Published in IEEE International Symposium on Signals, Circuits and Systems (ISSCS) 2, 581-584, 2005
which should be used for any reference to this work

A Fast Recursive Implementation of Gabor Filters

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Abstract— **This paper presents two methods, based on the rational approximation of functions, for designing Gabor like IIR filters with reduced computational complexity. The performance of the designed filters is discussed and comparisons with other approaches are made.**

I. INTRODUCTION

Gabor filters [1] are widely used as pre-processors in signal processing applications, especially in computer vision and image processing applications such as pattern recognition, motion analysis, stereoscopy etc. [2]-[5]. Their use in these applications has been motivated partially by the fact that they can model responses of orientation selective cells in the visual cortex. Another motivation of using such filters as feature extractors is that they are capable of extracting both local (spatial) and frequency information from an image with minimum uncertainty, by performing a local frequency analysis. A further important property is that a narrow-band Gabor function closely approximates analytic functions, allowing separate analysis of the magnitude (envelope) and phase characteristics in the spatial or temporal domains. A two-dimensional (2D) separable Gabor filter is described by the impulse response (a 2D complex exponential modulated by a 2D Gaussian window) and respectively the transfer function (a 2D Gaussian shifted at $[\Omega_x, \Omega_y]$ in the (ω_x, ω_y) frequency plane):

$$g(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right)\exp(j(\Omega_x x + \Omega_y y))$$

$$G(\omega_x, \omega_y) = \exp\left(-\frac{(\omega_x - \Omega_x)^2\sigma_x^2}{2} - \frac{(\omega_y - \Omega_y)^2\sigma_y^2}{2}\right) \quad (1)$$

Most applications require filter banks, each filter being described by different central frequency and selectivity. The choice of the filter parameters depends on the application, for example in [5] the parameters were specified as follows:

$$\Omega_x = \Omega_v \cos\varphi_u, \Omega_y = \Omega_v \sin\varphi_u, \Omega_v = 2^{-\frac{v+2}{2}}\pi, \varphi_u = u\pi / N \quad (2)$$
$$v = 1, ..., M,\ u = 1, ..., N$$

where u and v are frequency and orientation indexes, respectively. A wavelet property of $g_{u,v}(x,y)$ is often preferred i.e., $\sigma_v \Omega_v = k$ and the functions are self-similar (scaled, dilated and rotated versions of a mother function).

The major disadvantage of Gabor filtering approaches, when applied as a convolution of a discrete signal of finite length with a window whose elements are given by a discrete Gabor function, is that they are computationally intensive due to the large number of filter coefficients. The high number of operations required by FIR implementation, given by the product of the signal dimension and the filter coefficients of Gabor filters, is critical for those applications that require real-time processing. One possibility to circumvent this disadvantage is to filter the signal in the frequency domain using Fast Fourier Transform (FFT). This can be done by calculating the spectrum of the signal using FFT, multiplying it by the transfer function of the filter and using the inverse FFT to achieve the filtered signal in the original domain. This operation is equivalent to the circular convolution of a finite length signal with the filter's impulse response. The computational complexity of this filtering procedure is of the same order as for an FFT, i.e. $M\log_2(M)$ for an M – sized signal.

It is known that the number of operations required by Gabor FIR filters depends on the effective length of their impulse response. This drawback can be avoided by using IIR filters with a reduced number of coefficients designed by approximating the frequency characteristics of Gabor filters, as introduced in [6]. Since the desired filters are zero phase, the IIR filtering can be achieved only by using a forward-backward filtering approach as in [6]-[8]. Using this filtering procedure the computational complexity will be smaller than when using FFT. The filters presented by Young in [6] are designed by approximating the Gaussian transfer function using a rational approximation given in [7]. Their design procedure is limited to sixth-order filters, due the form of approximation found in the mentioned references, and it was done by transforming a differential equation represented by a Laplace transform in a set of difference equation considered in the Z-domain by means of backward difference technique.

We will present two different methods for designing the filters coefficients, using the Padé and Padé-Chebyshev approximations. Using these approximation methods, it is possible to design filters having arbitrary order and to obtain better approximation performances than obtained in [6]. Since 2D Gabor filters are separable, i.e., $G(\omega_x, \omega_y) = G(\omega_x)G(\omega_y)$, only the mono-dimensional case (1D) will be considered further on, since the 2D filters can be substituted by two cascaded 1D filters operating along each direction.

