{A}),$$

then either $\mu = n$ or $m_1 = \ldots = m_\ell$.

Let $\mu \in [n]$ and write $\mu = \sum_{i=1}^{j-1} n_i + \delta + 1$, $0 \le \delta \le n_j - 1$. Throughout the section, we denote

$$r_\mu = \max_{\mathcal{A}\in\mathbb{A}(\mu)} \dim(\mathcal{A}) = \sum_{i=1}^{j-1} m_i n_i + (\delta+1)m_j.$$

The Singleton Bound for rank-metric codes was first proved in [Del78, Theorem 5.4]. A Singleton Bound for sum-rank metric codes was established in [BGLR21, Theorem 3.2], for codes which are not necessarily linear. Our next theorem generalizes the previous results in the case of linear sum-rank metric codes.

Theorem 6.2. Let $C \subseteq \mathbb{M}$ be a code and let $r \in [\dim(C)]$. Let $j \in [\ell]$ and $0 \le \delta \le n_j - 1$ be such that

$$d_r(C) - 1 \ge \sum_{i=1}^{j-1} n_i + \delta.$$

Then

$$\dim(C) \le \sum_{i=j}^{\ell} m_i n_i - m_j\delta + r - 1. \tag{6.1.1}$$

Proof. Let $\mathcal{A}_i = \mathbb{F}_q^{m_i\times n_i}$ for $i \in [j-1]$, let $\mathcal{A}_j \subseteq \mathbb{F}_q^{m_j\times n_j}$ be an optimal anticode of dimension δm_j, and let $\mathcal{A}_i = 0$ for $j+1 \le i \le \ell$. Let $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_\ell$, then

$$\dim(C \cap \mathcal{A}) \le r - 1.$$

Therefore

$$\dim(C) + \sum_{i=1}^{j-1} m_i n_i + m_j\delta - r + 1 \le \dim(C) + \dim(\mathcal{A}) - \dim(C\cap\mathcal{A}) = \dim(C+\mathcal{A}) \le \sum_{i=1}^{\ell} m_i n_i.$$

□

Theorem 6.2 yields upper bounds on all the generalized weights of C.

Corollary 6.3. Let $C \subseteq \mathbb{M}$ be a code and let $r \in [\dim(C)]$, $j \in [\ell]$, and $0 \le \delta \le n_j - 1$ be such that $\dim(C) \ge \sum_{i=j}^{\ell} m_i n_i - m_j\delta + r$. Then

$$d_r(C) \le \sum_{i=1}^{j-1} n_i + \delta.$$

In particular, if $\dim(C) = \sum_{i=j}^{\ell} m_i n_i - m_j\delta$, then

$$d_1(C) \le \ldots \le d_{m_j}(C) \le \sum_{i=1}^{j-1} n_i + \delta + 1.$$

Corollary 6.3 suggests the following definition of MSRD code. The same definition was given in [BGLR21, Definition 3.3] for codes which are not necessarily linear.

Definition 6.4. A code C is MSRD if there exist $j \in [\ell]$ and $0 \le \delta \le n_j - 1$ such that

$$d(C) = \sum_{i=1}^{j-1} n_i + \delta + 1 \quad \text{and} \quad \dim(C) = \sum_{i=j}^{\ell} m_i n_i - \delta m_j.$$

Next we study some properties which are closely related to being MSRD.

$(C0)$ For any $\mathcal{A}$ optimal anticode of $\operatorname{maxsrk}(\mathcal{A}) = d(C) - 1$ and $\dim(\mathcal{A}) = r_{d(C)-1}$ one has $C + \mathcal{A} = \mathbb{M}$.

$(C1)$ The code C has $\dim(C) = \sum_{i=j}^{\ell} m_i n_i - m_j \delta$ and for any $\mathcal{A}$ optimal anticode of $\operatorname{maxsrk}(\mathcal{A}) \le \sum_{i=1}^{j-1} n_i + \delta$ one has $C \cap \mathcal{A} = 0$.

$(C2)$ For any $\mathcal{A} \in \mathbb{A}(d(C))$, let $k = \max\{i \in [\ell] \mid \mathcal{A}_i \neq 0\}$. Then

$$\dim(C \cap \mathcal{A}) \ge m_k.$$

$(C3)$ The code C has $d(C) + d(C^\perp) = n + 2$.

It is clear that being MSRD is equivalent to satisfying $(C0)$. We now show that it is also equivalent to satisfying $(C1)$.

Proposition 6.5. Let $j \in [\ell]$ and $0 \le \delta \le n_j - 1$. Let $0 \neq C \subseteq \mathbb{M}$ be a code. Then C is MSRD if and only if it satisfies $(C1)$.

Proof. Suppose that C is MSRD of $\dim(C) = \sum_{i=j}^{\ell} m_i n_i - \delta m_j$. Let $\mathcal{A}$ be an optimal anticode of $\operatorname{maxsrk}(\mathcal{A}) \le d(C) - 1$. Then $C \cap \mathcal{A} = 0$ since, for every $0 \neq C \in C$, one has $\operatorname{srk}(C) \ge d(C) > \operatorname{maxsrk}(\mathcal{A})$, so $C \notin \mathcal{A}$.

Suppose now that C satisfies $(C1)$. Then $d(C) \le \sum_{i=1}^{j-1} n_i + \delta + 1$ by Corollary 6.3. Let $C = (C_1, \dots, C_\ell) \in C$. For each $i \in [\ell]$, there is an optimal rank-metric anticode $\mathcal{A}_i \subseteq \mathbb{F}_q^{m_i \times n_i}$ of $\dim(\mathcal{A}_i) = m_i \operatorname{rk}(C_i)$ which contains C_i. Therefore $\mathcal{A} = \mathcal{A}_1 \times \ldots \times \mathcal{A}_\ell$ is an optimal sum-rank metric anticode of $\operatorname{maxsrk}(\mathcal{A}) = \operatorname{srk}(C)$ which contains C. Since $C \cap \mathcal{A} \neq 0$, it must be that $\operatorname{maxsrk}(\mathcal{A}) = \operatorname{srk}(C) \ge \sum_{i=1}^{j-1} n_i + \delta + 1$, therefore C is MSRD. □

Proposition 6.6. Let $0 \neq C \subseteq \mathbb{M}$ be a code and write its minimum distance as $d = d(C) = \sum_{i=1}^{j-1} n_i + \delta + 1$, where $j \in [\ell]$ and $0 \le \delta \le n_j - 1$. For $S \subseteq [n]$, denote by $\mathbb{F}_q[S]$ the set of elements of $\mathbb{M}$ which are zero outside of the columns indexed by S. For any $d \le h \le n$, let $S_h := [d-1] \cup \{h\}$. The following hold:

1. C is MSRD if and only if for any $d \le h \le n$ we have

$$\dim(C \cap \mathbb{F}_q[S_h]) = m_k$$

where $k = \max\{\nu \mid \sum_{i=1}^{\nu-1} n_i < h\}$.

2. If C satisfies $(C2)$, then C is MSRD.

Proof. 1. Assume that C is MSRD and let $d \leq h \leq n$. We have

$$\begin{aligned}\dim(C \cap \mathbb{F}_q[S_h]) &\geq \dim(C) + \dim(\mathbb{F}_q[S_h]) - \sum_{i=1}^{\ell} m_i n_i \\ &= \sum_{i=j}^{\ell} m_i n_i - \delta m_j + \sum_{i=1}^{j-1} m_i n_i + \delta m_j + m_k - \sum_{i=1}^{\ell} m_i n_i = m_k.\end{aligned}$$

Conversely, suppose that for $d \leq h \leq n$ one has $\dim(C \cap \mathbb{F}_q[S_h]) \geq m_k$. Let $d \leq h' \leq n$, $h \neq h'$. Then

$$\dim(C \cap \mathbb{F}_q[S_h] \cap \mathbb{F}_q[S_{h'}]) = \dim(C \cap \mathbb{F}_q[[d-1]]) = 0$$

hence

$$\dim(C) \geq \sum_{h=d}^{n} \dim(C \cap \mathbb{F}_q[S_h]) \geq \sum_{i=j}^{\ell} m_i n_i - \delta m_j. \tag{6.1.2}$$

Theorem 6.2 gives the reverse inequality, hence C is MSRD.

This proves that C is MSRD if and only if $\dim(C \cap \mathbb{F}_q[S_h]) \geq m_k$ for all $d \leq h \leq n$. Notice moreover that (6.1.2) and Theorem 6.2 imply that, if $\dim(C \cap \mathbb{F}_q[S_h]) \geq m_k$ for all $d \leq h \leq n$, then in fact $\dim(C \cap \mathbb{F}_q[S_h]) = m_k$ for all $d \leq h \leq n$. This concludes the proof of the first part of the statement.

2. Suppose that C satisfies (C2). For any $d \leq h \leq n$, letting $\mathcal{A} = \mathbb{F}_q[S_h] \in \mathbb{A}(d)$, one has that $\dim(C \cap \mathbb{F}_q[S_h]) \geq m_k$. As shown in 1., combining (6.1.2) and Theorem 6.2 one obtains that C is MSRD. □

The next examples show that there exist nontrivial codes which satisfy property $(C2)$ and that not every MSRD code satisfies $(C2)$.

Example 6.7. In $\mathbb{F}_2^{2\times 2} \times \mathbb{F}_2^{1\times 1}$, let

$$C = \left\langle \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 1 \right), \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right), \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1 \right) \right\rangle.$$

We have $d(C) = 2$ and C satisfies $(C2)$.

Example 6.8. Let $C \subseteq \mathbb{F}_2^{3\times 3} \times \mathbb{F}_2^{2\times 2} \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$ be

$$C = \left\langle \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1, 1, 0 \right), \left(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, 0, 1, 1 \right) \right\rangle.$$

The code C has dimension 2 with $d(C) = 7$, hence it is an MSRD code. Consider now the optimal anticode

$$\mathcal{A} = \langle E_{i,1}, E_{i,2} \mid i \in [3] \rangle \times \mathbb{F}_2^{2\times 2} \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2.$$

We have $\mathrm{maxsrk}(\mathcal{A}) = 7$ and $\mathcal{A} \cap C = 0$. Hence C does not satisfy $(C2)$.

Proposition 6.9. Let $C \subseteq \mathbb{M}$ be a non-trivial code. Then C satisfies $(C3)$ if and only if both C and $C^{\perp}$ are MSRD.

Proof. Write $\dim(C) = \sum_{i=j}^{\ell} m_i n_i - \delta m_j - s$ for some $j \in [\ell]$, $0 \le \delta \le n_j - 1$, and $0 \le s \le m_j - 1$. By Corollary 6.3

$$d_1(C) \le \sum_{i=1}^{j-1} n_i + \delta + 1. \tag{6.1.3}$$

Moreover, $\dim(C^\perp) = \dim(\mathbb{M}) - \dim(C) = \sum_{i=1}^{j-1} m_i n_i + \delta m_j + s$, which by Corollary 5.34 implies that

$$d_1(C^\perp) \le \sum_{i=j}^{\ell} n_j - \delta + \begin{cases} 1 & \text{if } s = 0 \\ 0 & \text{else.} \end{cases} \tag{6.1.4}$$

Therefore

$$d(C) + d(C^\perp) \le \begin{cases} n+2 & \text{if } s = 0 \\ n+1 & \text{else.} \end{cases}$$

If C satisfies $(C3)$, then $s = 0$ and both C and $C^\perp$ are MSRD. Conversely, if C and $C^\perp$ are MSRD, then $s = 0$ and both (6.1.3) and (6.1.4) are equalities. It follows that C satisfies $(C3)$. □

In the next proposition we prove that, if $m_1 = \ldots = m_\ell$, then properties $(C2)$ and $(C3)$ are equivalent to being MSRD.

Proposition 6.10. Let $C \subseteq \mathbb{M}$ be a non-trivial code. If $m_1 = \ldots = m_\ell = m$, then both $(C2)$ and $(C3)$ are equivalent to being MSRD. In particular, the dual of an MSRD code is MSRD.

Proof. Let $C \subseteq \mathbb{M}$ be a non-trivial code. If C is MSRD, then it satisfies $(C3)$ by [BGLR21, Theorem 6.1]. If C satisfies property $(C3)$, then it is MSRD by Proposition 6.9.

If C satisfies $(C2)$, then it is MSRD by Proposition 6.6. We now prove that if C is MSRD, then it satisfies $(C2)$. Let $\mathcal{A} \in \mathbb{A}(d(C))$, then

$$\dim(C) + \dim(\mathcal{A}) \le mn + \dim(C \cap \mathcal{A}).$$

Hence by Lemma 6.1 we have

$$mn + m \le mn + \dim(C \cap \mathcal{A}),$$

so C satisfies $(C2)$. □

Moreover, one can prove that $(C3)$ defines a trivial family of codes, unless $m_1 = \cdots = m_\ell$. Notice that this shows in particular that the dual of a non-trivial MSRD code can never be MSRD, unless $m_1 = \cdots = m_\ell$.

Proposition 6.11. If there exists a non-trivial code $C \subseteq \mathbb{M}$ that satisfies $(C3)$, then $m_1 = \cdots = m_\ell$.

Proof. Write $d(C^\perp) - 1 = \sum_{i=1}^{k-1} n_i + \varepsilon$ for some $k \in [\ell]$ and $0 \le \varepsilon \le n_k - 1$. Since $d(C) + d(C^\perp) - 2 = n$, one has

$$d(C) - 1 = \sum_{i=1}^{j-1} n_i + \delta = \sum_{i=k}^{\ell} n_i - \varepsilon \tag{6.1.5}$$

for some $j \in [\ell]$ and $0 \leq \delta \leq n_j - 1$. Since C and $C^\perp$ are MSRD by Proposition 6.9, one has

$$\dim(C) = \sum_{i=j}^{\ell} n_i m_i - \delta m_j = sum_{i=1}^{k-1} n_i m_i + \varepsilon m_k = \dim(\mathbb{M}) - \dim(C^\perp). \tag{6.1.6}$$

Lemma 6.1, together with (6.1.6), implies that

$$\max \dim \mathbb{A}(d(C) - 1) = \min \dim \mathbb{A}(d(C) - 1),$$

which by Lemma 6.1 implies that $m_1 = \cdots = m_\ell$. □

6.2. Generalized weights of MSRD codes

In this section, we study the generalized weights of MSRD codes and propose a definition of r-MSRD codes, analogous to that of r-MRD codes. The next theorem states that the generalized weights of an MSRD code are determined by its parameters. This generalizes similar results for MDS codes in the Hamming metric and MRD codes in the rank metric. We postpone the proof, since in Theorem 6.17 we will prove a more general result.

Theorem 6.12. Let $C \subseteq \mathbb{M}$ be an MSRD code and write $d(C) = \sum_{i=1}^{j-1} n_i + \delta + 1$ for some $j \in [\ell]$ and $0 \leq \delta \leq n_j - 1$. Let $d(C) \leq h \leq n$ and let $k = \max\{v \mid \sum_{i=1}^{v-1} n_i < h\}$. Let $r \in [\dim(C)]$ be of the form

$$r = r_h - r_{d(C)-1} - m_k + 1.$$

Then

$$d_r(C) = \cdots = d_{r+m_k-1}(C) = h.$$

Remark 6.13. One can also write down the generalized weights computed in Theorem 6.12 as follows. Let $j \in [\ell]$, $0 \leq \delta \leq n_j - 1$, and let $C \subseteq \mathbb{M}$ be an MSRD code with $d(C) = \sum_{i=1}^{j-1} n_i + \delta + 1$ and $\dim(C) = \sum_{i=j}^{\ell} m_i n_i - \delta m_j$. Write

$$h = \sum_{i=1}^{k-1} n_i + \varepsilon + 1$$

where $k \geq j$. Since $d(C) \leq h \leq n$, one has that $\delta \leq \varepsilon \leq n_j - 1$ if $k = j$, and $0 \leq \varepsilon \leq n_k - 1$ if $k > j$. Then

$$r = (\varepsilon - \delta) m_j + 1$$

if $k = j$ and $\delta \leq \varepsilon \leq n_j - 1$, and

$$r = (n_j - \delta) m_j + \sum_{i=j+1}^{k-1} m_i n_i + \varepsilon m_k + 1$$

if $j < k \leq \ell$ and $0 \leq \varepsilon \leq n_k - 1$.

Remark 6.14. It follows from Theorem 6.12 that both bounds in the statement of Theorem 6.2

are met for $r \in [\dim(C)]$ of the form $r = 1, m_j + 1, \ldots, (n_j - \delta - 1)m_j + 1$, and

$$r = (n_j - \delta)m_j + \sum_{i=j+1}^{k-1} m_i n_i + \varepsilon m_k + 1$$

with $j < k \leq \ell$ and $0 \leq \varepsilon \leq n_k - 1$.

Remark 6.15. Let $d_0(C) = 0$ and $d_{\dim(C)+1}(C) = n + 1$. Theorem 6.12 states that, for any $d(C) \leq h \leq n$ and r of the form $r = r_h - r_{d(C)-1} - m_k + 1$, we have

$$d_{r-1}(C) < d_r(C) = \ldots = d_{r+m_k-1}(C) < d_{r+m_k}(C).$$

Inspired by Remark 6.15 and by the definition of r-MRD codes, we define a notion of r-MSRD code as follows. Notice that being 1-MSRD is equivalent to being MSRD.

Definition 6.16. Let $j \in [\ell]$, $0 \leq \delta \leq n_j - 1$, and let $C \subseteq \mathbb{M}$ be a code of $\dim(C) = \sum_{i=j}^{\ell} m_i n_i - \delta m_j$. Define $d_{\max} = \sum_{i=1}^{j-1} n_i + \delta + 1$, let $d_{\max} \leq h \leq n$ and

$$r = r_h - r_{d_{\max}-1} - m_k + 1,$$

where $k = \max\{v \mid \sum_{i=1}^{v-1} n_i < h\}$. We say that C is r-MSRD if

$$d_r(C) = h.$$

We conclude this section by showing that, if C is r-MSRD, then C is r'-MSRD for all $r' \geq r$, where r, r' are integers of the form given in Definition 6.16. This observation allows us to compute the generalized weights of an r-MSRD code. Since an MSRD code is 1-MSRD, the proof of next theorem also proves Theorem 6.12.

Theorem 6.17. Let $j \in [\ell]$, $0 \leq \delta \leq n_j - 1$, and let $C \subseteq \mathbb{M}$ be a non-trivial code of $\dim(C) = \sum_{i=j}^{\ell} m_i n_i - \delta m_j$. Define $d_{\max} = \sum_{i=1}^{j-1} n_i + \delta + 1$, let $d_{\max} \leq h \leq n$ and

$$r = r_h - r_{d_{\max}-1} - m_k + 1,$$

where $k = \max\{v \mid \sum_{i=1}^{v-1} n_i < h\}$. If C is r-MSRD, then

$$d_r(C) = \ldots = d_{r+m_k-1}(C) = h.$$

Moreover, C is $(r + m_k)$-MSRD.

Proof. We have

$$h = d_r(C) \leq \ldots \leq d_{r+m_k-1}(C) \leq h,$$

where the equality follows from the definition of r-MSRD code, the first and second inequalities from Proposition 5.33, and the third from Corollary 6.3. Therefore $d_r(C) = \ldots = d_{r+m_k-1}(C) = h$.

Since $d_{r+m_k}(C) \geq d_r(C) = h > \sum_{i=1}^{k-1} n_i$, then by Lemma 5.35

$$d_{r+m_k}(C) \geq d_r(C) + 1 = h + 1.$$

The reverse inequality follows from Corollary 6.3, hence $d_{r+m_k}(C) = h+1$. Since

$$\max\left\{v : \sum_{i=1}^{v-1} n_i < h+1\right\} = \begin{cases} k & \text{if } \varepsilon < n_k - 1, \\ k+1 & \text{if } \varepsilon = n_k - 1, \end{cases}$$

we let

$$m' = \begin{cases} m_k & \text{if } \varepsilon < n_k - 1, \\ m_{k+1} & \text{if if } \varepsilon = n_k - 1. \end{cases}$$

Since $m' = r_{h+1} = r_h$, one has that $r + m_k = r_{h+1} - r_{d_{\max}-1} - m' + 1$, hence we proved that C is $(r+m_k)$-MSRD. □

Remark 6.18. We follow the notation of the last theorem. If a code C is such that $d_r < h$ but $d_{r+s}(C) = h$ for some $1 \leq s \leq m_k - 1$ then by Corollary 6.3 we have

$$d_{r+s}(C) = \cdots = d_{r+m_k-1}(C) = h.$$

However, this does not imply that C is an $(r+m_k)$-MSRD code, as the next example shows.

Example 6.19. An MSRD code $\mathcal{D}$ of dimension 4 in $\mathbb{F}_2^{4\times4} \times \mathbb{F}_2^{4\times2} \times \mathbb{F}_2^{2\times2}$ has weights $d_1(\mathcal{D}) = d_2(\mathcal{D}) = 7$, $d_3(\mathcal{D}) = d_4(\mathcal{D}) = 8$. Let C be generated by the following four elements

$$\left(\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&0\end{pmatrix}\right), \left(\begin{pmatrix}0&0&0&1\\1&0&0&1\\0&1&0&0\\0&0&1&0\end{pmatrix}, \begin{pmatrix}0&1\\1&1\\0&0\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}\right),$$
$$\left(\begin{pmatrix}0&0&1&0\\0&0&1&1\\1&0&0&1\\0&1&0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&0\\1&0\\0&1\end{pmatrix}, 0\right), \left(0, 0, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$$

The code C has dimension 4 and $d_1(C) = 1$, then C is not MSRD. We checked using the computer algebra system Macaulay2 [GS] that the only nonzero codewords of C of sum-rank less than 7 are the third and the fourth element in the previous list. Hence $d_2(C) = d_2(\mathcal{D}) = 7$. Taking $\mathcal{A} = \mathbb{F}_2^{4\times4} \times \mathbb{F}_2^{4\times2} \times \left\{\begin{pmatrix}a&0\\b&0\end{pmatrix} \mid a, b \in \mathbb{F}_2\right\}$ we can see that $d_3(C) = 7 < 8 = d_3(\mathcal{D})$. In particular, C is not 3-MSRD.

6.3. Constructions

We conclude this chapter by discussing the existence of MSRD codes. In [GMPS23, Apendix], one can find an explicit discussion on MSRD code constructions that are linear over an extension field, including linearized Reed-Solomon codes [MP18]. Such constructions satisfy the condition $m_1 = m_2 = \ldots = m_\ell$. Non-trivial constructions of MSRD codes such that not all $m_1, m_2, \ldots, m_\ell$ are equal are given in [BGLR21, Section VII]. In addition, the next results allow us to construct MSRD codes by puncturing and shortening MSRD codes.

Theorem 6.20 ([BGLR21, Theorem VI.14])**.** Suppose there exists an MSRD code $C \subseteq \mathbb{M}$ such that

$$d(C) = \sum_{i=1}^{j-1} n_i + \delta + 1 \quad \text{and} \quad \dim(C) = \sum_{i=j}^{\ell} m_i n_i - \delta m_j,$$

for some $j \in [\ell]$ and $0 \leq \delta \leq n_j - 1$.

1. Choose $s \in \{j, \dots, \ell\}$ and set

$$\widetilde{n}_i = \begin{cases} n_i, & \text{if } i \neq s, \\ n_s - 1, & \text{if } i = s. \end{cases}$$

   There exists an MSRD code with sum-rank distance d in $\widetilde{\mathbb{M}} := \mathbb{F}_q^{m_1 \times \widetilde{n}_1} \times \cdots \times \mathbb{F}_q^{m_\ell \times \widetilde{n}_\ell}$.

2. Choose $s \in \{j+1, \dots, \ell\}$ and set

$$\widetilde{m}_i = \begin{cases} m_i, & \text{if } i \neq s, \\ m_s - 1, & \text{if } i = s. \end{cases}$$

   There exists a linear MSRD code with sum-rank distance d in $\widetilde{\mathbb{M}} := \mathbb{F}_q^{\widetilde{m}_1 \times n_1} \times \cdots \times \mathbb{F}_q^{\widetilde{m}_\ell \times n_\ell}$.

Theorem 6.21 ([BGLR21, Theorem VI.15])**.** Suppose there exists an MSRD code $C \subseteq \mathbb{M}$ such that

$$d(C) = \sum_{i=1}^{j-1} n_i + \delta + 1 \quad \text{and} \quad \dim(C) = \sum_{i=j}^{\ell} m_i n_i - \delta m_j,$$

for some $j \in [\ell]$ and $0 \leq \delta \leq n_j - 1$. If $\delta > 0$, choose $s \in [j]$ and if $\delta = 0$, choose $s \in [j-1]$. Set

$$\widetilde{n}_i = \begin{cases} n_i, & \text{if } i \neq s, \\ n_s - 1, & \text{if } i = s. \end{cases}$$

There exists an MSRD code with sum-rank distance $d - 1$ in $\widetilde{\mathbb{M}} := \mathbb{F}_q^{m_1 \times \widetilde{n}_1} \times \cdots \times \mathbb{F}_q^{m_\ell \times \widetilde{n}_\ell}$.

Theorem 6.20 is obtained via shortening, whereas Theorem 6.21 is obtained via puncturing (or restriction). These operations may be performed e.g. on the MSRD constructed in [MP18], [BGLR21, Section VII], and in [GMPS23, Apendix] to obtain MSRD codes with new parameters. When $m_1 = \ldots = m_\ell$, the shortening and puncturing above can be done on any index s, which recovers [MP19, Corollary 7]. Notice however that the construction in [MP19, Corollary 7] preserves $\mathbb{F}_{q^m}$-linearity, as discussed in [GMPS23, Apendix].

The next is a non-existential results, in the form of a bound on the parameter ℓ for MSRD codes.

Theorem 6.22 ([BGLR21, Theorem VI.12])**.** Suppose $v = n_1 = \ldots = n_\ell$ and $m = m_1 = \ldots = m_\ell$, and suppose there exists an MSRD code $C \subseteq \mathbb{M}$ of minimum sum-rank distance $d \geq 3$. Then

$$\begin{aligned} \ell &\leq \left\lfloor \frac{d-3}{v} \right\rfloor + \left\lfloor \frac{q^v - q^{v\lfloor (d-3)/v \rfloor + v - d + 3} + (q-1)(q^m+1)}{q^v - 1} \right\rfloor \\ &\leq \left\lfloor \frac{d-3}{v} \right\rfloor + 1 + \left\lfloor \frac{q^m(q-1)}{q^v - 1} \right\rfloor . \end{aligned}$$

In particular, we have the following:

1. If $v \mid d-3$, then
$$\ell \leq \frac{d-3}{v} + \left\lfloor \frac{(q-1)(q^m+1)}{q^v-1} \right\rfloor .$$

2. If $d \leq v+2$, then
$$\ell \leq \left\lfloor \frac{q^v - q^{v-d+3} + (q-1)(q^m+1)}{q^v-1} \right\rfloor \leq 1 + \left\lfloor \frac{q^m(q-1)}{q^v-1} \right\rfloor .$$
If in addition $v = m$, then
$$\ell \leq \left\lfloor \frac{q^{v+1}-1}{q^v-1} \right\rfloor \leq q+1,$$
and if $v = m \leq 2$, then $\ell \leq q$.

3. If $d = 3$ and $v \mid m$, then
$$\ell \leq (q-1) \cdot \frac{q^m-1}{q^v-1}.$$

Assume that $v = n_1 = \ldots = n_\ell$ and $m = m_1 = \ldots = m_\ell$. In this case, the MDS Conjecture [HP03, page 265] (see also the discussion at the end of this section) implies that, if an MSRD code exists, then $\ell \leq (q^m+1)/n$. However, Theorem 6.22 gives a tighter bound on ℓ when $d \leq q^m(1 - n(q-1)/(q^n-1)) + 4 - n$. If $m = v$ and $d \leq v+2$, then Theorem 6.22 gives the tighter bound $\ell \leq q+1$. Notice that, for any $d \geq 3$, if $m = v = 1$, then the sum-rank metric coincides with the Hamming metric and Theorem 6.22 yields the bound $\ell \leq q+d-2$, which is known for MDS codes, see [HP03, Corollary 7.4.3(ii)].

Finally, the bound in item 3 in Theorem 6.22 is met with equality by the MSRD codes in [MP22, Section 4.4]. Moreover, following [MP18] one can construct linearized Reed-Solomon codes with $\ell = q-1$. When $m = n$ grows and d is constant, the value $\ell = q-1 = \Theta(q)$ attains asymptotically the general bound in Theorem 6.22.

We end this section with a few results on the existence of chain of MSRD codes. Our motivation comes mainly from Theorem 7.20 and Theorem 7.23, where we assume the existence of a chain of MSRD codes with given parameters over a field of size q. The next proposition relies on the existence of linearized Reed-Solomon codes.

Proposition 6.23. Let $j = \min\{i \in [t] \mid m_i = m_t\}$ and let $d = \sum_{i=1}^{j-1} n_i + 1$. Suppose that $n_i \leq m_t$ for all $i \in [t]$. If $q > t$, then there exist MSRD codes $C_n \subseteq \ldots \subseteq C_{d+1} \subseteq C_d \subseteq \mathbb{M}$ with $d(C_i) = i$, for $d \leq i \leq n = n_1 + \cdots + n_t$.

Proof. Since $q > t$ and $n_i \leq m_j$ for all $i \in [t]$, then by [MP18, Definition 31 and Theorem 4] there exists a chain of linearized Reed-Solomon codes $\mathcal{D}_n \subseteq \ldots \subseteq \mathcal{D}_{d+1} \subseteq \mathcal{D}_d \subseteq \mathbb{F}_q^{m_j \times n_1} \times \cdots \times \mathbb{F}_q^{m_j \times n_t}$. The thesis follows from observing that increasing m_j in the positions i with $i < j$ by adding $m_i - m_j$ rows of zeros to each matrix does not affect the property of being MSRD. □

The next result complements Proposition 6.23, under the assumption that one of the m_i's is large enough.

Proposition 6.24. Let $j = \min\{i \in [t] \mid m_i = m_t\}$ and let $d = \sum_{i=1}^{j-1} n_i + \delta + 1$ for some $0 \leq \delta \leq n_j - 1$. Let $h = \max\{i \in [t] \mid n_i > m_t\}$ and assume that $h \in [t]$ and $m_h \geq m_t \left(\sum_{i=j}^{t} n_i - \delta\right)$. If $q > t - h$, then there exist MSRD codes $C_n \subseteq \ldots \subseteq C_{d+1} \subseteq C_d \subseteq \mathbb{M}$ with $d(C_i) = i$, for $d \leq i \leq n = n_1 + \cdots + n_t$.

Proof. Since $n_j \leq m_j$, then $h < j$. By Proposition 6.23 there exists a chain of MSRD codes $\mathcal{D}_n \subseteq \ldots \subseteq \mathcal{D}_{d+1} \subseteq \mathcal{D}_d \subseteq \mathbb{F}_q^{m_{h+1} \times n_{h+1}} \times \cdots \times \mathbb{F}_q^{m_t \times n_t}$ with $d(\mathcal{D}_i) = i - \sum_{\ell=1}^{h} n_\ell$ for $d \leq i \leq n$. Moreover, we claim that there exists a code $\mathcal{D}_0 \subseteq \mathbb{F}_q^{m_1 \times n_1} \times \cdots \times \mathbb{F}_q^{m_h \times n_h}$ with minimum distance $\sum_{i=1}^{h} n_i$ and dimension m_h. In fact, for all $\ell \in [h]$ there exists an MRD code $\mathcal{U}_\ell \subseteq \mathbb{F}_q^{m_\ell \times n_\ell}$ with minimum distance n_ℓ and dimension m_ℓ. For any fixed $\ell \in [h]$, let $U_{i\ell} \in \mathcal{U}_\ell$ be linearly independent matrices with $i \in [m_h]$. Then $N_i = (U_{i1}, \ldots, U_{i,h})$, $i \in [m_h]$, are a basis of a code $\mathcal{D}_0$ with the required properties. Let $M_{i,1}, \ldots, M_{i,D_i}$ be a basis of $\mathcal{D}_i$, where $D_i = \dim(\mathcal{D}_i)$ and $d \leq i \leq n$. Finally, let $C_i \subseteq \mathbb{M}$ be the code with basis $(N_1, M_{i,1}), \ldots, (N_{D_i}, M_{i,D_i})$. The construction works, since $D_n \leq \ldots \leq D_d = \sum_{\ell=j}^{t} m_\ell n_\ell - m_j \delta \leq m_h$. The code C_i has minimum distance i and dimension D_i, hence it is MSRD, for $d \leq i \leq n$. By construction we have $C_n \subseteq \ldots \subseteq C_{d+1} \subseteq C_d \subseteq \mathbb{M}$. □

More of course is known if we restrict to linear block codes and the Hamming metric. It is well known that MDS codes exist whenever $q \geq n - 1$. The MDS Conjecture states that, if $2 \leq k \leq q - 1$, this sufficient condition is also necessary. The MDS Conjecture was proven in [Bal12] by Ball in several situations, including the case when q is prime. See also [BL19] for a recent survey. The problem of characterizing the parameter sets for which MDS or MSRD codes exist is a highly nontrivial one and remains open in general.

7. Integer sequences that are generalized weights of a linear code

Which integer sequences are sequences of generalized weights of a linear code? For linear block codes, this question appears as [TV95, Problem 3.2]. The question is answered in [HKY92] for binary linear block codes of dimension up to three and in [Kløø93] for binary linear block codes of dimension four. In this chapter, we fully answer this question for linear block codes, rank-metric codes, and more generally for sum-rank metric codes.

Necessary conditions for a sequence of positive integers to be the sequence of generalized weights of a linear code appear in Proposition 1.4 for block codes, in [Rav16a] for rank-metric codes, and in Proposition 5.33 for the more general situation of sum-rank metric codes. The goal of this chapter is proving that these necessary conditions are also sufficient, at least over a field of large enough size. We do so under an existence assumption for MDS and MSRD codes.

In addition to characterizing the integer sequences which are the sequence of generalized weights of a linear (block, rank-metric, or sum-rank metric) code, we prove that the same numerical sequences are also sequences of greedy weights of a linear code of the same kind. For the case of linear block codes, we also prove that the same numerical sequences are sequences of relative generalized Hamming weights and of relative greedy weights. Moreover, we discuss the related problem of which integer sequences are the sequence of generalized weights of a linear sum-rank metric subcode of a given MSRD code.

The chapter is structured as follows. In Section 7.1 we extend the definition of greedy weights and the related notion of chain condition to sum-rank metric codes. In Section 7.2 we discuss in detail the case of linear block codes. Theorem 7.5 shows that a sequence of positive integers is the sequence of generalized weights of a linear block code if and only if it is increasing. In Section 7.3 we discuss the general case of sum-rank metric codes. Theorem 7.20 is the main result of this chapter. In Theorem 7.26 we state the result for rank-metric codes. Notice that, due to the existence of Gabidulin codes, for rank-metric codes we do not need to assume the existence of a chain of nested MRD codes.

7.1. Greedy weights

In this section we briefly discuss some concepts related to generalized weights which appear in the literature on linear block codes. We start by recalling the definition of relative generalized Hamming weights given in [LMVC05] for linear block codes.

Definition 7.1. Let $C_2 \subsetneq C_1 \subseteq \mathbb{F}_q^n$ be nested codes with $k_j = \dim(C_j)$, $j = 1, 2$. The r-th relative generalized Hamming weight of the pair $C_2 \subsetneq C_1 \subseteq \mathbb{F}_q^n$ is

$$d_r(C_1, C_2) = \min\{|\mathrm{supp}(\mathcal{D})| : \mathcal{D} \text{ is a subcode of } C_1, \mathcal{D} \cap C_2 = 0 \text{ and } \dim(\mathcal{D}) \geq r\},$$

for $r \in [k_1 - k_2]$.

The concept of greedy weights was introduced in [CEZ98, CEZ99] for linear block codes and then modified to the definition that is generally used today in [CK99]. We extend the definition to sum-rank metric codes in the natural way.

Definition 7.2. Let $0 \neq C \subseteq \mathbb{M}$ be a k-dimensional code. A greedy 1-subcode is a 1-dimensional subcode $\mathcal{D}_1 \subseteq C$ such that $\mathrm{wt}(\mathcal{D}_1) = d_1(C)$. For $2 \leq r \leq k$, a greedy r-subcode is an r-dimensional subcode $\mathcal{D}_r \subseteq C$ of minimum weight among those that contain a greedy $(r-1)$-subcode. For $r \in [k]$, we define the r-th greedy weight of C as

$$g_r(C) = \mathrm{wt}(\mathcal{D}_r)$$

where $\mathcal{D}_r$ is a greedy r-subcode of C.

Finally, we extend the concept of chain condition to sum-rank metric codes. The chain condition was originally defined in [WY93] for linear block codes.

Definition 7.3. A k-dimensional code $0 \neq C \subseteq \mathbb{M}$ satisfies the chain condition if there exists a chain of subcodes $0 \subsetneq \mathcal{D}_1 \subsetneq \mathcal{D}_2 \subsetneq \ldots \subsetneq \mathcal{D}_k = C$ such that $d_r(C) = \mathrm{wt}(\mathcal{D}_r)$ for all $r \in [k]$.

The next simple proposition clarifies the relation between greedy weights and chain condition.

Proposition 7.4. Let $C \subseteq \mathbb{M}$ be a k-dimensional code. Then C satisfies the chain condition if and only if $d_r(C) = g_r(C)$ for $r \in [k]$.

Proof. If $d_r(C) = g_r(C)$ for $r \in [k]$, then let $0 \subsetneq \mathcal{D}_1 \subsetneq \ldots \subsetneq \mathcal{D}_k \subseteq C$ be a chain of greedy subcodes. This chain satisfies the chain condition, since $d_r(C) = g_r(C) = \mathrm{wt}(\mathcal{D}_r)$ for $r \in [k]$. Conversely, if the chain condition holds, then there exists a chain $0 \subsetneq \mathcal{D}_1 \subsetneq \ldots \subsetneq \mathcal{D}_k \subseteq C$ with $d_r(C) = \mathrm{wt}(\mathcal{D}_r)$ for $r \in [k]$. One easily proves by induction on r that each $\mathcal{D}_r$ is a greedy r-subcode, since $\mathcal{D}_r$ is an r-dimensional subcode of C with $\mathrm{wt}(\mathcal{D}_r) = d_r(C) \leq g_r(C)$. □

7.2. Generalized Hamming weights of linear block codes

In this section we focus on linear block codes and generalized Hamming weights. We start by showing that every increasing sequence of positive integers is the sequence of generalized weights of a linear block code. Then we establish similar results for relative and greedy weights.

Theorem 7.5. Any increasing sequence of positive integers is the sequence of generalized Hamming weights of a linear block code. In addition, there exists one such code in $\mathbb{F}_q^n$, provided that n is greater than or equal to the last integer in the sequence and $q \geq n$. Moreover, the code may always be chosen such that it satisfies the chain condition. In particular, any increasing sequence of positive integers is the sequence of greedy weights of a linear block code.

Proof. Any non-empty increasing sequence $d_1, \ldots, d_k$ of positive integers can be uniquely written as the juxtaposition of ℓ maximal subsequences with the property that any two consecutive entries in the same subsequence differ by one. Let a_i be the length of the i-th such subsequence. In other words, the sequence $d_1, \ldots, d_k$ has the form

$$\underbrace{d_1, d_1+1, \ldots, d_1+a_1-1}_{a_1}, \underbrace{d_{a_1+1}, \ldots, d_{a_1+1}+a_2-1}_{a_2}, \ldots, \underbrace{d_{a_1+\cdots+a_{\ell-1}+1}, \ldots, d_{a_1+\cdots+a_{\ell-1}+1}+a_\ell-1}_{a_\ell}$$

for some $\ell \geq 1$ and $a_1, \ldots, a_\ell \geq 1$. Then the length of the sequence is $k = a_1 + \cdots + a_\ell$ and we let $n = d_k$.

Denote by $\mathrm{RS}(n, h)$ a Reed-Solomon code of dimension h and minimum distance $n - h + 1$ in $\mathbb{F}_q^n$ for some $q \geq n$. By induction on $\ell \geq 1$, we prove that there exists a code $C \subseteq \mathrm{RS}(n, n - d_1 + 1)$ with the properties that its generalized weights are the given sequence of integers and that it has subcodes $0 \subsetneq C_1 \subsetneq \ldots \subsetneq C_k = C$ such that $\dim(C_i) = i$ and $\mathrm{supp}(C_i) = [d_i]$ for all $i \in [k]$.

If $\ell = 1$, then $k = a_1$, $n = d_k = d_1 + a_1 - 1$, and the sequence consists of a_1 consecutive positive integers: $d_1, d_1 + 1, \ldots, d_1 + a_1 - 1$. One can let $C = \mathrm{RS}(n, n - d_1 + 1)$. A chain of subcodes $C_1 \subseteq \ldots \subseteq C_k = C$ such that $\dim(C_i) = i$ and $\mathrm{supp}(C_i) = [d_i]$ for all $i \in [a_1]$ is constructed by letting C_i be the subcode of C supported on the first d_i entries, for $i \in [a_1]$.

By induction, suppose that we can construct a code $\mathcal{D} \subseteq \mathrm{RS}(n, n - d_{a_1+1} + 1)$, whose generalized weights are

$$\underbrace{d_{a_1+1}, \ldots, d_{a_1+1} + a_2 - 1}_{a_2}, \ldots, \underbrace{d_{a_1+\cdots+a_{\ell-1}+1}, \ldots, d_{a_1+\cdots+a_{\ell-1}+1} + a_\ell - 1}_{a_\ell}.$$

In addition, $\mathcal{D}$ contains a chain of subcodes $\mathcal{D}_1 \subseteq \ldots \subseteq \mathcal{D}_{a_2+\cdots+a_\ell} = \mathcal{D}$ with $\dim(\mathcal{D}_i) = i$ and $\mathrm{supp}(\mathcal{D}_i) = [d_i(\mathcal{D})] = [d_{i+a_1}]$, for all $i \in [a_2 + \cdots + a_\ell]$.

Denote by $e_1, \ldots, e_n$ the elements of the standard basis of $\mathbb{F}_q^n$. Consider nested codes $\mathrm{RS}(n, n - d_1 + 1) \supseteq \mathrm{RS}(n, n - d_{a_1+1} + 1) \supseteq \mathcal{D}$ and let

$$\mathcal{E}_i = \mathrm{RS}(n, n - d_1 + 1) \cap \langle e_1, \ldots, e_{d_1+i-1} \rangle$$

for $i \in [a_1]$. Regarded as a subcode of $\langle e_1, \ldots, e_{d_1+i-1} \rangle = \mathbb{F}_q^{d_1+i-1}$, each $\mathcal{E}_i$ is an MDS code of parameters $(d_1 + i - 1, i, d_1)$. In fact, $\mathcal{E}_i$ has minimum distance $d(\mathcal{E}_i) = d_1$, since $\mathrm{RS}(n, n - d_1 + 1)$ contains a codeword supported on the first $d_1 \leq d_1 + i - 1$ coordinates. Moreover, $\mathcal{E}_i$ is the subspace of $\mathrm{RS}(n, n - d_1 + 1)$ obtained by evaluating the polynomials of degree up to $n - d_1$ whose evaluation in the last $n - d_1 - i + 1$ points is zero. Therefore, $\dim(\mathcal{E}_i) \geq (n - d_1 + 1) - (n - d_1 - i + 1) = i$. Since $\mathcal{E}_i$ has length $d_1 + i - 1$, it is MDS by the Singleton Bound. Let $\mathcal{E} = \mathcal{E}_{a_1}$.

Let $C = \mathcal{D} + \mathcal{E} \subseteq \mathrm{RS}(n, n - d_1 + 1)$. Since $d(\mathcal{D}) = d_{a_1+1} > d_{a_1} = d_1 + a_1 - 1 \geq |\mathrm{supp}(\mathcal{E})|$, then $\mathcal{D} \cap \mathcal{E} = 0$ and

$$\dim(C) = \dim(\mathcal{D}) + \dim(\mathcal{E}) = (a_2 + \ldots + a_\ell) + a_1 = k.$$

Since $\mathcal{E} \subseteq C \subseteq \mathrm{RS}(n, n - d_1 + 1)$, then

$$d_1 + i - 1 = d_i(\mathrm{RS}(n, n - d_1 + 1)) \leq d_i(C) \leq d_i(\mathcal{E}) = d_1 + i - 1$$

for $i \in [a_1]$. It follows that the first a_1 generalized weights of C agree with the first a_1 elements of the integer sequence.

Since $C = \mathcal{D} + \mathcal{E}$ and $\mathcal{D} \cap \mathcal{E} = 0$, then any subcode of C of dimension $i + a_1$ with $i \geq 1$ contains a subcode of $\mathcal{D}$ of dimension $i + a_1 - \dim(\mathcal{E}) = i$. It follows that $d_{i+a_1}(C) \geq d_i(\mathcal{D}) = d_{i+a_1}$. By the induction hypothesis, for all $i \in [a_2 + \cdots + a_\ell]$ there exists $\mathcal{D}_i \subseteq \mathcal{D}$ of $\dim(\mathcal{D}_i) = i$, such that $\mathrm{supp}(\mathcal{D}_i) = [d_i(\mathcal{D})] = [d_{i+a_1}]$. Notice that $d_{i+a_1} \geq d_1 + i + a_1 - 1 \geq d_1 + a_1$, hence $[d_{i+a_1}] = \mathrm{supp}(\mathcal{D}_i) \supseteq \mathrm{supp}(\mathcal{E}) = [d_1 + a_1 - 1]$. Since $\mathcal{D}_i \cap \mathcal{E} \subseteq \mathcal{D} \cap \mathcal{E} = 0$, then $\mathcal{D}_i + \mathcal{E}$ is an $(i + a_1)$-dimensional subspace of C with

$$\mathrm{supp}(\mathcal{D}_i + \mathcal{E}) = \mathrm{supp}(\mathcal{D}_i) = [d_{i+a_1}],$$

so $d_{i+a_1}(C) \leq d_{i+a_1}$. This proves that $d_{i+a_1}(C) = d_{i+a_1}$ for $i \in [a_2 + \cdots + a_\ell]$ and concludes the proof that $d_1, \ldots, d_k$ are the generalized weights of C.

We now show that C contains a chain of codes as claimed. Recall that $\mathcal{E}$ contains a chain of codes

$$0 \subsetneq \mathcal{E}_1 \subsetneq \ldots \subsetneq \mathcal{E}_{a_1} = \mathcal{E}$$

such that $\dim(\mathcal{E}_i) = i$ and $\mathrm{supp}(\mathcal{E}_i) = [d_i]$, for $i \in [a_1]$. Moreover

$$\mathcal{E} \subsetneq \mathcal{D}_1 + \mathcal{E} \subsetneq \ldots \subsetneq \mathcal{D}_{a_2+\cdots+a_\ell} + \mathcal{E} = \mathcal{D} + \mathcal{E} = C,$$

with $\dim(\mathcal{D}_i + \mathcal{E}) = i + a_1$ and $\mathrm{supp}(\mathcal{D}_i + \mathcal{E}) = [d_{i+a_1}]$, for $i \in [a_2 + \cdots + a_\ell]$. This proves in addition that C satisfies the chain condition, since $d_i(C) = d_i(\mathcal{E}) = \mathrm{wt}(\mathcal{E}_i)$ for $i \in [a_1]$ and $d_i(C) = d_{i-a_1}(\mathcal{D}) = \mathrm{wt}(\mathcal{D}_{i-a_1} + \mathcal{E})$ for $i \in [k] \setminus [a_1]$. Therefore, the greedy weights of C coincide with its generalized weights by Proposition 7.4. □

We make the following observations regarding the construction of the code C in the proof of Theorem 7.5.

Remark 7.6. i) The construction yields a code C of length n defined over a field of cardinality $q \geq n$ and which is contained in a $\mathrm{RS}(n, n - d_1 + 1)$, where d_1 is the first entry of the sequence and n is the last. Notice that n points of evaluation in $\mathbb{F}_q$ are needed in order to have a sequence of nested Reed-Solomon codes, as Reed-Solomon codes using evaluation at infinity are not nested. See also Example 7.7.

ii) The length n of the code can be chosen to be larger than the last generalized weight. In this case, the construction yields a degenerate code.

While it is clear that classical Reed-Solomon codes are nested, this is not the case in general if one uses infinity as an evaluation point. We next show a concrete example of this phenomenon.

Example 7.7. Let q be a prime power and let $\mathbb{F}_q = \{\alpha_1, \ldots, \alpha_q\}$. The code $\mathrm{RS}(q + 1, k)$ is the image of the encoding map

$$\begin{array}{rcl} \mathbb{F}_q[x]_{<k} & \longrightarrow & \mathbb{F}_q^{q+1} \\ p(x) & \longmapsto & (p(\alpha_1), \ldots, p(\alpha_q), p(\infty)) \end{array}$$

where $\mathbb{F}_q[x]_{<k}$ is the space of univariate polynomials with coefficients in $\mathbb{F}_q$ of degree smaller than k and $p(\infty)$ is the coefficient of x^{k-1} in $p(x)$. One may have $\mathrm{RS}(q+1, k) \not\subseteq \mathrm{RS}(q+1, k+1)$. E.g., for any q one has

$$\mathrm{RS}(q + 1, 1) = \langle (1, \ldots, 2) \rangle \not\subseteq \mathrm{RS}(q + 1, 1) = \langle (1, \ldots, 1, 0), (\alpha_1, \ldots, \alpha_q, 1) \rangle.$$

Theorem 7.5 may fail over a field of small cardinality. This is connected to Remark 7.6 i), since over a field of small cardinality (chains of nested) MDS codes may not exist for all choices of the parameters. The next examples illustrate what can go wrong over $\mathbb{F}_2$.

Example 7.8. The integer sequence $n - 1, n$ satisfies the necessary conditions of Proposition 5.33. However, by [HKY92, Theorem 10], it is the sequence of a generalized weights of a two-dimensional subcode of $\mathbb{F}_2^n$ if and only if $3(n - 1) \leq 2n$, that is $n \leq 3$. This is related to the non-existence of a chain of nested MDS codes of dimension one and two in $\mathbb{F}_2^n$.

Example 7.9. The integer sequence 4, 5, 7 satisfies the necessary conditions of Proposition 5.33. However, [HKY92, Theorem 10] implies that it is not the sequence of generalized weights of a three-dimensional subcode of $\mathbb{F}_2^7$. This is related to the non-existence of a chain of binary nested MDS codes with parameters $(7, 1, 7)$ and $(7, 4, 4)$.

In the language of matroidal ideals, we can reformulate Theorem 7.5 as follows.

Corollary 7.10. Any increasing sequence of positive integers can be realized as the sequence of initial degrees of the free modules in a graded minimal free resolution of a matroidal ideal.

Proof. The thesis follows by combining [JV13, Theorem 2] and Theorem 7.5. □

The construction of the code C in the proof of Theorem 7.5 can be made explicit, by providing an example of a code which satisfies the chain condition and whose generalized weights are equal to any given increasing sequence of positive integers.

Example 7.11. Let $1 \le k \le n \le q$ and $1 \le d_1 < d_2 < \ldots < d_k \le n$. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}_q$ be distinct elements and consider the polynomials $f_i = (x - \alpha_{d_i+1})(x - \alpha_{d_i+2}) \cdots (x - \alpha_n) \in \mathbb{F}_q[x]$, $i \in [k]$. The k-dimensional code $C \subseteq \mathbb{F}_q^n$ with generator matrix

$$G = \begin{pmatrix} f_1(\alpha_1) & f_1(\alpha_2) & \ldots & f_1(\alpha_n) \\ f_2(\alpha_1) & f_2(\alpha_2) & \ldots & f_2(\alpha_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(\alpha_1) & f_k(\alpha_2) & \ldots & f_k(\alpha_n) \end{pmatrix}$$

satisfies $d_r(C) = d_r$, for $r \in [k]$. Moreover, $d_r(C)$ is realized by the subspace generated by the first r rows of G. In particular, C satisfies the chain condition and $g_r(C) = d_r$ for $r \in [k]$.

We now turn our attention to relative weights. The Singleton Bound for relative generalized Hamming weights [LMVC05, Section IV] provides a necessary condition for a sequence of integers to be the relative generalized Hamming weights of a pair of linear block codes.

Lemma 7.12. Let $C_2 \subsetneq C_1 \subseteq \mathbb{F}_q^n$ be codes with $k_j = \dim(C_j)$, $j = 1, 2$. Then

$$d_i(C_1, C_2) \le n - k_1 + i$$

for $i \in [k_1 - k_2]$.

In particular, the last relative weight satisfies $d_{k_1-k_2}(C_1, C_2) \le n - k_2$. Since monotonicity also holds by [LMVC05, Proposition 2], every sequence $d_1, \ldots, d_{k_1-k_2}$ of relative generalized Hamming weights of a pair of subcodes of $\mathbb{F}_q^n$ as above must satisfy

$$1 \le d_1 < d_2 < \cdots < d_{k_1-k_2} \le n - k_2.$$

Fix $0 \le k_2 < k_1 \le n$ and let $k = k_1 - k_2$. We next show that any sequence of generalized Hamming weights of a k-dimensional subcode of $\mathbb{F}_q^{n-k_2}$ is also the sequence of relative generalized Hamming weights of a pair of nested subcodes of $\mathbb{F}_q^n$ dimension k_1 and k_2.

Lemma 7.13. Let $0 \leq k_2 < k_1 \leq n$ be positive integers. Let $C \subseteq \mathbb{F}_q^{n-k_2}$ be a code of dimension $k_1 - k_2$. Define $C_1 = C \times \mathbb{F}_q^{k_2}$ and $C_2 = 0 \times \mathbb{F}_q^{k_2}$, where 0 denotes the zero code in $\mathbb{F}_q^{n-k_2}$. Then

$$d_i(C_1, C_2) = d_i(C),$$

for $i \in [k_1 - k_2]$.

Proof. Let $i \in [k_1 - k_2]$. Since $C_2 \subsetneq C_1 \subseteq \mathbb{F}_q^n$, then $d_i(C_1, C_2)$ is well-defined. We will prove that $d_i(C_1, C_2) \leq d_i(C)$ and $d_i(C) \leq d_i(C_1, C_2)$.

First, let $\mathcal{D} \subseteq C$ such that $|\mathrm{supp}(\mathcal{D})| = d_i(C)$ and $\dim(\mathcal{D}) = i$. Define $\mathcal{D}' = \mathcal{D} \times 0$, where 0 denotes the zero code in $\mathbb{F}_q^{k_2}$. Clearly $\mathcal{D}' \subseteq C_1$, $\mathcal{D}' \cap C_2 = 0$, and $\dim(\mathcal{D}') = i$. Hence

$$d_i(C_1, C_2) \leq |\mathrm{supp}(\mathcal{D}')| = |\mathrm{supp}(\mathcal{D})| = d_i(C).$$

Next, take $\mathcal{D} \subseteq C_1$ such that $\mathcal{D} \cap C_2 = 0$, $\dim(\mathcal{D}) = i$, and $d_i(C_1, C_2) = |\mathrm{supp}(\mathcal{D})|$. Consider the natural projection map $\pi : C_1 \longrightarrow \mathbb{F}_q^{n-k_2}$ onto the first $n - k_2$ coordinates, and define $\mathcal{D}' = \pi(\mathcal{D})$. Since $\ker(\pi) = C_2$ and $\mathcal{D} \cap C_2 = 0$, then $\dim(\mathcal{D}') = \dim(\mathcal{D}) = i$. Moreover, since $C_1 = C \times \mathbb{F}_q^{k_2}$, we have that $\mathcal{D}' \subseteq \pi(C_1) = C$. Therefore,

$$d_i(C) \leq |\mathrm{supp}(\mathcal{D}')| \leq |\mathrm{supp}(\mathcal{D})| = d_i(C_1, C_2)$$

for $i \in [k_1 - k_2]$. □

As a consequence we can characterize the sequences of positive integers that are the sequence of relative generalized Hamming weights of a pair of nested codes.

Theorem 7.14. Any increasing sequence of positive integers is the sequence of relative generalized Hamming weights of a pair of nested linear block codes. In addition, there exists a pair of nested codes of dimensions k_1 and k_2 in $\mathbb{F}_q^n$ with relative generalized weights $d_1, \ldots, d_{k_1-k_2}$ provided that $0 \leq k_2 < k_1 \leq n$ and $q \geq n - k_2 \geq d_{k_1-k_2}$. Moreover, the pair may always be chosen such that it satisfies the relative chain condition. In particular, any increasing sequence of positive integers is the sequence of relative greedy weights of a pair of nested linear block codes.

Proof. The result follows by combining Lemma 7.13, Theorem 7.5, and Proposition 7.4. □

7.3. Rank-metric and sum-rank-metric codes

In this section, we extend Theorem 7.5 to sum-rank-metric codes, and discuss in particular the case of rank-metric codes. We start by discussing a special situation in which the result can be easily proved. More precisely, we observe that any non-decreasing sequence of positive integers is the sequence of generalized weights of an $\mathbb{F}_q$-linear rank-metric code for any q and for large enough m, n.

Theorem 7.15. Fix a prime power q. Any non-decreasing sequence of positive integers is the sequence of generalized weights of a linear rank-metric code.

Proof. Let $d_1, \ldots, d_k$ be a non-decreasing sequence of positive integers. Consider the space $\mathbb{F}_q^{m \times n}$, where $m = \sum_{r=1}^{k} d_r$ and $n = d_k$. We denote by $E_{i,j}$ the matrix in $\mathbb{F}_q^{m \times n}$ whose entries are equal to zero, except for a one in position (i, j). For $r \in [k]$, we let

$$C_r = \sum_{t=1}^{d_r} E_{d_1+\cdots+d_{r-1}+t,t} \in \mathbb{F}_q^{m\times n}.$$

Let $C \subseteq \mathbb{F}_q^{m\times n}$ be the code generated by $C_1, \ldots, C_k$. Then, we have that $\dim(C) = k$ and $d_r(C) = d_r$ for all $r \in [k]$. Indeed, consider the optimal anticode $\mathcal{A}_r = \langle E_{i,j} \mid i \in [m], j \in [d_r]\rangle \subseteq \mathbb{F}_q^{m\times n}$. Then,

$$\dim(C \cap \mathcal{A}_r) \geq \dim\langle C_1, \ldots, C_r\rangle = r,$$

hence $d_r(C) \leq d_r$. Since by construction any matrix in C of rank strictly smaller than d_r is contained in $\langle C_1, \ldots, C_{r-1}\rangle$, we conclude that $d_r(C) = d_r$. □

In the rest of the section, we characterize the integer sequences which are the generalized weights of sum-rank and rank-metric subcodes of a given ambient space. We start by describing the sequence of generalized weights of the ambient space $\mathbb{M}$. The next result is a direct consequence of the definition of generalized weights.

Lemma 7.16. The sequence of generalized weights of $\mathbb{M} = \mathbb{F}_q^{m_1\times n_1} \times \ldots \times \mathbb{F}_q^{m_t\times n_t}$ is

$$\underbrace{1, \ldots, 1}_{m_1}, \underbrace{2, \ldots, 2}_{m_1}, \ldots, \underbrace{n_1, \ldots, n_1}_{m_1},$$

$$\underbrace{n_1 + 1, \ldots, n_1 + 1}_{m_2}, \underbrace{n_1 + 2, \ldots, n_1 + 2}_{m_2}, \ldots, \underbrace{n_1 + n_2, \ldots, n_1 + n_2}_{m_2},$$

$$\vdots$$

$$\underbrace{n_1 + \cdots + n_{t-1} + 1, \ldots, n_1 + \cdots + n_{t-1} + 1}_{m_t}, \ldots, \underbrace{n_1 + \cdots + n_t, \ldots, n_1 + \cdots + n_t}_{m_t}.$$

A sequence of integers is a subsequence of this sequence if and only if it is a non-decreasing sequence in $[n_1 + \cdots + n_t]$ such that the integer $n_1 + \cdots + n_{i-1} + j$ appears at most m_i times, for all $i \in [t]$ and $j \in [n_i]$.

The previous lemma allows us to characterize the subsequences of the sequence of generalized weights of $\mathbb{M}$.

Lemma 7.17. A non-decreasing sequence of positive integers $d_1, \ldots, d_k$ with $d_k \leq n = n_1 + \cdots + n_t$ is a subsequence of the sequence of generalized weights of $\mathbb{M}$ if and only if

$$d_{r+m_j} > \sum_{i=1}^{j-1} n_i \text{ implies } d_{r+m_j} \geq d_r + 1,$$

for all $j \in [t]$ and $r \in [k - m_j]$.

Proof. If $d_1, \ldots, d_k$ is a subsequence of the sequence of generalized weights of $\mathbb{M} = \mathbb{F}_q^{m_1\times n_1} \times \ldots \times \mathbb{F}_q^{m_t\times n_t}$, then for any pair of positive integers (r, j) such that $j \in [t]$ and $r \in [k - m_j]$, there exists an index h such that $d_{h+m_j}(\mathbb{M}) = d_{r+m_j}$. If $d_{r+m_j} > \sum_{i=1}^{j-1} n_i$, then

$$d_{h+m_j}(\mathbb{M}) = d_{r+m_j} > \sum_{i=1}^{j-1} n_i.$$

Hence by Lemma 5.35 we obtain

$$d_{h+m_j}(\mathbb{M}) \geq d_h(\mathbb{M}) + 1 \geq d_r + 1,$$

where the last inequality follows from the fact that $d_1, \dots, d_k$ is a subsequence of the sequence of the generalized weights of $\mathbb{M}$.

Suppose now that for each pair of positive integers (r, j) such that $j \in [t]$ and $r \in [k - m_j]$ we have that

$$d_{r+m_j} > \sum_{i=1}^{j-1} n_i \text{ implies } d_{r+m_j} \geq d_r + 1.$$

This implies that for any $j \in [t]$ and $\delta \in [n_j]$ we have at most m_j elements in the sequence $d_1, \dots, d_k$ that are equal to $\sum_{i=1}^{j-1} n_i + \delta$. Together with $d_k \leq n_1 + \cdots + n_t$, we deduce that $d_1, \dots, d_k$ is a subsequence of the sequence of generalized weights of $\mathbb{M}$ by Lemma 7.16. □

We give now a necessary condition for a sequence of integers to be the sequence of generalized weights of a subcode of an MSRD code. In Theorem 7.20, we will show that such a condition is also sufficient, if we assume the existence of a suitable chain of MSRD codes. Notice that the case $\mathcal{D} = \mathbb{M}$ is precisely Lemma 5.35.

Proposition 7.18. Let $C \subseteq \mathcal{D} \subseteq \mathbb{M}$ and assume that $\mathcal{D}$ is an MSRD code. Then the sequence of generalized weights of C is a subsequence of the sequence of generalized weights of $\mathcal{D}$.

Proof. By Theorem 6.12 the sequence of generalized weights of $\mathcal{D}$ is the subsequence consisting of the last $\dim(\mathcal{D})$ generalized weights of $\mathbb{M}$. By Lemma 5.35 and Lemma 7.17, the sequence of generalized weights of C is a subsequence of the sequence of generalized weights of $\mathbb{M}$, which is described in Lemma 7.16. The thesis now follows from the previous two facts and the fact that $d_r(C) \geq d_r(\mathcal{D})$ for $r \in [\dim(C)]$. □

The next example shows that the conclusion of Proposition 7.18 does not necessarily hold for an arbitrary code $\mathcal{D}$.

Example 7.19. For $q > 2$, let $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$ and let

$$\mathcal{D} = \langle (1,1,0,0,0,0), (0,0,1,1,1,0), (0,0,0,\alpha,1,1) \rangle \subseteq \mathbb{F}_q^6.$$

The generalized weights of $\mathcal{D}$ are $2, 4, 6$. However,

$$C = \langle (1,1,0,0,0,0), (0,0,1,1,1,0) \rangle \subseteq \mathcal{D}$$

has generalized weights $2, 5$.

Our main result is a characterization of the integer sequences that are the sequence of generalized weights of a sum-rank-metric code. We prove our result under the assumption that a suitable chain of MSRD codes exists. Such a chain exists for many choices of parameters, some of which are described in Proposition 6.23 and Proposition 6.24. In particular, it exists in the Hamming metric (corresponding to $m_i = n_i = 1$ for all $i \in [t]$) whenever $q \geq n$. In fact, if $q \geq n$, then Reed-Solomon codes form such a chain, see also Remark 7.6 ii). Hence the next theorem may be regarded as a generalization of Theorem 7.5.

Theorem 7.20. Let $\mathcal{D} \subseteq \mathbb{M}$ be an MSRD code of minimum distance d and assume that there exists a chain of MSRD codes $\mathcal{D} = \mathcal{D}_d \supseteq \mathcal{D}_{d+1} \supseteq \ldots \supseteq \mathcal{D}_n$ such that $d(\mathcal{D}_h) = h$, for $d \leq h \leq n$. A sequence of positive integers $d_1, \ldots, d_k$ is the sequence of generalized sum-rank weights of a code $C \subseteq \mathcal{D}$ if and only if $k \leq \dim(\mathcal{D})$ and $d_1, \ldots, d_k$ is a subsequence of the sequence of generalized weights of $\mathcal{D}$. Moreover, the code C may always be chosen such that it satisfies the chain condition. In particular, any subsequence of the sequence of generalized sum-rank weights of $\mathcal{D}$ is the sequence of greedy weights of a subcode of $\mathcal{D}$.

Proof. Necessity follows from Proposition 7.18. We now prove sufficiency: A sequence of integers as in the statement of the theorem can be uniquely written as the juxtaposition of ℓ maximal constant subsequences of length $a_2 - 1, a_3 - a_2, \ldots, a_{\ell+1} - a_\ell$ as

$$d_1, \ldots, d_1, d_{a_2}, \ldots, d_{a_2}, \ldots, d_{a_\ell}, \ldots, d_{a_\ell}, \tag{7.3.1}$$

for some $1 = a_1 < a_2 < \ldots < a_\ell \leq k$ and $d_1 < d_{a_2} < \ldots < d_{a_\ell} \leq n$, and where we set $a_{\ell+1} = k+1$. By assumption, the sequence (7.3.1) is a subsequence of the sequence of generalized weights of $\mathcal{D}$. For each $h \in [\ell]$ there exist $j \in [t]$ and $0 \leq \delta \leq n_j - 1$ such that $d_{a_h} = \sum_{i=1}^{j-1} n_i + \delta + 1$. By Proposition 7.18 and Lemma 7.16, this implies that $a_{h+1} - a_h \leq m_j$. Since $\mathcal{D}_{d_{a_h}}$ has dimension $\sum_{i=j}^{t} m_i n_i - \delta m_j$, then there exists a subcode $C_h \subseteq \mathcal{D}_{d_{a_h}}$ of dimension $a_{h+1} - a_h$ supported on the first d_{a_h} columns. In fact, let $\mathcal{A}$ be the optimal anticode supported on the first d_{a_h} columns. Then $\dim(\mathcal{A}) = \sum_{i=1}^{j-1} n_i m_i + (\delta+1) m_j$, hence $\dim(\mathcal{A} \cap \mathcal{D}_{d_{a_h}}) = \dim(\mathcal{A}) + \dim(\mathcal{D}_{d_{a_h}}) - \dim(\mathcal{A} + \mathcal{D}_{d_{a_h}}) \geq m_j$ and one can choose $C_h \subseteq \mathcal{A} \cap \mathcal{D}_{d_{a_h}}$. Notice that every nonzero element of C_h has sum-rank d_{a_h}. In addition, $\operatorname{maxsrk}(\sum_{i=1}^{h-1} C_i) \leq d_{a_{h-1}} < d_{a_h} = d(C_h)$, hence $(C_1 + \cdots + C_{h-1}) \cap C_h = 0$.

Let $C = C_1 + \cdots + C_\ell$. Then $C \subseteq \mathcal{D}_{d_1} \subseteq \mathcal{D}$ and

$$\dim(C) = \sum_{h=1}^{\ell} \dim(C_h) = \sum_{h=1}^{\ell} (a_{h+1} - a_h) = k.$$

To show that (7.3.1) is the sequence of generalized weights of C, we need to prove that $d_r(C) = d_{a_h}$ for every $h \in [\ell]$ and r such that $a_h \leq r < a_{h+1}$. As before, let $\mathcal{A}$ be the optimal anticode supported on the first d_{a_h} columns. By construction, $C_1 + \cdots + C_h \subseteq \mathcal{A} \cap C$, hence $\dim(\mathcal{A} \cap C) \geq a_{h+1} - 1 \geq r$, thus $d_r(C) \leq \operatorname{maxsrk}(\mathcal{A}) = d_{a_h}$. Now, let $\mathcal{A}'$ be an optimal anticode such that $d_r(C) = \operatorname{wt}(\mathcal{U}_r) = \operatorname{maxsrk}(\mathcal{A}')$, where $\mathcal{U}_r \subseteq \mathcal{A}' \cap C$ and $\dim(\mathcal{U}_r) \geq r$. Since $\dim(\mathcal{A}' \cap C) \geq r \geq a_h$ and $\dim(C_1 + \cdots + C_{h-1}) = a_h - 1$, we have that $\mathcal{A}' \cap (C_h + \cdots + C_\ell) \neq 0$. It follows that $d_r(C) = \operatorname{maxsrk}(\mathcal{A}') \geq d(C_h + \cdots + C_\ell) \geq d_{a_h}$, proving that $d_r(C) = d_{a_h}$.

Finally, to see that C satisfies the chain condition, let $\mathcal{U}_0 = 0$ and let $\mathcal{A}_h$ be the optimal anticode supported on the first d_{a_h} columns for $h \in [\ell]$. For $a_h \leq r < a_{h+1}$, let $\mathcal{U}_r \subseteq \mathcal{A}_h \cap C$ be an r-dimensional subspace that contains $\mathcal{U}_{r-1}$. Notice that $\operatorname{wt}(\mathcal{U}_r) \leq d_{a_h}$ since $\mathcal{U}_r \subseteq \mathcal{A}_h$ and $\operatorname{wt}(\mathcal{U}_r) \geq d_r(C) = d_{a_h}$ since $\mathcal{U}_r$ is an r-dimensional subcode of C. Therefore $\operatorname{wt}(\mathcal{U}_r) = d_{a_h} = d_r(C)$ and the chain $0 \subsetneq \mathcal{U}_1 \subsetneq \mathcal{U}_2 \subsetneq \ldots \subsetneq \mathcal{U}_k = C$ has the required properties. Since C satisfies the chain condition, then its greedy weights coincide with its generalized weights by Proposition 7.4. □

Example 7.19 also shows that Theorem 7.20 may fail, if the ambient code is not MSRD.

Example 7.21. For $q > 2$, let $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$ and let

$$\mathcal{D} = \langle (1, 1, 0, 0, 0, 0), (0, 0, 1, 1, 1, 0), (0, 0, 0, \alpha, 1, 1) \rangle \subseteq \mathbb{F}_q^6.$$

The generalized weights of $\mathcal{D}$ are $2, 4, 6$. However, there is no subcode $C \subseteq \mathcal{D}$ whose generalized weights are equal to $2, 4$.

Remark 7.22. The assumption on the sequence of nested MSRD codes is necessary for Theorem 7.20 to hold, in the following sense. Suppose that every MSRD code has the property that every subsequence of its generalized weights is realized by one of its subcodes. Consider an MSRD code $\mathcal{D} = \mathcal{D}_d$ with $d_1(\mathcal{D}) = d$ and the subsequence of its generalized weights consisting of all the generalized weights which are different from d. Let $\mathcal{D}_{d+1}$ be a subcode of $\mathcal{D}$ which realizes this subsequence. Then $\mathcal{D}_{d+1}$ is MSRD and we consider the subsequence of its generalized weights consisting of all the generalized weights which are different from $d + 1$. By assumption, there is a subcode $\mathcal{D}_{d+2}$ of $\mathcal{D}_{d+1}$ which realizes this subsequence of generalized weights. Proceeding in this fashion, we obtain a sequence of nested MSRD codes as in the statement of Theorem 7.20.

Theorem 7.20 allows us to characterize the integer sequences which are the sequence of generalized weights of a sum-rank metric code as follows.

Theorem 7.23. Assume that there exists a chain of MSRD codes $\mathbb{M} = \mathcal{D}_1 \supseteq \mathcal{D}_2 \supseteq \ldots \supseteq \mathcal{D}_n$ such that $d(\mathcal{D}_h) = h$ for $h \in [n]$. A sequence of positive integers $d_1, \ldots, d_k$ is the sequence of generalized sum-rank weights of a code $C \subseteq \mathbb{M}$ if and only if $d_1, \ldots, d_k$ is a non-decreasing sequence in $[n]$ such that the integer $n_1 + \cdots + n_{i-1} + j$ appears in it at most m_i times, for all $i \in [t]$ and $j \in [n_i]$. Moreover, the code C may always be chosen such that it satisfies the chain condition. In particular, any subsequence of the sequence of generalized sum-rank weights of $\mathbb{M}$ is the sequence of greedy weights of a sum-rank metric code $C \subseteq \mathbb{M}$.

Remark 7.24. The assumption that MSRD codes exist for every choice of parameters is necessary for Theorem 7.23 to hold, in the following sense. If it is true that every subsequence of the sequence of generalized weights of $\mathbb{M}$ is the sequence of generalized weights of a code $C \subseteq \mathbb{M}$, then for $i \in [t]$ and $j \in [n_i]$ consider the subsequence consisting of all generalized weights of $\mathbb{M}$ which are bigger than or equal to $n_1 + \cdots + n_{i-1} + j$. That is the subsequence

$$\underbrace{n_1 + \cdots + n_{i-1} + j, \ldots, n_1 + \cdots + n_{i-1} + j}_{m_i}, \ldots, \underbrace{n_1 + \cdots + n_i, \ldots, n_1 + \cdots + n_i}_{m_i},$$

$$\underbrace{n_1 + \cdots + n_{i-1} + j, \ldots, n_1 + \cdots + n_{i-1} + j}_{m_i}, \ldots, \underbrace{n_1 + \cdots + n_i, \ldots, n_1 + \cdots + n_i}_{m_i},$$

$$\vdots$$

$$\underbrace{n_1 + \cdots + n_{t-1} + 1, \ldots, n_1 + \cdots + n_{t-1} + 1}_{m_t}, \ldots, \underbrace{n_1 + \cdots + n_t, \ldots, n_1 + \cdots + n_t}_{m_t}.$$

A code $C \subseteq \mathbb{M}$ with those generalized weights is an MSRD code with minimum distance $n_1 + \cdots + n_{i-1} + j$. This proves that, if the conclusion of Theorem 7.23 holds, then MSRD codes exists for every possible choice of minimum distance.

In Theorem 7.23 we assume the existence of a maximal chain of MSRD codes. This is a stronger assumption than just assuming the existence of MSRD codes for every choice of parameters. The next example shows that it is possible that MSRD codes exist for every choice of parameters, but no maximal chain of MSRD codes exists. In such a situation, the example also shows that it is possible for the conclusion of Theorem 7.23 to hold, that is, every subsequence of the sequence of generalized weights of $\mathbb{M}$ is the sequence of generalized weights of a code in $\mathbb{M}$.

Example 7.25. Let $q = 2$, $t = 3$, and $n_i = m_i = 1$ for $i \in [3]$, that is $\mathbb{M} = \mathbb{F}_2^3$. The only MDS code in $\mathbb{F}_2^3$ with minimum distance 2 is the even-weight code. The only MDS code in $\mathbb{F}_2^3$ with minimum distance 3 is $\langle(1, 1, 1)\rangle$. Therefore, $\mathbb{F}_2^3$ contains MSD codes with every possible minimum distance, however it does not contain a chain of nested MDS codes with minimum distances 2 and 3. While Theorem 7.23 does not apply in this situation, it is easy to check by direct inspection that for every subsequence of the sequence $1, 2, 3$ of generalized weights of $\mathbb{F}_2^3$ there exists a code $C \subseteq \mathbb{F}_2^3$ whose generalized weights coincide with the chosen subsequence.

Finally, we state Theorem 7.23 in the generality of rank-metric codes. This provides a characterization of the integer sequences that are the sequence of generalized weights of a rank-metric subcode of $\mathbb{F}_q^{m\times n}$ for given q, n, m. Notice that Gabidulin codes [Del78, Gab85] form a chain of nested MRD codes of minimum rank distances $1, 2, \ldots, n$ for any m, n, and q with $n \leq m$.

Theorem 7.26. Fix $1 \leq n \leq m$ and q a prime power. A non-decreasing sequence of positive integers $d_1, \ldots, d_k$ is the sequence of generalized rank weights of a linear subcode of $\mathbb{F}_q^{m\times n}$ if and only if $k \leq mn$, $d_k \leq n$, and any constant subsequence has length at most m.

Remark 7.27. Fix any prime power q. Given a non-decreasing sequence of positive integers $d_1, \ldots, d_k$, any $C \subseteq \mathbb{F}_q^{m\times n}$ which has those generalized weights must have $n \geq d_k$. Moreover, one can always make $n = d_k$, since the code that we construct in the proof of Theorem 7.20 is supported on d_k columns. Theorem 7.15 motivates the question of what is the smallest m for which there exists a code $C \in \mathbb{F}_q^{m\times d_k}$ such that $d_i(C) = d_i$ for $i \in [k]$. Theorem 7.26 implies that the least m for which there exists $C \subseteq \mathbb{F}_q^{m\times n}$ with the desired sequence of generalized weights is the maximum between d_k and the maximum length of a constant subsequence of $d_1, \ldots, d_k$.

Since Gabidulin codes are $\mathbb{F}_{q^m}$-linear, we also have the following.

Theorem 7.28. Fix $1 \leq n < m$ and q a prime power. An increasing sequence of positive integers $d_1, \ldots, d_k$ is the sequence of generalized rank weights of an $\mathbb{F}_{q^m}$-linear subcode of $\mathbb{F}_q^{m\times n}$ if and only if $k, d_k \leq n$.

Notice that the previous theorem is not a straightforward consequence of Theorem 7.26. However, we omit the proof since it is very similar to the proof of Theorem 7.20. The interested reader can find a written proof in the Master's Thesis of Nellen [Nel24].

Using Gabidulin codes, we can make the codes that we constructed in the proof of Theorem 7.20 explicit in the case of rank metric codes and vector rank metric codes. Let $1 \leq d_1 < \ldots < d_k \leq n$, $n \leq m$. Let $\beta_1, \ldots, \beta_n \in \mathbb{F}_{q^m}$ be $\mathbb{F}_q$-linearly independent elements and consider the linearized polynomials

$$f_i = \prod_{\beta\in\mathcal{U}_i}(x - \beta),$$

where $\mathcal{U}_i = \langle \beta_{d_i+1}, \dots, \beta_n \rangle_{\mathbb{F}_q} \subseteq \mathbb{F}_{q^m}$. Then the k-dimensional $\mathbb{F}_{q^m}$-linear code in $\mathbb{F}_{q^m}^n$ with generator matrix

$$G = \begin{pmatrix} f_1(\beta_1) & f_1(\beta_2) & \dots & f_1(\beta_n) \\ f_2(\beta_1) & f_2(\beta_2) & \dots & f_2(\beta_n) \\ \vdots & \vdots & \ddots & \vdots \\ f_k(\beta_1) & f_k(\beta_2) & \dots & f_k(\beta_n) \end{pmatrix}$$

satisfies $d_r(C) = d_r$, for $r \in [k]$. Since $\mathbb{F}_{q^m}^n \cong \mathbb{F}_q^{m \times n}$, the same construction yields a code in $\mathbb{F}_q^{m \times n}$ with the same generalized weights.

8. Generalized weights for convolutional codes

In [For94], Forney suggests that generalized Hamming weights may be extended to convolutional codes. Motivated by this observation, Rosenthal and York in [RY97] introduce a notion of generalized Hamming weights for convolutional codes. Given a convolutional code C, they forget about its module structure and regard it as an infinite dimensional vector space endowed with the Hamming metric. Their definition of generalized Hamming weights is the natural extension of the usual definition to infinite dimensional codes.

The aim of this chapter is to introduce a new family of generalized weights for convolutional codes, which takes into account the module structure of the codes. Instead of looking at the size of supports of $\mathbb{F}_q$-linear subspaces of a convolutional code, we consider the size of supports of its subcodes, that is, its $\mathbb{F}_q[x]$-linear submodules. Since a convolutional code is an $\mathbb{F}_q[x]$-submodule of $\mathbb{F}_q[x]^n$, it is natural to consider the set of its submodules, since they carry the same mathematical structure as the code itself.

The chapter is organized as follows. In Section 8.1 we give our definition of generalized weights and we discuss their connection with the generalized weights defined in [RY97] and the generalized weights for linear block codes. We also prove by means of an example that these generalized weights do not satisfy Wei duality. In Section 8.2 we show that the computation of our generalized weights can be simplified by considering only submodules with certain properties. For instance, we prove that the generalized weights are realized by submodules generated by codewords of minimal support. In Section 8.3 we establish some upper bounds for the generalized weights of Maximum Distance Separable (MDS) and Maximum Distance Profile (MDP) codes. Our bounds imply that, depending on the code parameters, some or all of the generalized weights of MDS codes are determined by the code length, rank, and internal degree. In Section 8.4, we define the maximum weight of a convolutional code and prove an anticode bound. We define optimal convolutional anticodes as the codes which meet the anticode bound, we classify optimal anticodes, and compute their generalized weights.