II. THE DESIGN OF IIR APPROXIMATION OF GABOR FILTERS

To design an IIR approximation of a Gabor filter, it suffices to design an IIR approximation of a Gaussian filter, and then to shift its transfer function to a desired central frequency Ω_0. This is equivalent to multiplying the filter coefficients of the approximated Gaussian filter with a corresponding complex exponential:

$$G(\omega) = e^{-\frac{\omega^2\sigma^2}{2}} \approx H_N(\omega) = \left.\frac{p_0}{\sum_{k=-N}^{N} q_k z^k}\right|_{z=e^{j\omega}}, \ \omega \in (-\pi, \pi] \Rightarrow$$
$$G(\omega - \Omega_0) \approx H_N(\omega - \Omega_0) = \left.\frac{p_0}{Q(e^{-j\Omega_0} z)}\right|_{z=e^{j\omega}} \Leftrightarrow q_k \to q_k e^{-jk\Omega_0} \quad (3)$$

The transfer function of a Gaussian filter is real valued (zero phase) and since its approximation must have the same properties the coefficients of the polynomial $Q(z)$ must feature the symmetry: $q_k = q_{-k}$. Thus the transfer function of the approximated Gaussian filter has the following:

$$H_N(\omega) = p_0 \Big/ \left(q_0 + \sum_{k=1}^{N} 2q_k \cos(k\omega) \right) \quad (4)$$

A transfer function (4) can be obtained by making the change of variable:

$$\omega \to \arccos x, \ \ x \in [-1, 1]$$
$$G(\omega) \to G(x) = \exp\left(-\frac{(\arccos x)^2 \sigma^2}{2} \right) \approx R(x) = a_0 \Big/ \left(\sum_{k=0}^{N} b_k x^k \right) \quad (5)$$

and finding a rational approximation generically denoted for any degree N as $R(x)$ for $G(x)$ and making the reversed change of variable $x = \cos\omega$. Using the trigonometric identities for $\cos^n x$, the resulting transfer function will have the form (4), and the filter coefficients of the approximated Gaussian filter can be identified by inspection. For determining the rational approximation $R(x)$ of $G(x)$ and then the transfer function of the Gaussian filter, Padé and Padé -Chebyshev approximations can be used.

A. Design Using Padé Approximation

The theoretical background of Padé approximation can be found in [10]. In our case the approximation of $G(x)$ consists of a rational function $R(x)$ which is determined from the Taylor series expansion in $x = 1$ that corresponds to $\omega = 0$. This rational function can be found by constraining its value and its first N derivatives at the approximation point to be identical with $G(x)$ and its N derivatives, respectively:

$$G(x) = e^{-\frac{(\arccos x)^2 \sigma^2}{2}} = \sum_{n=0}^{\infty} \frac{1}{n!} G^{(n)}(x)\Big|_{x=1} (x-1)^n = \sum_{n=0}^{\infty} c_n x^n \approx R(x)$$
$$\frac{d^k}{dx^k} G(x)\Big|_{x=1} = \frac{d^k}{dx^k} R(x)\Big|_{x=1}, \ k = 0, ..., N \quad (6)$$

Using this approximation, the transfer functions for IIR Gaussian filters of order 2N (N=1, 2, 3) are given by:

$$H_1(\omega) = 1 \Big/ \left(1 + \sigma^2 - \sigma^2 \cos(\omega)\right)$$
$$H_2(\omega) = 12 \Big/ \left(12 + 15\sigma^2 + 9\sigma^4 - (16\sigma^2 + 12\sigma^4)\cos(\omega) + (\sigma^2 + 3\sigma^4)\cos(2\omega)\right)$$
$$H_3(\omega) = 360 \Big/ \Big(360 + 490\sigma^2 + 420\sigma^4 + 150\sigma^6 - (540\sigma^2 + 585\sigma^4 + 225\sigma^6)\cos(\omega) +$$
$$+ (54\sigma^2 + 180\sigma^4 + 90\sigma^6)\cos(2\omega) - (4\sigma^2 + 15\sigma^4 + 15\sigma^6)\cos(3\omega)\Big) \quad (7)$$