8.1. Generalized weights

In order to define the generalized weights of a convolutional code, we first wish to define a notion of weight of a code. This replaces the usual notion of cardinality of the support of a code.

Definition 8.1. Let $C \subseteq \mathbb{F}_q[x]^n$ be a convolutional code of rank $k \leq n$. The weight of C is

$$\mathrm{wt}(C) = \min\{|\mathrm{supp}(\langle c_1(x), \ldots, c_k(x)\rangle_{\mathbb{F}_q})| : C = \langle c_1(x), \ldots, c_k(x)\rangle_{\mathbb{F}_q[x]}\}.$$

Notice that the set $\{\mathrm{supp}(\langle c_1(x), \ldots, c_k(x)\rangle_{\mathbb{F}_q}) : C = \langle c_1(x), \ldots, c_k(x)\rangle_{\mathbb{F}_q[x]}\}$ may not have a minimum with respect to inclusion, as the next example shows. This explains why in Definition 8.1 we do not mention an inclusion-minimum support.

Example 8.2. Let $C = \langle (1+x^2, 0, 1), (1, 1, 0)\rangle \subseteq \mathbb{F}_2[x]^3$. Then C is a noncatastrophic $(3, 2, 2)$ binary code and

$$\begin{pmatrix} 1+x^2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & x^2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

are two row-reduced generator matrices for C whose $\mathbb{F}_q$-rowspaces have incomparable supports. We claim that these supports are minimal in the set

$$\{\text{supp}(c_1(x)) \cup \text{supp}(c_2(x)) : C = \langle c_1(x), c_2(x)\rangle_{\mathbb{F}_2[x]}\}.$$

In fact, if $c_1(x), c_2(x) \in \mathbb{F}_2[x]$ are generators of C, then

$$\text{supp}(c_1(x)) \cup \text{supp}(c_2(x)) \supseteq \{(1, 0), (2, 0), (3, 0)\}.$$

Moreover, we have $\text{supp}(c_1(x)) \cup \text{supp}(c_2(x)) \neq \{(1, 0), (2, 0), (3, 0)\}$, otherwise $\langle c_1(x), c_2(x)\rangle_{\mathbb{F}_2}$ would be a two-dimensional binary linear block code of length 3 and the largest dimension of a binary linear block code inside C is one.

We are now ready to define the generalized weights of a convolutional code.

Definition 8.3. Let C be an (n, k, δ) convolutional code. For $1 \leq r \leq k$, the r-th generalized weight of C is

$$d_r(C) = \min\{\text{wt}(\mathcal{D}) \mid \mathcal{D} \subseteq C \text{ is a convolutional subcode of } \text{rk}(\mathcal{D}) \geq r\}.$$

We say that $\mathcal{D} \subseteq C$ realizes the r-th generalized weight of C if $\text{rk}(\mathcal{D}) = r$ and $d_r(C) = \text{wt}(\mathcal{D})$.

In the next lemma we provide several equivalent formulations of Definition 8.3. In particular, we show that $d_r(C)$ is the minimum weight of a subcode of C of rank r.

Lemma 8.4. Let C be a convolutional code. Let $\mathcal{V}$ denote the set of $\mathbb{F}_q$-linear subspaces of C. The following quantities are equal to $d_r(C)$:

1. $\min\{|\text{supp}(U)| : U \subseteq C \text{ and } \text{rk}(\langle U\rangle_{\mathbb{F}_q[x]}) \geq r\}$,
2. $\min\{|\text{supp}(U)| : U \subseteq C \text{ and } \text{rk}(\langle U\rangle_{\mathbb{F}_q[x]}) = r\}$,
3. $\min\{|\text{supp}(U)| : U \in \mathcal{V} \text{ and } \text{rk}(\langle U\rangle_{\mathbb{F}_q[x]}) \geq r\}$,
4. $\min\{|\text{supp}(U)| : U \in \mathcal{V} \text{ and } \text{rk}(\langle U\rangle_{\mathbb{F}_q[x]}) = r\}$,
5. $\min\{\text{wt}(\mathcal{D}) \mid \mathcal{D} \subseteq C \text{ is a subcode of } \text{rk}(\mathcal{D}) = r\}$,

for $1 \leq r \leq k$.

Proof. We already observed that the support of a linear space is equal to the union of the supports of a system of generators. Therefore, for any set U, the support of U is the same as the support of the $\mathbb{F}_q$-linear space generated by U. This proves that definitions 1. and 3. are equivalent and 2. and 4. are equivalent. Moreover, by comparing the sets over which we minimize, one sees that the minimum in 4. is greater than or equal to that in 3., the minimum in 5. is greater than or equal to the r-th generalized weight of C, the minimum in 5. is greater than or equal to that in 4., and the r-th generalized weight of C is greater than or equal to the minimum in 3. In order to prove that all numbers coincide, it suffices to show that the minimum in 3. is greater than or equal to that in 5.

Let $U \subseteq C$ be an $\mathbb{F}_q$-linear subspace such that $\mathrm{rk}(\langle U\rangle_{\mathbb{F}_q[x]}) \geq r$. Then $\dim(U) \geq r$ and there exists $U' \subseteq U$ an $\mathbb{F}_q$-linear subspace such that $\dim(U') = \mathrm{rk}(\langle U'\rangle_{\mathbb{F}_q[x]}) = r$. We conclude, since $U' \subseteq U$ implies that $\mathrm{supp}(U') \subseteq \mathrm{supp}(U)$. □

Remark 8.5. Consider a noncatastrophic code C. One may also define the generalized weights as

$$\tilde{d}_r(C) = \min\{\mathrm{wt}(\mathcal{D}) \mid \mathcal{D} \subseteq C \text{ is a noncatastrophic subcode of } \mathrm{rk}(\mathcal{D}) \geq r\}$$

or equivalently as

$$\tilde{d}_r(C) = \min\{\mathrm{wt}(\mathcal{D}) \mid \mathcal{D} \subseteq C \text{ is a noncatastrophic subcode of } \mathrm{rk}(\mathcal{D}) = r\}$$

for $1 \leq r \leq k$. Notice that this definition is not equivalent to Definition 8.3. Indeed, it may happen that $\tilde{d}_1(C) \neq d_{\mathrm{free}}(C)$, while $d_1(C) = d_{\mathrm{free}}(C)$ for every C, as shown in Proposition 8.6. For example, let $C = \langle(1, 1 + x + x^2 + x^3)\rangle_{\mathbb{F}_2[x]} \subseteq \mathbb{F}_2[x]^2$. Then $d_{\mathrm{free}}(C) = \mathrm{wt}(1 + x, 1 + x^4) = 4$ and every element of minimum weight generates a catastrophic code. On the other side, $\mathrm{wt}(1, 1 + x + x^2 + x^3) = 5$ hence $\tilde{d}_1(C) = 5$. Since we want the first generalized weight to be equal to $d_{\mathrm{free}}(C)$, we will not discuss this definition further.

In the next proposition, we establish some basic properties of the generalized weights. In particular, we prove that they are strictly increasing and that the minimum distance coincides with the first generalized weight. Moreover, we provide an upper bound on each generalized weight.

Proposition 8.6. Let $C \subseteq \mathcal{B} \subseteq \mathbb{F}_q[x]^n$ be convolutional codes, let $k = \mathrm{rk}(C)$ and let δ_1 be the memory of C. Then:

1. $d_1(C) = d_{\mathrm{free}}(C)$.
2. $d_r(C) < d_{r+1}(C)$ for all $1 \leq r \leq k - 1$.
3. $d_r(\mathcal{B}) \leq d_r(C)$ for all $1 \leq r \leq k$.
4. $d_r(C) \leq n(\delta_1 + 1) - k + r$ for all $1 \leq r \leq k$.
5. $d_k(C) \leq \mathrm{wt}(C)$.

Proof. Items 1., 3., and 5. follow directly from the definition, while Item 4. follows by combining 2. and 5. For Item 2, let $\mathcal{U} = \langle u_1, \ldots, u_{r+1}\rangle_{\mathbb{F}_q[x]}$ be a subcode of C that realizes $d_{r+1}(C)$. After adding suitable multiples of u_1 to the other generators, we may suppose that $\mathrm{supp}(u_1) \not\subseteq \mathrm{supp}(\langle u_2, \ldots, u_{r+1}\rangle_{\mathbb{F}_q})$. Since $u_2, \ldots, u_{r+1}$ are still $\mathbb{F}_q[x]$-linearly independent, we conclude. □

Remark 8.7. Notice that, unlike what happens for linear block codes, one does not always have that $d_k(C) = \mathrm{wt}(C)$. For example, for the code C of Remark 8.5, one has $d_1(C) = 4 < 5 = \mathrm{wt}(C)$.

Remark 8.8. In Proposition 8.44 we prove that MDS codes meet the bound in 4. when $k \mid \delta$, showing that the bound is sharp. In Proposition 8.54 we prove that the generalized weights of optimal anticodes increase by one at each step, showing that the bound in 2. is sharp.

Generalized weights are invariant under isometries of convolutional codes. As a consequence, they are also invariant under strong isometries as defined in [GL09].

Proposition 8.9. If C_1 and C_2 are isometric convolutional codes, then they have the same weight and generalized weights.

Proof. Let $\phi : C_1 \to C_2$ be an isometry. Since $\phi^{-1} : C_2 \to C_1$ is also an isometry, it suffices to prove that the weight and the generalized weights of C_1 are greater than or equal to the corresponding invariants of C_2.

Let $c_1(x), \dots, c_k(x)$ be a basis of C_1 such that $\mathrm{wt}(C_1) = |\mathrm{supp}(\langle c_1(x), \dots, c_k(x)\rangle_{\mathbb{F}_q})|$. Then $\phi(c_1(x)), \dots, \phi(c_k(x))$ are a basis of C_2 and the restriction of ϕ is an $\mathbb{F}_q$-linear isometry of linear block codes between $\langle c_1(x), \dots, c_k(x)\rangle_{\mathbb{F}_q}$ and $\langle \phi(c_1(x)), \dots, \phi(c_k(x))\rangle_{\mathbb{F}_q}$ with respect to the Hamming distance. In particular,

$$\begin{aligned}\mathrm{wt}(C_1) =& |\mathrm{supp}(\langle c_1(x), \dots, c_k(x)\rangle_{\mathbb{F}_q})| = \\ =& |\mathrm{supp}(\langle \phi(c_1(x)), \dots, \phi(c_k(x))\rangle_{\mathbb{F}_q})| \geq \mathrm{wt}(C_2).\end{aligned}$$

Suppose now that $\mathcal{D} \subseteq C_1$ realizes $d_r(C_1)$. Since ϕ is an isomorphism of $\mathbb{F}_q[x]$-modules, then $\mathrm{rk}(\phi(\mathcal{D})) = r$. Moreover, ϕ induces an isometry between $\mathcal{D}$ and $\phi(\mathcal{D})$, hence

$$d_r(C_1) = \mathrm{wt}(\mathcal{D}) = \mathrm{wt}(\phi(\mathcal{D})) \geq d_r(C_2).$$ □

In [RY97], the authors introduce the concept of generalized Hamming weights for convolutional codes. They regard convolutional codes as $\mathbb{F}_q(x)$-linear subspaces of $\mathbb{F}_q(x)^n$, where $\mathbb{F}_q(x)$ denotes the field of rational functions in x with coefficients in $\mathbb{F}_q$. Since the only elements of finite weight of $\mathbb{F}_q(x)^n$ are the elements of $\mathbb{F}_q[x]^n$, they are lead to considering $\mathbb{F}_q$-linear subspaces of $\mathbb{F}_q[x]^n$. Here we recall their definition, stating it for convolutional codes defined as $\mathbb{F}_q[x]$-submodules of $\mathbb{F}_q[x]^n$. Notice that, if $\mathbb{C} \subseteq \mathbb{F}_q(x)^n$ is a convolutional code as considered in [RY97], its generalized weights as defined in [RY97] are equal to those that we define next for the $\mathbb{F}_q[x]$-module $C = \mathbb{C} \cap \mathbb{F}_q[x]^n$.

Definition 8.10. Let C be a convolutional code. For every positive integer r, the r-th generalized Hamming weight of C is

$$d'_r(C) = \min\{|\mathrm{supp}(U)| : U \text{ is an } \mathbb{F}_q\text{-linear subspace of } C \text{ and } \dim(U) = r\}.$$

Remark 8.11. It follows directly from the definitions that $d'_r(C) \leq d_r(C)$ for $1 \leq r \leq \mathrm{rk}(C)$.

From now on, we refer to the weights from Definition 8.3 as generalized weights and to those from Definition 8.10 as generalized Hamming weights. Even though Definition 8.10 appears to be similar to our Definition 8.3, the fact that we consider rank r subcodes in place of r-dimensional subspaces leads to a different set of invariants.

In the next examples we exhibit two pairs of non-isometric codes. The codes in the first example can be distinguished using the generalized weights, but not using the generalized Hamming weights, while the codes in the second example can be distinguished using the generalized Hamming weights, but not using the generalized weights.

Example 8.12. (a) Let $C_1, C_2 \in \mathbb{F}_q[x]^3$ be the convolutional codes generated respectively by $(1, 0, 0), (0, 1, 1 + x)$ and $(1, 0, 0), (0, 0, 1)$. Then, $d'_r(C_1) = d'_r(C_2) = r$ for every positive integer

r. By computing the generalized weights according to Definition 8.3, we can prove that the two codes are not isometric, since $d_1(C_1) = d_1(C_2) = 1$, $d_2(C_1) = 4$ and $d_2(C_2) = 2$.

(b) Let $C_1, C_2 \in \mathbb{F}_q[x]^3$ be the convolutional codes generated respectively by $(1, 1, 1)$ and $(1 + x, 0, 1)$. Then $d_1(C_1) = d_1(C_2)$, that is, the generalized weights of C_1 and C_2 coincide. However, $d'_2(C_1) = 6$ and $d'_2(C_2) = 5$, in particular the codes are not isometric, as generalized Hamming weights are invariant under isometry.

Notice that an (n, k, δ) convolutional code has exactly k generalized weights and an infinite number of generalized Hamming weights. In particular, one can recover the rank of a convolutional code from its generalized weights, but not from its generalized Hamming weights, as the next example shows.

Example 8.13. Let $C \subseteq \mathbb{F}_2[x]^n$ be such that $(1, 0, \dots, 0) \in C$. Then $d'_r(C) = r$ for any $r \geq 1$. Clearly, there exist codes of any rank $k \leq n$ which contain the codeword $(1, 0, \dots, 0)$.

It may happen that, in order to distinguish two non-isometric codes, one needs to compute an arbitrarily large number of generalized Hamming weights. For instance, in the next example we show that, for fixed n and k, there exist non-isometric convolutional codes with the same first N generalized weights for N arbitrary large as δ goes to infinity. We will use the following simple lemma.

Lemma 8.14. Let q be a prime number. For a polynomial $p(x) \in \mathbb{F}_q[x] \setminus \{0\}$ we have that

$$\mathrm{wt}\left(p(x)\sum_{t=0}^{N} x^{q^t}\right) + \mathrm{wt}(p(x)) \geq N + 2.$$

Proof. Let $p(x) = a_1x^{k_1} + \dots + a_\ell x^{k_\ell}$ be a polynomial with $k_1 < \dots < k_\ell$ and $\mathrm{wt}(p(x)) = \ell > 0$. If $\ell \geq N + 1$ or $\ell = 1$ the statement is trivially true. Suppose $\ell < N + 1$. Clearly,

$$p(x)\sum_{t=0}^{N} x^{q^t} = \sum_{t=0}^{N}\sum_{i=1}^{\ell} a_i x^{k_i+q^t}. \tag{8.1.1}$$

Two monomials have the same exponent if and only if there are i_1, i_2, t_1, t_2 such that $k_{i_1} + q^{t_1} = k_{i_2} + q^{t_2}$. Moreover if $(t_1, t_2) \neq (t_3, t_4)$ with $t_1 < t_2$ and $t_3 < t_4$ then $q^{t_2} - q^{t_1} \neq q^{t_4} - q^{t_3}$. Therefore, in (8.1.1) there are at most $\ell(\ell - 1)/2$ pairs of monomials with the same exponent. Since the number of monomials in the sum is $(N + 1)\ell$, we have that

$$\mathrm{wt}\left(p(x)\sum_{t=0}^{N} x^{q^t}\right) + \ell \geq (N + 1)\ell - \ell(\ell - 1) + \ell = \ell(N - \ell + 3).$$

Finally, since $1 < \ell < N + 1$, we have that $\ell(N - \ell + 3) \geq N + 2$. □

Example 8.15. Let q be a prime number and let $C_N = \langle (1, 1, 0), (0, \sum_{t=0}^{N} x^{q^t}, 1) \rangle_{\mathbb{F}_q[x]} \subseteq \mathbb{F}_q[x]^3$. We claim that $d'_r(C_N) = 2r$ for all $r \in \{1, \dots, N + 1\}$ and $d'_{N+2}(C_N) = 2(N + 1) + 1$. In particular, the first N generalized weights of C_{N-1} and C_N coincide and $d'_{N+1}(C_{N-1}) \neq d'_{N+1}(C_N)$.

Let $U = \langle p_{i,1}(x)(1, 1, 0) + p_{i,2}(x)(0, \sum_{t=0}^{N} x^{q^t}, 1) \mid 1 \leq i \leq r \rangle_{\mathbb{F}_q} \subseteq C_N$ be a linear subspace of dimension $r \leq N + 1$. Let $J \subseteq \{1, \dots, r\}$ be a maximal set of indices such that $\{p_{j,1}(x)\}_{j \in J}$ is an $\mathbb{F}_q$-linearly independent set. If $|J| = r$, then $d'_r(C_N) \geq 2r$. If $|J| < r$, we may assume without loss of generality that $p_{i,1}(x) = 0$ for every $i \notin J$. It is easy to show that the support of U has cardinality at least $|J|$, when restricted to the first component. The set $\{p_{j,2}(x)\}_{j\notin J}$ is $\mathbb{F}_q$-linearly independent by assumption. Moreover, after replacing the elements of the set with appropriate linear combinations, we may assume that there exists a $\bar{j} \notin J$ such that the last entry of the support of U has cardinality greater than or equal to $r - |J| - 1 + \mathrm{wt}(p_{\bar{j},2}(x))$. Since the cardinality of the second entry of the support of U is at least $\mathrm{wt}\left(p_{\bar{j},2}(x)\sum_{t=0}^{N} x^{q^t}\right)$, then

$$d_r'(C_N) \geq |J| + r - |J| - 1 + \mathrm{wt}(p_{\bar{j},2}(x)) + \mathrm{wt}\left(p_{\bar{j},2}(x)\sum_{t=0}^{N} x^{q^t}\right).$$

By Lemma 8.14, we conclude that

$$d_r'(C_N) \geq N + 1 + r \geq 2r.$$

Let $V = \langle (x^i, x^i, 0) \mid 1 \leq i \leq r\rangle_{\mathbb{F}_q}$. Then V is an r-dimensional subspace of C_N with $|\mathrm{supp}(V)| = 2r$, showing that $d_r'(C_N) = 2r$ for $1 \leq r \leq N+1$.

Moreover, $V = \langle (0, \sum_{t=0}^{N} x^{q^t}, 1), (x^{q^i}, x^{q^i}, 0) \mid\leq i \leq N\rangle_{\mathbb{F}_q}$ is an $(N+2)$-dimensional $\mathbb{F}_q$-linear subspace of C_N, showing that $d_{N+2}'(C_N) \leq 2(N+1)+1$. Since $d_{N+1}'(C_N) = 2(N+1)$, then $d_{N+2}'(C_N) = 2(N+1)+1$ by Proposition 8.6.

The next example is concerned with the relation between the generalized weights of a convolutional code and its erasure-correction properties.

Example 8.16. Let $C_1 = \langle (1,0,0),(0,1,0)\rangle_{\mathbb{F}_q[x]}$ and $C_2 = \langle (1,0,0),(0,1,1+x)\rangle_{\mathbb{F}_q[x]} \subseteq \mathbb{F}_q[x]^3$. Then $d_{\mathrm{free}}(C_1) = d_{\mathrm{free}}(C_2) = 1$ and accordingly for both codes there are single erasure patterns that cannot be repaired. For example, if $*$ denotes an erasure, for both codes a received word $(*,0,0)$ has distance one from both $(0,0,0)$ and $(1,0,0)$. In addition, the first code has the property that an erasure in the second coordinate cannot be repaired, while an erasure in the third can always be repaired. We claim that the second code has the property that any two erasures located in the second and third coordinate can be repaired. In fact, such a repair corresponds to a repair in the code $C = \langle (1,1+x)\rangle_{\mathbb{F}_q[x]}$ obtained from C_2 by projecting on the last two coordinates and $d_{\mathrm{free}}(C) = 3$. This is reflected by the generalized weights of C_1 and C_2, which are

$$d_1(C_1) = d_1(C_2) = 1, \quad d_2(C_1) = 2, \quad d_2(C_2) = 4.$$

In the next proposition we make an attempt at formalizing the phenomenon that we observe in the previous example.

Proposition 8.17. Let $C \subseteq \mathbb{F}_q[x]^n$ be an (n,k) convolutional code with generalized weights $d_1 < d_2 < \ldots < d_k$. Fix $1 \leq r < k$ and let $c_1(x),\ldots,c_r(x) \in C$ be such that $\langle c_1(x),\ldots,c_r(x)\rangle_{\mathbb{F}_q[x]}$ realizes the r-th generalized weight of C. Let $c_{r+1}(x),\ldots,c_k(x) \in C$ be such that $c_1(x),\ldots,c_k(x)$ generates a subcode of rank k. Then $\mathcal{D} = \langle c_{r+1}(x),\ldots,c_k(x)\rangle_{\mathbb{F}_q[x]}$ is an $(n,k-r)$ subcode of C with $d_{\mathrm{free}}(\mathcal{D}) = d_{r+1} - d_r$.

Proof. Let $c(x) \in \mathcal{D} \setminus \{0\}$. Then $C' = \langle c_1(x),\ldots,c_r(x),c(x)\rangle_{\mathbb{F}_q[x]}$ has rank $r+1$ and

$$|\mathrm{supp}\{c_1(x),\ldots,c_r(x),c(x)\}| \geq \mathrm{wt}(C') \geq d_{r+1}.$$

It follows that $\mathrm{wt}(c(x)) \geq d_{r+1} - |\mathrm{supp}\{c_1(x),\ldots,c_r(x)\}| = d_{r+1} - d_r$. □

The proposition shows in particular that a large difference between two consecutive generalized weights of a code reflects the fact that there is a subcode with large erasure-correction capability. In addition, the proposition provides a concrete description of such a subcode. In Example 8.16, the subcode $\mathcal{D}$ in the statement of the proposition is $\mathcal{D} = \langle(0, 1, 1 + x)\rangle_{\mathbb{F}_q[x]}$. Notice moreover that the codes of Example 8.16 have the same generalized Hamming weights. This shows that the generalized Hamming weights of a convolutional code do not capture this phenomenon.

In the next proposition, we collect some facts on $(n, k, 0)$ convolutional codes that we will use in the sequel.

Proposition 8.18. Let $C \subseteq \mathbb{F}_q[x]^n$ be an $(n, k, 0)$ convolutional code. Then $C = \langle C[0]\rangle_{\mathbb{F}_q[x]}$ and $C[0] \subseteq \mathbb{F}_q^n$ is a linear block code with $\dim(C[0]) = k$ and minimum distance $d_{\min}(C[0]) = d_{\text{free}}(C)$. Moreover $C^\perp = \langle C[0]^\perp\rangle_{\mathbb{F}_q[x]}$.

Proof. Since C is an $(n, k, 0)$ convolutional code, then $C[0] = C \cap \mathbb{F}_q^n$ and $C = \langle C[0]\rangle_{\mathbb{F}_q[x]}$. Since $C[0] \subseteq \mathbb{F}_q^n$, then

$$\dim(C[0]) = \mathrm{rk}(\langle C[0]\rangle_{\mathbb{F}_q[x]}) = \mathrm{rk}(C).$$

Since $C[0] \hookrightarrow C$ is an $\mathbb{F}_q$-linear isometry, $d_{\min}(C[0]) \geq d_{\text{free}}(C)$. However, for any $c(x) \in C$, $c[0] \in C[0]$ and $\mathrm{wt}(c[0]) \leq \mathrm{wt}(c(x))$, showing that $d_{\min}(C[0]) \leq d_{\text{free}}(C)$. Finally, the equality $C^\perp = \langle C[0]^\perp\rangle_{\mathbb{F}_q[x]}$ follows from the definition of dual code. □

The next proposition relates the generalized weights of an $(n, k, 0)$ convolutional code with the generalized Hamming weights of the linear block code generated by the same elements.

Proposition 8.19. Let C be an $(n, k, 0)$ convolutional code. Then

$$d_r(C) = d_r^H(C[0])$$

for $1 \leq r \leq \mathrm{rk}(C) = \dim(C[0])$, where $d_r^H(C[0])$ denotes the r-th generalized Hamming weight of $C[0]$. In particular

$$d_r(C) \leq n + k - r.$$

In addition, if $D \subseteq C[0]$ is an r-dimensional subspace such that $d_r^H(C[0]) = |\mathrm{supp}(D)|$, then $\mathcal{D} = \langle D\rangle_{\mathbb{F}_q[x]} \subseteq C$ realizes $d_r(C)$. In particular

$$d_{\mathrm{rk}(C)}(C) = \mathrm{wt}(C) = |\mathrm{supp}(C[0])|,$$

where $\mathrm{supp}(C[0])$ denotes the Hamming support of $C[0]$.

Proof. Since $C = \langle C[0]\rangle_{\mathbb{F}_q[x]}$ and $C[0] \subseteq \mathbb{F}_q^n$, then $\mathrm{rk}(C) = \dim(C[0])$ by Proposition 8.18. Fix $1 \leq r \leq \mathrm{rk}(C)$ and let $D \subseteq C[0]$ be an $\mathbb{F}_q$-linear subspace such that $\dim(D) = r$ and $d_r^H(C[0]) = |\mathrm{supp}(D)|$. Let $\mathcal{D} = \langle D\rangle_{\mathbb{F}_q[x]} \subseteq C$, then $\mathrm{rk}(\mathcal{D}) = r$ and $\mathrm{wt}(\mathcal{D}) \leq |\mathrm{supp}(D)| = d_r^H(C[0])$. This implies that

$$d_r^H(C[0]) \geq d_r(C).$$

To prove the reverse inequality, let $\mathcal{D} \subseteq C$ be a rank r subcode and let $v_1, \ldots, v_r$ be an $\mathbb{F}_q[x]$-basis of $\mathcal{D}$ such that $\mathrm{wt}(\mathcal{D}) = |\mathrm{supp}(\langle v_1, \ldots, v_r\rangle_{\mathbb{F}_q})|$. For $1 \leq i \leq r$ write $v_i = \sum_{j=0}^{t_i} v_i[j]x^j$. Since $C = \langle C[0]\rangle_{\mathbb{F}_q[x]}$, then $v_i[j] \in C[0]$ for all i, j. Let $U = \langle\{v_i[j]\}_{j,i}\rangle_{\mathbb{F}_q}$. By construction $\mathcal{D} \subseteq \langle U\rangle_{\mathbb{F}_q[x]}$

and $\mathrm{wt}(\mathcal{D}) = |\mathrm{supp}(\langle v_1, \ldots, v_r\rangle_{\mathbb{F}_q})| \geq |\mathrm{supp}(U)|$. Since $\dim(U) = \mathrm{rk}(\langle U\rangle_{\mathbb{F}_q[x]}) \geq \mathrm{rk}(\mathcal{D}) = r$, one can find $U' \subseteq U$ an $\mathbb{F}_q$-linear subspace with $\dim(U') = r$. Then $\mathrm{rk}(\langle U'\rangle_{\mathbb{F}_q[x]}) = r$ and

$$\mathrm{wt}(\langle U'\rangle_{\mathbb{F}_q[x]}) \leq |\mathrm{supp}(U')| \leq |\mathrm{supp}(U)| \leq \mathrm{wt}(\mathcal{D}).$$

Observe in addition that, if $\mathcal{D}$ has an $\mathbb{F}_q[x]$-basis $v'_1, \ldots, v'_r$ which consists of elements of $\mathbb{F}_q^n$, then $U = U' = \langle v'_1, \ldots, v'_r\rangle_{\mathbb{F}_q}$, showing that

$$\mathrm{wt}(\mathcal{D}) = |\mathrm{supp}(\langle v'_1, \ldots, v'_r\rangle_{\mathbb{F}_q})|.$$

Summarizing we have shown that, for every submodule $\mathcal{D} \subseteq C$ of rank r, one can find a submodule of C of the form $\langle U'\rangle_{\mathbb{F}_q[x]}$ for some $U' \subseteq C[0]$, with rank r and weight smaller than or equal to the weight of $\mathcal{D}$. Since $|\mathrm{supp}(U')| = \mathrm{wt}(\langle U'\rangle_{\mathbb{F}_q[x]})$, this implies that $d_r^H(C[0]) \leq d_r(C)$. □

The generalized Hamming weights of an $(n, k, 0)$ convolutional code may not coincide with the generalized Hamming weights of the linear block code generated by the same elements, as the next example shows.

Example 8.20. Let $C = \langle (1, 0, 0), (0, 1, 1)\rangle_{\mathbb{F}_2[x]} \subseteq \mathbb{F}_2^3$ and let $C[0] \subseteq \mathbb{F}_2^3$. The generalized Hamming weights of $C[0]$ are $d_1^H(C[0]) = 1$ and $d_2^H(C[0]) = 3$, while the generalized Hamming weights of C are $d'_r(C) = r$ for all $r \geq 1$.

The next result is obtained by combining Proposition 8.19 and Wei duality for the generalized Hamming weights of a linear block code.

Proposition 8.21. Let C be an $(n, k, 0)$ convolutional code. Then $C^\perp$ is an $(n, n-k, 0)$ convolutional code and its set of generalized weights is

$$\{d_r(C^\perp) \mid 1 \leq r \leq n-k\} = \{n + 1 - d_r(C) \mid 1 \leq r \leq k\}.$$

In particular, the generalized weights of C determine those of $C^\perp$.

Proof. By Proposition 8.18, C has the form $C = \langle C[0]\rangle_{\mathbb{F}_q[x]}$ and $\dim(C[0]) = k$. Moreover, $C^\perp = \langle C[0]^\perp\rangle_{\mathbb{F}_q[x]}$ is an $(n, n-k, 0)$ convolutional code. Wei duality states that

$$\{d_r^H(C[0]^\perp) \mid 1 \leq r \leq n-k\} = \{n + 1 - d_r^H(C[0]) \mid 1 \leq r \leq k\},$$

see [Wei91, Theorem 3]. We conclude by Proposition 8.19. □

For a catastrophic convolutional code, one may have that $C \subsetneq (C^\perp)^\perp$. In such a situation, one expects to be able to find a code such that C and $(C^\perp)^\perp$ have different generalized weights. This is indeed the case, as the next example shows.

Example 8.22. Let $C = \langle (1 + x, 0)\rangle_{\mathbb{F}_2[x]} \subseteq \mathbb{F}_2[x]^2$. Then $C^\perp = \langle (0, 1)\rangle$ and $(C^\perp)^\perp = \langle (1, 0)\rangle \supsetneq C$. In addition, $d_1(C) = 2$ while $d_1((C^\perp)^\perp) = 1$. In particular, C and $(C^\perp)^\perp$ have different generalized weights, while having the same dual code $C^\perp$.

Noncatastrophic convolutional codes coincide with their double dual. However, no result along the lines of Wei duality holds even when restricting to this class of codes. More precisely, the next example shows that there exist noncatastrophic convolutional codes with the same generalized weights and whose dual codes have different generalized weights.

Example 8.23. Let $C_1 = \langle(1+x, 1+x, 1, 0)\rangle_{\mathbb{F}_q[x]}$ and $C_2 = \langle(1+x, 1, 1, 1)\rangle_{\mathbb{F}_q[x]}$ be $(4, 1, 1)$ noncatastrophic convolutional codes. We have that $d_1(C_1) = d_1(C_2) = 5$, while $d_1(C_1^\perp) = 1$ and $d_1(C_2^\perp) = 2$.

The previous example also shows that Wei duality cannot hold for any set of generalized weights with the property that the first generalized weight is the free distance of the code. More precisely, we have the following.

Remark 8.24. Given any definition of generalized weights for convolutional codes such that the first generalized weight is the free distance of the code, the generalized weights of a code do not determine in general the generalized weights of its dual.

Although Wei duality does not hold for this type of dual, there are other types of duality that have been considered in the literature. For example, in [TRS12] the authors define the reverse of a convolutional code.

Definition 8.25. Let C be an (n, k, δ) convolutional code and let $\mathrm{rev} : \mathbb{F}_q[x]^n \to \mathbb{F}_q[x]^n$ be the map given by $\mathrm{rev}(0) = 0$ and

$$\mathrm{rev}(c(x)) = x^{\deg(c(x))} c\left(\frac{1}{x}\right)$$

if $c(x) \neq 0$. The reverse code of C is

$$\mathrm{rev}(C) = \langle \mathrm{rev}(c(x)) : c(x) \in C\rangle_{\mathbb{F}_q[x]}.$$

Remark 8.26. Let $c(x) \in \mathbb{F}_q[x]^n \setminus \{0\}$. The following are equivalent:

1. $\mathrm{rev}(\mathrm{rev}(c(x))) = c(x)$,
2. $\deg(c(x)) = \deg(\mathrm{rev}(c(x)))$,
3. $x \nmid c(x)$.

In addition, one has $x^d \mathrm{rev}(\mathrm{rev}(c(x))) = c(x)$ for $d = \max\{t \geq 0 : x^t \mid c(x)\}$.

In [TRS12] Tomás, Rosenthal, and Smarandache define the reverse of a convolutional code C as the code generated by the reverses of the rows of a row-reduced generator matrix of C. For the sake of completeness, we prove that Definition 8.25 is equivalent to the definition from [TRS12]. This implies in particular that if C is an (n, k, δ) convolutional code, then $\mathrm{rev}(C)$ is an (n, k, δ') convolutional code, for some δ'.

Proposition 8.27. Let C be an (n, k, δ) convolutional code and let G be a row-reduced generator matrix for C. Let $c_1, \dots, c_k$ be the rows of G. Then

$$\mathrm{rev}(C) = \langle \mathrm{rev}(c_1), \dots, \mathrm{rev}(c_k)\rangle_{\mathbb{F}_q[x]}.$$

In particular, $\mathrm{rk}(\mathrm{rev}(C)) = k$.

Proof. It is clear from the definition that $\mathrm{rev}(C) \supseteq \langle \mathrm{rev}(c_1), \dots, \mathrm{rev}(c_k)\rangle_{\mathbb{F}_q[x]}$. Therefore, it suffices to show that $\mathrm{rev}(c) \in \langle \mathrm{rev}(c_1), \dots, \mathrm{rev}(c_k)\rangle_{\mathbb{F}_q[x]}$ for all $c \in C$. Let $c \in C$ and let

$u_1, \ldots, u_k \in \mathbb{F}_q[x]$ such that $c = \sum u_i c_i$. Since G is row-reduced, we have that $\deg(c) = \max_i\{\deg(u_i) + \deg(c_i)\}$. Therefore

$$\mathrm{rev}(c) = x^{\deg(c)} \sum_{i=1}^{k} u_i\left(\frac{1}{x}\right) c_i\left(\frac{1}{x}\right) = \sum_{i=1}^{k} x^{\deg(c)-\deg(c_i)} u_i\left(\frac{1}{x}\right) \mathrm{rev}(c_i).$$

Since $\deg(c) = \max_i\{\deg(u_i) + \deg(c_i)\}$, then $x^{\deg(c)-\deg(c_i)} u_i\left(\frac{1}{x}\right) \in \mathbb{F}_q[x]$ for all i. □

Corollary 8.28. Let C be an (n, k, δ) convolutional code. The following hold:

1. $\mathrm{rev}(\mathrm{rev}(C)) \supseteq C$.
2. There exists a positive integer d, such that $x^d \mathrm{rev}(\mathrm{rev}(C)) \subseteq C$.
3. If C is noncatastrophic, then $\mathrm{rev}(\mathrm{rev}(C)) = C$.

Proof. 1. Let G be a row-reduced generator matrix for C and let $c_1, \ldots, c_k$ be the rows of G. For every $1 \le i \le k$ there exists $t_i \ge 0$ such that $x^{t_i}\mathrm{rev}(\mathrm{rev}(c_i)) = c_i$. Therefore, $\mathrm{rev}(\mathrm{rev}(C)) \supseteq C$.

2. Consider the ascending chain of modules

$$C \subseteq C : x \subseteq \ldots \subseteq C : x^t \subseteq \ldots \tag{8.1.2}$$

where $C : x^t = \{c(x) \in \mathbb{F}_q[x]^n \mid x^t c(x) \in C\}$ for $t \ge 0$. Since every submodule of $\mathbb{F}_q[x]^n$ is finitely generated, any ascending chain of submodules of $\mathbb{F}_q[x]^n$ is stationary. This means that there exists a d such that $C : x^d = C : x^t$ for any $t \ge d$. Let $c \in C$, then $c = x^t \mathrm{rev}(\mathrm{rev}(c))$ for some $t \ge 0$, hence $\mathrm{rev}(\mathrm{rev}(c)) \in C : x^t$. If $t \le d$, then $C : x^t \subseteq C : x^d$. If instead $t \ge d$, then $C : x^t = C : x^d$. In both cases, $\mathrm{rev}(\mathrm{rev}(c)) \in C : x^d$. It follows that $C : x^d \supseteq \mathrm{rev}(\mathrm{rev}(C))$. This implies that

$$x^d \mathrm{rev}(\mathrm{rev}(C)) \subseteq C.$$

3. For a noncatastrophic code one has $\{c(x) \in \mathbb{F}_q[x]^n \mid x^d c(x) \in C\} = C$ for all $d \ge 0$ by Proposition 2.14. Therefore, all the containments in (8.1.2) are equalities. In particular

$$\mathrm{rev}(\mathrm{rev}(C)) \subseteq C : x^d = C \subseteq \mathrm{rev}(\mathrm{rev}(C))$$

where the first containment follows from 2. and the second from 1. □

The next proposition proves that the generalized weights of C are the same as those of $\mathrm{rev}(C)$.

Proposition 8.29. Let C be an (n, k, δ) convolutional code. Then

$$d_r(C) = d_r(\mathrm{rev}(C)),$$

for $1 \le r \le k$.

Proof. Let $\mathcal{D} = \langle c_1, \ldots, c_r \rangle_{\mathbb{F}_q[x]}$ be a submodule of C that realizes $d_r(C)$ and such that $\mathrm{wt}(\mathcal{D}) = |\mathrm{supp}(\langle c_1, \ldots, c_r \rangle_{\mathbb{F}_q})|$. Let $s = \max\{\deg(c_i) : 1 \le i \le r\}$ and let

$$\mathcal{D}' = \langle x^{s-\deg(c_1)}\mathrm{rev}(c_1), \ldots, x^{s-\deg(c_r)}\mathrm{rev}(c_r)\rangle_{\mathbb{F}_q[x]}.$$

Since $\mathcal{D}' \subseteq \mathrm{rev}(C)$ and $\mathrm{rk}\,(\mathcal{D}') = \mathrm{rk}\,(\mathrm{rev}(\mathcal{D})) = r$, then

$$d_r(\mathrm{rev}(C)) \leq \mathrm{wt}(\mathcal{D}') \leq |\mathrm{supp}(\langle x^{s-\deg(c_1)}\mathrm{rev}(c_1), \dots, x^{s-\deg(c_r)}\mathrm{rev}(c_r)\rangle_{\mathbb{F}_q})| = \mathrm{wt}(\mathcal{D}) = d_r(C).$$

To prove the reverse inequality, let $\mathcal{D} \subseteq \mathrm{rev}(\mathrm{rev}(C))$ be a submodule that realizes $d_r(\mathrm{rev}(\mathrm{rev}(C)))$. By Corollary 8.28 there exists a positive integer d such that $x^d\mathcal{D} \subseteq C$. Since $\mathrm{rk}(x^d\mathcal{D}) = \mathrm{rk}(\mathcal{D}) = r$, then

$$d_r(C) \leq \mathrm{wt}(x^d\mathcal{D}) = \mathrm{wt}(\mathcal{D}) = d_r(\mathrm{rev}(\mathrm{rev}(C))).$$

Therefore

$$d_r(C) \leq d_r(\mathrm{rev}(\mathrm{rev}(C))) \leq d_r(\mathrm{rev}(C)) \leq d_r(C).$$

□

8.2. Minimal supports

In this chapter we study codewords of minimal support and submodules that realize the generalized weights of a convolutional code. We show that it is possible to calculate the generalized weights considering only subspaces with special properties. In particular, in Theorem 8.37 we prove that, in order to compute $d_r(C)$, we may restrict to subspaces generated by vectors, whose degree is bounded by a function of n, k, r and δ_1 only. Moreover, we show that the generalized weights are realized by subspaces generated by codewords of minimal support.

Some of the results in this section are similar to those obtained in [GR22, Section 3] for a large family of support functions and codes over rings. Notice however that the setup of [GR22] does not apply to our situation, as the Hamming support for convolutional codes is not a support according to Definition 3.4.

Definition 8.30. Let $C \subseteq \mathbb{F}_q[x]^n$ be a convolutional code. A codeword $c \in C$ is minimal if its support is minimal among the supports of the nonzero codewords of C.

It is easy to show that, for a given code, minimal supports correspond uniquely to minimal codewords, up to a nonzero scalar multiple.

Lemma 8.31. Let $C \subseteq \mathbb{F}_q[x]^n$ be a convolutional code. If two minimal codewords $u(x), v(x) \in C$ have the same support, then there exists $\alpha \in \mathbb{F}_q^*$ such that $u(x) = \alpha v(x)$.

Proof. If $u(x), v(x) \in C$ have the same support, then there exists $\alpha \in \mathbb{F}_q^*$ such that $\mathrm{supp}(u(x) - \alpha v(x)) \subsetneq \mathrm{supp}(u(x))$. By the minimality of the support of $u(x)$, we deduce that $u(x) - \alpha v(x) = 0$. □

In this section, we study the subcodes of C which realize its generalized weights. We start by showing that if C is a noncatastrophic convolutional code, then each of its generalized weights is realized by a subspace that contains an element that is not divisible by x.

Theorem 8.32. Let C be a noncatastrophic convolutional code of rank k. For $1 \leq r \leq k$, consider the set

$$\mathcal{U}_r = \{\mathcal{D} \subseteq C \text{ is a subcode of } \mathrm{rk}(\mathcal{D}) = r \text{ and } \exists\, c \in \mathcal{D} \text{ with } c[0] \neq 0\}.$$

Then

$$d_r(C) = \min\{\mathrm{wt}(\mathcal{D}) \mid \mathcal{D} \in \mathcal{U}_r\}.$$

Proof. Let $\mathcal{D} = \langle c_1, \dots, c_r \rangle_{\mathbb{F}_q[x]} \subseteq C$ be a subcode that realizes the r-th generalized weight and such that $\mathrm{wt}(\mathcal{D}) = |\mathrm{supp}(\langle c_1, \dots, c_r \rangle_{\mathbb{F}_q})|$. Let $\ell = \max\{d \geq 0 : x^d \mid c_i \text{ for } 1 \leq i \leq k\}$. Since C is noncatastrophic, by Proposition 2.14 we have that $\mathcal{D}' = \langle c_1/x^\ell, \dots, c_r/x^\ell \rangle \in \mathcal{U}_r$. This concludes the proof, since $\mathrm{wt}(\mathcal{D}) = |\mathrm{supp}(\langle c_1, \dots, c_r \rangle_{\mathbb{F}_q})| = |\mathrm{supp}(\langle c_1/x^\ell, \dots, c_r/x^\ell \rangle_{\mathbb{F}_q})| \geq \mathrm{wt}(\mathcal{D}')$. □

Let $\phi_{d,\delta} : \mathbb{F}_q[x] \to \mathbb{F}_q[x]_{\leq \delta}$ be the linear map given by

$$\phi_{d,\delta}(a_0 x + \dots + a_{d-\delta} x^{d-\delta} + \dots a_d x^d + \dots + a_n x^n) = a_{d-\delta} + a_{d-\delta+1} x + \dots a_d x^\delta,$$

where $\mathbb{F}_q[x]_{\leq \delta}$ denotes the set of polynomials of degree at most δ. We can extend this map to $\mathbb{F}_q[x]^{k \times n}$ by applying $\phi_{d,\delta}$ to each entry. This yields the map $\Phi_{d,\delta} : \mathbb{F}_q[x]^{k \times n} \to \mathbb{F}_q[x]^{k \times n}_{\leq \delta}$, given by

$$\Phi_{d,\delta}((m_{i,j}(x))_{i,j}) = (\phi_{d,\delta}(m_{i,j}(x)))_{i,j}.$$

In order to simplify the notation, given a matrix $M \in \mathbb{F}_q[x]^{k \times n}$, we write $M_t = \Phi_{t,t}(M)$. The next lemma follows directly from the definition.

Lemma 8.33. Let $M \in \mathbb{F}_q[x]^{k \times n}$ be a matrix with entries in $\mathbb{F}_q[x]$.

1. If each entry of M is divisible by x^t for some $t \in \mathbb{N}$, then

$$\Phi_{d,\delta}(x^{-t} M) = \Phi_{d+t,\delta}(M).$$

2. If $d - \delta \leq \deg(M) \leq d$, then

$$\Phi_{d,\delta}(M) = \Phi_{\deg(M), \delta - d + \deg(M)}(M).$$

The next lemma is crucial for the proof of Theorem 8.37.

Lemma 8.34. Let C be an (n, k, δ) convolutional code, G a row-reduced generator matrix for C with memory $\delta_1 \geq 1$, $M = (m_{i,j}(x))_{i,j} \in \mathbb{F}_q[x]^{r \times k}$, and $1 \leq s_1 < s_2 \leq d = \max_{i,j} \deg(m_{i,j}(x))$ natural numbers. If $s_2 - s_1 \geq q^{\delta_1 k r}$, then there exist $M' = (m'_{i,j}(x))_{i,j} \in \mathbb{F}_q[x]^{r \times k}$ and a natural number $t < d' = \max_{i,j} \deg(m'_{i,j}(x))$ such that:

1. $s_1 \leq t < t + d - d' \leq s_2$.

2. $(M'G)_t = (MG)_t$.

3. $\Phi_{d+\delta_1, d'+\delta_1-t-1}(MG) = \Phi_{d'+\delta_1, d'+\delta_1-t-1}(M'G)$.

Proof. Consider the set $F = \{\Phi_{s+\delta_1, \delta_1-1}(M_s G) : s_1 \leq s \leq s_2\}$ and denote by A the following set

$$A = \bigcup_{h=\delta_1}^{\infty} \left\{ \Phi_{h,\delta_1-1} \begin{pmatrix} c_1 \\ \vdots \\ c_r \end{pmatrix} : c_i \in C \text{ and } \deg(c_i) \leq h \right\}.$$

Since $s + \delta_1 > \delta_1$ and the rows of $M_s G$ are elements of C of degree smaller than or equal to $s + \delta_1$, then

$$|F| \leq |A| \leq q^{\delta_1 k r}.$$

The second inequality follows from observing that there are r rows, each row is a combination of k generators of C and there are δ_1 possible shifts.

Since $s_2 - s_1 \geq q^{\delta_1 kr}$, by the pigeonhole principle there exist $s_1 \leq t < t' \leq s_2$ such that $\Phi_{t+\delta_1,\delta_1-1}(M_t G) = \Phi_{t'+\delta_1,\delta_1-1}(M_{t'}G)$. Let

$$M' = M_t + (M - M_{t'})x^{t-t'}. \tag{8.2.3}$$

We claim that M' and t satisfy the statement. Since $d \geq t' > t$, it follows from equation (8.2.3) that $d' = d - t' + t$ and therefore $s_1 \leq t < t' = d + t - d' \leq s_2$. Moreover

$$(M'G)_t = (M_t G + (M - M_{t'})x^{t-t'}G)_t = (M_t G)_t = (MG)_t.$$

Finally,

$$\begin{aligned}
&\Phi_{d'+\delta_1,d'+\delta_1-t-1}(M'G) = \Phi_{d'+\delta_1,d'+\delta_1-t-1}((M_t + (M - M_{t'})x^{t-t'})G) = \\
&= \Phi_{d'+\delta_1,d'+\delta_1-t-1}(M_t G) + \Phi_{d'+\delta_1,d'+\delta_1-t-1}(((M - M_{t'})x^{t-t'})G) = \\
&= \Phi_{t+\delta_1,\delta_1-1}(M_t G) + \Phi_{d'+\delta_1+t'-t,d'+\delta_1-t-1}((M - M_{t'})G) = \\
&= \Phi_{t+\delta_1,\delta_1-1}(M_t G) + \Phi_{d+\delta_1,d'+\delta_1-t-1}(MG) - \Phi_{t'+\delta_1,\delta_1-1}(M_{t'}G) = \Phi_{d+\delta_1,d'+\delta_1-t-1}(MG),
\end{aligned}$$

where the second and third equalities follow from Lemma 8.33, while the last one follows from the fact that $\Phi_{t+\delta_1,\delta_1-1}(M_t G) = \Phi_{t'+\delta_1,\delta_1-1}(M_{t'}G)$. □

Remark 8.35. By Lemma 8.34 we have that

$$|\mathrm{supp}(M'G)| \leq |\mathrm{supp}(MG)|.$$

Indeed

$$\begin{aligned}
|\mathrm{supp}(M'G)| &= |\mathrm{supp}((M'G)_t)| + |\mathrm{supp}(\Phi_{d'+\delta_1,d'+\delta_1-t-1}(M'G))| \\
&= |\mathrm{supp}((MG)_t)| + |\mathrm{supp}(\Phi_{d+\delta_1,d'+\delta_1-t-1}(MG))| \leq |\mathrm{supp}(MG)|.
\end{aligned}$$

Since G is a generator matrix of C, the submodules spanned by the rows of MG and $M'G$ are subcodes of C. Suppose that the submodule generated by the rows of MG realizes $d_r(C)$. Since Lemma 8.34 implies that $|\mathrm{supp}(M'G)| \leq |\mathrm{supp}(MG)|$, if we had $\mathrm{rk}(MG) = \mathrm{rk}(M'G)$, then the submodule generated by the rows of $M'G$ would realize $d_r(C)$ as well. However, it may happen that $\mathrm{rk}(MG) \neq \mathrm{rk}(M'G)$, as the next example shows.

Example 8.36. Let C be an $(3, 2, 1)$ code in $\mathbb{F}_2[x]^3$ with row-reduced generator matrix

$$G = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $M \in \mathbb{F}_2[x]^{2\times 2}$ be the following matrix of degree $d = 38$

$$M = \begin{pmatrix} x^{38} & x^{17} \\ x^{17} & 1 \end{pmatrix}.$$

Let $s_1 = 17$ and $s_2 = 37$. Since $s_2 - s_1 = 20 \geq 16 = 2^4 = q^{\delta_1 kr}$, we can apply Lemma 8.34 to M

and G. Let $M' \in \mathbb{F}_2[x]^{2\times 2}$ be the matrix

$$M' = \begin{pmatrix} x^{34} & x^{17} \\ x^{17} & 1 \end{pmatrix}.$$

We claim that M' with $t = 20$ satisfies the conditions in Lemma 8.34. Indeed, we have that $s_1 = 17 < t = 20 < 24 = 38 + 20 - 34 = d + t - d' < 37$. Moreover,

$$(M'G)_{20} = \begin{pmatrix} 0 & x^{17} & 0 \\ x^{17} & 1 & x^{18} \end{pmatrix} = (MG)_{20}.$$

Finally, we have that

$$\Phi_{35,14}(M'G)_{20} = \begin{pmatrix} x^{13} & 0 & x^{14} \\ 0 & 0 & 0 \end{pmatrix} = \Phi_{39,14}(MG).$$

Notice that $\det(M') = 0$ while $\det(M) = x^{38} - x^{34}$, therefore $\mathrm{rk}(MG) \neq \mathrm{rk}(M'G)$. Notice moreover that $(M', 20)$ is not the only pair that satisfies the conditions in Lemma 8.34 and that other pairs may behave differently. For example, consider the matrix $M'' \in \mathbb{F}_2[x]^{2\times 2}$ given by

$$M'' = \begin{pmatrix} x^{35} & x^{17} \\ x^{17} & 1 \end{pmatrix}.$$

One can verify that the pair $(M'', 20)$ satisfies the conditions in Lemma 8.34 and $\mathrm{rk}(MG) = \mathrm{rk}(M''G)$.

For a matrix $M \in \mathbb{F}_q^{m\times n}$ and $S \subseteq \{1, \ldots, k\}$, $L \subseteq \{1, \ldots .n\}$ we let $M(S, L)$ denote the submatrix of M consisting of the rows indexed by S and the columns indexed by L. We are now ready to prove the main result of this section. It says that, in order to compute generalized weights, it suffices to look at a subcodes generated by codewords of bounded degree. A similar result for the generalized Hamming weights of a convolutional code is stated without proof in [RY97, Yor97].

Theorem 8.37. Let C be an (n, k, δ) convolutional code with memory δ_1. Then, $d_r(C)$ is realized by a subspace generated by codewords of degree at most $((r+2)q^{\delta_1 kr} + 1)(n(\delta_1 + 1) - k + r) + \delta_1$.

Proof. If $\delta_1 = 0$, then $d_r(C)$ is realized by a subspace generated by codewords of degree zero by Proposition 8.19. If $\delta_1 \geq 1$, let G be a generator matrix for C with memory δ_1. Suppose that $d_r(C)$ is realized by a subspace generated by the rows of MG, where $M = (m_{i,j}(x))_{i,j} \in \mathbb{F}_q[x]^{r\times k}$ is a matrix with $\max_{i,j} \deg(m_{i,j}(x)) \geq ((r+2)q^{\delta_1 kr} + 1)(n(\delta_1 + 1) - k + r) + 1$ and such that there exists $N \subseteq \{1, \ldots n\}$ with $|N| = r$ and $MG[R, N]$ has non-zero determinant, where $R = \{1, \ldots, r\}$. By Proposition 8.6

$$d_r(C) \leq n(\delta_1 + 1) - k + r.$$

Therefore, there exist $s_1 > s_2 > \cdots > s_{r+3}$ such that

$$s_h - s_{h+1} \geq q^{\delta_1 kr} \text{ and } \Phi_{s_1, s_1 - s_{r+3}}(MG) = 0.$$

By induction, we define a chain of matrices $M^{(1)}, \ldots, M^{(r+2)}$ as follows. Applying Lemma 8.34 to M, s_1, s_2, we obtain $M^{(1)} = (m^{(1)}_{i,j}(x))_{i,j} \in \mathbb{F}_q[x]^{r\times k}$. In the same way, applying Lemma 8.34 to

$M^{(h)}, s_h, s_{h+1}$, we obtain a matrix $M^{(h+1)} = (m_{i,j}^{(h+1)}(x))_{i,j} \in \mathbb{F}_q[x]^{r\times k}$. By Lemma 8.34, we have

$$(MG)_{s_{r+3}} = (M^{(1)}G)_{s_{r+3}} = \cdots = (M^{(r+2)}G)_{s_{r+3}}$$

and

$$|\mathrm{supp}(MG)| = |\mathrm{supp}(M^{(1)}G)| = \cdots = |\mathrm{supp}(M^{(r+2)}G)|.$$

If there exists $1 \le \bar{h} \le r+2$ such that $\det(M^{(\bar{h})}G[R,N]) \neq 0$, then $d_r(C)$ is also realized by $MG^{(\bar{h})}$ and we conclude since the maximum row degree of $MG^{(\bar{h})}$ is smaller than that of MG. Else, compute the determinant of $MG[R,N]$, indicating the monomial operations without solving them. We obtain

$$\det(MG[R,N]) = \sum_u a_u x^{\alpha_{1,u}} \ldots x^{\alpha_{r,u}}.$$

We rewrite this sum as

$$\det(MG[R,N]) = \sum_{d,\ell} S(MG,d,\ell),$$

where with $S(MG,d,\ell)$ we denote the partial sum of those terms $a_u x^{\alpha_{1,u}} \ldots x^{\alpha_{r,u}}$ that appear in $\det(MG[R,N])$ such that $\sum \alpha_{h,u} = d$ and the number of exponents that are greater or equal than s_{r+3} is exactly ℓ. Notice that there exists at least a pair (d,ℓ) such that $S(MG,d,\ell) \neq 0$ since $\det(MG[R,N]) \neq 0$. In the same way we define $S(M^{(h)}G,d,\ell)$ for each $1 \le h \le r+2$. By construction, we have that for each $1 \le h \le r+2$ there exists a natural number w_h such that

$$\det(M^{(h)}G[R,N]) = \sum_{d,\ell} S(M^{(h)}G,d,\ell) = \sum_{d,\ell} S(MG,d,\ell)x^{-w_h(\ell)}. \tag{8.2.4}$$

For each $1 \le h \le r+2$, let y_h be

$$y_h = \max\{d : \text{ there exists } \ell \text{ such that } S(M^{(h)}G,d,\ell) \neq 0\}$$

and z_h be

$$z_h = \min\{\ell : S(M^{(h)}G,y_h,\ell) \neq 0\}.$$

We prove now that for $0 \le h \le r+1$ we have that $z_h > z_{h+1}$. Since $\det(M^{(h+1)}G[R,N]) = 0$, there exists $\bar{\ell} > z_{h+1}$ such that $S(M^{(h+1)}G, y_{h+1}, \bar{\ell}) \neq 0$. Then, by equation (8.2.4) we have that

$$y_h - (w_{h+1} - w_h)\bar{\ell} \ge y_{h+1} \ge y_h - (w_{h+1} - w_h)z_h.$$

So, we obtain that $z_{h+1} < \bar{\ell} \le z_h$. Therefore, we have that $r \ge z_1 > z_2 > \cdots > z_{r+2} \ge 0$. Since this is a contradiction, we conclude that there exists $h \in \{1,\ldots,r\}$ such that $\det(M^{(h)}G[R,N]) \neq 0$. □

Even though we do not expect the bound of Theorem 8.37 to be sharp, the theorem implies that the generalized weights of a convolutional code can be computed by exhaustive search in a finite number of steps. However, since the upper bound in Theorem 8.37 is large, it does not lead to a practical algorithm. It would be interesting to better understand under which assumptions and by how much this bound can be improved.

Question 8.38. Is it possible to sharpen the bound in Theorem 8.37?

In this section we improve the bound of Theorem 8.37 for two families of codes. Proposition 8.19 shows that, if $\delta = 0$, then $d_r(C)$ is realized by a subspace generated by codewords of degree 0 for all r, while Proposition 8.48 improves the bound of Theorem 8.37 for certain MDS codes.

In order to further simplify the computation of the generalized weights, we show that they are realized by subspaces generated by elements of minimal support. This is analogous to what happens for generalized Hamming weights of linear block codes and in fact for a much larger class of codes and supports, as discussed in [GR22, Section 3].

Lemma 8.39. Let C be an (n, k, δ) convolutional code and let $u_1, \dots, u_r \in C$ be such that $\operatorname{rk}\left(\langle u_1, \dots, u_r\rangle_{\mathbb{F}_q[x]}\right) = r$. Then, there exist $u'_1, \dots, u'_r \in C$ minimal codewords such that

$$\operatorname{supp}(u'_i) \subseteq \operatorname{supp}(u_i) \text{ for } 1 \le i \le r \text{ and } \operatorname{rk}\left(\langle u'_1, \dots, u'_r\rangle_{\mathbb{F}_q[x]}\right) = r.$$

Proof. Let $e \in C$ be a minimal codeword with $\operatorname{supp}(e) \subseteq \operatorname{supp}(u_1)$. If

$$\operatorname{rk}(\langle e, u_2, \dots, u_r\rangle_{\mathbb{F}_q[x]}) = r,$$

then let $u'_1 = e$. Else, we claim that there exists $\alpha \in \mathbb{F}_q^*$ such that $\operatorname{supp}(u_1 - \alpha e) \subsetneq \operatorname{supp}(u_1)$ and

$$\operatorname{rk}(\langle u_1 - \alpha e, u_2, \dots, u_r\rangle_{\mathbb{F}_q[x]}) = r.$$

In fact, if $\operatorname{rk}(\langle u_1 - \alpha e, u_2, \dots, u_r\rangle_{\mathbb{F}_q[x]}) < r$, then there exist $p_1(x), \dots, p_r(x), q_1(x), \dots, q_r(x) \in \mathbb{F}_q[x]$ such that $p_1(x), q_1(x) \neq 0$ and

$$p_1(x)e = \sum_{i=2}^{r} p_i(x)u_i \text{ and } q_1(x)(u_1 - \alpha e) = \sum_{i=2}^{r} q_i(x)u_i.$$

Therefore

$$p_1(x)q_1(x)u_1 = \sum_{i=2}^{r}(p_1(x)q_i(x) + \alpha q_1(x)p_i(x))u_i,$$

contradicting the assumption that $\operatorname{rk}\left(\langle u_1, \dots, u_r\rangle_{\mathbb{F}_q[x]}\right) = r$. If $u_1 - \alpha e$ is a minimal codeword, let $u'_1 = u_1 - \alpha e$, otherwise we repeat this process. Notice that, since at each step the support becomes strictly smaller, we find a minimal codeword in a finite number of steps. Proceeding in the same way for $u_2, \dots, u_r$ we find minimal codewords $u'_1, \dots, u'_r \in C$ with the desired properties. □

The next theorem shows that, when computing generalized weights, we may restrict to subcodes generated by minimal codewords with the property that the support of each of them is not contained in the union of the supports of the others.

Theorem 8.40. Let C be an (n, k, δ) convolutional code, let $1 \le r \le k$. Then there exist r minimal codewords $u_1, \dots, u_r \in C$ such that $\langle u_1, \dots, u_r\rangle_{\mathbb{F}_q[x]}$ realizes $d_r(C)$ and $\operatorname{supp}(u_i) \not\subseteq \bigcup_{j\neq i} \operatorname{supp}(u_j)$ for $1 \le i \le r$.

Proof. Let $\mathcal{U} = \langle u_1, \dots, u_r\rangle_{\mathbb{F}_q[x]} \subseteq C$ realize $d_r(C) = |\operatorname{supp}(\{u_1, \dots, u_r\})|$. Up to performing Gaussian elimination, we can assume that $\operatorname{supp}(u_i) \not\subseteq \bigcup_{j\neq i} \operatorname{supp}(u_j)$ for $1 \le i \le r$. By Lemma 8.39 there exist $u'_1, \dots, u'_r$ elements of minimal support such that $\operatorname{supp}(u'_i) \subseteq \operatorname{supp}(u_i)$

for $1 \leq i \leq r$ and $\mathrm{rk}(\mathcal{U}') = r$, where $\mathcal{U}' = \langle u'_1, \ldots, u'_r \rangle_{\mathbb{F}_q[x]}$. By construction,

$$\mathrm{wt}(\mathcal{U}) = |\mathrm{supp}(\{u_1, \ldots, u_r\})| \geq |\mathrm{supp}(\{u'_1, \ldots, u'_r\})| \geq \mathrm{wt}(\mathcal{U}'). \tag{8.2.5}$$

Since $d_r(C) = \mathrm{wt}(\mathcal{U})$ and $\mathrm{rk}(\mathcal{U}') = r$, we conclude that $\mathrm{wt}(\mathcal{U}) = \mathrm{wt}(\mathcal{U}')$ and $\mathcal{U}'$ realizes $d_r(C)$. In addition, $\mathrm{supp}(\{u_1, \ldots, u_r\}) = \mathrm{supp}(\{u'_1, \ldots, u'_r\})$ by (8.2.5). Therefore

$$\bigcup_{j \neq i} \mathrm{supp}(u_j) \subsetneq \bigcup_{j=1}^{r} \mathrm{supp}(u_j) = \bigcup_{j=1}^{r} \mathrm{supp}(u'_j).$$

Finally, since by construction $\bigcup_{j \neq i} \mathrm{supp}(u'_j) \subseteq \bigcup_{j \neq i} \mathrm{supp}(u_j)$, we conclude that $\mathrm{supp}(u'_i) \not\subseteq \bigcup_{j \neq i} \mathrm{supp}(u'_j)$. □

8.3. MDS convolutional codes

This section focuses on the generalized weights of MDS and MDP convolutional codes. Proposition 8.6 yields lower bounds for the generalized weights of these two classes of codes. Indeed, for an MDS code C

$$d_r(C) \geq d_1(C) + r - 1 = d_{\mathrm{free}}(C) + r - 1 = (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + r \tag{8.3.6}$$

for $1 \leq r \leq \mathrm{rk}(C)$. Similarly, for an MDP code C

$$\begin{aligned} d_r(C) &\geq d_1(C) + r - 1 = d_{\mathrm{free}}(C) + r - 1 \geq d_L^c(C) + r - 1 \\ &= (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + \left\lfloor \frac{\delta}{n-k} \right\rfloor + 1\right) + r \end{aligned}$$

for $1 \leq r \leq \mathrm{rk}(C)$.

Proposition 8.6 also provides an upper bound on the generalized weights of any convolutional code. In this section, we show that this bound can be sharpened when C is MDS or MDP. We start by showing that, if C is an MDS or MDP convolutional code, then the row degrees of a row-reduced matrix of C can only take certain values.

Lemma 8.41. Let C be an (n, k, δ) a convolutional code, $G(x) = (p_{i,j}(x))_{i,j}$ be a row-reduced generator matrix for C and $\delta_i = \max_j \deg(p_{i,j}(x))$ for $1 \leq i \leq k$.

1. If C is MDP, then $\left\lfloor \frac{\delta}{k} \right\rfloor \leq \delta_i \leq \left\lfloor \frac{\delta}{k} \right\rfloor + k - a$, where $\delta = k \left\lceil \frac{\delta}{k} \right\rceil - a$ and $0 \leq a < k$.
2. If C is MDS, then $\left\lfloor \frac{\delta}{k} \right\rfloor \leq \delta_i \leq \left\lfloor \frac{\delta}{k} \right\rfloor + 1$.
3. If C is MDS or MDP and $k \mid \delta$, then $\delta_i = \frac{\delta}{k}$.

Proof. Suppose by contradiction that there exists an index i such that $\delta_i < \left\lfloor \frac{\delta}{k} \right\rfloor$. If $k = n$ and C is MDS, then

$$\mathrm{wt}(p_{i,1}(x), \ldots, p_{i,n}(x)) \leq n \left\lfloor \frac{\delta}{n} \right\rfloor < \delta + 1 = d_{\mathrm{free}}(C).$$

If $k < n$, then

$$\begin{aligned}\mathrm{wt}(p_{i,1}(x),\ldots,p_{i,n}(x)) &\leq n\left\lfloor\frac{\delta}{k}\right\rfloor = n\left(\left\lfloor\frac{\delta}{k}\right\rfloor+1\right)-n= \\ &= (n-k)\left(\left\lfloor\frac{\delta}{k}\right\rfloor+1\right)+k\left\lfloor\frac{\delta}{k}\right\rfloor+k-n \\ &< (n-k)\left(\left\lfloor\frac{\delta}{k}\right\rfloor+1\right)+(n-k)\left\lfloor\frac{\delta}{n-k}\right\rfloor+1.\end{aligned}$$

If C is MDP, then

$$(n-k)\left(\left\lfloor\frac{\delta}{k}\right\rfloor+1\right)+(n-k)\left\lfloor\frac{\delta}{n-k}\right\rfloor+1 = d_L^c(C) \leq d_{\mathrm{free}}(C).$$

If C is MDS, then

$$(n-k)\left(\left\lfloor\frac{\delta}{k}\right\rfloor+1\right)+(n-k)\left\lfloor\frac{\delta}{n-k}\right\rfloor+1 \leq d_{\mathrm{free}}(C).$$

In all cases, the weight of the i-th row is strictly smaller than the free distance of the code, a contradiction. This proves that $\delta_i \geq \left\lfloor\frac{\delta}{k}\right\rfloor$ for $1 \leq i \leq k$.

If $k \mid \delta$, then $\delta_i = \frac{\delta}{k}$, since $\sum \delta_i = \delta$ and $\delta_i \geq \frac{\delta}{k}$. Else, since $\delta_i \geq \left\lfloor\frac{\delta}{k}\right\rfloor$ and $\sum\delta_i = \delta$, then $\delta_i \leq \left\lfloor\frac{\delta}{k}\right\rfloor + k - a$.

Finally, let C be MDS and suppose that there exists an index ℓ such that $\delta_\ell > \left\lfloor\frac{\delta}{k}\right\rfloor + 1$. Consider the submatrix $G_\ell(x)$ obtained from $G(x)$ by deleting the ℓ-th row and let C_ℓ be the code associated to $G_\ell(x)$. Then $G_\ell(x)$ is row-reduced and $C_\ell \subseteq C$. By Proposition 8.6 we have that $d_{\mathrm{free}}(C_\ell) \geq d_{\mathrm{free}}(C)$. On the other hand, by Theorem 2.15

$$\begin{aligned}d_{\mathrm{free}}(C_\ell) &\leq (n-k+1)\left(\left\lfloor\frac{\delta-\left\lfloor\frac{\delta}{k}\right\rfloor-2}{k-1}\right\rfloor+1\right)+\delta-\left\lfloor\frac{\delta}{k}\right\rfloor+ \\ &-2+1=(n-k+1)\left(\left\lfloor\frac{\delta}{k}\right\rfloor+1\right)+\delta-\left\lfloor\frac{\delta}{k}\right\rfloor-1 \leq \\ &\leq (n-k)\left(\left\lfloor\frac{\delta}{k}\right\rfloor+1\right)+\delta = d_{\mathrm{free}}(C)-1,\end{aligned}$$

a contradiction. □

As a consequence of Lemma 8.41, we obtain an upper bound for the generalized weights of an MDS code, that improves bound 4. from Proposition 8.6 whenever $k \nmid \delta$.

Proposition 8.42. Let C be an (n,k,δ) convolutional code and write $\delta = k\left\lceil\frac{\delta}{k}\right\rceil - a$ with $0 \leq a < k$.

1. If C is MDS, then
$$d_k(C) \leq n\left(\left\lceil\frac{\delta}{k}\right\rceil+1\right)-a.$$

2. If C is MDP, then
$$d_k(C) \leq n\left(\left\lfloor\frac{\delta}{k}\right\rfloor+k-a+1\right)-k+1.$$

Proof. 1. If $a = 0$, then the bound follows from Proposition 8.6 and Lemma 8.41. If $a > 0$, let $G(x)$ be a row-reduced generator matrix for C and let $c_1(x), \ldots, c_k(x)$ be the rows of $G(x)$. Since C is MDS, then $\deg(c_1(x)) = \cdots = \deg(c_{k-a}(x)) = \left\lceil \frac{\delta}{k} \right\rceil$ and $\deg(c_{k-a+1}(x)) = \cdots = \deg(c_k) = \left\lfloor \frac{\delta}{k} \right\rfloor$. For $1 \leq i \leq k-a$ there exist $\alpha_{i,1}, \ldots, \alpha_{i,a} \in \mathbb{F}_q$ such that the last a entries of $\tilde{c}_i(x) = c_i(x) + \sum_{j=1}^{a} \alpha_{i,j} x c_{k-a+j}(x)$ have degree smaller or equal than $\left\lfloor \frac{\delta}{k} \right\rfloor$. Since

$$\operatorname{rk}\left(\langle \tilde{c}_1(x), \ldots, \tilde{c}_{k-a}(x), c_{k-a+1}(x), \ldots, c_k(x) \rangle_{\mathbb{F}_q[x]}\right) = k,$$

we obtain that

$$d_k(C) \leq |\operatorname{supp}(c_1(x), \ldots, \tilde{c}_{k-a}(x), c_{k-a+1}(x), \ldots, c_k(x))| \leq n\left(\left\lceil \frac{\delta}{k} \right\rceil + 1\right) - a.$$

2. The proof is similar to that of part 1. □

The next corollary is a rewriting of the bound for MDS codes from Proposition 8.42.

Corollary 8.43. Let C be an (n, k, δ) MDS convolutional code.

1. If $k \mid \delta$, then $d_k(C) \leq (n-k)\left(\frac{\delta}{k} + 1\right) + \delta + k$.
2. If $k \nmid \delta$, then $d_k(C) \leq (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + n$.
3. If $k = n$, then $d_k(C) \leq \delta + k$.

A consequence of Corollary 8.43 is that some, and in some cases all, of the generalized weights of an (n, k, δ) MDS code are determined by the code parameters. In particular, the generalized weights of an MDS code such that $k \mid \delta$ meet bound 4. from Proposition 8.6.

Proposition 8.44. Let C be an (n, k, δ) MDS convolutional code.

1. If $k = n$, then $d_r(C) = \delta + r$, for $1 \leq r \leq k$.
2. If $k \mid \delta$, then
$$d_r(C) = n\left(\frac{\delta}{k} + 1\right) - k + r,$$
for $1 \leq r \leq k$.
3. If $\delta = k\left\lceil \frac{\delta}{k} \right\rceil - a$ with $0 < a < k$, then
$$d_r(C) = (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + r,$$
for $1 \leq r \leq a$.

Proof. The equalities in 1. and 2. are equivalent to $d_k(C) \leq d_{\text{free}}(C) + k - 1$. The claims now follow from Proposition 8.6, more precisely from the fact that the generalized weights are increasing.

3. Let $G(x)$ be a row-reduced matrix for C. Let $C' \subseteq C$ the subcode generated by the a rows of $G(x)$ of degree $\left\lfloor \frac{\delta}{k} \right\rfloor$. Then C' is an MDS $\left(n, a, a\left\lfloor \frac{\delta}{k} \right\rfloor\right)$ convolutional code. Hence

$$(n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + r \leq d_r(C) \leq d_r(C') = n\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) - a + r$$

for $1 \leq r \leq a$, where the first inequality follows from (8.3.6), the second from Proposition 8.6, and the equality from part 2. Since the first and last quantities agree, the thesis follows. □

Proposition 8.44 shows in particular that, if $k \nmid \delta$, then the first $a = k\left\lceil \frac{\delta}{k} \right\rceil - \delta$ generalized weights of an (n, k, δ) MDS convolutional code are determined by its parameters. We now show that the other generalized weights are not in general determined by the parameters of the code. In the next example, we exhibit two MDS codes with parameters $(3, 2, 1)$ with different second generalized weight. In particular, by Proposition 8.9 this provides an example of MDS codes with the same parameters (n, k, δ), which are not isometric.

Example 8.45. (a) Let $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$ and let $C_1 \subseteq \mathbb{F}_q[x]^3$ with be the code with generator matrix

$$\begin{pmatrix} 2x & x+1 & x+1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The free distance of C_1 is $d_{\mathrm{free}}(C_1) = d_1(C_1) = 3$, as shown in [LP20, Example 3.2]. Since

$$|\mathrm{supp}(\langle (2x, x+1, x+1), (x, x, 2x) \rangle_{\mathbb{F}_q}| = 5,$$

we have that $d_2(C_1) \leq 5$. We claim that $d_2(C_1) = 5$. Assume by contradiction that there are $c_1, c_2 \in C_1$ such that $|\mathrm{supp}(\langle c_1, c_2 \rangle_{\mathbb{F}_q})| \leq 4$ and $\mathrm{rk}(\langle c_1, c_2 \rangle_{\mathbb{F}_q[x]}) = 2$. Then there exists $c \in \langle c_1, c_2 \rangle_{\mathbb{F}_q}$ such that $\mathrm{wt}(c) = 3$ and c has a zero entry. Let $p(x), q(x) \in \mathbb{F}_q[x]$ such that

$$c = (p(x) + 2xq(x), p(x) + q(x)(x+1), 2p(x) + q(x)(x+1)).$$

Since c has a zero entry, then $p(x), q(x) \neq 0$. Moreover:

- If $p(x) + 2xq(x) = 0$, then
$$\mathrm{wt}(c) = \mathrm{wt}(q(x)(-x+1)) + \mathrm{wt}(q(x)(1-3x)) = 4.$$
- If $p(x) + (x+1)q(x) = 0$, then
$$\mathrm{wt}(c) = \mathrm{wt}(q(x)(x-1)) + \mathrm{wt}(-q(x)(x+1)) = 4.$$
- If $2p(x) + q(x)(x+1) = 0$, then
$$\mathrm{wt}(c) = \mathrm{wt}\left(\frac{1}{2}q(x)(3x-1)\right) + \mathrm{wt}\left(\frac{1}{2}q(x)(x+1)\right) = 4.$$

Therefore $\mathrm{wt}(c) = 4$, which yields a contradiction. We conclude that $d_2(C_1) = 5$.

(b) Let $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$ and let $C_2 \in \mathbb{F}_q[x]^3$ be the code with generator matrix

$$\begin{pmatrix} 2x & x+1 & 0 \\ 1 & 1 & 2 \end{pmatrix}.$$

It is easy to check that $d_1(C_2) = 3$. Moreover

$$|\mathrm{supp}(\langle (2x, x+1, 0), (x, x, 2x) \rangle_{\mathbb{F}_q})| = 4,$$

hence $d_2(C_2) = 4$ by Proposition 8.6.

Next we produce a new upper bound for the last $k - a$ generalized weights of an MDS convolutional code, in the case when they are not determined by the parameters of the code. To an (n, k, δ) convolutional code $C = \langle c_1, \dots, c_k \rangle_{\mathbb{F}_q[x]}$ we associate the linear block code $C[0] = \langle c_1[0], \dots, c_k[0] \rangle_{\mathbb{F}_q}$. Notice that, if C is MDS, then $\dim(C[0]) = k$. In the next proposition, we establish a relation between the last $k - a$ generalized weights of C and the first $k - a$ generalized Hamming weights of $C[0]$.

Proposition 8.46. Let C be an (n, k, δ) MDS convolutional code. If $\delta = k \left\lceil \frac{\delta}{k} \right\rceil - a$ with $0 < a < k$, then

$$d_{a+r}(C) \le (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + a + \min\left\{ d_r^H(C[0]), d_r^H(\mathrm{rev}(C)[0]) \right\}$$

for $1 \le r \le k - a$.

Proof. We start by showing that

$$d_{a+r}(C) \le (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + a + d_r^H(C[0]).$$

Since C is MDS, by Lemma 8.41 there exist $c_1, \dots, c_k$ such that $C = \langle c_1, \dots, c_k \rangle_{\mathbb{F}_q[x]}$, $\deg(c_1) = \cdots = \deg(c_a) = \left\lfloor \frac{\delta}{k} \right\rfloor$ and any element of $\langle c_{a+1}, \dots, c_k \rangle_{\mathbb{F}_q[x]}$ has degree greater than or equal to $\left\lfloor \frac{\delta}{k} \right\rfloor + 1$. If $c \in \langle c_1, \dots, c_a \rangle_{\mathbb{F}_q}$, then

$$\mathrm{wt}(c[0]) \ge d_{\mathrm{free}}(C) - n \left\lfloor \frac{\delta}{k} \right\rfloor = n - a + 1.$$

Since $d_{k-a}^H(C[0]) \le n - a$, if $U \subseteq C[0]$ realizes $d_r^H(C[0])$, then $U \cap \langle c_1[0], \dots, c_a[0] \rangle_{\mathbb{F}_q} = 0$. Therefore, there exist $c'_1, \dots, c'_r \in C$ with $\deg(c'_1), \dots, \deg(c'_r) = \left\lceil \frac{\delta}{k} \right\rceil$ such that $U = \langle c'_1[0], \dots, c'_r[0] \rangle_{\mathbb{F}_q}$. Let $\mathcal{D} = \langle c'_1, \dots, c'_r, xc_1, \dots, xc_a \rangle_{\mathbb{F}_q[x]}$. Then

$$\begin{aligned} \mathrm{rk}\,(\mathcal{D}) = \mathrm{rk}\left(\langle c'_1, \dots, c'_r, c_1, \dots, c_a \rangle_{\mathbb{F}_q[x]}\right) \ge \\ \ge \dim\left(\langle c'_1[0], \dots, c'_r[0], c_1[0], \dots, c_a[0] \rangle_{\mathbb{F}_q}\right) = a + r. \end{aligned}$$

Therefore $\mathrm{rk}(\mathcal{D}) = a + r$ and

$$\begin{aligned} d_{a+r}(C) \le \mathrm{wt}(\mathcal{D}) \le |\mathrm{supp}(\{c'_1, \dots, c'_r, xc_1, \dots, xc_a\})| \le \\ \le (n-k)\left\lceil \frac{\delta}{k} \right\rceil + \delta + a + d_H^r(C[0]). \end{aligned}$$

In fact, $c'_1[0], \dots, c'_r[0]$ contribute $d_H^r(C[0])$ to the size of the support and the rest contributes at most $n \left\lceil \frac{\delta}{k} \right\rceil = (n-k)\left\lceil \frac{\delta}{k} \right\rceil + \delta + a$. Since the same inequality holds for $\mathrm{rev}(C)$, we conclude by Proposition 8.29. □

The next example shows that the bound from Proposition 8.46 is sharp, at least in some cases.