The quality of the approximations can be estimated by means of the root-squared error ($RSE(\sigma, \mathrm{N})$) and the maximum absolute error ($\varepsilon_{\max}(\sigma, \mathrm{N})$):

$$RSE(\sigma, \mathrm{N}) = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G(\omega) - H_N(\omega) \right|^2 d\omega}$$
$$\varepsilon_{\max}(\sigma, \mathrm{N}) = \max\{|G(\omega) - H_N(\omega)|\}, \ \omega \in (-\pi..\pi], \ N = 1, 2, 3 \quad (8)$$

which are represented in Figure 1 for $\sigma \in [1, 8]$. The errors for Gabor filters case are the same as for Gaussian filters.

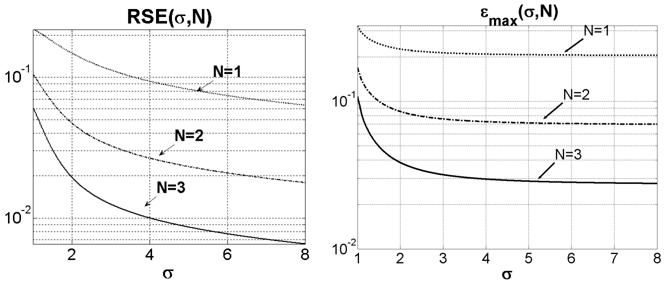

Figure 1. The errors for Padé approximation

B. Design using Padé-Chebyshev approximation

The Padé-Chebyshev approximation consists of a rational function which is determined from the Chebyshev series expansion of $G(x)$ over the interval $[a, 1]$ as detailed in (9), similarly to the Padé approximation:

$$G(x) = \exp\left(-\frac{(\arccos x)^2 \sigma^2}{2} \right) = \sum_{k=0}^{\infty} c_k T(k, \frac{x - (a+1)/2}{(1-a)/2}) \approx R(x)$$
$$R(x) = \frac{a_0}{\sum_{k=0}^{N} b_k T(k, \frac{x - (a+1)/2}{(1-a)/2})} = \frac{a_0}{\sum_{k=0}^{N} d_k x^k}, \ \ x \in [a, 1] \quad (9)$$

where $T(k, \frac{x - (a+1)/2}{(1-a)/2})$ are the Chebyshev polynomials normalized to the interval $[a, 1]$, c_k representing the corresponding coefficients of the Chebyshev series expansion, whereas the coefficients a_0, b_k can be computed by a modified Padé approximation [11].

We have observed that $G(x)$ is better approximated for the entire interval $[-1, 1]$ if its approximation is made for a smaller interval $[a, 1]$, $-1 < a < 1$. The optimal value of a can be obtained, for example, by minimizing the maximum absolute error $\varepsilon_{\max}(\sigma, \mathrm{N})$ from (8). In Figure 2 these optimum values of a calculated for the approximations of Gaussian or Gabor filters of order 2N (N=1, 2, 3) for $\sigma \in [2, 8]$ are represented.

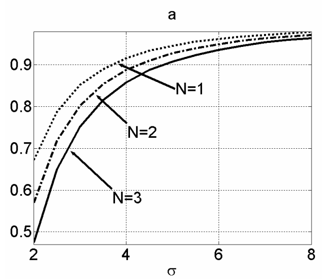

Figure 2. The optimum values of *a*

In Figure 3 the errors (8) are represented, for the optimal values of a.

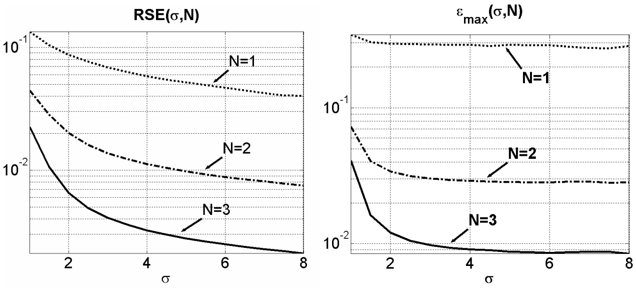

Figure 3. The errors for optimal values of *a* for Padé-Chebyshev approximation

Using this method, the resulted coefficients of the transfer functions (4) for approximations of Gaussian filters of order 2N (N=1, 2, 3) and σ =4 are:

N	p_0	q_0	q_1	q_2	q_3
1	0.2579193521	11.46847736	-11.26854398	0	0
2	0.331263456	158.9938557	-210.5180332	51.86497672	0
3	0.2297112244	1306.560690	-1957.451230	781.3962164	-130.277998

Compared to the Padé approximation, this method leads to better approximation performances from any considered error point of view. The main disadvantage of this method is that the coefficients of $R(x)$, and the coefficients of $H_N(\omega)$, are determined numerically and do not have an explicit dependence on the standard deviation of the approximated function as in the Padé approximation case.