Example 8.47. Let $n = 3$, $k = 2$, $\delta = 1$, hence $a = 1$. Proposition 8.44 states that the free distance of an MDS code C with these parameters is 3. By Proposition 8.46 the second generalized weight of C is at most 5, since $\dim(C[0]) = 2$ and a linear block code of dimension 2 has minimum distance at most 2. Therefore, $d_2(C) \in \{4, 5\}$.

Let $\mathbb{F}_q$ be a field of characteristic different from 2, 3 and let C_1 and C_2 be the codes from Example 8.45. Then $C_1[0] = \langle(0,1,1),(1,0,1)\rangle_{\mathbb{F}_q}$, $\mathrm{rev}(C_1)[0] = \langle(1,0,-1),(0,1,3)\rangle_{\mathbb{F}_q}$, $C_2[0] = \langle(0,1,0),(1,0,2)\rangle_{\mathbb{F}_q}$, and $\mathrm{rev}(C_2)[0] = \langle(2,1,0),(1,0,-2)\rangle_{\mathbb{F}_q}$. By Proposition 8.46, $d_2(C_1) \leq 5$ and $d_2(C_2) \leq 4$. Notice that not only both codes meet their respective bounds, but also all the possible values for the second generalized weight of a $(3,2,1)$ MDS convolutional codes predicted by Proposition 8.46 are realized by some code.

One may ask whether the upper bound from Proposition 8.46 is in fact an equality. If this were the case, then the generalized weights of an MDS code would be realized by subspaces generated by codewords of degree at most $\left\lceil \frac{\delta}{k} \right\rceil$. This would answer Question 8.38 for MDS codes. We conclude this section with a result in this direction. In particular, the next result applies to a reverse MDP code, such that $n-k \mid \delta$. See [TRS12] for the definition and a proof of existence of reverse MDP codes.

Proposition 8.48. Let C be an (n,k,δ) sMDS convolutional code such that $\mathrm{rev}(C)$ is MDP. Then, $d_r(C)$ is realized by a subcode generated by codewords of degree smaller than $\left\lfloor \frac{\delta}{k} \right\rfloor + \left\lceil \frac{\delta}{n-k} \right\rceil + 1$.

Proof. By Theorem 8.40 we have that $d_r(C)$ is realized by a subcode $\mathcal{D}$ generated by minimal codewords $c_1, \ldots, c_r$ such that $\mathrm{supp}(c_i) \subsetneq \bigcup_{j \neq i} \mathrm{supp}(c_j)$ for $1 \leq i \leq r$. If there exists i such that c_i has degree greater or equal to $\left\lfloor \frac{\delta}{k} \right\rfloor + \left\lceil \frac{\delta}{n-k} \right\rceil + 1$, then

$$\mathrm{wt}(c_i) \geq d_{\mathrm{free}}(C) + n - k + 1,$$

since C is sMDS and its reverse is MDP. We conclude that

$$\begin{aligned} \mathrm{wt}(\mathcal{D}) &\geq d_{\mathrm{free}}(C) + n - k + r = \\ &= (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + n + 1 + r - k. \end{aligned}$$

This contradicts bound 1. or 2. in Corollary 8.43, since $\mathcal{D}$ realizes $d_r(C)$. □

8.4. Optimal anticodes

In this section we prove an anticode bound for convolutional codes and define optimal convolutional anticodes as the codes that meet the anticode bound. We give a complete classification of optimal anticodes and we compute their generalized weights. We start by recalling the anticode bound for linear block codes with the Hamming metric that follows from Theorem 5.2.2. It also follows from direct inspection of the reduced row echelon form of a generator matrix of the code.

Theorem 8.49. Let $C \subseteq \mathbb{F}_q^n$ be an $\mathbb{F}_q$-linear code. Then

$$\dim(C) \leq \mathrm{maxwt}_H(C).$$

A convolutional code contains codewords of arbitrarily large weight. However, any finite dimensional subspace contains a finite number of codewords, therefore one can define its maximum weight in the usual way.

Definition 8.50. Let $C \subseteq \mathbb{F}_q[x]^n$ be an (n, k, δ) convolutional code and let $U \subseteq C$ be a finite dimensional vector space. The maximum weight of U is

$$\mathrm{maxwt}(U) = \max\{\mathrm{wt}(u) \mid u \in U\}.$$

The maximum weight of C is

$$\mathrm{maxwt}(C) = \min\{\mathrm{maxwt}(U) : U \text{ is an } \mathbb{F}_q\text{-linear subspace of } C \text{ and } \mathrm{rk}\left(\langle U\rangle_{\mathbb{F}_q[x]}\right) = \mathrm{rk}(C)\}.$$

Remark 8.51. For an (n, k, δ) convolutional code C one has

$$\mathrm{maxwt}(C) = \min\{\mathrm{maxwt}(U) : U \text{ is an } \mathbb{F}_q\text{-linear subspace of } C \text{ and } \dim(U) = k = \mathrm{rk}\left(\langle U\rangle_{\mathbb{F}_q[x]}\right)\}.$$

In fact, every $\mathbb{F}_q$-linear subspace $V \subseteq C$ which generates a subcode of rank k contains a k-dimensional $\mathbb{F}_q$-linear subspace $U \subseteq V$ such that U generates a subcode of rank k and $\mathrm{wt}(U) \leq \mathrm{wt}(V)$.

As a consequence of Theorem 8.49 we obtain a bound for convolutional codes that we call anticode bound, in analogy with the Hamming metric case and the sum-rank metric case.

Theorem 8.52 (Anticode bound)**.** Let $C \subseteq \mathbb{F}_q[x]^n$ be an (n, k, δ) convolutional code. Then

$$\mathrm{rk}(C) \leq \mathrm{maxwt}(C).$$

Proof. Let U be an $\mathbb{F}_q$-linear subspace of C such that $\mathrm{rk}(\langle U\rangle_{\mathbb{F}_q[x]}) = k$ and let $d = \max\{\deg(c) : c \in U\}$. There exists an $\mathbb{F}_q$-linear homomorphism $\varphi : U \to \mathbb{F}_q^{n(d+1)}$ such that $\mathrm{wt}(c) = \mathrm{wt}_H(\varphi(c))$ for all $c \in C$. Since $\dim(U) = \dim(\varphi(U))$ and $\mathrm{maxwt}(U) = \mathrm{maxwt}_H(\varphi(U))$, by Theorem 8.49 we conclude that

$$\mathrm{rk}(C) = \mathrm{rk}\left(\langle U\rangle_{\mathbb{F}_q[x]}\right) \leq \dim(U) \leq \mathrm{maxwt}_H(\varphi(U)) = \mathrm{maxwt}(U).$$

Since the inequality holds for every $\mathbb{F}_q$-linear subspace $U \subseteq C$, we conclude. □

Definition 8.53. A convolutional code $\mathcal{A}$ is an optimal (convolutional) anticode if

$$\mathrm{rk}(\mathcal{A}) = \mathrm{maxwt}(\mathcal{A}).$$

The next proposition shows that two optimal anticodes with the same rank have the same generalized weights, when $q \neq 2$.

Proposition 8.54. Let $q \neq 2$ and let $\mathcal{A} \subseteq \mathbb{F}_q[x]^n$ be an optimal anticode. Then $d_r(\mathcal{A}) = r$ for $1 \leq r \leq \mathrm{rk}(\mathcal{A})$.

Proof. Since $\mathcal{A}$ is an optimal anticode, there exists an $\mathbb{F}_q$-linear subspace U such that $\mathrm{rk}(\mathcal{A}) = \mathrm{rk}\left(\langle U\rangle_{\mathbb{F}_q[x]}\right) = \mathrm{maxwt}(U)$. Let φ be as in the proof of Theorem 8.52. By Theorem 8.49

$$\mathrm{rk}\left(\langle U\rangle_{\mathbb{F}_q[x]}\right) \leq \dim(\varphi(U)) \leq \mathrm{maxwt}_H(\varphi(U)) = \mathrm{maxwt}(U).$$

Then $\varphi(U)$ is an optimal anticode in the Hamming metric. Since $q \neq 2$, $\varphi(U)$ is generated by a subset of the standard basis of $\mathbb{F}_q^n$ by [Rav16a, Proposition 9]. In particular, $|\mathrm{supp}(\varphi(U))| =$

$\mathrm{maxwt}_H(\varphi(U))$. Since $|\mathrm{supp}(U)| = |\mathrm{supp}(\varphi(U))|$, we have that

$$|\mathrm{supp}(U)| = \mathrm{maxwt}(U) = \dim(U) = \mathrm{rk}(\mathcal{A}).$$

This implies that $d_{\mathrm{rk}(\mathcal{A})}(\mathcal{A}) \le \mathrm{rk}(\mathcal{A})$. Since $1 \le d_1(\mathcal{A}) < \ldots < d_{\mathrm{rk}(\mathcal{A})}(\mathcal{A}) \le \mathrm{rk}(\mathcal{A})$ by Proposition 8.6, we conclude. □

The assumption that $q \neq 2$ in the previous proposition is necessary, as the next example shows.

Example 8.55. Let $\mathcal{A} = \langle (1,1,0),(1,0,1)\rangle_{\mathbb{F}_2[x]}$. $\mathcal{A}$ is an optimal anticode, since $\mathrm{rk}(\mathcal{A}) = 2 = \mathrm{maxwt}_H(\langle (1,1,0),(1,0,1)\rangle_{\mathbb{F}_2}) \ge \mathrm{maxwt}(\mathcal{A})$. However, $d_1(\mathcal{A}) = 2$ and $d_2(\mathcal{A}) = 3$ by Proposition 8.19.

The next result is the converse of Proposition 8.54. Notice that, in this case, we do not need to assume that $q \neq 2$.

Proposition 8.56. Let $\mathcal{A} \subseteq \mathbb{F}_q[x]^n$ be an (n,k,δ) convolutional code. If $d_k(\mathcal{A}) = k$, then $\mathcal{A}$ is an optimal anticode.

Proof. Let U be an $\mathbb{F}_q$-linear subspace of $\mathcal{A}$ such that $d_k(\mathcal{A}) = |\mathrm{supp}(U)|$ and $\mathrm{rk}\left(\langle U\rangle_{\mathbb{F}_q[x]}\right) = k$. By Theorem 8.49

$$\mathrm{maxwt}(U) \le |\mathrm{supp}(U)| = k \le \dim(U) \le \mathrm{maxwt}(U).$$

Therefore $\mathrm{maxwt}(U) = k \ge \mathrm{maxwt}(\mathcal{A})$, so $\mathcal{A}$ is an optimal anticode by Theorem 8.52. □

Because of the generalized-weight-preserving correspondence between $(n,k,0)$ convolutional codes and (n,k) linear block codes, the next result is not surprising.

Proposition 8.57. An $(n,k,0)$ convolutional code $C \subseteq \mathbb{F}_q[x]^n$ is an optimal anticode if and only if $C[0] \subseteq \mathbb{F}_q^n$ is an optimal anticode with respect to the Hamming metric.

Proof. If $C[0] \subseteq \mathbb{F}_q^n$ is an optimal anticode with respect to the Hamming metric, then

$$\mathrm{maxwt}(C) \le \mathrm{maxwt}(C[0]) = \dim(C[0]) = k,$$

where the last equality follows from Proposition 8.18. Therefore, C is an optimal anticode.

Conversely, suppose that C is an optimal anticode. If $q \neq 2$, by Proposition 8.19 and Proposition 8.54 we have that

$$\dim(C[0]) = k = d_k(C) = d_k^H(C[0]) = |\mathrm{supp}(C[0])| \ge \mathrm{maxwt}(C[0]).$$

Therefore $\dim(C[0]) = \mathrm{maxwt}(C[0])$ by Theorem 8.52 and $C[0]$ is an optimal anticode. Let $q = 2$. Since C is an optimal anticode and by Theorem 8.40, there exists $U = \langle c_1(x),\ldots,c_k(x)\rangle_{\mathbb{F}_q}$ such that $k = \mathrm{rk}\left(\langle U\rangle_{\mathbb{F}_q[x]}\right) = \dim(U) = \mathrm{maxwt}(U)$ and $\mathrm{supp}(c_j) \not\subseteq \bigcup_{i\neq j}\mathrm{supp}(c_i(x))$ for $1 \le j \le k$. This implies that $|\mathrm{supp}(\sum_{i\neq j} c_i(x))| \ge k-1$ for $1 \le j \le k$. Moreover, $|\mathrm{supp}(\sum c_i(x))| \ge k$, therefore equality must hold. It follows that every element of $\mathrm{supp}(c_j)$, except for the one that belongs to no $\mathrm{supp}(c_i)$ for $i \neq j$, must belong to $\mathrm{supp}(\sum_{i\neq j} c_i(x))$. This implies that $\mathrm{wt}(c_j(x)) \le 2$ for $1 \le j \le k$. Let

$$V = \{c_i[0],\ldots,c_i[\deg(c_i)] : 1 \le i \le k\}.$$

Since $c_i(x) \in C$ for all i, then $c_i[j] \in C[0]$ for $0 \leq j \leq \deg(c_i)$. Therefore $V \subseteq C[0]$. Moreover,

$$k = \mathrm{rk}(\langle U\rangle_{\mathbb{F}_q[x]}) \leq \mathrm{rk}(\langle V\rangle_{\mathbb{F}_q[x]}) \leq \mathrm{rk}(C) = k$$

from which

$$k = \mathrm{rk}(\langle V\rangle_{\mathbb{F}_q[x]}) \leq \dim(\langle V\rangle_{\mathbb{F}_q}) \leq \dim(C[0]) = k.$$

It follows that $V = C[0]$. Therefore, there exist $a_1, \dots, a_k \in \langle V\rangle_{\mathbb{F}_q}$ that satisfy the following conditions:

- $\langle a_1, \dots, a_k\rangle_{\mathbb{F}_q} = C[0]$,
- $\mathrm{wt}(a_1) = \cdots = \mathrm{wt}(a_r) = 1$ and $d_1^H(\langle a_{r+1}, \dots, a_k\rangle_{\mathbb{F}_q}) = 2$ for some $1 \leq r \leq k$,
- $\bigcup_{i=1}^r \mathrm{supp}(a_i) \cap \bigcup_{i=r+1}^k \mathrm{supp}(a_i) = \emptyset$,
- for $r+1 \leq j \leq k$, there exist $1 \leq \bar{j} \leq k$ and $0 \leq \hat{j} \leq \deg(c_{\bar{j}})$ such that $a_j = c_{\bar{j}}\left[\hat{j}\right]$.

Since $\mathrm{wt}(a_j) = 2$ for $r+1 \leq j \leq k$, up to permuting $c_1(x), \dots, c_k(x)$, we may assume without loss of generality that $c_j(x) = c_j[\deg(c_j)]x^{\deg(c_j)}$ for $r+1 \leq j \leq k$. In particular, $a_j = c_j[\deg(c_j)]$. Fix $1 \leq i \leq r$. Suppose that $\mathrm{supp}(\langle\{c_i[0], \dots, c_i[\deg(c_i)]\}\rangle) \subseteq \mathrm{supp}(\langle a_{r+1}, \dots, a_k\rangle)$, then $\langle c_i(x), c_{r+1}(x), \dots, c_k(x)\rangle_{\mathbb{F}_q[x]} \subseteq \langle a_{r+1}, \dots, a_k\rangle_{\mathbb{F}_q[x]}$, but this is a contradiction since $\mathrm{rk}(\langle c_i(x), c_{r+1}(x), \dots, c_k(x)\rangle_{\mathbb{F}_q[x]}) = k - r + 1$ while $\mathrm{rk}(\langle a_{r+1}, \dots, a_k\rangle_{\mathbb{F}_q[x]}) = k - r$. If instead we assume that $|\mathrm{supp}(\langle\{c_i[0], \dots, c_i[\deg(c_i)]\}\rangle) \cap \mathrm{supp}(\langle a_{r+1}, \dots, a_k\rangle)| = 1$, we again find a contradiction since $d_1^H(\langle a_{r+1}, \dots, a_k\rangle_{\mathbb{F}_q}) = 2$. Therefore $\mathrm{supp}(\langle\{c_i[0], \dots, c_i[\deg(c_i)] : 1 \leq i \leq r\}\rangle) \subseteq \mathrm{supp}(\langle a_1, \dots, a_r\rangle)$, hence $\bigcup_{i=1}^r \mathrm{supp}(c_i(x)) \cap \bigcup_{i=r+1}^k \mathrm{supp}(c_i(x)) = \emptyset$.

Suppose by contradiction that $\mathrm{maxwt}(C[0]) \geq k+1$. Then there exists $I \subseteq \{1, \dots, k\}$ such that

$$k+1 \leq \mathrm{wt}\left(\sum_{i\in I} a_i\right) = \mathrm{wt}\left(\sum_{i\in S} a_i\right) + \mathrm{wt}\left(\sum_{i\in I\setminus S} a_i\right),$$

where $S = I \cap \{r+1, \dots, k\}$. Since $\mathrm{wt}(\sum_{i\in I\setminus S} a_i) = |I \setminus S| \leq r$, then $\mathrm{wt}(\sum_{i\in S} a_i) \geq k - r + 1$. Therefore

$$\mathrm{wt}\left(\sum_{i=1}^r c_i(x) + \sum_{i\in S} c_i(x)\right) = \mathrm{wt}\left(\sum_{i=1}^r c_i(x)\right) + \mathrm{wt}\left(\sum_{i\in S} c_i(x)\right) \geq r + \mathrm{wt}\left(\sum_{i\in S} a_i\right) \geq k+1,$$

contradicting the assumption that C is an optimal anticode of rank k. Therefore we conclude that $\mathrm{maxwt}(C[0]) = k$, that is, $C[0]$ is an optimal anticode. □

The rest of the section is devoted to classifying optimal anticodes. We start by introducing the concept of elementary optimal anticode.

Definition 8.58. A code $\mathcal{A} \subseteq \mathbb{F}_q[x]^n$ with $\mathrm{rk}(\mathcal{A}) = k$ is an elementary optimal anticode if there exist a set $J = \{j_1, \dots, j_k\}$ with $1 \leq j_1 < \cdots < j_k \leq n$ and non-negative integers $a_1, \dots, a_k$ such that $\mathcal{A} = \langle c_1(x), \dots, c_k(x)\rangle_{\mathbb{F}_q[x]}$, where for $1 \leq i \leq k$ the only nonzero entry of $c_i(x)$ is x^{a_i} in position j_i.

Lemma 8.59. Every elementary optimal anticode is an optimal anticode.

Proof. Let $\mathcal{A}$ be an elementary optimal anticode. By definition $\mathcal{A} = \langle c_1(x), \dots, c_k(x)\rangle_{\mathbb{F}_q[x]}$, where for $1 \le i \le k$ the only nonzero entry of $c_i(x)$ is x^{a_i} in position j_i. Then, $\mathrm{maxwt}(\mathcal{A}) \le \mathrm{maxwt}(U) = k = \mathrm{rk}(\mathcal{A})$, where $U = \langle c_1(x), \dots, c_k(x)\rangle_{\mathbb{F}_q}$. We conclude by Theorem 8.52. □

Theorem 8.60 (Characterization of optimal anticodes)**.** Let $q \neq 2$ and let $\mathcal{A} \subseteq \mathbb{F}_q[x]^n$ be a code. Then: $\mathcal{A}$ is an optimal anticode if and only if there exists an elementary optimal anticode $\mathcal{A}'$ such that $\mathrm{rk}(\mathcal{A}') = \mathrm{rk}(\mathcal{A})$ and $\mathcal{A}' \subseteq \mathcal{A}$.

Proof. $\Leftarrow$) If $\mathcal{A}$ contains an elementary optimal anticode $\mathcal{A}'$ of the same rank, then by Proposition 8.6, Lemma 8.59, and Proposition 8.54

$$d_{\mathrm{rk}(\mathcal{A})}(\mathcal{A}) \le d_{\mathrm{rk}(\mathcal{A}')}(\mathcal{A}') = \mathrm{rk}(\mathcal{A}') = \mathrm{rk}(\mathcal{A}).$$

Therefore $\mathcal{A}$ is an optimal anticode by Proposition 8.56.

$\Rightarrow$) Assume that $\mathcal{A}$ is an optimal anticode. By Proposition 8.54 we have that $d_{\mathrm{rk}(\mathcal{A})}(\mathcal{A}) = \mathrm{rk}(\mathcal{A})$. Then there exists an $\mathbb{F}_q$-linear subspace U of $\mathcal{A}$ such that

$$|\mathrm{supp}(U)| = \dim(U) = \mathrm{rk}\left(\langle U\rangle_{\mathbb{F}_q[x]}\right) = \mathrm{rk}(\mathcal{A}).$$

Therefore U is generated by elements of rank 1, which are supported on distinct entries. The same elements generate the elementary optimal anticode $\mathcal{A}'$ as an $\mathbb{F}_q[x]$-module. □

Notice that not all optimal anticodes are elementary, as the next example shows.

Example 8.61. Let $\mathcal{A} = \langle (1, x), (x, 0)\rangle_{\mathbb{F}_q[x]}$ be a code. It is easy to show that $\mathcal{A}$ is not an elementary optimal anticode. However $\mathcal{A}' = \langle (x, 0), (0, x^2)\rangle_{\mathbb{F}_q[x]} \subseteq \mathcal{A}$ is an elementary optimal anticode with generalized weights $d_1(\mathcal{A}') = 1$ and $d_2(\mathcal{A}') = 2$. By Proposition 8.6, $\mathcal{A}$ has the same generalized weights as $\mathcal{A}'$, so $\mathcal{A}$ is an optimal anticode by Proposition 8.56.

We conclude this section with a proof that the dual of an optimal anticode is an optimal anticode, provided that $q \neq 2$.

Lemma 8.62. Let $q \neq 2$. Every optimal anticode $\mathcal{A} \subseteq \mathbb{F}_q[x]^n$ with $\mathrm{rk}(\mathcal{A}) = r$ is contained in a code generated over $\mathbb{F}_q[x]$ by r vectors of the standard basis of $\mathbb{F}_q^n$.

Proof. Since $\mathcal{A}$ is an optimal anticode, by Theorem 8.60 there exists an elementary optimal anticode $\mathcal{A}' \subseteq \mathcal{A}$ such that $\mathrm{rk}(\mathcal{A}') = \mathrm{rk}(\mathcal{A}) = r$. We may assume without loss of generality that $\mathcal{A}'$ is maximal with respect to inclusion among the codes with those properties. By definition $\mathcal{A}'$ has a system of generators consisting of r vectors of weight 1, let $i_1, \dots, i_r$ be the positions of the nonzero entries in the generators of $\mathcal{A}'$. Suppose that there exists $c \in \mathcal{A} \setminus \mathcal{A}'$ which has a nonzero entry in a position different from $i_1, \dots, i_r$ and let $\mathcal{B} = \mathcal{A}' + \langle c\rangle_{\mathbb{F}_q[x]}$. Then $\mathcal{B} \subseteq \mathcal{A}$ and $\mathrm{rk}(\mathcal{B}) = \mathrm{rk}(\mathcal{A}) + 1$. This is a contradiction, showing that $\mathcal{A} \subseteq \langle e_{i_1}, \dots, e_{i_r}\rangle_{\mathbb{F}_q}[x]$, where e_i denotes the i-th standard basis vector of $\mathbb{F}_q^n$. □

Corollary 8.63. Let $q \neq 2$. The dual code $\mathcal{A}^\perp$ of an optimal anticode $\mathcal{A}$ is an elementary optimal anticode generated by vectors of the standard basis of $\mathbb{F}_q^n$.

Proof. Let $\mathrm{rk}(\mathcal{A}) = r$, then $\mathrm{rk}(\mathcal{A}^\perp) = n - r$. By Lemma 8.62, $\mathcal{A}$ is generated by $c_1, \dots, c_r$ with $\mathrm{wt}(c_j) = 1$ for $1 \le j \le r$. Let $i_j \in \{1, \dots, n\}$ be the position of the nonzero entry of c_j, $1 \le j \le r$ and let $J = \{1, \dots, n\} \setminus \{i_1, \dots, i_r\}$. Consider the elementary optimal anticode $\mathcal{B} = \langle \{e_j\}_{j\in J}\rangle_{\mathbb{F}_q[x]}$. It is easy to check that $\mathcal{B} \subseteq \mathcal{A}^\perp$. Moreover, any module that properly contains $\mathcal{B}$ has rank larger than $n - r$. We conclude that $\mathcal{A}^\perp = \mathcal{B}$. □

Corollary 8.64. Let $q \neq 2$. An optimal anticode $\mathcal{A} \subseteq \mathbb{F}_q[x]^n$ of rank r is noncatastrophic if and only if is generated by r vectors of the standard basis of $\mathbb{F}_q^n$.

Proof. A code generated by r vectors of the standard basis of $\mathbb{F}_q^n$ is an elementary optimal anticode and it is noncatastrophic by Proposition 2.14. Conversely, let $\mathcal{A}$ be a noncatastrophic optimal anticode. Then $\mathcal{A} = (\mathcal{A}^\perp)^\perp$ and we conclude by Corollary 8.63. □

The conclusions of Lemma 8.62, Corollary 8.63, and Corollary 8.64 do not hold over $\mathbb{F}_2$, as the next example shows.

Example 8.65. Let $\mathcal{A} = \langle (1,1,0),(1,0,1)\rangle_{\mathbb{F}_2[x]}$ be the optimal anticode from Example 8.55. It is clear that $\mathcal{A}$ is not contained in any subcode of $\mathbb{F}_2[x]^3$ generated by two vectors of the standard basis of $\mathbb{F}_2^3$. Moroever, it is easy to show that $\mathcal{A}^\perp = \langle (1,1,1)\rangle_{\mathbb{F}_2[x]}$, in particular $\mathcal{A}^\perp$ is not an elementary optimal anticode. Finally, $\mathcal{A}$ is noncatastrophic by Proposition 2.14, but it does not contain any vector of weight 1.

9. Generalized column distances

In this chapter, we continue our investigation of the mathematical structure of convolutional codes by introducing a new family of invariants: the r-generalized column weights. A notion of generalized column weight for the j-truncation of a convolutional code was given by Cardell, Firer, and Napp in [CFN17], for the special case of noncatastrophic convolutional codes. Later in [CNF19] the same authors modified their definition and introduced unrestricted generalized column distances of noncatastrophic convolutional codes, which they further studied in [CFN20].

Here, we extend the definition from [CFN17] to any convolutional code. For each r and j we obtain the (r, j)-generalized column weight of the code. By taking the limit of the (r, j)-generalized column weight of a code as j tends to infinity, we define its r-generalized column weight. This produces new invariants of a convolutional code, whose basic properties we study in this chapter. Notice that (r, j)-generalized column weights are not invariant under isometry, not even in the special case of noncatastrophic convolutional codes, as was already observed in [GL09, CFN17]. Motivated by this observation, we introduce the notions of j-equivalences and equivalences of convolutional codes and show that (r, j)-generalized column weights are invariant under j-equivalences. Moreover, we show that r-generalized column weights are invariant both under equivalence and under isometry. In addition, we investigate the relations between generalized column weights and other invariants of convolutional codes, including the unrestricted generalized column distances defined in [CNF19] and the generalized weights defined in Chapter 8.

This chapter is organized as follows. In Section 9.1 we introduce j-equivalences and equivalences and we study their basic properties, including their relation with isometries and strong isometries. In Section 9.2 we define (r, j)-generalized column weights and r-generalized column weights. We then establish some crucial properties of these invariants, e.g., we show that they are invariant under j-equivalences, equivalences, and isometries (see Proposition 9.22, Corollary 9.23, and Theorem 9.24 for the precise statements). In Section 9.3 we discuss the relation between generalized column weights and unrestricted generalized column distances and generalized weights.

9.1. Equivalences and j-equivalences

In this section we define j-equivalences and equivalences of convolutional codes. We study their main properties and their relation with isometries of convolutional codes. We start by recalling the definition of isometry and of strong isometry.

Definition 9.1. An $\mathbb{F}_q[x]$-isomorphism of convolutional codes $\phi : C_1 \to C_2$ is an isometry if $\mathrm{wt}(c) = \mathrm{wt}(\phi(c))$ for all $c \in C_1$. If in addition $\deg(c) = \deg(\phi(c))$ for all $c \in C_1$, then ϕ is a strong isometry.

Isometries of convolutional codes have been classified by Gluesing-Luerssen in [GL09].

Theorem 9.2 ([GL09, Theorem 4.1])**.** Let $C_1, C_2 \subseteq \mathbb{F}_q[x]^n$ be convolutional codes and let $\phi : C_1 \to C_2$ be an isometry. There exist a permutation matrix $P \in \mathrm{GL}_n(\mathbb{F}_q)$ and a diagonal matrix

$D = \mathrm{diag}(a_1x^{m_1}, \dots, a_nx^{m_n})$ where $a_1, \dots, a_n \in \mathbb{F}_q^*$ and $m_1, \dots, m_n \in \mathbb{Z}$ such that $\phi(c) = cPD$ for all $c \in C$.

Motivated by the observation that generalized column distances as defined in [CFN17] are not invariant under isometries or strong isometries, we propose a definition of j-equivalence of convolutional codes. In the next section, we show that j-th generalized column distances are invariant under j-equivalences.

Definition 9.3. For each $j \in \mathbb{N}_0$, an $\mathbb{F}_q[x]$-isomorphism of convolutional codes $\phi : C_1 \to C_2$ is called j-equivalence if $\phi : (C_1)_{[0,j]} \to (C_2)_{[0,j]}$ is a Hamming weight-equivalence, i.e., $\mathrm{wt}(c_{[0,j]}) = \mathrm{wt}(\phi(c)_{[0,j]})$ for all $c \in C_1$. We say that ϕ is an equivalence if it is a j-equivalence for all $j \in \mathbb{N}_0$.

The next proposition follows easily from the definition. It collects some of the basic properties of j-equivalences.

Proposition 9.4. Let $j \in \mathbb{N}_0$ and let $\phi : C_1 \to C_2$ be a j-equivalence. The following hold.

1. $\phi^{-1} : C_2 \to C_1$ is a j-equivalence.
2. $\dim(C_1)_{[h,i]} = \dim(C_2)_{[h,i]}$ for any $0 \le h \le i \le j$. In particular, $\dim(C_1[0]) = \dim(C_2[0])$.
3. Let $\mathcal{D} \subseteq C$, then $\phi \restriction_{\mathcal{D}}$ is a j-equivalence.
4. If $\psi : C_2 \to C_3$ is a j'-equivalence, then $\psi \circ \phi$ is a $\min\{j, j'\}$-equivalence.
5. If $\psi : C_3 \to C_1$ is a j'-equivalence, then $\phi \circ \psi$ is a $\min\{j, j'\}$-equivalence.
6. If $\psi : C_3 \to C_4$ is a j'-equivalence, then $\psi \times \phi$ is a $\min\{j, j'\}$-equivalence.

In the next lemma, we collect a few more facts on j-equivalences.

Lemma 9.5. Let $j \in \mathbb{N}_0$, and let $\phi : C_1 \to C_2$ be a j-equivalence. Then:

1. ϕ is a j'-equivalence for $0 \le j' \le j$.
2. ϕ induces a Hamming-weight equivalence $\phi : (C_1)_{[h,i]} \to (C_2)_{[h,i]}$ for $0 \le h \le i \le j$. In particular, $\mathrm{wt}(c_{[i,i]}) = \mathrm{wt}(\phi(c)_{[i,i]})$ for $0 \le i \le j$.

Proof. To prove the first part of the thesis, it suffices to show that if ϕ is a j-equivalence then it is also a $(j-1)$-equivalence. Let $c \in C_1$. Since $\mathrm{wt}(c_{[0,j-1]}) = \mathrm{wt}((xc)_{[0,j]})$, we have that

$$\mathrm{wt}(c_{[0,j-1]}) = \mathrm{wt}((xc)_{[0,j]}) = \mathrm{wt}(\phi(xc)_{[0,j]}) = \mathrm{wt}((x\phi(c))_{[0,j]}) = \mathrm{wt}(\phi(c)_{[0,j-1]}).$$

It follows that ϕ is a $(j-1)$-equivalence.

To prove that the restriction $\phi : (C_1)_{[h,i]} \to (C_2)_{[h,i]}$ is a Hamming-weight equivalence for $0 \le h \le i \le j$, observe that for any $c \in C_1$

$$\mathrm{wt}(c_{[h,i]}) = \mathrm{wt}(c_{[0,i]}) - \mathrm{wt}(c_{[0,h-1]}) = \mathrm{wt}(\phi(c)_{[0,i]}) - \mathrm{wt}(\phi(c)_{[0,h-1]}) = \mathrm{wt}(\phi(c)_{[h,i]}). \qquad \square$$

Remark 9.6. Fix $j \ge 0$. A j-equivalence may not be an isometry and vice versa, as the following examples show.

(a) Let $C_1 = \langle(1, x, 1)\rangle_{\mathbb{F}_q[x]}$ and $C_2 = \langle(1, x, x)\rangle_{\mathbb{F}_q[x]}$. The $\mathbb{F}_q[x]$-linear map $\phi : C_1 \to C_2$ given by $\phi((1, x, 1)) = (1, x, x)$ is an isometry, but not a 0-equivalence.

(b) Let $C_1 = \langle(1, x^2, x^3)\rangle_{\mathbb{F}_q[x]}$ and $C_2 = \langle(1, x^2, x^3+x^4)\rangle_{\mathbb{F}_q[x]}$. The $\mathbb{F}_q[x]$-linear map $\phi : C_1 \to C_2$ given by $\phi((1, x^2, x^3)) = (1, x^2, x^3 + x^4)$ is a 3-equivalence, but not an isometry.

While a j-equivalence for a fixed value of j may not be an isometry, every equivalence is a strong isometry.

Proposition 9.7. An equivalence between convolutional codes is a strong isometry.

Proof. Let $\phi : C_1 \to C_2$ be an equivalence and let $c \in C_1$. Since ϕ is a j-equivalence for all $j \geq 0$, we obtain that

$$\mathrm{wt}(c) = \lim_{j\to\infty} \mathrm{wt}(c_{[0,j]}) = \lim_{j\to\infty} \mathrm{wt}(\phi(c)_{[0,j]}) = \mathrm{wt}(\phi(c)),$$

i.e., ϕ is weight-preserving. Moreover, $\mathrm{wt}(c_{[i,i]}) = \mathrm{wt}(\phi(c)_{[i,i]})$ for all $i \geq 0$ by Lemma 9.5, hence ϕ is degree-preserving. □

Notice that a strong isometry may not be an equivalence, as the next example shows.

Example 9.8. Let $C_1 = \langle(1, x^2)\rangle$, $C_2 = \langle(x, x^2)\rangle$. Then $\phi : C_1 \to C_2$ defined as $\phi(p(x), x^2p(x)) = (xp(x), x^2p(x))$ is a strong isometry which is not a 0-equivalence, hence not an equivalence.

The next theorems provides us with a simple characterization of equivalences.

Theorem 9.9. Let $\phi : C_1 \to C_2$ be an equivalence of convolutional codes. There exist a permutation matrix $P \in \mathrm{GL}_n(\mathbb{F}_q)$ and a diagonal matrix $D = \mathrm{diag}(a_1, \ldots, a_n)$ with $a_1 \ldots, a_n \in \mathbb{F}_q^*$ such that $\phi(c) = cPD$ for all $c \in C_1$. In particular, every equivalence can be extended to an isometry of $\mathbb{F}_q[x]^n$.

Proof. The statement follows by combining Proposition 9.7, Theorem 9.2, and Lemma 9.5. □

The next proposition provides us with an effective criterion to check whether an isomorphism of convolutional codes is an equivalence.

Proposition 9.10. Let $\phi : C_1 \to C_2$ be an $\mathbb{F}_q[x]$-isomorphism of convolutional codes. Let $c_1, \ldots, c_k$ be a basis of C_1 and let $t = \max\{\deg(c_1), \ldots, \deg(c_k), \deg(\phi(c_1)), \ldots, \deg(\phi(c_k))\}$. If ϕ is a j-equivalence for some $j \geq t$, then it is an equivalence.

Proof. By Lemma 9.5 it suffices to prove that, if ϕ is a j-equivalence for some $j \geq t$, then it is also a $(j + 1)$-equivalence. This is equivalent to showing that $\mathrm{wt}(c_{[j+1,j+1]}) = \phi(c_{[j+1,j+1]})$ for all $c \in C$, since

$$\mathrm{wt}(c_{[0,j+1]}) = \mathrm{wt}(c_{[0,j]}) + \mathrm{wt}(c_{[j+1,j+1]}) = \mathrm{wt}(\phi(c)_{[0,j]}) + \mathrm{wt}(c_{[j+1,j+1]})$$

and

$$\mathrm{wt}(\phi(c)_{[0,j+1]}) = \mathrm{wt}(\phi(c)_{[0,j]}) + \mathrm{wt}(\phi(c)_{[j+1,j+1]}).$$

Let $c = p_1(x)c_1 + \cdots + p_k(x)c_k \in C$ and let

$$\bar{c} = \frac{1}{x}((p_1(x) - p_1(0))c_1 + \cdots + (p_k(x) - p_k(0))c_k) \in C.$$

Since $j \geq t = \max\{\deg(c_1), \ldots, \deg(c_k), \deg(\phi(c_1)), \ldots, \deg(\phi(c_k))\}$, then $c_{[j+1,j+1]} = x\bar{c}_{[j,j]}$ and $\phi(c)_{[j+1,j+1]} = x\phi(\bar{c})_{[j,j]}$. As a consequence, we obtain

$$\mathrm{wt}\left(c_{[j+1,j+1]}\right) = \mathrm{wt}\left(\bar{c}_{[j,j]}\right) = \mathrm{wt}\left(\phi(\bar{c})_{[j,j]}\right) = \mathrm{wt}\left(\phi(c)_{[j+1,j+1]}\right),$$

which concludes the proof. □

Corollary 9.11. Let $\phi : C_1 \to C_2$ be an isometry of convolutional codes. Let δ_1 be the memory of C_1. If ϕ is a δ_1-equivalence, then it is an equivalence.

Proof. Let $c_1, \ldots, c_k$ be a row reduced basis of C_1 such that $\delta_1 = \deg(c_1) \geq \deg(c_2) \geq \cdots \geq \deg(c_k)$. Since ϕ is an isometry and a δ_1-equivalence, we obtain that $\deg(\phi(c_i)) \leq \delta_1$ for $1 \leq i \leq k$. We conclude by Proposition 9.10. □

Remark 9.12. (a) Remark 9.6 (a) is an example of an isometry ϕ of codes with $\delta_1 = 1$, which is not an equivalence. Since ϕ is not a 0-equivalence, it is not a 1-equivalence.

(b) When $\delta_1 = 0$, Corollary 9.11 yields the MacWilliams Extension Theorem for linear block codes. Notice however that this is not a new proof of the MacWilliams Extension Theorem, as the proof of Theorem 9.2 relies on it.

The next theorem provides us with a description of j-equivalences.

Theorem 9.13. Let $\phi : C_1 \to C_2$ be a j-equivalence of convolutional codes. There exist a permutation matrix $P \in \mathrm{GL}_n(\mathbb{F}_q)$ and a diagonal matrix $D = \mathrm{diag}(a_1, \ldots, a_n)$ with $a_1 \ldots, a_n \in \mathbb{F}_q^*$ such that $\phi(c)_{[0,j]} = c_{[0,j]}PD$ for all $c \in C_1$.

Proof. Let $C_1 = \langle c_1, \ldots, c_k\rangle_{\mathbb{F}_q[x]}$ and define $\bar{C}_1, \bar{C}_2$ as

$$\bar{C}_1 = \langle (c_1)_{[0,j]}, \ldots, (c_k)_{[0,j]}\rangle_{\mathbb{F}_q[x]} \text{ and } \bar{C}_2 = \langle \phi(c_1)_{[0,j]}, \ldots, \phi(c_k)_{[0,j]}\rangle_{\mathbb{F}_q[x]}.$$

Notice that for every $c \in C_1$ there exists $\bar{c} \in \bar{C}_1$ such that $\bar{c}_{[0,j]} = c_{[0,j]}$. Indeed, we have that

$$\begin{aligned} c_{[0,j]} &= \left(\sum_{i=1}^{k} p_i(x)c_i\right)_{[0,j]} = \left(\sum_{i=1}^{k} p_i(x)((c_i)_{[0,j]} + (c_i)_{[j+1,\deg(c_i)]})\right)_{[0,j]} \\ &= \left(\sum_{i=1}^{k} p_i(x)((c_i)_{[0,j]})\right)_{[0,j]} = \bar{c}_{[0,j]}. \end{aligned} \tag{9.1.1}$$

Moreover, the j-equivalence ϕ induces an $\mathbb{F}_q[x]$-linear isomorphism $\bar{\phi}$ between $\bar{C}_1$ and $\bar{C}_2$ defined by $\bar{\phi}((c_i)_{[0,j]}) = \phi(c_i)_{[0,j]}$ for $1 \leq i \leq k$. Since ϕ is a j-equivalence, then

$$\begin{aligned} \mathrm{wt}\left(\bar{c}_{[0,j]}\right) &= \mathrm{wt}\left(\left(\sum_{i=1}^{k} p_i(x)(c_i)_{[0,j]}\right)_{[0,j]}\right) = \mathrm{wt}\left(\left(\sum_{i=1}^{k} p_i(x)c_i\right)_{[0,j]}\right) \\ &= \mathrm{wt}\left(\left(\sum_{i=1}^{k} p_i(x)\phi(c_i)\right)_{[0,j]}\right) = \mathrm{wt}\left(\left(\sum_{i=1}^{k} p_i(x)\phi(c_i)_{[0,j]}\right)_{[0,j]}\right) = \mathrm{wt}\left(\phi(\bar{c})_{[0,j]}\right), \end{aligned}$$

and therefore also $\bar{\phi}$ is a j-equivalence. Moreover, since

$$j \geq \max\{\deg((c_1)_{[0,j]}), \dots, \deg((c_k)_{[0,j]}), \deg(\phi(c_1)_{[0,j]}), \dots, \deg(\phi(c_k)_{[0,j]})\},$$

then $\bar{\phi}$ is an equivalence by Proposition 9.10. By Theorem 9.9 there exist a permutation matrix $P \in \mathrm{GL}_n(\mathbb{F}_q)$ and a diagonal matrix $D = \mathrm{diag}(a_1, \dots, a_n)$ with $a_1 \dots, a_n \in \mathbb{F}_q^*$ such that $\phi(\bar{c}) = \bar{c}PD$ for all $\bar{c} \in \bar{C}_1$. For $c \in C$, following the notation of (9.1.1), we have that

$$\begin{aligned}\phi(c)_{[0,j]} &= \left(\sum_{i=1}^{k} p_i(x)\phi(c_i)\right)_{[0,j]} = \left(\sum_{i=1}^{k} p_i(x)\phi(c_i)_{[0,j]}\right)_{[0,j]} \\ &= \phi(\bar{c})_{[0,j]} = (\bar{c}PD)_{[0,j]} = \bar{c}_{[0,j]}PD = c_{[0,j]}PD,\end{aligned}$$

which concludes the proof. □

9.2. Generalized column weights

A generator matrix G of an (n, k, δ) convolutional code C can be expressed as $G = \sum G_i x^i$ with $G_i \in \mathbb{F}^{k\times n}$. The j-truncated sliding generator matrix G_j^c is defined as

$$G_j^c = \begin{pmatrix} G_0 & G_1 & \dots & G_j \\ 0 & G_0 & \dots & G_{j-1} \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 0 & G_0 \end{pmatrix} \in \mathbb{F}_q^{k(j+1)\times n(j+1)}.$$

For $j \geq \delta_1$, we define the matrix $G_j^{c\prime}$ as

$$G_j^{c\prime} = \begin{pmatrix} G_0 & G_1 & \dots & G_{\delta_1} & 0 & \dots & 0 \\ 0 & G_0 & \dots & G_{\delta_1-1} & G_{\delta_1} & \ddots & 0 \\ \vdots & \ddots & & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & G_0 & G_1 & \dots & G_{\delta_1} \end{pmatrix} \in \mathbb{F}_q^{k(j+1)\times n(j+1+\delta_1)}.$$

The j-truncated code or j-truncation of C is

$$C(j) = \{(v^0, v^1, \dots, v^j)G_j^c : v^i \in \mathbb{F}_q^k, v^0 \neq 0\} \subseteq \mathbb{F}_q^{n(j+1)}.$$

Equivalently,

$$C(j) = \{(p(x)G)_{[0,j]} : p(x) \in \mathbb{F}_q[x]^k, p(0) \neq 0\} \subseteq C.$$

Notice that it may happen that $0 \notin C(j)$, so $C(j)$ is not a vector space in general. However, one always has $C(0) \cup \{0\} = C[0] \subseteq \mathbb{F}_q^n$.

We recall that for every $j \geq 0$ the j-th column distance of a convolutional code C is defined as

$$d_j^c(C) = \min \mathrm{wt}\, C(j) = \min \left\{\mathrm{wt}\left((v^0, v^1, \dots, v^j)G_j^c\right) : v^i \in \mathbb{F}_q^k, v^0 \neq 0\right\}.$$

It follows from the definition that $d_j^c(C) \leq d_{j+1}^c(C)$ for $j \geq 0$.

When G_0 has full rank, we have the equivalent formulation

$$d_j^c(C) = \min\left\{\mathrm{wt}\left(c_{[0,j]}(x)\right) : c(x) \in C \text{ and } c(0) \neq 0\right\}.$$

This is the case in particular whenever the code is noncatastrophic. It follows that $d_j(C) > 0$ if G_0 has full rank and $j \geq 0$. In the case of catastrophic convolutional codes and more precisely for codes such that G_0 does not have full rank, it may be that $d_j^c(C) = 0$ for some $j \geq 0$. However, the following proposition shows that for every C there exists a $\hat{j}$ such that $d_{\hat{j}}^c(C) > 0$.

Proposition 9.14. Let C be an (n, k, δ) convolutional code. There exists $\hat{j} \geq 0$ such that $d_{\hat{j}}^c(C) > 0$.

Proof. Let $c_1, \ldots, c_k$ be a row reduced basis for C. Since $d_j^c(C)$ is a weakly increasing function of j, the thesis is equivalent to $d_j^c(C) > 0$ for j sufficiently large. If $d_j^c(C) = 0$ for every $j \geq \delta_1$, then for every $j \geq \delta_1$ there exists $v_j \in \mathbb{F}_q^{k(j+1)}$ such that $v_j^0 \neq 0$ and $v_j G_j^c = 0$. By construction, the first $n(j+1)$ entries of $v_j G_j^{c\prime}$ are equal to 0. By the pigeonhole principle there exist $\delta_1 \leq j_1 < j_2$ such that the last $n\delta_1$ entries of $v_{j_1} G_{j_1}^{c\,\prime}$ coincide with the last $n\delta_1$ entries of $v_{j_2} G_{j_2}^{c\,\prime}$. This implies that there exists two elements $s_1, s_2 \in C$ (corresponding to $v_{j_1} G_{j_1}^{c\,\prime}$ and $v_{j_2} G_{j_2}^{c\,\prime}$), polynomials $p_1, \ldots, p_k, q_1, \ldots, q_k$, and two indices $1 \leq i_1, i_2 \leq k$ such that $s_1 = \sum p_i c_i$, $s_2 = \sum q_i c_i$, $p_{i_1}(0) \neq 0$, $q_{i_2}(0) \neq 0$, and $x^{j_2 - j_1} s_1 = s_2$. Since

$$x^{j_2-j_1} \sum_{i=1}^{k} p_i c_i = \sum_{i=1}^{k} q_i c_i, \quad \text{then} \quad \sum_{i=1}^{k} (x^{j_2-j_1} p_i - q_i) c_i = 0.$$

As $q_{i_2}(0) \neq 0$, we have that $x^{j_2-j_1} p_{i_2} - q_{i_2} \neq 0$. This contradicts the assumption that $c_1, \ldots, c_k$ is a basis of C. We conclude that there exists a $\hat{j}$ such that $d_{\hat{j}}^c(C) > 0$. □

In [CFN17], the authors introduce the concept of j-th generalized column distances for non-catastrophic convolutional codes. In this chapter, we extend their definition to arbitrary codes and establish some properties of these invariants. Notice that our terminology slightly differs from that of [CFN17] and we call (r, j)-generalized column weights the j-th generalized column distances of [CFN17].

Definition 9.15. Let C be an (n, k, δ) convolutional code and G a generator matrix for C. For every $j \geq 0$ and $1 \leq r \leq k$ we define the (r, j)-generalized column weight as

$$d_j^r(C) = \min\left\{|\mathrm{supp}\{v_1 G_j^c, \ldots, v_r G_j^c\}| : v_i \in \mathbb{F}_q^{k(j+1)} \text{ and } \dim\left(\langle v_1^0, \ldots, v_r^0 \rangle_{\mathbb{F}_q}\right) = r\right\}.$$

We say that $v_1, \ldots, v_r \in \mathbb{F}_q^{k(j+1)}$ realize the (r, j)-generalized column weight with respect to G if $\dim\left(\langle v_1^0, \ldots, v_r^0 \rangle_{\mathbb{F}_q}\right) = r$ and $d_j^r(C) = |\mathrm{supp}\{v_1 G_j^c, \ldots, v_r G_j^c\}|$.

Finally, we define the r-generalized column weight as

$$d^r(C) = \lim_{j \to \infty} d_j^r(C).$$

Well-definedness of the r-generalized column weight follows from items 4 and 7 in Proposition 9.19. We stress that the generalized column weights of a code do not depend on the choice of a generator matrix. Indeed, let $c_1 \dots, c_k$ be the rows of a generator matrix G of C. Then

$$d_j^r(C) = \min\{|\mathrm{supp}(V_{[0,j]})| : V \subseteq C \text{ is an } \mathbb{F}_q\text{-linear space with } \dim(V) = r \text{ and for all } v = p_1c_1 + \cdots + p_kc_k \in V \setminus \{0\} \text{ there is } i \text{ such that } p_i(0) \neq 0\}.$$

If $\tilde{G}$ is another generator matrix of C with rows $\tilde{c}_1, \dots, \tilde{c}_k$, then there exists a unimodular matrix $U \in \mathbb{F}_q[x]^{k\times k}$ such that $\tilde{G} = UG$, i.e., $\tilde{c}_i = u_{i,1}c_1 + \cdots + u_{i,k}c_k$ for $1 \leq i \leq k$. For $v = p_1c_1 + \dots p_kc_k = \tilde{p}_1\tilde{c}_1 + \dots \tilde{p}_k\tilde{c}_k \in C$, we have that $v = \tilde{p}_1(u_{1,1}c_1 + \cdots + u_{1,k}c_k) + \cdots + \tilde{p}_k(u_{k,1}c_1 + \cdots + u_{k,k}c_k)$, hence

$$p_i = \sum_{s=1}^{k} (\tilde{p}_s u_{s,i})$$

for $1 \leq i \leq k$. Therefore, if there exists i such that $p_i(0) \neq 0$, then there exists s such that $\tilde{p}_s(0) \neq 0$.

While the (r, j)-generalized column weight of a code C does not depend on the choice of a generator matrix G of C, the vectors $v_1, \dots, v_r$ that realize $d_j^r(C)$ depend on the choice of the matrix G, as the next example shows.

Example 9.16. Let C be the code generated by the matrix G whose rows are $(1, 0, 1)$ and $(0, 1, 0)$. Then $d_1^2(C) = 3$ and it is realized by the vectors $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ with respect to G. The matrix G' whose rows are $(1, x, 1)$ and $(0, 1, 0)$ is also a generator matrix of C and $d_1^2(C)$ is realized by $(1, 0, 0, -1)$ and $(0, 1, 0, 0)$ with respect to G'.

Remark 9.17. The (r, j)-generalized column weights of a noncatastrophic convolutional code are defined in [CFN17] as

$$\bar{d}_j^r(C) = \min\left\{|\mathrm{supp}\{(c_1)_{[0,j]}, \dots, (c_r)_{[0,j]}\}| : c_i \in C \text{ and } \dim\left(\langle c_1(0), \dots, c_r(0)\rangle_{\mathbb{F}_q}\right) = r\right\}.$$

The set of which we take the minimum is always nonempty, as $\dim(C[0]) = \mathrm{rk}(G_0) = k$ for a noncatastrophic C of $\mathrm{rk}(C) = k$. Moreover, for a noncatastrophic code C

$$\dim\left(\langle v_1^0, \dots, v_r^0\rangle_{\mathbb{F}_q}\right) = r \iff \dim\left(\langle v_1G[0], \dots, v_rG[0]\rangle_{\mathbb{F}_q}\right) = r,$$

where $G[0]$ denotes the matrix G_j^c evaluated at 0. In other words, $v_iG[0] = v_i^0G_0$ for $1 \leq i \leq r$. It follows that $d_j^r(C) = \bar{d}_j^r(C)$.

If the code C is catastrophic, then $d_j^r(C)$ is well-defined, while $\bar{d}_j^r(C)$ may not be.

Example 9.18. Let $C = \langle(1, 0), (0, x)\rangle_{\mathbb{F}_q[x]}$. By a straightforward computation, we have that $d_0^1(C) = 0$, $d_0^2(C) = 1$, $d_j^1(C) = 1$ and $d_j^2(C) = 2$ for all $j \geq 1$. Therefore, $d^1(C) = 1$ and $d^2(C) = 2$. Notice that in this case $\bar{d}_0^2(C)$ is not defined, as $\dim(C[0]) = 1$.

In the next proposition we collect several basic properties of generalized column weights. In particular, we have that the $(1, j)$-generalized column weight coincides with the j-th column distance of the code and that generalized column weights are non-decreasing in both r and j. Items 1, 2, and 6 were proved in the noncatastrophic case in [CFN17].

Proposition 9.19. Let C be an (n, k, δ) convolutional code and let $\mathcal{D}$ be a subcode of C. Then

1. $d_j^1(C) = d_j^c(C)$ for $j \geq 0$.

2. $d_j^r(C) \leq d_j^{r+1}(C)$ for $j \geq 0$ and $1 \leq r < k$ and the inequality is strict if C is noncatastrophic.
3. $d^r(C) < d^{r+1}(C)$ for $1 \leq r < k$.
4. $d_j^r(C) \leq d_{j+1}^r(C)$ for $j \geq 0$ and $1 \leq r \leq k$.
5. $d_j^r(C) \leq d_j^r(\mathcal{D})$ for $j \geq 0$ and $1 \leq r \leq \mathrm{rk}(\mathcal{D})$.
6. If C is noncatastrophic, then $d_j^r(C) \leq (j+1)(n-k) + r$ for $j \geq 0$ and $1 \leq r \leq k$.
7. $d_j^r(C) \leq d^r(C) \leq n(\delta_1 + 1)$ for $j \geq 0$ and $1 \leq r \leq k$, where δ_1 is the memory of C.

Proof. Items 1, 4, 5 and the first part of item 2 follow directly from the definition. The noncatastrophic case of item 2 is shown in [CFN17, Theorem 1]. For item 6, see [CFN17, Proposition 1]. To prove item 7, it suffices to compute $|\mathrm{supp}\{e_1 G_j^c, \ldots, e_k G_j^c\}|$, where $e_i \in \mathbb{F}_q^{k(j+1)}$ is the i-th vector of the canonical basis. In order to prove item 3, first notice that items 4 and 7 imply that r-generalized column weights are well-defined. Indeed, the limit always exists since the (r, j)-generalized column weights are non-decreasing in j, and it is finite by item 7. Moreover, by item 7 there exists a $\bar{j} \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ $d_{\bar{j}}^r(C) = d_{\bar{j}+m}^r(C) = d^r(C)$. Fix $r > 1$. Up to increasing $\bar{j}$, we may assume that $d_{\bar{j}}^{r-1}(C) = d^{r-1}(C)$. By Proposition 9.14 there exist $\hat{j}$ such that $d_{\hat{j}}^c(C) > 0$. Let $j \geq \max\{\bar{j}, \hat{j}\}$ and suppose that $v_1, \ldots, v_r$ realize $d_j^r(C)$ with respect to G, i.e., $d_j^r(C) = |\mathrm{supp}\{v_1 G_j^c, \ldots, v_r G_j^c\}|$ and $\dim\left(\langle v_1^0, \ldots, v_r^0 \rangle_{\mathbb{F}_q}\right) = r$. Since $j \geq \hat{j}$, we have that $v_i G_j^c \neq 0$ for all $1 \leq i \leq r$. Hence there exist $\alpha_2, \ldots, \alpha_r \in \mathbb{F}_q$ such that $d_j^r(C) > |\mathrm{supp}\{(v_2 - \alpha_2 v_1) G_j^c, \ldots, (v_r - \alpha_r v_1) G_j^c\}|$ with $\dim\left(\langle (v_2 - \alpha_2 v_1)^0, \ldots, (v_r - \alpha_r v_1)^0 \rangle_{\mathbb{F}_q}\right) = r - 1$. We conclude that $d^r(C) \geq d_j^r(C) > d_j^{r-1}(C) = d^{r-1}(C)$. □

Notice in particular that, while (r, j)-generalized columns distances are only weakly increasing in r, r-generalized columns distances are strictly increasing in r.

Example 9.20. (a) The (r, j)-generalized column weights may not be strictly increasing in r for a fixed j, if the code is catastrophic. For instance, the code $C = \langle (x, 0), (0, x) \rangle$ has $d_0^1(C) = d_0^2(C) = 0$. This is coherent with item 2 of Proposition 9.19. On the other side, item 3 of Proposition 9.19 implies that, for j large enough, the (r, j)-generalized column weights are strictly increasing with r, also in the catastrophic case. For example, the same code C has $d_j^1(C) = 1$ and $d_j^2(C) = 2$ for all $j \geq 1$.

(b) The bound in item 6 of Proposition 9.19 may not hold for catastrophic codes. For instance, let q be a prime and let $n < q$. Let $C = \langle (1, 1, \ldots, 1), (x, 2x, \ldots, nx) \rangle_{\mathbb{F}_q[x]} \subseteq \mathbb{F}_q[x]^n$. Then

$$d_1^2(C) = 2n - 1 > 2(n-2) + 2 = (j+1)(n-k) + r.$$

Remark 9.21. Specializing item 6 in Proposition 9.19 to $r = 1$ and using item 1, one obtains the classical bound for column weights

$$d_j^c(C) \leq (n-k)(j+1) + 1,$$

that was originally proved in [GLRS06]. Moreover, the bound is achieved for several triples of parameters. Indeed, if C is a noncatastrophic (n, k, δ) MDP code we immediately obtain by the definition and by item 3 that $d_j^r(C) = (j+1)(n-k) + r$ for $0 \leq j \leq \left\lfloor \frac{\delta}{k} \right\rfloor + \left\lfloor \frac{\delta}{n-k} \right\rfloor$.

It has been already noticed in [GL09] that column distances are preserved neither under isometries nor under strong isometries. Item 1 of Proposition 9.19 therefore implies that (r, j)-generalized column weights cannot be invariant under isometries either. However, in the next proposition we show that they are invariant under j-equivalences. Later in this section, we prove that r-generalized column weights are invariant under isometries.

Proposition 9.22. Let $\phi : C_1 \to C_2$ be a j'-equivalence. Then, $d^r_j(C_1) = d^r_j(C_2)$ for $1 \leq r \leq k$ and $0 \leq j \leq j'$.

Proof. Let G be a generator matrix of C_1 and suppose that $v_1, \dots, v_r \in \mathbb{F}_q^{k(j+1)}$ realize the (r, j)-generalized column weight of C with respect to G. Since ϕ is a j'-equivalence, by Theorem 9.13 there exist a permutation matrix $P \in \mathrm{GL}_n(\mathbb{F}_q)$ and a diagonal matrix $D = \mathrm{diag}(a_1, \dots, a_n)$ with $a_1 \dots, a_n \in \mathbb{F}_q^*$ such that $\phi(G)_i = G_i PD$ for all $i \leq j'$. In particular,

$$\phi(G)^c_j = \begin{pmatrix} G_0PD & G_1PD & \dots & G_jPD \\ 0 & G_0PD & \dots & G_{j-1}PD \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & G_0PD \end{pmatrix}.$$

Hence

$$d^r_j(C_2) \leq |\mathrm{supp}\{v_1\phi(G)^c_j, \dots, v_r\phi(G)^c_j\}| = |\mathrm{supp}\{v_1G^c_j, \dots, v_rG^c_j\}| = d^r_j(C_1).$$

The reverse inequality follows by looking at $\phi^{-1} : C_2 \to C_1$, which is a j-equivalence by item 1 in Propositon 9.4. □

Since $d^r(C) = \lim_{j\to\infty} d^r_j(C)$, the previous proposition implies that r-generalized column weights are preserved by equivalences.

Corollary 9.23. Let $\phi : C_1 \to C_2$ be an equivalence. Then, $d^r(C_1) = d^r(C_2)$ for $1 \leq r \leq k$.

We now extend the result of Corollary 9.23 to all isometries.

Theorem 9.24. Let $\phi : C_1 \to C_2$ be an isometry. Then, $d^r(C_1) = d^r(C_2)$ for $1 \leq r \leq k$.

Proof. By Theorem 9.2 there exist a permutation matrix $P \in \mathrm{GL}_n(\mathbb{F}_q)$ and a diagonal matrix $D = \mathrm{diag}(a_1x^{m_1}, \dots, a_nx^{m_n})$ where $a_1, \dots, a_n \in \mathbb{F}_q^*$ and $m_1, \dots, m_n \in \mathbb{Z}$ such that $\phi(c) = cPD$ for all $c \in C$. Let $m = \max\{0, -m_i : 1 \leq i \leq n\}$. Let G be a generator matrix of C_1 and suppose that $p_1, \dots, p_r \in \mathbb{F}_q[x]^k$ realize the $(r, j+m)$-generalized column weight of C with respect to G, i.e., $\dim(\langle p_1(0), \dots, p_r(0)\rangle_{\mathbb{F}_q} = r$ and $d^r_{j+m}(C_1) = |\mathrm{supp}\{(p_1G)_{[0,j+m]}, \dots, (p_rG)_{[0,j+m]}\}|$. Then

$$\begin{aligned} d^r_j(C_2) &\leq \left|\mathrm{supp}\left\{(p_1\phi(G))_{[0,j]}, \dots, (p_r\phi(G))_{[0,j]}\right\}\right| = \left|\mathrm{supp}\left\{(p_1GPD)_{[0,j]}, \dots, (p_rGPD)_{[0,j]}\right\}\right| \\ &\leq \left|\mathrm{supp}\left\{(p_1G)_{[0,j+m]}, \dots, (p_rG)_{[0,j+m]}\right\}\right| = d^r_{j+m}(C_1). \end{aligned} \tag{9.2.2}$$

In fact, the columns of GPD are equal to those of G up to permutation and multiplying by a constant and a power of x with exponent smaller than or equal to m. Since none of these operations affects supports and a monomial of degree t in the i-th column of GP corresponds to a monomial of degree $t + m_i$ in the i-th column of GPD, then $(pG)_{[0,j+m]}$ contains all the monomials that appear in $(pGPD)_{[0,j]}$ (and possibly more). This proves the inequality in (9.2.2). As j goes to infinity, we obtain

$$d^r(C_2) = \lim_{j\to\infty} d^r_j(C_2) \leq \lim_{j\to\infty} d^r_{j+m}(C_1) = d^r(C_1).$$

The reverse inequality follows by considering ϕ^{-1} instead of ϕ. Therefore, we conclude that the r-generalized column weights are invariant under isometries. □

It follows from Proposition 9.19 that the sequence $(d^r_j(C))_{j\in\mathbb{N}}$ stabilizes after some $\bar{j}$. In the last part of this section, we give an upper bound on $\bar{j}$.

Theorem 9.25. Let C be an (n, k, δ) convolutional code with memory δ_1. If $\bar{j} = \min\{j : d^r_j(C) = d^r(C)\}$, then

$$\bar{j} < [n(\delta_1 + 1) + 1]q^{\delta_1 kr}.$$

Proof. If $\delta_1 = 0$ then the thesis holds, since $d^r(C) = d^r_0(C)$. Hence assume $\delta_1 \geq 1$. Let $\bar{j} = \min\{j : d^r_j(C) = d^r(C)\}$ and let $v_1, \dots, v_r \in \mathbb{F}_q^{(\bar{j}+1)k}$ be vectors that realize $d^r_{\bar{j}}(C)$ with respect to a row reduced generator matrix G of C. For each $\delta_1 \leq j \leq \bar{j}$ let $\pi_j : \mathbb{F}_q^{(\bar{j}+1)k} \to \mathbb{F}_q^{(j+1)k}$ be the canonical projection on the first $(j+1)k$ entries. Define vector spaces D_j as

$$D_j = \langle \pi_j(v_1)G^{c\prime}_j, \dots, \pi_j(v_r)G^{c\prime}_j\rangle_{\mathbb{F}_q}.$$

First of all notice that $(D_j)_{[0,j]} = (D_{\bar{j}})_{[0,j]}$, hence $d^r_{\bar{j}}(C) \geq \left|\mathrm{supp}(D_j)_{[0,j]}\right|$. Consider now the sequence $\{(D_j)_{[j+1,j+\delta_1]}\}_{j\geq 0}$. Since there are at most $q^{\delta_1 kr}$ different $\mathbb{F}_q$-linear spaces in the sequence $\{(D_j)_{[j+1,j+\delta_1]}\}_{j\geq 0}$, for every $1 \leq s \leq n(\delta_1+1)$ we can find two indices j_1 and j_2 such that

- $\delta_1 + (s-1)q^{\delta_1 kr} \leq j_1 < j_2 \leq \delta_1 + sq^{\delta_1 kr}$
- $(D_{j_1})_{[j_1+1,j_1+\delta_1]} = (D_{j_2})_{[j_2+1,j_2+\delta_1]}$.

We observe that if

$$\left|\mathrm{supp}\left\{(D_{j_1})_{[0,j_1]}\right\}\right| = \left|\mathrm{supp}\left\{(D_{j_2})_{[0,j_2]}\right\}\right|,$$

then $d^r(C) = d^r_{j_1}(C)$, contradicting the minimality of $\bar{j}$. Indeed, for $j = j_2 + t(j_2 - j_1)$ with $t \geq 0$, the vectors

$$t(v_i) = (v_i^0, \dots, v_i^{j_1}, v_i^{j_1+1}, \dots, v_i^{j_2}, \underbrace{v_i^{j_1+1}, \dots, v_i^{j_2}, \dots, v_i^{j_1+1}, \dots, v_i^{j_2}}_{\text{t times}})$$

realize $d^r_j(C)$ and $d^r_{j_2+t(j_2-j_1)}(C) = d^r_{j_1}(C)$ for all t. So the support must increase by at least 1 every $q^{\delta_1 kr}$ steps. If $\bar{j} \geq [n(\delta_1+1)+1]q^{\delta_1 kr}$, then

$$d^r(C) = d^r_{\bar{j}}(C) > n(\delta_1 + 1),$$

which contradicts Proposition 9.19. Hence we conclude that $\bar{j} < [n(\delta_1+1)+1]q^{\delta_1 kr}$. □

Theorem 9.25 implies that the (r, j)-generalized column weights can be computed in a finite amount of time by exhaustive search. As the parameters grow, however, such a computation quickly becomes practically infeasible.

Remark 9.26. Notice that we do not expect the bound in Theorem 9.25 to be sharp. In fact, one gets a sharper bound by substituting the quantity $q^{\delta_1 kr}$ by $\frac{q^{\delta_1 kr}-1}{(q^r-1)(q^r-q)\cdots(q^r-q^{r-1})}$, which is a tighter bound for the number of vector spaces of the form $(D_j)_{[j+1,j+\delta_1]}$. One therefore obtains $\bar{j} < [n(\delta_1+1)+1]\frac{q^{\delta_1 kr}-1}{(q^r-1)(q^r-q)\cdots(q^r-q^{r-1})} \sim n\delta_1 q^{\delta_1 kr - r^2}$. We do not know whether the bound can be further improved by means of different arguments.

9.3. Related definitions and comparison

Definition 9.27. Let C be a noncatastrophic code. The r-th unrestricted generalized column distance of the j-truncated code of C is given by

$$d_r(C(j)) = \min\left\{\left|\mathrm{supp}(D)\right| : D \subseteq C(j),\ \dim\left(\langle D\rangle_{\mathbb{F}_q}\right) = r\right\} \tag{9.3.3}$$

for $1 \le r \le k(j+1)$.

The concept of unrestricted generalized column distances for noncatastrophic codes was introduced by Cardell, Firer, and Napp in [CNF19] and further studied in [CFN20] by the same authors. In particular they proved that the r-th unrestricted generalized column distance is strictly increasing as a function of r and they showed how to compute it from the truncated parity-check matrix.

If the code C is noncatastrophic, then G_0 has full rank, hence G_j^c has full rank for every $j \ge 0$. Therefore

$$d_r(C(j)) = \min\left\{\left|\mathrm{supp}\{v_1 G_j^c, \dots, v_r G_j^c\}\right| : v_i \in \mathbb{F}_q^{(j+1)k},\ v_1^0 \neq 0,\ \dim\left(\langle v_1,\dots,v_r\rangle_{\mathbb{F}_q}\right) = r\right\}. \tag{9.3.4}$$

Notice that, unlike (9.3.3), (9.3.4) allows to extend Definition 9.27 to the catastrophic case in a similar way as we have done for the generalized column weights. Moreover (9.3.4) implies that, for every code C and any $1 \le r \le k$ and $0 \le j$, we have $d_r(C(j)) \le d_j^r(C)$.

If two codes have the same unrestricted generalized column distances, they may not have the same generalized column weights and vice versa, as the next examples show. The first example also shows how the limit as j goes to infinity of the unrestricted generalized column distances may depend only on a submodule of C of rank strictly smaller than k.

Example 9.28. Consider the following two codes of rank 2

$$C_1 = \langle(1,1,0,0,0),(0,0,1,1,1)\rangle_{\mathbb{F}_q[x]} \text{ and } Cc_2 = \langle(1,1,0,0,0),(0,x,1,1,1)\rangle_{\mathbb{F}_q[x]}.$$

One obtains by direct computation that

$$d_r(C_1(j)) = d_r(C_2(j)) = \begin{cases} 2r & \text{if } 1 \le r \le j+1, \\ 2(j+1)+3(r-j-1) & \text{if } j+1 < r \le 2j+2. \end{cases}$$

On the other hand, one can check that $d_1^2(C_1) = 5$, while $d_1^2(C_2) = 6$. Moreover, $d^2(C_1) = 5$ and $d^2(C_2) = 6$, while for all $r \ge 1$

$$\lim_{j\to\infty} d_r(C_1(j)) = \lim_{j\to\infty} d_r(C_2(j)) = 2r.$$

Let $C = \langle(1,1,0,0,0)\rangle_{\mathbb{F}_q[x]} \subseteq C_1 \cap C_2$ be a rank 1 subcode. One can easily check that $d_r(C(j)) =$

$2r$ for $1 \leq r \leq j+1$, hence $\lim_{j\to\infty} d_r(C(j)) = 2r$.

Example 9.29. Let $C_1 = \langle(1+x, 1, 0)\rangle_{\mathbb{F}_q[x]}$ and $C_2 = \langle(1, 1, x)\rangle_{\mathbb{F}_q[x]}$. We have that $d_0^1(C_1) = d_0^1(C_2) = 2$ and $d_j^1(C_1) = d_j^1(C_2) = 3$ for all $j \geq 1$. On the other side, $d_2(C_1(1)) = 4$ and $d_2(C_2(1)) = 5$.

For a fixed code and a fixed r, the r-th unrestricted generalized column distance of the j-truncated code may be increasing, decreasing or constant in j. This is different from the behavior of (r, j)-column weights which are non-decreasing in j for a fixed r, as shown in item 4 in Proposition 9.19.

Example 9.30. Let $C_1 = \langle(1, 1, 0, 0, 0), (0, 0, 1, 1, 1)\rangle_{\mathbb{F}_q[x]}$ be the code of Example 9.28 and $C_2 = \langle(1, x, 0), (0, x, 1)\rangle_{\mathbb{F}_q[x]}$. It is easy to check that

$$d_2(C_1(0)) = 5 > 4 = d_2(C_1(1)) = d_2(C_1(2)) \text{ and } d_2(C_2(0)) = 2 < 3 = d_2(C_2(1)).$$

We now discuss the relation among generalized column weights and generalized weights. As in the previous chapter, in order to avoid ambiguity we denote by $d_r^H(\mathcal{L})$ the r-generalized Hamming weight for an $\mathbb{F}_q$-linear code $\mathcal{L}$.

Proposition 9.31. Let C be an (n, k, δ) convolutional code. Then

- $d_0^r(C) = d_r^H(C[0])$ for $1 \leq r \leq k$.
- If $\delta = 0$, then $d^r(C) = d_j^r(C) = d_r^H(C[0])$ for $1 \leq r \leq k$ and $j \geq 0$.

Proof. The statement follows from the fact that for every linear code $\mathcal{L} \subseteq \mathbb{F}_q^n$ one has $\dim(\mathcal{L}) = \operatorname{rk}\left(\langle\mathcal{L}\rangle_{\mathbb{F}_q[x]}\right)$. □

The next theorem gives an equivalent definition for the r-generalized column weights of a noncatastrophic code. It will be useful to establish an inequality between the generalized weights defined in the previous chapter and the generalized column weights.

Theorem 9.32. Let C be a noncatastrophic code. Then

$$d^r(C) = \min\left\{\left|\operatorname{supp}\left\{v_1 G_j^{c\prime}, \ldots, v_r G_j^{c\prime}\right\}\right| : j \geq 0,\ v_i \in \mathbb{F}_q^{(j+1)k} \text{ with } \dim(\langle v_1^0, \ldots, v_r^0\rangle) = r\right\}.$$

Proof. By definition of $d^r(C)$, the left-hand side is smaller than or equal to the right-hand side. So it suffices to prove the reverse inequality. By Proposition 9.19 there exists a $\bar{j}$ such that $d_{\bar{j}}^r(C) = d^r(C)$. Let G be a generator matrix for C and let $v_1, \ldots, v_r \in \mathbb{F}_q^{(A+\bar{j}+1)k}$ with $A = q^{\delta_1 kr}$ be vectors that realize $d_{\bar{j}+A}^r(C)$ with respect to G. Then there exist $\bar{j} \leq j_1 < j_2 \leq \bar{j}+A$ such that, for $1 \leq i \leq r$, $(v_i^0, \ldots v_i^{j_1})G_{j_1}^{c\prime}$ corresponds to a codeword of the form $c_i + x^m d_i$ and $(v_i^0, \ldots v_i^{j_2})G_{j_2}^{c\prime}$ corresponds to a codeword of the form $c_i + x^n d_i$, where $n > m > \bar{j}$, $\deg(c_i) \leq \bar{j}$ and $\deg(d_i) \leq \delta_1 - 1$. Therefore, $(x^n - x^m)d_i = c_i + x^n d_i - (c_i + x^m d_i) \in C$. Since the code is noncatastrophic, $d_i \in C$ by Proposition 2.14, so there are $w_1, \ldots, w_r \in \mathbb{F}_q^{(j_1+1)k}$ such that $w_1^0 = \cdots = w_r^0 = 0$ and $(v_i^0 - w_i^0, \ldots v_i^{j_1} - w_i^{j_1})G_{j_1}^{c}{}'$ corresponds to c_i. Hence

$$d^r(C) = d_{\bar{j}}^r(C) = \left|\operatorname{supp}\left\{(v_1^0 - w_1^0, \ldots v_1^{j_1} - w_1^{j_1})G_{j_1}^{c\prime}, \ldots, (v_r^0 - w_r^0, \ldots v_r^{j_1} - w_r^{j_1})G_{j_1}^{c\prime}\right\}\right|.$$

This concludes the proof, since $\dim(\langle v_1^0, \ldots, v_r^0\rangle) = r$ implies that $\dim(\langle(v_1 - w_1)^0, \ldots, (v_r - w_r)^0\rangle) = r$. □

If the code is catastrophic, the inequality

$$d^r(C) \leq \min\{|\mathrm{supp}\{v_1 G_j^{c'}, \ldots, v_r G_j^{c'}\}| : j \geq 0,\ v_i \in \mathbb{F}_q^{(j+1)k} \text{ with } \dim(v_1^0, \ldots, v_r^0) = r\},$$

still holds. However we do not always have equality, as the next example shows.

Example 9.33. Consider the convolutional code $C = \langle (1, x), (x, 1)\rangle_{\mathbb{F}_q[x]}$. We have that $(1 - x^2, 0) = (1, x) - x(x, 1) \in C$. Similarly, $(1 - x^{2n}, 0), (0, 1 - x^{2n}) \in C$. Therefore $d_j^r(C) = d^r(C) = r$ for all $j \geq 0$ and $r = 1, 2$. On the other side,

$$\min\left\{\left|\mathrm{supp}\left\{vG_j^{c'}\right\}\right| : j \geq 0,\ v \in \mathbb{F}_q^{(j+1)k} \text{ with } v_1^0 \neq 0\right\} = 2.$$

In particular, $d_1(C) = 2$.

For a noncatastrophic code, the r-th generalized weight is smaller than or equal to the r-generalized column weight. In addition, equality holds for $r = 1$.

Corollary 9.34. Let C be a noncatastrophic code. Then

$$d_r(C) \leq d^r(C).$$

Moreover, $d_1(C) = d^1(C) = d_{\mathrm{free}}(C)$.

Proof. Let G be a generator matrix for C and let $c_1, \ldots, c_k$ be the rows of G. By Theorem 9.32 there exist $j \geq 0$ and $v_1, \ldots, v_r \in \mathbb{F}_q^{(j+1)k}$ such that $d^r(C) = \left|\mathrm{supp}\left\{v_1 G_j^{c'}, \ldots, v_r G_j^{c'}\right\}\right|$ and $\dim(\langle v_1^0, \ldots, v_r^0\rangle) = r$. For each $1 \leq i \leq r$, let $d_i \in C$ be the element that corresponds to $v_i G_j^{c'}$. In order to conclude, it suffices to show that $\dim(\langle v_1^0, \ldots, v_r^0\rangle) = r$ implies $\mathrm{rk}\left(\langle d_1, \ldots, d_r\rangle_{\mathbb{F}_q[x]}\right) = r$. Since the code is noncatastrophic, it suffices to show that $\sum q_i d_i \neq 0$ for every set of polynomials $q_1, \ldots, q_r$ such that at least one of them is not divisible by x. Suppose that $x \nmid q_s$. Since $\dim(\langle v_1^0, \ldots, v_r^0\rangle) = r$, then

$$q_s(0) v_s^0 G_0 + \sum_{i \neq s} q_i(0) v_i^0 G_0 \neq 0,$$

which implies that

$$q_s(0) d_s(0) + \sum_{i \neq s} q_i(0) d_i(0) = \sum_{i=1}^{r} q_i(0) d_i(0) \neq 0.$$

In particular, $\sum q_i d_i \neq 0$. Finally,

$$d_{\mathrm{free}}(C) \leq d^1(C) \leq d_1(C) = d_{\mathrm{free}}(C),$$

where the first inequality follows from Theorem 9.32. □

Example 9.33 shows that for a catastrophic code one can have $d^r(C) < d_r(C)$. The next example shows that there are catastrophic codes for which $d_r(C) < d^r(C)$.

Example 9.35. Let $C = \langle (1, x, 0), (0, 1, 1)\rangle_{\mathbb{F}_q[x]}$. One can check that C is noncatastrophic and $d_2(C) = 3 < 4 = d^2(C)$.

We conclude this section by exhibiting families of codes whose r-th generalized weight coincides with their r-generalized column weight for all $1 \leq r \leq k$. The proof of the next proposition is similar to the one of Proposition 8.44.