III. Implementation Considerations of Gabor Filters Approximations by Recursive Filters

Assuming that the approximations of Gaussian filters were determined using Padé or Padé-Chebyshev approximations, their transfer function in the Z-domain can be expressed as:

$$H_N(z) = 1\Big/\sum_{k=-N}^{N} q_k z^k = \frac{z^N}{q_N z^{2N} + q_{N-1} z^{2N-1} + \ldots + q_{N-1} z^1 + q_N} \quad (10)$$

the denominator of $H(z)$ being invariant to the change of variable $z^k \to z^{-k}$, having its poles in reciprocal pairs $[\lambda_k, 1/\lambda_k]$, $k = 1, \ldots, N$ as depicted in Figure 4.

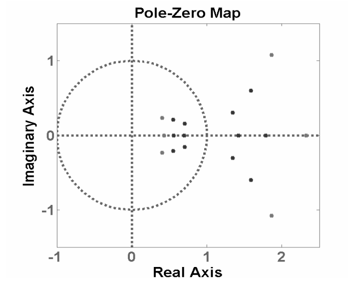

Figure 4. The pole-zero map of H(z) derived from Padé approximation for N=3, and $\sigma = 2, 4, 8$

The transfer function of a Gabor filter has the form (11) and its pole-zero map corresponds to a rotation of the pole-zero map of the approximated Gaussian filter with an angle Ω_0 referred to the origin of the complex plane.

$$H_N(z \cdot e^{-j\Omega_0}) = \frac{z^N e^{-jN\Omega_0}}{c\prod_{k=1}^{N}(z \cdot e^{-j\Omega_0} - \lambda_k)(z \cdot e^{-j\Omega_0} - 1/\lambda_k)}, |\lambda_k| < 1 \quad (11)$$

The transfer function of the form (10), which is non-causal, can always be generically expressed as the product or the sum of two transfer functions, which corresponds to a parallel or a serial implementation, respectively. These functions are involving causal and stable filters $H_{s,p+}(z)$, having assigned the poles confined to the open unit circle, complemented by an anti-causal and stable filters $H_{s,p-}(z)$, having assigned the poles located outside the unit circle:

$$\begin{aligned} H_N(z) &= H_{s+}(z) H_{s-}(z) = \frac{1}{Q_+(z)} \frac{1}{Q_-(z)} \\ H_N(z) &= H_{p+}(z) + H_{p-}(z) = \frac{P_1(z)}{Q_+(z)} + \frac{P_2(z)}{Q_-(z)} \\ Q_-(z) &= Q_+(z^{-1}) \end{aligned} \quad (12)$$

For Gabor filters approximations equation the implementation is similar to (12) by making the change of variable $z \to e^{-j\Omega_0} z$ and illustrated in Figure 5.

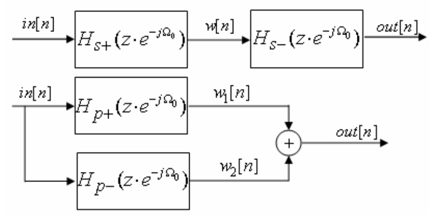

Figure 5. Serial and parallel implementation

A non-causal filter cannot be implemented in a recursive structure unless a forward-backward filtering technique is used, as in [6]-[8]. This technique is equivalent to the serial implementation in which the input sequence is first forward filtered using $H_{s+}(z \cdot e^{-j\Omega_0})$, followed by a signal time-reversal and the application of the filter $H_{s-}(z \cdot e^{-j\Omega_0})$, the final output sequence being time-reversed again. Using the basic properties of the Z-Transform, the time reversal corresponds to the mapping $z \to z^{-1}$, and $H_{s-}(z \cdot e^{-j\Omega_0})$ becomes $H_{s-}(z^{-1} \cdot e^{-j\Omega_0})$ featuring poles inside the unit circle. For the serial implementation where:

$$H_N(z \cdot \mathrm{e}^{-j\Omega_0}) = H_{s+}(z \cdot \mathrm{e}^{-j\Omega_0}) H_{s-}(z \cdot \mathrm{e}^{-j\Omega_0}) = \frac{1}{\left(\sum_{k=0}^{N} b_k e^{jk\Omega_0} z^{-k}\right)\left(\sum_{k=0}^{N} b_k e^{-jk\Omega_0} z^{k}\right)}$$

an input sequence of length M must be forward-backward filtered according to the following recurrence relations:

$$\begin{aligned} w[n] &= \left(in[n] + \sum_{k=1}^{N} b_k e^{jk\Omega_0} w[n-k] \right) / b_0, \; n = 1,\ldots,M \\ out[n] &= \left(w[n] + \sum_{k=1}^{N} b_k e^{-jk\Omega_0} out[n+k] \right) / b_0, \; n = M,\ldots,1 \end{aligned} \quad (13)$$

The parallel implementation can be described in a similar manner and gives an almost identical impulse response if the spatial and spectral aliasing is avoided by imposing the condition (14) presented in [6], thus, the effective length of the impulse response will be smaller than M and its spectrum will not be distorted.

$$1 \le \sigma \le M / 2\pi \quad (14)$$

Compared to parallel implementation the serial implementation does not generally give the same result when the two (backward and forward) filters are applied to an arbitrary signal, in opposite order as presented when zero initial condition of the filters are considered. To illustrate this effect, the difference between the output of a IIR Gaussian filter approximation, when a random signal was filtered forward-backward (FB) and backward-forward (BF), is presented in Figure 6. To minimize this difference, the initial condition must be determined using the approach presented in [8] at the cost of extra computational load, reducing the transient effects due to the finite signal length.

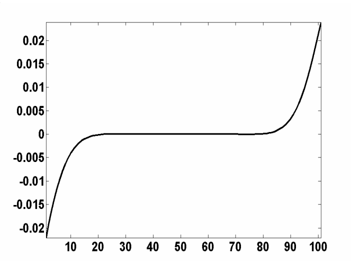

Figure 6. The difference between FB and BF filtering with zero initial conditions

In order to compare above results with those presented in [6], we have calculated the maximum absolute errors ($\varepsilon_{i\max}(\sigma, \mathrm{N})$) of the impulse response. These errors $\varepsilon_{i\max}(\sigma, \mathrm{N})$ are represented in Figure 7. The RSE of the impulse response being the same as those from frequency domain presented in Figure 1 and Figure 3. For the filters of order 2N when N=1,2 these errors are larger than those presented in [6]. For N=3 the errors for the filter designed using Padé approximation has almost the same performances and the filter designed using Padé-Chebyshev approximation has better performances. These approximation performances increase with the order of the filters.

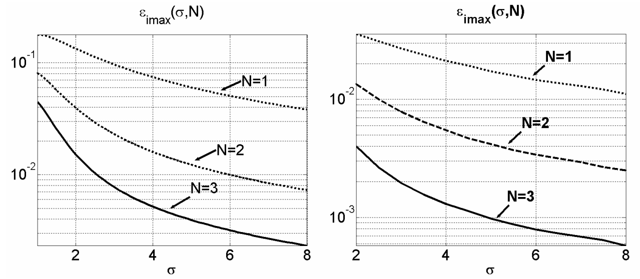

Figure 7. $\varepsilon_{i\max}(\sigma, \mathrm{N})$ for Padé (left) and Padé-Chebyshev (right) approximations

IV. Conclusions

Two methods for designing IIR filters with arbitrary order which approximate the response of the Gabor filters have been presented. Recursive implementation of such filters has the advantage of having low computational complexity i.e. of the order $O(2M(N+1))$ of complex additions and multiplications Based on these method analytical and numerical relations between filters coefficients and approximating Gabor filters parameters have been obtained. The approximations performances are comparable (Padé approximation) or better (Padé-Chebyshev approximation) than similar design approaches.

Acknowledgments

This paper was partly supported by COST 276 Action.

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