Proposition 9.36. Let C be a noncatastrophic MDS (n,k,δ) convolutional code. If $k \nmid \delta$, let $0 < a < k$ such that $\delta = k\left\lceil \frac{\delta}{k} \right\rceil - a$. Then, for $1 \le r \le a$

$$d^r(C) = d_r(C) = (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + r,$$

and for $a < r \le k$

$$d^r(C) \le d_r^H(C[0]) + (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta.$$

If $k \mid \delta$, then

$$d^r(C) = d_r(C) = (n-k)\left(\frac{\delta}{k} + 1\right) + \delta + r.$$

Proof. Let $c_1, \dots, c_k$ be a basis for C. Since C is MDS, if $k \nmid \delta$, we may assume that $\deg(c_1) = \cdots = \deg(c_a) = \left\lfloor \frac{\delta}{k} \right\rfloor$ and $\deg(c_{a+1}) = \cdots = \deg(c_k) = \left\lceil \frac{\delta}{k} \right\rceil$, see Proposition 8.44 for more details. For $r \le a$, we conclude by observing that $|\mathrm{supp}\{c_1, \dots, c_a\}| \le d_{\mathrm{free}}(C) + a$ and by Proposition 9.19. If $r > a$, let $d_1, \dots, d_r$ be linearly independent elements of $\langle c_1, \dots, c_k\rangle_{\mathbb{F}_q}$ such that $|\mathrm{supp}\{d_1(0), \dots, d_r(0)\}| = d_r^H(C[0])$ and $\dim(\langle d_1(0), \dots, d_r(0)\rangle_{\mathbb{F}_q}) = r$. Up to using $xc_1, \dots, xc_a$ to reduce the support, we may assume that $|\mathrm{supp}\{(d_1)_{[\delta_1,\delta_1]}, \dots, (d_r)_{[\delta_1,\delta_1]}\}| \le n - a$. Then

$$|\mathrm{supp}\{d_1, \dots, d_r\}| \le d_r^H(C[0]) + n\left(\left\lfloor \frac{\delta}{k} \right\rfloor\right) + n - a = d_r^H(C[0]) + (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta.$$

If $k \mid \delta$, then $\deg(c_1) = \cdots = \deg(c_k) = \frac{\delta}{k}$ and

$$\left|\mathrm{supp}\{c_1, \dots, c_k\}\right| \le n\left(\frac{\delta}{k} + 1\right) = (n-k)\left(\frac{\delta}{k} + 1\right) + \delta + k.$$

We conclude by items 1 and 3 of Proposition 9.19 and Proposition 8.44. □

Notice that the generalized column weights of an MDS code are not determined by its parameters, as the next example shows.

Example 9.37. Let $C = \langle (1,1,2), (2x, x+1, 0)\rangle_{\mathbb{F}_q[x]}$. The code C is MDS and one can check that $d^2(C) = 5$. The reverse code $\mathrm{rev}(C)$ of C is generated by $(1,1,2)$ and $(2, 1+x, 0)$. It is MDS, it has the same parameters as C, and $d^2(\mathrm{rev}(C)) = 4$.

Proposition 9.38. Let C be a noncatastrophic convolutional code. If C is MDS and MDP, then $d_r(C) = d^r(C)$ for $1 \le r \le k$.

Proof. If $k \mid \delta$, we conclude by Proposition 9.36. So suppose that $k \nmid \delta$ and let $0 < a < k$ be such that $\delta = k\left\lceil \frac{\delta}{k} \right\rceil - a$. Let $d_1, \dots, d_r$ be elements in C that realize $d_r(C)$, that is, $d_1, \dots, d_r$ generate a subcode of rank r and weight $d_r(C)$. Since C is noncatastrophic, we may assume without loss of generality that $d_1(0) \neq 0$. If $\dim(\langle d_1(0), \dots, d_r(0)\rangle_{\mathbb{F}_q}) = r$, then $d_r(C) \ge d^r(C)$ and we conclude by Corollary 9.34. Otherwise, suppose that there exists $d \in \langle d_1, \dots, d_r\rangle_{\mathbb{F}_q}$ such that $d(0) = 0$. Notice that $d_1^H(C[0]) = n - k + 1$ since C is MDP. Then

$$\left|\mathrm{supp}\{d_1, \dots, d_r\}\right| \ge (n-k+1) + (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta + r - 1$$

Therefore, by Proposition 9.36 and Corollary 9.34

$$d_r^H(C[0]) + (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta = d_r(C) \leq d^r(C) \leq d_r^H(C[0]) + (n-k)\left(\left\lfloor \frac{\delta}{k} \right\rfloor + 1\right) + \delta.$$

We conclude that in any case $d_r(C) = d^r(C)$. □

An (n, k, δ) strongly MDS code such that $(n-k) \mid \delta$ is both MDS and MDP. The next corollary then follows from Proposition 9.38.

Corollary 9.39. Let C be a noncatastrophic convolutional code. If C is strongly MDS and $(n-k) \mid \delta$, then $d_r(C) = d^r(C)$ for $1 \leq r \leq k$.

A. Length of path-reduction chains

In this appendix, we prove that every path-reduction chain of a matrix $M \in \mathbb{F}_q^{m\times n}$ has the same length.

Remark A.1. Let $M \in \mathbb{F}_q^{m\times n}$ and let $\sigma_1 = ((i_1, j_1), \dots, (i_k, j_k))$ and $\sigma_2 = ((i'_1, j'_1), \dots, (i'_h, j'_h))$ be two closed simple paths. Notice that if $\mathrm{supp}(\sigma_1) \neq \mathrm{supp}(\sigma_2)$, then $\mathrm{supp}(\sigma_1) \nsubseteq \mathrm{supp}(\sigma_2)$ and vice versa.

In the next lemma, we prove that if M contains two distinct closed single paths, than a path-reduction chain of M has length at least 3.

Lemma A.2. Let $M = (m_{ij}) \in \mathbb{F}_q^{m\times n}$, let $\sigma_1 = ((i_1, j_1), \dots, (i_k, j_k))$ and $\sigma_2 = ((i'_1, j'_1), \dots, (i'_h, j'_h))$ be two closed simple paths such that $\mathrm{supp}(\sigma_1) \neq \mathrm{supp}(\sigma_2)$. If $(i_1, j_1) = (i'_1, j'_1)$, then for each $(i_s, j_s) \in \mathrm{supp}(\sigma_1) \setminus \mathrm{supp}(\sigma_2)$ there is a closed simple path in $M - m_{i_1,j_1}E_{i_1,j_1}$ that contains (i_s, j_s).

Proof. Up to reversing the order of σ_2 and to a transposition, we may suppose without loss of generality that $j'_1 = j'_2 = j_k = j_1$. As a consequence, also $i_1 = i_2 = i'_h = i'_1$. Consider the list of positions

$$\gamma = (\gamma_0, \dots, \gamma_{h+k-3}) = ((i_2, j_2), \dots, (i_k, j_k), (i'_2, j'_2), \dots, (i'_h, j'_h)).$$

Notice that γ is not always a path, since it can contain repeated entries. Fix an s such that $(i_s, j_s) \in \mathrm{supp}(\sigma_1) \setminus \mathrm{supp}(\sigma_2)$ and let $\gamma_x = (i_s, j_s)$. We now recursively build a finite sequence of simple paths π_n, whose support is contained in that of γ and which start with γ_x. Let $\pi_1 = (\gamma_x)$. Suppose that we have constructed $\pi_{n-1} = (p_1, \dots, p_\ell)$ with $p_1 = \gamma_x$ and $p_\ell = \gamma_y$, with $y = x+n-2 \mod. h+k-2$ and $\ell \geq 2$. Let $z = y + 1 \mod. h + k - 2$ and define π_n as follows:

- If no two entries of π_{n-1} have either the first or the second coordinate in common with γ_z, then let $\pi_n = (p_1, \dots, p_\ell, \gamma_z)$.
- If there exists $1 \leq r < t \leq \ell$ such that p_r, p_t and γ_z share either the first or the second component, fix the smallest r and t with this property and let $\pi_n = (p_1, \dots, p_r, \gamma_z)$.

For $n \geq 2$, π_n is a simple path of length at least 2. If for some n we find a closed simple path, then we are done. Else $\pi_{h+k-2} = (\gamma_x, q_1, \dots, q_v, \gamma_{x-1})$ for some $q_1, \dots, q_v$. Since q_1 lies on the same line as γ_x and γ_{x+1} and γ_{x-1} does not lie on that line, then $v \geq 1$. Since γ_x and γ_{x-1} also lie on a common line, π_{h+k-2} is a closed simple path and $v \geq 2$. □

The next lemma shows that the length of a path-reduction chain is independent of the order of the reductions.

Lemma A.3. Let $M \in \mathbb{F}_q^{m\times n}$ and let $M, M_2, \dots, M_{k+1}$ be a path-reduction chain for M. Let $\alpha_1, \dots, \alpha_k$ be the ordered list of positions of the entries that we set to zero during the path-reduction chain. Any permutation of the sequence $\alpha_1, \dots, \alpha_k$ still yields a path-reduction chain for M.

Proof. Since the group of permutation of k elements is generated by the $k-1$ transpositions $(1,2),(2,3),\ldots,(k-1,k)$, it suffices to prove that setting to zero the entries in position

$$\alpha_1,\ldots,\alpha_{i-2},\alpha_i,\alpha_{i-1},\alpha_{i+1},\ldots,\alpha_k$$

in the given order gives a path-reduction chain for M, for $i=2,\ldots,k$. This corresponds to the sequence of matrices

$$M_1,M_2,\ldots,M_{i-1},\bar{M}_i,M_{i+1},M_{i+2},\ldots,M_{k+1}$$

where we let $M_1=M$. By assumption, M_{k+1} is irreducible and M_j is a reduction of M_{j-1} for $j=2,\ldots,i-1,i+2,\ldots,k$.

The matrix $\bar{M}_i$ is obtained from M_{i-1} by setting to zero the entry in position α_i. Since α_i belongs to a closed simple path π in M_i and every nonzero entry in M_i is also a nonzero entry in M_{i-1}, then π is also a closed simple path in M_{i-1}. Therefore, $\bar{M}_i$ is a reduction of M_{i-1}. In order to prove that M_{i+1} is a reduction of $\bar{M}_i$, we need to show that there is a closed simple path in $\bar{M}_i$ which contains α_{i-1}. Notice that $\bar{M}_i$ is equal to M_i, except for the entries in position α_{i-1} and α_i. By assumption, there are closed simple paths σ_1 and σ_2 in M_{i-1} such that σ_1 contains α_{i-1} and σ_2 contains α_i, but not α_{i-1}. If σ_1 does not contain α_i, then it is a closed simple path in $\bar{M}_i$ which contains α_{i-1}. If instead σ_1 contains α_i, then by Lemma A.2 there is a closed simple path in M_{i-1} which contains α_{i-1} but not α_i. This gives a closed simple path in $\bar{M}_i$ which contains α_{i-1}. □

We are now ready to prove that every path reduction chain of a given matrix has the same length.

Theorem A.4. Let $M\in\mathbb{F}_q^{m\times n}$ be a matrix. Every path-reduction chain of M has the same length.

Proof. We proceed by induction on the maximum length ℓ of a path-reduction chain of M. Notice that $\ell\geq 1$ and equality holds if and only if M is irreducible. If $\ell=2$, then M needs to have at least one closed simple path. Moreover, there is an α in the path such every closed simple path in M contains α. If M contains two distinct closed simple paths through α, then by Lemma A.2 it also contains a closed simple path that does not pass through α. It follows that M contains exactly one closed simple path and every path-reduction chain has length two and is obtained by replacing with zero one of the entries of M in one of the positions on the closed simple path.

Let $M,M_2,\ldots,M_\ell$ and $M,M_2',\ldots,M_k'$ be two path-reduction chains for M, $\ell\geq k$. Let $\alpha_1,\ldots,\alpha_{k-1}$ and $\beta_1,\ldots,\beta_{\ell-1}$ be the positions of the entries of M that we replace with zero to obtain the path-reduction chains $M,M_2',\ldots,M_k'$ and $M,M_2,\ldots,M_\ell$, respectively. Notice that $M_2,\ldots,M_\ell$ is a path-reduction chain for M_2 and, by the induction hypothesis, every path reduction chain for M_2 has length $\ell-1$. Starting from M_2, we construct a path-reduction chain $M_2,\bar{M}_3,\ldots,\bar{M}_\ell$ as follows. At each step $i=1,\ldots,k-1$, if there is a closed simple path that contains α_i, we replace the entry in position α_i by zero. We claim that we delete at most $k-2$ entries of M_2. In fact, if setting to zero the entries in position $\beta_1,\alpha_1,\ldots,\alpha_{k-1}$ in the prescribed order yields a path-reduction chain of M, by Lemma A.3 so does setting to zero the entries in position $\alpha_1,\ldots,\alpha_{k-1},\beta_1$. This contradicts the assumption that $M,M_2',\ldots,M_k'$ is a path-reduction chain. So we have obtained a path-reduction chain for M_2 of length $\ell-1\leq k-1$. It follows that $\ell=k$. □

B. Support spaces and information leakage

In this appendix we discuss an alternative notion of generalized weights that could be defined for linear sum-rank metric codes $\mathcal{C} \subseteq \mathbb{F}_q^{m_1 \times n_1} \times \mathbb{F}_q^{m_2 \times n_2} \times \cdots \times \mathbb{F}_q^{m_\ell \times n_\ell}$ as follows. Here, we do not assume that $n_i \leq m_i$, for $i = 1, 2, \ldots, \ell$. In Definition 2.10, for positive integers m and n, we defined the (row) support space $\mathcal{V}_\mathcal{S} \subseteq \mathbb{F}_q^{m \times n}$ associated to the vector space $\mathcal{S} \subseteq \mathbb{F}_q^n$ as

$$\mathcal{V}_\mathcal{S} = \{C \in \mathbb{F}_q^{m \times n} : \mathrm{rowsp}(C) \subseteq \mathcal{S}\}.$$

Clearly $\mathcal{V}_\mathcal{S}$ is an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^{m \times n}$ of dimension

$$\dim(\mathcal{V}_\mathcal{S}) = m \dim(\mathcal{S}).$$

Denote by $\mathcal{P}_{q,n}$ the collection of subspaces of $\mathbb{F}_q^n$. For a linear code $\mathcal{C} \subseteq \mathbb{F}_q^{m_1 \times n_1} \times \mathbb{F}_q^{m_2 \times n_2} \times \cdots \times \mathbb{F}_q^{m_\ell \times n_\ell}$, we may give an alternative definition of generalized weights as

$$\begin{aligned} d_r^{Supp}(\mathcal{C}) = \min \Bigg\{ \sum_{i=1}^{\ell} \dim(\mathcal{S}_i) : \ & \mathcal{S}_i \in \mathcal{P}_{q,n_i}, 1 \leq i \leq \ell, \\ & \dim\left(\mathcal{C} \cap \left(\mathcal{V}_{\mathcal{S}_1} \times \cdots \times \mathcal{V}_{\mathcal{S}_\ell}\right)\right) \geq r \Bigg\}. \end{aligned} \tag{B.0.1}$$

for $r \in [\dim(\mathcal{C})]$. In the case where,for all $i \in [\ell]$, $n_i < m_i$ or $n_i = m_i = 1$, then by [Rav16a, Theorem 26]

$$d_r^{Supp}(\mathcal{C}) = d_r(\mathcal{C}),$$

for all linear codes $\mathcal{C} \subseteq \mathbb{F}_q^{m_1 \times n_1} \times \mathbb{F}_q^{m_2 \times n_2} \times \cdots \times \mathbb{F}_q^{m_\ell \times n_\ell}$ and all $r \in [\dim(\mathcal{C})]$. In particular, both coincide with the generalized Hamming weights if $m_i = n_i = 1$ for $i \in [\ell]$.

Consider now arbitrary values of m_i and n_i, for $i \in [\ell]$. The weights in (B.0.1) present an advantage and a disadvantage with respect to using anticodes instead of support spaces. Their disadvantage is that such weights are not always invariant by arbitrary linear sum-rank isometries (simply notice that support spaces are not necessarily again support spaces after transposition of matrices, as we are only considering row supports). On the other hand, their advantage is that they measure information leakage to a wire-tapper in scenarios such as multishot linear network coding [MPK19].

More concretely, consider a linear code $\mathcal{C} \subseteq \mathbb{F}_q^{m_1 \times n_1} \times \mathbb{F}_q^{m_2 \times n_2} \times \cdots \times \mathbb{F}_q^{m_\ell \times n_\ell}$. Choose a complementary vector space $\mathcal{M} \oplus \mathcal{C} = \mathbb{F}_q^{m_1 \times n_1} \times \mathbb{F}_q^{m_2 \times n_2} \times \cdots \times \mathbb{F}_q^{m_\ell \times n_\ell}$. We may see $\mathcal{M}$ as our space of messages [MPK19, Definition 3]. A (random) message $M \in \mathcal{M}$ is encoded by choosing $C \in \mathcal{C}$ uniformly at random in $\mathcal{C}$, and finally setting $D = M + C$, in order to hide M. If we set $D = (D_1, D_2, \ldots, D_\ell)$, then $D_i \in \mathbb{F}_q^{m_i \times n_i}$ is sent through an $\mathbb{F}_q$-linearly coded network with n_i outgoing links from the source node, for $i \in [\ell]$. This could be the scenario in multishot network coding without delays (or treating delays as erasures), or in singleshot network coding where we know that the network has at least ℓ disconnected components (after removing the source and sink nodes), see [MPK19]. Assume we have no further knowledge of the network topology or linear network code, but we know that an adversary wire-taps μ arbitrary links of the network. Then the information contained in

$$W = (D_1B_1, D_2B_2, \ldots, D_\ell B_\ell)$$

is leaked to the wire-tapper, for matrices $B_i \in \mathbb{F}_q^{n_i \times \mu_i}$, for $i \in [\ell]$, such that

$$\sum_{i=1}^{\ell} \mathrm{rk}(B_i) \leq \sum_{i=1}^{\ell} \mu_i = \mu.$$

Equality may be attained, as we do not know nor have control over the matrices B_i. By [MPK19, Lemma 1], the amount of information on M leaked to the wire-tapper, measured in symbols in $\mathbb{F}_q$, is given by the mutual information in base q

$$I_q(M; W) \leq \dim\left(C^\perp \cap (\mathcal{V}_{\mathcal{S}_1} \times \cdots \times \mathcal{V}_{\mathcal{S}_\ell})\right),$$

where $\mathcal{S}_i = \mathrm{rowsp}(B_i^t) \in \mathcal{P}_{q,n_i}$, for $i \in [\ell]$. Furthermore, equality holds if M is chosen uniformly at random in $\mathcal{M}$. Thus, in such a situation, $d_r^{Supp}(C^\perp)$ represents the minimum number of links that an adversary needs to wire-tap in order to obtain at least r units of information (number of bits multiplied by $\log_2(q)$) of the sent message.

We conclude with some remarks on the properties of the generalized weights $d_r^{Supp}(C)$ as above. First, if $m = m_1 = m_2 = \ldots = m_\ell$, $n = n_1 + n_2 + \cdots + n_\ell$, and $C \subseteq \mathbb{F}_{q^m}^n \cong \mathbb{F}_q^{m_1 \times n_1} \times \mathbb{F}_q^{m_2 \times n_2} \times \cdots \times \mathbb{F}_q^{m_\ell \times n_\ell}$ is an $\mathbb{F}_{q^m}$-linear code, then

$$d_{rm-t}^{Supp}(C) = d_r^{SR}(C),$$

for $t \in \{0, \ldots, m-1\}$ and $r \in [\dim_{\mathbb{F}_{q^m}}(C)]$, where $d_r^{SR}(C)$ denotes the rth generalized weight considered in [MP19, Definition 10]. This equality is easy to prove and recovers [MPM17, Theorem 7] when $\ell = 1$. Moreover, this is the analogue of Theorem 5.41 for the case $m = m_1 = m_2 = \ldots = m_\ell$. Notice that the assumption that $n_i < m_i$ is not needed in this setting.

Finally, if we assume that $m_1 \geq m_2 \geq \ldots \geq m_\ell$, then all of the properties stated throughout this manuscript for the generalized weights $d_r(C)$ hold mutatis mutandis for the generalized weights $d_r^{Supp}(C)$, with similar proofs and without assuming that $n_i \leq m_i$ for $i \in [\ell]$. In order to prove this claim, it suffices to prove the analogue of Theorem 2.15, Proposition 5.33, and Lemma 5.35 for the generalized weights $d_r^{Supp}(C)$. In fact, the other properties follow from those results. We start with an analogue of Proposition 5.33.

Proposition B.1. Let $0 \neq C \subseteq \mathcal{D} \subseteq \mathbb{M}$, then:

1. $d_1^{Supp}(C) = d(C)$,
2. $d_r^{Supp}(C) \leq d_s^{Supp}(C)$ for $1 \leq r \leq s \leq \dim(C)$,
3. $d_r^{Supp}(C) \geq d_r^{Supp}(\mathcal{D})$ for $r \in [\dim(C)]$,
4. $d_{\dim(C)}^{Supp}(C) \leq n_1 + \cdots + n_\ell$,

5. $d^{Supp}_{r+n_1m_1+\cdots+n_{j-1}m_{j-1}+\delta m_j}(C) \geq d^{Supp}_r(C) + n_1 + \cdots + n_{j-1} + \delta$
for $j \in [\ell]$, $r \in [\dim(C) - (n_1m_1 + \cdots + n_{j-1}m_{j-1} + \delta m_j)]$, and $0 \leq \delta \leq n_j - 1$.

Proof. 1. Let $C = (C_1, \ldots, C_\ell) \in \mathcal{C}$ be an element of minimum sum-rank. Let $\mathcal{S}_i = \text{rowsp}(C_i)$ be a subspace such that $C_i \in \mathcal{V}_{\mathcal{S}_i}$ and $\dim(\mathcal{V}_{\mathcal{S}_i}) = m_i \text{rk}(C_i)$, for $i \in [\ell]$. Then $\mathcal{C} \cap (\mathcal{V}_{\mathcal{S}_1} \times \cdots \times \mathcal{V}_{\mathcal{S}_\ell}) \neq 0$, hence $d^{Supp}_1(\mathcal{C}) \leq d(\mathcal{C})$. To prove that they are equal, we observe that if $\mathcal{S}'_i \in \mathcal{P}_{q,n_i}$, for $i \in [\ell]$, are such that $\sum_{i=1}^{\ell} \dim(\mathcal{S}'_i) < d(\mathcal{C})$, then $\mathcal{C} \cap (\mathcal{V}_{\mathcal{S}'_1} \times \cdots \times \mathcal{V}_{\mathcal{S}'_\ell}) = 0$.

2., 3., and 4. follow directly from the definition.

5. Let $s = r + n_1m_1 + \cdots + n_{j-1}m_{j-1} + \delta m_j$. Let $\mathcal{S}_i \in \mathcal{P}_{q,n_i}$, for $i \in [\ell]$, such that $\dim(\mathcal{C} \cap (\mathcal{V}_{\mathcal{S}_1} \times \cdots \times \mathcal{V}_{\mathcal{S}_\ell})) \geq s$ and $d^{Supp}_s(\mathcal{C}) = \sum_{i=1}^{\ell} \dim(\mathcal{S}_i)$. For $i \in [\ell]$, let $u_i = \dim(\mathcal{S}_i)$. Since

$$\sum_{i=1}^{\ell} m_i u_i = \dim(\mathcal{V}_{\mathcal{S}_1} \times \cdots \times \mathcal{V}_{\mathcal{S}_\ell}) \geq \dim(\mathcal{C} \cap (\mathcal{V}_{\mathcal{S}_1} \times \cdots \times \mathcal{V}_{\mathcal{S}_\ell}))$$
$$\geq s > n_1m_1 + \cdots + n_{j-1}m_{j-1} + \delta m_j$$

and $m_1 \geq \cdots \geq m_\ell$, then $d^{Supp}_s(\mathcal{C}) = u_1 + \ldots + u_\ell > n_1 + \cdots + n_{j-1} + \delta$ by Lemma 5.32. Let $v_1, \ldots, v_\ell$ be such that $n_1 + \cdots + n_{j-1} + \delta = v_1 + \cdots + v_\ell$ and $v_i \leq u_i$ for $i \in [\ell]$. We have that $n_1m_1 + \cdots + n_{j-1}m_{j-1} + \delta m_j \geq v_1m_1 + \cdots + v_\ell m_\ell$, since $m_1 \geq \cdots \geq m_\ell$. For all $i \in [\ell]$ there exist vector spaces $\mathcal{S}'_i \subseteq \mathcal{S}_i$ of $\dim(\mathcal{S}'_i) = u_i - v_i$. Then

$$\dim(\mathcal{C} \cap (\mathcal{V}_{\mathcal{S}'_1} \times \cdots \times \mathcal{V}_{\mathcal{S}'_\ell})) \geq s - (v_1m_1 + \cdots + v_\ell m_\ell) \geq s - (n_1m_1 + \cdots + n_{j-1}m_{j-1} + \delta m_j) = r,$$

hence

$$d_r(\mathcal{C}) \leq \sum_{i=1}^{\ell} (u_i - v_i) = d_s(\mathcal{C}) - (n_1 + \cdots + n_{j-1} + \delta).$$ □

The Singleton-type bound from Corollary 5.34 is a direct consequence of Proposition 5.33. Notice however that it also follows directly from Proposition B.1 for the generalized weights $d^{Supp}_r(\mathcal{C})$. The next theorem is an analogue of Theorem 2.15 for the weights $d^{Supp}_r(\mathcal{C})$.

Theorem B.2. Let $\mathcal{C} \subseteq \mathbb{M}$ be a code and let $r \in [\dim(\mathcal{C})]$. Let $j \in [\ell]$ and $0 \leq \delta \leq n_j - 1$ be such that

$$d^{Supp}_r(\mathcal{C}) - 1 \geq \sum_{i=1}^{j-1} n_i + \delta.$$

Then

$$\dim(\mathcal{C}) \leq \sum_{i=j}^{\ell} m_i n_i - m_j \delta + r - 1. \tag{B.0.2}$$

Proof. Let $\mathcal{S}_i = \mathbb{F}_q^{n_i}$ for $i \in [j-1]$, let $\mathcal{S}_j \in \mathcal{P}_{q,n_j}$ be of dimension δ, and let $\mathcal{S}_i = 0$ for $j+1 \leq i \leq \ell$. Then

$$\dim(\mathcal{C} \cap (\mathcal{V}_{\mathcal{S}_1} \times \cdots \times \mathcal{V}_{\mathcal{S}_\ell})) \leq r - 1.$$

Therefore, we obtain that

$$\dim(C)+\sum_{i=1}^{j-1} m_i n_i + m_j\delta - r + 1 \leq \dim(C) + \dim(\mathcal{V}_{\mathcal{S}_1}\times\cdots\times\mathcal{V}_{\mathcal{S}_\ell})+$$

$$-\dim(C\cap(\mathcal{V}_{\mathcal{S}_1}\times\cdots\times\mathcal{V}_{\mathcal{S}_\ell})) = \dim(C+(\mathcal{V}_{\mathcal{S}_1}\times\cdots\times\mathcal{V}_{\mathcal{S}_\ell})) \leq \sum_{i=1}^{\ell} m_i n_i,$$

and this concludes the proof. □

A version of the Singleton Bound for the generalized weights $d_r^{Supp}(C)$ follows directly from Theorem B.2 and yields the same inequalities as in Corollary 6.3. Finally, we establish the analogue of Lemma 5.35 for the generalized weights $d_r^{Supp}(C)$.

Lemma B.3. Let $C \subseteq \mathbb{M}$ be a code and let $k \in [\ell]$, $r + m_k \in [\dim(C)]$. If

$$d_{r+m_k}^{Supp}(C) > \sum_{i=1}^{k-1} n_i$$

then

$$d_{r+m_k}^{Supp}(C) \geq d_r^{Supp}(C) + 1.$$

Proof. For $i \in [\ell]$, let $\mathcal{S}_i \in \mathcal{P}_{q,n_i}$ be such that $d_{r+m_k}^{Supp}(C) = \sum_{i=1}^{\ell}\dim(\mathcal{S}_i)$ and $\dim(C\cap(\mathcal{V}_{\mathcal{S}_1}\times\cdots\times\mathcal{V}_{\mathcal{S}_\ell})) \geq r + m_k$. We claim that there exists $k \leq j \leq \ell$ such that $\mathcal{S}_j \neq 0$. In fact, if this were not the case, then

$$\sum_{i=1}^{k-1} n_i \geq \sum_{i=1}^{\ell}\dim(\mathcal{S}_i) = d_{r+m_k}^{Supp}(C).$$

For $i \in [\ell]$, let $\mathcal{S}'_i \in \mathcal{P}_{q,n_i}$ be such that $\mathcal{S}'_i = \mathcal{S}_i$ if $i \neq j$, and $\mathcal{S}'_j \subseteq \mathcal{S}_j$ with $\dim(\mathcal{S}'_j) = \dim(\mathcal{S}_j) - 1$. One has

$$\dim(C\cap(\mathcal{V}_{\mathcal{S}'_1}\times\cdots\times\mathcal{V}_{\mathcal{S}'_\ell})) \geq \dim(C\cap(\mathcal{V}_{\mathcal{S}_1}\times\cdots\times\mathcal{V}_{\mathcal{S}_\ell})) - m_j \geq r + m_k - m_j \geq r,$$

hence

$$d_r^{Supp}(C) \leq \sum_{i=1}^{\ell}\dim(\mathcal{S}'_i) = d_{r+m_k}^{Supp}(C) - 1.$$ □

We conclude by noting that Wei's duality (Theorem 5.36) also holds for the generalized weights $d_r^{Supp}(C)$ with the same proof.

C. A mathematical reduction to modules over a polynomial ring

Denote by $\mathbb{F}_q[x]$ the polynomial ring, by $\mathbb{F}_q(x)$ the field of rational functions, and by $\mathbb{F}_q((x))$ the field of Laurent series, all with coefficients in $\mathbb{F}_q$. In other words

$$\mathbb{F}_q[x] = \left\{ \sum_{i=0}^{d} a_i x^i : d \in \mathbb{Z}, d \geq 0, a_i \in \mathbb{F}_q \right\},$$

$$\mathbb{F}_q(x) = \left\{ p(x)/q(x) : p(x), q(x) \in \mathbb{F}_q[x], q(x) \neq 0 \right\},$$

and

$$\mathbb{F}_q((x)) = \left\{ \sum_{i \geq e} a_i x^i : e \in \mathbb{Z}, a_i \in \mathbb{F}_q \right\}.$$

Notice that every nonconstant polynomial $q(x) \in \mathbb{F}_q[x] \setminus \mathbb{F}_q$ has an inverse $1/q(x) \in \mathbb{F}_q((x)) \setminus \mathbb{F}_q[x]$. Therefore, one has the chain of inclusions

$$\mathbb{F}_q[x] \subseteq \mathbb{F}_q(x) \subseteq \mathbb{F}_q((x)).$$

In Section 2.2, we define a convolutional code as an $\mathbb{F}_q[x]$-submodule of $\mathbb{F}_q[x]^n$. In the literature, however, convolutional codes are typically defined as $\mathbb{F}_q((x))$-subspaces of $\mathbb{F}_q((x))^n$ or $\mathbb{F}_q(x)$-subspaces of $\mathbb{F}_q(x)^n$ generated by elements of $\mathbb{F}_q[x]^n$. In this appendix we argue that, when studying generalized weights, it is natural to only consider $\mathbb{F}_q[x]$-submodule of $\mathbb{F}_q[x]^n$ and that our definition of generalized weights coincides with the natural definition for $\mathbb{F}_q((x))$-subspaces of $\mathbb{F}_q((x))^n$ and $\mathbb{F}_q(x)$-subspaces of $\mathbb{F}_q(x)^n$.

It is natural to define the weight of an element $p(x) \in \mathbb{F}_q((x))$ as the number of nonzero coefficients in the power-series expansion of $p(x)$ and the weight of a vector $c(x) \in \mathbb{F}_q((x))^n$ as the sum of the weights of its coordinates. It is clear that $c(x)$ has finite weight if and only if $c(x) \in \mathbb{F}_q[x]^n$.

In Definition 8.1 we define the weight of an $\mathbb{F}_q[x]$-submodule $C \subseteq \mathbb{F}_q[x]^n$ of rank $k \leq n$ as

$$\mathrm{wt}(C) = \min\{|\mathrm{supp}(\langle c_1(x), \ldots, c_k(x)\rangle_{\mathbb{F}_q})| : C = \langle c_1(x), \ldots, c_k(x)\rangle_{\mathbb{F}_q[x]}\}.$$

Similarly, if $K = \mathbb{F}_q(x)$ or $K = \mathbb{F}_q((x))$, the weight of a K-linear subspace $\mathbb{C} \subseteq K^n$ can be defined as

$$\mathrm{wt}(\mathbb{C}) = \min\{|\mathrm{supp}(\langle c_1(x), \ldots, c_k(x)\rangle_{\mathbb{F}_q})| : C = \langle c_1(x), \ldots, c_k(x)\rangle_{K}\}.$$

Notice that $\mathbb{C}$ has finite weight if and only if it has a system of generators which belong to $\mathbb{F}_q[x]^n$.

Similarly to the definition of generalized weights in the Hamming metric, one may define the generalized weights of a $\mathbb{C} \subseteq K^n$ of $\dim_K(\mathbb{C}) = k \leq n$ as

$$d_r(\mathbb{C}) = \min\{\mathrm{wt}(\mathbb{D}) \mid \mathbb{D} \subseteq \mathbb{C} \text{ is a } K\text{-linear subspace of } \dim_K(\mathbb{D}) \geq r\}.$$

If $\mathbb{C}$ has a basis which consists of elements of $\mathbb{F}_q[x]^n$, then $d_r(\mathbb{C}) < \infty$ for $1 \leq r \leq k$.

In Definition 8.3 we define the generalized weights of an $\mathbb{F}_q[x]$-submodule $C \subseteq \mathbb{F}_q[x]^n$ of rank $k \leq n$ as

$$d_r(C) = \min\{\mathrm{wt}(\mathcal{D}) \mid \mathcal{D} \subseteq C \text{ is an } \mathbb{F}_q[x]\text{-submodule of } \mathrm{rk}(\mathcal{D}) \geq r\}.$$

Let $\mathbb{C} \subseteq K^n$ of $\dim_K(\mathbb{C}) = k \leq n$ be generated by elements of $\mathbb{F}_q[x]^n$. We have argued that the K-linear subspaces $\mathbb{D} \subseteq \mathbb{C}$ of finite weight are exactly those generated by elements of $\mathbb{F}_q[x]^n$. Let $C = \mathbb{C} \cap \mathbb{F}_q[x]^n$. It is easy to check that $\mathrm{rk}(C) \geq k$. We now prove that that $\mathrm{rk}(C) = k$ and

$$d_r(C) = d_r(\mathbb{C})$$

for $1 \leq r \leq k$. This shows that defining convolutional codes as $\mathbb{F}_q[x]$-submodules of $\mathbb{F}_q[x]^n$ is not restrictive while dealing with generalized weights. In fact, given $c_1(x), \ldots, c_k(x) \in \mathbb{F}_q[x]^n$, the codes $\mathbb{C} = \langle c_1(x), \ldots, c_k(x)\rangle_K$ and $C = \mathbb{C} \cap \mathbb{F}_q[x]^n$ have the same generalized weights. Notice moreover that $\mathbb{C} \cap \mathbb{F}_q[x]^n$ is non-catastrophic by Proposition 2.14.

We conclude the appendix by proving the results claimed above.

Lemma C.1. Let $K = \mathbb{F}_q(x)$ or $K = \mathbb{F}_q((x))$. The following hold:

1. Let C be an $\mathbb{F}_q[x]$-submodule of $\mathbb{F}_q[x]^n$ of rank k. Then, $\dim_K\langle C\rangle_K = k$.
2. Let $\mathbb{C}$ be a K-linear subspace of K^n with basis $c_1(x), \ldots, c_k(x) \in \mathbb{F}_q[x]^n$. Then the module $\langle c_1(x), \ldots, c_k(x)\rangle_{\mathbb{F}_q[x]}$ has rank k.
3. Let $\mathbb{C}$ be a K-linear subspace of K^n of dimension k generated by elements in $\mathbb{F}_q[x]^n$. Then $\mathbb{C} \cap \mathbb{F}_q[x]^n$ has rank k.

Proof. 1. Let $c_1(x), \ldots, c_k(x)$ be a standard basis of C, i.e., a basis of C such that the matrix $G(x)$ with rows $c_1(x), \ldots, c_k(x)$ is in reduced row-echelon form over $\mathbb{F}_q[x]$. See [BN96, Section 3] for the definition and a discussion on standard bases, in particular [BN96, Theorem 3.3] for an algorithm to compute them. Since $G(x)$ is a generator matrix for $\mathbb{C}$ as a K-vector space and it is in reduced row-echelon form over $\mathbb{F}_q[x]$, then $\dim_K(\langle C\rangle_K) = \mathrm{rk}(G(x)) = k$.

2. Let $\mathbb{C}$ be a K-linear subspace of K^n with basis $c_1(x), \ldots, c_k(x) \in \mathbb{F}_q[x]^n$. Since $\mathbb{C} = \langle c_1(x), \ldots, c_k(x)\rangle_K$, by 1.

$$\mathrm{rk}(\langle c_1(x), \ldots, c_k(x)\rangle_{\mathbb{F}_q[x]}) = \dim_K(\langle\langle c_1(x), \ldots, c_k(x)\rangle_{\mathbb{F}_q[x]}\rangle_K) = \dim_K(\mathbb{C}) = k.$$

3. follows from 1. and 2., since $\mathbb{C} \cap \mathbb{F}_q[x]^n \supseteq \langle c_1(x), \ldots, c_k(x)\rangle_{\mathbb{F}_q[x]}$ and both $\mathbb{C} \cap \mathbb{F}_q[x]^n$ and $\langle c_1(x), \ldots, c_k(x)\rangle_{\mathbb{F}_q[x]}$ generate $\mathbb{C}$ over K. □

Proposition C.2. Let $\mathbb{C}$ be a K-linear subspace of K^n of dimension k generated by elements of $\mathbb{F}_q[x]^n$. Then

$$d_r(\mathbb{C}) = d_r(\mathbb{C} \cap \mathbb{F}_q[x]^n)$$

for $1 \leq r \leq k$.

Proof. Let $C = \mathbb{C} \cap \mathbb{F}_q[x]^n$. By Lemma C.1.3 $\dim_K(\mathbb{C}) = \mathrm{rk}(C)$, so C and $\mathbb{C}$ have the same number of generalized weights. Fix $1 \leq r \leq k$ and let $\mathcal{D} = \langle c_1(x), \ldots, c_r(x)\rangle_{\mathbb{F}_q[x]} \subseteq C$ be an $\mathbb{F}_q[x]$-submodule of $\mathrm{rk}(\mathcal{D}) = r$ such that $|\mathrm{supp}(c_1(x), \ldots, c_r(x))| = d_r(C)$. By Lemma C.1.1 we have that $\dim(\langle \mathcal{D} \rangle_K) = r$. Moreover, by the definition of weight $\mathrm{wt}(\langle \mathcal{D} \rangle_K) \leq d_r(C)$. Since $\langle \mathcal{D} \rangle_K$ is an r-dimensional K-linear subspace of $\mathbb{C}$ we conclude that $d_r(\mathbb{C}) \leq d_r(C)$.

Let $\mathbb{D}$ be a K-linear subspace of $\mathbb{C}$ of dimension r that realizes $d_r(\mathbb{C})$, i.e., one can find $c_1(x), \ldots, c_r(x) \in \mathbb{F}_q[x]^n$ such that $\langle c_1(x), \ldots, c_r(x) \rangle_K = \mathbb{D}$ and $|\mathrm{supp}(c_1(x), \ldots, c_r(x))| = d_r(\mathbb{C})$. By Lemma C.1.2 we have that $\mathrm{rk}(\langle c_1(x), \ldots, c_r(x) \rangle_{\mathbb{F}_q[x]}) = r$. Since $\langle c_1(x), \ldots, c_r(x) \rangle_{\mathbb{F}_q[x]} \subseteq C$, then

$$d_r(C) \leq \mathrm{wt}(\langle c_1(x), \ldots, c_r(x) \rangle_{\mathbb{F}_q[x]}) \leq \mathrm{wt}(\mathbb{D}) = d_r(\mathbb{C}),$$

which concludes the proof. □

D. Infinite latroids and convolutional codes

The aim of this appendix is to explain how to associate a combinatorial object with a convolutional code. As we have seen at the beginning of Chapter 3, it is possible to associate with a linear block code a matroid that captures most of its properties. If we regard a convolutional code C as an infinite dimensional linear block code, it is natural to look at finitary matroids, that are defined as follows.

Definition D.1. A finitary matroid is an ordered pair $(E, \mathcal{I})$ consisting of a set E and a collection $\mathcal{I}$ of subsets of E having the following four properties

1. $\emptyset \in \mathcal{I}$,
2. if $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$,
3. if $I_1, I_2 \in \mathcal{I}$ and $|I_2| < |I_1|$, then there exist an element $e \in I_1 \setminus I_2$ such that $I_2 \cup e \in \mathcal{I}$,
4. an infinite set is in $\mathcal{I}$ as soon as all its finite subsets are in $\mathcal{I}$.

The elements of $\mathcal{I}$ are the independent sets of the matroid. A subset of E that is not independent is called dependent. A minimal dependent set is called a circuit.

Remark D.2. If in the previous definition we consider a finite set E, we find the classical definition of a matroid. Indeed, the first three items are the axioms that define a matroid, see [Oxl11, Section 1.1], and when E is finite we can simply forget about the fourth item. In particular the last item implies that all the circuits of a finitary matroid are finite, but this is always the case when E is finite. Therefore, we can also define finitary matroids by the usual circuit axioms, together with the additional axiom that all circuits need to be finite.

Finitary matroids model infinite dimensional vector spaces, similarly to how usual matroids models finite dimensional ones. Let C be an (n, k, δ) convolutional code. Since C is an infinite dimensional $\mathbb{F}_q$-linear space, it is clear that $\mathscr{C} = \{\text{supp}(c) : c \text{ is a minimal codeword of } C\}$ is the set of circuits of a finitary matroid with ground set $E = [n] \times \mathbb{N}_0$, that we will denote by M_C. The rank function of M_C is given by $\text{rk}(X) = |X| - \dim_{\mathbb{F}_q}(C_X)$, where C_X is the set of codewords in C supported on X. In order to be able to recover some properties of the code from the associated finitary matroid, we need to remember the structure of its ground set.

Definition D.3. A convolutional matroid M is a finitary matroid with ground set of the form $[n] \times \mathbb{N}_0$.

The fact that the ground set of a convolutional matroid is of this form, it allows us to define the degree $\deg(C)$ of a circuit C as the smallest integer m for which C is contained in $E_m = [n] \times \{0, \dots, m\}$. In the next part of this appendix, we highlight some properties of M_C.

Lemma D.4. Let C be an (n,k,δ) convolutional code, and let $M_C=(E,\mathscr{C})$ be the convolutional matroid defined above. Then, for every $m\in\mathbb{N}_0$, the set $\{C\in\mathscr{C} : \mathrm{supp}(C)\subseteq E_m\}$ is the set of circuits of a matroid M_m supported on E_m.

Proof. The fact that M_m is a matroid is a consequence of the fact that C restricted to E_m is an $\mathbb{F}_q$-linear space and therefore the minimal codewords supported on E_m are the circuits of a matroid. □

Lemma D.5. Let $\tau : E\to E$ be the map given by $(i,j)\mapsto(i,j+1)$. Then, if C is noncatastrophic, $\tau(C)\in\mathscr{C}$ for all $C\in\mathscr{C}$.

Proof. This is a direct consequence of Proposition 2.14. □

A natural question at this point is which parameters of the code can we recover from the associated convolutional matroid. For instance it is clear that if we know the ground set E, we also know n. In the next two propositions, we show how one can recover the dimension and the degree of a convolutional code.

Proposition D.6. Let C be a convolutional code and let M be the convolutional matroid associated to C. We have that

$$\mathrm{rk}(C)=\lim_{m\to\infty}\frac{n(m+1)-\mathrm{rk}(M_m)}{m+1}.$$

Proof. Let C_m be the set of all the codewords in C with degree at most m, then $\dim(C_m)=n(m+1)-\mathrm{rk}(M_m)$. Let δ_1 be the memory of the code. For $m\geq\delta_1$, we have that $(m-\delta_1)k\leq\dim(C_m)\leq(m+1)k$. We conclude by dividing by $m+1$ and by taking the limit for m that goes to infinity. □

Proposition D.7. Let C be a noncatastrophic (n,k,δ) convolutional code and let M_C be the convolutional matroid associated to C. Then $\delta_1,\dots,\delta_k$ can be determined from M.

Proof. Let $C_1\dots,C_\ell$ all the circuits of the matroid M associated to $C[0]$. For each $i\in[\ell]$ we can select a circuit $\bar{C}_i$ in M_C such that $\mathrm{supp}(\bar{C}_i)\cap E_0=\mathrm{supp}(C_i)$ and with minimal degree among all the circuits with this property. There may be more than one circuit that satisfies this request, we just randomly choose one. We now go through to all the possible subsets $\{\bar{C}_{i_1},\dots\bar{C}_{i_k}\}$ of size k of $\{\bar{C}_1,\dots,\bar{C}_\ell\}$ such that $\mathrm{rk}(\mathrm{supp}(C_{i_1}),\dots,\mathrm{supp}(C_{i_k}))=k$. Among these sets we select one of those that minimize $\deg(C_{i_1})+\cdots+\deg(C_{i_k})$. Then, up to a permutation of the indices, we obtain $\delta_j=\deg(C_{i_j})$. □

The convolutional matroid associated to a code C, because of how it was defined, captures very well the $\mathbb{F}_q$-linear structure of C. So, it is not surprising the the generalized Hamming weights defined in Definition 8.10 are determined by M_C. In fact one has

$$d'_r(C)=\min\{|X| : \mathrm{rk}(C_X)=r\}.$$

Concerning the generalized weights defined in 8.3, it is not clear whether they are determined by M_C or not. In order to capture the module structure of a convolutional code, one could try to associate a latroid instead of a matroid as we already did in Section 3.4. Unfortunately, in the case of convolutional codes it is quite natural to work with lattices, for instance the set of finite subsets of E, that are not complete. This led us to give the following definition that extends the notion of latroid, see Definition 3.20.

Definition D.8. Let A be an ordered abelian group, and let $\mathcal{L}$ be a lattice with an unique minimal element $0_{\mathcal{L}}$. An A-semilatroid with rank function $\rho : \mathcal{L} \to A$ under length function $\|\cdot\| : \mathcal{L} \to A$ on the lattice $\mathcal{L}$, is a triple $(\rho, \|\cdot\|, \mathcal{L})$ such that:

1. $\rho(0_{\mathcal{L}}) = \|0_{\mathcal{L}}\| = 0_A$.
2. $\|\cdot\|$ is strictly increasing, that is, $\|L_1\| < \|L_2\|$ for all $L_1, L_2 \in \mathcal{L}$ with $L_1 < L_2$.
3. $\|\cdot\|$ is modular, that is, $\|L_1\| + \|L_2\| = \|L_1 \vee L_2\| + \|L_1 \wedge L_2\|$ for all $L_1, L_2 \in \mathcal{L}$.
4. ρ is bounded increasing, that is, $0 \leq \rho(L_2) - \rho(L_1) \leq \|L_2\| - \|L_1\|$ for all $L_1, L_2 \in \mathcal{L}$ with $L_1 < L_2$.
5. For every $M, L_1, L_2 \in L$ and $0 < a_1, a_2 \in A$ such that $\rho(L_i) + a_i = \|L_i\|$, $M < L_1 \vee L_2$ and $\|M\| > \|L_1 \vee L_2\| - (a_1 + a_2)$, we have that $\rho(M) + \rho(L_1 \wedge L_2) < \|M\| + \|L_1 \wedge L_2\|$.

It is clear that the first four items of the previous proposition coincide with the first four item of Definition 3.20. Therefore, in order to prove that a latroid is also a semi latroid, it is sufficient to verify that a bounded increasing submodular function verifies item 5. We show this in the next proposition.

Lemma D.9. A $(\rho, \|\cdot\|, \mathcal{L})$ A-latroid is an A-semilatroid.

Proof. Let $M, L_1, L_2 \in \mathcal{L}$ and $0 < a_1, a_2 \in A$ such that $\rho(L_i) + a_i = \|L_i\|$, $M < L_1 \vee L_2$ and $\|M\| > \|L_1 \vee L_2\| - (a_1 + a_2)$. Since ρ is submodular and bounded increasing we have that

$$\rho(M) + \rho(L_1 \wedge L_2) \leq \rho(L_1) + \rho(L_2) = \|L_1\| + \|L_2\| - a_1 - a_2 < \|M\| + \|L_1 \wedge L_2\|. \qquad \square$$

We already observed in Remark 3.21 that from a latroid we can obtain a matroid. This is also true for a semilatroid. The following result coincides with [Oxl11, Proposition 11.1.1] (see also [ER66]) when ρ is submodular and E is finite.

Proposition D.10. Let E be a set and let $R = (\rho, |\cdot|, \mathcal{P}_{\text{fin}}(E))$ be a $\mathbb{Z}$-semilatroid. Let $\mathscr{C}(R) = \{C \in \mathcal{P}_{\text{fin}}(E) :$ is minimal such that $\rho(C) < |C|\}$. Then $\mathscr{C}(R)$ is the collection of circuits of a finitary matroid M on E.

Proof. It is clear that $\emptyset \notin \mathscr{C}(R)$ and that for all $C_1, C_2 \in \mathscr{C}(R)$ such that $C_1 \subseteq C_2$ we have that $C_1 = C_2$. Let $C_1, C_2 \in \mathscr{C}(R)$ such that $C_1 \neq C_2$ and $C_1 \cap C_2 \neq \emptyset$. Fix $e \in C_1 \cap C_2$. Then, $|(C_1 \cup C_2) \setminus e| = |C_1 \cup C_2| - 1 > |C_1 \cup C_2| - |C_1| + \rho(C_1) - |C_2| + \rho(C_2)$. Since R is a $\mathbb{Z}$-semilatroid, we have that $\rho((C_1 \cup C_2) \setminus e) + \rho(C_1 \cap C_2) < |(C_1 \cup C_2) \setminus e| + |C_1 \cap C_2|$. Therefore, $\rho((C_1 \cup C_2) \setminus e) < |(C_1 \cup C_2) \setminus e| +$ and so there exists $C_3 \subseteq (C_1 \cup C_2) \setminus e$ such that $C_3 \in \mathscr{C}(R)$. $\square$

Remark D.11. Notice that ρ may not be the rank function of the finitary matroid $M = (E, \mathscr{C}(R))$. However, if X is a finite independent set of M, then $\text{rk}_M(X) = \rho(M)$.

We now show how to associate a semilatroid to a convolutional code. Let C be an (n, k, δ) convolutional code. We define $\rho_C : \mathcal{P}_{\text{fin}}(E) \to \mathbb{Z}$ as

$$\rho_C(X) = |X| - \text{rk}(\langle C_X \rangle_{\mathbb{F}_q[x]})$$

for all $X \in \mathcal{P}_{\text{fin}}(E)$.

Lemma D.12. For every $X \in \mathcal{P}_{\text{fin}}(E)$, $|X| - \dim(C_X)$ is non-decreasing.

Proof. If we remove one element from X, $|X|$ decreases by 1, while $\dim(C_X)$ decreases at most by 1. □

Lemma D.13. The triple $(\rho_C, |\cdot|, \mathcal{P}_{\text{fin}}(E))$ is a $\mathbb{Z}$-semilatroid.

Proof. For every $X_1 \subseteq X_2 \in \mathcal{P}_{\text{fin}}(E)$, we have that $C_{X_1} \subseteq C_{X_2}$. Therefore, one obtains

$$0 \leq \text{rk}(\langle C_{X_2}\rangle_{\mathbb{F}_q[x]}) - \text{rk}(\langle C_{X_1}\rangle_{\mathbb{F}_q[x]}) = |X_2| - |X_1| - \rho_C(X_2) + \rho_C(X_1).$$

Moreover, since $\text{rk}(\langle C_{X_2}\rangle_{\mathbb{F}_q[x]}) - \text{rk}(\langle C_{X_1}\rangle_{\mathbb{F}_q[x]}) \leq |X_2| - |X_1|$, we have that $0 \leq \rho_C(X_2) - \rho_C(X_1) \leq |X_2| - |X_1|$. So, ρ_C is bounded increasing.

Notice that $\rho(X) < |X|$ if and only if $\dim_{\mathbb{F}_q}(C_X) > 0$ for each $X \in \mathcal{P}_{\text{fin}}(E)$. Let $X_1, X_2, X_3 \in \mathcal{P}_{\text{fin}}(E)$ and $a_1, a_2 \in \mathbb{N}$ such that $\text{rk}(\langle C_{X_1}\rangle_{\mathbb{F}_q[x]}) = a_1$, $\text{rk}(\langle C_{X_2}\rangle_{\mathbb{F}_q[x]}) = a_2$, $X_3 \subseteq X_1 \cup X_2$ and $|X_3| > |X_1 \cup X_2| - a_1 - a_2$. We have that $\dim(C_{X_1}) \geq a_1$ and $\dim(C_{X_2}) \geq a_2$. If $\dim(C_{X_1 \cap X_2}) > 0$, we obtain $\rho_C(X_1 \cap X_2) < |X_1 \cap X_2|$. If $\dim(C_{X_1 \cap X_2}) = 0$, then $\dim(C_{X_1 \cup X_2}) \geq a_1 + a_2$. By Lemma D.12, since $|X_3| > |X_1 \cup X_2| - a_1 - a_2$, we have $\dim(C_{X_3}) > 0$. In both cases we conclude that $\rho(X_3) + \rho(X_1 \cap X_2) < |X_3| + |X_1 \cap X_2|$. □

Let C be a convolutional code and let $S_C = (\rho_C, |\cdot|, \mathcal{P}_{\text{fin}}(E))$ the associated $\mathbb{Z}$-semilatroid. We are interested in understanding which parameters of the code we can recover from S_C. Firstly, it is simple to express the generalized weights of Definition 8.3 as a function of S_C

$$d_r(C) = \min\{|X| : |X| - \rho(C_X) = r\}.$$

Secondly, notice that thanks to Proposition D.10 we can associate to C the matroid $M(S_C)$. Since $\dim(C_X) > 0$ if and only if $\text{rk}(\langle C_X\rangle) > 0$, we have that the circuits of $M(S_C)$ are exactly the circuits of the convolutional matroid M_C that we defined in the beginning of this appendix. This implies that $M_C = M(S_C)$. Therefore, we immediately obtain that every parameter of the code that is determined by M_C is also determined by S_C. It is not clear if the converse is true.

Question D.14. Let C_1 and C_2 be two convolutional codes such that $M_{C_{C_1}} = M_{C_2}$. Do we have that $S_{C_1} = S_{C_2}$?

A postive answer to the previous question would in particular tell us that the generalized weights of the convolutional codes are also determined by the convolutional matroid.

List of Symbols

$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_0$	the set of natural numbers with zero
$\mathbb{Z}$	the set of integers
$\mathcal{P}(E)$	the power set of a set E
$\mathcal{P}_{\text{fin}}(E)$	the set of all finite sets of E
$[n]$	the set $\{1,\ldots,n\}$ for $n \in \mathbb{N}$
q	a power of a prime number
$\mathbb{F}_q$	the field of cardinality q
$\mathbb{F}_q[x]$	the ring of univariate polynomials over $\mathbb{F}_q$
$\mathbb{F}_q^n$	a vector space of dimension n over $\mathbb{F}_q$
$\mathbb{S}$	$\mathbb{F}_q^{n_1} \times \cdots \times \mathbb{F}_q^{n_\ell}$
$\mathbb{F}_q^{m \times n}$	the space of $m \times n$ matrices over $\mathbb{F}_q$
$\mathbb{M}$	$\mathbb{F}_q^{m_1 \times n_1} \times \cdots \times \mathbb{F}_q^{m_\ell \times n_\ell}$
R	a ring
$R[\mathbf{x}]$	the ring of multivariate polynomials over R
C	a linear code
$C^\perp$	the dual of a linear code
C_A	the subcode of C supported on A
$C[0]$	the evaluation of a convolutional code C at 0
C^{rev}	the reverse of a linear code
rk	the rank of a matrix or of a module
srk	the sum-rank weight
wt	a weight
wt_H	the Hamming weight
maxsrk	the maximum sum-rank weight
d_{H}	the hamming distance
$d_{\min}$	the minimum distance
d_{free}	the free distance
d_j^c	the j-th column distance
d_r	the r-th generalized weight
$d_r(C_1, C_2)$	the r-th generalized relative weight
g_r	the r-th greedy weight
ρ	the covering radius or a rank function of a latroid
G(x)	a generator matrix
tr	the trace of a matrix
Tr	the trace of a vector of matrices
supp	a support function
rowsp	the rowspace of a matrix
colsp	the columnspace of a matrix
$\mu(C)$	the least cardinality of a system of generators of C

$M(C)$	$\mu(C_1) + \cdots + \mu(C_\ell)$ where $C = C_1 \times \cdots \times C_\ell$
$W_C(x, y)$	the homogeneous weight enumerator of C
$W_C(\mathbf{x}, \mathbf{y})$	the complete weight enumerator of C
$W_C^{(r)}(x, y)$	the r-th generalized weight enumerator of C
$T(\rho, x, y)$	the Tutte polynomial
$T(\rho, \lVert\cdot\rVert, \mathcal{L}, \mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v})$	the weighted Tutte-Whitney generating function
$\lVert\cdot\rVert$	a length function
$\lvert\cdot\rvert$	the cardinality of a set or the 1-norm of a vector
$\mathcal{L}$	a lattice
$\mathcal{R}^n$	the lattice of rectangular submodules of R^n
lt	the length of a module
I_C	the ideal associated with C
M_C	the matroid associated with C
$E_{i,j}$	an elementary matrix
$\mathbb{A}(\mu)$	the set of optimal anticodes with maximum sum-rank equal to μ
$\mathrm{RS}(n, k)$	a Reed-Solomon code with dimension k and length n
$\begin{bmatrix} a \\ b \end{bmatrix}_q$	the q-ary Gaussian coefficient

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Bibliography

[AM69] Michael F Atiyah and Ian G MacDonald. *Introduction to commutative algebra*. CRC Press, 1969.

[Bal12] Simeon Ball. On sets of vectors of a finite vector space in which every subset of basis size is a basis. *J. Eur. Math. Soc.*, 14:733–748, 2012.

[BCJR74] Lalit Bahl, John Cocke, Frederick Jelinek, and Josef Raviv. Optimal decoding of linear codes for minimizing symbol error rate (corresp.). *IEEE Transactions on information theory*, 20(2):284–287, 1974.

[BD18] Peter Beelen and Mrinmoy Datta. Generalized hamming weights of affine cartesian codes. *Finite Fields and Their Applications*, 51:130–145, 2018.

[BGG78] Kenneth Bogart, Don Goldberg, and Jean Gordon. An elementary proof of the MacWilliams theorem on equivalence of codes. *Information and control*, 37(1):19–22, 1978.

[BGL15] Aleams Barra and Heide Gluesing-Luerssen. MacWilliams Extension theorems and the local-global property for codes over Frobenius rings. *Journal of Pure and Applied Algebra*, 219(4):703–728, 2015.

[BGLR21] Eimear Byrne, Heide Gluesing-Luerssen, and Alberto Ravagnani. Fundamental properties of sum-rank-metric codes. *IEEE Transactions on Information Theory*, 67(10):6456–6475, 2021.

[BGLR22] Eimear Byrne, Heide Gluesing-Luerssen, and Alberto Ravagnani. Anticodes in the sum-rank metric. *Linear Algebra and its Applications*, 643:80–98, 2022.

[BL19] Simeon Ball and Michel Lavrauw. Arcs in finite projective spaces. *EMS Surv. Math. Sci.*, 6(1/2):133–172, 2019.

[BN96] Johannes Buchmann and Stefan Neis. Algorithms for linear algebra problems over principal ideal rings. Technical Report TI-7/96, Fachbereich Informatik Technische Hochschule Darmstadt, 1996.

[Bri10] Thomas Britz. Code enumerators and tutte polynomials. *IEEE transactions on information theory*, 56(9):4350–4358, 2010.

[BW23] Jessica Bariffi and Violetta Weger. Better bounds on the minimal lee distance. *arXiv preprint arXiv:2307.06079*, 2023.

[CEZ98] Gérard Cohen, Sylvia B. Encheva, and Gilles Zémor. Antichain codes. In *Proceedings. 1998 IEEE International Symposium on Information Theory*, page 232, 1998.

[CEZ99] Gérard Cohen, Sylvia B. Encheva, and Gilles Zémor. Antichain codes. *Des. Codes, Cryptogr.*, 18:71–80, 1999.

[CFN17] Sara D Cardell, Marcelo Firer, and Diego Napp. Generalized column distances for convolutional codes. In *2017 IEEE International Symposium on Information Theory (ISIT)*, pages 21–25. IEEE, 2017.

[CFN20] Sara D Cardell, Marcelo Firer, and Diego Napp. Generalized column distances. *IEEE Transactions on Information Theory*, 66(11):6863–6871, 2020.

[CGGU+14] Alain Couvreur, Philippe Gaborit, Valérie Gauthier-Umaña, Ayoub Otmani, and Jean-Pierre Tillich. Distinguisher-based attacks on public-key cryptosystems using reed–solomon codes. *Designs, Codes and Cryptography*, 73:641–666, 2014.

[CHH96] Ioana Constantinescu, Werner Heise, and Thomas Honold. Monomial extensions of isometries between codes over $\mathbb{Z}_m$. In *Proceedings of the 5th International Workshop on Algebraic and Combinatorial Coding Theory (ACCT'96), Unicorn Shumen*, pages 98–104, 1996.

[CK99] Wende Chen and Torleiv Kløve. On the second greedy weight for binary linear codes. In *Fossorier, M., Imai, H., Lin, S., Poli, A. (eds), Applied Algebra, Algebraic Algorithms and Error-Correcting Codes. AAECC 1999.*, volume 1719 of *Lecture Notes in Computer Science*, pages 131–141. Springer, Berlin, Heidelberg, 1999.

[CMGL+22] Eduardo Camps-Moreno, Elisa Gorla, Cristina Landolina, Elisa Lorenzo García, Umberto Martínez-Peñas, and Flavio Salizzoni. Optimal anticodes, MSRD codes, and generalized weights in the sum-rank metric. *IEEE Transactions on Information Theory*, 68(6):3806–3822, 2022.

[CMST21] Han Cai, Ying Miao, Moshe Schwartz, and Xiaohu Tang. A construction of maximally recoverable codes with order-optimal field size. *IEEE Transactions on Information Theory*, 68(1):204–212, 2021.

[CNF19] Sara D Cardell, Diego Napp, and Marcelo Firer. Unrestricted generalized column distances: A wider definition. In *2019 IEEE International Symposium on Information Theory (ISIT)*, pages 2783–2787. IEEE, 2019.

[Cra69] Henry H Crapo. The tutte polynomial. *Aequationes Mathematicae*, 3:211–229, 1969.

[CST+20] Susan M Cooper, Alexandra Seceleanu, Ştefan O Tohăneanu, Maria Vaz Pinto, and Rafael H Villarreal. Generalized minimum distance functions and algebraic invariants of geramita ideals. *Advances in Applied Mathematics*, 112:101940, 2020.

[Del78] Philippe Delsarte. Bilinear forms over a finite field, with applications to coding theory. *Journal of combinatorial theory, Series A*, 25(3):226–241, 1978.

[dlCKWW] Javier de la Cruz, Michael Kiermaier, Alfred Wassermann, and Wolfgang Willems. Algebraic structures of MRD codes. Preprint, available at https://arxiv.org/abs/1502.02711v1.

[dlCKWW16] Javier de la Cruz, Michael Kiermaier, Alfred Wassermann, and Wolfgang Willems. Algebraic structures of MRD codes. *Advances in Mathematics of Communications*, 10(3):499–510, 2016.

[dSP10] Clément de Seguins Pazzis. The affine preservers of non-singular matrices. *Archiv der Mathematik*, 95(4):333–342, 2010.

[Dys19] Sergey Dyshko. The extension theorem for Lee and Euclidean weight codes over integer residue rings. *Designs, Codes and Cryptography*, 87(6):1253–1269, 2019.

[Eis13] David Eisenbud. *Commutative algebra: with a view toward algebraic geometry*, volume 150. Springer Science & Business Media, 2013.

[ER66] Jack Edmonds and Gian-Carlo Rota. Submodular set functions. In *Waterloo combinatorics conference*. University of Waterloo Waterloo, Ontario, 1966.

[For94] G David Forney. Dimension/length profiles and trellis complexity of linear block codes. *IEEE Transactions on information theory*, 40(6):1741–1752, 1994.

[FTW92] Gui Liang Feng, Kenneth K Tzeng, and VK Wei. On the generalized hamming weights of several classes of cyclic cods. *IEEE transactions on information theory*, 38(3):1125–1130, 1992.

[Gab85] Ernest Mukhamedovich Gabidulin. Theory of codes with maximum rank distance. *Problemy peredachi informatsii*, 21(1):3–16, 1985.

[GGMPS23] Elisa Gorla, Elisa Lorenzo García, Umberto Martínez-Peñas, and Flavio Salizzoni. Integer sequences that are generalized weights of a linear code. *arXiv preprint arXiv:2307.06595*, 2023.

[GHM+14] Marcus Greferath, Thomas Honold, Cathy Mc Fadden, Jay A. Wood, and Jens Zumbrägel. MacWilliams' Extension Theorem for bi-invariant weights over finite principal ideal rings. *Journal of Combinatorial Theory, Series A*, 125:177–193, 2014.

[GL09] Heide Gluesing-Luerssen. On isometries for convolutional codes. *Advances in Mathematics of Communications*, 3:179–203, 2009.

[GLJ22] Heide Gluesing-Luerssen and Benjamin Jany. q-polymatroids and their relation to rank-metric codes. *Journal of Algebraic Combinatorics*, 56(3):725–753, 2022.

[GLRS06] Heide Gluesing-Luerssen, Joachim Rosenthal, and Roxana Smarandache. Strongly-MDS convolutional codes. *IEEE Transactions on Information Theory*, 52(2):584–598, 2006.

[GLS04] Heide Gluesing-Luerssen and Wiland Schmale. On cyclic convolutional codes. *Acta Applicandae Mathematica*, 82:183–237, 2004.

[GLS05] Leah Gold, John Little, and Hal Schenck. Cayley–bacharach and evaluation codes on complete intersections. *Journal of Pure and Applied Algebra*, 196(1):91–99, 2005.

[GMFZ13] Marcus Greferath, Cathy Mc Fadden, and Jens Zumbrägel. Characteristics of invariant weights related to code equivalence over rings. *Designs, codes and cryptography*, 66:145–156, 2013.

[GMM+14] Olav Geil, Stefano Martin, Ryutaroh Matsumoto, Diego Ruano, and Yuan Luo. Relative generalized hamming weights of one-point algebraic geometric codes. *IEEE Transactions on Information Theory*, 60(10):5938–5949, 2014.

[GMMCMMP22] Ignacio García-Marco, Irene Márquez-Corbella, Edgar Martínez-Moro, and Yuriko Pitones. Free resolutions and generalized Hamming weights of binary linear codes. *Mathematics*, 10(12):2079, 2022.

[GMPS23] Elisa Gorla, Umberto Martínez-Peñas, and Flavio Salizzoni. Sum-rank metric codes. *arXiv preprint arXiv:2304.12095*, 2023.

[Gor21] Elisa Gorla. Rank-metric codes. In *Concise Encyclopedia of Coding Theory*, pages 227–250. Chapman and Hall/CRC, 2021.

[GR22] Elisa Gorla and Alberto Ravagnani. Generalized weights of codes over rings and invariants of monomial ideals. *arXiv preprint arXiv:2201.05813*, 2022.

[Gre76] Curtis Greene. Weight enumeration and the geometry of linear codes. *Studies in Applied Mathematics*, 55(2):119–128, 1976.

[GS] Daniel R Grayson and Michael E Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[GS20] Sudhir R Ghorpade and Prasant Singh. Pure resolutions, linear codes, and betti numbers. *Journal of Pure and Applied Algebra*, 224(10):106385, 2020.

[GS23a] Elisa Gorla and Flavio Salizzoni. Generalized column weights and equivalences of convolutional codes. *Journal of Algebra and Its Applications*, page 2550072, 2023.

[GS23b] Elisa Gorla and Flavio Salizzoni. Generalized weights of convolutional codes. *IEEE Transactions on Information Theory*, 2023.

[GS24] Elisa Gorla and Flavio Salizzoni. Macwilliams' extension theorem for rank-metric codes. *Journal of Symbolic Computation*, 122:102263, 2024.

[GSJV20] Manuel González-Sarabia, Delio Jaramillo, and Rafael H Villarreal. On the generalized hamming weights of certain reed–muller-type codes. *Analele ştiinţifice ale Universităţii" Ovidius" Constanţa. Seria Matematică*, 28(1):205–217, 2020.

[GSMBVV19] Manuel González-Sarabia, José Martínez-Bernal, Rafael H Villarreal, and Carlos E Vivares. Generalized minimum distance functions. *Journal of Algebraic Combinatorics*, 50:317–346, 2019.

[Han03] Johan P Hansen. Linkage and codes on complete intersections. *Applicable Algebra in Engineering, Communication and Computing*, 14(3):175–185, 2003.

[HKM77] Tor Helleseth, Torleiv Kløve, and Johannes Mykkeltveit. The weight distribution of irreducible cyclic codes with block lengths $n_1((q^\ell - 1)/n)$. *Discrete Math.*, 18:179–211, 1977.

[HKY92] Tor Helleseth, Torleiv Kløve, and Øyvind Ytrehus. Generalized Hamming weights of linear codes. *IEEE Trans. Inform. Theory*, 38(3):1133–1140, 1992.

[HP98] Petra Heijnen and Ruud Pellikaan. Generalized hamming weights of q-ary reed-muller codes. *IEEE Transactions on Information Theory*, 44(1):181–196, 1998.

[HP03] W Cary Huffman and Vera Pless. *Fundamentals of Error-Correcting Codes*. Cambridge University Press, 2003.

[Hua51] Loo-Keng Hua. A theorem on matrices over a sfield and its applications. *Acta Math. Sinica*, 1(2):109–163, 1951.

[JL97] Heeralal Janwa and Arbind K. Lal. On the generalized hamming weights of cyclic codes. *IEEE Transactions on Information Theory*, 43(1):299–308, 1997.

[JP09a] Relinde Jurrius and Ruud Pellikaan. Codes, arrangemens and weight enumerators. *Soria Summer School on Computational Mathematics (S3CM): Applied Computational Algebraic Geomertric Modelling*, 2009.

[JP09b] Relinde Jurrius and Ruud Pellikaan. Extended and generalized weight enumerators. In *Proc. Int. Workshop on Coding and Cryptography WCC*, pages 76–91, 2009.

[JP13] Relinde Jurrius and Ruud Pellikaan. Codes, arrangements and matroids. In *Algebraic geometry modeling in information theory*, pages 219–325. World Scientific, 2013.

[JPV23] Trygve Johnsen, Rakhi Pratihar, and Hugues Verdure. Weight spectra of Gabidulin rank-metric codes and betti numbers. *São Paulo Journal of Mathematical Sciences*, 17(1):208–241, 2023.

[JRV16] Trygve Johnsen, Jan Roksvold, and Hugues Verdure. A generalization of weight polynomials to matroids. *Discrete Mathematics*, 339(2):632–645, 2016.

[JV13] Trygve Johnsen and Hugues Verdure. Hamming weights and Betti numbers of Stanley–Reisner rings associated to matroids. *Applicable Algebra in Engineering, Communication and Computing*, 24(1):73–93, 2013.

[JV14] Trygve Johnsen and Hugues Verdure. Stanley–Reisner resolution of constant weight linear codes. *Designs, codes and cryptography*, 72:471–481, 2014.

[JV21] Trygve Johnsen and Hugues Verdure. Möbius and coboundary polynomials for matroids. *Designs, Codes and Cryptography*, 89:2163–2177, 2021.

[Kai80] Thomas Kailath. *Linear systems*. Prentice-Hall information and system science series. Prentice-Hall, Englewood Cliffs, N.J, 1980.

[Kløv93] Torleiv Kløve. Minimum support weights of binary codes. *IEEE Trans. Inform. Theory*, 39(2):648–654, 1993.

[KMU15] Jun Kurihara, Ryutaroh Matsumoto, and Tomohiko Uyematsu. Relative generalized rank weight of linear codes and its applications to network coding. *IEEE Transactions On information theory*, 61(7):3912–3936, 2015.

[Kön31] Dénes König. Graphok és matrixok. *Mat. Fiz. Lapok*, 38:116–119, 1931.

[Lee15] Kwankyu Lee. Bounds for generalized Hamming weights of general AG codes. *Finite Fields and Their Applications*, 34:265–279, 2015.

[LK05] Hsiao-feng Lu and P Vijay Kumar. A unified construction of space-time codes with optimal rate-diversity tradeoff. *IEEE Transactions on Information Theory*, 51(5):1709–1730, 2005.

[LMVC05] Yuan Luo, Chaichana Mitrpant, A. J. Han Vinck, and Kefei Chen. Some new characters on the wire-tap channel of type II. *IEEE Trans. Inform. Theory*, 51(3):1222–1229, 2005.

[LP20] Julia Lieb and Raquel Pinto. Constructions of MDS convolutional codes using superregular matrices. *Journal of Algebra Combinatorics Discrete Structures and Applications*, 7(1):73–84, 2020.

[LPR21] Julia Lieb, Raquel Pinto, and Joachim Rosenthal. Convolutional codes. In *Concise Encyclopedia of Coding Theory*, pages 197–226. Chapman and Hall/CRC, 2021.

[LW19] Philippe Langevin and Jay Wood. The Extension Theorem for the Lee and Euclidean Weights over $\mathbb{Z}/p^k\mathbb{Z}$. *Journal of Pure and Applied Algebra*, 223(3):922–930, 2019.

[Mac62] Florence Jessie MacWilliams. *Combinatorial problems of elementary Abelian groups*. PhD thesis, Harvard University, 1962.

[Mes85] Roy Meshulam. On the maximal rank in a subspace of matrices. *The Quarterly Journal of Mathematics, Oxford, Second Series*, 36(2):225–229, 1985.

[MP16] Umberto Martínez-Peñas. On the similarities between generalized rank and Hamming weights and their applications to network coding. *IEEE Transactions on Information Theory*, 62(7):4081–4095, 2016.

[MP18] Umberto Martínez-Peñas. Skew and linearized reed–solomon codes and maximum sum rank distance codes over any division ring. *Journal of Algebra*, 504:587–612, 2018.

[MP19] Umberto Martínez-Peñas. Theory of supports for linear codes endowed with the sum-rank metric. *Designs, Codes and Cryptography*, 87(10):2295–2320, 2019.

[MP21] Christian Maire and Aurel Page. Codes from unit groups of division algebras over number fields. *Mathematische Zeitschrift*, 298:327–348, 2021.

[MP22] Umberto Martínez-Peñas. A general family of MSRD codes and PMDS codes with smaller field sizes from extended Moore matrices. *SIAM Journal on Discrete Mathematics*, 36(3):1868–1886, 2022.

[MPK19] Umberto Martínez-Peñas and Frank R Kschischang. Reliable and secure multishot network coding using linearized Reed-Solomon codes. *IEEE Transactions on Information Theory*, 65(8):4785–4803, 2019.

[MPM17] Umberto Martínez-Peñas and Ryutaroh Matsumoto. Relative generalized matrix weights of matrix codes for universal security on wire-tap networks. *IEEE Transactions on Information Theory*, 64(4):2529–2549, 2017.

[MR99] Carlos Munuera and Domingo Ramirez. The second and third generalized Hamming weights of Hermitian codes. *IEEE Transactions on Information Theory*, 45(2):709–712, 1999.

[Mun94] Carlos Munuera. On the generalized Hamming weights of geometric Goppa codes. *IEEE transactions on information theory*, 40(6):2092–2099, 1994.

[Nel24] Noemi Nellen. Integer sequences that are generalized weights of a vector rank metric code. Master's thesis, Universität Basel, 2024.

[Ner22] Alessandro Neri. Twisted linearized Reed-Solomon codes: A skew polynomial framework. *Journal of Algebra*, 609:792–839, 2022.

[NS18] Diego Napp and Filipa Santana. Multi-shot network coding. *Network Coding and Subspace Designs*, pages 91–104, 2018.

[NUF10] Roberto W Nóbrega and Bartolomeu F Uchôa-Filho. Multishot codes for network coding using rank-metric codes. In *2010 Third IEEE International Workshop on Wireless Network Coding*, pages 1–6. IEEE, 2010.

[OLWZ22] Cornelia Ott, Hedongliang Liu, and Antonia Wachter-Zeh. Covering properties of sum-rank metric codes. In *2022 58th Annual Allerton Conference on Communication, Control, and Computing (Allerton)*, pages 1–7. IEEE, 2022.

[OS12] Frederique Oggier and Adnen Sboui. On the existence of generalized rank weights. In *2012 International Symposium on Information Theory and its Applications*, pages 406–410. IEEE, 2012.

[Oxl11] James Oxley. *Matroid Theory*. Oxford University Press, 02 2011.

[PR97] Erez Petrank and Ron M Roth. Is code equivalence easy to decide? *IEEE Transactions on Information Theory*, 43(5):1602–1604, 1997.

[Rav16a] Alberto Ravagnani. Generalized weights: an anticode approach. *Journal of Pure and Applied Algebra*, 220(5):1946–1962, 2016.

[Rav16b] Alberto Ravagnani. Rank-metric codes and their duality theory. *Designs, Codes and Cryptography*, 80(1):197–216, 2016.

[Rav18] Alberto Ravagnani. Duality of codes supported on regular lattices, with an application to enumerative combinatorics. *Designs, Codes and Cryptography*, 86(9):2035–2063, 2018.

[Rot91] Ron M Roth. Maximum-rank array codes and their application to crisscross error correction. *IEEE transactions on Information Theory*, 37(2):328–336, 1991.

[Rotng] Joseph J Rotman. *Advanced Modern Algebra*. Prentice Hall, 2003 (second printing).

[RS99] Joachim Rosenthal and Roxana Smarandache. Maximum distance separable convolutional codes. *Applicable Algebra in Engineering, Communication and Computing*, 10:15–32, 1999.

[RY97] Joachim Rosenthal and Eric Von York. On the generalized Hamming weights of convolutional codes. *IEEE Transactions on Information Theory*, 43(1):330–335, 1997.

[SC95] Changshik Shim and Habong Chung. On the second generalized Hamming weight of the dual code of a double-error-correcting binary BCH code. *IEEE transactions on information theory*, 41(3):805–808, 1995.

[SK20] Mohannad Shehadeh and Frank R Kschischang. Rate-diversity optimal multiblock space-time codes via sum-rank codes. In *2020 IEEE International Symposium on Information Theory (ISIT)*, pages 3055–3060. IEEE, 2020.

[SZ19] Friedrich Martin Schneider and Jens Zumbrägel. MacWilliams' extension theorem for infinite rings. *Proceedings of the American Mathematical Society*, 147:947–961, 2019.

[Szá20] Gábor Szárnyas. Graphs and matrices: A translation of "Graphok és matrixok" by Dénes König (1931). Preprint, 2020. arXiv:2009.03780.

[TRS12] Virtudes Tomás, Joachim Rosenthal, and Roxana Smarandache. Decoding of convolutional codes over the erasure channel. *IEEE Transactions on Information Theory*, 58(1):90–108, 2012.

[Tut47] William T Tutte. A ring in graph theory. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 43, pages 26–40. Cambridge University Press, 1947.

[Tut54] William T Tutte. A contribution to the theory of chromatic polynomials. *Canadian journal of mathematics*, 6:80–91, 1954.

[TV95] Michael A Tsfasman and Serge G Vladut. Geometric approach to higher weights. *IEEE Transactions on Information Theory*, 41(6):1564–1588, 1995.

[TVT13] Ştefan O Tohăneanu and Adam Van Tuyl. Bounding invariants of fat points using a coding theory construction. *Journal of Pure and Applied Algebra*, 217(2):269–279, 2013.

[Ver04] Dirk Vertigan. Latroids and their representation by codes over modules. *Transactions of the American Mathematical Society*, 356(10):3841–3868, 2004.

[Vit67] Andrew Viterbi. Error bounds for convolutional codes and an asymptotically optimum decoding algorithm. *IEEE transactions on Information Theory*, 13(2):260–269, 1967.

[VL71] Jacobus Hendricus Van Lint. *Coding theory*, volume 201. Springer, 1971.

[vT93] Henk CA van Tilborg. Coding theory, a first course. *Eindhoven the Netherlands*, 1993.

[Wan62] Zhe-Xian Wan. A proof of the automorphisms of linear groups over a sfield of characteristic 2. *Sci. Sinica*, 11:1183–1194, 1962.

[Wei91] Victor K Wei. Generalized Hamming weights for linear codes. *IEEE Transactions on Information Theory*, 37(5):1412–1418, 1991.

[Woo99] Jay Wood. Duality for modules over finite rings and applications to coding theory. *American Journal of Mathematics*, 121(3):555–575, 1999.

[Woo08] Jay Wood. Code equivalence characterizes finite frobenius rings. *Proceedings of the American Mathematical Society*, 136(2):699–706, 2008.

[WY93] Victor K. Wei and Kyeongcheol Yang. On the generalized Hamming weights of product codes. *IEEE Trans. Inform. Theory*, 39(5):1709–1713, 1993.

[YLFL15] Minghui Yang, Jin Li, Keqin Feng, and Dongdai Lin. Generalized hamming weights of irreducible cyclic codes. *IEEE Transactions on Information Theory*, 61(9):4905–4913, 2015.

[Yor97] Eric Von York. *Algebraic description and construction of error correcting codes, a systems theory point of view*. PhD thesis, University of Notre Dame, Notre Dame, IN, 1997.